CHAPTER III

MAIN RESULTS

3.1 Linear Combination of Tripotent Matrix.

Lemma 3.1.1. [3, pp. 47-49]. For nonzero $c_1, c_2 \in \mathbb{C}$ and nonzero tripotent matrices $T_1, T_2 \in M_n(\mathbb{C})$ satisfying the commutativity property $T_1T_2 = T_2T_1$, let T be their linear combination of the form $T = c_1T_1 + c_2T_2$. Under the assumption that $T_2 \neq T_1$ and $T_2 \neq -T_1$, the matrix T is tripotent if and only if one of the following conditions holds:

(a) $c_1 = 1, c_2 = -1$ or $c_1 = -1, c_2 = 1$ and $T_1^2T_2 = T_1T_2^2$,

(b) $c_1 = 1, c_2 = -2$ or $c_1 = -1, c_2 = 2$ and $T_1^2T_2 = T_2 = T_1T_2^2$,

(c) $c_1 = 2, c_2 = -1$ or $c_1 = -2, c_2 = 1$ and $T_1^2T_2 = T_1 = T_1T_2^2$,

(d) $c_1 = 1, c_2 = 1$ or $c_1 = -1, c_2 = -1$ and $T_1^2T_2 = -T_1T_2^2$,

(e) $c_1 = 1, c_2 = 2$ or $c_1 = -1, c_2 = -2$ and $T_1^2T_2 = T_2 = -T_1T_2^2$,

(f) $c_1 = 2, c_2 = 1$ or $c_1 = -2, c_2 = -1$ and $T_1^2T_2 = -T_1 = -T_1T_2^2$,

(g $c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$ or $c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$ or $c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$ or $c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}$ and $T_1^2T_2 = T_2,\ T_1T_2^2 = T_1$.

Lemma 3.1.2. [3, pp. 47-49]. For nonzero $c_1, c_2 \in \mathbb{C}$ and nonzero tripotent matrices $T_0, T \in M_n(\mathbb{C})$ satisfying the commutativity property $T_0T = TT_0$, let A be their linear combination of the form $A = c_0T_0 + d_0T$. Under the assumption that $T \neq T_0$ and $T_0 \neq -T$, the matrix A is tripotent if and only if one of the following conditions holds:

(a) $c_0 = 1, d_0 = -1$ or $c_0 = -1, d_0 = 1$ and $T_0^2T = T_0T^2$,

(b) $c_0 = 1, d_0 = -2$ or $c_0 = -1, d_0 = 2$ and $T_0^2T = T = T_0T^2$,

(c) $c_0 = 2, d_0 = -1$ or $c_0 = -2, d_0 = 1$ and $T_0^2T = T_0 = T_0T^2$,

(d) $c_0 = 1, d_0 = 1$ or $c_0 = -1, d_0 = -1$ and $T_0^2T = -T_0T^2$,

(e) $c_0 = 1, d_0 = 2$ or $c_0 = -1, d_0 = -2$ and $T_0^2T = T = -T_0T^2$,

(f) $c_0 = 2, d_0 = 1$ or $c_0 = -2, d_0 = -1$ and $T_0^2T = -T_0 = -T_0T^2$,

(g) $c_0 = \frac{1}{2}, d_0 = \frac{1}{2}$ or $c_0 = \frac{1}{2}, d_0 = -\frac{1}{2}$ or $c_0 = -\frac{1}{2}, d_0 = \frac{1}{2}$
or $c_0 = -\frac{1}{2}, d_0 = -\frac{1}{2}$ and $T_0^2T = T, T_0T^2 = T_0$.

From Lemma 3.1.1 we have equation $T = c_1T_1 + c_2T_2$ and Lemma 3.1.2 we have equation $A = c_0T_0 + d_0T$, so we will bring T to replace in the linear combination of $A = c_0T_0 + d_0T$ and so we can get the equation as follows:

$$\begin{aligned} A &= c_0T_0 + d_0T \\ &= c_0T_0 + d_0(c_1T_1 + c_2T_2) \end{aligned} \tag{3.1.1}$$

By substituting $T = c_1T_1 + c_2T_2$ in the linear combination of the form $A = c_0T_0 + d_0T$, we have the following theorem.

Theorem 3.1.3. For nonzero $c_0, c_1, c_2 \in \mathbb{C}$, and nonzero tripotent matrices $T_0, T_1, T_2 \in M_n(\mathbb{C})$ satisfying the commutativity property, let A be their linear combination of the form

$$A = c_0T_0 + d_0(c_1T_1 + c_2T_2).$$

Under the assumption that $T_0 \neq (c_1T_1 + c_2T_2)$ and $(c_1T_1 + c_2T_2) \neq -T_0$, the matrix A is tripotent if and only if:

(a) $c_0 = 1, d_0 = -1, c_1 = 1, c_2 = -1$ or $c_0 = -1, d_0 = 1, c_1 = -1, c_2 = 1$ and $T_0^2(c_1T_1 + c_2T_2) = T_0(c_1T_1 + c_2T_2)^2$,

(b) $c_0 = 1, d_0 = -2, c_1 = 1, c_2 = -2$ or $c_0 = -1, d_0 = 2, c_1 = -1, c_2 = 2$ and $T_0^2(c_1T_1 + c_2T_2) = (c_1T_1 + c_2T_2) = T_0(c_1T_1 + c_2T_2)^2$,

(c) $c_0 = 2, d_0 = -1, c_1 = 2, c_2 = -1$ or $c_0 = -2, d_0 = 1, c_1 = -2, c_2 = 1$ and $T_0^2(c_1T_1 + c_2T_2) = T_0 = T_0(c_1T_1 + c_2T_2)^2$,

(d) $c_0 = 1, d_0 = 1, c_1 = 1, c_2 = 1$ or $c_0 = -1, d_0 = -1, c_1 = -1, c_2 = -1$ and $T_0^2(c_1T_1 + c_2T_2) = (c_1T_1 + c_2T_2) = -T_0(c_1T_1 + c_2T_2)^2$,

