CHAPTER V

COMMON FIXED POINTS OF NEW ITERATIONS WITH ERRORS FOR NONEXPANSIVE MAPPINGS AND ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN A BANACH SPACE

This chapter is to construct an iteration scheme for approximating common fixed points of three nonexpansive mappings and three asymptotically nonexpansive mappings to prove some strong and weak convergence theorems for such mappings in a uniformly convex Banach space.

5.1 Common Fixed Points of New Iterations with Errors for Nonexpansive Mappings in a Banach Space

Xu [39] introduced the following iterative schemes for nonexpansive mappings $T : C \longrightarrow C$ known as Mann iterative scheme with errors and Ishikawa iterative scheme with errors:

(1) The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1 - a_n - b_n)x_n + a_n T x_n + b_n u_n, n \geq 1, \end{aligned}$$

where $\{a_n\}, \{b_n\}$ are appropriate sequences in $[0, 1]$ and $\{u_n\}$ is a bounded sequence in C, is known as Mann iterative scheme with errors. This scheme reduces to Mann iterative scheme if $b_n = 0$.

(2) The sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= (1 - a_n - b_n)x_n + a_n T x_n + b_n u_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)x_n + \alpha_n T y_n + \beta_n v_n, n \geq 1, \end{aligned}$$

where $\{a_n\}, \{b_n\}, \{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0,1]$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C, is known as Ishikawa iterative scheme with errors. This scheme becomes Ishikawa iterative schemes if $b_n = \beta_n = 0$. Chidume and Moore [5] studied the above schemes in 1999. A generalization of Mann and Ishikawa iterative schemes was given by Das and Debata [8] and Takahashi and Tamura [34].

This scheme dealt with two nonexpansive mappings S and T:

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= (1 - b_n)x_n + b_n S x_n, \\ x_{n+1} &= (1 - a_n)x_n + a_n T y_n, n \geq 1. \end{aligned}$$

In 2005, Khan and Fukhar-ud-din [15] generalized this scheme to the one with errors for a pair of nonexpansive mappings S and T as follows.

(3) The sequence $\{x_n\}$, in this case, is defined by

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= (1 - a_n - b_n)x_n + a_n S x_n + b_n u_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)x_n + \alpha_n T y_n + \beta_n v_n, n \geq 1, \end{aligned}$$

where $\{a_n\}, \{b_n\}, \{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ and $\{u_n\}, \{v_n\}$ are bounded sequences in C. They proved weak and strong convergence of the iterative scheme (3) to common fixed point of two nonexpansive mapping S and T.

Recently, Khan and Fukhar-ud-din [16] introduced the three-step iterative scheme with errors for three nonexpanxive mappings as follows:

For a given $x_1 \in C$, compute the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= (1-a_n-b_n)x_n + a_nT_3x_n + b_nu_n, \\ y_n &= (1-c_n-d_n)x_n + a_nT_2z_n + d_nv_n, \\ x_{n+1} &= (1-\alpha_n-\beta_n)x_n + \alpha_nT_1y_n + \beta_nw_n, n \geq 1, \end{aligned}$$

where $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ and $\{u_n\}, \{v_n\}, \{v_n\}$ are bounded sequences in C, and they proved weak and strong convergence of this iterative scheme to common fixed point of T_1, T_2 and T_3.

Inspired and motivated by these facts, a new class of three-step iterative scheme is introduced and studied in this paper. We consider a new three-step iterative scheme defined as follows:

For a given $x_1 \in C$, compute the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ by the iterative scheme

$$\begin{aligned} z_n &= (1-a_n-b_n)x_n + a_nT_1x_n + b_nu_n, \\ y_n &= (1-c_n-d_n)z_n + c_nT_2z_n + d_nv_n, \qquad (5.1.1) \\ x_{n+1} &= (1-\alpha_n-\beta_n)y_n + \alpha_nT_3y_n + \beta_nw_n, \quad n \geq 1, \end{aligned}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0,1]$.

In order to prove this, the following lemmas are needed.

Lemma 5.1.1. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and $\{x_n\}$ be the sequences defined as in (5.1.1) and* $\sum_{n=1}^{\infty} b_n <$

∞, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. *If* $F \neq \emptyset$, *then* $\lim_{n\longrightarrow\infty} \|x_n - p\|$ *exists for all* $p \in F$.

Proof. Assume that $F \neq \emptyset$. Let $p \in F$ and

$$M_1 = \sup\{\|u_n - p\| : n \geq 1\},\ M_2 = \sup\{\|v_n - p\| :\geq 1\}$$

$$M_3 = \sup\{\|w_n - p\| : n \geq 1\}, \quad M = \max\{M_1, M_2, M_3\}.$$

We have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n - \beta_n)y_n + \alpha_n T_3 y_n + \beta_n w_n - p\| \\
&= \|(1 - \alpha_n - \beta_n)(y_n - p) + \alpha_n(T_3 y_n - p) + \beta_n(w_n - p)\| \\
&\leq (1 - \alpha_n - \beta_n)\|y_n - p\| + \alpha_n\|T_3 y_n - p\| + \beta_n\|w_n - p\| \\
&\leq \|y_n - p\| + \beta_n\|w_n - p\| \\
\|y_n - p\| &= \|(1 - c_n - d_n)z_n + c_n T_2 z_n + d_n v_n - p\| \\
&= \|(1 - c_n - d_n)(z_n - p) + c_n(T_2 z_n - p) + d_n(v_n - p)\| \\
&\leq (1 - c_n - d_n)\|z_n - p\| + c_n\|T_2 z_n - p\| + d_n\|v_n - p\| \\
&\leq \|z_n - p\| + d_n\|v_n - p\| \\
\|z_n - p\| &= \|(1 - a_n - b_n)x_n + b_n T_1 x_n + b_n u_n - p\| \\
&= \|(1 - a_n - b_n)(x_n - p) + a_n(T_1 x_n - p) + b_n(u_n - p)\| \\
&\leq (1 - a_n - b_n)\|x_n - p\| + b_n\|T_1 x_n - p\| + b_n\|u_n - p\| \\
&\leq \|x_n - p\| + b_n\|u_n - p\|
\end{aligned}$$

It follows that $\|x_{n+1} - p\| \leq \|x_n - p\| + b_n\|u_n - p\| + d_n\|v_n - p\| + \beta_n\|w_n - p\|$.

