CHAPTER V

CONCLUSION

A. Primitive polynomials over $\mathbb{F}_q$

1. Let $P(x) = p_k + p_{k-1}x + \cdots + p_0x^k \in \mathbb{F}_q[x]$ be irreducible over $\mathbb{F}_q$ and of degree $k \geq 2$. Let $m = q^k - 1,\ t = \frac{m}{q-1},\ y = x^{q-1}$ and

$$G(y) = \frac{y^t - 1}{(y-1)P(y)} = H(y) + \frac{r_0 + r_1y + \cdots + r_{k-1}y^{k-1}}{P(y)},$$

where $H(y) = \varepsilon_{t-k} + \varepsilon_{t-k-1}y + \cdots + \varepsilon_1 y^{t-k-1} \in \mathbb{F}_q[y]$. Then $P(x)$ is primitive over $\mathbb{F}_q$ if and only if the number of non-zero terms in $H(y)$, considered as polynomial in y over $\mathbb{F}_q$, is equal to $q^{k-1}(q-1) - 1 - N$, where N is the number of non-zero terms in the finite sequence $\varepsilon_{t-k+1}, \varepsilon_{t-k+2}, \ldots, \varepsilon_{m-1}, \varepsilon_m$ which is defined by

$$\varepsilon_{t-n} = r_n - \sum_{i=1}^{k-n-1} p_i \varepsilon_{t-n-i} \quad (n = 0, 1, \ldots, k-1)$$

where empty sum is interpreted as 0, and

$$\varepsilon_{t+n} = 1 - \sum_{i=1}^{k} p_i \varepsilon_{t+n-i} \quad (n = 1, 2, \ldots, m-t).$$

2. An irreducible polynomial $P(x) = p_k + p_{k-1}x + \cdots + p_1x^{k-1} + x^k \in \mathbb{F}_q[x]$ is primitive if and only if the finite sequence $\varepsilon_1,\ \varepsilon_2, \ldots,\ \varepsilon_m$, so defined as in 1, contains no two (identical) periodic subsequences.

B. Digit systems over $\mathbb{F}_q[x]$

1. Let $p(x,y) = y^n + b_{n-1}y^{n-1} + \cdots + b_1y - b_0 \in \mathbb{F}_q[x,y],\ b_i \in \mathbb{F}_q[x]$, $B_0 := \deg b_0,\ B := \max_{i=1,\ldots,n-1} \deg b_i$. Assume that $B < B_0,\ B_0 > 0$.

I. If $r = c_0 + c_1y + \cdots + c_{n-1}y^{n-1} \in \mathcal{R} \setminus \{0\},\ \ c_l \in \mathbb{F}_q[x]$, then r has a finite y-adic representation of length

$$L(r) \leq \max_{0 \leq l \leq n-1} L\left(c_l y^l\right),$$

where $L(T)$ denotes the length of the y-adic representation of $T \in \mathcal{R}$ having a finite expansion.

II. Let $c \in \mathbb{F}_q[x]\backslash\{0\}$. If $\deg c < B_0$, then $L(c) = 1$. If $\deg c = B_0$, then $L(c) = n+1$.

III. Let $c \in \mathbb{F}_q[x] \setminus \{0\}$, $C := \deg c$. If $C > B_0$, then

$$L(c) \leq 1 + \max\left\{ \left\lceil \frac{C - B_0 + 1 + jB}{B_0} \right\rceil n - j \ ; \ j = 0, 1, \ldots, \left\lceil \frac{C - B_0 + 1}{B_0 - B} \right\rceil \right\}$$

where $\lceil w \rceil$ denotes the least integer greater than or equal to $w \in \mathbb{R}$.

IV. For each $r = c_0 + c_1 y + \cdots + c_{n-1} y^{n-1} \in \mathcal{R} \setminus \{0\}$, we have

$$L(r) \leq n + \max\left\{ \left\lceil \frac{Q - B_0 + 1 + jB}{B_0} \right\rceil n - j \ ; \ j = 0, 1, \ldots, \left\lceil \frac{Q - B_0 + 1}{B_0 - B} \right\rceil \right\}$$

where $Q := \max_{0 \leq i \leq n-1} \deg c_i$.

2. Each $r = c_0 + c_1 y + \cdots + c_{n-1} y^{n-1} \in \mathcal{R} \setminus \{0\}$ has a finite y-adic representation of length not exceeding $\{Q + (n-1)B + 1\} n$.

3. For $p(x, y) = y - b_0 \in \mathbb{F}_q[x, y]$, $b_0 \in \mathbb{F}_q[x]$, $\deg b_0 > 0$, each $r = c \in \mathcal{R}\backslash\{0\}$ has a finite y-adic representation of length k, where k denotes the length of c considered as a polynomial in x written with respect to the base b_0. Furthermore, $k = 1 + \left\lfloor \frac{C}{B_0} \right\rfloor$, where $C := \deg c$ and $\lfloor \cdot \rfloor$ denotes the usual integer value function.

