CHAPTER III

(ORDERED) IDEAL EXTENSIONS IN (ORDERED) Γ-SEMIGROUPS

In this chapter, we divide into two sections, and many properties of extension $(\langle\langle A, I\rangle\rangle)$ $\langle A, I\rangle$ of an (ordered) ideal I by a subset A of a (ordered) Γ-semigroup M are provided. Moreover, the equivalence relation (Φ_I) ϕ_I on M defined by $((x,y) \in \Phi_I \Leftrightarrow \langle\langle x, I\rangle\rangle = \langle\langle y, I\rangle\rangle)$ $(x,y) \in \phi_I \Leftrightarrow \langle x, I\rangle = \langle y, I\rangle$ is considered. It will be shown that if M is commutative, and I is an (ordered) s-semiprime ideal of M, then (Φ_I) ϕ_I is a (ordered) semilattice congruence on M. In addition, if I is a (ordered) prime ideal of M, then (Φ_I) $\phi_I = \{(x,y) \mid x, y \in I \text{ or } x, y \notin I\}$.

3.1 Ideal Extensions in Γ-Semigroups

The following theorem is obtained in [13] and the following lemmas will be used frequently in this thesis.

Theorem 3.1.1. ([13]) *If M is a Γ-semigroup, then*

$$n = \bigcap_{I \in SP(M)} \sigma_I.$$

The next two lemmas are easy to verify.

Lemma 3.1.2. *If A is a subset of a Γ-semigroup M, then*

$$I(A) = A \cup M\Gamma A.$$

Lemma 3.1.3. *Let I be an ideal of a Γ-semigroup M and $A \subseteq B \subseteq M$. Then $\langle B, I\rangle \subseteq \langle A, I\rangle$.*

Lemma 3.1.4. *Let I be an ideal of a commutative Γ-semigroup M, $A \subseteq M$ and $\gamma \in \Gamma$. Then we have the following statements:*

(a) $\langle A, I\rangle$ *is an ideal of* M.

(b) $I \subseteq \langle A, I\rangle \subseteq \langle A\Gamma A, I\rangle \subseteq \langle A\gamma A, I\rangle$.

(c) *If* $A \subseteq I$, *then* $\langle A, I\rangle = M$.

Proof. (a) Let $x \in \langle A, I\rangle, y \in M$ and $\gamma \in \Gamma$. Then $A\Gamma(x\gamma y) = (A\Gamma x)\gamma y \subseteq I\Gamma M \subseteq I$, so $x\gamma y \in \langle A, I\rangle$. Hence $\langle A, I\rangle$ is an ideal of M.

(b) If $x \in I$, then $A\Gamma x \subseteq M\Gamma I \subseteq I$. Thus $x \in \langle A, I\rangle$. If $x \in \langle A, I\rangle$, then $(A\Gamma A)\Gamma x = A\Gamma(A\Gamma x) \subseteq M\Gamma I \subseteq I$. Thus $x \in \langle A\Gamma A, I\rangle$. If $x \in \langle A\Gamma A, I\rangle$, then $(A\gamma A)\Gamma x \subseteq (A\Gamma A)\Gamma x \subseteq I$. Thus $x \in \langle A\gamma A, I\rangle$. Hence $I \subseteq \langle A, I\rangle \subseteq \langle A\Gamma A, I\rangle \subseteq \langle A\gamma A, I\rangle$.

(c) Let $A \subseteq I$ and $x \in M$. Then $A\Gamma x \subseteq I\Gamma M \subseteq I$, so $x \in \langle A, I\rangle$. Hence $\langle A, I\rangle = M$.

Hence the proof is completed. □

Lemma 3.1.5. *Let* I *be an ideal of a* Γ*-semigroup* M *and* $A \subseteq M$. *Then*

$$\langle A, I\rangle = \bigcap_{a\in A}\langle a, I\rangle = \langle A \setminus I, I\rangle.$$

Proof. By Lemma 3.1.3, we have $\langle A, I\rangle \subseteq \bigcap_{a\in A}\langle a, I\rangle$. Now, let $x \in \bigcap_{a\in A}\langle a, I\rangle$. Then $a\Gamma x \subseteq I$ for all $a \in A$, so $A\Gamma x \subseteq I$. Thus $x \in \langle A, I\rangle$, so $\bigcap_{a\in A}\langle a, I\rangle \subseteq \langle A, I\rangle$. Hence $\langle A, I\rangle = \bigcap_{a\in A}\langle a, I\rangle$. By Lemma 3.1.4(c), we have $\langle A, I\rangle = \bigcap_{a\in A}\langle a, I\rangle = \langle A \setminus I, I\rangle$.