And so $\|x_{n+1} - p\| \leq \|x_n - p\| + M(b_n + d_n + \beta_n)$.

Since $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, by Lemma 2.1.32 implies that $\lim_{n\longrightarrow\infty} \|x_n - q\|$ exists. $\square$

Lemma 5.1.2. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined as in (5.1.1).*

(i) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, then $\lim_{n\longrightarrow\infty} \|T_3 y_n - y_n\| = 0$.

(ii) If $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, then $\lim_{n\longrightarrow\infty} \|T_2 z_n - z_n\| = 0$.

(iii) If $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, then $\lim_{n\longrightarrow\infty} \|T_1 x_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ and $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, then $\lim_{n\longrightarrow\infty} \|T_2 x_n - x_n\| = 0 = \lim_{n\longrightarrow\infty} \|T_3 x_n - x_n\|$.

Proof. (i) From Lemma 5.1.1, we have $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists for any $p \in F(T)$. It follows that $\{x_n - p\}, \{T_1 x_n - p\}, \{z_n - q\}, \{T_2 z_n - p\}$ and $\{y_n - p\}, \{T_3 y_n - p\}$ are all bounded. Also, $\{u_n - p\}, \{v_n - p\}$ and $\{w_n - p\}$ are bounded by the assumption. Now we set

$$
\begin{aligned}
r_1 &= \sup\{\|x_n - p\| : n \geq 1\}, \\
r_2 &= \sup\{\|T_1 x_n - p\| : n \geq 1\}, \\
r_3 &= \sup\{\|z_n - p\| : n \geq 1\}, \\
r_4 &= \sup\{\|T_2 z_n - p\| : n \geq 1\},
\end{aligned}
$$

$$\begin{aligned}
r_5 &= \sup\{\|y_n - p\| : n \geq 1\},\\
r_6 &= \sup\{\|T_3 y_n - p\| : n \geq 1\},\\
r_7 &= \sup\{\|u_n - p\| : n \geq 1\},\\
r_8 &= \sup\{\|v_n - p\| : n \geq 1\},\\
r_9 &= \sup\{\|w_n - p\| : n \geq 1\},\\
r &= \max\{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9\}.
\end{aligned}$$

By Lemma 2.1.36 we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1 - a_n - b_n)x_n + a_n T_1 x_n + b_n u_n - p\|^2\\
&= \|(1 - a_n - b_n)(x_n - p) + a_n(T_1 x_n - p) + b_n(u_n - p)\|^2\\
&\leq (1 - a_n - b_n)\|x_n - p\|^2 + a_n\|T_1 x_n - p\|^2 + b_n\|u_n - p\|^2\\
&\quad - a_n(1 - a_n - b_n)g(\|T_1 x_n - x_n\|)\\
&\leq (1 - a_n - b_n)\|x_n - p\|^2 + a_n\|x_n - p\|^2 + b_n\|u_n - p\|^2\\
&\leq \|x_n - p\|^2 + r^2 b_n - a_n(1 - a_n - b_n)g(\|T_1 x_n - x_n\|)\\
&\leq \|x_n - p\|^2 + r^2 b_n,\\
\|y_n - p\|^2 &= \|(1 - c_n - d_n)z_n + c_n T_2 z_n + d_n v_n - p\|^2\\
&= \|(1 - c_n - d_n)(z_n - p) + c_n(T_2 z_n - p) + d_n(v_n - p)\|^2\\
&\leq (1 - c_n - d_n)\|z_n - p\|^2 + c_n\|T_2 z_n - p\|^2 + d_n\|v_n - p\|^2\\
&\quad - c_n(1 - c_n - d_n)g(\|T_2 z_n - z_n\|)\\
&\leq (1 - c_n - d_n)\|z_n - p\|^2 + c_n\|z_n - p\|^2 + d_n\|v_n - p\|^2\\
&\quad - c_n(1 - c_n - d_n)g(\|T_2 z_n - z_n\|)\\
&\leq \|z_n - p\|^2 + r^2 d_n - c_n(1 - c_n - d_n)g(\|T_2 z_n - z_n\|)\\
&\leq \|z_n - p\|^2 + r^2 d_n
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1-\alpha_n-\beta_n)y_n + \alpha_n T_3 y_n + \beta_n w_n - p\|^2 \\
&= \|(1-\alpha_n-\beta_n)(y_n - p) + \alpha_n (T_3 y_n - p) + \beta_n (w_n - p)\|^2 \\
&\leq (1-\alpha_n-d_n)\|y_n - p\|^2 + \alpha_n\|T_3 y_n - p\|^2 + \beta_n\|w_n - p\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|T_3 y_n - y_n\|) \\
&\leq (1-\alpha_n-\beta_n)\|y_n - p\|^2 + \alpha_n\|y_n - p\|^2 + \beta_n\|w_n - p\|^2 \\
&\quad - \alpha_n(1-\alpha_n-\beta_n)g(\|T_3 y_n - y_n\|) \\
&\leq \|y_n - p\|^2 + r^2\beta_n - \alpha_n(1-\alpha_n-\beta_n)g(\|T_3 y_n - y_n\|),
\end{aligned}$$

which lead to the following:

$$\alpha_n(1-\alpha_n-\beta_n)g(\|T_3 y_n - y_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + r^2(b_n + d_n + \beta_n), \tag{5.1.2}$$

and

$$c_n(1-c_n-d_n)g(\|T_2 z_n - z_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + r^2(b_n + d_n + \beta_n), \tag{5.1.3}$$

and

$$a_n(1-a_n-b_n)g(\|T_1 x_n - x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + r^2(b_n + d_n + \beta_n), \tag{5.1.4}$$

(i) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, then there exist a positive integer n_0 and $\eta, \eta' \in (0,1)$ such that

$$0 < \eta < \alpha_n \text{ and } \alpha_n + \beta_n < \eta' < 1 \text{ for all } n \geq n_0.$$

This implies by (5.1.2) that

$$\eta(1-\eta')g(\|T_3 y_n - y_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + r^2(b_n + d_n + \beta_n), \tag{5.1.5}$$

for all $n \geq n_0$. It follows from (5.1.5) that for $m \geq n_0$

$$\begin{aligned}\sum_{n=n_0}^{m} g(\|T_3y_n - y_n\|) &\leq \frac{1}{\eta(1-\eta')}\Big(\sum_{n=n_0}^{m}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\ &\quad + r^2\sum_{n=n_0}^{m}(b_n + d_n + \beta_n)\Big) \\ &\leq \frac{1}{\eta(1-\eta')}\Big(\|x_{n_0} - p\|^2 \\ &\quad + r^2\sum_{n=n_0}^{m}(b_n + d_n + \beta_n)\Big). \qquad (5.1.6)\end{aligned}$$

By letting $m \longrightarrow \infty$ in inequality (5.1.6) we get that $\sum_{n=n_0}^{\infty} g(\|T_3y_n - y_n\|) < \infty$, which implies $\lim_{n\longrightarrow\infty} g(\|T_3y_n - y_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\longrightarrow\infty}\|T_3y_n - y_n\| = 0$.

