CHAPTER IV

LENGTHS, PERIODS AND EXPANSIONS OF ELEMENTS WRITTEN WITH RESPECT TO A DIGIT SYSTEM

Scheicher and Thuswaldner [20] proposed a new digit system for elements in a polynomial ring of two indeterminates over a polynomial ring of one indeterminate. In this chapter, we propose to answer the following questions.

1. Given $r \in \mathcal{R} \setminus \{0\}$ with finite y-adic representation, find its length.

2. Given $r \in \mathcal{R} \setminus \{0\}$ with ultimately periodic, but not finite, y-adic representation, find its period.

3. For a given $r \in \mathcal{R} \setminus \{0\}$ having a periodic y-adic representation, find a necessary and sufficient condition for r to have a prescribed period.

4. For $p(x,y)$ not being a DS-polynomial, what kind of expansions their elements can have.

4.1 Elements of finite lengths

In this section, bounds for elements of finite lengths are determined.

Theorem 4.1.1. *Let* $p(x,y) = y^n + b_{n-1}y^{n-1} + \cdots + b_1 y - b_0 \in \mathbb{F}_q[x,y]$, $b_i \in \mathbb{F}_q[x]$, $B_0 := \deg b_0$, $B := \max_{i=1,\ldots,n-1} \deg b_i$. *Assume that* $B < B_0$, $B_0 > 0$.

I. If $r = c_0 + c_1 y + \cdots + c_{n-1}y^{n-1} \in \mathcal{R} \setminus \{0\}$, $c_l \in \mathbb{F}_q[x]$, *then* r *has a finite* y*-adic representation of length*

$$L(r) \leq \max_{0 \leq l \leq n-1} L\left(c_l y^l\right),$$

where $L(T)$ *denotes the length of the* y*-adic representation of* $T \in \mathcal{R}$ *having a finite expansion.*

II. Let $c \in \mathbb{F}_q[x] \setminus \{0\}$*. If* $\deg c < B_0$*, then* $L(c) = 1$*. If* $\deg c = B_0$*, then* $L(c) = n + 1$*.*

III. Let $c \in \mathbb{F}_q[x] \setminus \{0\}$*,* $C := \deg c$*. If* $C > B_0$*, then*

$$L(c) \leq 1 + \max\left\{ \left\lceil \frac{C - B_0 + 1 + jB}{B_0} \right\rceil n - j \;\; ; \;\; j = 0, 1, \ldots, \left\lceil \frac{C - B_0 + 1}{B_0 - B} \right\rceil \right\}$$

where $\lceil w \rceil$ *denotes the least integer greater than or equal to* $w \in \mathbb{R}$*.*

IV. For each $r = c_0 + c_1 y + \cdots + c_{n-1} y^{n-1} \in \mathcal{R} \setminus \{0\}$*, we have*

$$L(r) \leq n + \max\left\{ \left\lceil \frac{Q - B_0 + 1 + jB}{B_0} \right\rceil n - j \;\; ; \;\; j = 0, 1, \ldots, \left\lceil \frac{Q - B_0 + 1}{B_0 - B} \right\rceil \right\}$$

where $Q := \max_{0 \leq i \leq n-1} \deg c_i$*.*

Proof. I. Since $p(x, y)$ is a DS-polynomial, each monomial $c_l y^l$ and r have unique finite y-adic representations and the result follows immediately from the fact that the length of the sum of two representations is not greater than the longer.

II. Let $r^{(0)} = c \in \mathbb{F}_q[x] \setminus \{0\}$ be such that $\deg c \leq B_0$. If $\deg c < B_0$, then $c \in \mathcal{N}$ and $r^{(0)}$ has a finite y-adic representation of length 1. If $\deg c = B_0$, then

$$r^{(0)} = c = \tilde{r}_0 b_0 + d_0 \quad (\tilde{r}_0 \in \mathbb{F}_q \setminus \{0\}),$$
$$r^{(1)} = \frac{r^{(0)} - d_0}{y} = \tilde{r}_0 b_1 + \tilde{r}_0 b_2 y + \cdots + \tilde{r}_0 b_{n-1} y^{n-2} + \tilde{r}_0 y^{n-1}.$$

Note that $\deg(\tilde{r}_0 b_1) < B_0$ and $\tilde{r}_0 b_1 = (0) b_0 + d_1$. Thus

$$r^{(2)} = \frac{r^{(1)} - d_1}{y} = \tilde{r}_0 b_2 + \cdots + \tilde{r}_0 b_{n-1} y^{n-3} + \tilde{r}_0 y^{n-2},$$

where, as before, $\deg(\tilde{r}_0 b_2) < B_0$ and $\tilde{r}_0 b_2 = (0) b_0 + d_2$. Proceeding in the same manner, we finally reach

$$r^{(n-1)} = \tilde{r}_0 b_{n-1} + \tilde{r}_0 y, \qquad r^{(n)} = \tilde{r}_0$$

which gives $r^{(n+1)} = 0$, showing that $r^{(0)}$ has a finite y-adic representation of length $n + 1$.

III. Using the division algorithm and (2.2.1), we get

$$r^{(0)} = c = \tilde{r}_0 b_0 + d_0 = d_0 + \tilde{r}_0 \left(y^n + b_{n-1}y^{n-1} + \cdots + b_1 y\right),$$

where $\deg \tilde{r}_0 = C - B_0$ and so $\deg(\tilde{r}_0 b_i) \le C - B_0 + B$. The strategy is to reduce the degrees (in x) of the coefficients $\tilde{r}_0 b_i$ until all coefficients belong to $\mathcal{N}$. Clearly we need to consider only the term with the highest possible power of y and $\deg(\tilde{r}_0 b_i)$, called the *highest term* for short, which in this case is of the form $\max\{\tilde{r}_0 y^n,\ \tilde{r}_0 b_i y^{n-1}\}$.

If we perform this reduction again, the two terms in the last maximum become

$$\tilde{r}_0 y^n = y^n \left(\tilde{r}_1{}' b_0 + d_1'\right) = y^n \left(d_1' + \tilde{r}_1{}' \left(y^n + b_{n-1}y^{n-1} + \cdots + b_1 y\right)\right)$$

and

$$\tilde{r}_0 b_i y^{n-1} = y^{n-1} \left(\tilde{r}_1 b_0 + d_1\right) = y^{n-1} \left(d_1 + \tilde{r}_1 \left(y^n + b_{n-1}y^{n-1} + \cdots + b_1 y\right)\right).$$

After two reductions, the highest term is thus of the form

$$\max\left\{a_0 y^{2n},\ a_1 y^{2n-1},\ a_2 y^{2n-2}\right\},$$

where we use the generic coefficients $a_i \in \mathbb{F}_q[x]$ with

$$\deg a_0 \le C - 2B_0,\ \ \deg a_1 \le C - 2B_0 + B,\ \ \deg a_2 \le C - 2B_0 + 2B.$$

Proceeding in the same manner, after j reductions the highest term is of the form

$$\max\left\{a_0 y^{jn},\ a_1 y^{jn-1},\ a_2 y^{jn-2}, \ldots,\ a_j y^{jn-j}\right\},$$

where $\deg a_0 \le C - jB_0$, $\deg a_1 \le C - jB_0 + B$, $\deg a_2 \le C - jB_0 + 2B, \ldots,$ $\deg a_j \le C - jB_0 + jB$.

