CHAPTER III

WEAK AND STRONG CONVERGENCE THEOREMS OF NEW ITERATIONS WITH ERRORS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

In this chapter, the modified Noor iterations with errors and new iterations with errors are defined by using an asymptotically nonexpansive self-mapping. Moreover, It is proved that if the mapping is completely continuous, then the modified Noor iterations with errors converges strongly to a fixed point of the mappings. And if the mapping satisfied condition (A) or completely continuous, then the new iterations with errors converges strongly to a fixed point of the mapping. It also shown that if the our space satisfies Opial' s condition, then we would have wake convergence theorem of our iteration.

3.1 Convergence Criteria of Modified Noor Iterations with Errors for Asymptotically Nonexpansive Mappings

Let X be a normed space, C be a nonempty convex subset of X, and $T : C \longrightarrow C$ be a given mapping. Then for a given $x_1 \in C$, compute the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n, \quad n \geq 1, \end{aligned} \tag{3.1.1}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}, \{\lambda_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}, \{v_n\}$ *and* $\{w_n\}$ are bounded sequences in C.

The iterative schemes (3.1.1) are called the modified Noor iterations with errors. Noor iterations include the Mann-Ishikawa iterations as spacial cases. If $\gamma_n = \mu_n = \lambda_n \equiv 0$, then (3.1.1) reduces to the modified Noor iterations defined by

$\gamma_n = \mu_n = \lambda_n \equiv 0$, then (3.1.1) reduces to the modified Noor iterations defined by Suantai [33]

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n) x_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n) x_n, \quad n \geq 1 \end{aligned} \tag{3.1.2}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0, 1]$.

We note that the usual Ishikawa and Mann iterations are special cases of (3.1.1) and if $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (3.1.1) reduces to the Noor iterations defined by Xu and Noor [40]

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n) x_n \\ y_n &= b_n T^n z_n + (1 - b_n) x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1 \end{aligned} \tag{3.1.3}$$

where $\{a_n\}, \{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

For $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (3.1.1) reduces to the usual Ishikawa iterative scheme

$$\begin{aligned} y_n &= b_n T^n x_n + (1 - b_n) x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1, \end{aligned} \tag{3.1.4}$$

where $\{b_n\}, \{\alpha_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, then (3.1.1) reduces to the usual Mann iterative scheme

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \quad n \geq 1, \tag{3.1.5}$$

where $\{\alpha_n\}$ is appropriate sequences in $[0, 1]$.

In the sequel, the following lemmas are needed to prove our main results.

Lemma 3.1.1. *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \longrightarrow [0, \infty), g(0) = 0$ such that*

$$\|\alpha x + \beta y + \mu z + \lambda w\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x - y\|),$$

for all $x, y, z, w \in B_r$, and all $\alpha, \beta, \mu, \lambda \in [0, 1]$ with $\alpha + \beta + \mu + \lambda = 1$.

Proof. We first observe that $(\mu/(1-\alpha-\beta))z + (\lambda/(1-\alpha-\beta))w \in B_r$ for all $z, w \in B_r$ and $\alpha, \beta, \mu, \lambda \in [0, 1]$ with $\alpha + \beta + \mu + \lambda = 1$. It follows from Lemma (2.1.36) and (2.1.35) that there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \longrightarrow [0, \infty), g(0) = 0$, such that

$$\begin{aligned}
\|\alpha x + \beta y + \mu z + \lambda w\|^2 &= \|\alpha x + \beta y \\
&\quad + (1 - \alpha - \beta)\Big[\frac{\mu}{(1 - \alpha - \beta)} z + \frac{\lambda}{(1 - \alpha - \beta)} w\Big]\|^2 \\
&\leq \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta g(\|x - y\|) \\
&\quad + (1 - \alpha - \beta)\Big\|\frac{\mu}{(1 - \alpha - \beta)} z + \frac{\lambda}{(1 - \alpha - \beta)} w\Big\|^2 \\
&\leq \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta g(\|x - y\|) \\
&\quad + (1 - \alpha - \beta)\Big[\frac{\mu}{(1 - \alpha - \beta)}\|z\|^2 + \frac{\lambda}{(1 - \alpha - \beta)}\|w\|^2\Big] \\
&= \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g(\|x - y\|).
\end{aligned}$$

□

Lemma 3.1.2. *If $\{b_n\}, \{c_n\}$ and $\{\mu_n\}$ are sequences in $[0, 1]$ such that $\limsup_{n\longrightarrow\infty}$ $(b_n + c_n + \mu_n) < 1$ and $\{k_n\}$ is a sequence of real number with $k_n \geq 1$ for all $n \geq 1$ and $\lim_{n\longrightarrow\infty} k_n = 1$, then there exist a positive integer N_1 and $\gamma \in (0, 1)$ such that $c_n k_n < \gamma$ for all $n \geq N_1$.*

Proof. By $\limsup_{n\longrightarrow\infty}(b_n + c_n + \mu_n) < 1$, there exists a positive integer N_0 and $\eta \in (0, 1)$ such that

$$c_n \leq b_n + c_n + \mu_n < \eta \quad \forall n \geq N_0.$$

Let $\eta' \in (0,1)$ with $\eta' > \eta$. From $\lim_{n\longrightarrow\infty} k_n = 1$, there exixts a positive integer $N_1 \geq N_0$ such that

$$k_n - 1 < \frac{1}{\eta'} - 1 \ \ \forall n \geq N_1,$$

from which we have $k_n < \frac{1}{\eta'}$ $\forall n \geq N_1$. Put $\gamma = \frac{\eta}{\eta'}$, then we have $c_n k_n < \gamma$ for all $n \geq N_1$. □

Lemma 3.1.3. *Let X be a uniformly convex Banach space, and let C be a nonempty closed, bounded and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be real sequences in $[0,1]$ such that $a_n + \gamma_n$, $b_n + c_n + \mu_n$ and $\alpha_n + \beta_n + \lambda_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$, and let $\{u_n\}, \{v_n\}$ and $\{w_n\}$ be the bounded sequences in C. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (3.1.1).*

(i) If q is a fixed point of T, then $\lim_{n\longrightarrow\infty} \|x_n - q\|$ exists.

(ii) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n$ and $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty}(b_n + c_n + \mu_n) < 1$, then $\lim_{n\longrightarrow\infty} \|T^n z_n - x_n\| = 0$.

(iii) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then $\lim_{n\longrightarrow\infty} \|T^n y_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty}(b_n + c_n + \mu_n) < 1$ and $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then $\lim_{n\longrightarrow\infty} \|T^n x_n - x_n\| = 0$.