Hence the proof is completed. □

Lemma 3.1.6. *Let* I *be an ideal of a* Γ*-semigroup* M. *Then* I *is a prime ideal of* M *if and only if* $\langle A, I\rangle = I$ *for all* $A \not\subseteq I$.

Proof. Assume that I is a prime ideal of M and $A \nsubseteq I$. Let $x \in \langle A, I\rangle$. Then $A\Gamma x \subseteq I$. By hypothesis and $A \nsubseteq I$, $x \in I$. Thus $\langle A, I\rangle \subseteq I$. By Lemma 3.1.4($b$), $\langle A, I\rangle = I$.

Conversely, assume that $\langle A, I\rangle = I$ for all $A \nsubseteq I$. Let $A, B \subseteq M$ be such that $A\Gamma B \subseteq I$ and $A \nsubseteq I$. Then $B \subseteq \langle A, I\rangle = I$. Hence I is a prime ideal of M.

Hence the proof is completed. □

We can easily prove the last lemma.

Lemma 3.1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subfamilies of $P(M)$ and $SP(M)$, respectively. Then we have the following statements:*

(a) $\bigcap_{P\in\mathcal{A}} P$ *is a semiprime ideal of M if* $\bigcap_{P\in\mathcal{A}} P \neq \emptyset$.

(b) $\bigcup_{P\in\mathcal{B}} P$ *is a prime ideal of M.*

(c) $\bigcap_{P\in\mathcal{B}} P$ *is an s-semiprime ideal of M if* $\bigcap_{P\in\mathcal{B}} P \neq \emptyset$.

(d) $\bigcup_{P\in\mathcal{B}} P$ *is an s-prime ideal of M.*

We now give some characterizations of extensions of ideals and the main theorems of this section as below.

Theorem 3.1.8. *Let P be a prime ideal of a commutative Γ-semigroup M and $A \subseteq M$. Then $\langle A, P\rangle$ is a prime ideal of M. Furthermore, $\langle A, \bigcap_{P\in P(M)} P\rangle$ is a semiprime ideal of M if $\bigcap_{P\in P(M)} P \neq \emptyset$.*

Proof. If $A \subseteq P$, then it follows from Lemma 3.1.4(c) that $\langle A, P\rangle = M$. If $A \nsubseteq P$, then it follows from Lemma 3.1.6 that $\langle A, P\rangle = P$. Hence $\langle A, P\rangle$ is a prime ideal of M. Now,

$$\begin{aligned} x \in \langle A, \bigcap_{P\in P(M)} P\rangle &\Leftrightarrow A\Gamma x \subseteq \bigcap_{P\in P(M)} P \\ &\Leftrightarrow A\Gamma x \subseteq P \text{ for all } P \in P(M) \\ &\Leftrightarrow x \in \langle A, P\rangle \text{ for all } P \in P(M) \\ &\Leftrightarrow x \in \bigcap_{P\in P(M)} \langle A, P\rangle. \end{aligned}$$

Hence

$$\langle A, \bigcap_{P\in P(M)} P\rangle = \bigcap_{P\in P(M)} \langle A, P\rangle.$$

By Lemma 3.1.7(a), $\langle A, \bigcap_{P\in P(M)} P\rangle$ is a semiprime ideal of M. □

Theorem 3.1.9. *Let A and B be subsets of a Γ-semigroup M and $A \subseteq M\Gamma A$. Then $I(A) \subseteq I(B)$ if and only if for every ideal J of M, $\langle B, J\rangle \subseteq \langle A, J\rangle$.*

