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รายงานวิจัยฉบับสมบูรณ์

การลู่เข้าแบบ Painlevé-Kuratowski ของ เซตผลเฉลยสำหรับปัญหาการผนวกเชิงแปรผัน

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สนับสนุนโดย

งบประมาณรายได้มหาวิทยาลัยนเรศวร

ปีงบประมาณ 2560

Executive Summary

1 ความสำคัญและที่มาของปัญหาที่ทำการวิจัย

Very recently inclusion problems were investigated as a generation of equilibrium problems, in order to include a wide class of problems in diverse field such as variational inequalities, vector optimization, game theory, fixed point and coincidence point problems, the Nash equilibrium problem, complementarity problems, trffic equlibria, etc. [6, 8, 9]. It should be noted here that the term "variational inclusion" is understood in different ways in several recent papers. In [5, 10] it means simply multivalued variational inequalities. Variational inclusion problems in [1, 3, 4] are problems of finding zeroes of maximal monotone mappings. In this note the terminology is similar to [6, 8, 9].

As for the stabile results investigated on the convergence of the sequence of map- pings, there are some results for the vector optimization, vector variational inequality problems and vector equilibrium problems with a sequence of sets converging in the sense of Painlevé-Kuratowski (see e.g., [9, 10, 13, 16, 21]. In [13], Huang discussed the convergence of the approximate efficient sets to the efficient sets of vector-valued and set-valued optimization problems in the sense of Painlevé-Kuratowski and Mosco. In [10], Fang et al. investigated the Painlevé-Kuratowski convergence of the solution sets of the perturbed set-valued weak vector variational inequality problems. In [16], Lalitha and Chatterjee investigated the Painlevé-Kuratowski set convergence of the solution sets of a nonconvex vector optimization problem. In [21], Peng and Yang investigated the Painlevé-Kuratowski set convergence of the solution sets of the perturbed vector equilib- rium problems without monotonicity in real linear metric spaces. Very recently, Li et al. [17] concerned with the stability for a generalized Ky Fan inequality when it is perturbed by vector-valued bifunction sequence and set sequence. By continuous convergence of the bifunction sequence and Painlevé-Kuratowski convergence of the set sequence, they established the Painlevé-Kuratowski convergence of the approximate solution mappings of a family of perturbed problems to the corresponding solution mapping of the original problem.

On the other hand, in [3], Anh et al. introduced and studied the parametric gen- eralized quasivariational inclusion problem (QVIP) which contains many kinds of prob- lems such as generalized quasivariational inclusion problems, quasioptimization problems, quasiequilibrium problems, quasivariational inequalities, complementarity problems, vec- tor minimization problems, Nash equilibria, fixed-point and coincidence-point problems, traffic networks, etc. It is well-known that a quasioptimization problem is more general than an optimization one as constraint sets depend on the decision variable as well. It is investigated in [3] the semicontinuity properties of solution maps to (QVIP). However, there are few results to obtain the well-posedness concerned with some special cases of (QVIP) (see, for example, [2, 21] and the references therein).

Motivated by the work reported in above, this paper we aim to establish some results for the solution set of a variational inclusion problem with setvalued mapping and we study Painlevé-Kuratowski convergence of the solution sets with a sequence of mapping converging continuously and sequence of set converging in the sense of Painlevé-Kuratowski.

2 วัตถุประสงค์ของโครงการวิจัย

2.1 To study the solution set of a variational inclusion problem with set valued mapping

2.2 To tudy Painlevé-Kuratowski convergence of the solution sets with a sequence of mapping converging continuously and sequence of set converging in the sense of Painlevé-Kuratowski.

3 วิธีการดำเนินการวิจัย และ สถานที่ทำการทดลอง/เก็บข้อมูล

3.1 วิธีการดำเนินการวิจัย

3.1.1 Find books and publications about the solution set of a variational inclusion problem with set valued mapping

3.1.2 To study the solution set of a variational inclusion problem with set valued mapping

3.1.3 To tudy Painlevé-Kuratowski convergence of the solution sets with a sequence of mapping converging continuously and sequence of set converging in the sense of Painlevé-Kuratowski.

3.2 สถานที่ทำการทดลอง/เก็บข้อมูล : Department of Mathematics, Faculty of science, Naresuan university

4 ระยะเวลาทำการวิจัย และแผนการดำเนินงานตลอดโครงการวิจัย

4.1 ระยะเวลาทำการวิจัย : 1 ปี

4.2 แผนการดำเนินงานตลอดโครงการวิจัย

- **1st Months**

Find books and publications about to study the solution set of a variational inclusion problem with set valued mapping

- **2nd-3rd Months**

Investigate and analyze the sufficient conditions for the solution set of a variational inclusion problem with set valued mapping

- **4th-5th Months**

Investigate and analyze the sufficient conditions for the solution set of a variational inclusion problem with set valued (Cont.)

- **6th Months**

Report the progress of the project to NU

- **7th-8th Months**

We shall give the appropriate and sufficient conditions for the Painlevé-Kuratowski convergence of the solution sets with a sequence of mapping converging continuously and sequence of set converging in the sense of Painlevé-Kuratowski.

- **9th-11th Months**

Prove the Painlevé-Kuratowski convergence of the solution sets with a sequence of mapping converging continuously and sequence of set converging in the sense of Painlevé-Kuratowski.

- **12th Months**

Write the paper and submit for publication in the ISI journal. Report the completed project to NU.

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1.1.1.2 ค่าตอบแทนการปฏิบัติงานนอกเวลาราชการ	10,000
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6.4 เกิดความร่วมมือและแลกเปลี่ยนทางวิชาการระหว่างนักคณิตศาสตร์ไทย และ นัก คณิตศาสตร์ต่างประเทศที่มีชื่อเสียงของโลก ซึ่งนำไปสู่การพัฒนาความเป็นเลิศทางวิชาการของวงการ คณิตศาสตร์ไทยและการพัฒนาประเทศชาติต่อไปในที่สุด

7 แผนการถ่ายทอดเทคโนโลยีหรือผลการวิจัยสู่กลุ่มเป้าหมาย

การเผยแพร่ผลงานทั้งในแง่ของการเข้าร่วมบรรยาย (oral presentation) ในการประชุมวิชาการทั้งในและต่างประเทศ รวมถึงส่งผลงานเพื่อพิจารณาตีพิมพ์ในวารสารระดับนานาชาติ เป็นผลให้ก่อให้เกิดการพัฒนาศึกษาวิจัยอย่างต่อเนื่องทั้งในด้านของกลุ่มนักวิจัยในเชิงทฤษฎี และการต่อยอดเพื่อนำไปประยุกต์ใช้อย่างต่อเนื่องในแขนงวิชาอื่นๆ

8 ผลสำเร็จและความคุ้มค่าของการวิจัยที่คาดว่าจะได้รับ

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9 เอกสารอ้างอิงของโครงการวิจัย

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บทคัดย่อ

ในงานวิจัยนี้เราจะใช้กระบวนการชนิดใหม่ของเสมือนคอนเวกซ์นัยทั่วไปสำหรับการส่งค่าเซต และการใช้ฟังก์ชันสเกลาร์แบบไม่เชิงเส้น ξ_q โดยปราศจากสมมติฐานเกี่ยวกับฟังก์ชันทางเดียวและความมีขอบเขตซึ่งทำให้ได้ผลลัพธ์ที่สามารถหาค่าได้ของผลเฉลยสำหรับปัญหาเชิงดุลยภาพเวกเตอร์แบบสมมาตรและปัญหาเชิงดุลยภาพสเกลาร์แบบสมมาตรสำหรับกระบวนการที่สร้างขึ้นพร้อมกันนี้ก็ยังได้เซตผลเฉลยของความเป็นคอนเวกซ์และได้สร้างตัวอย่างประกอบตัวทฤษฎีบทด้วย

Abstract

In this paper, by proposing a new type of Generalized C-quasiconvexity for the set-valued mappings and using the nonlinear scalarization function ξ_q and its properties, without assumption of monotonicity and boundedness, some existence results of the solutions for the symmetric vector equilibrium problems and symmetric scalar equilibrium problems are established. Moreover, the convexity of solution sets is also investigated. Finally, some examples in order to support our results are provided.

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CHAPTER 1

INTRODUCTION

In 1994, the equilibrium problem was proposed in Blum and Oettli [8].Then it has been intensively studied and extended. After that, more general equilibrium problems (see [8, 7]) have been extended to the case of vector-valued bifunctions, namely, vector equilibrium problems, which provides a unified model of several classes of problems, including, vector variational inequality problems, vector complementarity problems, vector optimization problems and vector saddle point problems, and so on (see, for example, [4, 6, 26, 25, 22, 12]).

The system of vector equilibrium problems, which is a family of equilibrium problems for vector-valued bifunctions defined on a product set was introduced in 2000, by Ansari et al.[5]. Furthermore, its applications in vector optimization problems and Nash equilibrium problem for vector-valued functions were presented by the authors. Nowadays, it is well known that the system of equilibrium problems, systems of vector variational inequalities, system of vector variational-like inequalities, system of optimization problems, fixed point problems and several related topics as special cases (see more [5, 2, 3, 15, 13, 27, 28, 30, 31, 32]) contained in the system of vector equilibrium problems.

On the other hand, the symmetric vector equilibrium problem which is a generalization of the equilibrium problem has been studied by many authors. A main topic of current research is to establish existence theorems (see, for example, [20, 16, 24, 18]). Another important topic is to study the topological properties of the solution sets, as it provides the possibility of continuously moving from one solution to any other solution.

Recently, reducing a vector optimization problem to a scalar optimization problem is a useful approach for analyzing it. The classical scalarization approaches using linear functionals have been already used for studying the existence of solutions of symmetric vector equilibrium problems (see [19, 36]). On the other hand, nonlinear scalarization functions play a vital role in this reduction. The nonlinear scalarization function ξ_q, which was commonly known as the Gerstewitz function in the theory of vector optimization [11, 33], have been used to studying many vector optimization problems. It is well known that the nonlinear scalarization function ξ_q has many good properties, such as continuity, sublinearity, convexity, (strict) monotonicity and so on. These properties have been fully exploited in the literature to deal with various nonconvex problems with vector objectives, such as existence of solutions, gap functions, duality, vector variational principles, well-posedness, vector minimax inequalities and vector network equilibrium problems. However, to the best of our knowledge, there is no paper dealing with the existence theorems for the symmetric vector equilibrium problem by using a nonlinear scalarization method. So, it is natural to raise and give an answer to the following question :

Question : Can one establish existence theorems for the symmetric vector equilibrium problem by using a nonlinear scalarization method ?

Motivated by the works mentioned above, by proposing a new type of C-quasiconvexity for a set-valued mapping together with using a nonlinear scalarization function and its properties, without assumption of monotonicity and boundedness, some existence results of the solutions for the symmetric vector equilibrium problems and symmetric scalar equilibrium problems are established. Moreover, the convexity of solution sets are investigated. Finally, some examples in order to support our results are provided.

CHAPTER 2

PRELIMINARIES

Throughout this paper, let X, Y, E and Z be real Hausdorff topological vector spaces. Let $A \subseteq X$ and $B \subseteq E$ be nonempty closed convex subsets, $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ be two set-valued mappings. Let $C \subseteq Y$ and $P \subseteq Z$ be two closed convex pointed cones with int$C \neq \emptyset$ and int$P \neq \emptyset$. Let Y^* and Z^* be the topological dual spaces of Y and Z, respectively. Let C^* and P^* be the dual cones of C and P, respectively, that is,

$$C^* = \{f \in Y^* : \langle f, y\rangle \geq 0, \text{ for all } y \in C\}$$

and

$$P^* = \{g \in Z^* : \langle g, y\rangle \geq 0, \text{ for all } y \in P\}.$$

The two symmetric vector equilibrium problems under our consideration are as follows: (SVEP_1) : find $(x, y) \in A \times B$ such that

$$\begin{cases} F(x,y,u) \not\subseteq (-\text{int}C), & \forall u \in A, \\ G(x,y,v) \not\subseteq (-\text{int}P), & \forall v \in B, \end{cases} \tag{SVEP$_1$}$$

and (SVEP_2) : find $(x, y) \in A \times B$ such that

$$\begin{cases} F(x,y,u) \cap (-\text{int}C) = \emptyset, & \forall u \in A, \\ G(x,y,v) \cap (-\text{int}P) = \emptyset, & \forall v \in B. \end{cases} \tag{SVEP$_2$}$$

It is clear that the solution set of (SVEP_2) is a subset of (SVEP_1). It is remark that (SVEP_1) is a special problem of the symmetric multivalued vector quasiequilibrium problems studied by Anh and Khan [1]. They obtained some sufficient conditions for the solution existence in topological vector spaces. However, in this paper, we will discuss for the solution existence by utilizing the nonlinear scalarization method.

Remark 2.1 (Special cases). (i) If $C = P, f : A \times B \longrightarrow Y$ and $g : A \times B \longrightarrow Z$ are two single-valued mappings,

$$F(x,y,u) = \{f(u,y) - f(x,y)\}, \forall(x,y,u) \in A \times B \times A$$

and

$$G(x,y,v) = \{g(x,v) - g(x,y)\}, \forall(x,y,v) \in A \times B \times B,$$

then the problem ($SVEP_2$) reduces to the single-valued symmetric vector equilibrium problem considered by [20, 16, 18]:

(ii) If $G \equiv 0$ and $F(x,y,u) = \{f(x,u)\}$ for any $(x,y,u) \in A \times B \times A$, then the problem ($SVEP_2$) is the equilibrium problem which was considered and studied by many authors (for example [21, 10, 5, 7]);

(iii) If $G \equiv 0$ and T is a mapping from A to $L(X,Y)$ where $L(X,Y)$ denotes the space of all continuous linear operators from X to Y, and $F(x,y,u) = \{\langle Tx, u - x\rangle\}$ for any $(x,y,u) \in A \times B \times A$, then the problem ($SVEP_2$) is the classic vector variational inequality problem which was introduced by Giannessi [21].

Now, we are going to recall the nonlinear scalarization fuction $\xi_q : Y \longrightarrow \mathbb{R}$, where $q \in \text{int}C$, as follows:

Definition 2.2. [12, 33] Given a fixed point $q \in \text{int}C$, the nonlinear scalarization function $\xi_q : Y \longrightarrow \mathbb{R}$ is defined by

$$\xi_q(y) = \min\{t \in \mathbb{R} : y \in tq - C\}.$$

In the special case of $Y = \mathbb{R}^l, C = \mathbb{R}^l_+$ and $q = (1,1,\ldots,1) \in \text{int}\mathbb{R}^l_+$, the nonlinear scalarization function can be expressed in the following equivalent form [12, Corollary 1.46]

$$\xi_q(y) = \max_{1\leq i\leq l}\{y_i\}, \forall y = (y_1, y_2, \ldots, y_l) \in \mathbb{R}^l.$$

The following results express some useful properties of the nonlinear scalarization function ξ_q.

Lemma 2.3. [12, Proposition 1.43] For any fixed $q \in \text{int}C$, $y \in Y$ and $r \in \mathbb{R}$. Then

(i) $\xi_q(y) < r \Leftrightarrow y \in rq - \text{intC}$ (*i.e.*, $\xi_q(y) \geq r \Leftrightarrow y \notin rq - \text{intC}$);

(ii) $\xi_q(y) \leq r \Leftrightarrow y \in rq - C$;

(iii) $\xi_q(y) = r \Leftrightarrow y \in rq - \partial C$, where ∂C denotes the boundary of C;

(iv) $\xi_q(rq) = r$;

(v) ξ_q is continuous, positive homogeneous, subadditive and convex on Y;

(vi) ξ_q is monotone (*i.e.*, $y_2 - y_1 \in C \Rightarrow \xi_q(y_1) \leq \xi_q(y_2)$) and strictly monotone (*i.e.*, $y_2 - y_1 \in -\text{intC} \Rightarrow \xi_q(y_1) < \xi_q(y_2)$) (see [12, 33]).