Proof. From [10], T has a fixed point $x^* \in C$. Choose a number $r > 1$ such that $C \subseteq B_r$ and $C - C \subseteq B_r$. By Lemma (2.1.36), there exists a continuous strictly increasing convex function $g_1 : [0,\infty) \longrightarrow [0,\infty), g_1(0) = 0$ such that

$$\|\lambda x + \beta y + \gamma z\|^2 \leq \lambda\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \lambda\beta g_1(\|x - y\|), \tag{3.1.6}$$

for all $x, y, z \in B_r$, and all $\lambda, \beta, \gamma \in [0,1]$ with $\lambda + \beta + \gamma = 1$. It follows from (3.1.6) that

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|a_n(T^n x_n - x^*) + (1 - a_n - \gamma_n)(x_n - x^*) + \gamma_n(u_n - x^*)\|^2 \\
&\leq a_n\|T^n x_n - x^*\|^2 + (1 - a_n - \gamma_n)\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\
&\quad - a_n(1 - a_n - \gamma_n)g_1(\|T^n x_n - x_n\|) \\
&\leq a_n k_n^2\|x_n - x^*\|^2 + (1 - a_n - \gamma_n)\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\
&\quad - a_n(1 - a_n - \gamma_n)g_1(\|T^n x_n - x_n\|) \\
&= \left(a_n k_n^2 + (1 - a_n - \gamma_n)\right)\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2 \\
&\quad - a_n(1 - a_n - \gamma_n)g_1(\|T^n x_n - x_n\|). \qquad (3.1.7)
\end{aligned}$$

By Lemma 3.1.1, there is a continuous, strictly increasing, and convex function $g_2 : [0, \infty) \longrightarrow [0, \infty)$, $g_2(0) = 0$ such that

$$\|\alpha x + \beta y + \mu z + \lambda w\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \mu\|z\|^2 + \lambda\|w\|^2 - \alpha\beta g_2(\|x - y\|) \qquad (3.1.8)$$

and all $\alpha, \beta, \mu, \lambda \in [0, 1]$ with $\alpha + \beta + \mu + \lambda = 1$, for all $x, y, z, w \in B_r$. It follows from (3.1.8) that

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|b_n(T^n z_n - x^*) + (1 - b_n - c_n - \mu_n)(x_n - x^*) \\
&\quad + c_n(T^n x_n - x^*) + \mu_n(v_n - x^*)\|^2 \\
&\leq b_n\|T^n z_n - x^*\|^2 + (1 - b_n - c_n - \mu_n)\|x_n - x^*\|^2 + c_n\|T^n x_n - x^*\|^2 \\
&\quad + \mu_n\|v_n - x^*\|^2 - b_n(1 - b_n - c_n - \mu_n)g_2(\|T^n z_n - x_n\|) \\
&\leq b_n k_n^2\|z_n - x^*\|^2 + (1 - b_n - c_n - \mu_n)\|x_n - x^*\|^2 + c_n k_n^2\|x_n - x^*\|^2 \\
&\quad + \mu_n\|v_n - x^*\|^2 - b_n(1 - b_n - c_n - \mu_n)g_2(\|T^n z_n - x_n\|). \qquad (3.1.9)
\end{aligned}$$

It follows from (3.1.7), (3.1.8) and (3.1.9) that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n(T^n y_n - x^*) + (1 - \alpha_n - \beta_n - \lambda_n)(x_n - x^*) \\
&\quad + \beta_n(T^n z_n - x^*) + \lambda_n(w_n - x^*)\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\|T^n y_n - x^*\|^2 + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - x^*\|^2 \\
&\quad + \beta_n\|T^n z_n - x^*\|^2 + \lambda_n\|w_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2(\|T^n y_n - x_n\|) \\
&\leq \alpha_n k_n^2\|y_n - x^*\|^2 + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - x^*\|^2 \\
&\quad + \beta_n k_n^2\|z_n - x^*\|^2 + \lambda_n\|w_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2(\|T^n y_n - x_n\|) \\
&\leq \alpha_n k_n^2\big(b_n k_n^2\|z_n - x^*\|^2 + c_n k_n^2\|x_n - x^*\|^2 \\
&\quad + (1-b_n-c_n-\mu_n)\|x_n - x^*\|^2 + \mu_n\|v_n - x^*\|^2 + \beta_n k_n^2\|z_n - x^*\|^2 \\
&\quad - b_n(1-b_n-c_n-\mu_n)g_2(\|T^n z_n - x_n\|)\big) \\
&\quad + (1-\alpha_n-\beta_n-\lambda_n)\|x_n - x^*\|^2 + \lambda_n\|w_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2(\|T^n y_n - x_n\|) \\
&= \|x_n - x^*\|^2 + \big(\alpha_n c_n k_n^4 + \alpha_n k_n^2(1-b_n-c_n-\mu_n) - \alpha_n - \beta_n - \lambda_n\big)\|x_n - x^*\|^2 \\
&\quad + \alpha_n\mu_n k_n^2\|v_n - x^*\|^2 + (\alpha_n b_n k_n^4 + \beta_n k_n^2)\|z_n - x^*\|^2 \\
&\quad - \alpha_n b_n k_n^2(1-b_n-c_n-\mu_n)g_2(\|T^n z_n - x_n\|) + \lambda_n\|w_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2(\|T^n y_n - x_n\|) \\
&\leq \|x_n - x^*\|^2 + \big(\alpha_n c_n k_n^4 + \alpha_n k_n^2(1-b_n-c_n-\mu_n) - \alpha_n - \beta_n - \lambda_n\big)\|x_n - x^*\|^2 \\
&\quad + \alpha_n\mu_n k_n^2\|v_n - x^*\|^2 \\
&\quad + (\alpha_n b_n k_n^4 + \beta_n k_n^2)\big((a_n k_n^2 + (1-a_n-\gamma_n))\|x_n - x^*\|^2 + \gamma_n\|u_n - x^*\|^2\big) \\
&\quad - \alpha_n b_n k_n^2(1-b_n-c_n-\mu_n)g_2(\|T^n z_n - x_n\|) + \lambda_n\|w_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2(\|T^n y_n - x_n\|) \\
&= \|x_n - x^*\|^2 + \big(\alpha_n c_n k_n^4 + \alpha_n k_n^2(1-b_n-c_n-\mu_n) - \alpha_n - \beta_n - \lambda_n\big)\|x_n - x^*\|^2 \\
&\quad + \alpha_n\mu_n k_n^2\|v_n - x^*\|^2 \\
&\quad + (\alpha_n b_n k_n^4 + \beta_n k_n^2)(a_n k_n^2 + (1-a_n-\gamma_n))\|x_n - x^*\|^2 \\
&\quad + (\alpha_n b_n k_n^4 + \beta_n k_n^2)\gamma_n\|u_n - x^*\|^2 \\
&\quad - \alpha_n b_n k_n^2(1-b_n-c_n-\mu_n)g_2(\|T^n z_n - x_n\|) + \lambda_n\|w_n - x^*\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2(\|T^n y_n - x_n\|)
\end{aligned}
$$