Proof. Assume that $I(A) \subseteq I(B)$. Let J be an ideal of M and $x \in \langle B, J\rangle$. By Lemma 3.1.2, we have $A \subseteq I(B) = B \cup M\Gamma B$. For any $a \in A$, if $a = y\alpha b$ for some $y \in M, b \in B$ and $\alpha \in \Gamma$, then $a\gamma x = (y\alpha b)\gamma x = y\alpha(b\gamma x) \in M\Gamma J \subseteq J$ for all $\gamma \in \Gamma$. Hence $a\gamma x \in J$ for all $\gamma \in \Gamma$, so $x \in \langle a, J\rangle$. If $a = b$ for some $b \in B$, then $a\gamma x = b\gamma x \in J$ for all $\gamma \in \Gamma$. Hence $a\gamma x \in J$ for all $\gamma \in \Gamma$, so $x \in \langle a, J\rangle$. Therefore $\langle B, J\rangle \subseteq \bigcap_{a\in A}\langle a, J\rangle$. It follows from Lemma 3.1.5 that $\langle B, J\rangle \subseteq \langle A, J\rangle$.

Conversely, assume that $\langle B, J\rangle \subseteq \langle A, J\rangle$ for all ideal J of M. Then $\langle B, I(B)\rangle \subseteq \langle A, I(B)\rangle$. Since $B \subseteq I(B)$, it follows from Lemma 3.1.4(c) that $\langle B, I(B)\rangle = M$. Thus $\langle A, I(B)\rangle = M$, so $M\Gamma A \subseteq I(B)$. Hence $A \subseteq M\Gamma A \subseteq I(B)$. This implies that $I(A) \subseteq I(B)$. □

Theorem 3.1.10. *If I is an s-semiprime ideal of a commutative Γ-semigroup M, then ϕ_I is a semilattice congruence on M.*

Proof. Let $(x, y) \in \phi_I, c \in M$ and $\gamma \in \Gamma$. Then $\langle x, I\rangle = \langle y, I\rangle$. Thus

$$\begin{aligned} a \in \langle x\gamma c, I\rangle &\Leftrightarrow (x\gamma c)\Gamma a \subseteq I \\ &\Leftrightarrow x\Gamma(c\gamma a) \subseteq I \\ &\Leftrightarrow c\gamma a \in \langle x, I\rangle \\ &\Leftrightarrow c\gamma a \in \langle y, I\rangle \\ &\Leftrightarrow y\Gamma(c\gamma a) \subseteq I \\ &\Leftrightarrow (y\gamma c)\Gamma a \subseteq I \\ &\Leftrightarrow a \in \langle y\gamma c, I\rangle. \end{aligned}$$

Hence $(x\gamma c, y\gamma c) \in \phi_I$. Similarly, we can show that $(c\gamma x, c\gamma y) \in \phi_I$. Hence ϕ_I is a congruence on M. Let $x \in M$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} a \in \langle x\gamma x, I\rangle &\Rightarrow (x\gamma x)\Gamma a \subseteq I \\ &\Rightarrow (x\gamma x\Gamma a)\Gamma a \subseteq I\Gamma M \subseteq I \\ &\Rightarrow (x\Gamma a)\gamma(x\Gamma a) \subseteq I \\ &\Rightarrow x\Gamma a \subseteq I \\ &\Rightarrow a \in \langle x, I\rangle. \end{aligned}$$

Thus $\langle x\gamma x, I\rangle \subseteq \langle x, I\rangle$. By Lemma 3.1.4($b$), we have $\langle x, I\rangle \subseteq \langle x\gamma x, I\rangle$. Hence $\langle x\gamma x, I\rangle = \langle x, I\rangle$, so $(x\gamma x, x) \in \phi_I$. Therefore ϕ_I is a semilattice congruence on M. □

Theorem 3.1.11. *If I is an s-prime ideal of a Γ-semigroup M, then $\phi_I = \sigma_I$ and $n \subseteq \phi_I$.*