The property (i) of Lemma 2.3 will play a vital role in scalarization. In fact, as the definition of ξ_q, the property (iv) of Lemma 2.3 could be strengthened to that

$$\xi_q(y + rq) = \xi_q(y) + r, \quad \forall y \in Y, r \in \mathbb{R}. \tag{2.1}$$

For any $q \in \text{int}C$, the set C^q defined by

$$C^q := \{y^* \in C^* : \langle y^*, q\rangle = 1\}$$

is a weak*-compact set of Y^* (see [12]). In addition, for the forms of ξ_q which were used in [29, Proposition 2.2] and [12, Corollary 2.1], the following equivalent form of ξ_q can be deduced from both of them.

Proposition 2.4. [9, Proposition 2.2] Let $q \in \text{int}C$. Then for $y \in Y$, $\xi_q(y) = \max_{y^* \in C^q} \langle y^*, y \rangle$.

Proposition 2.5. [9, Proposition 2.3] ξ_q is Lipschitz on Y, and its Lipschitz constant is

$$L := \sup_{y^* \in C^q} \|y^*\| \in \left[\frac{1}{\|q\|}, +\infty\right).$$

The following Example can be found in [[9], Example 2.1].

Example 2.6. (i) In the scalar case of $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, the Lipschitz constant of ξ_q is $L = \frac{1}{q}$ $(q > 0)$. Then,

$$|\xi_q(x) - \xi_q(y)| = \frac{1}{q}|x - y|.$$

for all $x, y \in \mathbb{R}$ and $q > 0$.
(ii) If $Y = \mathbb{R}^2$ and $C = \{(y_1, y_2) \in \mathbb{R}^2 : \frac{1}{4}y_1 \leq y_2 \leq 2y_1\}$. Take $q = (2,3) \in \text{int}C$. Then,

$$C^q = \{(y_1, y_2) \in \mathbb{R} : 2y_1 + 3y_2 = 1, y_1 \in [-0.1, 2]\}.$$

Then, Lipschitz constant is $L = \sup_{y^* \in C^q} \|y^*\| = \|(-2, 1)\| = \sqrt{5}$. Hence,

$$|\xi_q(y) - \xi_q(y')| = \sqrt{5}|y - y'|,$$

for all $y, y' \in \mathbb{R}$.

Definition 2.7. Let X and Y be real Hausdorff topological vector spaces. A set-valued mapping $T : X \longrightarrow 2^Y$ is said to be

(i) *closed* if its graph

$$Gr(T) = \{(x, y) \in X \times Y : y \in T(x)\}$$

is closed in $X \times Y$;

(ii) *upper semicontinuous* (u.s.c) if, for every $x \in X$ and every open set V satisfying $T(x) \subseteq V$, there exists a neighborhood U of x such that

$$T(U) = \bigcup_{y \in U} T(y) \subseteq V;$$

(iii) *lower semicontinuous* (l.s.c) if, for any $x \in X, y \in T(x)$ and any neighborhood V of y, there exists a neighborhood U of x such that

$$T(z) \cap V \neq \emptyset$$

for all $z \in U$.

Lemma 2.8. [34] A set-valued mapping $T : X \longrightarrow 2^Y$ is lower semicontinuous at $x \in X$ if and only if, for any net $\{x_i\}$ such that $x_i \longrightarrow x$ and $y \in T(x)$, there exists a net $\{y_i\}$ with $y_i \in T(x_i)$ such that $y_i \longrightarrow y$.

Now we recall some concepts related to the C-convexity for the set-valued mapping.

Definition 2.9. [36] Let $T : A \longrightarrow 2^Y$ be a set-valued mapping, where A is a nonempty convex subset of X. T is said to be

(i) *C-convex* if for every $z_1, z_2 \in A$ and $t \in [0,1]$,

$$tT(z_1) + (1-t)T(z_2) \subseteq T(tz_1 + (1-t)z_2) + C.$$

(ii) *C-quasiconvex* if for every $z_1, z_2 \in A$ and $t \in [0,1]$, either

$$T(z_1) \subseteq T(tz_1 + (1-t)z_2) + C;$$

or

$$T(z_2) \subseteq T(tz_1 + (1-t)z_2) + C.$$

In this paper, we introduce a new type of C-quasiconvexity for the given set-valued mapping which is a generalization of both C-convexity and C-quasiconvexity.

Definition 2.10. Let $T : A \longrightarrow 2^Y$ be a set-valued mapping, where A is a nonempty convex subset of X. Then T is said to be *Generalized C-quasiconvex* if for every $z_1, z_2 \in D$ and $t \in [0,1]$, either

$$T(z_1) \bigcap \Big(T(tz_1 + (1-t)z_2) + C\Big) \neq \emptyset;$$

or

$$T(z_2) \bigcap \Big(T(tz_1 + (1-t)z_2) + C\Big) \neq \emptyset.$$

Remark 2.11. It can be seen from the above definition that every C-quasiconvex mapping is a generalized C-quasiconvex mapping. However, the converse does not hold in general which can be found in Example 3.10 in Section 3.

The following lemma plays a key role in results reported in many works (for examples [36, 12]). Furthermore, we need it in the sequel.

Lemma 2.12. [14] Let $\{X_i\}_{i\in I}$ be a family of nonempty convex sets where each X_i is contained in a Hausdorff topological vector space E_i. Let $X = \prod_{i\in I} X_i$. For each $i \in I$, let $P_i : X \longrightarrow 2^{X_i}$ be a set-valued mapping such that

(i) for each $i \in I, P_i(x)$ is convex for all $x = (x_i)_{i\in I}$;

(ii) for each $x \in X, x_i \notin P_i(x)$;

(iii) for each $y_i \in X_i, P_i^{-1}(y_i) = \{x \in X : P_i(x) \supseteq \{y_i\}\}$ is open in X;

(iv) for each $i \in I$, there exist a nonempty compact subset N of X and a nonempty compact convex subset B_i of X_i such that for each $x \in X \setminus N$, there is an $i \in I$ satisfying $P_i(x) \cap B_i \neq \emptyset$.

Then there exists $x \in X$ such that $P_i(x) = \emptyset$ for all $i \in I$.

CHAPTER 3

MAIN RESULTS

3.1 Symmetric vector equilibrium problems

In this section, we present the scalar symmetric equilibrium problems which are equivalent to the symmetric vector equilibrium problems ($SVEP_1$) and ($SVEP_2$). The relationships between the solution sets and the existence results for them were established.

For any $q \in \text{int}C$ and $q' \in \text{int}P$, we also consider the following scalar symmetric equilibrium problems: ($SSEP_1(\xi)$): find $(x,y) \in A \times B$, such that

$$\begin{cases} \forall u \in A, \exists z \in F(x,y,u) : \xi_q(z) \geq 0, \\ \forall v \in B, \exists w \in G(x,y,v) : \xi_{q'}(w) \geq 0; \end{cases} \tag{$SSEP_1(\xi)$}$$

and ($SSEP_2(\xi)$) : find $(x,y) \in A \times B$, such that

$$\begin{cases} \xi_q(F(x,y,u)) \subseteq \mathbb{R}_+, \quad \forall u \in A, \\ \xi_{q'}(G(x,y,v)) \subseteq \mathbb{R}_+, \quad \forall v \in B. \end{cases} \tag{$SSEP_2(\xi)$}$$

We denote the solution sets of ($SVEP_1$), ($SVEP_2$), ($SSEP_1(\xi)$) and ($SSEP_2(\xi)$) by S_1, S_2, $S_1(\xi)$ and $S_2(\xi)$, respectively.

Before we give the existence of solutions for ($SVEP_1$) and ($SVEP_2$), we first need the following simple fact which illustrates the relationship between the solution sets S_1 and $S_1(\xi)$.

Lemma 3.1. For any fixed $q \in \text{int}C$ and $q' \in \text{int}P$, the following assertion is valid:

$$S_1 = S_1(\xi).$$

Proof. Firstly, we assume that $(x', y') \in S_1$. Hence for any $u \in A$, there exists $z \in F(x', y', u)$ such that

$$z \notin -\text{int}C.$$

Similarly, for any $v \in B$, there exist $w \in G(x', y', v)$ such that

$$w \notin -\text{int}P.$$

So, it follows from Lemma 2.3 (i) that for any $(u, v) \in A \times B$, there exists (z, w) such that

$$\xi_q(z) \geq 0 \text{ and } \xi_{q'}(w) \geq 0.$$

Therefore, we immediately get that $(x', y') \in S_1(\xi)$. Conversely, assume that $(x', y') \in S_1(\xi)$, then we can prove that $(x', y') \in S_1$ by using Lemma 2.3 with the reverse way of above part. □

Theorem 3.1. Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex pointed cone with $q \in \text{int}C \neq \emptyset$ and $q' \in \text{int}P \neq \emptyset$. Suppose $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ are two set-valued mappings satisfying the following conditions:

(i) for each $(x, y) \in A \times B, F(x, y, x) \cap C \neq \emptyset$, and $G(x, y, y) \cap P \neq \emptyset$;

(ii) for each $(x, y) \in A \times B, F(x, y, \cdot)$ is $C-$quasiconvex on A as well as $G(x, y, \cdot)$ is $P-$quasiconvex on B;

(iii) for each $u \in A$, $F(\cdot, \cdot, u)$ is lower semicontinuous on $A \times B$ and for each $v \in B$, $G(\cdot, \cdot, v)$ is lower semicontinuous on $A \times B$;

(iv) there exists nonempty compact convex sets $D_1 \subset A$ and $D_2 \subset B$ such that for each $(x, y) \in (A \times B) \setminus (D_1 \times D_2)$, there exist $x' \in D_1$ such that $F(x, y, x') \subseteq -\text{int}C$ or $y' \in D_2$ such that $G(x, y, y') \subseteq -\text{int}P$.

Then the set S_1 is nonempty.

Proof. For each $(x,y) \in A \times B$, define $P_1 : A \times B \longrightarrow 2^A$ and $P_2 : A \times B \longrightarrow 2^B$ as follow:

$$P_1(x,y) = \{u \in A : \forall z \in F(x,y,u), \xi_q(z) \notin \mathbb{R}_+\}$$

and

$$P_2(x,y) = \{v \in B : \forall w \in G(x,y,v), \xi_{q'}(w) \notin \mathbb{R}_+\}.$$

We will show that P_1 and P_2 satisfy all conditions of Lemma 2.12. Firstly, we prove that $P_1(x,y)$ and $P_2(x,y)$ are convex for all $(x,y) \in A \times B$. Suppose on the contrary that for some $(x,y) \in A \times B$, $P_1(x,y)$ is not convex. Then there exist $t_1, t_2 \in [0,1]$ with $t_1 + t_2 = 1$ and $u_1, u_2 \in P_1(x,y)$ such that $t_1u_1 + t_2u_2 \notin P_1(x,y)$. This means that

$$\xi_q(z) \in \mathbb{R}_+, \ \exists z \in F(x,y,t_1u_1 + t_2u_2).$$

By assumption (ii), we have either

$$F(x,y,u_1) \subseteq F(x,y,t_1u_1 + t_2u_2) + C,$$

or

$$F(x,y,u_2) \subseteq F(x,y,t_1u_1 + t_2u_2) + C.$$

Hence, we get either

$$\xi_q(F(x,y,u_1)) \subseteq \xi_q(F(x,y,t_1u_1 + t_2u_2)) + \xi_q(C) \subseteq \mathbb{R}_+,$$

or

$$\xi_q(F(x,y,u_2)) \subseteq \xi_q(F(x,y,t_1u_1 + t_2u_2)) + \xi_q(C) \subseteq \mathbb{R}_+,$$

which contradicts $u_1, u_2 \in P_1(x,y)$. Similarly, we can show that $P_2(x,y)$ is convex.

Next, we want to verify condition (ii) of Lemma 2.12, in fact we have to show that for each $(x,y) \in A \times B$, $x \notin P_1(x,y)$ and $y \notin P_2(x,y)$. For each $(x,y) \in$

$A \times B$, it follows from assumption (i) that $F(x,y,x) \cap C \neq \emptyset$ and $G(x,y,y) \cap P \neq \emptyset$. Thus, there exists $(z,w) \in F(x,y,x) \times G(x,y,y)$ such that

$$\xi_q(z) \in \mathbb{R}_+ \text{ and } \xi_{q'}(w) \in \mathbb{R}_+.$$

Invoking the definitions of $P_1(x,y)$ and $P_2(x,y)$, we have

$$x \notin P_1(x,y) \quad \text{and} \quad y \notin P_2(x,y).$$

To prove condition (iii) of Lemma 2.12, assume that $(u,v) \in A \times B$. Note that

$$\left(P_1^{-1}(u)\right)^c = \{(x,y) \in A \times B : \exists z \in F(x,y,u) \text{ s.t. } \xi_q(z) \in \mathbb{R}_+)\}. \tag{3.1}$$

Let $\{(x_i,y_i)\} \subseteq (P_1^{-1}(u))^c$ with $(x_i,y_i) \longrightarrow (x_0,y_0)$. As $F(x_0,y_0,u) \neq \emptyset$, we choose $z_0 \in F(x_0,y_0,u)$. By Lemma 2.8, there exists a net $\{z_i\} \subseteq F(x_i,y_i,u)$ such that $z_i \longrightarrow z_0$. Hence, by using the continuity of ξ_q we get

$$\xi_q(z_i) \longrightarrow \xi_q(z_0).$$

The condition (3.1) yields that $\xi_q(z_0) \geq 0$. Therefore, $(x_0,y_0) \in (P_1^{-1}(u))^c$ and so $(P_1^{-1}(u))^c$ is closed. Thus, we have that $P_1^{-1}(u)$ is open on A. Similarly, we can prove that $P_2^{-1}(v)$ is open on B. This completes the proof of condition (iii) of Lemma 2.12.

Finally, we have to show that condition (iv) of Lemma 2.12 holds. By assumption (iv), there exists nonempty compact set $D_1 \times D_2 \subseteq A \times B$ such that for any $(x,y) \in (A \times B) \setminus (D_1 \times D_2)$, there exits $x' \in D_1$ such that $F(x,y,x') \subseteq -\text{int}C$ or $y' \in D_2$ such that $G(x,y,y') \subseteq -\text{int}P$. Therefore, for each $(z,w) \in F(x,y,x') \times G(x,y,y'), \xi_q(z) \notin \mathbb{R}_+$, or $\xi_{q'}(w) \notin \mathbb{R}_+$. So, we immediately obtain, by the definitions of $P_1(x,y)$ and $P_2(x,y)$, that $x' \in P_1(x,y)$ or $y' \in P_2(x,y)$. This completes the proof of the condition (iv) of Lemma 2.12.

Consequently, the set-valued mappings P_1 and P_2 satisfy all conditions given in Lemma 2.12. So, there exists $(\bar{x},\bar{y}) \in A \times B$ such that

$$P_1(\bar{x},\bar{y}) = \emptyset \quad \text{and} \quad P_2(\bar{x},\bar{y}) = \emptyset.$$

Then, for each $(u,v) \in A \times B$, there exists $(z,w) \in F(\bar{x},\bar{y},u) \times G(\bar{x},\bar{y},v)$ such that

$$\xi_q(z) \in \mathbb{R}_+ \text{ and } \xi_{q'}(w) \in \mathbb{R}_+.$$

Therefore, we have $(\bar{x},\bar{y}) \in S_1(\xi)$. Using Lemma 3.1, we conclude that S_1 is nonempty. □

Remark 3.2. Comparing Theorem 3.1 and the results obtained in Anh and Khan [1], we can see that the main difference is that our techniques is based on the utilizing the nonlinear scalarization method while the mentioned work employed the relaxed quasiconvexities of the multi-valued mappings $F(\cdot,y,\cdot)$ and $G(\cdot,x,\cdot)$ as the main tools.

Now, we give the following example to illustrate Theorem 3.1.

Example 3.3. Let $X = Y = Z = \mathbb{R}, A = B = [0,1], C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ by, for any $(x,y,u) \in A \times B \times A$ and $(x,y,v) \in A \times B \times B$,

$$F(x,y,u) = [x-u,u] \text{ and } G(x,y,v) = [y-v,v].$$

It is clear that (i) given in Theorem 3.1 is satisfied. To establish the assumption (ii) of Theorem 3.1, let $u_1, u_2 \in A$ and $t_1, t_2 \in [0,1]$ with $t_1 + t_2 = 1$. Assume that $u_1 \leq u_2$, then for each $z \in F(x,y,u_1)$,

$$x - u_1 \leq z \leq u_1.$$

Then, we can get that

$$x - t_1u_1 - t_2u_2 \leq z \leq t_1u_1 + t_2u_2,$$

which means

$$F(x,y,u_1) \subseteq F(x,y,t_1u_1+t_2u_2) \subseteq F(x,y,t_1u_1+t_2u_2)+C$$

and so $F(x,y,\cdot)$ is C-quasiconvex on A. By the same fashion, we can show that $G(x,y,\cdot)$ also is P-quasiconvex on B.