$$
\begin{aligned}
&= \|x_n - x^*\|^2 + \big(\alpha_n c_n k_n^4 + \alpha_n k_n^2(1 - b_n - c_n - \mu_n) - \alpha_n - \beta_n - \lambda_n\big)\|x_n - x^*\|^2 \\
&\quad + \alpha_n \mu_n k_n^2 \|v_n - x^*\|^2 \\
&\quad + (\alpha_n b_n k_n^4 + \beta_n k_n^2 + a_n \alpha_n b_n k_n^6 + a_n \beta_n k_n^4 \\
&\quad - \alpha_n \gamma_n b_n k_n^4 - \gamma_n \beta_n k_n^2 - a_n \alpha_n b_n k_n^4 - a_n \beta_n k_n^2)\|x_n - x^*\|^2 \\
&\quad + (\alpha_n b_n k_n^4 + \beta_n k_n^2)\gamma_n \|u_n - x^*\|^2 \\
&\quad - \alpha_n b_n k_n^2 (1 - b_n - c_n - \mu_n) g_2(\|T^n z_n - x_n\|) + \lambda_n \|w_n - x^*\|^2 \\
&\quad - \alpha_n (1 - \alpha_n - \beta_n - \lambda_n) g_2(\|T^n y_n - x_n\|) \\
&\leq \|x_n - x^*\|^2 + \big(\alpha_n c_n k_n^2 (k_n^2 - 1) + \alpha_n (k_n^2 - 1) + \alpha_n b_n k_n^2 (k_n^2 - 1) + \beta_n (k_n^2 - 1) \\
&\quad + a_n \alpha_n b_n k_n^4 (k_n^2 - 1) + a_n \beta_n k_n^2 (k_n^2 - 1)\big)\|x_n - x^*\|^2 \\
&\quad + \alpha_n \mu_n k_n^2 \|v_n - x^*\|^2 + (\alpha_n b_n k_n^4 + \beta_n k_n^2)\gamma_n \|u_n - x^*\|^2 \\
&\quad - \alpha_n b_n k_n^2 (1 - b_n - c_n - \mu_n) g_2(\|T^n z_n - x_n\|) + \lambda_n \|w_n - x^*\|^2 \\
&\quad - \alpha_n (1 - \alpha_n - \beta_n - \lambda_n) g_2(\|T^n y_n - x_n\|) \\
&\leq \|x_n - x^*\|^2 + \big(\alpha_n c_n k_n^2 + \alpha_n + \alpha_n b_n k_n^2 + \beta_n \\
&\quad + a_n \alpha_n b_n k_n^4 + a_n \beta_n k_n^2\big)(k_n^2 - 1)\|x_n - x^*\|^2 + (k_n^4 + k_n^2)\gamma_n \|u_n - x^*\|^2 \\
&\quad + \mu_n k_n^2 \|v_n - x^*\|^2 \\
&\quad - \alpha_n b_n k_n^2 (1 - b_n - c_n - \mu_n) g_2(\|T^n z_n - x_n\|) + \lambda_n \|w_n - x^*\|^2 \\
&\quad - \alpha_n (1 - \alpha_n - \beta_n - \lambda_n) g_2(\|T^n y_n - x_n\|).
\end{aligned}
$$

Since $\{k_n\}$ and C are bounded, there exists a constant $M > 0$ such that

$$
\big(\alpha_n c_n k_n^2 + \alpha_n + \alpha_n b_n k_n^2 + \beta_n + a_n \alpha_n b_n k_n^4 + a_n \beta_n k_n^2\big)\|x_n - x^*\|^2 \leq M
$$

for all $n \geq 1$. It follows that

$$
\begin{aligned}
\alpha_n b_n k_n^2 (1 - b_n - c_n - \mu_n) g_2(\|T^n z_n - x_n\|) &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + M(k_n^2 - 1) + L\gamma_n + A\mu_n + r^2 \lambda_n
\end{aligned}
$$

and

$$\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2(\|T^n y_n - x_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ +M(k_n^2-1) + L\gamma_n + A\mu_n + r^2\lambda_n,$$

where $L = sup\{(k_n^4 + k_n^2)\|u_n - x^*\|^2 : n \geq 1\}$ *and* $A = sup\{k_n^2\|v_n - x^*\|^2 : n \geq 1\}$.

Now, if we let $K = max\{M, L, A, r^2\}$ then we get that

$$\alpha_n b_n k_n^2(1-b_n-c_n-\mu_n)g_2(\|T^n z_n - x_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ +K\big((k_n^2-1) + \gamma_n + \mu_n + \lambda_n\big) \quad (3.1.10)$$

and

$$\alpha_n(1-\alpha_n-\beta_n-\lambda_n)g_2(\|T^n y_n - x_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ +K\big((k_n^2-1) + \gamma_n + \mu_n + \lambda_n\big). \quad (3.1.11)$$

(i) If $q \in F(T)$, by taking $x^* = q$ in the inequality (3.1.10) we have $\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 + K\big((k_n^2-1) + \gamma_n + \mu_n + \lambda_n\big)$. Since $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$, it follows from Lemma 2.1.32 that $\lim_{n\longrightarrow\infty}\|x_n - q\|$ exists.

(ii) If $0 < \liminf_{n\longrightarrow\infty}\alpha_n$ and $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty}(b_n + c_n + \mu_n) < 1$, then there exists a positive integer n_0 and $\nu, \eta, \eta' \in (0,1)$ such that

$$0 < \nu < \alpha_n \text{ and } 0 < \eta < b_n \text{ and } b_n + c_n + \mu_n < \eta' < 1 \text{ for all } n \geq n_0.$$

This implies by (3.1.10) that

$$\nu\eta(1-\eta')g_2(\|T^n z_n - x_n\|) \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ +K\big((k_n^2-1) + \gamma_n + \mu_n + \lambda_n\big), \quad (3.1.12)$$

for all $n \geq n_0$. It follows from (3.2.12) that for $m \geq n_0$

$$\begin{aligned}\sum_{n=n_0}^{m} g_2(\|T^n z_n - x_n\|) &\leq \frac{1}{\nu\eta(1-\eta')}\Bigg(\sum_{n=n_0}^{m}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) \\ &\quad + K\sum_{n=n_0}^{m}((k_n^2 - 1) + \gamma_n + \mu_n + \lambda_n)\Bigg) \\ &\leq \frac{1}{\nu\eta(1-\eta')}\Bigg(\|x_{n_0} - x^*\|^2 \\ &\quad + K\sum_{n=n_0}^{m}((k_n^2 - 1) + \gamma_n + \mu_n + \lambda_n)\Bigg). \qquad (3.1.13)\end{aligned}$$

