CHAPTER III

MAIN RESULTS

3.1 Generalized Quotient Theorem

In this section, we prove the generalized quotient theorem.

Theorem 3.1.1. *Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ where $a_n \neq 0$, and $g(x) = x^m - b_{m-1}x^{m-1} - \cdots - b_2x^2 - b_1x - b_0$, be polynomials in $\mathbb{C}[x]$ and suppose that $m = \deg g(x) \leq \deg f(x) = n$, and the quotient on dividing $f(x)$ by $g(x)$ is $q(x) = d_{n-m}x^{n-m} + d_{n-m-1}x^{n-m-1} + \cdots + d_2x^2 + d_1x + d_0$ then*

$$d_{n-m-k} = \sum_{i=0}^{k} s_{k+1-i}a_{n-i}, \text{ for } k = 0, 1, \ldots, n-m \tag{3.1}$$

where $\{s_r\}$ is the linear recurrent sequence, $s_1 = 1$ and $s_r = \sum_{i=1}^{r-1} b_{m-i}s_{r-i}$, $r = 2, 3, \ldots$.

Proof. We will prove this theorem by mathematical induction on k.

In case $k = 0$, we have that $d_{n-m-0} = d_{n-m}$. Since $b_m = 1$, from (2.6) asserts that

$$d_{n-m} = a_n.$$

Since $s_1 = 1$, we have that

$$d_{n-m} = a_n = a_{n-0} = s_1a_{n-0} = \sum_{i=0}^{k} s_{0+1-i}a_{n-i}.$$

Thus the theorem is true for $k = 0$. Assume that the theorem is true for all $v < k \leq n - m$, such that

$$d_{n-m-v} = \sum_{i=0}^{v} s_{v+1-i}a_{n-i}, \text{ for all } v < n-m.$$

We will prove that the theorem is true for d_{n-m-k}. From (2.7), we have

$$d_{n-m-k} = a_{n-k} + b_{m-k}d_{n-m} + b_{m-(k-1)}d_{n-m-1)} + \cdots + b_{m-1}d_{n-m-(k-1)}.$$

By the first step and induction hypothesis, we have

$$\begin{aligned}
d_{n-m-k} &= s_1a_{n-k} + b_{m-k}\sum_{i=0}^{0} s_{0+1-i}a_{n-i} + b_{m-(k-1)}\sum_{i=0}^{1} s_{1+1-i}a_{n-i} \\
&\quad + b_{m-(k-2)}\sum_{i=0}^{2} s_{2+1-i}a_{n-i} + \cdots + b_{m-1}\sum_{i=0}^{k-1} s_{(k-1)+1-i}a_{n-i} \\
&= s_1a_{n-k} + b_{m-k}(s_1a_{n-0}) + b_{m-(k-1)}(s_1a_{n-1} + \sum_{i=0}^{1-1} s_{1+1-i}a_{n-i}) \\
&\quad + b_{m-(k-2)}(s_1a_{n-2} + \sum_{i=0}^{2-1} s_{2+1-i}a_{n-i}) + \cdots \\
&\quad + b_{m-1}(s_1a_{n-(k-1)} + \sum_{i=0}^{(k-1)-1} s_{(k-1)+1-i}a_{n-i}) \\
&= s_1a_{n-k} + (b_{m-k}s_1a_n) + (b_{m-(k-1)}s_1a_{n-1} + b_{m-(k-1)}\sum_{i=0}^{0} s_{1+1-i}a_{n-i}) \\
&\quad + (b_{m-(k-2)}s_1a_{n-2} + b_{m-(k-2)}\sum_{i=0}^{1} s_{2+1-i}a_{n-i}) + \cdots \\
&\quad + (b_{m-1}s_1a_{n-(k-1)} + b_{m-1}\sum_{i=0}^{k-2} s_{(k-1)+1-i}a_{n-i}) \\
&= s_1a_{n-k} + (b_{m-k}s_1a_n + b_{m-(k-1)}s_{1+1-0}a_{n-0} + b_{m-(k-2)}s_{2+1-0}a_{n-0} \\
&\quad + \cdots + b_{m-1}s_{(k-1)+1-0}a_{n-0}) + (b_{m-(k-1)}s_1a_{n-1}) \\
&\quad + (b_{m-(k-2)}s_1a_{n-2} + b_{m-(k-2)}\sum_{i=1}^{1} s_{2+1-i}a_{n-i}) + \cdots \\
&\quad + (b_{m-1}s_1a_{n-(k-1)} + b_{m-1}\sum_{i=1}^{k-2} s_{(k-1)+1-i}a_{n-i}) \\
&= s_1a_{n-k} + (b_{m-k}s_1 + b_{m-(k-1)}s_2 + b_{m-(k-2)}s_3 + \cdots + b_{m-1}s_k)a_n \\
&\quad + (b_{m-(k-1)}s_1a_{n-1}) + (b_{m-(k-2)}s_1a_{n-2} + b_{m-(k-2)}\sum_{i=1}^{1} s_{2+1-i}a_{n-i}) \\
&\quad + \cdots + (b_{m-1}s_1a_{n-(k-1)} + b_{m-1}\sum_{i=1}^{k-2} s_{(k-1)+1-i}a_{n-i}).
\end{aligned}$$