(ii) If $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, then by using the same argument as above with the inequality (5.1.3), it can be show that $\lim_{n\longrightarrow\infty}\|T_2z_n - z_n\| = 0$.

(iii) If $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, then by using (5.1.4) and the same argument as in (i), it can be show that $\lim_{n\longrightarrow\infty}\|T_1x_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ and $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, by (i), (ii) and (iii) we have

$$\lim_{n\longrightarrow\infty}\|T_3y_n - y_n\| = \lim_{n\longrightarrow\infty}\|T_2z_n - z_n\| = \lim_{n\longrightarrow\infty}\|T_1x_n - x_n\| = 0. \qquad (5.1.7)$$

From $z_n = (1 - a_n - b_n)x_n + a_nT_1x_n + b_nu_n$ and $y_n = (1 - c_n - d_n)z_n + c_nT_2z_n + d_nv_n$, we have

$$\begin{aligned}\|z_n - x_n\| &= \|(1 - a_n - b_n)x_n + a_nT_1x_n + b_nu_n - x_n\| \\ &= \|a_n(T_1x_n - x_n) + b_n(u_n - x_n)\| \\ &\leq a_n\|T_1x_n - x_n\| + b_n\|u_n - x_n\| \\ &\leq \|T_1x_n - x_n\| + 2rb_n,\end{aligned}$$

and $\|y_n - x_n\| \leq \|T_2 z_n - z_n\| + \|T_1 x_n - x_n\| + 2rb_n + 2rd_n$. Hence

$$\begin{aligned}\|T_2 x_n - x_n\| &\leq \|x_n - T_2 z_n\| + \|T_2 z_n - T_2 x_n\| \\ &\leq \|x_n - z_n\| + \|T_2 z_n - z_n\| + \|z_n - x_n\| \\ &\leq \|T_2 z_n - z_n\| + 2\|z_n - x_n\| \\ &\leq \|T_2 z_n - z_n\| + 2\|T_1 x_n - x_n\| + 4rb_n,\end{aligned}$$

and

$$\begin{aligned}\|T_3 x_n - x_n\| &\leq \|x_n - T_3 y_n\| + \|T_3 y_n - T_3 x_n\| \\ &\leq \|x_n - y_n\| + \|T_3 y_n - y_n\| + \|y_n - x_n\| \\ &\leq \|T_3 y_n - y_n\| + 2\|y_n - x_n\| \\ &\leq \|T_3 y_n - y_n\| + 2\|T_2 z_n - z_n\| \\ &\quad + 2\|T_1 x_n - x_n\| + 4rb_n + 4rd_n.\end{aligned}$$

It follows from (5.1.7) that $\lim_{n\longrightarrow\infty} \|T_2 x_n - x_n\| = \lim_{n\longrightarrow\infty} \|T_3 x_n - x_n\| = 0$. □

Theorem 5.1.3. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty} (\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty} (c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty} (a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined as in (5.1.1). If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3.

Proof. Let $p \in F$. By Lemma 5.1.1, $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F. To prove this, let p_1 and p_2

be weak limits of subsequence $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Lemma 5.1.2(iii), we have $\lim_{n\longrightarrow\infty}\|T_1x_n-x_n\|=0$. By Lemma 2.1.33, we have $I-T_1, I-T_2$ and $I-T_3$ are demiclosed with respect to zero, therefore $T_1p_i=p_i, T_2p_i=p_i$ and $T_3p_i=p_i$, (i=1, 2) hence $p_1,p_2\in F$. By Lemma 5.1.1 $\lim_{n\longrightarrow\infty}\|x_n-p_1\|$ and $\lim_{n\longrightarrow\infty}\|x_n-p_2\|$ exist. Using Lemma 2.1.34 we obtain that $p_1=p_2$. Hence $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3. □

Theorem 5.1.4. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C satisfying condition (A'') and let $\{a_n\},\{b_n\},\{c_n\},\{d_n\}\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that a_n+b_n, c_n+d_n and $\alpha_n+\beta_n$ are in $[0,1]$ for all $n\geq 1$ and $\sum_{n=1}^{\infty}b_n<\infty$, $\sum_{n=1}^{\infty}d_n<\infty$, $\sum_{n=1}^{\infty}\beta_n<\infty$ and*

(i) $0<\liminf_{n\longrightarrow\infty}\alpha_n\leq\limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n)<1$,

(ii) $0<\liminf_{n\longrightarrow\infty}c_n\leq\limsup_{n\longrightarrow\infty}(c_n+d_n)<1$, *and*

(iii) $0<\liminf_{n\longrightarrow\infty}a_n\leq\limsup_{n\longrightarrow\infty}(a_n+b_n)<1$.

For a given $x_1\in C$, let $\{x_n\},\{y_n\},\{z_n\}$ be the sequences defined as in (5.1.1). If $F\neq\emptyset$, then $\{x_n\},\{y_n\},\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. By Lemma 5.1.2, we have

$$\lim_{n\longrightarrow\infty}\|T_3x_n-x_n\|=\lim_{n\longrightarrow\infty}\|T_2x_n-x_n\|=\lim_{n\longrightarrow\infty}\|T_1x_n-x_n\|=0.$$

By condition (A''), we have

$$f(d(x_n,F))\leq\tfrac{1}{3}(\|x_n-T_1x_n\|+\|x_n-T_2x_n\|+\|x_n-T_3x_n\|).$$

It follows that $\lim_{n\longrightarrow\infty}f(d(x_n,F))\leq 0$, hence $\lim_{n\longrightarrow\infty}f(d(x_n,F))=0$.