(e) $c_0 = 1, d_0 = 2, c_1 = 1, c_2 = 2$ or $c_0 = -1, d_0 = -2, c_1 = -1, c_2 = -2$ and $T_0^2(c_1T_1 + c_2T_2) = (c_1T_1 + c_2T_2) = -T_0(c_1T_1 + c_2T_2)^2$,

(f) $c_0 = 2, d_0 = 1, c_1 = 2, c_2 = 1$ or $c_0 = -2, d_0 = -1, c_1 = -2, c_2 = -1$ and $T_0^2(c_1T_1 + c_2T_2) = -T_0 = -T_0(c_1T_1 + c_2T_2)^2$,

(g) $c_0 = \frac{1}{2}, d_0 = \frac{1}{2}, c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$ or $c_0 = \frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$ or $c_0 = -\frac{1}{2}, d_0 = \frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$ or $c_0 = -\frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}$ and $T_0^2(c_1T_1 + c_2T_2) = T_3. T_0(c_1T_1 + c_2T_2)^2 = T_0$.

Theorem 3.1.4. For nonzero $a_0, a_1, a_2 \in \mathbb{C}$, and nonzero tripotent matrices $T_0, T_1, T_2 \in M_n(\mathbb{C})$ satisfying the commutativity property, let A be their linear combination of the form

$$A = a_0T_0 + a_1T_1 + a_2T_2. \tag{3.1.2}$$

Then the matrix A is tripotent if and only if one of the following conditions holds:

(a) $a_0 = 1, a_1 = -1, a_2 = -1$ or $a_0 = -1, a_1 = -1, a_2 = 1$ and $T_0^2(T_1 - T_2) = T_0(T_1 - T_2)^2 = -T_0^2(T_1 - T_2)$,

(b) $a_0 = 1, a_1 = -2, a_2 = 4$ or $a_0 = -1, a_1 = -2, a_2 = 4$ and $T_0^2(T_1 - 2T_2) = (T_1 - 2T_2) = T_0(T_1 - 2T_2)^2$,

(c) $a_0 = 2, a_1 = -2, a_2 = 1$ or $a_0 = -2, a_1 = -2, a_2 = 1$ and $T_0^2(T_2 - T_1) = T_0 = T_0(T_2 - T_1)^2$,

(d) $a_0 = 1, a_1 = 1, a_2 = 1$ or $a_0 = -1, a_1 = 1, a_2 = 1$ and $T_0^2(T_1 + T_2) = -T_0(T_1 + T_2)^2$,

(e) $a_0 = 1, a_1 = 2, a_2 = 4$ or $a_0 = -1, a_1 = 2, a_2 = 4$ and $T_0^2(T_1 + T_2) = (T_1 + T_2) = -T_0(T_1 + T_2)^2$,

(f) $a_0 = 2, a_1 = 2, a_2 = 1$ or $a_0 = -2, a_1 = 2, a_2 = 1$ and
$T_0^2(2T_1 + T_2) = -T_0 = -T_0^2(2T_1 + T_2)$,

(g) $a_0 = \frac{1}{2}, a_1 = \frac{1}{4}, a_2 = \frac{1}{4}$ or $a_0 = -\frac{1}{2}, a_1 = \frac{1}{4}, a_2 = \frac{1}{4}$ and
$T_0^2(\frac{1}{2}T_1 + \frac{1}{2}T_2) = (\frac{1}{2}T_1 + \frac{1}{2}T_2), T_0(\frac{1}{2}T_1 + \frac{1}{2}T_2)^2 = T_0$,
$a_0 = \frac{1}{2}, a_1 = -\frac{1}{4}, a_2 = \frac{1}{4}$ or $a_0 = -\frac{1}{2}, a_1 = -\frac{1}{4}, a_2 = \frac{1}{4}$ and
$T_0^2(\frac{1}{2}T_1 - \frac{1}{2}T_2) = (\frac{1}{2}T_1 - \frac{1}{2}T_2), T_0(\frac{1}{2}T_1 - \frac{1}{2}T_2)^2 = T_0$.

Proof. Let

$$A = (c_0T_0 + d_0T) \quad \text{where} \quad T = c_1T_1 + c_2T_2 \tag{3.1.3}$$

Direct calculations show that A of from (3.1.1) is tripotent if and only if

$$c_0^3T_0 + 3c_0^2d_0T_0^2T + 3c_0d_0^2T_0T^2 + d_0^3T = c_0T_0 + d_0T$$

or equality

$$(c_0^3 - c_0)T_0 + 3c_0^2d_0T_0^2T + 3c_0d_0^2T_0T^2 + (d_0^3 - d_0)T = 0 \tag{3.1.4}$$

Substituting $T = c_1T_1 + c_2T_2$ to (3.1.4) show that

$$(c_0^3 - c_0)T_0 + 3c_0^2d_0T_0^2(c_1T_1 + c_2T_2) + 3c_0d_0^2T_0(c_1T_1 + c_2T_2)^2 + (d_0^3 - d_0)(c_1T_1 + c_2T_2) = 0 \tag{3.1.5}$$

By (3.1.3), we can rewrite equation:

$$\begin{aligned} A &= c_0T_0 + d_0T \\ &= c_0T_0 + d_0(c_1T_1 + c_2T_2) \\ &= c_0T_0 + d_0c_1T_1 + d_0c_2T_2 \end{aligned}$$

Let $a_0 = c_0, a_1 = d_0c_1$ and $a_2 = d_0c_2$. Thus

$$A = a_0T_0 + a_1T_1 + a_2T_2 \tag{3.1.6}$$

By Theorem 3.1.3 together with (3.1.5) we consider the following case:

