CHAPTER II

PRELIMINARIES

In this chapter, we give precise definitions, notations and basic results which will be used in Chapter III, Chapter IV, Chapter V, Chapter VI and Chapter VII. Moreover, many examples are provided.

We give the following definitions of a Γ-semigroup.

Let M and Γ be any two nonempty sets. A set M is called a ***Γ-semigroup*** [1] if there exists a mapping $\cdot : M \times \Gamma \times M \to M$, written as $(a, \gamma, b) \mapsto a\gamma b$, satisfying the following identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. A Γ-semigroup M is called ***commutative*** if $a\gamma b = b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$. A nonempty subset K of a Γ-semigroup M is called a ***sub-Γ-semigroup*** of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. For subsets A and B of M, let $A\Gamma B := \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma\}$. We also write $a\Gamma B, A\Gamma b$ and $a\Gamma b$ for $\{a\}\Gamma B, A\Gamma\{b\}$ and $\{a\}\Gamma\{b\}$, respectively. For $\gamma \in \Gamma$, let $A\gamma B := \{a\gamma b \mid a \in A \text{ and } b \in B\}$. Also, $a\gamma B$ and $A\gamma b$ are defined similarly.

A partially ordered Γ-semigroup M is called an ***ordered Γ-semigroup*** (some authors called po-Γ-semigroup) if for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$. If $(M; \leq)$ is an ordered Γ-semigroup, and K is a sub-Γ-semigroup of M, then $(K; \leq)$ is an ordered Γ-semigroup. For an ordered Γ-semigroup M and $a \in M$, define $(a] := \{t \in M \mid t \leq a\}$, and for a subset H of M, define $(H] := \bigcup_{h \in H} (h]$, that is, $(H] = \{t \in M \mid t \leq h \text{ for some } h \in H\}$. Clearly, $H \subseteq (H] = ((H]]$.

For more examples of Γ-semigroups and ordered Γ-semigroups can be seen in [13, 14, 15, 16] and [17], respectively.

A nonempty subset I of a Γ-semigroup M is called a ***left (right) ideal*** of

M if $M\Gamma I \subseteq I$ ($I\Gamma M \subseteq I$). A nonempty subset I of M is called an ***ideal*** of M if it is both a left ideal and a right ideal. An ideal I of M is called

a ***prime ideal*** of M if $a\Gamma b \subseteq I$ implies $a \in I$ or $b \in I$,

an ***s-prime ideal*** of M if $a\gamma b \in I$ implies $a \in I$ or $b \in I$,

a ***semiprime ideal*** of M if $a\Gamma a \subseteq I$ implies $a \in I$,

an ***s-semiprime ideal*** of M if $a\gamma a \in I$ implies $a \in I$

for all $a, b \in M$ and $\gamma \in \Gamma$. Then we have that I is a prime ideal of M if for any $A, B \subseteq M$, $A\Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. Analogous results hold if I is an s-prime ideal, a semiprime ideal and an s-semiprime ideal of M. For a Γ-semigroup, the following implications hold:

$$\begin{array}{ccc} s\text{-prime ideal} & \Rightarrow & \text{prime ideal} \\ \Downarrow & & \Downarrow \\ s\text{-semiprime ideal} & \Rightarrow & \text{semiprime ideal} \end{array}$$

Let

$$P(M) := \{A \mid A \text{ is a prime ideal of } M\},$$
$$SP(M) := \{A \mid A \text{ is an } s\text{-prime ideal of } M\}.$$

The intersection of all ideals of a Γ-semigroup M containing a nonempty subset A of M is called the ***ideal of M generated by A***, and denoted by $I(A)$. We also write $I(x)$ for $I(\{x\})$. A relation ρ on a Γ-semigroup M is called ***compatible*** if for any $a, b, c \in M$ and $\gamma \in \Gamma$, $(a, b) \in \rho$ implies $(a\gamma c, b\gamma c) \in \rho$ and $(c\gamma a, c\gamma b) \in \rho$. An equivalence relation ρ on a Γ-semigroup M is called a ***congruence*** on M if for any $a, b, c \in M$ and $\gamma \in \Gamma$, $(a, b) \in \rho$ implies $(a\gamma c, b\gamma c) \in \rho$ and $(c\gamma a, c\gamma b) \in \rho$. If ρ is a congruence on a Γ-semigroup M, then $M/\rho := \{(x)_\rho \mid x \in M\}$ is a Γ-semigroup with $(x)_\rho\gamma(y)_\rho = (x\gamma y)_\rho$ for all $x, y \in M$ and $\gamma \in \Gamma$ where $(x)_\rho := \{x' \in M \mid (x, x') \in \rho\}$. A congruence ρ on M is called a ***band congruence*** on