Since

$$s_1 = 1 \quad \text{and} \quad s_r \;=\; \sum_{i=1}^{r-1} b_{m-i}s_{r-i},$$
$$s_{k+1} \;=\; \sum_{i=1}^{k+1-1} b_{m-i}s_{k+1-i}$$
$$= \sum_{i=1}^{k} b_{m-i}s_{k+1-i},$$

we have

$$\begin{aligned} d_{n-m-k} &= s_1a_{n-k} + (\sum_{i=1}^{k} b_{m-i}s_{k+1-i})a_n + (b_{m-(k-1)}s_1a_{n-1}) \\ &\quad + (b_{m-(k-2)}s_1a_{n-2} + b_{m-(k-2)}\sum_{i=1}^{1} s_{(2+1-i}a_{n-i}) + \dots \\ &\quad + (b_{m-1}s_1a_{n-(k-1)} + b_{m-1}\sum_{i=1}^{k-2} s_{(k-1)+1-i}a_{n-i}) \\ &= s_1a_{n-k} + (s_{k+1})a_n + (b_{m-(k-1)}s_1a_{n-1}) \\ &\quad + (b_{m-(k-2)}s_1a_{n-2} + b_{m-(k-2)}\sum_{i=1}^{1} s_{(2+1-i}a_{n-i}) + \dots \\ &\quad + (b_{m-1}s_1a_{n-(k-1)} + b_{m-1}\sum_{i=1}^{k-2} s_{(k-1)+1-i}a_{n-i}). \end{aligned}$$

Similarly,

$$\begin{aligned} d_{n-m-k} &= s_1a_{n-k} + (s_{k+1})a_n \\ &\quad + (b_{m-(k-1)}s_1a_{n-1} + b_{m-(k-2)}s_{2+1-1}a_{n-1} + \dots + b_{m-1}s_{(k-1)+1-1}a_{n-1}) \\ &\quad + (b_{m-(k-2)}s_1a_{n-2}) + \dots + (b_{m-1}s_1a_{n-(k-1)} + b_{m-1}\sum_{i=2}^{k-2} s_{(k-1)+1-i}a_{n-i}) \\ &= s_1a_{n-k} + (s_{k+1})a_n \\ &\quad + (b_{m-(k-1)}s_1 + b_{m-(k-2)}s_2 + \dots + b_{m-1}s_{k-1})a_{n-1} \\ &\quad + (b_{m-(k-2)}s_1a_{n-2}) + \dots + (b_{m-1}s_1a_{n-(k-1)} + b_{m-1}\sum_{i=2}^{k-2} s_{(k-1)+1-i}a_{n-i}) \\ &= s_1a_{n-k} + (s_{k+1})a_n + (s_k)a_{n-1} + (b_{m-(k-2)}s_1a_{n-2}) + \dots \\ &\quad + (b_{m-1}s_1a_{n-(k-1)} - b_{m-1}\sum_{i=2}^{k-2} s_{(k-1)+1-i}a_{n-i}). \end{aligned}$$

Continuing in this processes, we get

$$\begin{aligned} d_{n-m-k} &= s_1 a_{n-k} + s_{k+1}a_n + s_k a_{n-1} + s_{k-1}a_{n-2} + \cdots + s_2 a_{n-k+1} \\ &= \sum_{i=0}^{k} s_{k+1-i}a_{n-i}. \end{aligned}$$

The theorem is proved. □

Theorem 3.1.2.(***Generalized Quotient Theorem***) *Let* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ *and* $g(x) = b_m x^m - b_{m-1}x^{m-1} - \cdots - b_2 x^2 - b_1 x - b_0$ *where* $a_n \neq 0$ *and* $b_m \neq 0$, *be polynomials in* $\mathbb{C}[x]$ *and suppose that* $m = \deg g(x) \leq \deg f(x) = n$, *and the quotient on dividing* $f(x)$ *by* $g(x)$ *is* $q(x) = d_{n-m}x^{n-m} + d_{n-m-1}x^{n-m-1} + \cdots + d_2 x^2 + d_1 x + d_0$ *then*

$$d_{n-m-k} = \sum_{j=0}^{k} t_{k+1-j}a_{n-j}, \tag{3.2}$$

and

$$q(x) = \sum_{i=0}^{n-m}\left(\sum_{j=0}^{n-m-i} t_{n-m-i+1-j}a_{n-j}\right)x^i. \tag{3.3}$$

where $\{t_r\}$ *is the linear recurrent sequence,* $t_1 = \frac{1}{b_m}$ *and* $t_r = \frac{1}{b_m}\sum_{i=1}^{r-1} b_{m-i}t_{r-i}$ *for* $r = 2, 3, 4, \ldots$.