Since f is a nondecreasing function and $f(0)=0$, we obtain that $\lim_{n\longrightarrow\infty}d(x_n,F)$ $=0$. Thus we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $(y_k)\subset F$

such that $\|x_{n_k} - y_k\| < 2^{-k}$ for all $k \geq 1$. Then following the method of proof of Tan and Xu [?], we get that $\{y_k\}$ is a Cauchy sequence in F and so it converges. Let $y_k \longrightarrow y$. Since F is closed, $y \in F$ and then $x_{n_k} \longrightarrow y$. By Lemma (5.1.1) $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exits, $\forall p \in F$. It implies that $\lim_{n\longrightarrow\infty} \|x_n - y\| = 0$. Hence $\{x_n\}$ converses strongly to a common fixed point of T_1, T_2 and T_3.

Since $\|y_n - x_n\| \leq \|T_2 z_n - z_n\| + \|T_1 x_n - x_n\| + 2rb_n + 2rd_n$ and $\|z_n - x_n\| \leq \|T_1 x_n - x_n\| + 2rb_n$, it follows from Lemma 5.1.2 (ii) and (iii) that $\|y_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ and $\|z_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, which imply that $\lim_{n\longrightarrow\infty} \|y_n - y\| = 0$ and $\lim_{n\longrightarrow\infty} \|z_n - y\| = 0$. □

Theorem 5.1.5. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty} (\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty} (c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty} (a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (5.1.1). If $F \neq \emptyset$ and one of T_1, T_2 and T_3 is completely continuous, then $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. By Lemma 5.1.1, $\{x_n\}$ is bounded. In addition, by Lemma 5.1.2, $lim_{n\longrightarrow\infty} \|x_n - T_1 x_n\| = 0$, $lim_{n\longrightarrow\infty} \|x_n - T_2 x_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|x_n - T_3 x_n\| = 0$, then $\{T_1 x_n\}$, $\{T_2 x_n\}$ and $\{T_3 x_n\}$ are also bounded. If T_1 is completely continuous, there exists subsequence $\{T_1 x_{n_j}\}$ of $\{T_1 x_n\}$ such that $T_1 x_{n_j} \longrightarrow p$ as $j \longrightarrow \infty$. Thus $\lim_{j\longrightarrow\infty} \|x_{n_j} - T_1 x_{n_j}\| = \lim_{j\longrightarrow\infty} \|x_{n_j} - T_2 x_{n_j}\| = \lim_{j\longrightarrow\infty} \|x_{n_j} - T_3 x_{n_j}\| = 0$. It follows that $\lim_{j\longrightarrow\infty} \|x_{n_j} - p\| = 0$. This implies by Lemma 2.1.33 that $p \in$

F. Furthermore, by Lemma 5.1.1, we get that $\lim_{n\longrightarrow\infty}\|x_n - p\|$ exists. Thus $lim_{n\longrightarrow\infty}\|x_n - p\| = 0$. The proof is completed. □

Theorem 5.1.6. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three nonexpansive self-mappings of C and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (5.1.1). If $F \neq \emptyset$ and one of T_1, T_2 and T_3 is demicompact, then $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. Suppose that T_1 is semicompact. By Lemma 5.1.2, $\lim_{n\longrightarrow\infty}\|x_n - T_1x_n\| = \lim_{n\longrightarrow\infty}\|x_n - T_2x_n\| = \lim_{n\longrightarrow\infty}\|x_n - T_3x_n\| = 0$. Then there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $p \in C$. It follows from Lemma 2.1.33 that $p \in F$. Since $\lim_{n\longrightarrow\infty}\|x_n - p\|$ exists by Lemma 5.1.1, it follows that $\lim_{n\longrightarrow\infty}\|x_n - p\| = 0$. □

5.2 Common Fixed Points of New Iterations with Errors for Asymptotically Nonexpansive Mappings in a Banach Space

Takahashi and Tamuro [34] introduced the following iterative schemes known as Ishikawa iterative schemes for a pair of nonexpansive mappings:

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= (1-a_n)x_n + a_nT_1x_n, \\ x_{n+1} &= (1-b_n)x_n + b_nT_2y_n, n \geq 1, \end{aligned} \tag{5.2.1}$$

where $\{a_n\}, \{b_n\} \in [0,1]$.

Khan and Hafiz [15] generalized the scheme (5.2.1) to the one with errors for a pair of nonexpansive mappings as follows:

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= (1-a_n-b_n)x_n + a_nT_1x_n + b_nu_n, \\ x_{n+1} &= (1-\alpha_n-\beta_n)x_n + \alpha_nT_2y_n + \beta_nv_n, n \geq 1, \end{aligned} \tag{5.2.2}$$

where $\{a_n\}, \{b_n\}, \{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0,1]$ $0 < \delta \leq a_n+b_n, \alpha_n+\beta_n \leq 1-\delta < 1$, and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

In 2006, Jeong and Kim [14] generalized this scheme (5.2.2) for a pair of asymptotically nonexpansive mappings as follows:

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= (1-a_n-b_n)x_n + a_nT_1^nx_n + b_nu_n, \\ x_{n+1} &= (1-\alpha_n-\beta_n)x_n + \alpha_nT_2^ny_n + \beta_nv_n, n \geq 1, \end{aligned} \tag{5.2.3}$$

where $\{a_n\}, \{b_n\}, \{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0,1]$ with $0 < \delta \leq a_n + b_n, \alpha_n + \beta_n \leq 1-\delta < 1$, and $\{u_n\}, \{v_n\}$ are bounded sequences in C.

Inspired and motivated by these facts, a new class of three-step iterative scheme is introduced and studied in this paper. We consider a new iterative scheme defined as follows. For a given $x_1 \in C$, compute the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$

by the iterative scheme

$$\begin{aligned} z_n &= (1 - a_n - b_n)x_n + a_n T_1^n x_n + b_n u_n, \\ y_n &= (1 - c_n - d_n)z_n + c_n T_2^n z_n + d_n v_n, \\ x_{n+1} &= (1 - \alpha_n - \beta_n)y_n + \alpha_n T_3^n y_n + \beta_n w_n, \quad n \geq 1, \end{aligned} \tag{5.2.4}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$, $\{\beta_n\}$ are appropriate sequences in $[0, 1]$.