Case (a). $c_0 = 1, d_0 = -1, c_1 = 1, c_2 = -1$ and $T_0^2(c_1T_1 + c_2T_2) = T_0(c_1T_1 + c_2T_2)^2$,

$$\begin{aligned} -3T_0^2(T_1 - T_2) + 3T_0(T_1 - T_2)^2 &= 0, \\ -T_0^2(T_1 - T_2) + T_0(T_1 - T_2)^2 &= 0, \\ -T_0^2(T_1 - T_2) + T_0^2(T_1 - T_2) &= 0. \end{aligned}$$

or $c_0=-1, d_0=1, c_1=-1, c_2=1$ and $T_0^2(c_1T_1+c_2T_2)=T_0(c_1T_1+c_2T_2)^2$,

$$\begin{aligned} 3T_0^2(T_2-T_1)-3T_0(T_2-T_1)^2 &= 0,\\ T_0^2(T_2-T_1)-T_0(T_2-T_1)^2 &= 0,\\ T_0^2(T_2-T_1)-T_0^2(T_2-T_1) &= 0. \end{aligned}$$

In this case, we have $a_0=1, a_1=-1, a_2=-1$ or $a_0=-1, a_1=-1, a_2=1$ and $T_0^2(T_1-T_2)=T_0(T_1-T_2)^2=-T_0^2(T_1-T_2)$.

Case (b). $c_0=1, d_0=-2, c_1=1, c_2=-2$ and

$$T_0^2(c_1T_1+c_2T_2)=(c_1T_1+c_2T_2)=T_0(c_1T_1+c_2T_2)^2,$$

$$\begin{aligned} 3(-2)T_0^2(T_1-2T_2)+3(-2)^2T_0(T_1-2T_2)^2+(-8+2)(T_1-2T_2) &= 0,\\ -6(T_1-2T_2)+12T_0(T_1-2T_2)^2-6(T_1-2T_2) &= 0,\\ -12(T_1-2T_2)+12(T_1-2T_2) &= 0, \end{aligned}$$

or $c_0=-1, d_0=2, c_1=-1, c_2=2$ and

$$T_0^2(c_1T_1+c_2T_2)=(c_1T_1+c_2T_2)=T_0(c_1T_1+c_2T_2)^2,$$

$$\begin{aligned} 3(-1)^2(2)T_0^2(2T_2-T_1)+3(-1)(2)^2T_0(2T_2-T_1)^2+(8-2)(2T_2-T_1) &= 0,\\ 6(2T_2-T_1)-12T_0(2T_2-T_1)^2+6(2T_2-T_1) &= 0,\\ 12(2T_2-T_1)-12(2T_2-T_1) &= 0. \end{aligned}$$

In this case, we have $a_0=1, a_1=-2, a_2=4$ or $a_0=-1, a_1=-2, a_2=4$ and $T_0^2(T_1-2T_2)=(T_1-2T_2)=T_0(T_1-2T_2)^2$.

Case (c). $c_0=2, d_0=-1, c_1=2, c_2=-1$ and

$$T_0^2(c_1T_1+c_2T_2)=T_0=T_0(c_1T_1+c_2T_2)^2$$

$$\begin{aligned} 6T_0+3(2)^2(-1)T_0^2(2T_1-T_2)+3(2)(-1)^2T_0(2T_1-T_2)^2 &= 0,\\ 6T_0-6T_0^2(2T_1-T_2)+6T_0(2T_1-T_2)^2 &= 0,\\ 6T_0-12T_0+6T_0 &= 0, \end{aligned}$$

or $c_0=-2, d_0=1, c_1=-2, c_2=1$ and

$$T_0^2(c_1T_1+c_2T_2)=T_0=T_0(c_1T_1+c_2T_2)^2$$

$$
\begin{aligned}
-6T_0 + 3(-2)^2(1)T_0^2(T_2 - 2T_1) + 3(-2)(1)^2T_0(2T_2 - 2T_1)^2 &= 0,\\
-6T_0 + 12T_0^2(T_2 - 2T_1) - 6T_0(2T_2 - 2T_1)^2 &= 0,\\
-6T_0 + 12T_0 - 6T_0 &= 0.
\end{aligned}
$$

In this case, we have $a_0 = 2, a_1 = -2, a_2 = 1$ or $a_0 = -2, a_1 = -2, a_2 = 1$ and $T_0^2(T_2 - T_1) = T_0 = T_0(T_2 - T_1)^2$.

Case (d). $c_0 = 1, d_0 = 1, c_1 = 1, c_2 = 1$ and $T_0^2(c_1T_1 + c_2T_2) = -T_0(c_1T_1 + c_2T_2)^2$,

$$
\begin{aligned}
3T_0^2(T_1 + T_2) + 3T_0(T_1 + T_2)^2 &= 0,\\
3T_0^2(T_1 + T_2) - 3T_0^2(T_1 + T_2) &= 0,
\end{aligned}
$$

or $c_0 = -1, d_0 = -1, c_1 = -1, c_2 = -1$ and $T_0^2(c_1T_1 + c_2T_2) = -T_0(c_1T_1 + c_2T_2)^2$,

$$
\begin{aligned}
-3T_0^2(-T_1 - T_2) - 3T_0(-T_1 - T_2)^2 &= 0,\\
-3T_0^2(-T_1 - T_2) + 3T_0^2(-T_1 - T_2) &= 0.
\end{aligned}
$$

In this case, we have $a_0 = 1, a_1 = 1, a_2 = 1$ or $a_0 = -1, a_1 = 1, a_2 = 1$ and $T_0^2(T_1 + T_2) = -T_0(T_1 + T_2)^2$.