For $j = \left\lceil \frac{C - B_0 + 1}{B_0} \right\rceil$, the first term in the $\max\{.\}$ cannot be further reduced and gives the highest power of y as $\left\lceil \frac{C - B_0 + 1}{B_0} \right\rceil n$.

For $j = \left\lceil \frac{C-B_0+1+B}{B_0} \right\rceil$, the second term in the $\max\{.\}$ cannot be further reduced and gives the highest power of y as $\left\lceil \frac{C-B_0+1+B}{B_0} \right\rceil n-1$.

For $j = \left\lceil \frac{C-B_0+1+2B}{B_0} \right\rceil$, the third term in the $\max\{.\}$ cannot be further reduced and gives the highest power of y as $\left\lceil \frac{C-B_0+1+2B}{B_0} \right\rceil n-2$.

Finally, for $j = \left\lceil \frac{C-B_0+1+jB}{B_0} \right\rceil$, i.e., $j = \left\lceil \frac{C-B_0+1}{B_0-B} \right\rceil$, the last term in the $\max\{.\}$ cannot be further reduced and gives the highest power of y as $\left\lceil \frac{C-B_0+1}{B_0-B} \right\rceil n - \left\lceil \frac{C-B_0+1}{B_0-B} \right\rceil$.

IV. This is immediate from I and III. □

A weaker bound, with respect to n, than that in Theorem 4.1.1 IV can also be obtained based on the work of Scheicher and Thuswaldner, which runs as follows:

Proposition 4.1.2. *Each $r = c_0 + c_1 y + \cdots + c_{n-1}y^{n-1} \in \mathcal{R} \setminus \{0\}$ has a finite y-adic representation of length not exceeding $\{Q + (n-1)B + 1\}\, n$.*

Proof. Recall from Lemma 2.2.1 that each $r^{(0)} \in \mathcal{R}\setminus\{0\}$ has a unique representation

$$r^{(0)} = r_0^{(0)} + r_1^{(0)} y + \cdots + r_{n-1}^{(0)} y^{n-1} := \left(r_0^{(0)}, r_1^{(0)}, \ldots, r_{n-1}^{(0)}\right)_s$$

whose coefficients can be transformed uniquely to another sequence, $\left(\varepsilon_1^{(0)}, \varepsilon_2^{(0)}, \ldots, \varepsilon_n^{(0)}\right)_\varepsilon$, via

$$r_j^{(0)} = \sum_{i=1}^{n} \varepsilon_i^{(0)} b_{i+j}$$

with $r_j^{(0)}, \varepsilon_i^{(0)} \in \mathbb{F}_q[x] \quad (j = 0, 1, \ldots, n-1)$ and $r^{(0)} := (\varepsilon_1^{(0)}, \varepsilon_2^{(0)}, \ldots, \varepsilon_n^{(0)})_\varepsilon$ is called the *ε-representation* of $r^{(0)}$. The map $\left(r_0^{(0)}, r_1^{(0)}, \ldots, r_{n-1}^{(0)}\right)_s \to \left(\varepsilon_1^{(0)}, \varepsilon_2^{(0)}, \ldots, \varepsilon_n^{(0)}\right)_\varepsilon$ is a bijection. Inductively, for one cycle we get

$$r^{(n)} = (\varepsilon_1^{(n)}, \varepsilon_1^{(n-1)}, \ldots, \varepsilon_1^{(1)})_\varepsilon \ ,$$

where

$$\max_{i=1,\dots,n} \deg \varepsilon_1^{(i)} \le \max_{i=1,\dots,n} (\deg \varepsilon_i^{(0)} - 1).$$

Continuing for another cycle, we get

$$r^{(2n)} = (\varepsilon_1^{(2n)}, \varepsilon_1^{(2n-1)}, \dots, \varepsilon_1^{(n+1)})_\varepsilon$$

where

$$\max_{i=n+1,\dots,2n} \deg \varepsilon_1^{(i)} \le \max_{i=1,\dots,n} (\deg \varepsilon_i^{(0)} - 2).$$

Iterating for $M := \max_{i=1,\dots,n} \deg \varepsilon_i^{(0)}$ times, we arrive at

$$r^{(Mn)} = (\varepsilon_1^{(Mn)}, \varepsilon_1^{(Mn-1)}, \dots, \varepsilon_1^{((M-1)n+1)})_\varepsilon$$

with $\varepsilon_1^{((M-1)n+1)}, \dots, \varepsilon_1^{(Mn-1)}, \varepsilon_1^{(Mn)}$ being constant, and so

$$r^{((M+1)n)} = (\varepsilon_1^{((M+1)n)}, \varepsilon_1^{((M+1)n-1)}, \dots, \varepsilon_1^{(Mn+1)})_\varepsilon = (0, 0, \dots, 0)_\varepsilon.$$

Hence, each $r^{(0)} \in \mathcal{R} \setminus \{0\}$ has a finite y-adic representation of length $\le (M+1)n$. It remains to show that $M \le Q + (n-1)B$. From Lemma 2.2.1, we have

$$r_j = \sum_{i=1}^{n} \varepsilon_i b_{i+j} \quad (j = 0, 1, \dots, n-1)$$

where $r_j, \varepsilon_i \in \mathbb{F}_q[x]$. Solving for ε_j, we get

$$\varepsilon_1 = r_{n-1}, \quad \varepsilon_2 = r_{n-2} - \varepsilon_1 b_{n-1}, \quad \varepsilon_3 = r_{n-3} - \varepsilon_1 b_{n-2} - \varepsilon_2 b_{n-1},$$

and generally,

$$\varepsilon_j = r_{n-j} - \sum_{i=1}^{j-1} \varepsilon_i b_{n-j+i} \quad (j = 1, 2, \dots, n),$$

where the summation is taken to be 0 should the upper limit of the sum is 0. Observe that

$$\deg \varepsilon_1 = \deg r_{n-1} \le Q,$$

$$\deg \varepsilon_2 = \deg (r_{n-2} - \varepsilon_1 b_{n-1}) \le Q + B,$$

$$\deg \varepsilon_3 = \deg (r_{n-3} - \varepsilon_1 b_{n-2} - \varepsilon_2 b_{n-1}) \le Q + 2B,$$

and the desired result follows easily by induction. □

The following example shows that the bound in Theorem 4.1.1 IV is sharp.