The next lemma is crucial for proving the main theorems.

Lemma 5.2.1. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $k_n, l_n, m_n \in [0, \infty)$, $\lim_{n\longrightarrow\infty} k_n = 1, \lim_{n\longrightarrow\infty} l_n = 1, \lim_{n\longrightarrow\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, respectively and $\{x_n\}$ be the sequence defined as in (5.2.4) and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. If $F \neq \emptyset$, then $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists for all $p \in F$.*

Proof. Assume that $F \neq \emptyset$. Let $p \in F$ and

$$M_1 = \sup\{\|u_n - p\| : n \geq 1\}, \; M_2 = \sup\{\|v_n - p\| : \geq 1\}$$

$$M_3 = \sup\{\|w_n - p\| : n \geq 1\}, \quad M = \max\{M_1, M_2, M_3\}.$$

We have

$$\begin{aligned} \|z_n - p\| &= \|(1 - a_n - b_n)x_n + b_n T_1^n x_n + b_n u_n - p\| \\ &= \|(1 - a_n - b_n)(x_n - p) + a_n(T_1^n x_n - p) + b_n(u_n - p)\| \\ &\leq (1 - a_n - b_n)\|x_n - p\| + b_n\|T_1^n x_n - p\| + b_n\|u_n - p\| \\ &\leq (1 - a_n - b_n)\|x_n - p\| + b_n k_n\|T_1^n x_n - p\| + b_n\|u_n - p\| \\ &\leq (1 + a_n(k_n - 1))\|x_n - p\| b_n\|u_n - p\| \\ &\leq k_n\|x_n - p\| + b_n\|u_n - p\| \end{aligned}$$

$$
\begin{aligned}
\|y_n - p\| &= \|(1-c_n-d_n)z_n + c_nT_2^nz_n + d_nv_n - p\| \\
&= \|(1-c_n-d_n)(z_n-p) + c_n(T_2^nz_n - p) + d_n(v_n-p)\| \\
&\leq (1-c_n-d_n)\|z_n-p\| + c_n\|T_2^nz_n - p\| + d_n\|v_n-p\| \\
&\leq (1-c_n-d_n)\|z_n-p\| + c_nl_n\|T_2^nz_n - p\| + d_n\|v_n-p\| \\
&\leq (1+c_n(l_n-1))\|z_n-p\| + d_n\|v_n-p\| \\
&\leq l_n\|z_n-p\| + d_n\|v_n-p\| \\
\|x_{n+1}-p\| &= \|(1-\alpha_n-\beta_n)y_n + \alpha_nT_3^ny_n + \beta_nw_n - p\| \\
&= \|(1-\alpha_n-\beta_n)(y_n-p) + \alpha_n(T_3^ny_n - p) + \beta_n(w_n-p)\| \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\| + \alpha_n\|T_3^ny_n - p\| + \beta_n\|w_n-p\| \\
&\leq (1-\alpha_n-\beta_n)\|y_n-p\| + \alpha_nm_n\|T_3^ny_n - p\| + \beta_n\|w_n-p\| \\
&\leq (1+\alpha_n(m_n-1))\|y_n-p\| + \beta_n\|w_n-p\| \\
&\leq m_n\|y_n-p\| + \beta_n\|w_n-p\| \\
&\leq m_n[l_n[k_n\|x_n-p\| + b_n\|u_n-p\|] + d_n\|v_n-p\|] + \beta_n\|w_n-p\| \\
&= m_nl_nk_n\|x_n-p\| + m_nl_nb_n\|u_n-p\|] + m_nd_n\|v_n-p\|] \\
&\quad + \beta_n\|w_n-p\| \\
&\leq h_n^3\|x_n-p\| + M(b_n+d_n+\beta_n) \\
&= (1+(h_n^3-1))\|x_n-p\| + M(b_n+d_n+\beta_n),
\end{aligned}
$$

where M is some positive constant and $h_n = \max\{k_n, l_n, M_n\}$. Notice that $\sum_{n=1}^{\infty}(h_n - 1) < \infty$ is equivalent to $\sum_{n=1}^{\infty}(h_n^3 - 1) < \infty$. Since $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, by Lemma 2.1.32 implies that $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists. □

Lemma 5.2.2. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $k_n, l_n, m_n \in [0,\infty)$, $\lim_{n\longrightarrow\infty} k_n = 1, \lim_{n\longrightarrow\infty} l_n = 1, \lim_{n\longrightarrow\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, respectively and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such* *that* $a_n + b_n$, $c_n + d_n$ *and* $\alpha_n + \beta_n$ *are in* $[0,1]$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. *For a given* $x_1 \in C$, *let* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences defined as in (5.2.4).*

(i) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n) < 1$, *then* $\lim_{n\longrightarrow\infty} \|T_3^n y_n - y_n\| = 0$.

(ii) If $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n+d_n) < 1$, *then* $\lim_{n\longrightarrow\infty} \|T_2^n z_n - z_n\| = 0$.

(iii) If $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n+b_n) < 1$, *then* $\lim_{n\longrightarrow\infty} \|T_1^n x_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n) < 1$, $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ *and* $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, *then* $\lim_{n\longrightarrow\infty} \|T_2^n x_n - x_n\| = 0 = \lim_{n\longrightarrow\infty} \|T_3^n x_n - x_n\|$.

Proof. (i) From Lemma 5.2.1, we know that $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists for any $p \in F$. It follow that $\{x_n - p\}, \{T_1^n x_n - p\}, \{z_n - q\}, \{T_2^n z_n - p\}$ and $\{y_n - p\}, \{T_3^n y_n - p\}$ are all bounded. Also, $\{u_n - p\}, \{v_n - p\}$ and $\{w_n - p\}$ are bounded by the assumption. Now we set

$$
\begin{aligned}
r_1 &= \sup\{\|x_n - p\| : n \geq 1\},\\
r_2 &= \sup\{\|T_1^n x_n - p\| : n \geq 1\},\\
r_3 &= \sup\{\|z_n - p\| : n \geq 1\},\\
r_4 &= \sup\{\|T_2^n z_n - p\| : n \geq 1\},\\
r_5 &= \sup\{\|y_n - p\| : n \geq 1\},\\
r_6 &= \sup\{\|T_3^n y_n - p\| : n \geq 1\},\\
r_7 &= \sup\{\|u_n - p\| : n \geq 1\},\\
r_8 &= \sup\{\|v_n - p\| : n \geq 1\},
\end{aligned}
$$