M if for any $a \in M$ and $\gamma \in \Gamma, (a\gamma a, a) \in \rho$. A band congruence ρ on M is called a ***semilattice congruence*** on M if $(a\gamma b, b\gamma a) \in \rho$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Example 2.1.([13]) Let $M = \{a, b, c, d\}$ and $\Gamma = \{\gamma\}$ with

$$x\gamma y = \begin{cases} b & \text{if } x, y \in \{a, b\}, \\ c & \text{otherwise.} \end{cases}$$

Then M is a Γ-semigroup. We can easily get all ideals of M as follows:

$$P_1 = M, P_2 = \{c, d\}, P_3 = \{b, c\}, P_4 = \{c\}, P_5 = \{a, b, c\}, P_6 = \{b, c, d\}.$$

It is easy to see that P_1 and P_2 are s-prime ideals of M, so P_1 and P_2 are semiprime ideals of M. Let

$$\begin{aligned} \rho_1 &= M \times M, \\ \rho_2 &= \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\}. \end{aligned}$$

It is easy to see that ρ_1 and ρ_2 are semilattice congruences on M.

Example 2.2. For $n \in \{1, 2\}$, let $M = \{n, n+1, n+2, ...\}$ and $\Gamma = \{-n\}$. Then M is a Γ-semigroup under usual addition. Let $I = \{2n, 2n+1, 2n+2, ...\}$. It is easy to verify that I is a semiprime ideal of M, and $\rho = \{(n, n)\}$ is a semilattice congruence on M.

If I is an ideal of a Γ-semigroup M and $A \subseteq M$, then

$$\langle A, I \rangle \quad := \quad \{x \in M \mid A\Gamma x \subseteq I\} \tag{2.1}$$

contains I and it is called the ***extension*** of I by A. Also, let $\langle a, I \rangle$ stand for $\langle \{a\}, I \rangle$. For an ideal I of a Γ-semigroup M, define the equivalence relation ϕ_I on M as follows:

$$\phi_I \quad := \quad \{(x, y) \mid \langle x, I \rangle = \langle y, I \rangle\}. \tag{2.2}$$

An ideal I of an ordered Γ-semigroup M is called an ***ordered ideal*** of M if for any $b \in M$ and $a \in I, b \leq a$ implies $b \in I$. An ordered ideal I of M is called

an ***ordered prime ideal*** of M if I is a prime ideal of M,

an ***ordered s-prime ideal*** of M if I is an s-prime ideal of M,

an ***ordered semiprime ideal*** of M if I is a semiprime ideal of M,

an ***ordered s-semiprime ideal*** of M if I is an s-semiprime ideal of M.

Then we have that I is an ordered prime ideal of M if for any $A, B \subseteq M, A\Gamma B \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. Analogous results hold if I is an ordered s-prime ideal, an ordered semiprime ideal and an ordered s-semiprime ideal of M. For an ordered Γ-semigroup, the following implications hold:

$$\begin{array}{ccc} \text{ordered } s\text{-prime ideal} & \Rightarrow & \text{ordered prime ideal} \\ \Downarrow & & \Downarrow \\ \text{ordered } s\text{-semiprime ideal} & \Rightarrow & \text{ordered semiprime ideal} \end{array}$$

Let

$$\begin{aligned} OP(M) &:= \{A \mid A \text{ is an ordered prime ideal of } M\}, \\ OSP(M) &:= \{A \mid A \text{ is an ordered } s\text{-prime ideal of } M\}. \end{aligned}$$