Example 4.1.3. Let

$$p(x, y) = y^3 + (x^3 + 2x^2 + 1)y^2 + (2x^3 + x^2 + x)y - (x^5 + 2x^3 + x + 1) \in \mathbb{F}_3[x, y].$$

Take

$$r = (2x^4+1)+(x^7+2x^5+x^4+x+2)y+(2x^9+2x^8+x^7+x^5+2x^3+x+1)y^2 \in \mathcal{R}\setminus\{0\},$$

so that $B = 3$, $B_0 = 5$, $Q = 9$. The bound for the length obtained from Theorem 4.1.1 IV is 10, while that obtained in Proposition 4.1.2 is 48. Here the y-adic representation is

$$\begin{aligned} r = {} & (2x^4 + 1) + (x^4 + 2x^3 + 2x^2 + x + 2)y + (2x^4 + 2x^3 + x^2 + x)y^2 + (2x^4 + x)y^3 \\ & + (2x^4 + 2x^3 + x^2 + 2)y^4 + (x^3 + 2x + 2)y^5 + (x^4 + 2x^3)y^6 + (2x^2 + 2x + 1)y^7 \\ & + (2x^3 + x^2 + 1)y^8 + 2y^9, \end{aligned}$$

which is of length 10.

Specializing the DS-polynomial $p(x, y)$ to be monic of degree 1 in y, we get a particularly attractive result, which is exactly the special case treated in [16], relating length with the usual base representation.

Corollary 4.1.4. *Let $p(x, y) = y - b_0 \in \mathbb{F}_q[x, y]$, $b_0 \in \mathbb{F}_q[x]$, $\deg b_0 > 0$. Each $r = c \in \mathcal{R} \setminus \{0\}$ has a finite y-adic representation of length k, where k denotes the length of c considered as a polynomial in x written with respect to the base b_0. Furthermore, $k = 1 + \left\lfloor \frac{C}{B_0} \right\rfloor$, where $C := \deg c$ and $\lfloor \cdot \rfloor$ denotes the usual integer value function.*

Proof. In this case we have $\mathcal{R} = \mathbb{F}_q[x]$ and $c = r^{(0)} = d_0 + d_1 y + \cdots + d_{k-1} y^{k-1}$ has a finite y-adic representation of length k if and only if the following division steps hold

$$r^{(0)} = c = r_0^{(0)} = \tilde{r}_0 b_0 + d_0 \ ,$$

$$\cdots$$

$$r^{(k-1)} = \tilde{r}_{k-2} = r_0^{(k-1)} = \tilde{r}_{k-1} b_0 + d_{k-1} \ ,$$

$$r^{(k)} = 0.$$

This merely says that the base b_0 representation of c is $(d_{k-1} \ \ d_{k-2} \ \ \ldots \ \ d_1 \ \ d_0)_{b_0}$, which has length k. The value of the length k follows easily from the above division algorithm. □

The next example illustrates the finding of Corollary 4.1.4.

Example 4.1.5. Let $p(x,y) = y - (x^3 + 2x + 1) \in \mathbb{F}_3[x,y]$, so that $B_0 = 3$. Take

$$r = c = 2x^8 + x^7 + x^5 + 2x^3 + x + 2 \in \mathbb{F}_3[x] \setminus \{0\},$$

so that $C = 8$. The division algorithm gives

$$\begin{aligned} 2x^8 + x^7 + x^5 + 2x^3 + x + 2 &= (2x^5 + x^4 + 2x^3 + x)(x^3 + 2x + 1) + (x^2 + 2), \\ 2x^5 + x^4 + 2x^3 + x &= (2x^2 + x + 1)(x^3 + 2x + 1) + (2x^2 + x + 2), \\ 2x^2 + x + 1 &= (0)(x^3 + 2x + 1) + (2x^2 + x + 1), \end{aligned}$$

yielding the base b_0 representation as

$$\begin{aligned} c &= (2x^2 + x + 1)(x^3 + 2x + 1)^2 + (2x^2 + x + 2)(x^3 + 2x + 1) + (x^2 + 2) \\ &= (x^2 + 2) + (2x^2 + x + 2)y + (2x^2 + x + 1)y^2, \end{aligned}$$

which is also its y-adic representation. The length is $k = 3 = 1 + \left\lfloor \frac{8}{3} \right\rfloor$.

4.2 Periods of periodic representations

In this section, we keep the same notation as in the first section, in particular, $B := \max_{i=1,\ldots,n-1} \deg b_i$, $B_0 := \deg b_0$. Assume that

$$B = B_0 > 0.$$

Here, each element $r \in \mathcal{R} \setminus \{0\}$ has an ultimately periodic y-adic representation with period denoted by $\mathrm{Per}(r)$. This is equivalent to the fact that the sequence $\mathcal{U}_r = \left(r^{(0)}, r^{(1)}, r^{(2)}, \ldots\right)$, $r := r^{(0)}$, is ultimately periodic with period $\mathrm{Per}(r)$.

Theorem 4.2.1. *Let $p(x,y) = y^n + b_{n-1}y^{n-1} + \cdots + b_1 y - b_0 \in \mathbb{F}_q[x,y]$, $B := \max_{i=1,\ldots,n-1} \deg b_i$, $B_0 := \deg b_0$. Assume that $B = B_0 > 0$.*

I. If $r = c_0 + c_1 y + \cdots + c_{n-1}y^{n-1} \in \mathcal{R} \setminus \{0\}$, $c_j \in \mathbb{F}_q[x]$, then the sequence $\mathcal{U}_r$ is ultimately periodic with period

$$\mathrm{Per}(r) \leq \text{l.c.m.}\{\mathrm{Per}(c_l)\ ;\ 0 \leq l \leq n-1\},$$

where l.c.m. refers to the least common multiple.

II. If $r = c \in \mathbb{F}_q[x] \setminus \{0\}$ and $C < B_0$ where $C := \deg c$, then the sequence $\mathcal{U}_r$ is finite, which may also be interpreted as ultimately periodic with period $Per(c) = 1$.

III. If $r = c \in \mathbb{F}_q[x] \setminus \{0\}$ and $C \geq B_0$ where $C := \deg c$, then the sequence $\mathcal{U}_r$ is ultimately periodic with period

$$\mathrm{Per}(c) \leq q^{(C+1)n-2B_0} - 1.$$

Proof. I. This follows immediately from the fact that the sum of ultimately periodic sequences is ultimately periodic with period less than or equal to the least common multiple of $\mathrm{Per}(c_0), \mathrm{Per}(c_1), \ldots, \mathrm{Per}(c_{n-1})$.

II. We have in this case $r^{(0)} = c = (0)b_0 + d_0$, and so $r^{(1)} = \frac{r^{(0)} - d_0}{y} = 0$, which implies that $r^{(n)} = 0$ for all $n \geq 1$.