Case (e). $c_0 = 1, d_0 = 2, c_1 = 1, c_2 = 2$

and $T_0^2(c_1T_1 + c_2T_2) = (c_1T_1 + c_2T_2) = -T_0(c_1T_1 + c_2T_2)^2$,

$$
\begin{aligned}
6T_0^2(T_1 + 2T_2) + 12T_0(T_1 + 2T_2)^2 + 6(T_1 + 2T_2) &= 0,\\
6(T_1 + 2T_2) - 12(T_1 + 2T_2) + 6(T_1 + 2T_2) &= 0,
\end{aligned}
$$

or $c_0 = -1, d_0 = -2, c_1 = -1, c_2 = -2$ and

$$T_0^2(c_1T_1 + c_2T_2) = (c_1T_1 + c_2T_2) = -T_0(c_1T_1 + c_2T_2)^2$$

$$
\begin{aligned}
-6T_0^2(-T_1 - 2T_2) - 12T_0(-T_1 - 2T_2)^2 - 6(-T_1 - 2T_2) &= 0,\\
-6T_0^2(-T_1 - 2T_2) + 12T_0(-T_1 - 2T_2) - 6(-T_1 - 2T_2) &= 0.
\end{aligned}
$$

In this case, we have $a_0 = 1, a_1 = 2, a_2 = 4$ or $a_0 = -1, a_1 = 2, a_2 = 4$ and $T_0^2(T_1 + T_2) = (T_1 + T_2) = -T_0(T_1 + T_2)^2$.

Case (f). $c_0 = 2, d_0 = 1, c_1 = 2, c_2 = 1$ and

$$T_0^2(c_1T_1 + c_2T_2) = -T_0 = -T_0(c_1T_1 + c_2T_2)^2,$$

$$\begin{aligned} 6T_0 + 12T_0^2(2T_1 + T_2) + 6T_0(2T_1 + T_2)^2 &= 0, \\ 6T_0 - 12T_0 + 6T_0 &= 0, \end{aligned}$$

or $c_0 = -2, d_0 = -1, c_1 = -2, c_2 = -1$ and

$$T_0^2(c_1T_1 + c_2T_2) = -T_0 = -T_0(c_1T_1 + c_2T_2)^2,$$

$$\begin{aligned} -6T_0 - 12T_0^2(-2T_1 - T_2) - 6T_0(-2T_1 - T_2)^2 &= 0, \\ -6T_0 + 12T_0 - 6T_0 &= 0. \end{aligned}$$

In this case, we have $a_0 = 2, a_1 = 2, a_2 = 1$ or $a_0 = -2, a_1 = 2, a_2 = 1$ and $T_0^2(2T_1 + T_2) = -T_0 = -T_0^2(2T_1 + T_2)$.

Case (g). $c_0 = \frac{1}{2}, d_0 = \frac{1}{2}, c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$ and

$$T_0^2(c_1T_1 + c_2T_2) = (c_1T_1 + c_2T_2),\ T_0(c_1T_1 + c_2T_2)^2 = T_0$$

$$\begin{aligned} (\tfrac{1}{8} - \tfrac{1}{2})T_0 + 3(\tfrac{1}{4})(\tfrac{1}{2})T_0^2(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) + 3(\tfrac{1}{2})(\tfrac{1}{4})T_0(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2)^2 & \\ + (\tfrac{1}{8} - \tfrac{1}{2})(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) &= 0, \\ -\tfrac{3}{8}T_0 + \tfrac{3}{8}T_0^2(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) + \tfrac{3}{8}T_0(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2)^2 - \tfrac{3}{8}(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) &= 0, \\ -\tfrac{3}{8}T_0 + \tfrac{3}{8}(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) + \tfrac{3}{8}T_0 - \tfrac{3}{8}(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) &= 0, \end{aligned}$$

or $c_0 = \frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$ and

$$T_0^2(c_1T_1 + c_2T_2) = (c_1T_1 + c_2T_2),\ T_0(c_1T_1 + c_2T_2)^2 = T_0,$$

$$\begin{aligned} -\tfrac{3}{8}T_0 - \tfrac{3}{8}T_0^2(\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) + \tfrac{3}{8}T_0(\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2)^2 + \tfrac{3}{8}(\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) &= 0, \\ -\tfrac{3}{8}T_0 - \tfrac{3}{8}(\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) + \tfrac{3}{8}T_0 + \tfrac{3}{8}(\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) &= 0. \end{aligned}$$

or $c_0 = -\frac{1}{2}, d_0 = \frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$ and

$$T_0^2(c_1T_1 + c_2T_2) = (c_1T_1 + c_2T_2),\ T_0(c_1T_1 + c_2T_2)^2 = T_0,$$

$$\begin{aligned} \tfrac{3}{8}T_0 + \tfrac{3}{8}T_0^2(-\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) - \tfrac{3}{8}T_0(-\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2)^2 - \tfrac{3}{8}(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) &= 0, \\ \tfrac{3}{8}T_0 + \tfrac{3}{8}(-\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) - \tfrac{3}{8}T_0 - \tfrac{3}{8}(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) &= 0, \end{aligned}$$

or $c_0 = -\frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}$ and

$$T_0^2(c_1T_1 + c_2T_2) = (c_1T_1 + c_2T_2),\ T_0(c_1T_1 + c_2T_2)^2 = T_0,$$

$$\tfrac{3}{8}T_0 - \tfrac{3}{8}T_0^2(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) - \tfrac{3}{8}T_0(-\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2)^2 + \tfrac{3}{8}(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) = 0,$$
$$\tfrac{3}{8}T_0 - \tfrac{3}{8}(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) - \tfrac{3}{8}T_0 + \tfrac{3}{8}(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) = 0.$$

In this case, we have

$$a_0 = \frac{1}{2}, a_1 = \frac{1}{4}, a_2 = \frac{1}{4} \quad \text{or} \quad a_0 = -\frac{1}{2}, a_1 = \frac{1}{4}, a_2 = \frac{1}{4}$$

and $T_0^2(\frac{1}{2}T_1 + \frac{1}{2}T_2) = (\frac{1}{2}T_1 + \frac{1}{2}T_2), T_0(\frac{1}{2}T_1 + \frac{1}{2}T_2)^2 = T_0$,

$$a_0 = \frac{1}{2}, a_1 = -\frac{1}{4}, a_2 = \frac{1}{4} \quad \text{or} \quad a_0 = -\frac{1}{2}, a_1 = -\frac{1}{4}, a_2 = \frac{1}{4}$$

and $T_0^2(\frac{1}{2}T_1 - \frac{1}{2}T_2) = (\frac{1}{2}T_1 - \frac{1}{2}T_2), T_0(\frac{1}{2}T_1 - \frac{1}{2}T_2)^2 = T_0$.

