CHAPTER II

PRELIMINARIES

This chapter contains the basic definitions, notations, and some known results needed for later chapter.

2.1 Some Auxiliary Results

A linear combination of column (or row) vectors $v_1, v_2, \ldots, v_m \in \mathbb{C}^n$ is

$$c_1v_1 + c_2v_2 + \cdots + c_mv_m$$

where the scalars $c_1, c_2, \ldots, c_m \in \mathbb{C}$. We shall define a linear combination of square complex matrices by the following.

Definition 2.1.1. If $A_1, A_2, \ldots, A_n \in M_n(\mathbb{C})$ is a square matrix of order n and $c_1, c_2, \ldots, c_n \in \mathbb{C}$. A linear combination of $A_1, A_2, \ldots, A_n$ is define by

$$c_1A_1 + c_2A_2 + \cdots + c_nA_n.$$

Definition 2.1.2. If $A \in M_n(\mathbb{C})$ is a square matrix of order n then we define $A^0 = I$ where I is the identity matrix. For $k > 0$, we define

$$A^k = \overbrace{A \ldots A}^{k-\text{times}} = A^{k-1}A = AA^{k-1}$$

Definition 2.1.3. A matrix $A \in M_n(\mathbb{C})$, is *idempotent* if $A^2 = A$.

For example, if

$$A_1 = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix},$$

which are idempotent whenever $\alpha, \mu, \lambda \in \{0, 1\}$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are idempotent matrices.

Definition 2.1.4. A matrix $A \in M_n(\mathbb{C})$, is *tripotent* if $A^3 = A$.

For example, if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Definition 2.1.5. A matrix $A \in M_n(\mathbb{C})$, is **t-potent** if $A^t = A$.

Definition 2.1.6. If $A \in M_n(\mathbb{C})$ then its *transpose* A^T is obtained by converting rows to columns, while the *conjugate transpose* $A^* \in M_n(\mathbb{C})$ is obtained by converting rows to columns and, in addition, conjugating the entries,

$$A^T = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}, \quad A^* = \begin{bmatrix} \overline{a}_{11} & \cdots & \overline{a}_{1n} \\ \vdots & & \vdots \\ \overline{a}_{m1} & \cdots & \overline{a}_{mn} \end{bmatrix}.$$

Definition 2.1.7. A matrix $A \in M_n(\mathbb{C})$, is *Hermitian* if $A^* = A$.

2.2 Idempotency of Linear Combinations of Two Idempotent Matrices.

A complete solution is established to the problem of characterizing all situations, where a linear combination of two different idempotent matrices P_1 and P_2 is also an idempotent matrix. Including naturally three such situations known in the literature, viz., if the combination is either the sum $P_1 + P_2$ or one of the differences $P_1 - P_2, P_2 - P_1$ the solution asserts that in the particular case where P_1 and P_2 are complex matrices such that $P_1 - P_2$ is Hermitian, these three situations exhaust the list of all possibilities and that this list extends when the above assumption on P_1 and P_2 is violated.

It is assumed throughout that c_1, c_2 are any nonzero elements of a field $\mathbb{C}$ P_1, P_2 are two different nonzero idempotent matrices over $\mathbb{C}$, i.e., $P_1 = P_1^2, P_2 = P_2^2$, and $P_1 \neq P_2$.

The purpose of this head is to establish a complete solution to the problem of characterizing all situations, where the operation of combining linearly P_1 and

P_2 preserves the idempotency property. Three such situations are known in the literature, viz., if the combination is either the sum $P_1 + P_2$ or one of the differences between P_1 and P_2, and appropriate additional conditions are fulfilled.

Theorem 2.2.1. [2, pp. 4-5]. Given two different nonzero idempotent matrices P_1 and P_2, let P be their linear combination of the form

$$P = c_1 P_1 + c_2 P_2,$$

(a) $P_1 P_2 = P_2 P_1$ holds along with either one of the following sets of conditions:

(i) $c_1 = 1, c_2 = 1, P_1 P_2 = 0,$

(ii) $c_1 = 1, c_2 = -1, P_1 P_2 = P_2,$

(iii) $c_1 = -1, c_2 = 1, P_1 P_2 = P_1,$

(b) $P_1 P_2 \neq P_2 P_1$ holds along with the condition $c_1 \in \mathbb{C} \setminus \{0, 1\}, c_2 = 1 - c_1,$ $(P_1 - P_2)^2 = 0.$

Corollary 2.2.2. [2, p.6]. Under the assumptions of the theorem, a necessary condition for $P = c_1 P_1 + c_2 P_2$ to be an idempotent matrix is that each of the products $P_1 P_2$ and $P_2 P_1$ is an idempotent matrix.

