CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapter.

Throughout this thesis, we let $\mathbb{R}$ stand for the set of all real numbers and $\mathbb{N}$ the set of all natural numbers.

2.1 Basic results.

Definition 2.1.1. Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : E \longrightarrow \mathbb{R}$ is said to be *a norm on X* if it satisfies the following conditions:

(1) $\|x\| \geq 0, \forall x \in E$;

(2) $\|x\| = 0 \Leftrightarrow x = 0$;

(3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in E$;

(4) $\|\alpha x\| = |\alpha|\|x\|, \forall x \in E$ and $\forall \alpha \in \mathbb{K}$.

Definition 2.1.2. Let $(E, \|\cdot\|)$ be a normed space.

(1) A sequence $\{x_n\} \subset E$ is said to *converge strongly* in X if there exists $x \in E$ such that $\lim_{n \longrightarrow \infty} \|x_n - x\| = 0$. That is, if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \epsilon, \forall n \geq N$. We often write $\lim_{n \longrightarrow \infty} x_n = x$ or $x_n \longrightarrow x$ to mean that x is the limit of the sequence $\{x_n\}$.

(2) A sequence $\{x_n\} \subset E$ is said to be *a Cauchy sequence* if for any $\epsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \epsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in B if and only if $\|x_m - x_n\| \longrightarrow 0$ as $m, n \longrightarrow \infty$.

Definition 2.1.3. A normed space X is called *complete* if every Cauchy sequence in X converges to an element in X.

Definition 2.1.4. A complete normed linear space over field $\mathbb{K}$ is called *a Banach space over* $\mathbb{K}$

Definition 2.1.5. Let C be a nonempty subset of normed space X. A mapping $T : C \longrightarrow C$ is said to be *lipschitzian* if there exists a constant $k \geq 0$ such that for all $x, y \in C$

$$\|Tx - Ty\| \leq k\|x - y\|. \tag{2.1.1}$$

The smallest number k for which 2.1.1 holds is called the *Lipschitz constant* of T.

Definition 2.1.6. A lipschitzian mapping $T : C \longrightarrow C$ with Lipschitz constant $k < 1$ is said to be a *contraction mapping.*

Definition 2.1.7. An element $x \in C$ is said to be a *fixed point* of a mapping $T : C \longrightarrow C$ iff $Tx = x$.

Definition 2.1.8. [Banach's contraction mapping principle] Let (M, d) be a complete metric spaces and let $T : M \longrightarrow M$ be a contraction. Then T has a unique fixed point x_0.

Definition 2.1.9. Let F and E be linear spaces over the field $\mathbb{K}$.

(1) A mapping $T : F \longrightarrow E$ is called *a linear operator* if $T(x+y) = Tx+Ty$ and $T(\alpha x) = \alpha Tx, \forall x, y \in F$, and $\forall \alpha \in \mathbb{K}$.

(2) A mapping $T : F \longrightarrow \mathbb{K}$ is called *a linear functional on* F if T a is linear operator.

Definition 2.1.10. A sequence $\{x_n\}$ in a normed spaces is said to *converge weakly* to some vector x if $\lim_{n\longrightarrow\infty} f(x_n) = f(x)$ holds for every continuous linear functional f. We often write $x_n \rightharpoonup x$ to mean that $\{x_n\}$ converge weakly to x.

Definition 2.1.11. Let F and E be normed spaces over the field $\mathbb{K}$ and $T : F \longrightarrow E$ a linear operator. T is said to be *bounded* on F, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in F$.

Definition 2.1.12. Sequence $\{x_n\}_{n=1}^{\infty}$ in a normed linear space X is said to be *a bounded sequence* if there exists $M > 0$; such that $\|x_n\| \leq M, \forall n \in \mathbb{N}$.

