



## รายงานวิจัยฉบับสมบูรณ์

โครงการ : ความต่อเนื่องแบบโฮเดอร์ของการส่งผลเฉลยสำหรับปัญหาเชิงดุลยภาพ  
แบบอ่อนวางนัยทั่วไป เชิงพารามิเตอร์โดยความเป็นคอนเวกซ์อย่างเข้ม

Holder Continuity of solution maps to Parametric Weak  
Generalized equilibrium problems via the strong convexity

โดย รองศาสตราจารย์ ดร.ระเบียน วังคีรี

กันยายน 2559



อธิบดีมหาวิทยาลัย

สัญญาเลขที่ R2559C045



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คณะผู้วิจัย สังกัด

รองศาสตราจารย์ ดร.ระเบียน วังศิริ คณะวิทยาศาสตร์

สำนักหอสมุด มหาวิทยาลัยนเรศวร
วันลงทะเบียน ๕-๗ ส.ค. 2562
เลขทะเบียน 1021032
เลขเรียกหนังสือ จ. 0A
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สนับสนุนโดยกองทุนวิจัยมหาวิทยาลัยนเรศวร  
ปีงบประมาณ 2558

## Executive Summary

The aim of this note is to establish the characterization of nonemptiness and boundedness of the solution set of equilibrium problem with stably pseudomonotone mappings. Our result extends and improves recent results in literature for monotone equilibrium problems.

**Lemma :** Let  $K$  be a nonempty, closed, and convex subset of a real reflexive Banach space  $X$ . suppose that  $f$  satisfies the conditions  $(f_1)$ – $(f_4)$ . Then for any given  $\xi \in K^*$  the solution of the problem  $x \in K$  is a solution of the problem

$$\text{find } x \in K \text{ such that } f(x, y) - \langle \xi, y - x \rangle \geq 0, \forall y \in K \quad (0.0.1)$$

if and only if it is a solution of the problem

$$\text{find } x \in K \text{ such that } f(y, x) - \langle \xi, x - y \rangle \leq 0, \forall y \in K \quad (0.0.2)$$

In addition, if  $K$  is bounded, then  $S^P$  and  $S^D$  are solvable.

**Theorem :** Let  $K$  be a nonempty, closed, and convex subset of a real reflexive Banach space  $X$  with  $\text{int } K^* \neq \emptyset$ . Suppose that  $f$  satisfies the conditions  $(f_1)$ – $(f_4)$ . Then the following statements are equivalent:

1.  $S^P$  is a nonempty and bounded.
2.  $S^D$  is a nonempty and bounded.
3.  $\mathcal{F}_K^+ \neq \emptyset$ .

**Remark :** Theorem 3 discusses the characterization of nonemptiness and boundedness for  $S^P$  and  $S^D$ , which is more general than the result in Theorem 3.1 of [7] in the case that  $f$  is relaxed to the stably pseudomonotone mapping.

## บทคัดย่อ

จุดประสงค์ของงานวิจัยนี้เพื่อ เพื่อให้ได้ให้คุณลักษณะของเซตผลเฉลยที่ไม่เป็นเซตว่างและเป็นเซตที่มีขอบเขตของปัญหาเชิงดุลยภาพที่มีการส่งส่งแบบทางเดียวเทียม งานวิจัยนี้ได้ขยายและปรับปรุงผลงานวิจัยเดิมที่ใช้การส่งส่งแบบทางเดียว

## Abstract

The aim of this note is to establish the characterization of nonemptiness and boundedness of the solution set of equilibrium problem with stably pseudomonotone mappings. Our result extends and improves recent results in literature for monotone equilibrium problems.



# Contents

1	Chapter 1 : Introduction	1
2	Main results	5
3	ภาคผนวก	11





# Chapter 1

## Introduction

Let  $K$  be a nonempty, closed and convex subset of a real reflexive Banach space  $X$  and let  $f : K \times K \rightarrow \mathbb{R} \cup \{+\infty\}$  be a bifunction. The equilibrium problem [1] is to find  $\bar{x} \in K$  such that

$$f(\bar{x}, y) \geq 0, \forall y \in K \quad (\text{EP})$$

and its dual problem is to find  $\bar{x} \in K$  such that

$$f(y, \bar{x}) \leq 0, \forall y \in K. \quad (\text{DEP})$$

The solutions sets to (EP) and (DEP) are denoted by  $S^P$  and  $S^D$ , respectively.

It is well known that the equilibrium problem provides a unified model of several classes of problems, including variational inequality problems, complementarity problems, optimization problems. There are many papers which have discussed the existence solution to the equilibrium problem (see [1, 3, 4, 2] and references therein). In order to solve the problem when  $K$  is unbounded, many authors studies the coercive assumption. Strict feasibility condition [5, 6] is one of the most useful tool to characterize the nonemptiness and boundedness for the solution set of variational inequality problem. Recently, the concept of strict feasibility was extended to equilibrium problem by Hu and Fang [7]. They proved that under suitable conditions, the monotone equilibrium problem has a nonempty and bounded solution set if and only if it is strictly feasible.

Motivated and inspired by Hu and Fang [7] and the idea of stable pseudomonotonicity in [8, 9], the purpose of this paper is to establish the characterization of nonemptiness and boundedness of the solution set of equilibrium problems with stably pseudomonotone mappings.

