CHAPTER I

INTRODUCTION

A primitive polynomial over $\mathbb{F}_q$ is a polynomial which is the minimal polynomial of a primitive element of an extension of $\mathbb{F}_q$. An important problem in the theory of finite fields is that of the determination of primitive polynomials. There have appeared a number of recent results about primitive polynomials such as those dealing with the existence of primitive polynomials with prescribed coefficients, see e.g. [7, 8, 9, 10, 11, 13]; with prescribed or arbitrary traces, see e.g. [4, 5, 15]; with the criterion for primitive polynomials, see e.g. [22]; and with the computation of the number of primitive polynomials, see e.g. [3]. In 2003 Fitzgerald [12] gave an attractive characterization of primitive polynomials, among irreducible polynomials, over finite fields $\mathbb{F}_q$. Fitzgerald's proof starts by equating the coefficients in a suitable quotient of the polynomial to be characterized, which yields a linear recurring sequence over a finite field. Using known results about the number of occurrences of periodic elements in such a sequence, the concept closely related to the polynomial being primitive, a striking characterization is deduced. In the first part of this thesis, Fitzgerald's proof is modified in such a way that the quotient of the polynomial to be characterized is purely rational, not integral as in Fitzgerald's original proof. This leads to an infinite linear recurring sequence, and known results about the occurrences of certain periodic elements in such sequence allow us to obtain the desired characterization.

The second theme of this thesis deals with a new digit system, invented by Scheicher and Thuswaldner, for elements in a polynomial ring of two variables. In 2003, Scheicher and Thuswaldner [20] proposed a new digit system for elements $p(x, y)$ in a polynomial ring of two variables over a polynomial ring $\mathbb{F}_q[x]$, called digit systems. The construction of their system resembles to similar algorithm in the case of polynomials in one variable, which is known in the literature as the CNS system. Scheicher and Thuswaldner gave a complete characterization of those $p(x, y)$ which produce finite and periodic y-adic expansions. There have been many other related results in the literature, e.g., in [16], special cases of the Scheicher-Thuswaldner digit systems are defined and discussed; the preprint [2] deals with the same digit systems and associated tilings; in [14] and [19] related number systems in finite fields are studied; in [17] metric properties of the digit systems considered here are studied. In the second part of this thesis, we determine of the bounds on the lengths of those elements with finite Scheicher-Thuswaldner's expansion, on the periods of elements with periodic expansions. In addition, a necessary and sufficient condition for periodic elements to have a prescribed period is proved, and an analysis of the simplest case, where the Scheicher-Thuswaldner's characterization is violated, is carried out. The tools used in this part are based on the analysis of Scheicher-Thuswaldner's algorithm.

This thesis is organized into five chapters. In Chapter II, we collect definitions and results, mainly without proof, to be used throughout the entire thesis. Chapter III gives the proof of our characterization for primitive polynomials modifying that of Fitzgerald which is the main result in the first part. Examlpes are also provided in order to compare the number of non-zero terms involved in our main theorem with that of Fitzgerald. Chapter IV contains the main results of the second part of the thesis involving Scheicher-Thuswaldner expansions. This chapter has four sections, namely, elements of finite lengths, periods of periodic representations, periodic representations with a prescribed period, and infinite and non-periodic expansions. In the first two sections, bounds on the lengths of elements whose expansions are finite and on the periods of elements with periodic expansions are derived. In the third section, conditions for a periodic expansion to have a prescribed period is given. The last section contains an analysis of the simplest case where the degree condition of Scheicher-Thuswalder is violated. Chapter V summarizes the main findings of the thesis.