Proof. Let $(x, y) \in \phi_I$. Then $\langle x, I\rangle = \langle y, I\rangle$. Suppose that $(x, y) \notin \sigma_I$. Without loss of generality, we may assume that $x \in I$ but $y \notin I$. By Lemmas 3.1.4(c) and 3.1.6, we have $\langle x, I\rangle = M$ and $\langle y, I\rangle = I$. Thus $I = M$, so $y \notin M$. This is a contradiction. Hence $(x, y) \in \sigma_I$, so $\phi_I \subseteq \sigma_I$. Let $(x, y) \in \sigma_I$. If $x \in I$, then $y \in I$. By Lemma 3.1.4(c), $\langle x, I \rangle = M = \langle y, I \rangle$. If $x \notin I$, then $y \notin I$. By Lemma 3.1.6, $\langle x, I \rangle = I = \langle y, I \rangle$. Hence $(x, y) \in \phi_I$, so $\sigma_I \subseteq \phi_I$. Therefore $\phi_I = \sigma_I$. It follows from Theorem 3.1.1 that

$$n = \bigcap_{J \in SP(M)} \sigma_J = \bigcap_{J \in SP(M)} \phi_J \subseteq \phi_I.$$

Hence the proof is completed. □

3.2 Ordered Ideal Extensions in Ordered Γ-Semigroups

Our purpose is to provide various properties of extensions of ordered ideals of an ordered Γ-semigroup M. The following lemma is evident.

Lemma 3.2.1. *Let I be an ordered ideal of an ordered Γ-semigroup M and $A, B \subseteq M$. Then the following statements hold:*

(a) *If $A \subseteq B$, then $\langle\langle B, I \rangle\rangle \subseteq \langle\langle A, I \rangle\rangle$.*

(b) *If $A \subseteq I$, then $\langle\langle A, I \rangle\rangle = M$.*

(c) $\langle\langle A, I \rangle\rangle \subseteq \langle\langle A \setminus I, I \rangle\rangle$.

Proposition 3.2.2. *Let I be an ordered ideal of an ordered Γ-semigroup M. Then for any $a \in M$,*

$$\langle\langle (a], I \rangle\rangle = \langle\langle a, I \rangle\rangle \subseteq \langle\langle x, I \rangle\rangle \text{ for all } x \leq a.$$

Proof. By Lemma 3.2.1(a), $\langle\langle (a], I \rangle\rangle \subseteq \langle\langle a, I \rangle\rangle$. If $y \in \langle\langle a, I \rangle\rangle$ and $z \in (a]$, then $z \leq a$, so $z\gamma y \leq a\gamma y \in a\Gamma y \subseteq I$ for all $\gamma \in \Gamma$. Hence $y \in \langle\langle (a], I \rangle\rangle$. Therefore we have $\langle\langle (a], I \rangle\rangle = \langle\langle a, I \rangle\rangle$. If $x \leq a$, then $x \in (a]$, so by Lemma 3.2.1(a), $\langle\langle (a], I \rangle\rangle \subseteq \langle\langle x, I \rangle\rangle$. □

Proposition 3.2.3. *If I is an ordered ideal of an ordered Γ-semigroup M and $A \subseteq M$, then*

$$\langle\!\langle A, I \rangle\!\rangle \subseteq \langle\!\langle A\Gamma A, I \rangle\!\rangle \subseteq \langle\!\langle A\Gamma' A, I \rangle\!\rangle \text{ for all } \Gamma \subseteq \Gamma'.$$

Proof. Since $(A\Gamma A)\Gamma\langle\!\langle A, I\rangle\!\rangle = A\Gamma(A\Gamma\langle\!\langle A, I\rangle\!\rangle) \subseteq A\Gamma I \subseteq I$ and $(A\Gamma' A)\Gamma\langle\!\langle A\Gamma A, I\rangle\!\rangle \subseteq (A\Gamma A)\Gamma\langle\!\langle A\Gamma A, I\rangle\!\rangle \subseteq I$, it follows that $\langle\!\langle A, I\rangle\!\rangle \subseteq \langle\!\langle A\Gamma A, I\rangle\!\rangle$ and $\langle\!\langle A\Gamma A, I\rangle\!\rangle \subseteq \langle\!\langle A\Gamma' A, I\rangle\!\rangle$, respectively. □

Proposition 3.2.4. *Let I and I_i be ordered ideals of an ordered Γ-semigroup M and $A, A_i \subseteq M$ for all $i \in \Lambda$. Then*

(a) $\langle\!\langle A, \bigcap_{i\in\Lambda} I_i \rangle\!\rangle = \bigcap_{i\in\Lambda} \langle\!\langle A, I_i \rangle\!\rangle$ *and*

(b) $\langle\!\langle \bigcup_{i\in\Lambda} A_i, I \rangle\!\rangle = \bigcap_{i\in\Lambda} \langle\!\langle A_i, I \rangle\!\rangle$.