Let  $K$  be a nonempty, closed and convex subset of a real reflexive Banach space  $X$  with dual space  $X^*$ . The dual cone  $K^*$  of  $K$  is defined as

$$K^* := \{\xi \in X^* : \langle \xi, x \rangle \geq 0, \forall x \in K\}.$$

The barrier cone  $\text{barr } K$  of  $K$  is defined as

$$\text{barr } K := \left\{ \xi \in X^* : \sup_{x \in K} \langle \xi, x \rangle < +\infty \right\}.$$

It is well-known that  $-K^* \subset \text{barr } K$ . The asymptotic cone of  $K$ , denoted by  $K_\infty$ , is defined by

$$K_\infty := \{d \in X : \exists \lambda_k \rightarrow 0, \exists x_k \in K \text{ with } \lambda_k x_k \rightarrow d\},$$

where " $\rightarrow$ " means weak convergence. It is known that, for given  $x \in K$ ,

$$K_\infty := \{d \in X : x + td \in K, \forall t > 0\}.$$

The following interesting results can be founded in [10, 6].

Let  $K$  be a nonempty, closed and convex subset of a reflexive Banach space  $X$ . Then  $K$  is well-positioned if and only if  $\text{int}(\text{barr } K) \neq \emptyset$ . Furthermore, if  $K$  is well-positioned, then there is no sequence  $\{x_n\} \subset K$  with  $\|x_n\| \rightarrow \infty$  such that  $x_n/\|x_n\| \rightarrow 0$ .

Let  $K$  be a well-positioned, closed, convex subset of a real reflexive Banach space  $X$  and  $\{A_n\}$  be a decreasing sequence of closed convex subsets of  $K$  with  $A := \bigcap_{n=1}^{\infty} A_n$  nonempty and bounded. Then  $A_n$  is bounded for some  $n$ .

Let  $K$  be a well-positioned, closed, convex subset of a real reflexive Banach space  $X$ ,  $h : K \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex lower semicontinuous function, and  $r_1, r_2 \in \mathbb{R}$  with  $r_1 \leq r_2$ . If the level set  $\{x \in K : h(x) \leq r_1\}$  is nonempty and bounded, then so is the set  $\{x \in K : h(x) \leq r_2\}$ .

The concepts of feasibility and strict feasibility for equilibrium problems can be introduces by means of asymptotic cone.

We say that (EP) (or (DEP)) is **feasible** [11] if the set  $\mathcal{F}_K := \{x \in K : f(x, x+d) \geq 0, \forall d \in K_{\infty}\} \neq \emptyset$ . We say that (EP) (or (DEP)) is **strictly feasible** [7] if the set  $\mathcal{F}_K : \{x \in K : f(x, x+d) > 0, \forall d \in K_{\infty} \setminus \{0\}\}$ .

A bifunction  $f : K \times K \rightarrow \mathbb{R} \cup \{+\infty\}$  is said to be **monotone** on  $K$ , if  $\forall x, y \in K$ ,  $f(x, y) + f(y, x) \leq 0$ ; **pseudo-monotone** on  $K$ , if  $\forall x, y \in K$ ,  $f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0$ . A pseudomonotone bifunction  $f$  is said to be **stably pseudomonotone** on  $K$  w.r.t.  $U \subset X^*$ , due to [9, 8], if  $\forall x, y \in K$  and  $\forall u^* \in U$ ,  $f(x, y) - \langle \xi, y - x \rangle \geq 0 \Rightarrow f(y, x) - \langle \xi, x - y \rangle \leq 0$ . It is well-known that every monotone mapping is a stably pseudomonotone mapping.

The following example illustrate that the converse inclusion may not be true.

**Example 1.** Let  $X = \mathbb{R}$  with  $X^* \equiv \mathbb{R}$  and  $K = [3, 4]$ . Define  $f : K \times K \rightarrow \mathbb{R} \cap \{+\infty\}$  by

$$f(x, y) = y(x - y), \forall x, y \in K. \quad (1.0.1)$$

Then  $f$  is stable pseudomonotone on  $K$  w.r.t.  $[1, 2]$  but not monotone.

*Proof.* We first show that  $f$  is stable pseudomonotone on  $K$  w.r.t.  $[1, 2]$ . For any  $x, y \in K$  and  $\xi \in [1, 2]$ , If

$$\begin{aligned} 0 \leq f(x, y) - \langle \xi, y - x \rangle &= y(y - x) - \xi(y - x) \\ &= (y - \xi)(y - x). \end{aligned}$$

Then  $y - x \geq 0$ , since  $z - \xi > 0$  for all  $z \in K$  and  $\xi \in [1, 2]$ . It then follows that

$$\begin{aligned} f(y, x) - \langle \xi, x - y \rangle &= x(x - y) - \xi(x - y) \\ &= (x - \xi)(x - y) \leq 0. \end{aligned}$$

□

Hence,  $f$  is stable pseudomonotone on  $K$  w.r.t.  $[1, 2]$ .

Finally, it easily seen that  $f$  is not monotone. Indeed, for any  $x, y \in K$

$$f(x, y) + f(y, x) = y(y - x) + x(x - y) = (x - y)^2 \geq 0.$$

We collect the following well-known KKM-Fan lemma.

[12] Let  $M$  be a nonempty, closed and convex subset of  $X$  and  $F : M \rightarrow 2^M$  be a set-valued map. Suppose that for any finite set  $\{x_1, \dots, x_m\} \subseteq M$ , one has



1.  $\text{conv}\{x_1, \dots, x_m\} \subset \bigcup_{i=1}^m F(x_i)$  (i.e.,  $F$  is a KKM map on  $M$ );
2.  $F(x)$  is closed for every  $x \in M$ ; and
3.  $F(x)$  compact for some  $x \in M$ .

Then  $\bigcap_{x \in M} F(x) \neq \emptyset$ .





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## Chapter 2

### Main results

In this section, we present the characterization of the solution for (EP) and (DEP). Before proving our results, we list the following assumptions hold.

- (f<sub>1</sub>) For any  $x \in K$ ,  $f(x, x) \geq 0$ .
- (f<sub>2</sub>) For any  $x, y \in K$  and  $\alpha \in [0, 1]$ ,  $\lim_{\alpha \rightarrow +0} \frac{f(x + t(y - x), x)}{t} = -f(x, y)$ .
- (f<sub>3</sub>) For any  $x \in K$ , the map  $y \mapsto f(x, y)$  is convex and lower semicontinuous.
- (f<sub>4</sub>)  $f$  is stably pseudomonotone on  $K$  w.r.t.  $K^*$ .