By Lemma ??, the tripotent matrix A, in Theorem 3.1.4 can uniquely be represented as a difference of two idempotent matrices A_1 and A_2 which are disjoint in the sense that $A_1A_2 = 0$ and $A_2A_1 = 0$. Thus, $A = A_1 - A_2$ where A_1 and A_2 are an idempotent matrices.

Given two different nonzero idempotent matrices A_1 and A_2, let C be their linear combination of the form

$$C = c_1A_1 + (-c_2)A_2 \tag{3.1.7}$$

Direct calculations show that, in view of $C^2 = C$, a matrix C of the form (3.1.7)

$$\begin{aligned} C &= c_1A_1 + (-c_2)A_2 \\ C^2 &= c_1^2A_1^2 - 2c_1c_2A_1A_2 + c_2^2A_2^2 \\ c_1A_1 + (-c_2)A_2 &= c_1^2A_1^2 - 2c_1c_2A_1A_2 + c_2^2A_2^2 \\ (c_1^2 - c_1)A_1^2 - 2c_1c_2A_1A_2 + (c_2^2 + c_2)A_2^2 &= 0 \end{aligned}$$

Then, in view of the Theorem 2.2.1, there is one case such that the matrix $C = c_1A_1 + c_2A_2$ (now equal to $C = c_1A_1 + (-c_2)A_2$) is idempotent where

$$c_1 = 1, c_2 = -1, \quad A_1A_2 = 0 = A_2A_1 \tag{3.1.8}$$

Hence A is an idempotent matrix, under this condition, criterion (3.1.8). The proof is complete. □

Corollary 3.1.5. For nonzero $c_0, d_0 \in \mathbb{C}$ and nonzero tripotent matrices $T_0, T_1, T_2 \in M_n(\mathbb{C})$ satisfying the commutativity property, let A be their linear combination of the form

$$A = a_0T_0 + a_1T_1 + a_2T_2.$$

Moreover, let $T_i \neq T_j$ and $T_i \neq -T_j$, $\forall i, j = 1, 2, 3$. Then:

(a) in cases where $T_iT_j = 0$, $\forall i, j = 1, 2, 3$ a matrix A is tripotent if and only if $(a_0, a_1, a_2) \in \{(1, 1, 1), (1, -1, 1), (-1, -1, 1), (-1, 1, 1)\}$,

(b) in cases $T_0T = T$, where $T = c_1T_1 + c_2T_2$ a matrix A is tripotent if and only if T is idempotent and $(a_0, a_1, a_2) \in \{(1, -1, 1), (-1, -1, 1), (1, -2, 4), (-1, -2, 4)\}$ or $-T$ is idempotent and $(a_0, a_1, a_2) \in \{(1, 1, 1), (-1, 1, 1), (1, 2, 4), (-1, 2, 4)\}$ or T_0 is idempotent equal to T and the pairs (a_0, a_1, a_2) are as in Theorem 3.1.4 (g),

(c) in cases $T_0T = T_0$, where $T = c_1T_1 + c_2T_2$ a matrix A is tripotent if and only if T_0 is idempotent and $(a_0, a_1, a_2) \in \{(1, -1, 1), (-1, -1, 1), (2, -2, 1), (-2, -2, 1)\}$ or $-T_0$ is idempotent and $(a_0, a_1, a_2) \in \{(1, 1, 1), (-1, 1, 1), (2, 2, 1), (-2, , 2, 1)\}$ or T is idempotent equal to T_0^2 and the pairs (a_0, a_1, a_2) are as in Theorem 3.1.4 (g).

Proof. Direct calculations show that, a matrix A of the from (3.1.5)

$$(c_0^3 - c_0)T_0 + 3c_0^2d_0T_0^2(c_1T_1 + c_2T_2) + 3c_0d_0^2T_0(c_1T_1 + c_2T_2)^2 + (d_0^3 - d_0)(c_1T_1 + c_2T_2) = 0$$

By Theorem 3.1.3 together with (3.1.5) we consider the following case:

Case(a). In cases where $T_iT_j = 0$, $\forall i, j = 1, 2, 3$ a matrix A is tripotent if and only if $(a_0, a_1, a_2) \in \{(1, 1, 1), (1, -1, 1), (-1, -1, 1), (-1, 1, 1)\}$,
Hence it is clear that if situation (a_0, a_1, a_2) in (3.1.5).
Case(b). In cases $T_0T = T$, where $T = c_1T_1 + c_2T_2$ a matrix A is tripotent if and only if T is idempotent and $c_0 = 1, d_0 = -1, c_1 = 1, c_2 = -1$ or $c_0 = -1, d_0 = 1, c_1 = -1, c_2 = 1$ or $c_0 = 1, d_0 = -2, c_1 = 1, c_2 = -2$ or $c_0 = -1, d_0 = 2, c_1 = -1, c_2 = 2$.