Proof. We will prove this theorem by mathematical induction on k.

In case $k = 0$, we have that $d_{n-m-0} = d_{n-m}$. From (2.6) assert that

$$b_m d_{n-m} = a_n \text{ implies } d_{n-m} = \frac{a_n}{b_m} = \frac{a_{n-0}}{b_m} = \frac{1}{b_m}a_{n-0} = t_1 a_{n-0}.$$

Thus the theorem is true for $k = 0$.

Assume the theorem is true for d_{n-m-v}, for all $v < k \leq n - m$. Thus

$$d_{n-m-v} = \sum_{i=1}^{v} t_{v+1-i}a_{n-i}, \quad \text{for all } v < n - m.$$

We will show that it is true for d_{n-m-k}. From (2.6), we have

$$d_{n-m-k} = \frac{a_{n-k} + b_{m-k}d_{n-m} + b_{m-(k-1)}d_{n-m-1} + \cdots + b_{m-1}d_{n-m-(k-1)}}{b_m}$$

by the first step and induction hypothesis we have

$$\begin{aligned}
d_{n-m-k} &= \frac{1}{b_m}a_{n-k} + \frac{b_{m-k}}{b_m}\sum_{i=0}^{0} t_{0+1-i}a_{n-i} + \frac{b_{m-(k-1)}}{b_m}\sum_{i=0}^{1} t_{1+1-i}a_{n-i} \\
&\quad + \frac{b_{m-(k-2)}}{b_m}\sum_{i=0}^{2} t_{2+1-i}a_{n-i} + \cdots + \frac{b_{m-1}}{b_m}\sum_{i=0}^{k-1} t_{(k-1)+1-i}a_{n-i} \\
&= t_1a_{n-k} + \frac{b_{m-k}}{b_m}(t_1a_{n-0}) + \frac{b_{m-(k-1)}}{b_m}(t_1a_{n-1} + \sum_{i=0}^{1-1} t_{1+1-i}a_{n-i}) \\
&\quad + \frac{b_{m-(k-2)}}{b_m}(t_1a_{n-2} + \sum_{i=0}^{2-1} t_{2+1-i}a_{n-i}) + \cdots + \frac{b_{m-1}}{b_m}(t_1a_{n-(k-1)} + \\
&\quad + \sum_{i=0}^{(k-1)-1} t_{(k-1)+1-i}a_{n-i}) \\
&= t_1a_{n-k} + \frac{1}{b_m}(b_{m-k}t_1a_n) + \frac{1}{b_m}(b_{m-(k-1)}t_1a_{n-1} \\
&\quad + b_{m-(k-1)}\sum_{i=0}^{0} t_{1+1-i}a_{n-i}) + \frac{1}{b_m}(b_{m-(k-2)}t_1a_{n-2} \\
&\quad + b_{m-(k-2)}\sum_{i=0}^{1} t_{2+1-i}a_{n-i}) + \cdots + \frac{1}{b_m}(b_{m-1}t_1a_{n-(k-1)} \\
&\quad + b_{m-1}\sum_{i=0}^{k-2} t_{(k-1)+1-i}a_{n-i}) \\
&= t_1a_{n-k} + \frac{1}{b_m}(b_{m-k}t_1a_n + b_{m-(k-1)}t_{1+1-0}a_{n-0} + b_{m-(k-2)}t_{2+1-0}a_{n-0} \\
&\quad + \cdots + b_{m-1}t_{(k-1)+1-0}a_{n-0}) + \frac{1}{b_m}(b_{m-(k-1)}t_1a_{n-1}) + \\
&\quad \frac{1}{b_m}(b_{m-(k-2)}t_1a_{n-2} + b_{m-(k-2)}\sum_{i=1}^{1} t_{2+1-i}a_{n-i}) + \cdots + \\
&\quad \frac{1}{b_m}(b_{m-1}t_1a_{n-(k-1)} + b_{m-1}\sum_{i=1}^{k-2} t_{(k-1)+1-i}a_{n-i})
\end{aligned}$$

$$
\begin{aligned}
d_{n-m-k} &= t_1a_{n-k} + \tfrac{1}{b_m}(b_{m-k}t_1 + b_{m-(k-1)}t_2 + b_{m-(k-2)}t_3 + \cdots + b_{m-1}t_k)a_n \\
&\quad + \frac{1}{b_m}(b_{m-(k-1)}t_1a_{n-1}) + \frac{1}{b_m}(b_{m-(k-2)}t_1a_{n-2}+ \\
&\quad b_{m-(k-2)}\sum_{i=1}^{1} t_{2+1-i}a_{n-i}) + \cdots + \frac{1}{b_m}(b_{m-1}t_1a_{n-(k-1)}+ \\
&\quad b_{m-1}\sum_{i=1}^{k-2} t_{(k-1)+1-i}a_{n-i}).
\end{aligned}
$$