III. Carrying out the first few steps of the Scheicher-Thuswaldner's construction in the preliminaries, we get

$$r^{(0)} = c = \tilde{r}_0 b_0 + d_0 = d_0 + \tilde{r}_0 \left(b_1 y + \cdots + b_{n-1} y^{n-1} + y^n\right)$$
$$= d_0 + y\left(\tilde{r}_0 b_1 + \tilde{r}_0 b_2 y + \cdots + \tilde{r}_0 b_{n-1} y^{n-2} + \tilde{r}_0 y^{n-1}\right) \left(= d_0 + y r^{(1)}\right)$$
$$= d_0 + y\left((\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 b_2 y + \cdots + \tilde{r}_0 b_{n-1} y^{n-2} + \tilde{r}_0 y^{n-1}\right)$$
$$= d_0 + d_1 y + y\left(\tilde{r}_1 b_0 + \tilde{r}_0 b_2 y + \cdots + \tilde{r}_0 b_{n-1} y^{n-2} + \tilde{r}_0 y^{n-1}\right)$$
$$= d_0 + d_1 y + y T_1(y),$$

where $T_1(y) := \tilde{r}_1 b_0 + \tilde{r}_0 b_2 y + \cdots + \tilde{r}_0 b_{n-1} y^{n-2} + \tilde{r}_0 y^{n-1}$,

$$= d_0 + d_1 y + y\left(\tilde{r}_1\left(b_1 y + \cdots + b_{n-1} y^{n-1} + y^n\right) + \tilde{r}_0 b_2 y + \cdots + \tilde{r}_0 b_{n-1} y^{n-2} + \tilde{r}_0 y^{n-1}\right)$$
$$= d_0 + d_1 y + y^2\left((\tilde{r}_1 b_1 + \tilde{r}_0 b_2) + (\tilde{r}_1 b_2 + \tilde{r}_0 b_3) y + \cdots + (\tilde{r}_1 b_{n-1} + \tilde{r}_0) y^{n-2} + \tilde{r}_1 y^{n-1}\right)$$
$$\left(= d_0 + d_1 y + y^2 r^{(2)}\right)$$
$$= d_0 + d_1 y + y^2\left((\tilde{r}_2 b_0 + d_2) + (\tilde{r}_1 b_2 + \tilde{r}_0 b_3) y + \cdots + (\tilde{r}_1 b_{n-1} + \tilde{r}_0) y^{n-2} + \tilde{r}_1 y^{n-1}\right)$$
$$= d_0 + d_1 y + d_2 y^2 + y^2\left(\tilde{r}_2 b_0 + (\tilde{r}_1 b_2 + \tilde{r}_0 b_3) y + \cdots + (\tilde{r}_1 b_{n-1} + \tilde{r}_0) y^{n-2} + \tilde{r}_1 y^{n-1}\right)$$
$$= d_0 + d_1 y + d_2 y^2 + y^2 T_2(y),$$

where $T_2(y) := \tilde{r}_2 b_0 + (\tilde{r}_1 b_2 + \tilde{r}_0 b_3) y + \cdots + (\tilde{r}_1 b_{n-1} + \tilde{r}_0) y^{n-2} + \tilde{r}_1 y^{n-1}$,

$\deg \tilde{r}_i \leq C - B_0$ and $\deg(\tilde{r}_0 b_i) \leq C - B_0 + B = C$.

Continuing this procedure, the expressions representing $T_1(y), T_2(y), \ldots$ are of the form

$$A_0 b_0 + A_1 y + A_2 y^2 + \cdots + A_{n-2} y^{n-2} + A_{n-1} y^{n-1} \tag{4.2.1}$$

where $A_i \in \mathbb{F}_q[x]$ are subject to the constraints

$$\max\{\deg A_0, \deg A_{n-1}\} \leq C - B_0, \quad \max\{\deg A_1, \ldots, \deg A_{n-2}\} \leq C.$$

Since the sequence $\mathcal{U}_r$ is infinite, the polynomials $A_0, \ldots, A_{n-1}$ are not all zero. Consequently, the number of possible expressions in (4.2.1), which is also the largest possible value of $\mathrm{Per}(c)$, is not greater than $\left(q^{C-B_0+1}\right)^2 \left(q^{C+1}\right)^{n-2} - 1$. □

From the proof of Theorem 4.2.1, we obtain two further remarks.

(i) From the proof of Part III and the results of the first two parts, we easily deduce that the pre-period of any element

$$r = c_0 + c_1 y + \cdots + c_{n-1} y^{n-1} \in \mathcal{R} \setminus \{0\},\ C_i := \deg c_i \quad (i = 0, 1, \ldots, n-1),$$

is at most

$$\text{l.c.m.}\left\{\max\left(q^{(C_i+1)n-2B_0} - 1, 1\right);\ i = 0, 1, \ldots, n-1\right\}.$$

(ii) In the proof of Part III, since $A_1, \ldots, A_{n-2}$ are sums of polynomials in $\mathbb{F}_q[x]$ whose shapes are difficult to determine exactly, it is unlikely that the bound on the period so obtained is best possible. However in some simple cases, such as when $n = 2$, a best possible bound can be attained as shown in the following proposition.

Proposition 4.2.2. *Let* $p(x, y) = y^2 + b_1 y - b_0 \in \mathbb{F}_q[x, y]$, $r := c \in \mathbb{F}_q[x] \setminus \{0\}$, $C := \deg c$, $B_0 := \deg b_0$, $B := \deg b_1$. *If* $C = B_0 = B$, *then the sequence* $\mathcal{U}_r$ *is ultimately periodic with period* $\text{Per}(c) \leq (q-1)^2$.

Proof. Here, $b_0 = b_1 y + y^2$. Carrying out a few steps of construction as in the proof of Theorem 4.2.1, we get

$$\begin{aligned}
r^{(0)} = c = \tilde{r}_0 b_0 + d_0 &= d_0 + \tilde{r}_0(b_1 y + y^2) = d_0 + y\left(\tilde{r}_0 b_1 + \tilde{r}_0 y\right) \\
&= d_0 + y\left((\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 y\right) = d_0 + d_1 y + y\left(\tilde{r}_1 b_0 + \tilde{r}_0 y\right) \\
&= d_0 + d_1 y + y T_1(y), \quad \text{where } T_1(y) := \tilde{r}_1 b_0 + \tilde{r}_0 y, \\
&= \ldots \\
&= d_0 + d_1 y + d_2 y^2 + y^2\left(\tilde{r}_2 b_0 + \tilde{r}_1 y\right) \\
&= d_0 + d_1 y + d_2 y^2 + y^2 T_2(y), \quad \text{where } T_2(y) := \tilde{r}_2 b_0 + \tilde{r}_1 y.
\end{aligned}$$

The general expressions for $T_1(y), T_2(y), \ldots$ are of the form $s b_0 + t y$; $s, t \in \mathbb{F}_q \setminus \{0\}$. Consequently, $\text{Per}(c) \leq (q-1)^2$. □

The following example shows the sharpness of the bound in Proposition 4.2.2.

Example 4.2.3. Let $p(x,y) = y^2 + (x^2+x+1)y - (x^2+x) \in \mathbb{F}_2[x,y]$. Take $r^{(0)} = c = x^2+1$. Then $C = B_0 = B$ and

$$r^{(1)} = (x^2+x+1)+y, \quad r^{(2)} = (x^2+x)+y, \quad r^{(3)} = (x^2+x)+y = r^{(2)}.$$

The sequence $\mathcal{U}_r$ is ultimately periodic with period $1 = (q-1)^2$.