2.3 Idempotency of Linear Combinations of an Idempotent Matrix and Tripotent Matrix.

The problem of characterizing situations, in which a linear combination $C = c_1 A + c_2 B$ of an idempotent matrix A and a tripotent matrix B is an idempotent matrix, when either B or $-B$ is an idempotent matrix, a complete solution and B is an essentially tripotent matrix in the sense that both idempotent matrices B_1 and B_2 constituting its unique decomposition $B = B_1 - B_2$ are nonzero. The problem is considered also under the additional assumption that the differences $A - B_1$ and $A - B_2$ are Hermitian matrices.

Lemma 2.3.1 [4, p. 22]. Every tripotent matrix can uniquely be represented as a difference of two idempotent matrices B_1 and B_2 which are disjoint in the sense that $B_1 B_2 = 0$ and $B_2 B_1 = 0$.

If B_1 and B_2 are nonzero idempotent matrices,then $B = B_1 - B_2$ will be called an *essentially tripotent matrix.* The first of them reveals how the idempotency of a linear combination $C = c_1A + c_2(B_1 - B_2)$ is related to the idempotency of two possible products of idempotent matrices A and $B_1 + B_2$. The second additional result shows how the set of solutions reduces when the differences $A - B_1$ and $A - B_2$ are Hermitian.

Theorem 2.3.2. [4, pp. 23-26] Let $A \in M_n(\mathbb{C})$ be a nonzero idempotent matrix and let $B \in M_n(\mathbb{C})$ be and essentially tripotent matrix uniquely decomposed as

$$B = B_1 - B_2,$$

where $B_1 \in M_n(\mathbb{C})$ and $B_2 \in M_n(\mathbb{C})$, both nonzero, are such that

$$B_1 = B_1^2, B_2 = B_2^2, B_1B_2 = 0 = B_2B_1$$

Let C be a linear combination of A and B of the form $C = c_1A + c_2B$, i.e.,

$$C = c_1A + c_2B_1 - c_2B_2$$

with nonzero $c_1, c_2 \in \mathbb{C}$. Then the following list comprises characterizations of all cases where C is an idempotent matrix:

(a) six cases, denote by $(a_1) - (a_6)$, in which

$$AB_1 = B_1A, \quad AB_2 = B_2A$$

(a_1) $c_1 = 1, c_2 = 1, AB_1 = 0, AB_2 = B_2,$

(a_2) $c_1 = 2, c_2 = 1, A = B_2,$

(a_3) $c_1 = 1, c_2 = -1, AB_1 = B_1, AB_2 = 0,$

(a_4) $c_1 = 2, c_2 = -1, A = B_1,$

(a_5) $c_1 = 1/2, c_2 = 1/2, A = B_1 + B_2,$

(a_6) $c_1 = 1/2, c_2 = -1/2, A = B_1 + B_2,$

(b) two cases, denote by (b_1) and (b_2), in which

$$AB_1 = B_1A,\ AB_2 \neq B_2A$$

(b_1) $c_1 = 2, c_2 = 1, (A - B_2)^2 = 0,$

(b_2) $c_1 = 1/2, c_2 = -1/2, (A - B_2)^2 = B_1$

(c) two cases, denote by (c_1) and (c_2), in which

$$AB_1 \neq B_1A,\ AB_2 = B_2A$$

and either one of the following sets of additional conditions holds:

(c_1) $c_1 = 2, c_2 = -1, (A - B_1)^2 = 0,$

(c_2) $c_1 = 1/2, c_2 = 1/2, (A - B_1)^2 = B_2$

(d) all case , in which

$$AB_1 \neq B_1A,\ AB_2 \neq B_2A$$

and $c_1 = 1$, while c_2 is an solution to the equation

$$(A - B_1)^2 - (A - B_2)^2 = c_2(B_1 + B_2).$$

2.4 A Note on Linear Combinations of Commuting Tripotent Matrices.

The purpose of this note is to characterize all situations in which a linear combination of two commuting tripotent matrices is also a tripotent matrix. It is assumed throughout that $c_1, c_2 \in \mathbb{C}$ are nonzero complex numbers and $T_1, T_2 \in M_n(\mathbb{C})$ are nonzero commuting tripotent complex matrices of order n, i.e., $T_1 = T_1^3, T_2 = T_2^3$, and $T_1T_2 = T_2T_1$. The purpose of this note is to characterize all situations in which a linear combination of T_1 and T_2 of the form

$$T = c_1T_1 + c_2T_2$$

is also a tripotent matrix. A similar problem, concerning the question of when a linear combination

$$P = c_1P_1 + c_2P_2$$

of nonzero idempotent matrices $P_1 = P_1^2$ and $P_2 = P_2^2$ is also idempotent, has been solved by J. K. Baksalary and O. M. Baksalary.