4. Let $p(x, y) = y^n + b_{n-1} y^{n-1} + \cdots + b_1 y - b_0 \in \mathbb{F}_q[x, y]$, $B := \max_{i=1,\ldots,n-1} \deg b_i$, $B_0 := \deg b_0$. Assume that $B = B_0 > 0$.

I. If $r = c_0 + c_1 y + \cdots + c_{n-1} y^{n-1} \in \mathcal{R} \setminus \{0\}$, $c_j \in \mathbb{F}_q[x]$, then the sequence $\mathcal{U}_r$ is ultimately periodic with period

$$\mathrm{Per}(r) \leq \text{l.c.m.}\{\mathrm{Per}(c_l) \ ; \ 0 \leq l \leq n-1\},$$

where l.c.m. refers to the least common multiple.

II. If $r = c \in \mathbb{F}_q[x] \setminus \{0\}$ and $C < B_0$ where $C := \deg c$, then the sequence $\mathcal{U}_r$ is finite, which may also be interpreted as ultimately periodic with period $\mathrm{Per}(c) = 1$.

III. If $r = c \in \mathbb{F}_q[x] \setminus \{0\}$ and $C \geq B_0$ where $C := \deg c$, then the sequence $\mathcal{U}_r$ is ultimately periodic with period

$$\text{Per}(c) \leq q^{(C+1)n-2B_0} - 1.$$

5. Let $p(x,y) = y^2 + b_1 y - b_0 \in \mathbb{F}_q[x,y]$, $r := c \in \mathbb{F}_q[x] \setminus \{0\}$, $C := \deg c$, $B_0 := \deg b_0$, $B := \deg b_1$. If $C = B_0 = B$, then the sequence $\mathcal{U}_r$ is ultimately periodic with period $\text{Per}(c) \leq (q-1)^2$.

6. Let $p(x,y) = y^n + b_{n-1}y^{n-1} + \cdots + b_1 y - b_0 \in \mathbb{F}_q[x,y]$, $B_0 := \deg b_0$, $B := \max_{i=1,\ldots,n-1} \deg b_i$, and

$$r := r^{(0)} = r_0^{(0)} + r_1^{(0)} y + \cdots + r_{n-1}^{(0)} y^{n-1} \in \mathcal{R} \setminus \{0\}.$$

Assume that $B_0 = B$. Then the sequence $\mathcal{U}_r$ is purely periodic of period $\pi \geq 1$ if and only if

$$r_{n-l}^{(0)} = \tilde{r}_{\pi-1} b_{n-l+1} + \tilde{r}_{\pi-2} b_{n-l+2} + \cdots + \tilde{r}_0 b_{n-l+\pi} + r_{n-l+\pi}^{(0)} \quad (l = 1, 2, \ldots, n)$$

where $\tilde{r}_i$ are as defined in (2.2.2)-(2.2.5); $b_n := 1$, and b_{j+1}, $r_j^{(0)}$ are taken to be 0 should $j \geq n$.

7. Let the notation be as in 6 and let $C_i := \deg r_i^{(0)}$ $(i = 0, 1, \ldots, n-1)$. The sequence $\mathcal{U}_r$ is ultimately periodic of period $\pi \geq 1$ if and only if there is an index

$$s < \text{l.c.m.} \left\{ \max\left(q^{(C_i+1)n-2B_0} - 1, 1\right);\ i = 0, 1, \ldots, n-1 \right\}$$

such that

$$r_{n-l}^{(s)} = \tilde{r}_{s+\pi-1} b_{n-l+1} + \tilde{r}_{s+\pi-2} b_{n-l+2} + \cdots + \tilde{r}_s b_{n-l+\pi} + r_{n-l+\pi}^{(s)} \quad (l = 1, 2, \ldots, n).$$

8. Let $p(x,y) = y^2 + b_1 y - b_0 \in \mathbb{F}_q[x,y]$, $B := \deg b_1$, $B_0 := \deg b_0$. If $B > B_0 > 0$, then each $r \in \mathcal{R} \setminus \{0\} := \{c_0 + c_1 y\ ;\ c_i \in \mathbb{F}_q[x]\} \setminus \{0\}$ either has a

finite or an infinite but non-periodic Scheicher-Thuswaldner representations.

More precisely, $r \in \mathcal{R} \setminus \{0\}$ has a finite expansion if and only if there exists a non-negative integer k such that

$$\max\left(\deg r_0^{(k)}, \deg r_1^{(k)}\right) < B_0,$$

where $r^{(k)} := r_0^{(k)} + r_1^{(k)} y$; moreover, the sequence $\mathcal{U}_r$ is finite of length $k+1$ if $r_1^{(k)} = 0$ and of length $k+2$ if $r_1^{(k)} \neq 0$.