$$r_9 = \sup\{\|w_n - p\| : n \geq 1\},$$

$$r = \max\{r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9\}.$$

By using Lemma 2.1.36 we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1 - a_n - b_n)x_n + a_n T_1^n x_n + b_n u_n - p\|^2 \\
&= \|(1 - a_n - b_n)(x_n - p) + a_n(T_1^n x_n - p) + b_n(u_n - p)\|^2 \\
&\leq (1 - a_n - b_n)\|x_n - p\|^2 + a_n\|T_1^n x_n - p\|^2 + b_n\|u_n - p\|^2 \\
&\quad - a_n(1 - a_n - b_n)g(\|T_1^n x_n - x_n\|) \\
&\leq (1 - a_n - b_n)\|x_n - p\|^2 + a_n k_n^2\|x_n - p\|^2 + b_n\|u_n - p\|^2 \\
&\leq (1 + a_n(k_n^2 - 1))\|x_n - p\|^2 + r^2 b_n \\
&\leq k_n^2\|x_n - p\|^2 + r^2 b_n, \\
\|y_n - p\|^2 &= \|(1 - c_n - d_n)z_n + c_n T_2^n z_n + d_n v_n - p\|^2 \\
&= \|(1 - c_n - d_n)(z_n - p) + c_n(T_2^n z_n - p) + d_n(v_n - p)\|^2 \\
&\leq (1 - c_n - d_n)\|z_n - p\|^2 + c_n\|T_2^n z_n - p\|^2 + d_n\|v_n - p\|^2 \\
&\quad - c_n(1 - c_n - d_n)g(\|T_2^n z_n - z_n\|) \\
&\leq (1 - c_n - d_n)\|z_n - p\|^2 + c_n l_n^2\|z_n - p\|^2 + d_n\|v_n - p\|^2 \\
&\quad - c_n(1 - c_n - d_n)g(\|T_2^n z_n - z_n\|) \\
&\leq (1 + c_n(l_n^2 - 1))\|z_n - p\|^2 + r^2 d_n \\
&\leq l_n^2\|z_n - p\|^2 + r^2 d_n
\end{aligned}$$

and so

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1 - \alpha_n - \beta_n)y_n + \alpha_n T_3^n y_n + \beta_n w_n - p\|^2 \\
&= \|(1 - \alpha_n - \beta_n)(y_n - p) + \alpha_n(T_3^n y_n - p) + \beta_n(w_n - p)\|^2 \\
&\leq (1 - \alpha_n - d_n)\|y_n - p\|^2 + \alpha_n\|T_3^n y_n - p\|^2 + \beta_n\|w_n - p\|^2 \\
&\quad - \alpha_n(1 - \alpha_n - \beta_n)g(\|T_3^n y_n - y_n\|)
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n-\beta_n)\|y_n-p\|^2+\alpha_n m_n^2\|y_n-p\|^2+\beta_n\|w_n-p\|^2 \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&\leq (1+\alpha_n(m_n^2-1))\|y_n-p\|^2+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&\leq m_n^2\|y_n-p\|^2+r^2\beta_n-\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&\leq m_n^2[l_n^2\|z_n-p\|^2+r^2 d_n]+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&\leq m_n^2[l_n^2[k_n^2\|x_n-p\|^2+r^2 b_n]+r^2 d_n]+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&= m_n^2 l_n^2 k_n^2\|x_n-p\|^2+m_n^2 l_n^2 r^2 b_n+m_n^2 r^2 d_n+r^2\beta_n \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&\leq h_n^3\|x_n-p\|^2+M(b_n+d_n+\beta_n) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \\
&= (1+(h_n^3-1))\|x_n-p\|^2+M(b_n+d_n+\beta_n) \\
&\quad -\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|),
\end{aligned}
$$

which leads to the following:

$$
\begin{aligned}
\alpha_n(1-\alpha_n-\beta_n)g(\|T_3^n y_n-y_n\|) \leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&+M((h_n^3-1)+b_n+d_n+\beta_n), \qquad (5.2.5)
\end{aligned}
$$

and

$$
\begin{aligned}
m_n^2 c_n(1-c_n-d_n)g(\|T_2^n z_n-z_n\|) \leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&+M((h_n^3-1)+b_n+d_n+\beta_n), \qquad (5.2.6)
\end{aligned}
$$

and

$$
\begin{aligned}
m_n^2 l_n^2 a_n(1-a_n-b_n)g(\|T_1^n x_n-x_n\|) \leq\ & \|x_n-p\|^2-\|x_{n+1}-p\|^2 \\
&+M((h_n^3-1)+b_n+d_n+\beta_n), \qquad (5.2.7)
\end{aligned}
$$

(i) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, then there exist a positive integer n_0 and $\eta, \eta' \in (0,1)$ such that

$$0 < \eta < \alpha_n \text{ and } \alpha_n + \beta_n < \eta' < 1 \text{ for all } n \geq n_0.$$

This implies by (5.2.5) that

$$\begin{aligned} \eta(1-\eta')g(\|T_3^n y_n - y_n\|) &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\quad + M((h_n^3 - 1) + b_n + d_n + \beta_n), \end{aligned} \tag{5.2.8}$$

for all $n \geq n_0$. It follows from (5.2.8) that for $m \geq n_0$

$$\begin{aligned} \sum_{n=n_0}^{m} g(\|T_3^n y_n - y_n\|) &\leq \frac{1}{\eta(1-\eta')}\Big(\sum_{n=n_0}^{m}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\ &\quad + M\sum_{n=n_0}^{m}((h_n^3 - 1) + b_n + d_n + \beta_n)\Big) \\ &\leq \frac{1}{\eta(1-\eta')}\Big(\|x_{n_0} - p\|^2 \\ &\quad + M\sum_{n=n_0}^{m}((h_n^3 - 1) + b_n + d_n + \beta_n)\Big). \end{aligned} \tag{5.2.9}$$

Let $m \longrightarrow \infty$ in inequality (5.2.9) we get that $\sum_{n=n_0}^{\infty} g(\|T_3^n y_n - y_n\|) < \infty$, and therefore $\lim_{n\longrightarrow\infty} g(\|T_3^n y_n - y_n\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\longrightarrow\infty} \|T_3^n y_n - y_n\| = 0$.