Since

$$t_1 = 1 \quad \text{and} \quad t_r = \frac{1}{b_m}\sum_{i=1}^{r-1} b_{m-1}t_{r-i},$$

we obtain

$$t_{k+1} = \frac{1}{b_m}\sum_{i=1}^{k+1-1} b_{m-1}t_{k+1-i} = \frac{1}{b_m}\sum_{i=1}^{k} b_{m-1}t_{k+1-i}.$$

Thus

$$
\begin{aligned}
d_{n-m-k} &= t_1a_{n-k} + (\tfrac{1}{b_m}\textstyle\sum_{i=1}^{k} b_{m-1}t_{k+1-i})a_n + \tfrac{1}{b_m}(b_{m-(k-1)}t_1a_{n-1}) \\
&\quad + \frac{1}{b_m}(b_{m-(k-2)}t_1a_{n-2} + b_{m-(k-2)}\sum_{i=1}^{1} t_{2+1-i}a_{n-i}) + \cdots \\
&\quad + \frac{1}{b_m}(b_{m-1}t_1a_{n-(k-1)} + b_{m-1}\sum_{i=1}^{k-2} t_{(k-1)+1-i}a_{n-i}) \\
&= t_1a_{n-k} + (t_{k+1})a_n + \frac{1}{b_m}(b_{m-(k-1)}t_1a_{n-1}) + \frac{1}{b_m}(b_{m-(k-2)}t_1a_{n-2} \\
&\quad + b_{m-(k-2)}\sum_{i=1}^{1} t_{2+1-i}a_{n-i}) + \cdots + \frac{1}{b_m}(b_{m-1}t_1a_{n-(k-1)} \\
&\quad + b_{m-1}\sum_{i=1}^{k-2} t_{(k-1)+1-i}a_{n-i}).
\end{aligned}
$$

Similarly,

$$\begin{aligned}
d_{n-m-k} &= t_1 a_{n-k} + t_{k+1} a_n + \frac{1}{b_m}(b_{m-(k-1)} t_1 a_{n-1} + b_{m-(k-2)} t_{2+1-1} a_{n-1} + \cdots \\
&\quad + b_{m-1} t_{(k-1)+1-1} a_{n-1}) + \frac{1}{b_m}(b_{m-(k-2)} t_1 a_{n-2}) + \cdots \\
&\quad + \frac{1}{b_m}(b_{m-1} t_1 a_{n-(k-1)} + b_{m-1} \sum_{i=2}^{k-2} t_{(k-1)+1-i} a_{n-i}) \\
&= t_1 a_{n-k} + t_{k+1} a_n + \frac{1}{b_m}(b_{m-(k-1)} t_1 + b_{m-(k-2)} t_2 + \cdots + b_{m-1} t_{k-1}) a_{n-1} \\
&\quad + \frac{1}{b_m}(b_{m-(k-2)} t_1 a_{n-2}) + \cdots + \frac{1}{b_m}(b_{m-1} t_1 a_{n-(k-1)} \\
&\quad + b_{m-1} \sum_{i=2}^{k-2} t_{(k-1)+1-i} a_{n-i}) \\
&= t_1 a_{n-k} + t_{k+1} a_n + t_k a_{n-1} + \tfrac{1}{b_m}(b_{m-(k-2)} t_1 a_{n-2}) + \cdots \\
&\quad + \frac{1}{b_m}(b_{m-1} t_1 a_{n-(k-1)} - b_{m-1} \sum_{i=2}^{k-2} t_{(k-1)+1-i} a_{n-i}).
\end{aligned}$$

Continuing in this processes, we get

$$\begin{aligned}
d_{n-m-k} &= t_1 a_{n-k} + t_{k+1} a_n + t_k a_{n-1} + t_{k-1} a_{n-2} + \cdots + t_2 a_{n-k-1} \\
&= \sum_{j=0}^{k} t_{k+1-j} a_{n-j}.
\end{aligned}$$

If $i = n - m - k$ then $k = n - m - i$, it follows that

$$d_i = \sum_{j=0}^{n-m-i} t_{n-m-i+1-j} a_{n-j}.$$

Since $q(x) = \sum_{i=0}^{n-m} d_i x^i$, we have

$$q(x) = \sum_{i=0}^{n-m} \left(\sum_{j=0}^{n-m-i} t_{n-m-i+1-j} a_{n-j} \right) x^i.$$

The proof is complete. □

Note: The coefficients $a_0, a_1, \ldots, a_{m-1}$ of $f(x)$ are not related to the quotient polynomial.