Next, we prove the assumption (iii) of Theorem 3.1. Let $u \in A$ be arbitrary fixed. Let $(x',y') \in A \times B, z \in F(x',y',u)$ and U be any neighborhood of z. Then, for each (x,y) in a neighborhood $[x',1] \times B$ of (x',y'), we have

$$F(x,y,u) = [x-u,u] \supseteq [x'-u,u].$$

Thus, $F(x,y,u) \cap U \supseteq \{z\} \neq \emptyset, \forall (x,y) \in [x',1] \times B$ and so the first statement of assumption (iii) of Theorem 3.1 is true. Similarly, we can check that the second one is also true.

Finally, take $D_1 = [\frac{1}{2},1] \subseteq A$ and $D_2 = [\frac{1}{2},1] \subseteq B$. Then, for each $(x,y) \in (A \times B) \setminus (D_1 \times D_2)$, there exit $x' = 1 \in D_1$ and $y' = 1 \in D_2$ such that

$$F(x,y,x') = [x-1,1] \text{ and } G(x,y,y') = [y-1,1].$$

Thus, we have

$$F(x,y,x') \subsetneq [-1,0) \subset -\text{int}C \text{ and } P(x,y,y') \subsetneq [-1,0) \subset -\text{int}P,$$

for all $(x,y) \in (A \times B) \setminus (D_1 \times D_2)$. The assumption (iv) of Theorem 3.1 is proved.

Now, we will show that $S_1 \neq \emptyset$. Taking $(x,y) = (1,1) \in A \times B$ leads to

$$F(x,y,u) = F(1,1,u) = [1-u,u],$$

and

$$G(x,y,v) = G(1,1,v) = [1-v,v],$$

which respectively follows that

$$F(1,1,u) = [1-u,u] \nsubseteq -\text{int}\mathbb{R}_+ = -\text{int}C,\ \forall u \in A,$$

and

$$G(1,1,u) = [1-v,v] \nsubseteq -\text{int}\mathbb{R}_+ = -\text{int}P,\ \forall v \in B.$$

This yields $(1,1) \in \mathrm{S}_1$. □

We give the following examples to show that all of the assumptions of Theorem 3.1 are essential and cannot be dropped.

Example 3.4. (Assumption (i) of Theorem 3.1 is essential.) Let $X = Y = Z = \mathbb{R}, A = B = [0,1], C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ as

$$F(x,y,u) = \Big(-u-\frac{1}{2},u\Big) \text{ and } G(x,y,v) = (-v-\frac{1}{2},v).$$

Firstly, to show that assumption (i) does not hold, take $x = y = 0$. So, we have that

$$F(x,y,x) \cap C = F(0,0,0) \cap \mathbb{R}_+ = (-\frac{1}{2},0) \cap \mathbb{R}_+ = \emptyset.$$

and

$$G(x,y,y) \cap P = G(0,0,0) \cap \mathbb{R}_+ = (-\frac{1}{2},0) \cap \mathbb{R}_+ = \emptyset.$$

We can verify all of the other assumptions of Theorem 3.1. However, the problem SVEP_1 has no solution, i.e. $\mathrm{S}_1(F,G) = \emptyset$ since for each $(x,y) \in A \times B$, there exists $(u,v) = (0,0) \in A \times B$ such that

$$F(x,y,u) = F(x,y,0) = (-\frac{1}{2},0) \subseteq -\text{int}\mathbb{R}_+ = -\text{int}C,$$

and

$$G(x,y,v) = G(x,y,0) = (-\frac{1}{2},0) \subseteq -\text{int}\mathbb{R}_+ = -\text{int}P.$$

The reason is assumption (i) of Theorem 3.1 is violated.

Example 3.5. (Assumption (ii) of Theorem 3.1 is essential) Let $X = Y = Z = \mathbb{R}, A = B = [0,1], C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ by

$$F(x,y,u) = \begin{cases} \{\frac{1}{2}\}, & u = x, \\ (-u-1, u-1], & \text{otherwise,} \end{cases}$$

and

$$G(x,y,v) = \begin{cases} \{\frac{1}{2}\}, & v = y, \\ (-v-1, v-1], & \text{otherwise.} \end{cases}$$

It is clear that assumptions (i), (iii) and (iv) of Theorem 3.1 are satisfied. However, assumption (ii) of Theorem 3.1 is violated. Indeed, let $x = y = \frac{1}{2}$, $t = \frac{1}{2}$, $u_1 = 1$ and $u_2 = 0$. So, we have that

$$F(x,y,u_1) = F(\frac{1}{2}, \frac{1}{2}, 1) = (-2, 1],$$
$$F(x,y,u_2) = F(\frac{1}{2}, \frac{1}{2}, 0) = (-1, 0],$$

and

$$F(x,y,t_1u_1 + t_2u_2) = F(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \{\frac{1}{2}\}.$$

Thus, we have that

$$F(x,y,u_1) \nsubseteq F(x,y,t_1u_1 + t_2u_2) + C,$$

and

$$F(x,y,u_2) \nsubseteq F(x,y,t_1u_1 + t_2u_2) + C.$$

Note that $S_1(F,G) = \emptyset$. Since for each $(x,y) \in A \times B$, there exists $(u,v) = (0,0) \in A \times B$ such that

$$F(x,y,u) = (-2,-1] \subseteq -\text{int}\mathbb{R}_+ = -\text{int}C,$$

and

$$G(x,y,v) = (-2,-1] \subseteq -\text{int}\mathbb{R}_+ = -\text{int}P.$$

Thus, assumption (ii) of Theorem 3.1 cannot be dropped. □

Example 3.6. (Assumption (iii) of Theorem 3.1 is essential) Let $X = Y = Z = \mathbb{R}, A = B = [-1,1], C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ as

$$F(x,y,u) = \begin{cases} \{x-u\}, & x \leq 0, \\ [-\frac{1}{2}, \frac{u}{2}), & \text{otherwise}, \end{cases}$$

and

$$G(x,y,v) = \begin{cases} \{y-v\}, & y \leq 0, \\ [-\frac{1}{2}, \frac{v}{2}), & \text{otherwise}, \end{cases}$$

To show that assumption (iii) of Theorem 3.1 is not satisfied, take $x' = y' = 0, u = 1$. Then, we have $F(x',y',u) = \{-1\}$. Let $z \in F(x',y',u)$, then $(-\frac{3}{2}, -\frac{1}{2})$ is a neighborhood of z. Thus, for each neighborhood V of (x',y') we have

$$(-\frac{3}{2}, -\frac{1}{2}) \cap V = (-\frac{3}{2}, -\frac{1}{2}) \cap [-\frac{1}{2}, \frac{1}{2}) = \emptyset,$$

for all $(x,y) \in V$ with $x > x' = 0$. In fact, it is not hard to show that all of other assumptions in Theorem 3.1 are satisfied, especially assumption (i) and (ii), which are clear by the definitions of F and G. However, $S_1(F,G) = \emptyset$. For each $(x,y) \in A \times B$, consider the following two cases:
if $x \leq 0$, then $F(x,y,u) = \{x-u\} \subseteq -\text{int}\mathbb{R}_+, \forall u \in (0,-1]$,
if $x > 0$, then $F(x,y,u) = [-\frac{1}{2}, \frac{u}{2}) \subseteq -\text{int}\mathbb{R}_+, \forall u \in [-1,0]$. The reason is assumption (iii) of Theorem 3.1 is dropped. □

Example 3.7. (Assumption (iv) of Theorem 3.1 is essential) Let $X = Y = Z = \mathbb{R}, A = B = [0,1], C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ as

$$F(x,y,u) = \begin{cases} (-xu-x, xu), & x = y \neq 0, \\ [-1, xu), & \text{otherwise}, \end{cases}$$

and

$$G(x,y,v) = \begin{cases} (-yv - y, yv), & x = y \neq 0, \\ [-1, yv), & \text{otherwise.} \end{cases}$$

We can show that almost all of the assumptions of Theorem 3.1 are satisfied, unless assumption (iv). To show that assumption (iv) of Theorem 3.1 is violated, for any nonempty compact set $D_1 \times D_2 \subseteq A \times B$, we take $(x,y) = (1,1) \in (A \times B) \setminus (D_1 \times D_2)$. Then, for each $(x', y') \in (D_1 \times D_2)$, we have

$$F(x,y,x') = (-x' - 1, x') \nsubseteq -\text{int}\mathbb{R}_+,$$
$$G(x,y,y') = (-y' - 1, y') \nsubseteq -\text{int}\mathbb{R}_+.$$

Then, the problem SVEP_1 has no solution since for each $(x,y) \in A \times B$, there exists $(u,v) = (0,0) \in A \times B$, such that

$$F(x,y,u) = \begin{cases} (-x, 0) \subseteq -\text{int}\mathbb{R}_+, & x = y \neq 0, \\ [-1, 0) \subseteq -\text{int}\mathbb{R}_+, & \text{otherwise,} \end{cases}$$

and

$$G(x,y,v) = \begin{cases} (-y, 0) \subseteq -\text{int}\mathbb{R}_+, & x = y \neq 0, \\ [-1, 0) \subseteq -\text{int}\mathbb{R}_+, & \text{otherwise.} \end{cases}$$

Hence, assumption (iii) of Theorem 3.1 is essential. □

Now we shall discuss about a link between the solution sets S_2 and $S_2(\zeta)$ for (SVEP_2).

Lemma 3.8. For any fixed $q \in -\text{int}C$ and $q' \in -\text{int}P$,

$$S_2 = S_2(\zeta).$$

Proof. Firstly, we assume that $(x', y') \in S_2(F, G)$, which means

$$F(x', y', u) \cap (-\text{int}C) = \emptyset, \quad \text{for all } u \in A,$$

and

$$G(x', y', v) \cap (-\text{int}P) = \emptyset, \quad \text{for all } v \in B.$$

So, by Lemma 2.3 we obtain that for any $(u, v) \in A \times B$,

$$z \notin -\text{int}C \text{ and } w \notin -\text{int}P$$

for all $(z, w) \in F(x', y', u) \times G(x', y', v)$. So, it follows that, for any pair $(u, v) \in A \times B$,

$$\xi_q(z) \in \mathbb{R}_+ \text{ and } \xi_{q'}(w) \in \mathbb{R}_+$$

for all $(z, w) \in F(x', y', u) \times G(x', y', v)$. Therefore, we get by the definition of ξ_q and $\xi_{q'}$ that

$$\xi_q(F(x', y', u)) \subseteq \mathbb{R}_+, \ \forall u \in A$$

and

$$\xi_{q'}(G(x', y', v)) \subseteq \mathbb{R}_+, \ \forall v \in B.$$

Hence $(x', y') \in S_2(\xi)$. Conversely, assume that $(x', y') \in S_2(\xi)$, then we can prove that $(x', y') \in S_2$ by using the same argument given in the proof of Lemma 2.3. □

Now a result on existence of solutions of the ($SVEP_2$) is verified by making use of the nonlinear scalarization function.

Theorem 3.2. *Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex cones with $q \in intC \neq \emptyset$ and $q' \in intP \neq \emptyset$. Suppose $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ are two set-valued mappings which satisfy the following conditions :*

(i) *for each $(x, y) \in A \times B, F(x, y, x) \subseteq C$ and $G(x, y, y) \subseteq P$;*

(ii) *for each $(x, y) \in A \times B, F(x, y, \cdot)$ is generalized C-quasiconvex on A as well as $G(x, y, \cdot)$ is generalized C-quasiconvex on B*

(iii) *for each* $(x,y,u) \in A \times B \times A$ *with* $F(x,y,u) \cap -intC \neq \emptyset$,

$$z \in F(x,y,u) \Rightarrow z - C \subseteq -intC,$$

and also for each $(x,y,v) \in A \times B \times B$ *with* $G(x,y,v) \cap -intP \neq \emptyset$,

$$w \in G(x,y,v) \Rightarrow w - P \subseteq -intP;$$

(iv) *for each* $u \in A$, $F(\cdot,\cdot,u)$ *is lower semicontinuous on* $A \times B$ *and for each* $v \in B$, $G(\cdot,\cdot,v)$ *is lower semicontinuous on* $A \times B$;

(v) *there exists nonempty compact convex sets* $D_1 \subseteq A$ *and* $D_2 \subseteq B$ *such that for each* $(x,y) \in (A \times B) \setminus (D_1 \times D_2)$, *there exist* $x' \in D_1$ *such that* $F(x,y,x') \cap -intC \neq \emptyset$ *or* $y' \in D_2$ *such that* $G(x,y,y') \cap -intP \neq \emptyset$.

Then the solution set S_2 *is nonemty.*

Proof. Let the set-valued mappings $P_1 : A \times B \longrightarrow 2^A$ and $P_2 : A \times B \longrightarrow 2^B$ be defined by, for any $(x,y) \in A \times B$,

$$P_1(x,y) = \{u \in A : \xi_q(F(x,y,u)) \not\subseteq \mathbb{R}_+\}$$

and

$$P_2(x,y) = \{v \in B : \xi_{q'}(G(x,y,v)) \not\subseteq \mathbb{R}_+\}.$$

We first show that P_1 and P_2 satisfy all the conditions given in Lemma 2.12. Firstly, we prove that $P_1(x,y), P_2(x,y)$ are convex for all $(x,y) \in A \times B$. Assume on the contrary that $P_1(x,y)$ is not convex. Then there exist $t_1, t_2 \in [0,1]$ with $t_1 + t_2 = 1$ and $u_1, u_2 \in P_1(x,y)$ such that $t_1u_1 + t_2u_2 \notin P_1(x,y)$, which gives that

$$\xi_q(F(x,y,t_1u_1 + t_2u_2)) \subseteq \mathbb{R}_+.$$

By assumption (ii), we have either

$$F(x,y,u_1) \bigcap \Big(F(x,y,t_1u_1 + t_2u_2) + C\Big) \neq \emptyset,$$

or

$$F(x,y,u_2)\bigcap\Big(F(x,y,t_1u_1+t_2u_2)+C\Big)\neq\emptyset.$$

It follows that, there is $z\in F(x,y,t_1u_1+t_2u_2)$ such that either

$$z=z_1-c,\ \exists z_1\in F(x,y,u_1),\ \exists c\in C$$

or

$$z=z_2-c',\ \exists z_2\in F(x,y,u_2),\ \exists c'\in C$$

Thus, by assumption (iii), we have either

$$\xi_q(z)=\xi_q(z_1-c)<0,$$

or

$$\xi_q(z)=\xi_q(z_2-c')<0.$$

This contradicts to $t_1u_1+t_2u_2\notin P_1(x,y)$. Similarly, we can show hat $P_2(x,y)$ is convex.