4.3 Periodic representations with a prescribed period

In this section, we first derive necessary and sufficient conditions for the sequence $\mathcal{U}_r$ to be purely periodic with prescribed period $\pi \geq 1$. We again keep the same notation as in the first section.

Theorem 4.3.1. *Let* $p(x,y) = y^n + b_{n-1}y^{n-1} + \cdots + b_1 y - b_0 \in \mathbb{F}_q[x,y]$, $B_0 := \deg b_0$, $B := \max_{i=1,\ldots,n-1} \deg b_i$, *and*

$$r := r^{(0)} = r_0^{(0)} + r_1^{(0)}y + \cdots + r_{n-1}^{(0)}y^{n-1} \in \mathcal{R} \setminus \{0\}.$$

Assume that $B_0 = B$. *Then the sequence* $\mathcal{U}_r$ *is purely periodic of period* $\pi \geq 1$ *if and only if*

$$r_{n-l}^{(0)} = \tilde{r}_{\pi-1}b_{n-l+1} + \tilde{r}_{\pi-2}b_{n-l+2} + \cdots + \tilde{r}_0 b_{n-l+\pi} + r_{n-l+\pi}^{(0)} \quad (l = 1,2,\ldots,n)$$

where $\tilde{r}_i$ *are as defined in (2.2.2)-(2.2.5);* $b_n := 1$, *and* $b_{j+1}, r_j^{(0)}$ *are taken to be* 0 *should* $j \geq n$.

Proof. If $\pi < n$, then let $n = \pi + j \quad (j \in \mathbb{N})$. By (2.2.3) and (2.2.4), we have

$$\begin{aligned} r^{(\pi)} &= r_0^{(\pi)} + r_1^{(\pi)} y + \cdots + r_{n-2}^{(\pi)} y^{n-2} + r_{n-1}^{(\pi)} y^{n-1} \\ &= \left(\tilde{r}_{\pi-1} b_1 + \tilde{r}_{\pi-2} b_2 + \cdots + \tilde{r}_0 b_\pi + r_\pi^{(0)}\right) + \cdots \\ &\quad + \left(\tilde{r}_{\pi-1} b_j + \tilde{r}_{\pi-2} b_{j+1} + \cdots + \tilde{r}_0 b_{\pi+(j-1)} + r_{\pi+(j-1)}^{(0)}\right) y^{j-1} \\ &\quad + \left(\tilde{r}_{\pi-1} b_{j+1} + \tilde{r}_{\pi-2} b_{j+2} + \cdots + \tilde{r}_0 b_{\pi+j}\right) y^j + \cdots \\ &\quad + \left(\tilde{r}_{\pi-1} b_{n-1} + \tilde{r}_{\pi-2} b_n\right) y^{n-2} + \left(\tilde{r}_{\pi-1} b_n\right) y^{n-1}. \end{aligned}$$

If $\pi \geq n$, using (2.2.4) we have

$$\begin{aligned} r^{(\pi)} = &\left(\tilde{r}_{\pi-1} b_1 + \tilde{r}_{\pi-2} b_2 + \cdots + \tilde{r}_{\pi-n} b_n\right) + \left(\tilde{r}_{\pi-1} b_2 + \tilde{r}_{\pi-2} b_3 + \cdots + \tilde{r}_{\pi-n+1} b_n\right) y \\ &+ \cdots + \left(\tilde{r}_{\pi-1} b_{n-1} + \tilde{r}_{\pi-2} b_n\right) y^{n-2} + \left(\tilde{r}_{\pi-1} b_n\right) y^{n-1}. \end{aligned}$$

The result follows from equating the coefficients of the powers of y in $r^{(0)} = r^{(\pi)}$. □

Immediate from Theorem 4.3.1 and Remark (i) after Theorem 4.2.1 is:

Corollary 4.3.2. *Let the notation be as in Theorem 4.3.1 and let* $C_i := \deg r_i^{(0)}$ $(i = 0, 1, \ldots, n-1)$. *The sequence* $\mathcal{U}_r$ *is ultimately periodic of period* $\pi \geq 1$ *if and only if there is an index*

$$s < l.c.m. \left\{\max\left(q^{(C_i+1)n-2B_0} - 1, 1\right) ; \ i = 0, 1, \ldots, n-1\right\}$$

such that

$$r_{n-l}^{(s)} = \tilde{r}_{s+\pi-1} b_{n-l+1} + \tilde{r}_{s+\pi-2} b_{n-l+2} + \cdots + \tilde{r}_s b_{n-l+\pi} + r_{n-l+\pi}^{(s)} \quad (l = 1, 2, \ldots, n).$$

The next example shows that the bound for the pre-period s in Corollary 4.3.2 is sharp.

Example 4.3.3. Let $p(x, y) = y^2 + (x+1)y - x \in \mathbb{F}_2[x, y]$. Note that $n = 2$, $B = \deg b_1 = \deg b_0 = B_0 = 1$. Take

$$r = r^{(0)} = x + 1 \in \mathcal{R} = \{a_0 + a_1 y \ ; \ a_i \in \mathbb{F}_2[x]\}$$

where $r_0^{(0)} = x + 1,\ C_0 = \deg r_0^{(0)} = 1$ and $C_1 = \deg r_1^{(0)},\ r_1^{(0)} = 0$. Then

$$r^{(1)} = (x+1) + y,\ r^{(2)} = x + y,\ r^{(3)} = x + y = r^{(2)}.$$

Thus, the sequence $\mathcal{U}_r$ is ultimately periodic with period $\pi = 1$ and pre-period

$$2 = s < \text{l.c.m.}\left\{\max\left(2^{(C_i+1)2-2B_0} - 1, 1\right);\ i = 0, 1\right\} = 3.$$

Example 4.3.4. Let $p(x, y) = y^2 + xy - x \in \mathbb{F}_3[x, y]$. Take

$$\begin{aligned} r = r^{(0)} &= (x^2+1) + (x+2)y \in \mathcal{R} \\ &= \{xx + 1\} + (x+2)y. \end{aligned}$$

Then

$$\begin{aligned} r^{(1)} &= (x^2 + x + 2) + xy = \{(x+1)x + 2\} + xy, \\ r^{(2)} &= (x^2 + 2x) + (x+1)y = \{(x+2)x + 0\} + (x+1)y, \\ r^{(3)} &= (x^2+1) + (x+2)y = r^{(0)}. \end{aligned}$$

The sequence $\mathcal{U}_r$ is purely periodic with period $\pi = 3$ and $\tilde{r}_0 = x,\ \tilde{r}_1 = x + 1$, $\tilde{r}_2 = x + 2$. Moreover, $r^{(0)}$ satisfies the following relations

$$r_{n-1}^{(0)} = r_1^{(0)} = x + 2 = \tilde{r}_2 b_2 = \tilde{r}_{\pi-1} b_{(n-1)+1}$$

and

$$r_{n-2}^{(0)} = r_0^{(0)} = x^2 + 1 = (x+2)x + (x+1) = \tilde{r}_2 b_1 + \tilde{r}_1 b_2 = \tilde{r}_{\pi-1} b_{(n-2)+1} + \tilde{r}_{\pi-2} b_{(n-2)+2}.$$