The following lemma shows that under suitable conditions, the solution set of (EP) and (DEP) are the same.

**Lemma 2.** *Let  $K$  be a nonempty, closed, and convex subset of a real reflexive Banach space  $X$ . suppose that  $f$  satisfies the conditions (f<sub>1</sub>)-(f<sub>4</sub>). Then for any given  $\xi \in K^*$  the solution of the problem  $x \in K$  is a solution of the problem*

$$\text{find } x \in K \text{ such that } f(x, y) - \langle \xi, y - x \rangle \geq 0, \forall y \in K \quad (2.0.1)$$

*if and only if it is a solution of the problem*

$$\text{find } x \in K \text{ such that } f(y, x) - \langle \xi, x - y \rangle \leq 0, \forall y \in K \quad (2.0.2)$$

*In addition, if  $K$  is bounded, then  $S^P$  and  $S^D$  are solvable.*

*Proof.* Let  $x$  be a solution of (2.0.1). Then  $f(x, y) - \langle \xi, y - x \rangle \geq 0 \forall y \in K$ . By condition (f<sub>4</sub>),  $x$  is a solution of (2.0.2). Conversely, let  $x$  be a solution of (2.0.2). Then  $f(y, x) - \langle \xi, x - y \rangle \geq 0 \forall y \in K$ . For any  $z \in K$ , set  $z_t := x + t(z - x)$ , for any  $t \in (0, 1)$ . Then  $z_t \in K$ , because of the convexity of  $K$ . It follows from (f<sub>1</sub>) that

$$f(z_t, z) \leq \langle \xi, x - z_t \rangle = t \langle \xi, x - z \rangle.$$

Condition (f<sub>2</sub>) give that  $-f(z, x) \leq \langle \xi, x - z \rangle$ . Hence,  $x$  solve (2.0.1).

Next, we prove that  $S^P$  and  $S^D$  are nonempty provided that  $K$  is bounded. Define  $F, G : K \rightarrow 2^K$  by

$$F(y) = \{x \in K : f(x, y) \geq 0\}, \forall y \in K \text{ and } G(y) = \{x \in K : f(y, x) \leq 0\}, \forall y \in K.$$

Then,  $G(y)$  is closed, convex and compact for all  $x \in K$ . It follows from Lemma 1 that

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \emptyset.$$

Then there exists  $\bar{x} \in K$  which solve (EP) and (DEP).  $\square$

**Theorem 3.** Let  $K$  be a nonempty, closed, and convex subset of a real reflexive Banach space  $X$  with  $\text{int } K^* \neq \emptyset$ . Suppose that  $f$  satisfies the conditions  $(f_1)$ – $(f_4)$ . Then the following statements are equivalent:

1.  $S^P$  is a nonempty and bounded.
2.  $S^D$  is a nonempty and bounded.
3.  $\mathcal{F}_K^+ \neq \emptyset$ .

*Proof.* It follows from Lemma 2, (i)  $\Leftrightarrow$  (ii) in the case where  $\xi = 0$ . Define a function  $g : K \times K^* \rightarrow \mathbb{R}$  by

$$g(x, \xi) := \sup_{y \in K} \frac{f(y, x) - \langle \xi, x - y \rangle}{\max\{1, \|y\|\}}, \quad \forall x \in K, \xi \in K^*.$$

It follows from  $(f_3)$  that  $g(\cdot, \xi)$  is convex and lower semicontinuous function and  $g(x) \geq 0$  for all  $x \in K$  and  $\xi \in K^*$ . Define the set  $A := \{x \in K : g(x) \leq 0\}$ . Then by Lemma 1,  $A$  is nonempty, closed and convex set. Clearly,  $A$  is the solution set of (DEP). By Lemma 2,  $A$  is also the solution set of (EP). Let  $\xi' \in \text{int } K^*$ . For every positive integer  $n$ , define

$$A_n := \left\{ x \in K : g(x, 0) \leq \frac{1}{n} \langle \xi', x \rangle \right\}.$$

Then  $\{A_n\}$  is a decreasing sequence as  $n \rightarrow +\infty$  of closed and convex subsets of  $K$  and  $A = \bigcap_{i=1}^{\infty} A_i$ . Notice that  $\text{barr } K$  has a nonempty interior, then  $K$  is well-positioned.

(ii)  $\Rightarrow$  (iii) Suppose that  $A$  is nonempty and bounded. It follows from Lemma 1 that there exists  $n_0$  such that  $A_{n_0}$  is nonempty and bounded. Setting  $r := g\left(z_0, \frac{1}{n} \xi'\right)$  for some  $z_0 \in A_{n_0}$ . We then have  $r \geq 0$ . We now consider the following set  $C_0 := \left\{ x \in K : g(x, 0) \leq \frac{1}{n_0} \langle \xi, x \rangle + r \right\}$ . Then by Lemma 1,  $C_0$  is nonempty and bounded. After calculating we have, for any  $y \in K$ ,

$$\frac{f(y, x)}{\max\{1, \|y\|\}} - \frac{f(y, x) - \left\langle \frac{1}{n_0} \xi, x - y \right\rangle}{\max\{1, \|y\|\}} = \frac{1}{n_0} \frac{\langle \xi, x - y \rangle}{\max\{1, \|y\|\}} \leq \frac{1}{n_0} \langle \xi, x \rangle,$$

because of  $\xi' \in \text{int } K^*$ . Thus  $g(x, 0) - g\left(x, \frac{1}{n_0} \xi'\right) \leq \frac{1}{n_0} \langle \xi', x \rangle$ , and so the set

$$C := \left\{ x \in K : g\left(x, \frac{1}{n_0} \xi'\right) \leq r \right\} \subset C_0$$

is nonempty, closed, convex and bounded.