Since $0 \leq t^2 - 1 \leq 2t(t-1)$ for all $t \geq 1$, the assumption $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $m \longrightarrow \infty$ in inequality (3.2.13) we get $\sum_{n=n_0}^{\infty} g_2(\|T^n z_n - x_n\|) < \infty$, and therefore $\lim_{n\longrightarrow\infty} g_2(\|T^n z_n - x_n\|) = 0$. Since g_2 is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\longrightarrow\infty}\|T^n z_n - x_n\| = 0$.

(iii) If $0 < \liminf_{n\longrightarrow\infty}\alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then by using a similar method, together with inequality (3.2.11), it can be shown that $\lim_{n\longrightarrow\infty}\|T^n y_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty}(b_n + c_n + \mu_n) < 1$ and $0 < \liminf_{n\longrightarrow\infty}\alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, by (ii) and (iii) we have

$$\lim_{n\longrightarrow\infty}\|T^n y_n - x_n\| = 0 \text{ and } \lim_{n\longrightarrow\infty}\|T^n z_n - x_n\| = 0. \qquad (3.1.14)$$

From $y_n = b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n$, we have

$$\|y_n - x_n\| \leq b_n\|T^n z_n - x_n\| + c_n\|T^n x_n - x_n\| + \mu_n\|v_n - x_n\|.$$

Thus

$$\begin{aligned}
\|T^n x_n - x_n\| &\leq \|T^n x_n - T^n y_n\| + \|T^n y_n - x_n\| \\
&\leq k_n\|x_n - y_n\| + \|T^n y_n - x_n\| \\
&\leq k_n(b_n\|T^n z_n - x_n\| + c_n\|T^n x_n - x_n\| + \mu_n\|v_n - x_n\|) \\
&\quad + \|T^n y_n - x_n\| \\
&= k_n b_n\|T^n z_n - x_n\| + c_n k_n\|T^n x_n - x_n\| \\
&\quad + \mu_n k_n\|v_n - x_n\| + \|T^n y_n - x_n\|. \qquad (3.1.15)
\end{aligned}$$

By Lemma 3.1.2, there exists positive integer n_1 and $\gamma \in (0,1)$ such that $c_n k_n < \gamma$ for all $n \geq n_1$. This together with (3.1.15) implies that for $n \geq n_1$

$$\begin{aligned}
(1-\gamma)\|T^n x_n - x_n\| &< (1 - c_n k_n)\|T^n x_n - x_n\| \\
&\leq k_n b_n\|T^n z_n - x_n\| + \mu_n k_n\|v_n - x_n\| + \|T^n y_n - x_n\|.
\end{aligned}$$

It follows from (3.2.14) that $\lim_{n\longrightarrow\infty}\|T^n x_n - x_n\| = 0$. □

Theorem 3.1.4. *Let X be a uniformly convex Banach space, and C a nonempty closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$ and $\{\lambda_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n + \mu_n \in [0,1]$ and $\alpha_n + \beta_n + \lambda_n \in [0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\lambda_n < \infty$ and*

(i) $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty}(b_n + c_n + \mu_n) < 1$, *and*

(ii) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n + \lambda_n) < 1$.

Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined by the modified Noor iterations with errors (3.1.1). Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

Proof. By Lemma 3.1.3 , we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|T^n y_n - x_n\| &= 0,\\ \lim_{n\longrightarrow\infty}\|T^n z_n - x_n\| &= 0, \qquad (3.1.16)\\ \lim_{n\longrightarrow\infty}\|T^n x_n - x_n\| &= 0.\end{aligned}$$

Since $x_{n+1} - x_n = \alpha_n(T^n y_n - x_n) + \beta_n(T^n z_n - x_n) + \lambda_n(w_n - x_n)$, we have

$$\begin{aligned}\|x_{n+1} - T^n x_{n+1}\| &\leq \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\|\\ &\leq \|x_{n+1} - x_n\| + k_n\|x_{n+1} - x_n\| + \|T^n x_n - x_n\|\\ &= (1 + k_n)\|x_{n+1} - x_n\| + \|T^n x_n - x_n\|\\ &\leq (1 + k_n)\alpha_n\|T^n y_n - x_n\| + (1 + k_n)\beta_n\|T^n z_n - x_n\|\\ &\quad + (1 + k_n)\lambda_n\|w_n - x_n\| + \|T^n x_n - x_n\|,\end{aligned}$$

This together with (3.1.16) implies that

$$\|x_{n+1} - T^n x_{n+1}\| \longrightarrow 0 \text{ (as } n \longrightarrow \infty).$$

Thus

$$\begin{aligned}\|x_{n+1} - T x_{n+1}\| &\leq \|x_{n+1} - T^{n+1} x_{n+1}\| + \|T x_{n+1} - T^{n+1} x_{n+1}\|\\ &\leq \|x_{n+1} - T^{n+1} x_{n+1}\| + k_1\|x_{n+1} - T^n x_{n+1}\| \longrightarrow 0,\end{aligned}$$

which implies

$$\lim_{n\longrightarrow\infty}\|T x_n - x_n\| = 0. \qquad (3.1.17)$$

Since T is completely continuous and $\{x_n\} \subseteq C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{T x_{n_k}\}$ converges. Therefore from (3.1.17), $\{x_{n_k}\}$ converges. Let $\lim_{n\longrightarrow\infty} x_{n_k} = q$. By continuity of T and (3.1.17) we have that $Tq = q$, so q is a fixed point of T. By Lemma 3.1.3 (i), $\lim_{n\longrightarrow\infty}\|x_n - q\|$ exists. But $\lim_{k\longrightarrow\infty}\|x_{n_k} - q\| = 0$. Thus $\lim_{n\longrightarrow\infty}\|x_n - q\| = 0$.

Since

$$\|y_n - x_n\| \leq b_n\|T^n z_n - x_n\| + c_n\|T^n x_n - x_n\| + \mu_n\|v_n - x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

and

$$\|z_n - x_n\| \leq a_n\|T^n x_n - x_n\| + \gamma_n\|u_n - x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

it follows that $\lim_{n\longrightarrow\infty} y_n = q$ and $\lim_{n\longrightarrow\infty} z_n = q$. □

For $\gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.1.4 , we obtain the following result.