If we take another element

$$\begin{aligned} r = r^{(0)} &= (x^2 + x + 2) + y \in \mathcal{R} \\ &= \{(x+1)x + 2\} + y, \end{aligned}$$

then

$$r^{(1)} = (x^2 + x + 1) + (x + 1)y = \{(x + 1)x + 1\} + (x + 1)y,$$

$$r^{(2)} = (x^2 + 2x + 1) + (x + 1)y = \{(x + 2)x + 1\} + (x + 1)y,$$

$$r^{(3)} = (x^2 + 1) + (x + 2)y = \{xx + 1\} + (x + 2)y,$$

$$r^{(4)} = (x^2 + x + 2) + xy = \{(x + 1)x + 2\} + xy,$$

$$r^{(5)} = (x^2 + 2x) + (x + 1)y = \{(x + 2)x + 0\} + (x + 1)y,$$

$$r^{(6)} = (x^2 + 1) + (x + 2)y = r^{(3)}.$$

The sequence $\mathcal{U}_r$ is ultimately periodic of period $\pi = 3$ with pre-period $s = 3$, $\tilde{r}_0 = x + 1,\ \tilde{r}_1 = x + 1,\ \tilde{r}_2 = x + 2,\ \tilde{r}_3 = x,\ \tilde{r}_4 = x + 1,\ \tilde{r}_5 = x + 2$. Note that $r^{(s)}$ satisfies the following relations

$$r^{(s)}_{n-1} = r^{(3)}_{n-1} = r^{(3)}_1 = x + 2 = \tilde{r}_5 b_2 = \tilde{r}_{s+\pi-1} b_{(n-1)+1}$$

and

$$\begin{aligned} r^{(s)}_{n-2} = r^{(3)}_{n-2} = r^{(3)}_0 = x^2 + 1 = (x + 2)x + (x + 1) &= \tilde{r}_5 b_1 + \tilde{r}_4 b_2 \\ &= \tilde{r}_{s+\pi-1} b_{(n-2)+1} + \tilde{r}_{s+\pi-2} b_{(n-2)+2}. \end{aligned}$$

4.4 Infinite, non-periodic expansions

In this section we investigate the simplest case where the degree condition of Scheicher and Thuswaldner is violated. This corresponds to the case where

$$p(x, y) = y^2 + b_1 y - b_0 \in \mathbb{F}_q[x, y],$$

with $\deg b_1 > \deg b_0 > 0$. For brevity, we keep the notation $B := \deg b_1$, $B_0 := \deg b_0$. Here, $\mathcal{R} = \{c_0 + c_1 y\ ;\ c_i \in \mathbb{F}_q[x]\}$ and we set $B = B_0 + \beta\ \ (\beta \in \mathbb{N})$. Take any starting element $r := r^{(0)} \in \mathcal{R} \setminus \{0\}$, i.e.,

$$r^{(0)} = r^{(0)}_0 + r^{(0)}_1 y = (\tilde{r}_0 b_0 + d_0) + r^{(0)}_1 y.$$

Following the Scheicher-Thuswaldner's construction in the preliminaries, we find

$$r^{(1)} = r_0^{(1)} + r_1^{(1)} y = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y = (\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 y,$$

$$r^{(2)} = r_0^{(2)} + r_1^{(2)} y = (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y.$$

Case 1. $\deg r_0^{(0)} < B_0, \quad \deg r_1^{(0)} < B_0.$

Clearly, in this case $\mathcal{U}_r = \left(r^{(0)}, r^{(1)}, r^{(2)} = 0, 0, \ldots\right)$ $\left(r := r^{(0)}\right)$, is a finite sequence of length ≤ 2. If $r_1^{(0)} \neq 0$, then the sequence $\mathcal{U}_r$ has length 2.

Case 2. $\deg r_0^{(0)} < B_0, \quad \deg r_1^{(0)} \geq B_0.$

Putting $\deg r_1^{(0)} = B_0 + j \qquad (j \in \mathbb{N}_0)$, we get

$$r = r^{(0)} = r_0^{(0)} + r_1^{(0)} y = (\tilde{r}_0 b_0 + d_0) + r_1^{(0)} y \quad (\tilde{r}_0 = 0),$$

$$r^{(1)} = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y = (\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 y,$$

where $\deg(\tilde{r}_0 b_1 + r_1^{(0)}) = \deg r_1^{(0)} = B_0 + j$ and so $\deg \tilde{r}_1 = j$. Next,

$$r^{(2)} = (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y,$$

where

$$\deg(\tilde{r}_1 b_1 + \tilde{r}_0) = \deg(\tilde{r}_1 b_1) = B_0 + \beta + j > B_0 + j = \deg(\tilde{r}_0 b_1 + r_1^{(0)}),$$

which implies $\deg \tilde{r}_2 > \deg \tilde{r}_1$. Proceeding in the same manner, in general we have

$$r^{(k)} = r_0^{(k)} + r_1^{(k)} y \quad (k \geq 2)$$

with $\deg r_0^{(k)} > \deg r_0^{(k-1)}$ and $\deg r_1^{(k)} > \deg r_1^{(k-1)}$, rendering the sequence $\mathcal{U}_r$ to be infinite and non-periodic.

Case 3. $\deg r_0^{(0)} \geq B_0, \quad \deg r_1^{(0)} < B_0.$

The same analysis as in Case 2 shows that the sequence $\mathcal{U}_r$ is infinite and non-periodic.

Case 4. $\deg r_0^{(0)} \geq B_0, \quad \deg r_1^{(0)} \geq B_0.$

Subcase 4.1. $\deg r_0^{(0)} = B_0, \quad \deg r_1^{(0)} = B_0.$

From $r = r^{(0)} = r_0^{(0)} + r_1^{(0)}y = (\tilde{r}_0 b_0 + d_0) + r_1^{(0)}y$, we see that $\tilde{r}_0 \in \mathbb{F}_q \setminus \{0\}$. Next,

$$r^{(1)} = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y = (\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 y,$$

with $\deg(\tilde{r}_0 b_1 + r_1^{(0)}) = \deg(\tilde{r}_0 b_1) = B_0 + \beta \quad (\beta \in \mathbb{N})$. Thus, $\deg \tilde{r}_1 = \beta$. Similarly,

$$r^{(2)} = (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y,$$

with $\deg(\tilde{r}_1 b_1 + \tilde{r}_0) = \deg(\tilde{r}_1 b_1) = B_0 + 2\beta$ and so $\deg \tilde{r}_2 = 2\beta$. Proceeding in the same manner, we have in general

$$r^{(k)} = r_0^{(k)} + r_1^{(k)}y \quad (k \geq 1)$$

with $\deg r_0^{(k)} = B_0 + k\beta$ and $\deg r_1^{(k)} = (k-1)\beta$ showing that the sequence $\mathcal{U}_r$ is infinite and non-periodic.