(ii) If $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, then by the using a similar method, together with inequality (5.2.6), it can be show that $\lim_{n\longrightarrow\infty} \|T_2^n z_n - z_n\| = 0$.

(iii) If $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, then by the using a similar method, together with inequality (5.2.7), it can be show that $\lim_{n\longrightarrow\infty} \|T_1^n x_n - x_n\| = 0$.

(iv) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n+\beta_n) < 1$, and $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ and $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$ by (i), (ii) and (iii) we have

$$\lim_{n\longrightarrow\infty} \|T_3^n y_n - y_n\| = \lim_{n\longrightarrow\infty} \|T_2^n z_n - z_n\| = \lim_{n\longrightarrow\infty} \|T_1^n x_n - x_n\| = 0. \tag{5.2.10}$$

From $z_n = (1-a_n-b_n)x_n + a_nT_1^nx_n + b_nu_n$ and $y_n = (1-c_n-d_n)z_n + c_nT_2^nz_n + d_nv_n$, we have

$$\begin{aligned}\|z_n - x_n\| &= \|(1-a_n-b_n)x_n + a_nT_1^nx_n + b_nu_n - x_n\| \\ &= \|a_n(T_1^nx_n - x_n) + b_n(u_n - x_n)\| \\ &\leq a_n\|T_1^nx_n - x_n\| + b_n\|u_n - x_n\| \\ &\leq \|T_1^nx_n - x_n\| + 2rb_n,\end{aligned}$$

and $\|y_n - x_n\| \leq \|T_2^nz_n - z_n\| + \|T_1^nx_n - x_n\| + 2rb_n + 2rd_n$. Hence

$$\begin{aligned}\|T_2^nx_n - x_n\| &\leq \|x_n - T_2^nz_n\| + \|T_2^nz_n - T_2^nx_n\| \\ &\leq \|x_n - z_n\| + \|T_2^nz_n - z_n\| + l_n\|z_n - x_n\| \\ &\leq \|T_2^nz_n - z_n\| + (1+l_n)\|z_n - x_n\| \\ &\leq \|T_2^nz_n - z_n\| + (1+l_n)[\|T_1^nx_n - x_n\| + 2rb_n],\end{aligned}$$

and

$$\begin{aligned}\|T_3^nx_n - x_n\| &\leq \|x_n - T_3^ny_n\| + \|T_3^ny_n - T_3^nx_n\| \\ &\leq \|x_n - y_n\| + \|T_3^ny_n - y_n\| + m_n\|y_n - x_n\| \\ &\leq \|T_3^ny_n - y_n\| + (1+m_n)\|y_n - x_n\| \\ &\leq \|T_3^ny_n - y_n\| + (1+m_n)[\|T_2^nz_n - z_n\| \\ &\quad + \|T_1^nx_n - x_n\| + 2rb_n + 2rd_n]\end{aligned}$$

It follows from (5.2.10) that $\lim_{n\longrightarrow\infty}\|T_2x_n - x_n\| = \lim_{n\longrightarrow\infty}\|T_3x_n - x_n\| = 0$. □

Lemma 5.2.3. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $k_n, l_n, m_n \in [0,\infty)$,* $\lim_{n\longrightarrow\infty} k_n = 1, \lim_{n\longrightarrow\infty} l_n = 1, \lim_{n\longrightarrow\infty} m_n = 1, \sum_{n=1}^{\infty}(k_n-1) < \infty, \sum_{n=1}^{\infty}(l_n-1) < \infty, \sum_{n=1}^{\infty}(m_n-1) < \infty,$

respectively and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0,1]$ *such that* $a_n + b_n$, $c_n + d_n$ *and* $\alpha_n + \beta_n$ *are in* $[0,1]$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. *For a given* $x_1 \in C$, *let* $\{x_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences defined as in (5.2.4).*

If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$, $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$ *and* $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$, *then* $\lim_{n\longrightarrow\infty} \|T_1x_n - x_n\| = \lim_{n\longrightarrow\infty} \|T_2x_n - x_n\| = \lim_{n\longrightarrow\infty} \|T_3x_n - x_n\| = 0$.

Proof. By Lemma 5.2.2, we have

$$\lim_{n\longrightarrow\infty} \|T_3^n y_n - y_n\| = \lim_{n\longrightarrow\infty} \|T_2^n z_n - z_n\| = \lim_{n\longrightarrow\infty} \|T_1^n x_n - x_n\| = 0,$$
$$\text{and} \quad \lim_{n\longrightarrow\infty} \|T_3^n x_n - x_n\| = \lim_{n\longrightarrow\infty} \|T_2^n x_n - x_n\| = 0. \quad (5.2.11)$$

Also we obtain that $\|y_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$ by the proof of lemma 5.2.2.

Since $x_{n+1} - y_n = \alpha_n(T_n^3 y_n - y_n) + \beta_n(w_n - y_n)$, we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \|x_{n+1} - y_n\| + \|y_n - x_n\| \\ &\leq \|T_n^3 y_n - y_n\| + \beta_n\|w_n - y_n\| + \|y_n - x_n\| \end{aligned}$$

And so

$$\begin{aligned} \|x_{n+1} - T_1^n x_{n+1}\| &\leq \|x_{n+1} - x_n\| + \|T_1^n x_{n+1} - T_1^n x_n\| + \|T_1^n x_n - x_n\| \\ &\leq \|x_{n+1} - x_n\| + k_n\|x_{n+1} - x_n\| + \|T_1^n x_n - x_n\| \\ &= (1 + k_n)\|x_{n+1} - x_n\| + \|T_1^n x_n - x_n\| \\ &\leq (1 + k_n)[\|T_n^3 y_n - y_n\| + \beta_n\|w_n - y_n\| + \|y_n - x_n\|] \\ &\quad + \|T_1^n x_n - x_n\|. \end{aligned}$$

This together with (5.2.11) implies that $\|x_{n+1} - T_1^n x_{n+1}\| \longrightarrow 0$ (as $n \longrightarrow \infty$).