The intersection of all ordered ideals of an ordered Γ-semigroup M containing a nonempty subset A of M is called the ***ordered ideal of M generated by A***, and denoted by $OI(A)$. We also write $OI(x)$ for $OI(\{x\})$. A semilattice congruence ρ on an ordered Γ-semigroup M is called an ***ordered semilattice congruence*** on M if for any $a, b \in M$ and $\gamma \in \Gamma, a \leq b$ implies $(a, a\gamma b) \in \rho$. From (2.1), if I is an ordered ideal of an ordered Γ-semigroup M and $A \subseteq M$, then the extension of I by A will be denoted by $\langle\langle A, I\rangle\rangle$. Notice that $A\Gamma\langle\langle A, I\rangle\rangle \subseteq I$ and for $B \subseteq M, A\Gamma B \subseteq I$ implies $B \subseteq \langle\langle A, I\rangle\rangle$. In fact, $\langle\langle A, I\rangle\rangle$ is an ordered ideal of M containing I if M is commutative. Since $A\Gamma(\langle\langle A, I\rangle\rangle\Gamma M) = (A\Gamma\langle\langle A, I\rangle\rangle)\Gamma M \subseteq I\Gamma M \subseteq I$, it follows that $\langle\!\langle A, I\rangle\!\rangle \Gamma M \subseteq \langle\!\langle A, I\rangle\!\rangle$. If $x \in \langle\!\langle A, I\rangle\!\rangle$ and $y \in M$ are such that $y \leq x$, then $a\gamma y \leq a\gamma x \in I$ for all $a \in A$ and $\gamma \in \Gamma$ which implies that $A\Gamma y \subseteq I$. Thus $y \in \langle\!\langle A, I\rangle\!\rangle$. From (2.2), if I is an ordered ideal of an ordered Γ-semigroup M, then the equivalence relation Φ_I on M is

$$\Phi_I \;:=\; \{(x,y) \mid \langle\!\langle x, I\rangle\!\rangle = \langle\!\langle y, I\rangle\!\rangle\}.$$

Let n be any integer such that $n \geq 2$. For any subsets $A_1, A_2, \ldots, A_{n-1}$ and A_n of a Γ-semigroup M, and let i be an integer such that $2 \leq i \leq n-1$. Define

$$\begin{aligned} \widehat{A}_{(1;n)} &:= A_2\Gamma A_3 \ldots A_{n-1}\Gamma A_n, \\ \widehat{A}_{(i;n)} &:= A_1\Gamma A_2 \ldots A_{i-1}\Gamma A_{i+1}\Gamma A_{i+2} \ldots A_{n-1}\Gamma A_n, \\ \widehat{A}_{(n;n)} &:= A_1\Gamma A_2 \ldots A_{n-2}\Gamma A_{n-1}. \end{aligned}$$

An ideal I of a Γ-semigroup M is called an ***n-prime ideal*** of M if for any subsets $A_1, A_2, \ldots, A_{n-1}$ and A_n of M, $A_1\Gamma A_2 \ldots A_{n-1}\Gamma A_n \subseteq I$ implies that there exists an integer i $(1 \leq i \leq n)$ such that

$$\widehat{A}_{(1;n)}, \widehat{A}_{(2;n)}, \ldots, \widehat{A}_{(i-1;n)}, \widehat{A}_{(i+1;n)}, \widehat{A}_{(i+2;n)}, \ldots, \widehat{A}_{(n;n)} \subseteq I.$$

An ideal I of a Γ-semigroup M is called an ***n-semiprime ideal*** of M if for any subsets $A_1, A_2, \ldots, A_{n-1}$ and A_n of M with $A_1 = A_2 = \ldots = A_n$, $A_1\Gamma A_2 \ldots A_{n-1}\Gamma A_n \subseteq I$ implies $\widehat{A}_{(n;n)} \subseteq I$. For a Γ-semigroup, we have the following statements:

(*a*) Every prime ideal is a semiprime ideal.

(*b*) Every n-prime ideal is an n-semiprime ideal.

(*c*) The prime ideals and the 2-prime ideals coincide.

(*d*) The semiprime ideals and the 2-semiprime ideals coincide.

An ordered ideal I of an ordered Γ-semigroup M is called an ***ordered n-prime ideal*** of M if for any subsets $A_1, A_2, \ldots, A_{n-1}$ and A_n of M, $A_1\Gamma A_2 \ldots A_{n-1}\Gamma A_n \subseteq I$ implies that there exists an integer i $(1 \leq i \leq n)$ such that

$$\widehat{A}_{(1;n)}, \widehat{A}_{(2;n)}, \ldots, \widehat{A}_{(i-1;n)}, \widehat{A}_{(i+1;n)}, \widehat{A}_{(i+2;n)}, \ldots, \widehat{A}_{(n;n)} \subseteq I.$$

An ordered ideal I of an ordered Γ-semigroup M is called an ***ordered n-semiprime ideal*** of M if for any subsets $A_1, A_2, \ldots, A_{n-1}$ and A_n of M with $A_1 = A_2 = \ldots = A_n$, $A_1\Gamma A_2 \ldots A_{n-1}\Gamma A_n \subseteq I$ implies $\widehat{A}_{(n;n)} \subseteq I$. For an ordered Γ-semigroup, we have the following statements:

(*a*) Every ordered prime ideal is an ordered semiprime ideal.