Subcase 4.2. $\deg r_0^{(0)} > B_0, \quad \deg r_1^{(0)} = B_0.$

Putting $\deg r_0^{(0)} = B_0 + l \quad (l \in \mathbb{N})$, we have

$$r = r^{(0)} = r_0^{(0)} + r_1^{(0)}y = (\tilde{r}_0 b_0 + d_0) + r_1^{(0)}y$$

with $\deg \tilde{r}_0 = l$. Similarly,

$$r^{(1)} = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y = (\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 y,$$

$$\deg(\tilde{r}_0 b_1 + r_1^{(0)}) = \deg(\tilde{r}_0 b_1) = B_0 + \beta + l,$$

yielding $\deg \tilde{r}_1 = \beta + l$, and

$$r^{(2)} = (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y,$$

$$\deg(\tilde{r}_1 b_1 + \tilde{r}_0) = \deg(\tilde{r}_1 b_1) = B_0 + 2\beta + l,$$

yielding $\deg \tilde{r}_2 = 2\beta + l$. In general,

$$r^{(k)} = r_0^{(k)} + r_1^{(k)}y \quad (k \geq 1),$$

$$\deg r_0^{(k)} = B_0 + k\beta + l, \quad \deg r_1^{(k)} = (k-1)\beta + l,$$

showing that the sequence $\mathcal{U}_r$ is infinite and non-periodic.

Subcase 4.3. $\deg r_0^{(0)} = B_0, \quad \deg r_1^{(0)} > B_0.$

Putting $\deg r_1^{(0)} = B_0 + j \quad (j \in \mathbb{N})$, we get

$$r = r^{(0)} = r_0^{(0)} + r_1^{(0)} y = (\tilde{r}_0 b_0 + d_0) + r_1^{(0)} y, \quad \tilde{r}_0 \in \mathbb{F}_q \setminus \{0\}\,.$$

Next,

$$r^{(1)} = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y = (\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 y, \ \deg(\tilde{r}_0 b_1) = B_0 + \beta.$$

There are two possibilities.

A. $\deg(\tilde{r}_0 b_1) \neq \deg r_1^{(0)}$.

If $\deg(\tilde{r}_0 b_1) > \deg r_1^{(0)}$, then $\deg(\tilde{r}_0 b_1 + r_1^{(0)}) = \deg(\tilde{r}_0 b_1) = B_0 + \beta$, and so $\deg \tilde{r}_1 = \beta$. By the same proof as in Subcase 4.1, we deduce that the sequence $\mathcal{U}_r$ is infinite and non-periodic.

If $\deg(\tilde{r}_0 b_1) < \deg r_1^{(0)}$, then $\deg(\tilde{r}_0 b_1 + r_1^{(0)}) = B_0 + j$ and so $\deg \tilde{r}_1 = j$. Next,

$$r^{(2)} = (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y,$$

$$\deg(\tilde{r}_1 b_1 + \tilde{r}_0) = \deg(\tilde{r}_1 b_1) = B_0 + \beta + j,$$

and so $\deg \tilde{r}_2 = \beta + j$. Proceeding in the same manner, we have generally

$$r^{(k)} = r_0^{(k)} + r_1^{(k)} y \quad (k \geq 2),$$

$$\deg r_0^{(k)} = B_0 + (k-1)\beta + j, \quad \deg r_1^{(k)} = (k-2)\beta + j,$$

yielding the sequence $\mathcal{U}_r$ infinite and non-periodic.

B. $\deg(\tilde{r}_0 b_1) = \deg r_1^{(0)}$.

We have $r^{(1)} = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y$, with $\deg \tilde{r}_0 = 0$. Treating $r^{(1)}$ as the starting element in $\mathcal{R} \setminus \{0\}$ and using the results of Cases 1 and 3, we deduce that:

if $\deg(\tilde{r}_0 b_1 + r_1^{(0)}) < B_0$, then the sequence $\mathcal{U}_r$ is finite of length 3, noting that $r^{(2)} = \tilde{r}_0$ and $r^{(3)} = 0$, while if $\deg(\tilde{r}_0 b_1 + r_1^{(0)}) \geq B_0$, then the sequence $\mathcal{U}_r$ is infinite and non-periodic.

Subcase 4.4. $\deg r_0^{(0)} > B_0, \quad \deg r_1^{(0)} > B_0$.

Putting $\deg r_0^{(0)} = B_0 + l, \quad \deg r_1^{(0)} = B_0 + j \quad (l, j \in \mathbb{N})$, we get

$$r = r^{(0)} = r_0^{(0)} + r_1^{(0)} y = (\tilde{r}_0 b_0 + d_0) + r_1^{(0)} y, \quad \deg \tilde{r}_0 = l,$$

and

$$r^{(1)} = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y = (\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 y, \quad \deg(\tilde{r}_0 b_1) = B_0 + \beta + l.$$

Again there are two possibilities.

A. $\deg(\tilde{r}_0 b_1) \neq \deg r_1^{(0)}$.

If $\deg(\tilde{r}_0 b_1) > \deg r_1^{(0)}$, then

$$\deg(\tilde{r}_0 b_1 + r_1^{(0)}) = \deg(\tilde{r}_0 b_1) = B_0 + \beta + l,$$

and so $\deg \tilde{r}_1 = \beta + l$. Using the result of Subcase 4.2, the sequence $\mathcal{U}_r$ is infinite and non-periodic.

If $\deg(\tilde{r}_0 b_1) < \deg r_1^{(0)}$, then

$$B_0 + \beta + l = \deg(\tilde{r}_0 b_1) < \deg(\tilde{r}_0 b_1 + r_1^{(0)}) = \deg r_1^{(0)} = B_0 + j,$$

which implies $l < j, \quad \deg \tilde{r}_1 = j$. Next,

$$r^{(2)} = (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y,$$

$$\deg(\tilde{r}_1 b_1 + \tilde{r}_0) = \deg(\tilde{r}_1 b_1) = B_0 + \beta + j,$$

and so $\deg \tilde{r}_2 = \beta + j$. The result of Subcase 4.3 shows that the sequence $\mathcal{U}_r$ is infinite and non-periodic.

B. $\deg(\tilde{r}_0 b_1) = \deg r_1^{(0)}$.

If $\deg(\tilde{r}_0 b_1 + r_1^{(0)}) < B_0$, then consider $r^{(1)} = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y$, $\deg \tilde{r}_0 = l$ as the starting element and using the results of Cases 1 and 2, we deduce that: if $l < B_0$, then the sequence $\mathcal{U}_r$ is finite of length 3 since $r^{(2)} = \tilde{r}_0$ and $r^{(3)} = 0$, while if $l \geq B_0$, then the sequence $\mathcal{U}_r$ is infinite and non-periodic.