If $c_0 = 1, d_0 = -1, c_1 = 1, c_2 = -1$, then

$$\begin{aligned} -3T_0^2(T_1 - T_2) + 3T_0(T_1 - T_2)^2 &= 0, \\ -T_0^2(T_1 - T_2) + T_0(T_1 - T_2)^2 &= 0, \\ -(T_1 - T_2) + (T_1 - T_2) &= 0. \end{aligned}$$

If $c_0 = -1, d_0 = 1, c_1 = -1, c_2 = 1$, then

$$\begin{aligned} 3T_0^2(T_2 - T_1) - 3T_0(T_2 - T_1)^2 &= 0, \\ T_0^2(T_2 - T_1) - T_0(T_2 - T_1)^2 &= 0, \\ (T_2 - T_1) - (T_2 - T_1) &= 0. \end{aligned}$$

If $c_0 = 1, d_0 = -2, c_1 = 1, c_2 = -2$, then

$$\begin{aligned} 3(-2)T_0^2(T_1 - 2T_2) + 3(-2)^2T_0(T_1 - 2T_2)^2 + (-8+2)(T_1 - 2T_2) &= 0, \\ -6(T_1 - 2T_2) + 12T_0(T_1 - 2T_2)^2 - 6(T_1 - 2T_2) &= 0, \\ -12(T_1 - 2T_2) + 12(T_1 - 2T_2) &= 0. \end{aligned}$$

If $c_0 = -1, d_0 = 2, c_1 = -1, c_2 = 2$, then

$$\begin{aligned} 3(-1)^2(2)T_0^2(2T_2 - T_1) + 3(-1)(2)^2T_0(2T_2 - T_1)^2 + (8-2)(2T_2 - T_1) &= 0, \\ 6(2T_2 - T_1) - 12T_0(2T_2 - T_1)^2 + 6(2T_2 - T_1) &= 0, \\ 12(2T_2 - T_1) - 12(2T_2 - T_1) &= 0. \end{aligned}$$

In this case, we have $(a_0, a_1, a_2) \in \{(1, -1, 1), (-1, -1, 1), (1, -2, 4), (-1, -2, 4)\}$.

In cases $T_0T = T$, where $T = c_1T_1 + c_2T_2$ a matrix A is tripotent if and only if $-T$ is idempotent and $c_0 = 1, d_0 = 1, c_1 = 1, c_2 = 1$ or $c_0 = -1, d_0 = -1, c_1 = -1, c_2 = -1$ or $c_0 = 1, d_0 = 2, c_1 = 1, c_2 = 2$ or $c_0 = -1, d_0 = -2, c_1 = -1, c_2 = -2$. If $c_0 = 1, d_0 = 1, c_1 = 1, c_2 = 1$, then

$$\begin{aligned} 3T_0^2(T_1 + T_2) + 3T_0(T_1 + T_2)^2 &= 0, \\ (T_1 + T_2) - (T_1 + T_2) &= 0. \end{aligned}$$

If $c_0 = -1, d_0 = -1, c_1 = -1, c_2 = -1$, then

$$\begin{aligned} -3T_0^2(-T_1 - T_2) - 3T_0(-T_1 - T_2)^2 &= 0, \\ -(-T_1 - T_2) + (-T_1 - T_2) &= 0. \end{aligned}$$

If $c_0 = 1, d_0 = 2, c_1 = 1, c_2 = 2$, then

$$\begin{aligned} 6T_0^2(T_1+2T_2)+12T_0(T_1+2T_2)^2+6(T_1+2T_2) &= 0, \\ 6(T_1+2T_2)-12(T_1+2T_2)+6(T_1+2T_2) &= 0. \end{aligned}$$

If $c_0 = -1, d_0 = -2, c_1 = -1, c_2 = -2$, then

$$\begin{aligned} -6T_0^2(-T_1-2T_2)-12T_0(-T_1-2T_2)^2-6(-T_1-2T_2) &= 0, \\ -6T_0^2(-T_1-2T_2)+12T_0(-T_1-2T_2)-6(-T_1-2T_2) &= 0. \end{aligned}$$

In this case, we have $(a_0, a_1, a_2) \in \{(1,1,1), (-1,1,1), (1,2,4), (-1,2,4)\}$

In cases $T_0T = T$, where $T = c_1T_1 + c_2T_2$ a matrix A is tripotent if and only if $T_0 = T^2$ and $c_0 = \frac{1}{2}, d_0 = \frac{1}{2}, c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$ or $c_0 = \frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$ or $c_0 = -\frac{1}{2}, d_0 = \frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$ or $c_0 = -\frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}$.
If $c_0 = \frac{1}{2}, d_0 = \frac{1}{2}, c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$, then

$$\begin{aligned} (\tfrac{1}{8}-\tfrac{1}{2})T_0+3(\tfrac{1}{4})(\tfrac{1}{2})T_0^2(\tfrac{1}{2}T_1+\tfrac{1}{2}T_2)+3(\tfrac{1}{2})(\tfrac{1}{4})T_0(\tfrac{1}{2}T_1+\tfrac{1}{2}T_2)^2 & \\ +(\tfrac{1}{8}-\tfrac{1}{2})(\tfrac{1}{2}T_1+\tfrac{1}{2}T_2) &= 0, \\ -\tfrac{3}{8}T_0+\tfrac{3}{8}T_0^2(\tfrac{1}{2}T_1+\tfrac{1}{2}T_2)+\tfrac{3}{8}T_0(\tfrac{1}{2}T_1+\tfrac{1}{2}T_2)^2-\tfrac{3}{8}(\tfrac{1}{2}T_1+\tfrac{1}{2}T_2) &= 0, \\ -\tfrac{3}{8}T_0+\tfrac{3}{8}(\tfrac{1}{2}T_1+\tfrac{1}{2}T_2)+\tfrac{3}{8}T_0-\tfrac{3}{8}(\tfrac{1}{2}T_1+\tfrac{1}{2}T_2) &= 0. \end{aligned}$$

If $c_0 = \frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$, then

$$\begin{aligned} -\tfrac{3}{8}T_0-\tfrac{3}{8}T_0^2(\tfrac{1}{2}T_1-\tfrac{1}{2}T_2)+\tfrac{3}{8}T_0(\tfrac{1}{2}T_1-\tfrac{1}{2}T_2)^2+\tfrac{3}{8}(\tfrac{1}{2}T_1-\tfrac{1}{2}T_2) &= 0, \\ -\tfrac{3}{8}T_0-\tfrac{3}{8}(\tfrac{1}{2}T_1-\tfrac{1}{2}T_2)+\tfrac{3}{8}T_0+\tfrac{3}{8}(\tfrac{1}{2}T_1-\tfrac{1}{2}T_2) &= 0. \end{aligned}$$