Theorem 3.1.5. *[33] Let X be a uniformly convex Banach space, and C a nonempty closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences of real numbers in $[0,1]$ with $b_n + c_n \in [0,1]$ and $\alpha_n + \beta_n \in [0,1]$ for all $n \geq 1$, and*

(i) $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty}(b_n + c_n) < 1$, *and*

(ii) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$.

Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined by the modified Noor iterations (3.1.2). Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

For $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.1.4 , we obtain the following result.

Theorem 3.1.6. *[40] Let X be a uniformly convex Banach space, and let C be a closed, bounded and convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be real sequences in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty} b_n < 1$, *and*

(ii) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty} \alpha_n < 1$,

For a given $x_1 \in C$, *define*

$$\begin{aligned} z_n &= a_n T^n x_n + (1-a_n)x_n \\ y_n &= b_n T^n z_n + (1-b_n)x_n, \quad n \geq 1 \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n. \end{aligned}$$

Then $\{x_n\}, \{y_n\}$, $\{z_n\}$ *converges strongly to a fixed point of* T.

When $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.1.4 , we can obtain Ishikawa-type convergence result which is a generalization of Theorem 3 in [28].

Theorem 3.1.7. *Let* X *be a uniformly convex Banach space, and let* C *be a closed, bounded and convex subset of* X. *Let* T *be a completely continuous asymptotically nonexpansive self-map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{b_n\}, \{\alpha_n\}$ *be a real sequence in* $[0,1]$ *satisfying*

(i) $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty} b_n < 1$, *and*

(ii) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty} \alpha_n < 1$.

For a given $x_1 \in C$, *define*

$$\begin{aligned} y_n &= b_n T^n z_n + (1-b_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to a fixed point of* T.

Theorem 3.1.8. *Let* X *be a uniformly convex Banach space which satisfies Opial's condition, and* C *a nonempty closed, bounded and convex subset of* X. *Let* T *be an asymptotically nonexpansive self-map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\lambda_n\}$ *be sequences of real numbers in* $[0,1]$ *with* $a_n + \gamma_n, b_n + c_n + \mu_n$ *and* $\alpha_n + \beta_n + \lambda_n$ *are in* $[0,1]$ *for all* $n \geq 1$, *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and*

(i) $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty}(b_n + c_n + \mu_n) < 1$, *and* *(ii)* $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n + \lambda_n) < 1$.

Let $\{x_n\}$ *be the sequence defined by modified Noor iterations with errors (3.1.1). Then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Proof. By the same proof as in Theorem 3.1.4, it can be show that $\lim_{n\longrightarrow\infty} \|Tx_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \longrightarrow u$ weakly as $n \longrightarrow \infty$, without loss of generality. By Lemma 2.1.33, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 2.1.33, $u, v \in F(T)$. By Lemma 3.1.3 (i), $\lim_{n\longrightarrow\infty} \|x_n - u\|$ and $\lim_{n\longrightarrow\infty} \|x_n - v\|$ exist. It follows from Lemma 2.1.34 that $u = v$. Therefore $\{x_n\}$ converges weakly to fixed point of T. □

For $\gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.1.8, we obtain the following result.

Corollary 3.1.9. *[33] Let* X *be a uniformly convex Banach space which satisfies Opial's condition, and* C *a nonempty closed, bounded and convex subset of* X. *Let* T *be an asymptotically nonexpansive self-map of* C *with* $\{k_n\}$ *satisfying* $k_n \geq 1$ *and* $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. *Let* $\{a_n\}, \{b_n\}, \{c_n\}, \{\alpha_n\}, \{\beta_n\}$ *be sequences of real numbers in* $[0, 1]$ *with* $b_n + c_n \in [0, 1]$ *and* $\alpha_n + \beta_n \in [0, 1]$ *for all* $n \geq 1$ *, and*

(i) $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty}(b_n + c_n) < 1$ *, and*

(ii) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$.

Let $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences defined by the the modified Noor iterations (3.1.2). Then* $\{x_n\}$ *converges weakly to a fixed point of* T.

For $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.1.8, we obtain the following result.

Corollary 3.1.10. *Let* X *be a uniformly convex Banach space which satisfies Opial's condition, and let* C *be a nonempty closed, bounded and convex subset* *of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}, \{b_n\}, \{\alpha_n\}$ be real sequences in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty} b_n < 1$, *and*

(ii) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty} \alpha_n < 1$,

For a given $x_1 \in C$, define

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n)x_n \\ y_n &= b_n T^n z_n + (1 - b_n)x_n, \quad n \geq 1 \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n. \end{aligned}$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

When $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 3.1.8, we can obtain Ishikawa-type convergence result which is a generalization of Theorem 3 in [28].

Corollary 3.1.11. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and let C be a nonempty closed, bounded and convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{b_n\}, \{\alpha_n\}$ be real sequences in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\longrightarrow\infty} b_n \leq \limsup_{n\longrightarrow\infty} b_n < 1$,*and*

(ii) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty} \alpha_n < 1$.

For a given $x_1 \in C$, define

$$\begin{aligned} y_n &= b_n T^n z_n + (1 - b_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n, \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

3.2 Weak and Strong Convergence Theorems of New Iterations with Errors for Asymptotically Nonexpansive Mappings

Let X be a normed space, C be a nonempty convex subset of X, and $T : C \longrightarrow C$ be a given mapping. Then for a given $x_1 \in C$, compute the sequence $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= (1 - a_n - b_n)x_n + a_n T^n x_n + b_n u_n, \\ y_n &= (1 - c_n - d_n)z_n + c_n T^n x_n + d_n v_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_n T^n x_n + \beta_n w_n, \quad n \geq 1, \end{aligned} \tag{3.2.1}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

If $a_n = b_n = c_n = d_n = \beta_n \equiv 0$, then (3.2.1) reduces to the usual Mann iterative scheme

$$x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n)x_n, \quad n \geq 1, \tag{3.2.2}$$

where $\{\alpha_n\}$ is appropriate sequences in $[0, 1]$.

Let $\{x_n\}$ be given sequence in C. Recall that a mapping $T : C \longrightarrow C$ with the nonempty fixed point set $F(T)$ in C is said to satisfy *Condition (A)* with respect to the sequence $\{x_n\}$ [7] if there is a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x_n - Tx_n\| \geq f(d(x_n, F(T))),$$

for all $n \geq 1$.

Lemma 3.2.1. *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let*

$\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (3.2.1).

(i) If p is a fixed point of T, then $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists.