The problem considered in this note becomes trivial when T_2 is a scalar multiple of T_1. Under the assumption that T_1 and T_2 are nonzero tripotent matrices, this is possible exclusively in two cases: when $T_2 = T_1$ or $T_2 = -T_1$. In the first of them, a linear combination $T = c_1T_1 + c_2T_2$ takes the form $T = (c_1 + c_2)T_1$ and hence is tripotent if and only if

$$c_2 = -c_1 \text{ or } c_2 = -c_1 + 1 \text{ or } c_2 = -c_1 - 1,$$

which corresponds $T = 0, T = T_1$, and $T = -T_1$, respectively. In the second case, $T = (c_1 - c_2)T_1$, and thus $T = T_3$ if and only if

$$c_2 = c_1 \text{ or } c_2 = c_1 - 1 \text{ or } c_2 = c_1 + 1,$$

which also corresponds to $T = 0, T = T_1$, and $T = -T_1$. Consequently, the case where T_2 a scalar multiple of T_1 is excluded from further considerations in this note.

Theorem 2.4.1. [3, pp. 47-49]. For nonzero $c_1, c_2 \in \mathbb{C}$ and nonzero tripotent matrices $T_1, T_2 \in M_n(\mathbb{C})$ satisfying the commutativity property $T_1T_2 = T_2T_1$, let T be their linear combination of the form $T = c_1T_1 + c_2T_2$. Under the assumption that $T_2 \neq T_1$ and $T_2 \neq -T_1$, the matrix T is tripotent if and only if:

(a) $c_1 = 1, c_2 = -1$ or $c_1 = -1, c_2 = 1$ and $T_1^2T_2 = T_1T_2^2$,

(b) $c_1 = 1, c_2 = -2$ or $c_1 = -1, c_2 = 2$ and $T_1^2T_2 = T_2 = T_1T_2^2$,

(c) $c_1 = 2, c_2 = -1$ or $c_1 = -2, c_2 = 1$ and $T_1^2T_2 = T_1 = T_1T_2^2$,

(d) $c_1 = 1, c_2 = 1$ or $c_1 = -1, c_2 = -1$ and $T_1^2T_2 = -T_1T_2^2$,

(e) $c_1 = 1, c_2 = 2$ or $c_1 = -1, c_2 = -2$ and $T_1^2T_2 = T_2 = -T_1T_2^2$,

(f) $c_1 = 2, c_2 = 1$ or $c_1 = -2, c_2 = -1$ and $T_1^2T_2 = -T_1 = -T_1T_2^2$,

(g) $c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$ or $c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$ or $c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$ or $c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}$ and $T_1^2T_2 = T_2, T_1T_2^2 = T_1$.

2.5 Idempotency of Linear Combinations of Three Idempotent Matrices, Two of which are Disjoint.

Given nonzero idempotent matrices A_1, A_2, A_3 such that A_2 and A_3 are disjoint, i.e., $A_2A_3 = 0 = A_3A_2$, the problem of characterizing all situations, in which a linear combination $C = c_1A_1 + c_2A_2 + c_3A_3$ is an idempotent matrix, was studied, by [1, pp. 3-7].

Theorem 2.5.1. [1, pp. 68-73]. Let $A_1, A_2, A_3 \in M_n(\mathbb{C})$ be nonzero matrices satisfying conditions A_1, A_2 and A_3 are an idempotent matrices and $A_2A_3 = 0 = A_3A_2$ and let C be a linear combination of them of the form $C = c_1A_1 + c_2A_2 + c_3A_3$, corresponding to $\gamma = (c_1, c_2, c_3)$. Moreover, let $\mathbb{C}_0$ and $\mathbb{C}_1$ stand for $\mathbb{C} \setminus \{0\}$ and $\mathbb{C} \setminus \{1\}$, respectively. Then the following list comprises characteristics of all cases where C is an idempotent matrix.