Next, we verify condition (ii) of Lemma 2.12. In fact, we have to show that $x\notin P_1(x,y)$ and $y\notin P_2(x,y)$. Let $(x,y)\in A\times B$. By assumption (i), for each $(z,w)\in F(x,y,x)\times G(x,y,y)$. This says $z\in C$ and $w\in P$, and so

$$z\notin -\text{int}C \text{ and } w\notin -\text{int}P.$$

Hence, by Lemma 2.3 (i), we get that

$$\xi_q(z)\in\mathbb{R}_+ \text{ and } \xi_{q'}(w)\in\mathbb{R}_+$$

for all $(z,w)\in F(x,y,x)\times G(x,y,y)$, which means

$$\xi_q(F(x,y,x))\subseteq\mathbb{R}_+ \quad\text{and}\quad \xi_{q'}(F(x,y,x))\subseteq\mathbb{R}_+.$$

It follows that, for all $(x,y)\in A\times B$,

$$x\notin P_1(x,y) \quad\text{and}\quad y\notin P_2(x,y).$$

To verify condition (iii) of Lemma 2.12, assume that $(u,v) \in A \times B$. Note that

$$\left(P_1^{-1}(u)\right)^c = \{(x,y) \in A \times B : \xi_q(F(x,y,u)) \subseteq \mathbb{R}_+\}.$$

Let $\{(x_i,y_i)\} \in (P_1^{-1}(u))^c$ with $(x_i,y_i) \longrightarrow (x_0,y_0)$. By assumption (iv), for each $z_0 \in F(x_0,y_0,u)$, there exist $z_i \in F(x_i,y_i,u)$ such that $z_i \longrightarrow z_0$. Since $\xi_q(F(x_i,y_i,u)) \subseteq \mathbb{R}_+$, $\xi_q(z_i) \in \mathbb{R}_+$. By the continuity of ξ_q, we get $\xi_q(z_0) \in \mathbb{R}_+$. As z_0 is an arbitrary, we obtain $\xi_q(F(x_0,y_0,u)) \subseteq \mathbb{R}_+$. Thus $(x_0,y_0) \in (P_1^{-1}(u))^c$, and so $(P_1^{-1}(u))^c$ is closed. Hence, we have that $P_1^{-1}(u)$ is open on A. Similarly, we can prove that $P_2^{-1}(v)$ is open on B. Finally, we have to show that condition(iv) of Lemma 2.12 is satisfied. By assumption (v), there exist nonempty compact sets $D_1 \times D_2 \subseteq A \times B$ such that for any $(x,y) \in (A \times B) \setminus (D_1 \times D_2)$, there exists $x' \in D_1$ such that $F(x,y,x') \cap -\text{int}C \neq \emptyset$ or there exits $y' \in D_2$ such that $G(x,y,y') \cap -\text{int}P \neq \emptyset$. Thus, for any $(x,y) \in (A \times B) \setminus (D_1 \times D_2)$, we obtain that $\xi_q(F(x,y,x')) \nsubseteq \mathbb{R}_+$, for some $x' \in D_1$ or $\xi'_q(G(x,y,y')) \nsubseteq \mathbb{R}_+$, for some $y' \in D_2$. So, we immediately obtain by the definition of $P_1(x,y)$ that

$$x' \in P_1(x,y), \text{ for some } x' \in D_1$$

or

$$y' \in P_2(x,y), \text{ for some } y' \in D_2.$$

Therefore, we proved condition (iv) of Lemma 2.12 and so P_1 and P_2 satisfy all conditions of Lemma 2.12. Hence, we can conclude that there exists $(\bar{x},\bar{y}) \in A \times B$ such that

$$P_1(\bar{x},\bar{y}) = \emptyset \quad \text{and} \quad P_2(\bar{x},\bar{y}) = \emptyset.$$

This means there exists $(\bar{x},\bar{y}) \in A \times B$ such that

$$\xi_q(F(\bar{x},\bar{y},u)) \subseteq \mathbb{R}_+, \ \forall u \in A$$

and

$$\xi_{q'}(G(\bar{x},\bar{y},v)) \subseteq \mathbb{R}_+, \ \forall v \in B.$$

Therefore $(\bar{x}, \bar{y}) \in S_2(\xi)$ and so by Lemma 3.1 we completes the proof that S_2 is nonempty. □

Remark 3.9. Comparing Theorem 3.2 and the results obtained in Anh and Khan [1] and Lemma 2.3 in Zhong, Huang and Wong [36], we can see that the main difference is that our techniques is based on the utilizing the nonlinear scalarization method. Further, the C-quasiconvexity of the mapping $F(x,y,\cdot)$ and $G(x,y,\cdot)$ are weakened by generalized C-quasiconvexity. Hence, Theorem 3.2 can be applicable in the following situation while the aforecited results do not work as in the following example.

Example 3.10. Let $X = Y = Z = \mathbb{R}$, $A = B = [0,1]$, $C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ as

$$F(x,y,u) = \begin{cases} (u, u+1), & u \leq x; \\ [-u, 1), & x < u; \end{cases}$$

and

$$G(x,y,v) = \begin{cases} (v, v+1), & v \leq y; \\ [-v, 1), & y < v. \end{cases}$$

Firstly, we show that F is not C-quasiconvex. Taking $x = \frac{1}{2}, u_1 = 1, u_2 = 0$, and $t_1 = t_2 = \frac{1}{2}$, we have the following relations

$$\begin{aligned} F(x,y,u_2) = (0,1) &\not\subseteq \left(\frac{1}{2}, +\infty\right) \\ &= \left(\frac{1}{2}, \frac{3}{2}\right) + \mathbb{R}_+ = F(x,y,t_1u_2 + t_2u_2) + C \end{aligned}$$

and

$$F(x,y,u_1) = [-1,1) \not\subseteq \left(\frac{1}{2}, +\infty\right) = F(x,y,t_1u_2 + t_2u_2) + C.$$

Hence, F is not C-quasiconvex. However, all assumptions given in Theorem 3.2 are satisfied. Firstly, it is clear that the assumption (i) given in Theorem 3.2 is satisfied. Next, we shall establish the assumption (ii). To this end, for fixed

$(x,y) \in A \times B$, let $u_1, u_2 \in A$ and $t_1, t_2 \in [0,1]$ with $t_1 + t_2 = 1$. Assume that $u_1 \leq u_2$. Then, we have the following three cases :

Case I : If $u_1 \leq u_2 \leq x$, then $t_1u_1 + t_2u_2 \leq u_2 \leq x$ and

$$F(x,y,u_2) \cap \Big(F(x,y,t_1u_1+t_2u_2)+C\Big)$$

$$= (u_2, u_2+1) \cap (t_1u_1+t_2u_2, +\infty) \neq \emptyset.$$

Case II : If $u_1 \leq x < u_2$, then we have either $t_1u_1 + t_2u_2 > x$ or $t_1u_1 + t_2u_2 \leq x$. Thus, we have either

$$F(x,y,u_2) \cap \Big(F(x,y,t_1u_1+t_2u_2)+C\Big)$$

$$= [-u_2, 1) \cap [-t_1u_1 - t_2u_2, +\infty) \neq \emptyset,$$

or

$$F(x,y,u_2) \cap \Big(F(x,y,t_1u_1+t_2u_2)+C\Big)$$

$$= [-u_2, 1) \cap (t_1u_1+t_2u_2, +\infty) \neq \emptyset,$$

Case III : If $x < u_1 \leq u_2$, then $t_1u_1 + t_2u_2 > x$, and hence

$$F(x,y,u_2) \cap \Big(F(x,y,t_1u_1+t_2u_2)+C\Big) = [-u_2, 1) \cap [-t_1u_1 - t_2u_2, +\infty) \neq \emptyset.$$

Hence, we have that F is generalized C-quasiconvex. Similarly, we can show that G is generalized C-quasiconvex.

In order to verify assumption (iii), notice that for each element $u \in A$, $F(x,y,u) \cap -\text{int}C \neq \emptyset$ if $u > x$. Assume that $z \in F(x,y,u)$, then z also belongs $[-u,1) \subseteq [-1,1)$. It is not hard to see that $z - C \subseteq -\text{int}C$. Similarly, we can show that G also satisfies this assumption.

Next, to verify assumption (iv) of Theorem 3.2, let $(x', y') \in A \times B$ and $z \in F(x', y', u)$.

Case I : If $u \leq x'$, then $z \in (x', x'+1)$. Let U be arbitrary neighborhood of z. For each (x,y) belongs to neighborhood $(u, x'] \times B$ of (x', y'), we have

$$F(x,y,u) \supseteq (x', x'+1),\ \forall x \in (u, x'].$$

Hence,$F(x,y,u) \cap U \neq \emptyset,\ \forall (x,y) \in (u,x'] \times B$.

Case II : If $u > x'$, then $z \in [-u, 1)$. Let U be arbitrary neighborhood of z. For each (x,y) belongs to neighborhood $[x', 1] \times B$ of (x', y'), we have

$$F(x,y,u) = [-u, 1) \ni z.$$

Hence, $F(x,y,u) \cap U \neq \emptyset,\ \forall (x,y) \in [x',1] \times B$. Therefore, $F(\cdot,\cdot,u)$ satisfies the condition (iv) on A. Similarly, $G(\cdot,\cdot,v)$ satisfies the condition (iv) on B.

Finally, we show that the assumption (iv) of Theorem 3.2 holds, take $D_1 = [\frac{1}{2}, 1] \subset A$ and $D_2 = [\frac{1}{2}, 1] \subset B$. Then, for each element (x,y) belongs $(A \times B) \setminus (D_1 \times D_2)$, there exit $x' = 1 \in D_1$ and $y' = 1 \in D_2$ such that

$$F(x,y,x') \cap -\text{int}C = [-1, \infty) \cap -\text{int}\mathbb{R}_+ = [-1, 0) \neq \emptyset.$$

Therefore, all assumptions in Theorem 3.2 are satisfied. In fact, it is easy to see that $(1,1) \in S_2$.

□

3.2 Convexity of the solution set of Symmetric Vector Equilibrium Problem.

In this section we study the convexity of the solution set S_2. The sufficient conditions for the convexity of S_2 were established. Now, we recall the following useful features, which lead us to obtain our results in the sequel.

Definition 3.11. [17] Let K be a subset of a topological vector space E. A set-valued mapping $F : K \longrightarrow 2^E \setminus \{\emptyset\}$ is said to be a *KKM-mapping* if for any $\{x_1, x_2, ..., x_n\} \subset K$,

$$\text{co}\{x_1, x_2, ..., x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i),$$

where $2^E \setminus \{\emptyset\}$ stands for the family of all nonempty subsets of E, while the notion $\text{co}\{x_1, x_2, ...x_n\}$ denotes the convex hull of $\{x_1, x_2, ..., x_n\}$.

The following well- known lemma plays vital role in our results in this section.

Lemma 3.12. [17] Let K be a subset of a topological vector space E. A set-valued mapping $F : K \longrightarrow 2^X$ be a KKM-mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x \in B} F(x)$ is compact. Then,

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

Theorem 3.3. Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex pointed cone with $q \in \text{int}C \neq \emptyset$ and $q' \in \text{int}P \neq \emptyset$. Suppose $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ are two set-valued mappings which satisfy the following conditions:

(i) for each $(x, y) \in A \times B, F(x, y, x) \subseteq C$ and $G(x, y, y) \subseteq P$;

(ii) for each $(x, y) \in A \times B, F(x, y, \cdot)$ is $C-$ convex on A as well as $G(x, y, \cdot)$ is $P-$ convex on B.

(iii) for each $u \in A$, $F(\cdot, \cdot, u)$ is lower semicontinuous on $A \times B$ and for each $v \in B$, $G(\cdot, \cdot, v)$ is lower semicontinuous on $A \times B$;

(iv) there exists nonempty compact convex set $D_1 \times D_2 \subseteq A \times B$ and compact set $M_1 \times M_2 \subseteq A \times B$ such that for each $(x, y) \in (A \times B) \setminus (M_1 \times M_2)$, there exist $(x', y') \in D_1 \times D_2$ such that $F(x, y, x') \cap -\text{int}C \neq \emptyset$ or $y' \in D_2$ such that $G(x, y, y') \cap -\text{int}P \neq \emptyset$.

Then, the solution set $S_2(\xi)$ is a nonempty compact subset of $A \times B$. Furthermore, S_2 is convex.

Proof. let $q \in -\text{int}C$, and $q' \in -\text{int}P$. Define a set-valued mapping $T : A \times B \longrightarrow A \times B$ by

$$T(z, w) = \{(x, y) \in A \times B : \xi_q(F(x, y, z)) \subseteq \mathbb{R}_+, \ \xi_{q'}(G(x, y, w)) \subseteq \mathbb{R}_+\}.$$

Note that $S_2(\xi) = \bigcap_{(z,w) \in A \times B} T(z, w)$. We assert that the set-valued mapping T fulfils all the assumptions of Lemma 3.12. Firstly, we will show that is a KKM-mapping. Suppose on the contrary, then there exists a subset $\{(x_1, y_1), ..., (x_n, y_n)\}$ of $A \times B$ and $(z, w) \in A \times B$ such that

$$(z, w) \in \text{co}\{(x_1, y_1), ..., (x_n, y_n)\} \setminus \bigcup_{i=1}^{n} T(x_i, y_i).$$

Hence, there exist $\alpha_1, ..., \alpha_n \in \mathbb{R}_+$ such that

$$\sum_{i=1}^{n} \alpha_i = 1 \quad \text{and} \quad (z, w) = \sum_{i=1}^{n} \alpha_i (x_i, y_i).$$

Thus, for all $i = 1, 2, ..., n$, we have

$$\xi_q(F(z, w, x_i)) \nsubseteq \mathbb{R}_+ \ \text{ or } \ \xi_{q'}(G(z, w, y_i)) \nsubseteq \mathbb{R}_+. \tag{3.2}$$

By assumption (ii), the C-quasiconvexity and P-quasiconvexity of F and G are fulfilled respectively, and so

$$F(z, w, x_i) \subseteq F(z, w, z) + C, \quad \text{for some } i = 1, 2, ..., n,$$

and

$$G(z, w, y_i) \subseteq G(z, w, w) + C, \quad \text{for some } i = 1, 2, ..., n.$$

Hence, there is $i \in \{1, 2, ..., n\}$ such that

$$\xi_q(F(z, w, x_i)) \subseteq \xi_q(F(z, w, z)) + \xi_q(C) \subseteq \mathbb{R}_+,$$

and

$$\xi_{q'}G((z, w, y_i)) \subseteq \xi_{q'}(G(z, w, w)) + \xi_{q'}(C) \subseteq \mathbb{R}_+.$$

This contradicts 3.2, and so T is a KKM mapping. Next, we will show that for each $(z, w) \in A \times B$, the set $T(z, w)$ is closed. Let $(z, w) \in A \times B$ and $\{(z_i, w_i)\} \subseteq T(z, w)$ be a net converges to (z_1, w_2). Since $(z_i, w_i) \in T(z, w)$ for all i, we have

$$\xi_q(F(z_i, w_i, z)) \subseteq \mathbb{R}_+ \text{ and } \xi_{q'}(G(z_i, w_i, w)) \subseteq \mathbb{R}_+, \forall i.$$

Let $(h_1, h_2) \in \xi_q(F(z_1, w_2, z)) \times \xi_{q'}(G(z_1, w_2, w))$. Then there exists the pair $(z_2, w_3) \in F(z_1, w_2, z) \times G(z_1, w_2, w)$ such that

$$(h_1, h_2) = (\xi_q(z_2), \xi_{q'}(w_3)).$$

By assumption (iii), there is $(t_i, s_i) \in F(z_i, w_i, z) \times G(z_i, w_i, w)$ such that

$$(t_i, s_i) \longrightarrow (z_2, w_3).$$

Since $(t_i, s_i) \in F(z_i, w_i, z) \times G(z_i, w_i, w)$ for all i, we have

$$\xi_q(t_i) \geq 0 \text{ and } \xi_{q'}(s_i) \geq 0 \text{ for all } i.$$

Therefore, by the continuity of ξ_q and $\xi'_{q'}$, we get

$$h_1 \geq 0 \text{ and } h_2 \geq 0.$$

Since (h_1, h_2) is arbitrary element belongs to $\xi_q(F(z_1, w_2, z)) \times \xi_{q'}(G(z_1, w_2, w))$, we get

$$\xi_q(F(z_1, w_2, z)) \subseteq \mathbb{R}_+ \text{ and } \xi_{q'}(G(z_1, w_2, w)) \subseteq \mathbb{R}_+.$$

Hence, $(z_1, w_2) \in T(z, w)$ and so $T(z, w)$ is closed for any $(z, w) \in A \times B$. Now, all the assumptions of Lemma 3.12 are fulfilled and so $S_2(\xi)$ is nonempty. Further, it follows from assumption (iv) that

$$S_2(\xi) \subseteq M_1 \times M_2,$$

and so it completes the proof that $S_2(\xi)$ is a nonempty compact subset of $A \times B$. By Lemma 3.8, S_2 is also nonempty and compect. Finally, the C-convexity of $F(x,y,\cdot)$ on A and the P-convexity of $G(x,y,\cdot)$ on B imply the set $T(z,w)$ is convex for all $(z,w) \in A \times B$. Hence, the set $S_2(\xi)$ is convex (The intersection of the convex sets is convex.). Therefore, by Lemma 3.8, S_2 is also convex. This completes the proof. □

CHAPTER 4

CONCLUSION

The following results are all results of this research:

1. Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex pointed cone with $q \in \text{int}C \neq \emptyset$ and $q' \in \text{int}P \neq \emptyset$. Suppose $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ are two set-valued mappings satisfying the following conditions:

   (i) for each $(x,y) \in A \times B, F(x,y,x) \cap C \neq \emptyset$, and $G(x,y,y) \cap P \neq \emptyset$;

   (ii) for each $(x,y) \in A \times B, F(x,y,\cdot)$ is $C-$quasiconvex on A as well as $G(x,y,\cdot)$ is $P-$quasiconvex on B;

   (iii) for each $u \in A$, $F(\cdot,\cdot,u)$ is lower semicontinuous on $A \times B$ and for each $v \in B$, $G(\cdot,\cdot,v)$ is lower semicontinuous on $A \times B$;

   (iv) there exists nonempty compact convex sets $D_1 \subset A$ and $D_2 \subset B$ such that for each $(x,y) \in (A \times B) \setminus (D_1 \times D_2)$, there exist $x' \in D_1$ such that $F(x,y,x') \subseteq -\text{int}C$ or $y' \in D_2$ such that $G(x,y,y') \subseteq -\text{int}P$.