Proof. For $x \in M$,

$$\begin{aligned} x \in \langle\!\langle A, \bigcap_{i\in\Lambda} I_i \rangle\!\rangle &\Leftrightarrow A\Gamma x \subseteq \bigcap_{i\in\Lambda} I_i \\ &\Leftrightarrow A\Gamma x \subseteq I_i \text{ for all } i \in \Lambda \\ &\Leftrightarrow x \in \langle\!\langle A, I_i \rangle\!\rangle \text{ for all } i \in \Lambda \\ &\Leftrightarrow x \in \bigcap_{i\in\Lambda} \langle\!\langle A, I_i \rangle\!\rangle, \end{aligned}$$

and

$$\begin{aligned} x \in \langle\!\langle \bigcup_{i\in\Lambda} A_i, I \rangle\!\rangle &\Leftrightarrow (\bigcup_{i\in\Lambda} A_i)\Gamma x \subseteq I \\ &\Leftrightarrow A_i\Gamma x \subseteq I \text{ for all } i \in \Lambda \\ &\Leftrightarrow x \in \langle\!\langle A_i, I \rangle\!\rangle \text{ for all } i \in \Lambda \\ &\Leftrightarrow x \in \bigcap_{i\in\Lambda} \langle\!\langle A_i, I \rangle\!\rangle. \end{aligned}$$

□

Proposition 3.2.5. *Let I be an ordered ideal of an ordered Γ-semigroup M. Then I is an ordered prime ideal of M if and only if $\langle\langle A, I\rangle\rangle = I$ for all $A \subseteq M$ with $A \not\subseteq I$.*

Proof. Assume that I is an ordered prime ideal of M, and let $A \not\subseteq I$. Since $A\Gamma\langle\langle A, I\rangle\rangle \subseteq I, A \not\subseteq I$ and I is an ordered prime ideal of M, it follows that $\langle\langle A, I\rangle\rangle \subseteq I$ which implies that $\langle\langle A, I\rangle\rangle = I$.

Conversely, assume that $\langle\langle A, I\rangle\rangle = I$ for all $A \not\subseteq I$. To show that I is an ordered prime ideal of M, let $A, B \subseteq M$ be such that $A\Gamma B \subseteq I$ and $A \not\subseteq I$. Then $B \subseteq \langle\langle A, I\rangle\rangle = I$. □

Recall that for an ordered ideal I of an ordered Γ-semigroup M and $A \subseteq M$, $\langle\langle A, I\rangle\rangle$ is an ordered ideal of M if M is commutative.

Corollary 3.2.6. *Assume that M is a commutative ordered Γ-semigroup. If I is an ordered prime ideal of M and $A \subseteq M$, then so is $\langle\langle A, I\rangle\rangle$.*

Proof. This follows directly from Lemma 3.2.1(b) and Proposition 3.2.5. □

It is obviously seen that a nonempty intersection of ordered prime ideals of an ordered Γ-semigroup M is an ordered semiprime ideal of M.

Corollary 3.2.7. *Assume that M is a commutative ordered Γ-semigroup and $A \subseteq M$. If $\{I_i \mid i \in \Lambda\}$ is a collection of ordered prime ideals of M such that $\bigcap_{i\in\Lambda} I_i \neq \emptyset$, then $\langle\langle A, \bigcap_{i\in\Lambda} I_i\rangle\rangle$ is an ordered semiprime ideal of M.*

Proof. By Corollary 3.2.6, $\langle\langle A, I_i\rangle\rangle$ is an ordered prime ideal of M for all $i \in \Lambda$. But $\langle\langle A, \bigcap_{i\in\Lambda} I_i\rangle\rangle = \bigcap_{i\in\Lambda}\langle\langle A, I_i\rangle\rangle$ by Proposition 3.2.4(a). It follows that $\langle\langle A, \bigcap_{i\in\Lambda} I_i\rangle\rangle$ is an ordered semiprime ideal of M. □

Proposition 3.2.8. *For $A, B \subseteq M$ where M is an ordered Γ-semigroup, if $OI(A) \subseteq OI(B)$, then $\langle\langle B, I\rangle\rangle \subseteq \langle\langle A, I\rangle\rangle$ for every ordered ideal I of M.*