The remainder theorem asserts that the remainder on dividing $f(x)$ by $x - b$ is $f(b)$, now we can prove the Quotient Theorem as follows:

Corollary 3.1.1.(*Quotient Theorem*) *If* $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, $a_n \neq 0$, *then the quotient on dividing* $f(x)$ *by* $x - b$ *is*

$$q(x) = \sum_{i=0}^{n-1}\sum_{j=1}^{n-1} b^{j-1}a_{i+j}x^i.$$

Proof. From the well known Remainder Theorem asserts that

$$f(x) = (x - b)q(x) + f(b).$$

Thus

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$
$$= (x - b)(d_{n-1}x^{n-1} + d_{n-2}x^{n-2} + \cdots + d_2x^2 + d_1x + d_0) + f(b).$$

Since $m = 1$, by Theorem 3.1.1, we have

$$d_{n-1-k} = \sum_{i=0}^{k} s_{k+1-i}a_{n-i},$$

where $\{s_r\}$ is the linear recurrent sequence, $s_1 = 1$ and $s_r = \sum_{i=1}^{r-1} b_{1-i}s_{r-i}$ for $r = 2, 3, \ldots$.

For $n - 1 - k = 0$ implies that $k = n - 1$. Thus

$$d_0 = \sum_{i=0}^{n-1} s_{n-1+1-i}a_{n-i} = \sum_{i=0}^{n-1} s_{n-i}a_{n-i} = s_1a_1 + s_2a_2 + \cdots + s_na_n,$$
$$d_1 = \sum_{i=0}^{n-2} s_{n-2+1-i}a_{n-i} = \sum_{i=0}^{n-2} s_{n-1-i}a_{n-i} = s_1a_2 + s_2a_3 + \cdots + s_{n-1}a_n,$$
$$\vdots \qquad (3.4)$$
$$d_{n-2} = \sum_{i=0}^{1} s_{1+1-i}a_{n-i} = \sum_{i=0}^{1} s_{2-i}a_{n-i} = s_1a_{n-1} + s_2a_n,$$
$$d_{n-1} = \sum_{i=0}^{0} s_{0+1-i}a_{n-i} = \sum_{i=0}^{0} s_{1-i}a_{n-i} = s_1a_n.$$

Since $s_1 = 1$ and $s_r = \sum_{i=1}^{r-1} b_{1-i}s_{r-i}$, we have

$$s_1 = 1,$$
$$s_2 = \sum_{i=1}^{2-1} b_{1-i}s_{2-i} = \sum_{i=1}^{1} b_{1-i}s_{2-i} = b_0s_1 = b_01 = b_0 = b,$$
$$s_3 = \sum_{i=1}^{3-1} b_{1-i}s_{3-i} = \sum_{i=1}^{2} b_{1-i}s_{3-i} = b_0s_2 = bb = b^2,$$
$$s_4 = \sum_{i=1}^{4-1} b_{1-i}s_{4-i} = \sum_{i=1}^{3} b_{1-i}s_{4-i} = b_0s_3 = bb^2 = b^3,$$
$$\vdots$$
$$s_n = \sum_{i=1}^{n-1} b_{1-i}s_{n-i} = b_0s_{n-1} = bb^{n-2} = b^{n-1}. \qquad (3.5)$$

From (3.4) and (3.5), we get

$$d_0 = 1a_1 + ba_2 + \cdots + b^{n-2}a_{n-1} + b^{n-1}a_n,$$
$$d_1 = 1a_2 + ba_3 + \cdots + b^{n-2}a_n,$$
$$\vdots$$
$$d_{n-2} = 1a_{n-1} + ba_n,$$
$$d_{n-1} = 1a_n.$$

Hence

$$d_i = \sum_{j=1}^{n-i} b^{j-1} a_{i+j}.$$

Since

$$q(x) = d_{n-1}x^{n-1} + d_{n-2}x^{n-2} + \cdots + d_1 x + d_0 = \sum_{i=0}^{n-1} d_i x^i,$$

we have

$$q(x) = \sum_{i=0}^{n-1} d_i x^i = \sum_{i=0}^{n-1} \sum_{j-1}^{n-1} b^{j-1} a_{i+j} x^i.$$

This completes the proof. □

3.2 Generalized Remainder Theorem

Theorem 3.2.1.(*Generalized Remainder Theorem*) *Let* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ *and* $g(x) = b_m x^m - b_{m-1}x^{m-1} - \cdots - b_2 x^2 - b_1 x - b_0$, *where* $a_n \neq 0$ *and* $b_m \neq 0$, *be polynomials in* $\mathbb{C}[x]$ *and suppose that* $m = \deg g(x) \leq \deg f(x) = n$, *then the remainder on dividing* $f(x)$ *by* $g(x)$ *is*

$$r(x) = \sum_{k=0}^{m-1} \left(a_k + \sum_{i+j=k} b_i \sum_{v=0}^{n-m-j} t_{n-m-j+1-v} a_{n-v} \right) x^k \tag{3.6}$$

where $\{t_r\}$ *is the linear recurrent sequence,* $t_1 = \frac{1}{b_m}$, *and* $t_r = \frac{1}{b_m} \sum_{i=1}^{r-1} b_{m-i} t_{r-i}$ *for* $r = 2, 3, \ldots$.