   Then the set S_1 is nonempty.

2. Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex cones with $q \in \text{int}C \neq \emptyset$ and $q' \in \text{int}P \neq \emptyset$. Suppose $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ are two set-valued mappings which satisfy the following conditions :

   (i) for each $(x,y) \in A \times B, F(x,y,x) \subseteq C$ and $G(x,y,y) \subseteq P$;

   (ii) for each $(x,y) \in A \times B, F(x,y,\cdot)$ is generalized C-quasiconvex on A as well as $G(x,y,\cdot)$ is generalized C-quasiconvex on B

(iii) for each $(x,y,u) \in A \times B \times A$ with $F(x,y,u) \cap -\text{int}C \neq \emptyset$,

$$z \in F(x,y,u) \Rightarrow z - C \subseteq -\text{int}C,$$

and also for each $(x,y,v) \in A \times B \times B$ with $G(x,y,v) \cap -\text{int}P \neq \emptyset$,

$$w \in G(x,y,v) \Rightarrow w - P \subseteq -\text{int}P;$$

(iv) for each $u \in A$, $F(\cdot,\cdot,u)$ is lower semicontinuous on $A \times B$ and for each $v \in B$, $G(\cdot,\cdot,v)$ is lower semicontinuous on $A \times B$;

(v) there exists nonempty compact convex sets $D_1 \subseteq A$ and $D_2 \subseteq B$ such that for each $(x,y) \in (A \times B) \setminus (D_1 \times D_2)$, there exist $x' \in D_1$ such that $F(x,y,x') \cap -\text{int}C \neq \emptyset$ or $y' \in D_2$ such that $G(x,y,y') \cap -\text{int}P \neq \emptyset$.

Then the solution set S_2 is nonemty.

3. Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex pointed cone with $q \in \text{int}C \neq \emptyset$ and $q' \in \text{int}P \neq \emptyset$. Suppose $F : A \times B \times A \longrightarrow 2^Y$ and $G : A \times B \times B \longrightarrow 2^Z$ are two set-valued mappings which satisfy the following conditions:

   (i) for each $(x,y) \in A \times B, F(x,y,x) \subseteq C$ and $G(x,y,y) \subseteq P$;

   (ii) for each $(x,y) \in A \times B, F(x,y,\cdot)$ is $C-$ convex on A as well as $G(x,y,\cdot)$ is $P-$ convex on B.

   (iii) for each $u \in A$, $F(\cdot,\cdot,u)$ is lower semicontinuous on $A \times B$ and for each $v \in B$, $G(\cdot,\cdot,v)$ is lower semicontinuous on $A \times B$;

   (iv) there exists nonempty compact convex set $D_1 \times D_2 \subseteq A \times B$ and compact set $M_1 \times M_2 \subseteq A \times B$ such that for each $(x,y) \in (A \times B) \setminus (M_1 \times M_2)$, there exist $(x',y') \in D_1 \times D_2$ such that $F(x,y,x') \cap -\text{int}C \neq \emptyset$ or $y' \in D_2$ such that $G(x,y,y') \cap -\text{int}P \neq \emptyset$.

Then, the solution set $S_2(\xi)$ is a nonempty compact subset of $A \times B$. Furthermore, S_2 is convex.

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APPENDIX

On the existence of solutions of symmetric vector equilibrium problems via nonlinear scalarization

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Abstract

In this paper, by proposing a new type of Generalized C-quasiconvexity for the set-valued mappings and using the nonlinear scalarization function ξ_q and its properties, without assumption of monotonicity and boundedness, some existence results of the solutions for the symmetric vector equilibrium problems and symmetric scalar equilibrium problems are established. Moreover, the convexity of solution sets is also investigated. Finally, some examples in order to support our results are provided.

Keywords

Symmetric vector equilibrium problem; nonlinear scalarization function; generalized C-quasiconvexity; upper and lower semicontinuity

Main Subjects

49-XX Calculus of variations and optimal control; optimization

ON THE EXISTENCE OF SOLUTIONS OF SYMMETRIC VECTOR EQUILIBRIUM PROBLEMS VIA NONLINEAR SCALARIZATION

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ABSTRACT. In this paper, by proposing a new type of Generalized C-quasiconvexity for the set-valued mappings and using the nonlinear scalarization function ξ_q and its properties, without assumption of monotonicity and boundedness, some existence results of the solutions for the symmetric vector equilibrium problems and symmetric scalar equilibrium problems are established. Moreover, the convexity of solution sets is also investigated. Finally, some examples in order to support our results are provided.
Keywords: Symmetric vector equilibrium problem, Nonlinear scalarization function, Generalized C-quasiconvexity, Upper and lower semicontinuity.
MSC(2010): 49K40, 90C33, 91B50.

1. Introduction

In 1994, the equilibrium problem was proposed in Blum and Oettli [8].Then it has been intensively studied and extended. After that, more general equilibrium problems (see [8, 7]) have been extended to the case of vector-valued bifunctions, namely, vector equilibrium problems, which provides a unified model of several classes of problems, including, vector variational inequality problems, vector complementarity problems, vector optimization problems and vector saddle point problems, and so on (see, for example, [4, 6, 26, 25, 22, 12]).

The system of vector equilibrium problems, which is a family of equilibrium problems for vector-valued bifunctions defined on a product set was introduced in 2000, by Ansari et al.[5]. Furthermore, its applications in vector optimization problems and Nash equilibrium problem for vector-valued functions were presented by the authors. Nowadays, it is well known that the system of equilibrium problems, systems of vector variational inequalities, system of vector variational-like inequalities, system of optimization problems, fixed point problems and several related topics as special cases (see more [5, 2, 3, 15, 13, 27, 28, 30, 31, 32]) contained in the system of vector equilibrium problems.

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On the other hand, the symmetric vector equilibrium problem which is a generalization of the equilibrium problem has been studied by many authors. A main topic of current research is to establish existence theorems (see, for example, [20, 16, 24, 18]). Another important topic is to study the topological properties of the solution sets, as it provides the possibility of continuously moving from one solution to any other solution.

Recently, reducing a vector optimization problem to a scalar optimization problem is a useful approach for analyzing it. The classical scalarization approaches using linear functionals have been already used for studying the existence of solutions of symmetric vector equilibrium problems (see [19, 36]). On the other hand, nonlinear scalarization functions play a vital role in this reduction. The nonlinear scalarization function ξ_q, which was commonly known as the Gerstewitz function in the theory of vector optimization [11, 33], have been used to studying many vector optimization problems. It is well known that the nonlinear scalarization function ξ_q has many good properties, such as continuity, sublinearity, convexity, (strict) monotonicity and so on. These properties have been fully exploited in the literature to deal with various nonconvex problems with vector objectives, such as existence of solutions, gap functions, duality, vector variational principles, well-posedness, vector minimax inequalities and vector network equilibrium problems. However, to the best of our knowledge, there is no paper dealing with the existence theorems for the symmetric vector equilibrium problem by using a nonlinear scalarization method. So, it is natural to raise and give an answer to the following question :

Question : Can one establish existence theorems for the symmetric vector equilibrium problem by using a nonlinear scalarization method ?

Motivated by the works mentioned above, by proposing a new type of C-quasiconvexity for a set-valued mapping together with using a nonlinear scalarization function and its properties, without assumption of monotonicity and boundedness, some existence results of the solutions for the symmetric vector equilibrium problems and symmetric scalar equilibrium problems are established. Moreover, the convexity of solution sets are investigated. Finally, some examples in order to support our results are provided.

2. Preliminaries

Throughout this paper, let X, Y, E and Z be real Hausdorff topological vector spaces. Let $A \subseteq X$ and $B \subseteq E$ be nonempty closed convex subsets, $F : A \times B \times A \to 2^Y$ and $G : A \times B \times B \to 2^Z$ be two set-valued mappings. Let $C \subseteq Y$ and $P \subseteq Z$ be two closed convex pointed cones with $\text{int}C \neq \emptyset$ and $\text{int}P \neq \emptyset$. Let Y^* and Z^* be the topological dual spaces of Y and Z, respectively. Let C^* and P^* be the dual cones of C and P, respectively, that is,

$$C^* = \{f \in Y^* : \langle f, y \rangle \geq 0, \text{ for all } y \in C\}$$

and

$$P^* = \{g \in Z^* : \langle g, y\rangle \geq 0, \text{ for all } y \in P\}.$$

The two symmetric vector equilibrium problems under our consideration are as follows: (SVEP_1) : find $(x, y) \in A \times B$ such that

$$(\mathrm{SVEP}_1) \qquad \begin{cases} F(x,y,u) \not\subseteq (-\mathrm{int}C), & \forall u \in A, \\ G(x,y,v) \not\subseteq (-\mathrm{int}P), & \forall v \in B, \end{cases}$$

and (SVEP_2) : find $(x, y) \in A \times B$ such that

$$(\mathrm{SVEP}_2) \qquad \begin{cases} F(x,y,u) \cap (-\mathrm{int}C) = \emptyset, & \forall u \in A, \\ G(x,y,v) \cap (-\mathrm{int}P) = \emptyset, & \forall v \in B. \end{cases}$$

It is clear that the solution set of (SVEP_2) is a subset of (SVEP_1). It is remark that (SVEP_1) is a special problem of the symmetric multivalued vector quasiequilibrium problems studied by Anh and Khan [1]. They obtained some sufficient conditions for the solution existence in topological vector spaces. However, in this paper, we will discuss for the solution existence by utilizing the nonlinear scalarization method.

Remark 2.1 (Special cases). (i) If $C = P, f : A \times B \to Y$ and $g : A \times B \to Z$ are two single-valued mappings,

$$F(x,y,u) = \{f(u,y) - f(x,y)\}, \forall (x,y,u) \in A \times B \times A$$

and

$$G(x,y,v) = \{g(x,v) - g(x,y)\}, \forall (x,y,v) \in A \times B \times B,$$

then the problem (SVEP_2) reduces to the single-valued symmetric vector equilibrium problem considered by [20, 16, 18]:

(ii) If $G \equiv 0$ and $F(x,y,u) = \{f(x,u)\}$ for any $(x,y,u) \in A \times B \times A$, then the problem (SVEP_2) is the equilibrium problem which was considered and studied by many authors (for example [21, 10, 5, 7]);

(iii) If $G \equiv 0$ and T is a mapping from A to $L(X,Y)$ where $L(X,Y)$ denotes the space of all continuous linear operators from X to Y, and $F(x,y,u) = \{\langle Tx, u-x\rangle\}$ for any $(x,y,u) \in A \times B \times A$, then the problem (SVEP_2) is the classic vector variational inequality problem which was introduced by Giannessi [21].

Now, we are going to recall the nonlinear scalarization fuction $\xi_q : Y \to \mathbb{R}$, where $q \in \mathrm{int}C$, as follows:

Definition 2.2. [12, 33] Given a fixed point $q \in \mathrm{int}C$, the nonlinear scalarization function $\xi_q : Y \to \mathbb{R}$ is defined by

$$\xi_q(y) = \min\{t \in \mathbb{R} : y \in tq - C\}.$$

In the special case of $Y = \mathbb{R}^l, C = \mathbb{R}^l_+$ and $q = (1, 1, \dots, 1) \in \text{int}\mathbb{R}^l_+$, the nonlinear scalarization function can be expressed in the following equivalent form [12, Corollary 1.46]

$$\xi_q(y) = \max_{1\leq i\leq l}\{y_i\}, \forall y = (y_1, y_2, \dots, y_l) \in \mathbb{R}^l.$$

The following results express some useful properties of the nonlinear scalarization function ξ_q.

Lemma 2.3. [12, Proposition 1.43] For any fixed $q \in \text{int}C$, $y \in Y$ and $r \in \mathbb{R}$. Then

(i) $\xi_q(y) < r \Leftrightarrow y \in rq - \text{int}C$ (*i.e.*, $\xi_q(y) \geq r \Leftrightarrow y \notin rq - \text{int}C$);
(ii) $\xi_q(y) \leq r \Leftrightarrow y \in rq - C$;
(iii) $\xi_q(y) = r \Leftrightarrow y \in rq - \partial C$, where ∂C denotes the boundary of C;
(iv) $\xi_q(rq) = r$;
(v) ξ_q is continuous, positive homogeneous, subadditive and convex on Y;
(vi) ξ_q is monotone (*i.e.*, $y_2 - y_1 \in C \Rightarrow \xi_q(y_1) \leq \xi_q(y_2)$) and strictly monotone (*i.e.*, $y_2 - y_1 \in -\text{int}C \Rightarrow \xi_q(y_1) < \xi_q(y_2)$) (see [12, 33]).

The property (i) of Lemma 2.3 will play a vital role in scalarization. In fact, as the definition of ξ_q, the property (iv) of Lemma 2.3 could be strengthened to that

$$\xi_q(y + rq) = \xi_q(y) + r, \quad \forall y \in Y,\ r \in \mathbb{R}. \tag{2.1}$$

For any $q \in \text{int}C$, the set C^q defined by

$$C^q := \{y^* \in C^* : \langle y^*, q\rangle = 1\}$$

is a weak*-compact set of Y^* (see [12]). In addition, for the forms of ξ_q which were used in [29, Proposition 2.2] and [12, Corollary 2.1], the following equivalent form of ξ_q can be deduced from both of them.

Proposition 2.4. [9, Proposition 2.2] Let $q \in \text{int}C$. Then for $y \in Y$, $\xi_q(y) = \max_{y^*\in C^q}\langle y^*, y\rangle$.

Proposition 2.5. [9, Proposition 2.3] ξ_q is Lipschitz on Y, and its Lipschitz constant is

$$L := \sup_{y^*\in C^q} \|y^*\| \in \left[\frac{1}{\|q\|}, +\infty\right).$$

The following Example can be found in [[9], Example 2.1].

Example 2.6. (i) In the scalar case of $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, the Lipschitz constant of ξ_q is $L = \frac{1}{q}\,(q > 0)$. Then,

$$|\xi_q(x) - \xi_q(y)| = \frac{1}{q}|x - y|.$$

for all $x, y \in \mathbb{R}$ and $q > 0$.
(ii) If $Y = \mathbb{R}^2$ and $C = \{(y_1, y_2) \in \mathbb{R}^2 : \frac{1}{4}y_1 \leq y_2 \leq 2y_1\}$. Take $q = (2, 3) \in \text{int}C$. Then,

$$C^q = \{(y_1, y_2) \in \mathbb{R} : 2y_1 + 3y_2 = 1, y_1 \in [-0.1, 2]\}.$$

Then, Lipschitz constant is $L = \sup_{y^* \in C^q} \|y^*\| = \|(-2,1)\| = \sqrt{5}$. Hence,

$$|\xi_q(y) - \xi_q(y')| = \sqrt{5}|y - y'|,$$

for all $y, y' \in \mathbb{R}$.