Definition 2.1.13. Let F and E be normed spaces over the field $\mathbb{K}$, $T : F \longrightarrow E$ an operator and $c \in F$. We say that T is *continuous at c* if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(c)\| < \epsilon$ whenever $\|x - c\| < \delta$ and $x \in F$. If T is continuous at each $x \in F$, then T is said to be *continuous on F*.

Definition 2.1.14. Let X and Y be normed spaces. The mapping $T : X \longrightarrow Y$ is said to be *completely continuous* if and only if $T(C)$ is a compact subset of Y for every bounded subset C of X.

Definition 2.1.15. A mapping $T : C \longrightarrow C$ is said to be *semicompact* if, for any sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x \in C$.

Definition 2.1.16. A subset C of a normed linear space X is said to be *convex set in X* if $\lambda x + (1 - \lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

Definition 2.1.17. Let X be a real normed space and C a nonempty subset of X. A mapping $T : C \longrightarrow C$ is said to be

(a) *nonexpansive* whenever $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$;

(b) *asymptotically nonexpansive* on C if there exists a sequence $\{k_n\}$ in $[1, \infty)$, with $\lim_{n \longrightarrow \infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{2.1.2}$$

for all $x, y \in C$ and each $n \geq 1$;

(c) *asymptotically nonexpansive in the intermediate sense* [4] provided T is uniformly continuous and

$$\limsup_{n \longrightarrow \infty} \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{2.1.3}$$

From the above definitions, it follows that asymptotically nonexpansive mapping must be asymptotically nonexpansive mapping in the intermediate sense but the converges does not hold as the following example:

Example 2.1.18. [18] *Let* $X = \mathbb{R}$, $C = [\frac{-1}{\pi}, \frac{1}{\pi}]$ *and* $|k| < 1$. *For each* $x \in C$, *define*

$$T(x) = \begin{cases} kx\sin\frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then T *is an asymptotically nonexpansive in the intermediate sense.* It is well known in [17] that $T^n x \longrightarrow 0$ uniformly, but is not a Lipschitzian mapping so that it is not asymptotically nonexpansive mapping.

Definition 2.1.19. [10] A Banach space X is said to be *uniformly convex* if for each $0 < \epsilon \leq 2$, there is $\delta > 0$ such that $\forall x, y \in X$, the condition $\|x\| = \|y\| = 1$, and $\|x - y\| \geq \epsilon$ imply $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Definition 2.1.20. [10] Let X be a Banach space. Then *the modulus of convexity of* X $\delta : [0, 2] \longrightarrow [0, 1]$ defined as follows:

$$\delta(\epsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\}.$$

Theorem 2.1.21. [10] *Let* X *be a Banach space. Then* X *is uniformly convex if and only if* $\delta(\epsilon) > 0$ *for all* $\epsilon > 0$.

Theorem 2.1.22. [2] *Let* C *be a nonempty, closed, convex and bounded subset of uniformly convex Banach space* E *and let* $T : C \longrightarrow C$ *be nonexpansive mapping. Then* T *has a fixed point.*

Theorem 2.1.23. [10] *Let* C *be a nonempty, closed, convex and bounded subset of uniformly convex Banach space* E *and let* $T : C \longrightarrow C$ *be asymptotically nonexpansive mapping. Then* T *has a fixed point.*

Theorem 2.1.24. [19] *Let* C *be a nonempty, closed, convex and bounded subset of uniformly convex Banach space* E *and let* $T : C \longrightarrow C$ *be asymptotically nonexpansive mapping in the intermediate sense. Then* T *has a fixed point.*

Definition 2.1.25. [26] A Banach space X is said to satisfy *Opial's condition* if any sequence $\{x_n\}$ in C, $x_n \longrightarrow x$ weakly as $n \longrightarrow \infty$ implies that $\limsup_{n\longrightarrow\infty} \|x_n - x\| < \limsup_{n\longrightarrow\infty} \|x_n - y\|$ for all $y \in C$ with $y \neq x$.

Definition 2.1.26. [32] The mapping $T : C \longrightarrow C$ with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) if there exists a nondecreasing function $f : [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T)))$$

for all $x \in C$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.