If $c_0 = -\frac{1}{2}, d_0 = \frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$, then

$$\begin{aligned} \tfrac{3}{8}T_0+\tfrac{3}{8}T_0^2(-\tfrac{1}{2}T_1+\tfrac{1}{2}T_2)-\tfrac{3}{8}T_0(-\tfrac{1}{2}T_1+\tfrac{1}{2}T_2)^2-\tfrac{3}{8}(-\tfrac{1}{2}T_1-\tfrac{1}{2}T_2) &= 0, \\ \tfrac{3}{8}T_0+\tfrac{3}{8}(-\tfrac{1}{2}T_1+\tfrac{1}{2}T_2)-\tfrac{3}{8}T_0-\tfrac{3}{8}(-\tfrac{1}{2}T_1-\tfrac{1}{2}T_2) &= 0. \end{aligned}$$

If $c_0 = -\frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}$, then

$$\begin{aligned} \tfrac{3}{8}T_0-\tfrac{3}{8}T_0^2(-\tfrac{1}{2}T_1-\tfrac{1}{2}T_2)-\tfrac{3}{8}T_0(-\tfrac{1}{2}T_1+\tfrac{1}{2}T_2)^2+\tfrac{3}{8}(-\tfrac{1}{2}T_1-\tfrac{1}{2}T_2) &= 0, \\ \tfrac{3}{8}T_0-\tfrac{3}{8}(-\tfrac{1}{2}T_1-\tfrac{1}{2}T_2)-\tfrac{3}{8}T_0+\tfrac{3}{8}(-\tfrac{1}{2}T_1-\tfrac{1}{2}T_2) &= 0. \end{aligned}$$

In this case, we have $(a_0, a_1, a_2) \in \{(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, (-\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}), (-\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\}$.

Case(c). In cases $T_0T = T_0$, where $T = c_1T_1 + c_2T_2$ a matrix A is tripotent if and only if T_0 is idempotent and $c_0 = 1, d_0 = -1, c_1 = 1, c_2 = -1$ or $c_0 = -1, d_0 = 1, c_1 = -1, c_2 = 1$ or $c_0 = 2, d_0 = -1, c_1 = 2, c_2 = -1$ or $c_0 = -2, d_0 = 1, c_1 = -2, c_2 = 1$.
If $c_0 = 1, d_0 = -1, c_1 = 1, c_2 = -1$, then

$$\begin{aligned} -3T_0^2(T_1 - T_2) + 3T_0(T_1 - T_2)^2 &= 0, \\ -T_0^2(T_1 - T_2) + T_0(T_1 - T_2)^2 &= 0, \\ -T_0 + T_0 &= 0. \end{aligned}$$

If $c_0 = -1, d_0 = 1, c_1 = -1, c_2 = 1$, then

$$\begin{aligned} 3T_0^2(T_2 - T_1) - 3T_0(T_2 - T_1)^2 &= 0, \\ T_0^2(T_2 - T_1) - T_0(T_2 - T_1)^2 &= 0, \\ T_0 - T_0 &= 0, \end{aligned}$$

If $c_0 = 2, d_0 = -1, c_1 = 2, c_2 = -1$, then

$$\begin{aligned} 6T_0 + 3(2)^2(-1)T_0^2(2T_1 - T_2) + 3(2)(-1)^2T_0(2T_1 - T_2)^2 &= 0, \\ 6T_0 - 6T_0^2(2T_1 - T_2) + 6T_0(2T_1 - T_2)^2 &= 0, \\ 6T_0 - 12T_0 + 6T_0 &= 0. \end{aligned}$$

If $c_0 = -2, d_0 = 1, c_1 = -2, c_2 = 1$, then

$$\begin{aligned} -6T_0 + 3(-2)^2(1)T_0^2(T_2 - 2T_1) + 3(-2)(1)^2T_0(2T_2 - 2T_1)^2 &= 0, \\ -6T_0 + 12T_0^2(T_2 - 2T_1) - 6T_0(2T_2 - 2T_1)^2 &= 0, \\ -6T_0 + 12T_0 - 6T_0 &= 0. \end{aligned}$$

In this case, we have $(a_0, a_1, a_2) \in \{(1, -1, 1), (-1, -1, 1), (2, -2, 1), (-2, -2, 1)\}$.

In cases $T_0T = T_0$, where $T = c_1T_1 + c_2T_2$ a matrix A is tripotent if and only if $-T_0$ is idempotent and $c_0 = 1, d_0 = 1, c_1 = 1, c_2 = 1$ or $c_0 = -1, d_0 = -1, c_1 = -1, c_2 = -1$ or $c_0 = 2, d_0 = 1, c_1 = 2, c_2 = 1$ or $c_0 = -2, d_0 = -1, c_1 = -2, c_2 = -1$.
If $c_0 = 1, d_0 = 1, c_1 = 1, c_2 = 1$, then

$$\begin{aligned} 3T_0^2(T_1 + T_2) + 3T_0(T_1 + T_2)^2 &= 0, \\ T_0 - T_0 &= 0. \end{aligned}$$

If $c_0 = -1, d_0 = -1, c_1 = -1, c_2 = -1$, then

$$\begin{aligned} -3T_0^2(-T_1 - T_2) - 3T_0(-T_1 - T_2)^2 &= 0, \\ -T_0 + T_0 &= 0. \end{aligned}$$

If $c_0 = 2, d_0 = 1, c_1 = 2, c_2 = 1$, then

$$\begin{aligned} 6T_0 + 12T_0^2(2T_1 + T_2) + 6T_0(2T_1 + T_2)^2 &= 0, \\ 6T_0 - 12T_0 + 6T_0 &= 0. \end{aligned}$$

If $c_0 = -2, d_0 = -1, c_1 = -2, c_2 = -1$, then

$$\begin{aligned} -6T_0 - 12T_0^2(-2T_1 - T_2) - 6T_0(-2T_1 - T_2)^2 &= 0, \\ -6T_0 + 12T_0 - 6T_0 &= 0. \end{aligned}$$

In this case, we have $(a_0, a_1, a_2) \in \{(1,1,1), (-1,1,1), (2,2,1), (-2,,2,1)\}$.