(*b*) Every n-ordered prime ideal is an n-ordered semiprime ideal.

(*c*) The ordered prime ideals and the 2-ordered prime ideals coincide.

(*d*) The ordered semiprime ideals and the 2-ordered semiprime ideals coincide.

A Γ-semigroup M is called ***r (l)-archimedean*** if for any $a, b \in M, b \in a\Gamma M$ ($b \in M\Gamma a$) or there exists an integer $m \geq 2$ such that $b\gamma_1 b\gamma_2 b \ldots b\gamma_{m-1} b \in a\Gamma M$ ($b\gamma_1 b\gamma_2 b \ldots b\gamma_{m-1} b \in M\Gamma a$) for some $\gamma_1, \gamma_2, \ldots, \gamma_{m-1} \in \Gamma$. A Γ-semigroup M is called ***archimedean*** if it is both l-archimedean and r-archimedean. A sub-Γ-semigroup (left ideal) T of M is called an ***r-archimedean sub-Γ-semigroup (left ideal)*** of M if T is r-archimedean. A Γ-semigroup M is called a ***band of r-archimedean sub-Γ-semigroups (left ideals) of M*** if there exists a band congruence ρ on M such that the ρ-class $(x)_\rho$ of M containing x is an r-archime- dean sub-Γ-semigroup (left ideal) of M for all $x \in M$. A sub-Γ-semigroup (left ideal) T of M is called a ***weakly r-archimedean sub-Γ-semigroup (left ideal)*** of M if for any $a, b \in T, b \in a\Gamma M$ or there exists an integer $m \geq 2$ such that $b\gamma_1 b\gamma_2 b \ldots b\gamma_{m-1} b \in a\Gamma M$ for some $\gamma_1, \gamma_2, \ldots, \gamma_{m-1} \in \Gamma$. A Γ-semigroup M is called a ***band of weakly r-archimedean sub-Γ-semigroups (left ideals) of M*** if there exists a band congruence ρ on M such that the ρ-class $(x)_\rho$ of M containing x is a weakly r-archimedean sub-Γ-semigroup (left ideal) of M for all $x \in M$.

In the sequel, the following relations on M are used frequently:

$$\eta_r := \{(a,b) \mid b \in a \cup a\Gamma M \text{ or there exists an integer } m \geq 2 \text{ such that } b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in a \cup a\Gamma M \text{ for some } \alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma\},$$

$$\eta_l := \{(a,b) \mid b \in a \cup M\Gamma a \text{ or there exists an integer } m \geq 2 \text{ such that } b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in a \cup M\Gamma a \text{ for some } \alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma\}.$$

An ordered Γ-semigroup M is called ***negative*** if for any $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma b \leq a$ and $a\gamma b \leq b$. An ordered Γ-semigroup M is called ***r (l)-archimedean*** if for any $a, b \in M, b \in (a\Gamma M]$ ($b \in (M\Gamma a]$) or there exists an integer $m \geq 2$ such that $b\gamma_1 b\gamma_2 b \ldots b\gamma_{m-1} b \in (a\Gamma M]$ ($b\gamma_1 b\gamma_2 b \ldots b\gamma_{m-1} b \in (M\Gamma a]$) for some $\gamma_1, \gamma_2, \ldots, \gamma_{m-1} \in \Gamma$. An ordered Γ-semigroup M is called ***archimedean*** if it is both l-archimedean and r-archimedean. A sub-Γ-semigroup T of M is called an ***r-archimedean sub-Γ-semigroup*** of M if T is r-archimedean. An ordered Γ-semigroup M is called a ***band of r-archimedean sub-Γ-semigroups of M*** if there exists a band congruence ρ on M such that the ρ-class $(x)_\rho$ of M containing x is an r-archimedean sub-Γ-semigroup of M for all $x \in M$. A sub-Γ-semigroup T of M is called a ***weakly r-archimedean sub-Γ-semigroup*** of M if for any $a, b \in T, b \in (a\Gamma M]$ or there exists an integer $m \geq 2$ such that $b\gamma_1 b\gamma_2 b \ldots b\gamma_{m-1} b \in (a\Gamma M]$ for some $\gamma_1, \gamma_2, \ldots, \gamma_{m-1} \in \Gamma$. An ordered Γ-semigroup M is called a ***band of weakly r-archimedean sub-Γ-semigroups of M*** if there exists a band congruence ρ on M such that the ρ-class $(x)_\rho$ of M containing x is a weakly r-archimedean sub-Γ-semigroup of M for all $x \in M$.