(ii) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n) < 1$, then $\lim_{n\longrightarrow\infty} \|T^n x_n - y_n\| = 0$.

(iii) If $0 < \limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n) < 1$ and $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty} (c_n + d_n) < 1$, then $\lim_{n\longrightarrow\infty} \|T^n x_n - z_n\| = 0$.

(iv) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n) < 1$, and $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ and $\limsup_{n\longrightarrow\infty} a_n < 1$ then $\lim_{n\longrightarrow\infty} \|T^n x_n - x_n\| = 0$.

Proof. (i) Let $p \in F(T)$, and

$$M_1 = \sup\{\|u_n - p\| : n \geq 1\}, M_2 = \sup\{\|v_n - p\| : n \geq 1\},$$

$$M_3 = \sup\{\|w_n - p\| : n \geq 1\}, M = \max\{M_i : i = 1, 2, 3\}.$$

Using (3.2.1), we have

$$\begin{aligned}
\|z_n - p\| &= \|(1 - a_n - b_n)(x_n - p) + a_n(T^n x_n - p) + b_n(u_n - p)\| \\
&\leq (1 - a_n - b_n)\|x_n - p\| + a_n k_n \|x_n - p\| + b_n \|u_n - p\| \\
&\leq (1 + a_n(k_n - 1))\|x_n - p\| + M b_n \\
&\leq k_n \|x_n - p\| + M b_n, \\
\|y_n - p\| &= \|(1 - c_n - d_n)(z_n - p) + c_n(T^n x_n - p) + d_n(v_n - p)\| \\
&\leq (1 - c_n - d_n)\|z_n - p\| + c_n k_n \|x_n - p\| + d_n \|v_n - p\| \\
&\leq (1 - c_n - d_n)[k_n \|x_n - p\| + M b_n] + c_n k_n \|x_n - p\| + M d_n \\
&\leq k_n \|x_n - p\| + M b_n + M d_n,
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\alpha_n-\beta_n)(y_n-p)+\alpha_n(T^nx_n-p)+\beta_n(w_n-p)\| \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\|+\alpha_nk_n\|x_n-p\|+\beta_n\|w_n-p\| \\
&\leq (1-\alpha_n-\beta_n)[k_n\|x_n-p\|+Mb_n+Md_n]+\alpha_nk_n\|x_n-p\|+M\beta_n \\
&\leq k_n\|x_n-p\|+M(b_n+d_n+\beta_n) \\
&= \big(1+(k_n-1)\big)\|x_n-p\|+M(b_n+d_n+\beta_n),
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n-1)<\infty$, the assertion (i) follows from Lemma 2.1.32.

(ii) By (i), we know that $\lim_{n\longrightarrow\infty}\|x_n-p\|$ exists for any $p\in F(T)$. It follow that $\{x_n-p\},\{T^nx_n-p\}$ and $\{y_n-p\}$ are all bounded. Also, $\{u_n-p\},\{v_n-p\}$ and $\{w_n-p\}$ are bounded by the assumption. Now we set

$$\begin{aligned}
r_1 &= \sup\{\|x_n-p\|:n\geq 1\}, \\
r_2 &= \sup\{\|T^nx_n-p\|:n\geq 1\}, \\
r_3 &= \sup\{\|y_n-p\|:n\geq 1\}, \\
r_4 &= \sup\{\|z_n-p\|:n\geq 1\}, \\
r_5 &= \sup\{\|u_n-p\|:n\geq 1\}, \\
r_6 &= \sup\{\|v_n-p\|:n\geq 1\}, \\
r_7 &= \sup\{\|w_n-p\|:n\geq 1\}, \\
r &= \max\{r_i:i=1,2,3,4,5,6,7\}. \qquad (3.2.3)
\end{aligned}$$

By using Lemma 2.1.36 we have

$$
\begin{aligned}
\|z_n - p\|^2 &= \|(1-a_n-b_n)(x_n-p) + a_n(T^n x_n - p) + b_n(u_n - p)\|^2 \\
&\leq (1-a_n-b_n)\|x_n-p\|^2 + a_n\|T^n x_n - p\|^2 + b_n\|u_n-p\|^2 \\
&\quad - a_n(1-a_n-b_n)g(\|T^n x_n - x_n\|) \\
&\leq (1-a_n-b_n)\|x_n-p\|^2 + a_n k_n^2\|x_n-p\|^2 + b_n\|u_n-p\|^2 \\
&\leq (1-a_n+a_n {k_n}^2)\|x_n-p\|^2 + r^2 b_n \\
&\leq (1+a_n(k_n^2-1))\|x_n-p\|^2 + r^2 b_n \\
&\leq (1+(k_n^2-1))\|x_n-p\|^2 + r^2 b_n \\
&\leq k_n^2\|x_n-p\|^2 + r^2 b_n, \\
\|y_n - p\|^2 &= \|(1-c_n-d_n)(z_n-p) + c_n(T^n x_n - p) + d_n(v_n - p)\|^2 \\
&\leq (1-c_n-d_n)\|z_n-p\|^2 + c_n\|T^n x_n - p\|^2 + d_n\|v_n-p\|^2 \\
&\quad - c_n(1-c_n-d_n)g(\|T^n x_n - z_n\|) \\
&\leq (1-c_n-d_n)\|z_n-p\|^2 + c_n k_n^2\|x_n-p\|^2 + d_n\|v_n-p\|^2 \\
&\quad - c_n(1-c_n-d_n)g(\|T^n x_n - z_n\|) \\
&\leq (1-c_n-d_n)(k_n^2\|x_n-p\|^2 + r^2 b_n) + c_n k_n^2\|x_n-p\|^2 + r^2 d_n \\
&\quad - c_n(1-c_n-d_n)g(\|T^n x_n - z_n\|) \\
&\leq \left((1-c_n-d_n)k_n^2 + c_n k_n^2\right)\|x_n-p\|^2 + r^2 b_n + r^2 d_n \\
&\quad - c_n(1-c_n-d_n)g(\|T^n x_n - z_n\|) \\
&= (1-d_n)k_n^2\|x_n-p\|^2 + r^2 b_n + r^2 d_n \\
&\quad - c_n(1-c_n-d_n)g(\|T^n x_n - z_n\|) \\
&\leq k_n^2\|x_n-p\|^2 + r^2 b_n + r^2 d_n,
\end{aligned}
$$