Set

$$K_i = \{x \in K : \|x\| \leq i\}, \quad i = 1, 2, \dots$$



and

$$\hat{g}_i(x) := \sup_{y \in K_i} \frac{f(y, x) - \frac{1}{n_0} \langle \xi', x - y \rangle}{\max\{1, \|y\|\}}, \quad \forall x \in X.$$

Then  $\hat{g}_i(x)$  is convex and lower semicontinuous on bounded subset  $K_i$  for all  $i \in \mathbb{N}$  and  $\hat{g}_i(x) \geq 0$  for all  $x \in K_i$ . For every  $i \in \mathbb{N}$ , it follows from the prove in Lemma 2 that there are  $x_i \in K_i$  such that  $f(y, x_i) - \frac{1}{n_0} \langle \xi', x_i - y \rangle \leq 0$ ,  $\forall y \in K_i$ . This implies that  $\hat{g}_i(x_i) = 0$ . Define  $D_i := \{x \in K : \hat{g}_i(x) \leq r\}$ . It is not hard to check that  $\{D_i\}_{i=1}^\infty$  is a decreasing sequence of nonempty, closed and convex subsets of  $K$  and  $C = \bigcap_{i=1}^\infty D_i$ . Since  $C$  is nonempty and bounded, it follows from Lemma 1 that there exists  $i_0 \in \mathbb{N}$  such that  $D_{i_0}$  is nonempty and bounded. Then there exists a positive integer  $L$  such that  $\sup_{x \in D_{i_0}} \|x\| < L$  and  $i_0 \leq L$ . since  $\hat{g}_L(x_L) = 0$ ,  $x_L \in D_L \subset D_{i_0}$ , and so  $x_L \in K_L$ .

For any  $y \in K$  and  $t \in (0, 1)$ , setting  $y_t := (1 - t)x_L + ty \in K_L$ , we have  $y_t \rightarrow x_L$  as  $t \rightarrow +0$ . It follows from linearity that, for all  $y \in K$

$$f(y_t, x_L) - \frac{1}{n_0} \langle \xi', x_L - y_t \rangle \leq 0 \Rightarrow \frac{f(y_t, x_L)}{t} - \frac{1}{n_0} \langle \xi', x_L - y \rangle \leq 0.$$

By  $(f_3)$ , we have  $f(x_L, y) \geq \frac{1}{n_0} \langle \xi', y - x_L \rangle$  for all  $y \in K$ .

For any  $d \in K_\infty \setminus \{0\}$ , we known that  $x_L + d \in K$ , and so  $f(x_L, x_L + d) \geq \frac{1}{n_0} \langle \xi', d \rangle > 0$ , since  $\xi' \in \text{int } K^*$ . Therefore,  $x_L \in \mathcal{F}_K^+$ .

(iii)  $\Rightarrow$  (i) Suppose that  $\mathcal{F}_K^+ \neq \emptyset$ . Then there exists  $x_0 \in K$  such that  $f(x_0, x_0 + d) > 0$ ,  $\forall d \in K_\infty \setminus \{0\}$ . Setting  $D := \{x \in K : f(x_0, x) \leq 0\}$ . It is not hard to check that  $D$  is nonempty, closed and convex. We claim that  $D$  is bounded. If not, there exists  $x_n \in D$  with  $\|x_n\| \rightarrow +\infty$ . Without loss of generality, we can suppose that  $x_n / \|x_n\| \rightharpoonup d \in K_\infty$ . Since  $K$  is well-positioned, from Lemma 1 we have  $d \neq 0$ . It follows from  $x_n \in D$  that  $f(x_0, x_n) \leq 0$ . Since  $f$  is convex and lower semicontinuous at second variable, it follows that

$$f(x_0, x_0 + d) \leq \liminf_{n \rightarrow \infty} f\left(x_0, x_0 + \frac{1}{\|x_n\|}(x_n - x_0)\right) \leq 0,$$

which contradicts  $x_0 \in \mathcal{F}_K^+$ . Thus  $D$  is nonempty, convex and weak compact. Define

$$D_z := \{x \in D : f(x, z) \geq 0\}, \quad \forall z \in K.$$

Thus

$$\bar{x} \in S^P \Leftrightarrow \bar{x} \in \bigcap_{z \in K} D_z.$$

Define

$$D'_z = \{x \in D : f(z, x) \leq 0, \forall z \in K\}.$$

Then  $D'_z$  is nonempty, closed and convex. By a similar argument as in the proof of Lemma 2, we can show that

$$\bigcap_{z \in K} D'_z = \bigcap_{z \in K} D_z.$$

For any finite set  $\{z_i : i = 1, 2, \dots, n\} \subset K$ , let  $M = \text{conv}\{D \cup \{z_1, z_2, \dots, z_n\}\}$ . It is known that  $M$  is weakly compact. By Lemma 2, there exists  $\hat{x} \in M$  such that

$$f(y, \hat{x}) \leq 0, \quad \forall y \in M.$$



From  $x_0 \in D \subset M$ , we have  $f(x_0, \hat{x}) \leq 0$  and so  $\hat{x} \in D$ . Furthermore, we have

$$f(z_i, \hat{x}) \leq 0, \quad i = 1, 2, \dots, n,$$

because  $z_i \in M$ . Therefore  $\hat{x} \in \bigcap_{i=1}^n D'_{z_i}$ , and so  $\{D'_z : z \in K\}$  has the finite intersection property. Since  $D$  is weakly compact, we get  $S^P = \bigcap_{z \in K} D_z = \bigcap_{z \in K} D'_z \neq \emptyset$ . Then  $S^P$  is nonempty. Since  $D$  is bounded and  $S^P = \bigcap_{z \in K} D_z \subset D$ , we have  $S^P$  is bounded. The proof is complete.  $\square$

**Remark 4.** Theorem 3 discusses the characterization of nonemptiness and boundedness for  $S^P$  and  $S^D$ , which is more general than the result in Theorem 3.1 of [7] in the case that  $f$  is relaxed to the stably pseudomonotone mapping.