In cases $T_0T = T$, where $T = c_1T_1 + c_2T_2$ a matrix A is tripotent if and only if $T = T_0^2$ and $c_0 = \frac{1}{2}, d_0 = \frac{1}{2}, c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$ or $c_0 = \frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$ or $c_0 = -\frac{1}{2}, d_0 = \frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$ or $c_0 = -\frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}$.
If $c_0 = \frac{1}{2}, d_0 = \frac{1}{2}, c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$, then

$$\begin{aligned} (\tfrac{1}{8} - \tfrac{1}{2})T_0 + 3(\tfrac{1}{4})(\tfrac{1}{2})T_0^2(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) + 3(\tfrac{1}{2})(\tfrac{1}{4})T_0(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2)^2 & \\ + (\tfrac{1}{8} - \tfrac{1}{2})(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) &= 0, \\ -\tfrac{3}{8}T_0 + \tfrac{3}{8}T_0^2(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) + \tfrac{3}{8}T_0(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2)^2 - \tfrac{3}{8}(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) &= 0, \\ -\tfrac{3}{8}T_0 + \tfrac{3}{8}(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) + \tfrac{3}{8}T_0 - \tfrac{3}{8}(\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) &= 0. \end{aligned}$$

If $c_0 = \frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}$, then

$$\begin{aligned} -\tfrac{3}{8}T_0 - \tfrac{3}{8}T_0^2(\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) + \tfrac{3}{8}T_0(\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2)^2 + \tfrac{3}{8}(\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) &= 0, \\ -\tfrac{3}{8}T_0 - \tfrac{3}{8}(\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) + \tfrac{3}{8}T_0 + \tfrac{3}{8}(\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) &= 0. \end{aligned}$$

If $c_0 = -\frac{1}{2}, d_0 = \frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = \frac{1}{2}$, then

$$\begin{aligned} \tfrac{3}{8}T_0 + \tfrac{3}{8}T_0^2(-\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) - \tfrac{3}{8}T_0(-\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2)^2 - \tfrac{3}{8}(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) &= 0, \\ \tfrac{3}{8}T_0 + \tfrac{3}{8}(-\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2) - \tfrac{3}{8}T_0 - \tfrac{3}{8}(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) &= 0. \end{aligned}$$

If $c_0 = -\frac{1}{2}, d_0 = -\frac{1}{2}, c_1 = -\frac{1}{2}, c_2 = -\frac{1}{2}$, then

$$\begin{aligned} \tfrac{3}{8}T_0 - \tfrac{3}{8}T_0^2(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) - \tfrac{3}{8}T_0(-\tfrac{1}{2}T_1 + \tfrac{1}{2}T_2)^2 + \tfrac{3}{8}(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) &= 0, \\ \tfrac{3}{8}T_0 - \tfrac{3}{8}(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) - \tfrac{3}{8}T_0 + \tfrac{3}{8}(-\tfrac{1}{2}T_1 - \tfrac{1}{2}T_2) &= 0. \end{aligned}$$

In this case, we have $(a_0, a_1, a_2) \in \{(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}), (\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, (-\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}), (-\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\}$. Under this condition, criterion (3.1.5) can be the proof is complete. □

3.2 Linear Combination of Idempotent and Tripotent Matrices.

Theorem 3.2.1. Let c_1 and c_2 be nonzero complex number. Let A and B be nonzero complex matrices and $c_1A + c_2B = C$ satisfy $A^3 = A$, $B^s = B$, $AB = BA$, $A \neq B$, and $C^2 = C$. Then $B^2 = B$ or $B^3 = B$.

Proof. Let A be idempotent and B be s-potent. If $c_1A + c_2B$ is idempotent, then

$$\begin{aligned} c_1A + c_2B &= c_1(A_1 - A_2) + c_2B, \quad \text{(by Lemma ??)} \\ &= (c_1A_1 - c_1A_2) + c_2B \\ &= (c_1A_1 + dA_2) + c_2B \text{ (where } d = -c_1) \\ &= c_1A_1 + (dA_2 + c_2B) \end{aligned}$$

By Theorem 2.6.1, B must be idempotent or tripotent.

If B is an idempotent matrix, then $c_1A_1 + (dA_2 + c_2B)$ is idempotent. From Theorem 2.2.1 asserts that there are d, c_2 such that $(dA_2 + c_2B)$ is idempotent. Now, let $E = (dA_2 + c_2B)$ be idempotent. From Theorem 2.2.1, we can find some scalar k_1, k_2 such that $k_1A_1 + k_2E$ is idempotent. Thus, there exists some scalars for which combination of A and B is idempotent.

If B is a tripotent matrix, then $c_1A_1 + (dA_2 + c_2B)$ is idempotent. From Theorem 2.3.2 asserts that there are d, c_2 such that $(dA_2 + c_2B)$ is idempotent. Now, let $Q = (dA_2 + c_2B)$ be idempotent. From Theorem 2.2.1, we can find some scalar l_1, l_2 such that $l_1A_1 + l_2Q$ is idempotent. Thus, there exists some scalar for which combination of A and B is idempotent. □

Corollary 3.2.2. If one of A or B or C is idempotent (or tripotent) and the combination A, B and C is idempotent (or tripotent), then the others matrices must be idempotent (or tripotent).

Proof. It follows directly from Theorem 3.2.1 □

Theorem 3.2.3. Let $A_1, \ldots, A_n$ be $t_1, \ldots, t_n$-potent respectively. If combination of $A_1, \ldots, A_n$ is idempotent and one of $A_1, \ldots, A_n$ is idempotent (or tripotent) and the combination $A_1, \ldots, A_n$ is idempotent, then $A_1, \ldots, A_n$ each the other matrices must be idempotent (or tripotent).