Proof. The division algorithm in the polynomial ring $\mathbb{C}[x]$ asserts that there are polynomials $q(x) = d_{n-m}x^{n-m} + d_{n-m-1}x^{n-m-1} + \cdots + d_2 x^2 + d_1 x + d_0 = \sum_{k=0}^{n-m} d_k x^k$ and $r(x) = c_{m-1}x^{m-1} + c_{m-2}x^{m-2} + \cdots + c_2 x^2 + c_1 x + c_0 = \sum_{k=0}^{m-1} c_k x^k$ in $\mathbb{C}[x]$ such that:

$$f(x) = g(x)q(x) + r(x),$$

that is

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$
$$= (b_m x^m - b_{m-1}x^{m-1} - \cdots - b_1x - b_0)(d_{n-m}x^{n-m} + d_{n-m-1}x^{n-m-1}$$
$$+ \cdots + d_1x + d_0) + (c_{m-1}x^{m-1} + c_{m-2}x^{m-2} + \cdots + c_2x^2 + c_1x + c_0).$$

From the system of linear equation (2.4),

$$c_k - \sum_{i+j=k} b_i d_j = a_k$$

where $k = 0, 1, \ldots, m-1$, $i = 0, 1, \ldots, m$, and $j = 0, 1, \ldots, n-m$ implies that

$$c_k = a_k + \sum_{i+j=k} b_i d_j \tag{3.7}$$

where $k = 0, 1, \ldots, m-1$, $i = 0, 1, \ldots, m$; and $j = 0, 1, \ldots, n-m$. From (3.2),

$$d_{n-m-k} = \sum_{v=0}^{k} t_{k+1-v} a_{n-v}, \tag{3.8}$$

where $\{t_r\}$ is the linear recurrent sequence, $t_1 = \frac{1}{b_m}$, and $t_r = \frac{1}{b_m}\sum_{i=1}^{r-1} b_{m-i}t_{r-i}$ for $r = 2, 3, \ldots$.

If $j = n-m-k$ then $k = n-m-j$, hence (3.8) become

$$d_j = \sum_{v=0}^{n-m-j} t_{n-m-j+1-v} a_{n-v}. \tag{3.9}$$

Substitute d_j in equation (3.7), we get

$$c_k = a_k + \sum_{i+j=k} b_i \sum_{v=0}^{n-m-j} t_{n-m-j+1-v} a_{n-v}$$
$$= a_k + \sum_{i+j=k} \sum_{v=0}^{n-m-j} t_{n-m-j+1-v} a_{n-v} b_i. \tag{3.10}$$

Since $r(x) = c_{m-1}x^{m-1} + c_{m-2}x^{m-2} + \cdots + c_2x^2 + c_1x + c_0 = \sum_{k=0}^{m-1} c_k x^k$, we have

$$r(x) = \sum_{k=0}^{m-1} \left(a_k + \sum_{i+j=k} b_i \sum_{v=0}^{n-m-j} t_{n-m-j+1-v} a_{n-v} \right) x^k.$$

The theorem is proved. □

3.3 The Division Algorithm

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ and $g(x) = b_mx^m - b_{m-1}x^{m-1} - \cdots - b_2x^2 - b_1x - b_0$, where $a_n \neq 0$ and $b_m \neq 0$, be polynomials in $\mathbb{C}[x]$ and suppose that $m = \deg g(x) \leq \deg f(x) = n$, then by the division algorithm in $\mathbb{C}[x]$, there exists unique $q(x)$ and $r(x)$ in $\mathbb{C}[x]$ such that

$$f(x) = g(x)q(x) + r(x) \text{ where } r(x) = 0 \text{ or } \deg r(x) \leq \deg g(x).$$

From (3.4),

$$q(x) = \sum_{i=0}^{n-m} \left(\sum_{j=0}^{n-m-i} t_{n-m-i+1-j}a_{n-j} \right) x^i$$

and from (3.7),

$$r(x) = \sum_{k=0}^{m-1} \left(a_k + \sum_{i+j=k} b_i \sum_{v=0}^{n-m-j} t_{n-m-j+1-v}a_{n-v} \right) x^k.$$

Therefore

$$\sum_{i=0}^{n} a_ix^i = (\sum_{i=0}^{n-m}(\sum_{j=0}^{n-m-i} t_{n-m-i+1-j})x^i)(\sum_{i=1}^{m} b_ix^i) + \sum_{k=0}^{m-1}(a_k + \sum_{i+j=k} b_i \sum_{v=0}^{n-m-j} t_{n-m-j+1-v}a_{n-v})x^k$$

where $\{t_r\}$ is the linear recurrent sequence, $t_1 = \frac{1}{b_m}$, and $t_r = \frac{1}{b_m}\sum_{i=1}^{r-1} b_{m-i}t_{r-i}$ for $r = 2, 3, \ldots$.