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## Chapter 3

### ภาคผนวก

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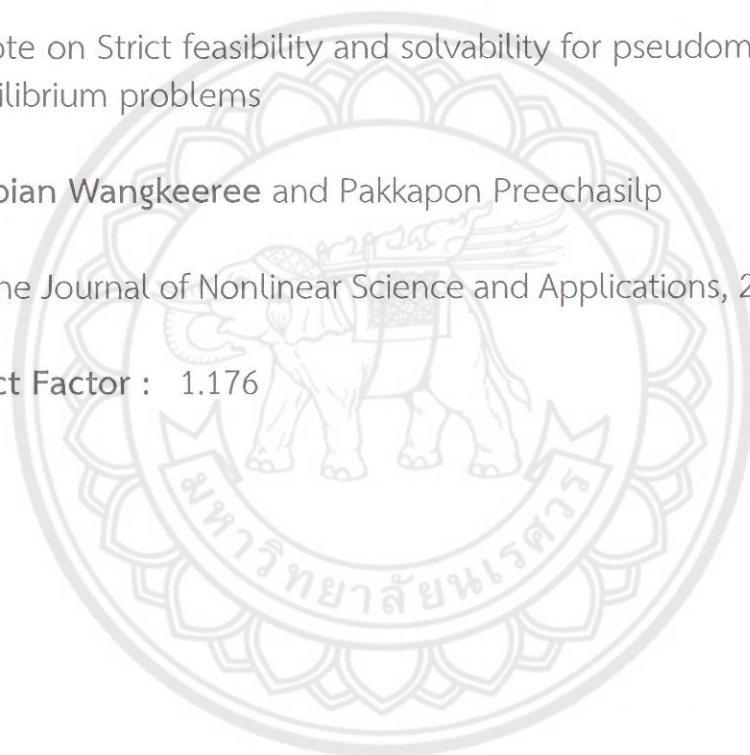
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เรื่อง : A note on Strict feasibility and solvability for pseudomonotone equilibrium problems

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# A note on Strict feasibility and solvability for pseudomonotone equilibrium problems<sup>☆</sup>

Rabian Wangkeeree<sup>a</sup>, Pakkapon Preechasilp<sup>\*,b</sup>

<sup>a</sup>Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

<sup>b</sup>Program in Mathematics, Faculty of Education, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand

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## Abstract

The aim of this note is to establish the characterization of nonemptiness and boundedness of the solution set of equilibrium problem with stably pseudomonotone mappings. Our result extends and improves recent results in literature for monotone equilibrium problems.

**Keywords:** Equilibrium problem, Strict feasibility, Stably pseudomonotone mapping.

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## 1. Introduction

Let  $K$  be a nonempty, closed and convex subset of a real reflexive Banach space  $X$  and let  $f : K \times K \rightarrow \mathbb{R} \cup \{+\infty\}$  be a bifunction. The equilibrium problem [1] is to find  $\bar{x} \in K$  such that

$$f(\bar{x}, y) \geq 0, \forall y \in K \quad (\text{EP})$$

and its dual problem is to find  $\bar{x} \in K$  such that

$$f(y, \bar{x}) \leq 0, \forall y \in K. \quad (\text{DEP})$$

The solutions sets to (EP) and (DEP) are denoted by  $S^P$  and  $S^D$ , respectively.

It is well known that the equilibrium problem provides a unified model of several classes of problems, including variational inequality problems, complementarity problems, optimization problems. There are many papers which have discussed the existence solution to the equilibrium problem (see [1–4] and references therein). In order to solve the problem when  $K$  is unbounded, many authors studies the coercive assumption. Strict feasibility condition [5, 6] is one of the most useful tool to characterize the nonemptiness and boundedness for the solution set of variational inequality problem. Recently, the concept of strict feasibility was extended to equilibrium problem by Hu and Fang [7]. They proved ~~prove~~ that under suitable conditions, the monotone equilibrium problem has a nonempty and bounded solution set if and only if it is strictly feasible.

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\*Corresponding author.

Email addresses: preechasilpp@gmail.com (P. Preechasilp), rabianw@nu.ac.th (R. Wangkeeree).

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Motivated and inspired by Hu and Fang [7] and the idea of stable pseudomonotonicity in [8, 9], the purpose of this paper is to establish the characterization of nonemptiness and boundedness of the solution set of equilibrium problems with stably pseudomonotone mappings.

## 2. Preliminaries

Let  $K$  be a nonempty, closed and convex subset of a real reflexive Banach space  $X$  with dual space  $X^*$ . The dual cone  $K^*$  of  $K$  is defined as

$$K^* := \{\xi \in X^* : \langle \xi, x \rangle \geq 0, \forall x \in K\}.$$

The **barrier cone**  $\text{barr } K$  of  $K$  is defined as

$$\text{barr } K := \left\{ \xi \in X^* : \sup_{x \in K} \langle \xi, x \rangle < +\infty \right\}.$$

It is well-known that  $-K^* \subset \text{barr } K$ . The **asymptotic cone** of  $K$ , denoted by  $K_\infty$ , is defined by

$$K_\infty := \{d \in X : \exists \lambda_k \rightarrow 0, \exists x_k \in K \text{ with } \lambda_k x_k \rightharpoonup d\},$$

where “ $\rightharpoonup$ ” means weak convergence. It is known that, for given  $x \in K$ ,

$$K_\infty := \{d \in X : x + td \in K, \forall t > 0\}.$$

The following interesting results can be founded in [6, 10].