and so

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\alpha_n-\beta_n)(y_n-p)+\alpha_n(T^nx_n-p)+\beta_n(w_n-p)\|^2\\
&\le (1-\alpha_n-d_n)\|y_n-p\|^2+\alpha_n\|T^nx_n-p\|^2+\beta_n\|w_n-p\|^2\\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T^nx_n-y_n\|)\\
&\le (1-\alpha_n-\beta_n)\|y_n-p\|^2+\alpha_nk_n^2\|x_n-p\|^2+\beta_n\|w_n-p\|^2\\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T^nx_n-y_n\|)\\
&\le (1-\alpha_n-\beta_n)(k_n^2\|x_n-p\|^2+r^2b_n+r^2d_n)+\alpha_nk_n^2\|x_n-p\|^2\\
&\quad +r^2\beta_n-\alpha_n(1-\alpha_n-\beta_n)g(\|T^nx_n-y_n\|)\\
&\le \big((1-\alpha_n-\beta_n)k_n^2+\alpha_nk_n^2\big)\|x_n-p\|^2+r^2b_n+r^2d_n+r^2\beta_n\\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T^nx_n-y_n\|)\\
&= (1-\beta_n)k_n^2\|x_n-p\|^2+r^2b_n+r^2d_n+r^2\beta_n\\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T^nx_n-y_n\|)\\
&\le k_n^2\|x_n-p\|^2+r^2b_n+r^2d_n+r^2\beta_n\\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T^nx_n-y_n\|)\\
&\le k_n^2\|x_n-p\|^2+r^2(b_n+d_n+\beta_n)-\alpha_n(1-\alpha_n-\beta_n)g(\|T^nx_n-y_n\|)\\
&= \|x_n-p\|^2+(k_n^2-1)\|x_n-p\|^2+r^2(b_n+d_n+\beta_n)\\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T^nx_n-y_n\|)\\
&\le \|x_n-p\|^2+r^2(b_n+d_n+\beta_n)+r^2(k_n^2-1)\\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T^nx_n-y_n\|),
\end{aligned}$$

which leads to the following:

$$\begin{aligned}
\alpha_n(1-\alpha_n-\beta_n)g(\|T^nx_n-y_n\|) \le\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&+r^2(k_n^2-1)+r^2(b_n+d_n+\beta_n), \qquad (3.2.4)
\end{aligned}$$

and

$$\begin{aligned}
(1-\alpha_n-\beta_n)c_n(1-c_n-d_n)g(\|T^nx_n-z_n\|) \le\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&+r^2(k_n^2-1)+r^2(b_n+d_n+\beta_n), (3.2.5)
\end{aligned}$$

If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \le \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, then there exist a positive integer n_0 and $\eta, \eta' \in (0,1)$ such that

$$0 < \eta < \alpha_n \ \text{ and } \ \alpha_n + \beta_n < \eta' < 1 \ \text{ for all } \ n \ge n_0.$$

This implies by (3.2.4) that

$$\begin{aligned}\eta(1-\eta')g(\|T^n x_n - y_n\|) \le\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ & + r^2(k_n^2 - 1) + r^2(b_n + d_n + \beta_n),\end{aligned} \tag{3.2.6}$$

for all $n \ge n_0$. It follows from (3.2.6) that for $m \ge n_0$

$$\begin{aligned}\sum_{n=n_0}^{m} g_2(\|T^n x_n - y_n\|) \ \le\ & \frac{1}{\eta(1-\eta')}\Bigg(\sum_{n=n_0}^{m}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\ & + r^2\sum_{n=n_0}^{m}(b_n + d_n + \beta_n + (k_n^2 - 1))\Bigg) \\ \le\ & \frac{1}{\eta(1-\eta')}\Bigg(\|x_{n_0} - p\|^2 \\ & + r^2\sum_{n=n_0}^{m}(b_n + d_n + \beta_n + (k_n^2 - 1))\Bigg).\end{aligned} \tag{3.2.7}$$

Since $0 \le t^2 - 1 \le 2t(t-1)$ for all $t \ge 1$, the assumption $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $m \longrightarrow \infty$ in inequality (3.2.7) we get that $\sum_{n=n_0}^{\infty} g_2(\|T^n x_n - y_n\|) < \infty$, and therefore $\lim_{n\longrightarrow\infty} g_2(\|T^n x_n - y_n\|) = 0$. Since g_2 is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\longrightarrow\infty}\|T^n x_n - y_n\| = 0$.

(iii) If $0 < \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$ and $0 < \liminf_{n\longrightarrow\infty} c_n \le \limsup_{n\longrightarrow\infty}$ $(c_n + d_n) < 1$, then by the using a similar method, together with inequality (3.2.5), it can be show that $\lim_{n\longrightarrow\infty}\|T^n x_n - z_n\| = 0$.

(iv) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \le \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, and $0 < \liminf_{n\longrightarrow\infty}$ $c_n \le \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ and $\limsup_{n\longrightarrow\infty} a_n < 1$, by (ii) and (iii) we have

$$\lim_{n\longrightarrow\infty}\|T^n x_n - y_n\| = 0 \ \text{ and } \ \lim_{n\longrightarrow\infty}\|T^n x_n - z_n\| = 0. \tag{3.2.8}$$

From $y_n = (1-c_n-d_n)z_n + c_nT^nx_n + d_nv_n$, we have

$$\begin{aligned}
\|y_n - x_n\| &= \|(1-c_n-d_n)z_n + c_nT^nx_n + d_nv_n - x_n\| \\
&= \|(z_n - x_n) + c_n(T^nx_n - z_n) + d_n(v_n - z_n)\| \\
&\leq \|z_n - x_n\| + c_n\|T^nx_n - z_n\| + d_n\|v_n - z_n\| \\
&= \|(1-a_n-b_n)x_n + a_nT^nx_n + b_nu_n - x_n\| + c_n\|T^nx_n - z_n\| \\
&\quad + d_n\|v_n - z_n\| \\
&= \|a_n(T^nx_n - x_n) + b_n(u_n - x_n)\| + c_n\|T^nx_n - z_n\| + d_n\|v_n - z_n\| \\
&\leq a_n\|T^nx_n - x_n\| + b_n\|u_n - x_n\| + c_n\|T^nx_n - z_n\| + d_n\|v_n - z_n\| \\
&\leq a_n\|T^nx_n - x_n\| + c_n\|T^nx_n - z_n\| + 2rb_n + 2rd_n, \qquad (3.2.9)
\end{aligned}$$

where r is defined by (3.2.3). Thus

$$\begin{aligned}
\|T^nx_n - x_n\| &\leq \|T^nx_n - y_n\| + \|y_n - x_n\| \\
&\leq \|T^nx_n - y_n\| + a_n\|T^nx_n - x_n\| + c_n\|T^nx_n - z_n\| \\
&\quad + 2rb_n + 2rd_n, \qquad (3.2.10)
\end{aligned}$$

and so

$$(1-a_n)\|T^nx_n - x_n\| \leq \|T^nx_n - y_n\| + c_n\|T^nx_n - z_n\| + 2rb_n + 2rd_n. \quad (3.2.11)$$

Since $\limsup_{n\longrightarrow\infty} a_n < 1$ and $\lim_{n\longrightarrow\infty} b_n = \lim_{n\longrightarrow\infty} d_n = 0$, it follows from (3.2.8) that $\lim_{n\longrightarrow\infty} \|T^nx_n - x_n\| = 0$. □

Theorem 3.2.2. *Let X be a uniformly convex Banach space, and C a nonempty closed convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be sequences of real numbers in $[0,1]$ with $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, *and*

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ *and* $\limsup_{n\longrightarrow\infty} a_n < 1$.