If $\deg(\tilde{r}_0 b_1 + r_1^{(0)}) \geq B_0$, then consider $r^{(1)} = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y$, $\deg \tilde{r}_0 = l$ as the starting element and using the results of Case 3 and Subcases 4.1, 4.2, we deduce that: if $l < B_0$ or $l = B_0$, then the sequence $\mathcal{U}_r$ is infinite and non-periodic.

If $\deg(\tilde{r}_0 b_1 + r_1^{(0)}) = B_0$, $l = \deg \tilde{r}_0 > B_0$, then $r^{(2)} = (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y$.
If $\deg(\tilde{r}_1 b_1) \neq \deg \tilde{r}_0$, then the sequence $\mathcal{U}_r$ is infinite and non-periodic by Subcase 4.3.
If $\deg(\tilde{r}_1 b_1) = \deg \tilde{r}_0$, $\deg(\tilde{r}_1 b_1 + \tilde{r}_0) < B_0$, then the sequence $\mathcal{U}_r$ is finite of length 4, noting that $r^{(3)} = \tilde{r}_1$, $r^{(4)} = 0$, by Subcase 4.3.
If $\deg(\tilde{r}_1 b_1 + \tilde{r}_0) \geq B_0$, then the sequence $\mathcal{U}_r$ is infinite and non-periodic by Subcase 4.3.

There remains the case

$$\deg(\tilde{r}_0 b_1 + r_1^{(0)}) > B_0, \quad \deg \tilde{r}_0 > B_0,$$

in which we repeat the process of Subcase 4.4. Recall that here $\deg r_0^{(0)} > B_0$, $\deg r_1^{(0)} > B_0$. Putting

$$\deg r_0^{(0)} := B_0 + l_0, \ \deg r_1^{(0)} := B_0 + j_0 \quad (l_0, j_0 \in \mathbb{N}),$$

we get

$$r^{(1)} = r_0^{(1)} + r_1^{(1)} y = (\tilde{r}_0 b_1 + r_1^{(0)}) + \tilde{r}_0 y = (\tilde{r}_1 b_0 + d_1) + \tilde{r}_0 y,$$

$$\deg(\tilde{r}_0 b_1) = B_0 + \beta + l_0, \ \deg \tilde{r}_0 = l_0.$$

Since $\deg(\tilde{r}_0 b_1 + r_1^{(0)}) > B_0$ and $\deg \tilde{r}_0 > B_0$, setting

$$\deg(\tilde{r}_0 b_1 + r_1^{(0)}) = B_0 + l_1 \leq B_0 + j_0, \quad l_0 = \deg \tilde{r}_0 = B_0 + j_1,$$

we get $\deg \tilde{r}_1 = l_1 \le j_0 = \beta + l_0,\ \ j_1 = l_0 - B_0$. Observe that the total degree in x of $r^{(1)}$ is equal to

$$\deg r_0^{(1)} + \deg r_1^{(1)} = \deg(\tilde{r}_0 b_1 + r_1^{(0)}) + \deg \tilde{r}_0 = B_0 + l_1 + l_0.$$

Next consider

$$r^{(2)} = r_0^{(2)} + r_1^{(2)} y = (\tilde{r}_1 b_1 + \tilde{r}_0) + \tilde{r}_1 y = (\tilde{r}_2 b_0 + d_2) + \tilde{r}_1 y.$$

If $\deg r_0^{(2)} < B_0$ or $\deg r_1^{(2)} < B_0$, then the results of Cases 1, 2 and 3 show that the sequence $\mathcal{U}_r$ is finite of length 4, noting that $r^{(3)} = \tilde{r}_1$ and $r^{(4)} = 0$, or infinite, non-periodic.

If $\deg r_0^{(2)} \ge B_0$ and $\deg r_1^{(2)} \ge B_0$, then by the proof of Subcase 4.4 the sequence $\mathcal{U}_r$ is finite of length 5 if $\max\{\deg r_0^{(3)}, \deg r_1^{(3)}\} < B_0$ and of length 6 if $\max\{\deg r_0^{(4)}, \deg r_1^{(4)}\} < B_0$, or infinite non-periodic in every case except when $\deg(\tilde{r}_1 b_1) = \deg \tilde{r}_0$, $\deg(\tilde{r}_1 b_1 + \tilde{r}_0) > B_0$ and $\deg \tilde{r}_1 > B_0$.

In the latter situation, we repeat the process of Subcase 4.4 for the third time, keeping in mind that $\beta + l_1 = j_1$. Since $\deg(\tilde{r}_1 b_1 + \tilde{r}_0) > B_0$ and $\deg \tilde{r}_1 > B_0$, setting

$$\deg(\tilde{r}_1 b_1 + \tilde{r}_0) = B_0 + l_2 \le B_0 + j_1,\ \ l_1 = \deg \tilde{r}_1 = B_0 + j_2,$$

we get $\deg \tilde{r}_2 = l_2 \le j_1 = \beta + l_1,\ j_2 = l_1 - B_0$. Consequently, the total degree in x of $r^{(2)}$ is equal to

$$\deg r_0^{(2)} + \deg r_1^{(2)} = \deg(\tilde{r}_1 b_1 + \tilde{r}_0) + \deg \tilde{r}_1 = B_0 + l_2 + l_1$$

i.e., the total degree of $r^{(2)}$ is reduced from that of $r^{(1)}$ by at least $l_0 - l_2 = B_0$. Continuing this process till we reach the stage where

$$r^{(t)} = r_0^{(t)} + r_1^{(t)} y, \quad \deg r_0^{(t)} < B_0 \text{ or } \deg r_1^{(t)} < B_0.$$

Appealing to the results of Cases 1, 2 and 3, we conclude that the sequence $\mathcal{U}_r$ can only be

(i) finite of length $k+1$ if there exists a non-negative integer k such that $r_1^{(k)} = 0,\ \deg r_0^{(k)} < B_0$ or

(ii) finite of length $k+2$ if there exists a non-negative integer k such that $r_1^{(k)} \neq 0,\ \max\left\{\deg r_0^{(k)}, \deg r_1^{(k)}\right\} < B_0$ or

(iii) infinite, non-periodic.

Summarizing, we have:

Theorem 4.4.1. *Let $p(x,y) = y^2 + b_1 y - b_0 \in \mathbb{F}_q[x,y]$, $B := \deg b_1$, $B_0 := \deg b_0$. If $B > B_0 > 0$, then each $r \in \mathcal{R} \setminus \{0\} := \{c_0 + c_1 y \ ;\ c_i \in \mathbb{F}_q[x]\} \setminus \{0\}$ either has a finite or an infinite but non-periodic Scheicher-Thuswaldner representations. More precisely, $r \in \mathcal{R} \setminus \{0\}$ has a finite expansion if and only if there exists a non-negative integer k such that*

$$\max\left(\deg r_0^{(k)}, \deg r_1^{(k)}\right) < B_0,$$

where $r^{(k)} := r_0^{(k)} + r_1^{(k)} y$; moreover, the sequence $\mathcal{U}_r$ is finite of length $k+1$ if $r_1^{(k)} = 0$ and of length $k+2$ if $r_1^{(k)} \neq 0$.