**Lemma 2.1.** *Let  $K$  be a nonempty, closed and convex subset of a reflexive Banach space  $X$ . Then  $K$  is well-positioned if and only if  $\text{int}(\text{barr } K) \neq \emptyset$ . Furthermore, if  $K$  is well-positioned, then there is no sequence  $\{x_n\} \subset K$  with  $\|x_n\| \rightarrow \infty$  such that  $x_n/\|x_n\| \rightharpoonup 0$ .*

**Lemma 2.2.** *Let  $K$  be a well-positioned, closed, convex subset of a real reflexive Banach space  $X$  and  $\{A_n\}$  be a decreasing sequence of closed convex subsets of  $K$  with  $A := \bigcap_{n=1}^\infty A_n$  nonempty and bounded. Then  $A_n$  is bounded for some  $n$ .*

**Lemma 2.3.** *Let  $K$  be a well-positioned, closed, convex subset of a real reflexive Banach space  $X$ ,  $h : K \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper convex lower semicontinuous function, and  $r_1, r_2 \in \mathbb{R}$  with  $r_1 \leq r_2$ . If the level set  $\{x \in K : h(x) \leq r_1\}$  is nonempty and bounded, then so is the set  $\{x \in K : h(x) \leq r_2\}$ .*

The concepts of feasibility and strict feasibility for equilibrium problems can be introduced by means of asymptotic cone.

We say that (EP) (or (DEP)) is **feasible** [11] if the set  $\mathcal{F}_K := \{x \in K : f(x, x+d) \geq 0, \forall d \in K_\infty\} \neq \emptyset$ . We say that (EP) (or (DEP)) is **strictly feasible** [7] if the set  $\mathcal{F}_K : \{x \in K : f(x, x+d) > 0, \forall d \in K_\infty \setminus \{0\}\}$ .

A bifunction  $f : K \times K \rightarrow \mathbb{R} \cup \{+\infty\}$  is said to be **monotone** on  $K$ , if  $\forall x, y \in K$ ,  $f(x, y) + f(y, x) \leq 0$ ; **pseudo-monotone** on  $K$ , if  $\forall x, y \in K$ ,  $f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0$ . A pseudomonotone bifunction  $f$  is said to be **stably pseudomonotone** on  $K$  w.r.t.  $U \subset X^*$ , due to [8, 9], if  $\forall x, y \in K$  and  $\forall u^* \in U$ ,  $f(x, y) - \langle \xi, y - x \rangle \geq 0 \Rightarrow f(y, x) - \langle \xi, x - y \rangle \leq 0$ . It is well-known that every monotone mapping is a stably pseudomonotone mapping.

The following example illustrate that the converse inclusion may not be true.

**Example 2.4.** Let  $X = \mathbb{R}$  with  $X^* \equiv \mathbb{R}$  and  $K = [3, 4]$ . Define  $f : K \times K \rightarrow \mathbb{R} \cap \{+\infty\}$  by

$$f(x, y) = y(x - y), \quad \forall x, y \in K. \quad (2.1)$$

Then  $f$  is stable pseudomonotone on  $K$  w.r.t.  $[1, 2]$  but not monotone.

**Proof.** We first show that  $f$  is stable pseudomonotone on  $K$  w.r.t.  $[1, 2]$ . For any  $x, y \in K$  and  $\xi \in [1, 2]$ , If

$$\begin{aligned} 0 \leq f(x, y) - \langle \xi, y - x \rangle &= y(y - x) - \xi(y - x) \\ &= (y - \xi)(y - x). \end{aligned}$$

Then  $y - x \geq 0$ , since  $z - \xi > 0$  for all  $z \in K$  and  $\xi \in [1, 2]$ . It then follows that

$$\begin{aligned} f(y, x) - \langle \xi, x - y \rangle &= x(x - y) - \xi(x - y) \\ &= (x - \xi)(x - y) \leq 0. \end{aligned}$$

Hence,  $f$  is stable pseudomonotone on  $K$  w.r.t.  $[1, 2]$ .

Finally, it easily seen that  $f$  is not monotone. Indeed, for any  $x, y \in K$

$$f(x, y) + f(y, x) = y(y - x) + x(x - y) = (x - y)^2 \geq 0.$$

We collect the following well-known KKM-Fan lemma.

**Lemma 2.5.** [12] Let  $M$  be a nonempty, closed and convex subset of  $X$  and  $F : M \rightarrow 2^M$  be a set-valued map. Suppose that for any finite set  $\{x_1, \dots, x_m\} \subseteq M$ , one has

- (i)  $\text{conv}\{x_1, \dots, x_m\} \subset \bigcup_{i=1}^m F(x_i)$  (i.e.,  $F$  is a KKM map on  $M$ );
- (ii)  $F(x)$  is closed for every  $x \in M$ ; and
- (iii)  $F(x)$  compact for some  $x \in M$ .

Then  $\bigcap_{x \in M} F(x) \neq \emptyset$ .

### 3. Main results

In this section, we present the characterization of the solution for (EP) and (DEP). Before proving our results, we list the following assumptions hold.

- (f<sub>1</sub>) For any  $x \in K$ ,  $f(x, x) \geq 0$ .
- (f<sub>2</sub>) For any  $x, y \in K$  and  $\alpha \in [0, 1]$ ,  $\lim_{\alpha \rightarrow +0} \frac{f(x + \alpha(y - x), x)}{\alpha} = -f(x, y)$ .
- (f<sub>3</sub>) For any  $x \in K$ , the map  $y \mapsto f(x, y)$  is convex and lower semicontinuous.
- (f<sub>4</sub>)  $f$  is stably pseudomonotone on  $K$  w.r.t.  $K^*$ .

The following lemma shows that under suitable conditions, the solution set of (EP) and (DEP) are the same.