Let $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences defined by the three-step iterative scheme (3.2.1). If* T *satisfies Condition*(A) *with respect to the sequence* $\{x_n\}$*, then* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *converge strongly to a fixed point of* T.

Proof. By Lemma 3.2.1 , we have

$$\begin{aligned} \lim_{n\longrightarrow\infty} \|T^n x_n - y_n\| &= 0, \\ \lim_{n\longrightarrow\infty} \|T^n x_n - z_n\| &= 0, \\ \lim_{n\longrightarrow\infty} \|T^n x_n - x_n\| &= 0. \end{aligned} \tag{3.2.12}$$

It follows from (3.2.9) and (3.2.12) that $\lim_{n\longrightarrow\infty} \|y_n - x_n\| = 0$.

Since

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(1 - \alpha_n - \beta_n)y_n + \alpha_n T^n x_n + \beta_n w_n - x_n\| \\ &= \|(y_n - x_n) + \alpha_n(T^n x_n - y_n) + \beta_n(w_n - y_n)\| \\ &\leq \|y_n - x_n\| + \alpha_n\|T^n x_n - y_n\| + \beta_n\|w_n - y_n\|, \end{aligned} \tag{3.2.13}$$

it follows that $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$.

Thus

$$\begin{aligned} \|x_{n+1} - T^n x_{n+1}\| &\leq \|x_{n+1} - x_n\| + \|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\ &\leq \|x_{n+1} - x_n\| + k_n\|x_{n+1} - x_n\| + \|T^n x_n - x_n\| \longrightarrow 0, \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| + \|Tx_{n+1} - T^{n+1}x_{n+1}\| \\ &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| + k_1\|x_{n+1} - T^n x_{n+1}\| \longrightarrow 0, \end{aligned}$$

which implies

$$\lim_{n\longrightarrow\infty} \|Tx_n - x_n\| = 0. \tag{3.2.14}$$

Since T satisfies $Condition(A)$ with respect to the sequence $\{x_n\}$, there is a nondecreasing function $f : [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x_n - Tx_n\| \geq f(d(x_n, F(T))),$$

for all $n \geq 1$. This together with (3.2.14) imply $\lim_{n\longrightarrow\infty} d(x_n, F(T)) = 0$.

Then $\lim_{n\longrightarrow\infty} x_n = q \in F(T)$ by Theorem 2.1 [41]. Thus $\lim_{n\longrightarrow\infty} \|x_n - q\| = 0$. Since $\|y_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, and

$$\begin{aligned} \|z_n - x_n\| &= \|(1 - a_n - b_n)x_n + a_nT^nx_n + b_nu_n - x_n\| \\ &\leq a_n\|T^nx_n - x_n\| + b_n\|u_n - x_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty, \end{aligned}$$

it follows that $\lim_{n\longrightarrow\infty} y_n = q$ and $\lim_{n\longrightarrow\infty} z_n = q$. □

From Theorem 3.2.2, we have the following.

Theorem 3.2.3. *Let X be a uniformly convex Banach space, and C a nonempty closed convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be sequences of real numbers in $[0, 1]$ with $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, *and*

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ *and* $\limsup_{n\longrightarrow\infty} a_n < 1$.

Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined by the three-step iterative scheme (3.2.1). Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a fixed point of T.

Proof. It is well known that every continuous and demicompact mapping must satisfy $Condition(A)$ (see [32]). Since $T : C \longrightarrow C$ is completely continuous, it is continuous and demicompact, so T satisfies $Condition(A)$ on C, and so the conclusion of theorem follows from Theorem 3.2.2 □

For $a_n = b_n = c_n = d_n \equiv 0$, then Theorem 3.2.3 reduces to the Mann iteration with errors.

Corollary 3.2.4. *Let X be a uniformly convex Banach space, and let C be a closed convex subset of X. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ be real sequences in $[0,1]$ such that $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$.*

For a given $x_1 \in C$, define

$$x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \alpha_n T^n x_n + \beta_n w_n, \quad n \geq 1,$$

where $\{w_n\}$ is bounded sequence in C. Then $\{x_n\}$ converges strongly to a fixed point of T.

Theorem 3.2.5. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ be sequences of real numbers in $[0,1]$ with $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, *and*

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ *and* $\limsup_{n\longrightarrow\infty} a_n <$ 1.

Let $\{x_n\}$ be the sequence defined by three-step iterative scheme (3.2.1). Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. By using the same proof as in Theorem 3.2.3, it can be shown that $\lim_{n\longrightarrow\infty}$ $\|Tx_n - x_n\| = 0$. Since X is uniformly convex and $\{x_n\}$ is bounded, we may assume that $x_n \longrightarrow u$ weakly as $n \longrightarrow \infty$, without loss of generality. By Lemma 2.1.33, we have $u \in F(T)$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. From Lemma 2.1.33, $u, v \in F(T)$. By Lemma 3.2.1 (i), $\lim_{n\longrightarrow\infty}\|x_n - u\|$ and $\lim_{n\longrightarrow\infty}\|x_n - v\|$ exist. It follows from Lemma 2.1.34 that $u = v$. Therefore $\{x_n\}$ converges weakly to fixed point of T. □

When $a_n = b_n \equiv 0$ in Theorem 3.2.5, we obtain weak convergence theorem of the two-step iteration with errors as follows:

Corollary 3.2.6. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and C a nonempty closed convex subset of X. Let T be an asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \emptyset$. Let $\{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\}$ be sequences of real numbers in $[0,1]$ such that*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, *and*

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$.

For a given $x_1 \in C$, define

$$\begin{aligned} y_n &= (1 - c_n - d_n)x_n + c_n T^n x_n + d_n v_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_n T^n x_n + \beta_n w_n, \quad n \geq 1, \end{aligned}$$

where $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. Then $\{x_n\}$ converges weakly to a fixed point of T.