Thus

$$\begin{aligned} \|x_{n+1} - T_1 x_{n+1}\| &\leq \|x_{n+1} - T_1^{n+1} x_{n+1}\| + \|T_1 x_{n+1} - T_1^{n+1} x_{n+1}\| \\ &\leq \|x_{n+1} - T_1^{n+1} x_{n+1}\| + k_1\|x_{n+1} - T_1^n x_{n+1}\| \longrightarrow 0, \end{aligned}$$

which implies that $\lim_{n\longrightarrow\infty}\|T_1x_n - x_n\| = 0$.

Similarly, we obtain that $\lim_{n\longrightarrow\infty}\|T_2x_n - x_n\| = \lim_{n\longrightarrow\infty}\|T_3x_n - x_n\| = 0$ □

Now we give a weak convergence theorem for iteration (5.2.4).

Theorem 5.2.4. *Let X be a uniformly convex Banach space which satisfies Opial's condition, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $k_n, l_n, m_n \in [0,\infty)$, $\lim_{n\longrightarrow\infty} k_n = 1, \lim_{n\longrightarrow\infty} l_n = 1, \lim_{n\longrightarrow\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, respectively and $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) If $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$,

(ii) If $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$,

(iii) If $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined as in (5.2.4). If $F \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3.

Proof. Let $p \in F$. By Lemma 5.2.1, $\lim_{n\longrightarrow\infty}\|x_n - p\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F. To prove this, let p_1 and p_2 be weak limits of subsequence $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Lemma 5.2.3, we have $\lim_{n\longrightarrow\infty}\|T_1x_n - x_n\| = 0$. By Lemma 2.1.33, we have $I - T_1, I - T_2$ and $I - T_3$ are demiclosed with respect to zero, therefore $T_1p_i = p_i, T_2p_i = p_i$ and $T_3p_i = p_i$, (i=1, 2) hence $p_1, p_2 \in F$. By Lemma 5.2.1 $\lim_{n\longrightarrow\infty}\|x_n - p_1\|$ and $\lim_{n\longrightarrow\infty}\|x_n - p_2\|$ exist. Using Lemma 2.1.34 we obtain that $p_1 = p_2$. Hence $\{x_n\}$ converges weakly to a common fixed point of T_1, T_2 and T_3. □

Our next goal is to prove a strong convergence theorem:

Theorem 5.2.5. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $k_n, l_n, m_n \in [0, \infty)$, $\lim_{n\longrightarrow\infty} k_n = 1, \lim_{n\longrightarrow\infty} l_n = 1, \lim_{n\longrightarrow\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, respectively and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ such that $a_n + b_n$, $c_n + d_n$ and $\alpha_n + \beta_n$ are in $[0,1]$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences defined as in (5.2.4). If $F \neq \emptyset$ and one of T_1, T_2 and T_3 is completely continuous, then $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to a common fixed point of T_1, T_2 and T_3.

Proof. By Lemma 5.2.1, $\{x_n\}$ is bounded. In addition, by Lemma 5.2.3, $lim_{n\longrightarrow\infty} \|x_n - T_1x_n\| = 0$, $lim_{n\longrightarrow\infty}\|x_n - T_2x_n\| = 0$ and $\lim_{n\longrightarrow\infty} \|x_n - T_3x_n\| = 0$, then $\{T_1x_n\}$, $\{T_2x_n\}$ and $\{T_3x_n\}$ are also bounded. If T_1 is completely continuous, there exists subsequence $\{T_1x_{n_j}\}$ of $\{T_1x_n\}$ such that $T_1x_{n_j} \longrightarrow p$ as $j \longrightarrow \infty$. Thus $\lim_{j\longrightarrow\infty} \|x_{n_j} - T_1x_{n_j}\| = \lim_{j\longrightarrow\infty} \|x_{n_j} - T_2x_{n_j}\| = \lim_{j\longrightarrow\infty} \|x_{n_j} - T_3x_{n_j}\| = 0$. It follows that $\lim_{j\longrightarrow\infty} \|x_{n_j} - p\| = 0$. This implies by Lemma 2.1.33 that $p \in F$. Furthermore, by Lemma 5.2.1, we get that $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists. Thus $lim_{n\longrightarrow\infty}\|x_n - p\| = 0$. □

Theorem 5.2.6. *Let X be a uniformly convex Banach space, and let C be a nonempty closed and convex subset of X. Let T_1, T_2, T_3 be three asymptotically nonexpansive self-mapping of C with $k_n, l_n, m_n \in [0, \infty)$, $\lim_{n\longrightarrow\infty} k_n = 1, \lim_{n\longrightarrow\infty} l_n = 1, \lim_{n\longrightarrow\infty} m_n = 1$, $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, $\sum_{n=1}^{\infty}(m_n - 1) < \infty$, respectively and let $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in*

$[0,1]$ *such that* $a_n + b_n$, $c_n + d_n$ *and* $\alpha_n + \beta_n$ *are in* $[0,1]$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$ *and*

(i) $0 < \liminf_{n\longrightarrow\infty} \alpha_n \leq \limsup_{n\longrightarrow\infty}(\alpha_n + \beta_n) < 1$,

(ii) $0 < \liminf_{n\longrightarrow\infty} c_n \leq \limsup_{n\longrightarrow\infty}(c_n + d_n) < 1$, *and*

(iii) $0 < \liminf_{n\longrightarrow\infty} a_n \leq \limsup_{n\longrightarrow\infty}(a_n + b_n) < 1$.

For a given $x_1 \in C$, *let* $\{x_n\}, \{y_n\}, \{z_n\}$ *be the sequences defined as in (5.2.4). If* $F \neq \emptyset$ *and one of* T_1, T_2 *and* T_3 *is demicompact, then* $\{x_n\}, \{y_n\}, \{z_n\}$ *converge strongly to a common fixed point of* T_1, T_2 *and* T_3.

Proof. Suppose that T_1 is semicompact. By Lemma 5.2.3, $\lim_{n\longrightarrow\infty} \|x_n - T_1x_n\| = \lim_{n\longrightarrow\infty} \|x_n - T_2x_n\| = \lim_{n\longrightarrow\infty} \|x_n - T_3x_n\| = 0$. Then there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $p \in C$. It follows from Lemma 2.1.33 that $p \in F$. Since $\lim_{n\longrightarrow\infty} \|x_n - p\|$ exists by Lemma 5.2.1, it follows that $\lim_{n\longrightarrow\infty} \|x_n - p\| = 0$. □