(a) The cases denoted by (a1)-(a10), in which

$$A_1A_2 = A_2A_1, A_1A_3 = A_3A_1,$$

and any of the following sets of additional conditions holds:

(a1) $A_1A_2 = 0, A_1A_3 = 0$, and $\gamma = (1, 1, 1)$;

(a2) $A_1A_2 = A_2, A_1A_3 = 0$, and $\gamma = (1, .1, 1)$;

(a3) $A_1A_2 = 0, A_1A_3 = A_3$, and $\gamma = (1, 1, .1)$;

(a4) $A_1A_2 = A_2, A_1A_3 = A_34$, and $\gamma = (1, -1, -1)$;

(a5) $A_1A_2 + A_1A_3 = A_1$ and $\gamma = (-1, 1, 1)$;

(a6) $A_1A_2 = A_1 - A_3$ and either $\gamma = (-1, 1, 1)$ or $\gamma = (-1, 1, 2)$;

(a7) $A_1A_3 = A_1 - A_2$ and either $\gamma = (-1, 1, 1)$ or $\gamma = (-1, 2, 1)$;

(a8) $A_2 = A_1$ and either $\gamma = (c_1, -c_1, 1), c_1 \in \mathbb{C}_0$, or $\gamma = (c_1, 1-c_1, 1), c_1 \in \mathbb{C}_1$;

(a9) $A_3 = A_1$ and either $\gamma = (c_1, 1, -c_1), c_1 \in \mathbb{C}_0$, or $\gamma = (c_1, 1, 1-c_1), c_1 \in \mathbb{C}_1$;

(a10) $A_2 + A_3 = A_1$ and either $\gamma = (c_1, -c_1, -c_1). c_1 \in \mathbb{C}_0$, or
$\gamma = (c_1, -c_1, 1 - c_1), c_1 \in \mathbb{C}_1$, or $\gamma = (c_1, 1 - c_1, -c1), c_1 \in \mathbb{C}_1$, or
$\gamma = (c_1, 1 - c_1, 1 - c_1), c_1 \in \mathbb{C}_1$.

(b) The cases denoted by (b1)-(b3), in which

$$A_1A_2 = A_2A_1, A_1A_3 \neq A_3A_1,$$

and any of the following sets of additional conditions holds:

(b1) $(A_1 - A_3)^2 = A_1A_2$ and $\gamma = (-1, 1, 2)$;

(b2) $(A_1 - A_3)^2 = 0$ and $= (c_1, 1, 1 - c_1), c_1 \in \mathbb{C}_1$;

(b3) $(A_1 - A_3)^2 = A_2$ and either $\gamma = (c_1, 1 - c_1, 1 - c_1), c_1 \in \mathbb{C}_1$, or

$\gamma = (c_1, -c_1, 1 - c_1), c_1 \in \mathbb{C}_1$.

(c) The cases denoted by (c1)-(c3), in which

$$A_1A_2 \neq A_2A_1, A_1A_3 = A_3A_1,$$

and any of the following sets of additional conditions holds:

(c1) $(A_1 - A_2)^2 = A_1A_3$ and $\gamma = (-1, 2, 1)$;

(c2) $(A_1 - A_2)^2 = 0$ and $\gamma = (c_1, 1 - c_1, 1), c_1 \in \mathbb{C}_1$;

(c3) $(A_1 - A_2)^2 = A_3$ and either $\gamma = (c_1, 1 - c_1, -c_1), c_1 \in \mathbb{C}_1$, *or*

$\gamma = (c_1, 1 - c_1, 1 - c_1), c_1 \in \mathbb{C}_1$.

(d) The cases denoted by (d1) and (d2), in which

$$A_1A_2 \neq A_2A_1, \quad A_1A_3 \neq A_3A_1,$$

and any of the following sets of additional conditions holds:

(d1) $(A_1 - A_2)^2 + (A_1 - A_3)^2 = A_1$ and $\gamma = (c_1, 1 - c_1, 1 - c_1), c_1 \in \mathbb{C}_1$;

(d2) $c_1c_2(A_1 - A_2)^2 + c_1c_3(A_1 - A_3)^2 + c_2c_3(A_2 + A_3) = 0$

for any γ such that $c_1 + c_2 + c_3 = 1$.

2.6 Idempotency of Linear Combinations of an Idempotent Matrix and a t-Potent Matrix that Commute.

We consider the following problem: to describe all pairs (c_1, c_2) of nonzero complex numbers for which there exist an idempotent complex matrix A (i.e., $A^2 = A$) and a t-potent complex matrix B (i.e., $B^t = B$) such that their linear combination $c_1A + c_2B$ is an idempotent matrix. This problem was studied in [2] and [4] for $t = 2$ and $t = 3$, respectively.We solve it for all $t > 1$, but only if A and B commute. We suppose that B has at least two distinct nonzero eigenvalues since otherwise $B = \lambda P$, where $P^2 = P$, that is, $c_1A + c_2B = c_1A + c_2\lambda P$ is a linear combination of idempotent matrices

Theorem 2.6.1. [5, pp. 417] Let c_1 and c_2 be nonzero real numbers. Let A and B be nonzero complex matrices and $c_1A + c_2B = C$ satisfy $A^2 = A, B^{k+1} = B, AB = BA, A \neq B$, and $C^2 = C$. Then $B^2 = B$ or $B^3 = B$.