3.4 Related Topics

In this section, we find remainder in case $g(x) = b(x-b_1)^{r_1}(x-b_2)^{r_2}\ldots(x-b_s)^{r_s}$, $r_1 + r_2 + \cdots + r_s = m$. By [5], let $\mathbb{C}$ be the complex field, $g(x)$ is a nonconstant polynomial in $\mathbb{C}[x]$,

$$g(x) = b(x-b_1)^{r_1}(x-b_2)^{r_2}\ldots(x-b_s)^{r_s} \tag{3.11}$$

where $r_1 + r_2 + \cdots + r_s = m$, $\deg g(x) = m$, if $f(x) \in \mathbb{C}[x]$ and $\deg f(x) > m$ then the remainder

$$r(x) = c_1 + c_2 x + \cdots + c_{m-1}x^{m-2} + c_m x^{m-1}$$

on dividing $f(x)$ by $g(x)$ is

$$r(x) = \frac{\det({}^1V)}{\det V} + \frac{\det({}^2V)}{\det V}x + \cdots + \frac{\det({}^{m-1}V)}{\det V}x^{m-2} + \frac{\det({}^mV)}{\det V}x^{m-1} \qquad (3.12)$$

where $\det V = \left(\prod_{i=1}^{s}\prod_{k=1}^{m_i-1} k!\right) \prod_{1\leq i<j\leq s} (b_j - b_i)^{m_i m_j} \neq 0$, V is the confluence Vandermonde matrix and $({}^jV)$ are matrices obtained form V by successively replacing the 1^{st}, 2^{nd}, $\ldots, m^{th}$ columns by the block column vector

$$[f^{(0)}(b_1) \;\; f^{(1)}(b_1) \;\; \ldots \;\; f^{(r_1-1)}(b_1) \;\vdots\; \cdots \;\vdots\; f^{(0)}(b_s) \;\; f^{(1)}(b_s) \;\; \ldots \;\; f^{(r_s-1)}(b_s)]^T. \qquad (3.13)$$

The proof of this theorem by produce a system of liner equations in the form $V\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = [c_1, c_2, \ldots, c_n]^T$ and $\mathbf{b}$ is the block column matrix in (3.13) that is,

$$\left[\begin{array}{cccc}
(1)^{(0)} & (b_1)^{(0)} & \ldots & (b_1^{m-2})^{(0)} & (b_1^{m-1})^{(0)} \\
(1)^{(1)} & (b_1)^{(1)} & \ldots & (b_1^{m-2})^{(1)} & (b_1^{m-1})^{(1)} \\
\vdots & \vdots & & \vdots & \vdots \\
(1)^{(r_1-1)} & (b_1)^{(r_1-1)} & \ldots & (b_1^{m-2})^{(r_1-1)} & (b_1^{m-1})^{(r_1-1)} \\ \hline
(1)^{(0)} & (b_2)^{(0)} & \ldots & (b_2^{m-2})^{(0)} & (b_2^{m-1})^{(0)} \\
(1)^{(1)} & (b_2)^{(1)} & \ldots & (b_2^{m-2})^{(1)} & (b_2^{m-1})^{(1)} \\
\vdots & \vdots & & \vdots & \vdots \\
(1)^{(r_2-1)} & (b_2)^{(r_2-1)} & \ldots & (b_2^{m-2})^{(r_2-1)} & (b_2^{m-1})^{(r_2-1)} \\ \hline
\vdots & \vdots & & \vdots & \vdots \\ \hline
(1)^{(0)} & (b_s)^{(0)} & \ldots & (b_s^{m-2})^{(0)} & (b_s^{m-1})^{(0)} \\
(1)^{(1)} & (b_s)^{(1)} & \ldots & (b_s^{m-2})^{(1)} & (b_s^{m-1})^{(1)} \\
\vdots & \vdots & & \vdots & \vdots \\
(1)^{(r_s-1)} & (b_s)^{(r_s-1)} & \ldots & (b_s^{m-2})^{(r_s-1)} & (b_s^{m-1})^{(r_s-1)}
\end{array}\right]
\left[\begin{array}{c}
c_1 \\ c_2 \\ \vdots \\ c_{r_1} \\ c_{r_1+1} \\ \vdots \\ c_{r_1+r_2} \\ c_{r_1+r_2+1} \\ \vdots \\ \vdots \\ \vdots \\ c_m
\end{array}\right]
=
\left[\begin{array}{c}
f^{(0)}(b_1) \\ f^{(1)}(b_1) \\ \vdots \\ f^{(r_1-1)}(b_1) \\ \hline
f^{(0)}(b_2) \\ f^{(1)}(b_2) \\ \vdots \\ f^{(r_2-1)}(b_2) \\ \hline
\vdots \\ \hline
f^{(0)}(b_s) \\ f^{(1)}(b_s) \\ \vdots \\ f^{(r_s-1)}(b_s)
\end{array}\right]. \qquad (3.14)$$

In solving this system of linear equations by Cramer's rule, the solution is

$$c_j = \frac{\det({}^jV)}{\det V}, \quad j = 1, 2, \ldots, m$$

and by [4], we can find inverse of V. Then we get $\mathbf{x} = V^{-1}\mathbf{b}$, the column matrix $\mathbf{x}$ is the coefficient matrix of the remainder polynomial on dividing $f(x)$ by $g(x)$.