A congruence ρ on an ordered Γ-semigroup M is called a ***regular congruence*** on M if there exists an order "$\preceq$" on M/ρ such that:

(*i*) $(M/\rho; \preceq)$ is an ordered Γ-semigroup with $(x)_\rho\gamma(y)_\rho = (x\gamma y)_\rho$ for all $x, y \in M$ and $\gamma \in \Gamma$.

(*ii*) The mapping $\varphi : M \to M/\rho$ defined by $\varphi(x) = (x)_\rho$ for all $x \in M$ is isotone.

We call the order " $\preceq$ " on M/ρ a ***regular order*** with respect to a regular congruence ρ on M.

In the sequel, the following relations on M are used frequently:

$$\begin{aligned}
\bar{\eta}_r &:= \{(a,b) \mid b \in (a \cup a\Gamma M] \text{ or there exists an integer } m \geq 2 \text{ such that} \\
&\qquad b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in (a \cup a\Gamma M] \text{ for some } \alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma\}, \\
\bar{\eta}_l &:= \{(a,b) \mid b \in (a \cup M\Gamma a] \text{ or there exists an integer } m \geq 2 \text{ such that} \\
&\qquad b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in (a \cup M\Gamma a] \text{ for some } \alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma\}.
\end{aligned}$$

A sub-Γ-semigroup F of a Γ-semigroup M is called a ***filter*** of M if for any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in F$ implies $a, b \in F$. The intersection of all filters of a Γ-semigroup M containing a nonempty subset A of M is the ***filter of M generated by A***. For $A = \{x\}$, let $n(x)$ denote the filter of M generated by $\{x\}$.

For a nonempty subset A of a Γ-semigroup M, we define equivalence relations on M as follows:

$$\begin{aligned}
\sigma_A &:= \{(x,y) \mid x, y \in A \text{ or } x, y \notin A\}, \\
n &:= \{(x,y) \mid n(x) = n(y)\}.
\end{aligned}$$

For any congruence ρ on a Γ-semigroup M and $x \in M$, let

$f(x)_\rho$ denote the filter of M generated by ρ-class $(x)_\rho$,

t denote the filter of M generated by $\bigcup_{y \in (x)_\rho} n(y)$.

A Γ-semigroup M is called a ***semilattice of archimedean sub-Γ-semigroups of M*** if there exists a semilattice congruence ρ on M such that the ρ-class $(x)_\rho$ is an archimedean sub-Γ-semigroup of M for all $x \in M$. Let K be a sub-Γ-semigroup of a Γ-semigroup M. For $a, b \in K$, we define $a \mid_K b$ if $b = a$ or $b = a\gamma x$ for some $x \in K$ and $\gamma \in \Gamma$. If $K = M$, then we also write $a \mid_M b$ as $a \mid b$.

We define relations on a Γ-semigroup M as follows:

$$\begin{aligned}
\delta &:= \{(x,y) \mid y \mid x\},\\
\mu &:= \{(x,y) \mid x \mid y \text{ or } x \mid y\gamma_1 y\gamma_2 y...y\gamma_m y \text{ for some } m \in \mathbb{N}\\
&\quad \text{and } \gamma_1,\gamma_2,...,\gamma_m \in \Gamma\},\\
\eta &:= \mu \cap \mu^{-1}.
\end{aligned}$$

From the definition of archimedean Γ-semigroups, we see that a Γ-semigroup M is archimedean if for any $a, b \in M$, $a \mid b$ or $a \mid b\gamma_1 b\gamma_2 b...b\gamma_m b$ for some $m \in \mathbb{N}$ and $\gamma_1, \gamma_2, ..., \gamma_m \in \Gamma$, and $b \mid a$ or $b \mid a\beta_1 a\beta_2 a...a\beta_n a$ for some $n \in \mathbb{N}$ and $\beta_1, \beta_2, ..., \beta_n \in \Gamma$. Equivalently, $M \times M = \eta$.