Definition 2.7. Let X and Y be real Hausdorff topological vector spaces. A set-valued mapping $T : X \to 2^Y$ is said to be

(i) *closed* if its graph

$$Gr(T) = \{(x,y) \in X \times Y : y \in T(x)\}$$

is closed in $X \times Y$;

(ii) *upper semicontinuous* (u.s.c) if, for every $x \in X$ and every open set V satisfying $T(x) \subseteq V$, there exists a neighborhood U of x such that

$$T(U) = \bigcup_{y \in U} T(y) \subseteq V;$$

(iii) *lower semicontinuous* (l.s.c) if, for any $x \in X, y \in T(x)$ and any neighborhood V of y, there exists a neighborhood U of x such that

$$T(z) \cap V \neq \emptyset$$

for all $z \in U$.

Lemma 2.8. [34] A set-valued mapping $T : X \to 2^Y$ is lower semicontinuous at $x \in X$ if and only if, for any net $\{x_i\}$ such that $x_i \to x$ and $y \in T(x)$, there exists a net $\{y_i\}$ with $y_i \in T(x_i)$ such that $y_i \to y$.

Now we recall some concepts related to the C-convexity for the set-valued mapping.

Definition 2.9. [36] Let $T : A \to 2^Y$ be a set-valued mapping, where A is a nonempty convex subset of X. T is said to be

(i) *C-convex* if for every $z_1, z_2 \in A$ and $t \in [0,1]$,

$$tT(z_1) + (1-t)T(z_2) \subseteq T(tz_1 + (1-t)z_2) + C.$$

(ii) *C-quasiconvex* if for every $z_1, z_2 \in A$ and $t \in [0,1]$, either

$$T(z_1) \subseteq T(tz_1 + (1-t)z_2) + C;$$

or

$$T(z_2) \subseteq T(tz_1 + (1-t)z_2) + C.$$

In this paper, we introduce a new type of C-quasiconvexity for the given set-valued mapping which is a generalization of both C-convexity and C-quasiconvexity.

Definition 2.10. Let $T : A \to 2^Y$ be a set-valued mapping, where A is a nonempty convex subset of X. Then T is said to be *Generalized C-quasiconvex* if for every $z_1, z_2 \in D$ and $t \in [0,1]$, either

$$T(z_1) \bigcap \Big(T(tz_1 + (1-t)z_2) + C\Big) \neq \emptyset;$$

or

$$T(z_2)\bigcap\Big(T(tz_1+(1-t)z_2)+C\Big)\neq\emptyset.$$

Remark 2.11. It can be seen from the above definition that every C-quasiconvex mapping is a generalized C-quasiconvex mapping. However, the converse does not hold in general which can be found in Example 3.12 in Section 3.

The following lemma plays a key role in results reported in many works (for examples [36, 12]). Furthermore, we need it in the sequel.

Lemma 2.12. [14] Let $\{X_i\}_{i\in I}$ be a family of nonempty convex sets where each X_i is contained in a Hausdorff topological vector space E_i. Let $X=\prod_{i\in I}X_i$. For each $i\in I$, let $P_i:X\to 2^{X_i}$ be a set-valued mapping such that

(i) for each $i\in I, P_i(x)$ is convex for all $x=(x_i)_{i\in I}$;
(ii) for each $x\in X, x_i\notin P_i(x)$;
(iii) for each $y_i\in X_i, P_i^{-1}(y_i)=\{x\in X : P_i(x)\supseteq\{y_i\}\}$ is open in X;
(iv) for each $i\in I$, there exist a nonempty compact subset N of X and a nonempty compact convex subset B_i of X_i such that for each $x\in X\backslash N$, there is an $i\in I$ satisfying $P_i(x)\cap B_i\neq\emptyset$.

Then there exists $x\in X$ such that $P_i(x)=\emptyset$ for all $i\in I$.

3. Symmetric vector equilibrium problems

In this section, we present the scalar symmetric equilibrium problems which are equivalent to the symmetric vector equilibrium problems (SVEP_1) and (SVEP_2). The relationships between the solution sets and the existence results for them were established.

For any $q\in\mathrm{int}C$ and $q'\in\mathrm{int}P$, we also consider the following scalar symmetric equilibrium problems: ($\mathrm{SSEP}_1(\xi)$): find $(x,y)\in A\times B$, such that

$$(\mathrm{SSEP}_1(\xi))\qquad\begin{cases}\forall u\in A,\exists z\in F(x,y,u):\xi_q(z)\geq 0,\\ \forall v\in B,\exists w\in G(x,y,v):\xi_{q'}(w)\geq 0;\end{cases}$$

and ($\mathrm{SSEP}_2(\xi)$) : find $(x,y)\in A\times B$, such that

$$(\mathrm{SSEP}_2(\xi))\qquad\begin{cases}\xi_q(F(x,y,u))\subseteq\mathbb{R}_+, & \forall u\in A,\\ \xi_{q'}(G(x,y,v))\subseteq\mathbb{R}_+, & \forall v\in B.\end{cases}$$

We denote the solution sets of (SVEP_1), (SVEP_2), ($\mathrm{SSEP}_1(\xi)$) and ($\mathrm{SSEP}_2(\xi)$) by S_1, S_2, $\mathrm{S}_1(\xi)$ and $\mathrm{S}_2(\xi)$, respectively.

Before we give the existence of solutions for (SVEP_1) and (SVEP_2), we first need the following simple fact which illustrates the relationship between the solution sets S_1 and $\mathrm{S}_1(\xi)$.

Lemma 3.1. For any fixed $q\in\mathrm{int}C$ and $q'\in\mathrm{int}P$, the following assertion is valid:

$$\mathrm{S}_1=\mathrm{S}_1(\xi).$$

Proof. Firstly, we assume that $(x', y') \in \mathrm{S}_1$. Hence for any $u \in A$, there exists $z \in F(x', y', u)$ such that

$$z \notin -\mathrm{int}C.$$

Similarly, for any $v \in B$, there exist $w \in G(x', y', v)$ such that

$$w \notin -\mathrm{int}P.$$

So, it follows from Lemma 2.3 (i) that for any $(u, v) \in A \times B$, there exists (z, w) such that

$$\xi_q(z) \geq 0 \text{ and } \xi_{q'}(w) \geq 0.$$

Therefore, we immediately get that $(x', y') \in \mathrm{S}_1(\xi)$. Conversely, assume that $(x', y') \in \mathrm{S}_1(\xi)$, then we can prove that $(x', y') \in \mathrm{S}_1$ by using Lemma 2.3 with the reverse way of above part. □

Theorem 3.2. Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex pointed cone with $q \in \mathrm{int}C \neq \emptyset$ and $q' \in \mathrm{int}P \neq \emptyset$. Suppose $F : A \times B \times A \to 2^Y$ and $G : A \times B \times B \to 2^Z$ are two set-valued mappings satisfying the following conditions:

(i) for each $(x, y) \in A \times B, F(x, y, x) \cap C \neq \emptyset$, and $G(x, y, y) \cap P \neq \emptyset$;
(ii) for each $(x, y) \in A \times B, F(x, y, \cdot)$ is $C-$quasiconvex on A as well as $G(x, y, \cdot)$ is $P-$quasiconvex on B;
(iii) for each $u \in A$, $F(\cdot, \cdot, u)$ is lower semicontinuous on $A \times B$ and for each $v \in B$, $G(\cdot, \cdot, v)$ is lower semicontinuous on $A \times B$;
(iv) there exists nonempty compact convex sets $D_1 \subset A$ and $D_2 \subset B$ such that for each $(x, y) \in (A \times B) \setminus (D_1 \times D_2)$, there exist $x' \in D_1$ such that $F(x, y, x') \subseteq -\mathrm{int}C$ or $y' \in D_2$ such that $G(x, y, y') \subseteq -\mathrm{int}P$.

Then the set S_1 is nonempty.

Proof. For each $(x, y) \in A \times B$, define $P_1 : A \times B \to 2^A$ and $P_2 : A \times B \to 2^B$ as follow:

$$P_1(x, y) = \{u \in A : \forall z \in F(x, y, u), \xi_q(z) \notin \mathbb{R}_+\}$$

and

$$P_2(x, y) = \{v \in B : \forall w \in G(x, y, v), \xi_{q'}(w) \notin \mathbb{R}_+\}.$$

We will show that P_1 and P_2 satisfy all conditions of Lemma 2.12. Firstly, we prove that $P_1(x, y)$ and $P_2(x, y)$ are convex for all $(x, y) \in A \times B$. Suppose on the contrary that for some $(x, y) \in A \times B$, $P_1(x, y)$ is not convex. Then there exist $t_1, t_2 \in [0, 1]$ with $t_1 + t_2 = 1$ and $u_1, u_2 \in P_1(x, y)$ such that $t_1u_1 + t_2u_2 \notin P_1(x, y)$. This means that

$$\xi_q(z) \in \mathbb{R}_+, \ \exists z \in F(x, y, t_1u_1 + t_2u_2).$$

By assumption (ii), we have either

$$F(x, y, u_1) \subseteq F(x, y, t_1u_1 + t_2u_2) + C,$$

or

$$F(x, y, u_2) \subseteq F(x, y, t_1u_1 + t_2u_2) + C.$$

Hence, we get either

$$\xi_q(F(x,y,u_1)) \subseteq \xi_q(F(x,y,t_1u_1+t_2u_2)) + \xi_q(C) \subseteq \mathbb{R}_+,$$

or

$$\xi_q(F(x,y,u_2)) \subseteq \xi_q(F(x,y,t_1u_1+t_2u_2)) + \xi_q(C) \subseteq \mathbb{R}_+,$$

which contradicts $u_1, u_2 \in P_1(x,y)$. Similarly, we can show that $P_2(x,y)$ is convex.

Next, we want to verify condition (ii) of Lemma 2.12, in fact we have to show that for each $(x,y) \in A \times B$, $x \notin P_1(x,y)$ and $y \notin P_2(x,y)$. For each $(x,y) \in A \times B$, it follows from assumption (i) that $F(x,y,x) \cap C \neq \emptyset$ and $G(x,y,y) \cap P \neq \emptyset$. Thus, there exists $(z,w) \in F(x,y,x) \times G(x,y,y)$ such that

$$\xi_q(z) \in \mathbb{R}_+ \text{ and } \xi_{q'}(w) \in \mathbb{R}_+.$$

Invoking the definitions of $P_1(x,y)$ and $P_2(x,y)$, we have

$$x \notin P_1(x,y) \quad \text{and} \quad y \notin P_2(x,y).$$

To prove condition (iii) of Lemma 2.12, assume that $(u,v) \in A \times B$. Note that

$$\left(P_1^{-1}(u)\right)^c = \{(x,y) \in A \times B : \exists z \in F(x,y,u) \text{ s.t. } \xi_q(z) \in \mathbb{R}_+\}. \tag{3.1}$$

Let $\{(x_i,y_i)\} \subseteq (P_1^{-1}(u))^c$ with $(x_i,y_i) \to (x_0,y_0)$. As $F(x_0,y_0,u) \neq \emptyset$, we choose $z_0 \in F(x_0,y_0,u)$. By Lemma 2.8, there exists a net $\{z_i\} \subseteq F(x_i,y_i,u)$ such that $z_i \to z_0$. Hence, by using the continuity of ξ_q we get

$$\xi_q(z_i) \to \xi_q(z_0).$$

The condition (3.1) yields that $\xi_q(z_0) \geq 0$. Therefore, $(x_0,y_0) \in (P_1^{-1}(u))^c$ and so $(P_1^{-1}(u))^c$ is closed. Thus, we have that $P_1^{-1}(u)$ is open on A. Similarly, we can prove that $P_2^{-1}(v)$ is open on B. This completes the proof of condition (iii) of Lemma 2.12.

Finally, we have to show that condition (iv) of Lemma 2.12 holds. By assumption (iv), there exists nonempty compact set $D_1 \times D_2 \subseteq A \times B$ such that for any $(x,y) \in (A \times B) \setminus (D_1 \times D_2)$, there exits $x' \in D_1$ such that $F(x,y,x') \subseteq -\text{int}C$ or $y' \in D_2$ such that $G(x,y,y') \subseteq -\text{int}P$. Therefore, for each $(z,w) \in F(x,y,x') \times G(x,y,y'), \xi_q(z) \notin \mathbb{R}_+$, or $\xi_{q'}(w) \notin \mathbb{R}_+$. So, we immediately obtain, by the definitions of $P_1(x,y)$ and $P_2(x,y)$, that $x' \in P_1(x,y)$ or $y' \in P_2(x,y)$. This completes the proof of the condition (iv) of Lemma 2.12.

Consequently, the set-valued mappings P_1 and P_2 satisfy all conditions given in Lemma 2.12. So, there exists $(\bar{x},\bar{y}) \in A \times B$ such that

$$P_1(\bar{x},\bar{y}) = \emptyset \quad \text{and} \quad P_2(\bar{x},\bar{y}) = \emptyset.$$

Then, for each $(u,v) \in A \times B$, there exists $(z,w) \in F(\bar{x},\bar{y},u) \times G(\bar{x},\bar{y},v)$ such that

$$\xi_q(z) \in \mathbb{R}_+ \text{ and } \xi_{q'}(w) \in \mathbb{R}_+.$$

Therefore, we have $(\bar{x},\bar{y}) \in \mathrm{S}_1(\xi)$. Using Lemma 3.1, we conclude that S_1 is nonempty. □

Remark 3.3. Comparing Theorem 3.2 and the results obtained in Anh and Khan [1], we can see that the main difference is that our techniques is based on the utilizing the nonlinear scalarization method while the mentioned work employed the relaxed quasiconvexities of the multi-valued mappings $F(\cdot,y,\cdot)$ and $G(\cdot,x,\cdot)$ as the main tools.

Now, we give the following example to illustrate Theorem 3.2.

Example 3.4. Let $X = Y = Z = \mathbb{R}, A = B = [0,1], C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \to 2^Y$ and $G : A \times B \times B \to 2^Z$ by, for any $(x,y,u) \in A \times B \times A$ and $(x,y,v) \in A \times B \times B$,

$$F(x,y,u) = [x-u,u] \text{ and } G(x,y,v) = [y-v,v].$$

It is clear that (i) given in Theorem 3.2 is satisfied. To establish the assumption (ii) of Theorem 3.2, let $u_1,u_2 \in A$ and $t_1,t_2 \in [0,1]$ with $t_1+t_2=1$. Assume that $u_1 \leq u_2$, then for each $z \in F(x,y,u_1)$,

$$x-u_1 \leq z \leq u_1.$$

Then, we can get that

$$x-t_1u_1-t_2u_2 \leq z \leq t_1u_1+t_2u_2,$$

which means

$$F(x,y,u_1) \subseteq F(x,y,t_1u_1+t_2u_2) \subseteq F(x,y,t_1u_1+t_2u_2)+C$$

and so $F(x,y,\cdot)$ is C-quasiconvex on A. By the same fashion, we can show that $G(x,y,\cdot)$ also is P-quasiconvex on B.

Next, we prove the assumption (iii) of Theorem 3.2. Let $u \in A$ be arbitrary fixed. Let $(x',y') \in A \times B, z \in F(x',y',u)$ and U be any neighborhood of z. Then, for each (x,y) in a neighborhood $[x',1] \times B$ of (x',y'), we have

$$F(x,y,u) = [x-u,u] \supseteq [x'-u,u].$$

Thus, $F(x,y,u)\cap U \supseteq \{z\} \neq \emptyset, \forall (x,y) \in [x',1]\times B$ and so the first statement of assumption (iii) of Theorem 3.2 is true. Similarly, we can check that the second one is also true.

Finally, take $D_1 = [\frac{1}{2},1] \subseteq A$ and $D_2 = [\frac{1}{2},1] \subseteq B$. Then, for each $(x,y) \in (A\times B)\setminus(D_1\times D_2)$, there exit $x'=1 \in D_1$ and $y'=1 \in D_2$ such that

$$F(x,y,x') = [x-1,1] \text{ and } G(x,y,y') = [y-1,1].$$

Thus, we have

$$F(x,y,x') \subseteq [-1,0) \subset -\text{int}C \text{ and } P(x,y,y') \subseteq [-1,0) \subset -\text{int}P,$$

for all $(x,y) \in (A\times B)\setminus(D_1\times D_2)$. The assumption (iv) of Theorem 3.2 is proved.