For example, let $f(x)$ and $g(x)$ be any complex polynomial where $g(x) = b(x-b_1)^3(x-b_2)$ form (3.14), we get

$$V = \begin{bmatrix} 1 & b_1 & b_1^2 & b_1^3 \\ 0 & 1 & 2b_1 & 3b_1^2 \\ 1 & b_2 & b_2^2 & b_2^3 \\ 0 & 1 & 2b_2 & 3b_2^2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} f^{(0)}(b_1) \\ f^{(1)}(b_1) \\ f^{(0)}(b_2) \\ f^{(1)}(b_2) \end{bmatrix}$$

Now

$$V^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ b_1 & 1 & b_2 & 1 \\ b_1^2 & 2b_1 & b_2^2 & 2b_2 \\ b_1^3 & 3b_1^2 & b_2^3 & 3b_2^2 \end{bmatrix} =: W$$

Thus, form [4, p. 1546], we get

$$W^{-1} = \begin{bmatrix} \frac{b_2^2(b_2-3b_1)}{(b_2-b_1)^3} & \frac{6b_1b_2}{(b_2-b_1)^3} & \frac{-3(b_1+b_2)}{(b_2-b_1)^3} & \frac{2}{(b_2-b_1)^3} \\ \frac{-b_2^2b_1}{(b_2-b_1)^2} & \frac{b_2(b_2+2b_1)}{(b_2-b_1)^2} & \frac{-(2b_2+b_1)}{(b_2-b_1)^2} & \frac{1}{(b_2-b_1)^2} \\ \frac{b_1^2(b_1-3b_2)}{(b_1-b_2)^3} & \frac{6b_1b_2}{(b_1-b_2)^3} & \frac{-3(b_1+b_2)}{(b_1-b_2)^3} & \frac{2}{(b_1-b_2)^3} \\ \frac{-b_1^2b_2}{(b_1-b_2)^2} & \frac{b_1(b_1+2b_2)}{(b_1-b_2)^2} & \frac{-(2b_1+b_2)}{(b_1-b_2)^2} & \frac{1}{(b_1-b_2)^2} \end{bmatrix}. \tag{3.15}$$

In fact

$$(V^T)^{-1} = (V^{-1})^T.$$

Therefore form (3.15), we have

$$V^{-1} = \begin{bmatrix} \frac{b_2^2(b_2-3b_1)}{(b_2-b_1)^3} & \frac{-b_2^2b_1}{(b_2-b_1)^2} & \frac{b_1^2(b_1-3b_2)}{(b_1-b_2)^3} & \frac{-b_1^2b_2}{(b_1-b_2)^2} \\ \frac{6b_1b_2}{(b_2-b_1)^3} & \frac{b_2(b_2+2b_1)}{(b_2-b_1)^2} & \frac{6b_1b_2}{(b_1-b_2)^3} & \frac{b_1(b_1+2b_2)}{(b_1-b_2)^2} \\ \frac{-3(b_1+b_2)}{(b_2-b_1)^3} & \frac{-(2b_2+b_1)}{(b_2-b_1)^2} & \frac{-3(b_1+b_2)}{(b_1-b_2)^3} & \frac{-(2b_1+b_2)}{(b_1-b_2)^2} \\ \frac{2}{(b_2-b_1)^3} & \frac{1}{(b_2-b_1)^2} & \frac{2}{(b_1-b_2)^3} & \frac{1}{(b_1-b_2)^2} \end{bmatrix}.$$

Finally, we obtain

$$\mathbf{x} = V^{-1}\mathbf{b} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} \frac{b_2^2(b_2-3b_1)}{(b_2-b_1)^3} & \frac{-b_2^2 b_1}{(b_2-b_1)^2} & \frac{b_1^2(b_1-3b_2)}{(b_1-b_2)^3} & \frac{-b_1^2 b_2}{(b_1-b_2)^2} \\ \frac{6b_1 b_2}{(b_2-b_1)^3} & \frac{b_2(b_2+2b_1)}{(b_2-b_1)^2} & \frac{6b_1 b_2}{(b_1-b_2)^3} & \frac{b_1(b_1+2b_2)}{(b_1-b_2)^2} \\ \frac{-3(b_1+b_2)}{(b_2-b_1)^3} & \frac{-(2b_2+b_1)}{(b_2-b_1)^2} & \frac{-3(b_1+b_2)}{(b_1-b_2)^3} & \frac{-(2b_1+b_2)}{(b_1-b_2)^2} \\ \frac{2}{(b_2-b_1)^3} & \frac{1}{(b_2-b_1)^2} & \frac{2}{(b_1-b_2)^3} & \frac{1}{(b_1-b_2)^2} \end{bmatrix} \begin{bmatrix} f^{(0)}(b_1) \\ f^{(1)}(b_1) \\ f^{(0)}(b_2) \\ f^{(1)}(b_2) \end{bmatrix}.$$