Definition 2.1.27. [15] Two mappings $T_1, T_2 : C \longrightarrow C$ where C a nonempty subset of X, is said to satisfy *condition* (A') if there exists a nondecreasing function $f : [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\frac{1}{2}(\|x - T_1x\| + \|x - T_2x\|) \geq f(d(x, F))$$

for all $x \in C$ where $F := F(T_1) \cap F(T_2)$ and $d(x, F) = \inf\{\|x - x^*\| : x^* \in F\}$.

Remark 2.1.28. Note that condition (A') reduces to condition (A) when $T_1 = T_2$.

We modify this condition for three mappings $T_1, T_2, T_3 : C \longrightarrow C$ as follows:

Definition 2.1.29. Three mappings $T_1, T_2, T_3 : C \longrightarrow C$ where C is a subset of X, is said to satisfy condition (A'') if there exists a nondecreasing function $f : [0,\infty) \longrightarrow [0,\infty)$ with $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\frac{1}{3}(\|x - T_1x\| + \|x - T_2x\| + \|x - T_3x\|) \geq f(d(x, F))$$

for all $x \in C$ where $F := F(T_1) \cap F(T_2) \cap F(T_3)$.

Remark 2.1.30. Note that condition (A'') reduces to condition (A) when $T_1 = T_2 = T_3$.

Remark 2.1.31. [32] It is well known that every continuous and demicompact mapping must satisfy condition (A). Since every completely continuous $T : C \longrightarrow C$ is continuous and demicompact so that it satisfies condition (A).

Lemma 2.1.32. [21] *Let* $\{a_n\}, \{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \forall n = 1, 2, ...$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then*

(i) $\lim_{n \longrightarrow \infty} a_n$ *exists;*

(ii) $\lim_{n \longrightarrow \infty} a_n = 0$, *whenever* $\liminf_{n \longrightarrow \infty} a_n = 0$.

Lemma 2.1.33. [7] *Let* X *be a uniformly convex Banach space,* C *a nonempty closed convex subset of* X, *and* $T : C \longrightarrow C$ *be an asymptotically nonexpansive mapping. Then* $I - T$ *is demiclosed at* 0, *i.e., if* $x_n \longrightarrow x$ *weakly and* $x_n - Tx_n \longrightarrow 0$ *strongly, then* $x \in F(T)$, *where* $F(T)$ *is the set of fixeds point of* T.

Lemma 2.1.34. [33] *Let* X *be a Banach space which satisfies Opial's condition and let* $\{x_n\}$ *be a sequence in* X. *Let* $u, v \in X$ *be such that* $\lim_{n \longrightarrow \infty} \|x_n - u\|$ *and* $\lim_{n \longrightarrow \infty} \|x_n - v\|$ *exist. If* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *are subsequence of* $\{x_n\}$ *which converge weakly to* u *and* v, *respectively, then* $u = v$.

Lemma 2.1.35. [37] *Let* $p > 1$, $r > 0$ *be two fixed numbers. Then a Banach space* X *is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function* $g : [0, \infty) \longrightarrow [0, \infty)$, $g(0) = 0$ *such that*

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda\|x\|^p + (1 - \lambda)\|y\|^p - w_p(\lambda)g(\|x - y\|),$$

for all x, y *in* $B_r = \{x \in X : \|x\| \leq r\}$, $\lambda \in [0, 1]$, *where*

$$w_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda).$$

Lemma 2.1.36. [7] *Let* X *be a uniformly convex Banach space and* $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. *Then there exists a continuous, strictly increasing, and convex function* $g : [0, \infty) \longrightarrow [0, \infty), g(0) = 0$ *such that*

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g(\|x - y\|),$$

for all $x, y, z \in B_r$, *and all* $\lambda, \beta, \gamma \in [0, 1]$ *with* $\lambda + \beta + \gamma = 1$.