Now, we will show that $\mathrm{S}_1 \neq \emptyset$. Taking $(x,y) = (1,1) \in A \times B$ leads to

$$F(x,y,u) = F(1,1,u) = [1-u,u],$$

and

$$G(x,y,v) = G(1,1,v) = [1-v, v],$$

which respectively follows that

$$F(1,1,u) = [1-u, u] \nsubseteq -\text{int}\mathbb{R}_+ = -\text{int}C, \ \forall u \in A,$$

and

$$G(1,1,u) = [1-v, v] \nsubseteq -\text{int}\mathbb{R}_+ = -\text{int}P, \ \forall v \in B.$$

This yields $(1,1) \in \mathrm{S}_1$. □

We give the following examples to show that all of the assumptions of Theorem 3.2 are essential and cannot be dropped.

Example 3.5. (Assumption (i) of Theorem 3.2 is essential.) Let $X = Y = Z = \mathbb{R}, A = B = [0,1], C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \to 2^Y$ and $G : A \times B \times B \to 2^Z$ as

$$F(x,y,u) = \Big(-u - \frac{1}{2}, u \Big) \text{ and } G(x,y,v) = (-v - \frac{1}{2}, v).$$

Firstly, to show that assumption (i) does not hold, take $x = y = 0$. So, we have that

$$F(x,y,x) \cap C = F(0,0,0) \cap \mathbb{R}_+ = (-\frac{1}{2}, 0) \cap \mathbb{R}_+ = \emptyset.$$

and

$$G(x,y,y) \cap P = G(0,0,0) \cap \mathbb{R}_+ = (-\frac{1}{2}, 0) \cap \mathbb{R}_+ = \emptyset.$$

We can verify all of the other assumptions of Theorem 3.2. However, the problem SVEP_1 has no solution, i.e. $\mathrm{S}_1(F,G) = \emptyset$ since for each $(x,y) \in A \times B$, there exists $(u,v) = (0,0) \in A \times B$ such that

$$F(x,y,u) = F(x,y,0) = (-\frac{1}{2}, 0) \subseteq -\text{int}\mathbb{R}_+ = -\text{int}C,$$

and

$$G(x,y,v) = G(x,y,0) = (-\frac{1}{2}, 0) \subseteq -\text{int}\mathbb{R}_+ = -\text{int}P.$$

The reason is assumption (i) of Theorem 3.2 is violated.

Example 3.6. (Assumption (ii) of Theorem 3.2 is essential) Let $X = Y = Z = \mathbb{R}, A = B = [0,1], C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \to 2^Y$ and $G : A \times B \times B \to 2^Z$ by

$$F(x,y,u) = \begin{cases} \{\frac{1}{2}\}, & u = x, \\ (-u-1, u-1], & \text{otherwise}, \end{cases}$$

and

$$G(x,y,v) = \begin{cases} \{\frac{1}{2}\}, & v = y, \\ (-v-1, v-1], & \text{otherwise}. \end{cases}$$

It is clear that assumptions (i), (iii) and (iv) of Theorem 3.2 are satisfied. However, assumption (ii) of Theorem 3.2 is violated. Indeed, let $x = y = \frac{1}{2}$, $t = \frac{1}{2}$, $u_1 = 1$ and $u_2 = 0$. So, we have that

$$F(x, y, u_1) = F(\frac{1}{2}, \frac{1}{2}, 1) = (-2, 1],$$
$$F(x, y, u_2) = F(\frac{1}{2}, \frac{1}{2}, 0) = (-1, 0],$$

and

$$F(x, y, t_1u_1 + t_2u_2) = F(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \{\frac{1}{2}\}.$$

Thus, we have that

$$F(x, y, u_1) \nsubseteq F(x, y, t_1u_1 + t_2u_2) + C,$$

and

$$F(x, y, u_2) \nsubseteq F(x, y, t_1u_1 + t_2u_2) + C.$$

Note that $S_1(F, G) = \emptyset$. Since for each $(x, y) \in A \times B$, there exists $(u, v) = (0, 0) \in A \times B$ such that

$$F(x, y, u) = (-2, -1] \subseteq -\text{int}\mathbb{R}_+ = -\text{int}C,$$

and

$$G(x, y, v) = (-2, -1] \subseteq -\text{int}\mathbb{R}_+ = -\text{int}P.$$

Thus, assumption (ii) of Theorem 3.2 cannot be dropped. □

Example 3.7. (Assumption (iii) of Theorem 3.2 is essential) Let $X = Y = Z = \mathbb{R}, A = B = [-1, 1], C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \to 2^Y$ and $G : A \times B \times B \to 2^Z$ as

$$F(x, y, u) = \begin{cases} \{x - u\}, & x \leq 0, \\ [-\frac{1}{2}, \frac{u}{2}), & \text{otherwise}, \end{cases}$$

and

$$G(x, y, v) = \begin{cases} \{y - v\}, & y \leq 0, \\ [-\frac{1}{2}, \frac{v}{2}), & \text{otherwise}, \end{cases}$$

To show that assumption (iii) of Theorem 3.2 is not satisfied, take $x' = y' = 0, u = 1$. Then, we have $F(x', y', u) = \{-1\}$. Let $z \in F(x', y', u)$, then $(-\frac{3}{2}, -\frac{1}{2})$ is a neighborhood of z. Thus, for each neighborhood V of (x', y') we have

$$(-\frac{3}{2}, -\frac{1}{2}) \cap V = (-\frac{3}{2}, -\frac{1}{2}) \cap [-\frac{1}{2}, \frac{1}{2}) = \emptyset,$$

for all $(x, y) \in V$ with $x > x' = 0$. In fact, it is not hard to show that all of other assumptions in Theorem 3.2 are satisfied, especially assumption (i) and (ii), which are clear by the definitions of F and G. However, $S_1(F, G) = \emptyset$. For each $(x, y) \in A \times B$, consider the following two cases:
if $x \leq 0$, then $F(x, y, u) = \{x - u\} \subseteq -\text{int}\mathbb{R}_+, \forall u \in (0, -1]$,
if $x > 0$, then $F(x, y, u) = [-\frac{1}{2}, \frac{u}{2}) \subseteq -\text{int}\mathbb{R}_+, \forall u \in [-1, 0]$. The reason is assumption (iii) of Theorem 3.2 is dropped. □

Example 3.8. (Assumption (iv) of Theorem 3.2 is essential) Let $X = Y = Z = \mathbb{R}, A = B = [0,1], C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \to 2^Y$ and $G : A \times B \times B \to 2^Z$ as

$$F(x,y,u) = \begin{cases} (-xu - x, xu), & x = y \neq 0, \\ [-1, xu), & \text{otherwise}, \end{cases}$$

and

$$G(x,y,v) = \begin{cases} (-yv - y, yv), & x = y \neq 0, \\ [-1, yv), & \text{otherwise}. \end{cases}$$

We can show that almost all of the assumptions of Theorem 3.2 are satisfied, unless assumption (iv). To show that assumption (iv) of Theorem 3.2 is violated, for any nonempty compact set $D_1 \times D_2 \subseteq A \times B$, we take $(x,y) = (1,1) \in (A \times B) \setminus (D_1 \times D_2)$. Then, for each $(x', y') \in (D_1 \times D_2)$, we have

$$F(x,y,x') = (-x' - 1, x') \nsubseteq -\text{int}\mathbb{R}_+,$$
$$G(x,y,y') = (-y' - 1, y') \nsubseteq -\text{int}\mathbb{R}_+.$$

Then, the problem SVEP_1 has no solution since for each $(x,y) \in A \times B$, there exists $(u,v) = (0,0) \in A \times B$, such that

$$F(x,y,u) = \begin{cases} (-x, 0) \subseteq -\text{int}\mathbb{R}_+, & x = y \neq 0, \\ [-1, 0) \subseteq -\text{int}\mathbb{R}_+, & \text{otherwise}, \end{cases}$$

and

$$G(x,y,v) = \begin{cases} (-y, 0) \subseteq -\text{int}\mathbb{R}_+, & x = y \neq 0, \\ [-1, 0) \subseteq -\text{int}\mathbb{R}_+, & \text{otherwise}. \end{cases}$$

Hence, assumption (iii) of Theorem 3.2 is essential. □

Now we shall discuss about a link between the solution sets S_2 and $\text{S}_2(\xi)$ for (SVEP_2).

Lemma 3.9. For any fixed $q \in -\text{int}C$ and $q' \in -\text{int}P$,

$$\text{S}_2 = \text{S}_2(\xi).$$

Proof. Firstly, we assume that $(x', y') \in \text{S}_2(F, G)$, which means

$$F(x', y', u) \cap (-\text{int}C) = \emptyset, \quad \text{for all } u \in A,$$

and

$$G(x', y', v) \cap (-\text{int}P) = \emptyset, \quad \text{for all } v \in B.$$

So, by Lemma 2.3 we obtain that for any $(u,v) \in A \times B$,

$$z \notin -\text{int}C \text{ and } w \notin -\text{int}P$$

for all $(z, w) \in F(x', y', u) \times G(x', y', v)$. So, it follows that, for any pair $(u,v) \in A \times B$,

$$\xi_q(z) \in \mathbb{R}_+ \text{ and } \xi_{q'}(w) \in \mathbb{R}_+$$

for all $(z,w) \in F(x',y',u) \times G(x',y',v)$. Therefore, we get by the definition of ξ_q and $\xi_{q'}$ that

$$\xi_q(F(x',y',u)) \subseteq \mathbb{R}_+, \ \forall u \in A$$

and

$$\xi_{q'}(G(x',y',v)) \subseteq \mathbb{R}_+, \ \forall v \in B.$$

Hence $(x',y') \in \mathrm{S}_2(\xi)$. Conversely, assume that $(x',y') \in \mathrm{S}_2(\xi)$, then we can prove that $(x',y') \in \mathrm{S}_2$ by using the same argument given in the proof of Lemma 2.3. □

Now a result on existence of solutions of the (SVEP_2) is verified by making use of the nonlinear scalarization function.

Theorem 3.10. *Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex cones with $q \in intC \neq \emptyset$ and $q' \in intP \neq \emptyset$. Suppose $F : A \times B \times A \to 2^Y$ and $G : A \times B \times B \to 2^Z$ are two set-valued mappings which satisfy the following conditions :*

(i) *for each $(x,y) \in A \times B, F(x,y,x) \subseteq C$ and $G(x,y,y) \subseteq P$;*

(ii) *for each $(x,y) \in A \times B, F(x,y,\cdot)$ is generalized C-quasiconvex on A as well as $G(x,y,\cdot)$ is generalized C-quasiconvex on B*

(iii) *for each $(x,y,u) \in A \times B \times A$ with $F(x,y,u) \cap -intC \neq \emptyset$,*

$$z \in F(x,y,u) \Rightarrow z - C \subseteq -intC,$$

and also for each $(x,y,v) \in A \times B \times B$ with $G(x,y,v) \cap -intP \neq \emptyset$,

$$w \in G(x,y,v) \Rightarrow w - P \subseteq -intP;$$

(iv) *for each $u \in A$, $F(\cdot,\cdot,u)$ is lower semicontinuous on $A \times B$ and for each $v \in B$, $G(\cdot,\cdot,v)$ is lower semicontinuous on $A \times B$;*

(v) *there exists nonempty compact convex sets $D_1 \subseteq A$ and $D_2 \subseteq B$ such that for each $(x,y) \in (A \times B) \setminus (D_1 \times D_2)$, there exist $x' \in D_1$ such that $F(x,y,x') \cap -intC \neq \emptyset$ or $y' \in D_2$ such that $G(x,y,y') \cap -intP \neq \emptyset$.*

Then the solution set S_2 is nonemty.

Proof. Let the set-valued mappings $P_1 : A \times B \to 2^A$ and $P_2 : A \times B \to 2^B$ be defined by, for any $(x,y) \in A \times B$,

$$P_1(x,y) = \{u \in A : \xi_q(F(x,y,u)) \nsubseteq \mathbb{R}_+\}$$

and

$$P_2(x,y) = \{v \in B : \xi_{q'}(G(x,y,v)) \nsubseteq \mathbb{R}_+\}.$$

We first show that P_1 and P_2 satisfy all the conditions given in Lemma 2.12. Firstly, we prove that $P_1(x,y), P_2(x,y)$ are convex for all $(x,y) \in A \times B$. Assume on the contrary that $P_1(x,y)$ is not convex. Then there exist $t_1,t_2 \in [0,1]$ with $t_1+t_2 = 1$ and $u_1,u_2 \in P_1(x,y)$ such that $t_1u_1+t_2u_2 \notin P_1(x,y)$, which gives that

$$\xi_q(F(x,y,t_1u_1+t_2u_2)) \subseteq \mathbb{R}_+.$$

By assumption (ii), we have either

$$F(x,y,u_1)\bigcap\Big(F(x,y,t_1u_1+t_2u_2)+C\Big)\neq\emptyset,$$

or

$$F(x,y,u_2)\bigcap\Big(F(x,y,t_1u_1+t_2u_2)+C\Big)\neq\emptyset.$$

It follows that, there is $z\in F(x,y,t_1u_1+t_2u_2)$ such that either

$$z=z_1-c,\ \exists z_1\in F(x,y,u_1),\ \exists c\in C$$

or

$$z=z_2-c',\ \exists z_2\in F(x,y,u_2),\ \exists c'\in C$$

Thus, by assumption (iii), we have either

$$\xi_q(z)=\xi_q(z_1-c)<0,$$

or

$$\xi_q(z)=\xi_q(z_2-c')<0.$$

This contradicts to $t_1u_1+t_2u_2\notin P_1(x,y)$. Similarly, we can show hat $P_2(x,y)$ is convex.

Next, we verify condition (ii) of Lemma 2.12. In fact, we have to show that $x\notin P_1(x,y)$ and $y\notin P_2(x,y)$. Let $(x,y)\in A\times B$. By assumption (i), for each $(z,w)\in F(x,y,x)\times G(x,y,y)$. This says $z\in C$ and $w\in P$, and so

$$z\notin -\text{int}C \text{ and } w\notin -\text{int}P.$$

Hence, by Lemma 2.3 (i), we get that

$$\xi_q(z)\in\mathbb{R}_+ \text{ and } \xi_{q'}(w)\in\mathbb{R}_+$$

for all $(z,w)\in F(x,y,x)\times G(x,y,y)$, which means

$$\xi_q(F(x,y,x))\subseteq\mathbb{R}_+ \quad\text{and}\quad \xi_{q'}(F(x,y,x))\subseteq\mathbb{R}_+.$$

It follows that, for all $(x,y)\in A\times B$,

$$x\notin P_1(x,y) \quad\text{and}\quad y\notin P_2(x,y).$$

To verify condition (iii) of Lemma 2.12, assume that $(u,v)\in A\times B$. Note that

$$\Big(P_1^{-1}(u)\Big)^c=\{(x,y)\in A\times B:\xi_q(F(x,y,u))\subseteq\mathbb{R}_+\}.$$

Let $\{(x_i,y_i)\}\in(P_1^{-1}(u))^c$ with $(x_i,y_i)\to(x_0,y_0)$. By assumption (iv), for each $z_0\in F(x_0,y_0,u)$, there exist $z_i\in F(x_i,y_i,u)$ such that $z_i\to z_0$. Since $\xi_q(F(x_i,y_i,u))\subseteq\mathbb{R}_+$, $\xi_q(z_i)\in\mathbb{R}_+$. By the continuity of ξ_q, we get $\xi_q(z_0)\in\mathbb{R}_+$. As z_0 is an arbitrary, we obtain $\xi_q(F(x_0,y_0,u))\subseteq\mathbb{R}_+$. Thus $(x_0,y_0)\in(P_1^{-1}(u))^c$, and so $(P_1^{-1}(u))^c$ is closed. Hence, we have that $P_1^{-1}(u)$ is open on A. Similarly, we can prove that $P_2^{-1}(v)$ is open on B. Finally, we have to show that condition(iv) of Lemma 2.12 is satisfied. By assumption (v), there exist nonempty compact sets $D_1\times D_2\subseteq A\times B$ such that for any $(x,y)\in(A\times B)\setminus(D_1\times D_2)$, there exists $x'\in D_1$ such that $F(x,y,x')\cap -\text{int}C\neq\emptyset$ or there exits $y'\in D_2$ such that $G(x,y,y')\cap-\text{int}P\neq\emptyset$. Thus, for any $(x,y)\in(A\times B)\setminus(D_1\times D_2)$, we obtain that $\xi_q(F(x,y,x'))\nsubseteq$

$\mathbb{R}_+$, for some $x' \in D_1$ or $\xi'_q(G(x,y,y')) \nsubseteq \mathbb{R}_+$, for some $y' \in D_2$. So, we immediately obtain by the definition of $P_1(x,y)$ that

$$x' \in P_1(x,y), \text{ for some } x' \in D_1$$

or

$$y' \in P_2(x,y), \text{ for some } y' \in D_2.$$

Therefore, we proved condition (iv) of Lemma 2.12 and so P_1 and P_2 satisfy all conditions of Lemma 2.12. Hence, we can conclude that there exists $(\bar{x},\bar{y}) \in A \times B$ such that

$$P_1(\bar{x},\bar{y}) = \emptyset \quad \text{and} \quad P_2(\bar{x},\bar{y}) = \emptyset.$$

This means there exists $(\bar{x},\bar{y}) \in A \times B$ such that

$$\xi_q(F(\bar{x},\bar{y},u)) \subseteq \mathbb{R}_+, \ \forall u \in A$$

and

$$\xi_{q'}(G(\bar{x},\bar{y},v)) \subseteq \mathbb{R}_+, \ \forall v \in B.$$

Therefore $(\bar{x},\bar{y}) \in \mathrm{S}_2(\xi)$ and so by Lemma 3.1 we completes the proof that S_2 is nonempty. □

Remark 3.11. Comparing Theorem 3.10 and the results obtained in Anh and Khan [1] and Lemma 2.3 in Zhong, Huang and Wong [36], we can see that the main difference is that our techniques is based on the utilizing the nonlinear scalarization method. Further, the C-quasiconvexity of the mapping $F(x,y,\cdot)$ and $G(x,y,\cdot)$ are weakened by generalized C-quasiconvexity. Hence, Theorem 3.10 can be applicable in the following situation while the aforecited results do not work as in the following example.