**Lemma 3.1.** *Let  $K$  be a nonempty, closed, and convex subset of a real reflexive Banach space  $X$ . suppose that  $f$  satisfies the conditions  $(f_1)$ – $(f_4)$ . Then for any given  $\xi \in K^*$  the solution of the problem  $x \in K$  is a solution of the problem*

$$\text{find } x \in K \text{ such that } f(x, y) - \langle \xi, y - x \rangle \geq 0, \forall y \in K \quad (3.1)$$

*if and only if it is a solution of the problem*

$$\text{find } x \in K \text{ such that } f(y, x) - \langle \xi, x - y \rangle \leq 0, \forall y \in K \quad (3.2)$$

*In addition, if  $K$  is bounded, then  $S^P$  and  $S^D$  are solvable.*

**Proof.** Let  $x$  be a solution of (3.1). Then  $f(x, y) - \langle \xi, y - x \rangle \geq 0 \forall y \in K$ . By condition  $(f_4)$ ,  $x$  is a solution of (3.2). Conversely, let  $x$  be a solution of (3.2). Then  $f(y, x) - \langle \xi, x - y \rangle \geq 0 \forall y \in K$ . For any  $z \in K$ , set  $z_t := x + t(z - x)$ , for any  $t \in (0, 1)$ . Then  $z_t \in K$ , because of the convexity of  $K$ . It follows from  $(f_1)$  that

$$f(z_t, z) \leq \langle \xi, x - z_t \rangle = t \langle \xi, x - z \rangle.$$

Condition  $(f_2)$  give that  $-f(z, x) \leq \langle \xi, x - z \rangle$ . Hence,  $x$  solve (3.1).

Next, we prove that  $S^P$  and  $S^D$  are nonempty provided that  $K$  is bounded. Define  $F, G : K \rightarrow 2^K$  by

$$F(y) = \{x \in K : f(x, y) \geq 0\}, \forall y \in K \text{ and } G(y) = \{x \in K : f(y, x) \leq 0\}, \forall y \in K.$$

Then,  $G(y)$  is closed, convex and compact for all  $x \in K$ . It follows from Lemma 2.5 that

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \emptyset.$$

Then there exists  $\bar{x} \in K$  which solve (EP) and (DEP).  $\square$

**Theorem 3.2.** *Let  $K$  be a nonempty, closed, and convex subset of a real reflexive Banach space  $X$  with  $\text{int } K^* \neq \emptyset$ . Suppose that  $f$  satisfies the conditions  $(f_1)$ – $(f_4)$ . Then the following statements are equivalent:*

- (i)  $S^P$  is a nonempty and bounded.
- (ii)  $S^D$  is a nonempty and bounded.
- (iii)  $\mathcal{F}_K^+ \neq \emptyset$ .

**Proof.** It follows from Lemma 3.1, (i)  $\Leftrightarrow$  (ii) in the case where  $\xi = 0$ . Define a function  $g : K \times K^* \rightarrow \mathbb{R}$  by

$$g(x, \xi) := \sup_{y \in K} \frac{f(y, x) - \langle \xi, x - y \rangle}{\max\{1, \|y\|\}}, \forall x \in K, \xi \in K^*.$$

It follows from  $(f_3)$  that  $g(\cdot, \xi)$  is convex and lower semicontinuous function and  $g(x) \geq 0$  for all  $x \in K$  and  $\xi \in K^*$ . Define the set  $A := \{x \in K : g(x) \leq 0\}$ . Then by Lemma 2.5,  $A$  is nonempty, closed and convex set. Clearly,  $A$  is the solution set of (DEP). By



Lemma 3.1,  $A$  is also the solution set of (EP). Let  $\xi' \in \text{int } K^*$ . For every positive integer  $n$ , define

$$A_n := \left\{ x \in K : g(x, 0) \leq \frac{1}{n} \langle \xi', x \rangle \right\}.$$

Then  $\{A_n\}$  is a decreasing sequence as  $n \rightarrow +\infty$  of closed and convex subsets of  $K$  and  $A = \bigcap_{i=1}^{\infty} A_i$ . Notice that  $\text{barr } K$  has a nonempty interior, then  $K$  is well-positioned.

(ii)  $\Rightarrow$  (iii) Suppose that  $A$  is nonempty and bounded. It follows from Lemma 2.2 that there exists  $n_0$  such that  $A_{n_0}$  is nonempty and bounded. Setting  $r := g\left(z_0, \frac{1}{n_0} \xi'\right)$  for some  $z_0 \in A_{n_0}$ . We then have  $r \geq 0$ . We now consider the following set  $C_0 := \left\{ x \in K : g(x, 0) \leq \frac{1}{n_0} \langle \xi, x \rangle + r \right\}$ . Then by Lemma 2.3,  $C_0$  is nonempty and bounded. After calculating we have, for any  $y \in K$ ,

$$\frac{f(y, x)}{\max\{1, \|y\|\}} - \frac{f(y, x) - \left\langle \frac{1}{n_0} \xi, x - y \right\rangle}{\max\{1, \|y\|\}} = \frac{1}{n_0} \frac{\langle \xi, x - y \rangle}{\max\{1, \|y\|\}} \leq \frac{1}{n_0} \langle \xi, x \rangle,$$

because of  $\xi' \in \text{int } K^*$ . Thus  $g(x, 0) - g\left(x, \frac{1}{n_0} \xi'\right) \leq \frac{1}{n_0} \langle \xi', x \rangle$ , and so the set

$$C := \left\{ x \in K : g\left(x, \frac{1}{n_0} \xi'\right) \leq r \right\} \subset C_0$$

is nonempty, closed, convex and bounded.