A filter F of an ordered Γ-semigroup M is called an ***ordered filter*** of M if for any $b \in M$ and $a \in F, a \leq b$ implies $b \in F$. The intersection of all ordered filters of an ordered Γ-semigroup M containing a nonempty subset A of M is the ***ordered filter of M generated by A***. For $A = \{x\}$, let $N(x)$ denote the ordered filter of M generated by $\{x\}$. A relation ρ on an ordered Γ-semigroup M is called a ***pseudoorder*** on M if $\leq \subseteq \rho$, and ρ is transitive and compatible. A congruence ρ on an ordered Γ-semigroup M is called a ***complete semilattice congruence*** on M if for all $a, b \in M$ and $\gamma \in \Gamma, (a\gamma b, b\gamma a) \in \rho$, and for any $a, b \in M$ and $\gamma \in \Gamma, a \leq b$ implies $(a, a\gamma b) \in \rho$. If ρ is a complete semilattice congruence on an ordered Γ-semigroup M, then $a \leq a$ for all $a \in M$. Thus $(a, a\gamma a) \in \rho$ for all $a \in M$ and $\gamma \in \Gamma$. Hence complete semilattice congruences and ordered semilattice congruences coincide.

For an ordered Γ-semigroup M, we define equivalence relations on M as follows:

$$\mathcal{N} := \{(x,y) \mid N(x) = N(y)\}.$$

For any congruence ρ on an ordered Γ-semigroup M and $x \in M$, let

$F(x)_\rho$ denote the ordered filter of M generated by ρ-class $(x)_\rho$,

T denote the ordered filter of M generated by $\bigcup_{y \in (x)_\rho} N(y)$.

An ordered Γ-semigroup M is called an ***ordered semilattice of archimedean sub-Γ-semigroups of M*** if there exists an ordered semilattice congruence ρ on M such that the ρ-class $(x)_\rho$ is an archimedean sub-Γ-semigroup of M for all $x \in M$. Let K be a sub-Γ-semigroup of an ordered Γ-semigroup M. For $a, b \in K$, we define $a \parallel_K b$ if $b \leq a$ or $b \leq a\gamma x$ for some $x \in K$ and $\gamma \in \Gamma$. If $K = M$, then we also write $a \parallel_M b$ as $a \parallel b$. We define relations on an ordered Γ-semigroup M as follows:

$$\begin{aligned} \bar{\delta} &:= \{(x, y) \mid y \parallel x\}, \\ \bar{\mu} &:= \{(x, y) \mid x \parallel y \text{ or } x \parallel y\gamma_1 y\gamma_2 y ... y\gamma_m y \text{ for some } m \in \mathbb{N} \\ &\quad\ \text{and } \gamma_1, \gamma_2, ..., \gamma_m \in \Gamma\}, \\ \bar{\eta} &:= \bar{\mu} \cap \bar{\mu}^{-1}. \end{aligned}$$

From the definition of archimedean ordered Γ-semigroups, we see that an ordered Γ-semigroup M is archimedean if for any $a, b \in M, a \parallel b$ or $a \parallel b\gamma_1 b\gamma_2 b...b\gamma_m b$ for some $m \in \mathbb{N}$ and $\gamma_1, \gamma_2, ..., \gamma_m \in \Gamma$, and $b \parallel a$ or $b \parallel a\beta_1 a\beta_2 a...a\beta_n a$ for some $n \in \mathbb{N}$ and $\beta_1, \beta_2, ..., \beta_n \in \Gamma$. Equivalently, $M \times M = \bar{\eta}$.

Let $(M; \leq)$ and $(N; \leq')$ be ordered Γ-semigroups. A bijection $\varphi : M \to N$ is called an ***isomorphism*** if it satisfies the following conditions:

(*i*) For any $x, y \in M$, $x \leq y$ if and only if $\varphi(x) \leq' \varphi(y)$.

(*ii*) For any $x, y \in M$ and $\gamma \in \Gamma$, $\varphi(x\gamma y) = \varphi(x)\gamma\varphi(y)$.

We write in this case, $M \cong N$.

Now, let

$$\begin{aligned} SC(M) &:= \{\rho \mid \rho \text{ is a semilattice congruence on } M\}, \\ OSC(M) &:= \{\rho \mid \rho \text{ is an ordered semilattice congruence on } M\}, \\ RC(M) &:= \{\rho \mid \rho \text{ is a regular congruence on } M\}. \end{aligned}$$