Example 3.12. Let $X = Y = Z = \mathbb{R}, A = B = [0,1], C = P = \mathbb{R}_+$ and define the mappings $F : A \times B \times A \to 2^Y$ and $G : A \times B \times B \to 2^Z$ as

$$F(x,y,u) = \begin{cases} (u, u+1), & u \le x; \\ [-u, 1), & x < u; \end{cases}$$

and

$$G(x,y,v) = \begin{cases} (v, v+1), & v \le y; \\ [-v, 1), & y < v. \end{cases}$$

Firstly, we show that F is not C-quasiconvex. Taking $x = \frac{1}{2}, u_1 = 1, u_2 = 0,$ and $t_1 = t_2 = \frac{1}{2}$, we have the following relations

$$\begin{aligned} F(x,y,u_2) = (0,1) &\nsubseteq \left(\frac{1}{2}, +\infty\right) \\ &= \left(\frac{1}{2}, \frac{3}{2}\right) + \mathbb{R}_+ = F(x,y,t_1u_2 + t_2u_2) + C \end{aligned}$$

and

$$F(x,y,u_1) = [-1,1) \nsubseteq \left(\frac{1}{2}, +\infty\right) = F(x,y,t_1u_2 + t_2u_2) + C.$$

Hence, F is not C-quasiconvex. However, all assumptions given in Theorem 3.10 are satisfied. Firstly, it is clear that the assumption (i) given in Theorem 3.10 is satisfied. Next, we shall establish the assumption (ii). To this end, for fixed $(x, y) \in A \times B$, let $u_1, u_2 \in A$ and $t_1, t_2 \in [0, 1]$ with $t_1 + t_2 = 1$. Assume that $u_1 \leq u_2$. Then, we have the following three cases :

Case I : If $u_1 \leq u_2 \leq x$, then $t_1u_1 + t_2u_2 \leq u_2 \leq x$ and

$$F(x, y, u_2) \cap \Big(F(x, y, t_1u_1 + t_2u_2) + C\Big)$$
$$= (u_2, u_2 + 1) \cap (t_1u_1 + t_2u_2, +\infty) \neq \emptyset.$$

Case II : If $u_1 \leq x < u_2$, then we have either $t_1u_1 + t_2u_2 > x$ or $t_1u_1 + t_2u_2 \leq x$. Thus, we have either

$$F(x, y, u_2) \cap \Big(F(x, y, t_1u_1 + t_2u_2) + C\Big)$$
$$= [-u_2, 1) \cap [-t_1u_1 - t_2u_2, +\infty) \neq \emptyset,$$

or

$$F(x, y, u_2) \cap \Big(F(x, y, t_1u_1 + t_2u_2) + C\Big)$$
$$= [-u_2, 1) \cap (t_1u_1 + t_2u_2, +\infty) \neq \emptyset,$$

Case III : If $x < u_1 \leq u_2$, then $t_1u_1 + t_2u_2 > x$, and hence

$$F(x, y, u_2) \cap \Big(F(x, y, t_1u_1 + t_2u_2) + C\Big) = [-u_2, 1) \cap [-t_1u_1 - t_2u_2, +\infty) \neq \emptyset.$$

Hence, we have that F is generalized C-quasiconvex. Similarly, we can show that G is generalized C-quasiconvex.

In order to verify assumption (iii), notice that for each element $u \in A$, $F(x, y, u) \cap -\text{int}C \neq \emptyset$ if $u > x$. Assume that $z \in F(x, y, u)$, then z also belongs $[-u, 1) \subseteq [-1, 1)$. It is not hard to see that $z - C \subsetneq -\text{int}C$. Similarly, we can show that G also satisfies this assumption.

Next, to verify assumption (iv) of Theorem 3.10, let $(x', y') \in A \times B$ and $z \in F(x', y', u)$.

Case I : If $u \leq x'$, then $z \in (x', x' + 1)$. Let U be arbitrary neighborhood of z. For each (x, y) belongs to neighborhood $(u, x'] \times B$ of (x', y'), we have

$$F(x, y, u) \supseteq (x', x' + 1),\ \forall x \in (u, x'].$$

Hence,$F(x, y, u) \cap U \neq \emptyset,\ \forall (x, y) \in (u, x'] \times B$.

Case II : If $u > x'$, then $z \in [-u, 1)$. Let U be arbitrary neighborhood of z. For each (x, y) belongs to neighborhood $[x', 1] \times B$ of (x', y'), we have

$$F(x, y, u) = [-u, 1) \ni z.$$

Hence, $F(x, y, u) \cap U \neq \emptyset,\ \forall (x, y) \in [x', 1] \times B$. Therefore, $F(\cdot, \cdot, u)$ satisfies the condition (iv) on A. Similarly, $G(\cdot, \cdot, v)$ satisfies the condition (iv) on B.

Finally, we show that the assumption (iv) of Theorem 3.10 holds, take $D_1 = [\frac{1}{2}, 1] \subset A$ and $D_2 = [\frac{1}{2}, 1] \subset B$. Then, for each element (x, y) belongs $(A \times B) \setminus (D_1 \times D_2)$, there exit $x' = 1 \in D_1$ and $y' = 1 \in D_2$ such that

$$F(x, y, x') \cap -\text{int}C = [-1, \infty) \cap -\text{int}\mathbb{R}_+ = [-1, 0) \neq \emptyset.$$

Therefore, all assumptions in Theorem 3.10 are satisfied. In fact, it is easy to see that $(1,1) \in \mathrm{S}_2$.

□

4. Convexity of the solution set of Symmetric Vector Equilibrium Problem.

In this section we study the convexity of the solution set S_2. The sufficient conditions for the convexity of S_2 were established. Now, we recall the following useful features, which lead us to obtain our results in the sequel.

Definition 4.1. [17] Let K be a subset of a topological vector space E. A set-valued mapping $F : K \to 2^E \setminus \{\emptyset\}$ is said to be a *KKM-mapping* if for any $\{x_1, x_2, ..., x_n\} \subset K$,

$$\mathrm{co}\{x_1, x_2, ..., x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i),$$

where $2^E \setminus \{\emptyset\}$ stands for the family of all nonempty subsets of E, while the notion $\mathrm{co}\{x_1, x_2, ...x_n\}$ denotes the convex hull of $\{x_1, x_2, ..., x_n\}$.

The following well- known lemma plays vital role in our results in this section.

Lemma 4.2. [17] Let K be a subset of a topological vector space E. A set-valued mapping $F : K \to 2^X$ be a KKM-mapping with closed values in K. Assume that there exists a nonempty compact convex subset B of K such that $\bigcap_{x \in B} F(x)$ is compact. Then,

$$\bigcap_{x \in K} F(x) \neq \emptyset.$$

Theorem 4.3. Let $A \subseteq X$ and $B \subseteq E$ be nonempty convex subsets, let $C \subseteq Y$ and $P \subseteq Z$ be closed convex pointed cone with $q \in \mathrm{int}C \neq \emptyset$ and $q' \in \mathrm{int}P \neq \emptyset$. Suppose $F : A \times B \times A \to 2^Y$ and $G : A \times B \times B \to 2^Z$ are two set-valued mappings which satisfy the following conditions:

(i) for each $(x,y) \in A \times B, F(x,y,x) \subseteq C$ and $G(x,y,y) \subseteq P$;

(ii) for each $(x,y) \in A \times B, F(x,y,\cdot)$ is $C-$ convex on A as well as $G(x,y,\cdot)$ is $P-$ convex on B.

(iii) for each $u \in A$, $F(\cdot,\cdot,u)$ is lower semicontinuous on $A \times B$ and for each $v \in B$, $G(\cdot,\cdot,v)$ is lower semicontinuous on $A \times B$;

(iv) there exists nonempty compact convex set $D_1 \times D_2 \subseteq A \times B$ and compact set $M_1 \times M_2 \subseteq A \times B$ such that for each $(x,y) \in (A \times B) \setminus (M_1 \times M_2)$, there exist $(x',y') \in D_1 \times D_2$ such that $F(x,y,x') \cap -\mathrm{int}C \neq \emptyset$ or $y' \in D_2$ such that $G(x,y,y') \cap -\mathrm{int}P \neq \emptyset$.

Then, the solution set $\mathrm{S}_2(\xi)$ is a nonempty compact subset of $A \times B$. Furthermore, S_2 is convex.

Proof. let $q \in -\text{int}C$, and $q' \in -\text{int}P$. Define a set-valued mapping $T : A \times B \to A \times B$ by

$$T(z,w) = \{(x,y) \in A \times B : \xi_q(F(x,y,z)) \subseteq \mathbb{R}_+,\ \xi_{q'}(G(x,y,w)) \subseteq \mathbb{R}_+\}.$$

Note that $\mathrm{S}_2(\xi) = \bigcap_{(z,w)\in A\times B} T(z,w)$. We assert that the set-valued mapping T fulfils all the assumptions of Lemma 4.2. Firstly, we will show that is a KKM-mapping. Suppose on the contrary, then there exists a subset $\{(x_1,y_1),...,(x_n,y_n)\}$ of $A \times B$ and $(z,w) \in A \times B$ such that

$$(z,w) \in \text{co}\{(x_1,y_1),...,(x_n,y_n)\} \setminus \bigcup_{i=1}^{n} T(x_i,y_i).$$

Hence, there exist $\alpha_1,...,\alpha_n \in \mathbb{R}_+$ such that

$$\sum_{i=1}^{n} \alpha_i = 1 \quad \text{and} \quad (z,w) = \sum_{i=1}^{n} \alpha_i(x_i,y_i).$$

Thus, for all $i = 1,2,...,n$, we have

$$\xi_q(F(z,w,x_i)) \not\subseteq \mathbb{R}_+ \ \text{ or } \ \xi_{q'}(G(z,w,y_i)) \not\subseteq \mathbb{R}_+. \tag{4.1}$$

By assumption (ii), the C-quasiconvexity and P-quasiconvexity of F and G are fulfilled respectively, and so

$$F(z,w,x_i) \subseteq F(z,w,z) + C, \quad \text{for some } i = 1,2,...,n,$$

and

$$G(z,w,y_i) \subseteq G(z,w,w) + C, \quad \text{for some } i = 1,2,...,n.$$

Hence, there is $i \in \{1,2,...,n\}$ such that

$$\xi_q(F(z,w,x_i)) \subseteq \xi_q(F(z,w,z)) + \xi_q(C) \subseteq \mathbb{R}_+,$$

and

$$\xi_{q'}G((z,w,y_i)) \subseteq \xi_{q'}(G(z,w,w)) + \xi_{q'}(C) \subseteq \mathbb{R}_+.$$

This contradicts 4.1, and so T is a KKM mapping. Next, we will show that for each $(z,w) \in A \times B$, the set $T(z,w)$ is closed. Let $(z,w) \in A \times B$ and $\{(z_i,w_i)\} \subseteq T(z,w)$ be a net converges to (z_1,w_2). Since $(z_i,w_i) \in T(z,w)$ for all i, we have

$$\xi_q(F(z_i,w_i,z)) \subseteq \mathbb{R}_+ \text{ and } \xi_{q'}(G(z_i,w_i,w)) \subseteq \mathbb{R}_+, \forall i.$$

Let $(h_1,h_2) \in \xi_q(F(z_1,w_2,z)) \times \xi_{q'}(G(z_1,w_2,w))$. Then there exists the pair $(z_2,w_3) \in F(z_1,w_2,z) \times G(z_1,w_2,w)$ such that

$$(h_1,h_2) = (\xi_q(z_2), \xi_{q'}(w_3)).$$

By assumption (iii), there is $(t_i,s_i) \in F(z_i,w_i,z) \times G(z_i,w_i,w)$ such that

$$(t_i,s_i) \to (z_2,w_3).$$

Since $(t_i,s_i) \in F(z_i,w_i,z) \times G(z_i,w_i,w)$ for all i, we have

$$\xi_q(t_i) \geq 0 \text{ and } \xi_{q'}(s_i) \geq 0 \text{ for all } i.$$

Therefore, by the continuity of ξ_q and ξ'_q, we get

$$h_1 \geq 0 \text{ and } h_2 \geq 0.$$

Since (h_1, h_2) is arbitrary element belongs to $\xi_q(F(z_1, w_2, z)) \times \xi_{q'}(G(z_1, w_2, w))$, we get

$$\xi_q(F(z_1, w_2, z)) \subseteq \mathbb{R}_+ \quad \text{and} \quad \xi_{q'}(G(z_1, w_2, w)) \subseteq \mathbb{R}_+.$$

Hence, $(z_1, w_2) \in T(z, w)$ and so $T(z, w)$ is closed for any $(z, w) \in A \times B$. Now, all the assumptions of Lemma 4.2 are fulfilled and so $S_2(\xi)$ is nonempty. Further, it follows from assumption (iv) that

$$S_2(\xi) \subseteq M_1 \times M_2,$$

and so it completes the proof that $S_2(\xi)$ is a nonempty compact subset of $A \times B$. By Lemma 3.9, S_2 is also nonempty and compect. Finally, the C-convexity of $F(x, y, \cdot)$ on A and the P-convexity of $G(x, y, \cdot)$ on B imply the set $T(z, w)$ is convex for all $(z, w) \in A \times B$. Hence, the set $S_2(\xi)$ is convex (The intersection of the convex sets is convex.). Therefore, by Lemma 3.9, S_2 is also convex. This completes the proof. □

5. Conclusions

In this paper, we considered the problems (SVEP$_1$), (SVEP$_2$), (SSEP$_1(\xi)$) and (SSEP$_2(\xi)$). By introducing the new type of C-quasiconvexity for a set-valued mapping and using a nonlinear scalarization function ξ_q and its properties, we obtianed some existence results of the solutions for the symmetric vector equilibrium problems and symmetric scalar equilibrium problems. In fact, our studying is without assumption of monotonicity and boundedness. Moreover, the convexity of solution sets are investigated. Finally, some examples in order to support our results are provided.

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