Set

$$K_i = \{x \in K : \|x\| \leq i\}, \quad i = 1, 2, \dots$$

and

$$\hat{g}_i(x) := \sup_{y \in K_i} \frac{f(y, x) - \frac{1}{n_0} \langle \xi', x - y \rangle}{\max\{1, \|y\|\}}, \quad \forall x \in X.$$

Then  $\hat{g}_i(x)$  is convex and lower semicontinuous on bounded subset  $K_i$  for all  $i \in \mathbb{N}$  and  $\hat{g}_i(x) \geq 0$  for all  $x \in K_i$ . For every  $i \in \mathbb{N}$ , it follows from the prove in Lemma 3.1 that there are  $x_i \in K_i$  such that  $f(y, x_i) - \frac{1}{n_0} \langle \xi', x_i - y \rangle \leq 0, \forall y \in K_i$ . This implies that  $\hat{g}_i(x_i) = 0$ . Define  $D_i := \{x \in K : \hat{g}_i(x) \leq r\}$ . It is not hard to check that  $\{D_i\}_{i=1}^{\infty}$  is a decreasing sequence of nonempty, closed and convex subsets of  $K$  and  $C = \bigcap_{i=1}^{\infty} D_i$ . Since  $C$  is nonempty and bounded, it follows from Lemma 2.2 that there exists  $i_0 \in \mathbb{N}$  such that  $D_{i_0}$  is nonempty and bounded. Then there exists a positive integer  $L$  such that  $\sup_{x \in D_{i_0}} \|x\| < L$  and  $i_0 \leq L$ . since  $\hat{g}_L(x_L) = 0, x_L \in D_L \subset D_{i_0}$ , and so  $x_L \in K_L$ .

For any  $y \in K$  and  $t \in (0, 1)$ , setting  $y_t := (1 - t)x_L + ty \in K_L$ , we have  $y_t \rightarrow x_L$  as  $t \rightarrow +0$ . It follows from linearity that, for all  $y \in K$

$$f(y_t, x_L) - \frac{1}{n_0} \langle \xi', x_L - y_t \rangle \leq 0 \Rightarrow \frac{f(y_t, x_L)}{t} - \frac{1}{n_0} \langle \xi', x_L - y \rangle \leq 0.$$

By  $(f_3)$ , we have  $f(x_L, y) \geq \frac{1}{n_0} \langle \xi', y - x_L \rangle$  for all  $y \in K$ .

For any  $d \in K_\infty \setminus \{0\}$ , we know that  $x_L + d \in K$ , and so  $f(x_L, x_L + d) \geq \frac{1}{n_0} \langle \xi', d \rangle > 0$ , since  $\xi' \in \text{int } K^*$ . Therefore,  $x_L \in \mathcal{F}_K^+$ .

(iii)  $\Rightarrow$  (i) Suppose that  $\mathcal{F}_K^+ \neq \emptyset$ . Then there exists  $x_0 \in K$  such that  $f(x_0, x_0 + d) > 0$ ,  $\forall d \in K_\infty \setminus \{0\}$ . Setting  $D := \{x \in K : f(x_0, x) \leq 0\}$ . It is not hard to check that  $D$  is nonempty, closed and convex. We claim that  $D$  is bounded. If not, there exists  $x_n \in D$  with  $\|x_n\| \rightarrow +\infty$ . Without loss of generality, we can suppose that  $x_n/\|x_n\| \rightarrow d \in K_\infty$ . Since  $K$  is well-positioned, from Lemma 2.1 we have  $d \neq 0$ . It follows from  $x_n \in D$  that  $f(x_0, x_n) \leq 0$ . Since  $f$  is convex and lower semicontinuous at second variable, it follows that

$$f(x_0, x_0 + d) \leq \liminf_{n \rightarrow \infty} f\left(x_0, x_0 + \frac{1}{\|x_n\|}(x_n - x_0)\right) \leq 0,$$

which contradicts  $x_0 \in \mathcal{F}_K^+$ . Thus  $D$  is nonempty, convex and weak compact. Define

$$D_z := \{x \in D : f(x, z) \geq 0\}, \forall z \in K.$$

Thus

$$\bar{x} \in S^P \Leftrightarrow \bar{x} \in \bigcap_{z \in K} D_z.$$

Define

$$D'_z = \{x \in D : f(z, x) \leq 0, \forall z \in K\}.$$

Then  $D'_z$  is nonempty, closed and convex. By a similar argument as in the proof of Lemma 3.1, we can show that

$$\bigcap_{z \in K} D'_z = \bigcap_{z \in K} D_z.$$

For any finite set  $\{z_i : i = 1, 2, \dots, n\} \subset K$ , let  $M = \text{conv}\{D \cup \{z_1, z_2, \dots, z_n\}\}$ . It is known that  $M$  is weakly compact. By Lemma 3.1, there exists  $\hat{x} \in M$  such that

$$f(y, \hat{x}) \leq 0, \forall y \in M.$$

From  $x_0 \in D \subset M$ , we have  $f(x_0, \hat{x}) \leq 0$  and so  $\hat{x} \in D$ . Furthermore, we have

$$f(z_i, \hat{x}) \leq 0, i = 1, 2, \dots, n,$$

because  $z_i \in M$ . Therefore  $\hat{x} \in \bigcap_{i=1}^n D'_{z_i}$ , and so  $\{D'_z : z \in K\}$  has the finite intersection property. Since  $D$  is weakly compact, we get  $S^P = \bigcap_{z \in K} D_z = \bigcap_{z \in K} D'_z \neq \emptyset$ . Then  $S^P$  is nonempty. Since  $D$  is bounded and  $S^P = \bigcap_{z \in K} D_z \subset D$ , we have  $S^P$  is bounded. The proof is complete.  $\square$

**Remark 3.3.** Theorem 3.2 discusses the characterization of nonemptiness and boundedness for  $S^P$  and  $S^D$ , which is more general than the result in Theorem 3.1 of [7] in the case that  $f$  is relaxed to the stably pseudomonotone mapping.

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