Proof. Assume that $OI(A) \subseteq OI(B)$ and let I be an ordered ideal of M. If $x \in \langle\langle B, I\rangle\rangle$, then $B\Gamma x \subseteq I$. Since $A \subseteq OI(B)$, it follows from Lemma 3.2.1(a) that $\langle\langle OI(B), I\rangle\rangle \subseteq \langle\langle A, I\rangle\rangle$. By Lemma 3.2.1(a) and Propositions 3.2.2 and 3.2.4(b),

$$\begin{aligned}
\langle\langle B, I\rangle\rangle &\subseteq \langle\langle B, I\rangle\rangle \cap \langle\langle M\Gamma B, I\rangle\rangle \\
&\subseteq \langle\langle B, I\rangle\rangle \cap \langle\langle (M\Gamma B], I\rangle\rangle \\
&= \bigcap_{b\in B} \langle\langle b, I\rangle\rangle \cap \langle\langle (M\Gamma B], I\rangle\rangle \\
&= \bigcap_{b\in B} \langle\langle (b], I\rangle\rangle \cap \langle\langle (M\Gamma B], I\rangle\rangle \\
&= \langle\langle \bigcup_{b\in B} (b], I\rangle\rangle \cap \langle\langle (M\Gamma B], I\rangle\rangle \\
&= \langle\langle (B], I\rangle\rangle \cap \langle\langle (M\Gamma B], I\rangle\rangle \\
&= \langle\langle (B] \cup (M\Gamma B], I\rangle\rangle \\
&= \langle\langle (B \cup M\Gamma B], I\rangle\rangle \\
&= \langle\langle OI(B), I\rangle\rangle \subseteq \langle\langle A, I\rangle\rangle.
\end{aligned}$$

Hence $\langle\langle B, I\rangle\rangle \subseteq \langle\langle A, I\rangle\rangle$. □

Lemma 3.2.9. *If M is a commutative ordered Γ-semigroup, and I is an ordered ideal of M, then Φ_I is a congruence on M.*

Proof. Let $(x, y) \in \Phi_I, c \in M$ and $\gamma \in \Gamma$. Then $\langle\langle x, I\rangle\rangle = \langle\langle y, I\rangle\rangle$. For $a \in M$,

$$\begin{aligned}
a \in \langle\langle x\gamma c, I\rangle\rangle &\Leftrightarrow (x\gamma c)\Gamma a \subseteq I \\
&\Leftrightarrow x\Gamma(c\gamma a) \subseteq I \\
&\Leftrightarrow c\gamma a \in \langle\langle x, I\rangle\rangle \\
&\Leftrightarrow c\gamma a \in \langle\langle y, I\rangle\rangle \\
&\Leftrightarrow y\Gamma(c\gamma a) \subseteq I \\
&\Leftrightarrow (y\gamma c)\Gamma a \subseteq I \\
&\Leftrightarrow a \in \langle\langle y\gamma c, I\rangle\rangle,
\end{aligned}$$

and

$$\begin{aligned} a \in \langle\!\langle c\gamma x, I\rangle\!\rangle &\Leftrightarrow (c\gamma x)\Gamma a \subseteq I \\ &\Leftrightarrow x\Gamma(c\gamma a) \subseteq I \\ &\Leftrightarrow c\gamma a \in \langle\!\langle x, I\rangle\!\rangle \\ &\Leftrightarrow c\gamma a \in \langle\!\langle y, I\rangle\!\rangle \\ &\Leftrightarrow y\Gamma(c\gamma a) \subseteq I \\ &\Leftrightarrow (c\gamma y)\Gamma a \subseteq I \\ &\Leftrightarrow a \in \langle\!\langle c\gamma y, I\rangle\!\rangle. \end{aligned}$$

Thus $(x\gamma c, y\gamma c) \in \Phi_I$ and $(c\gamma x, c\gamma y) \in \Phi_I$. Hence Φ_I is a congruence on M. □

Proposition 3.2.10. *If M is a commutative ordered Γ-semigroup, and I is an ordered s-semiprime ideal of M, then Φ_I is an ordered semilattice congruence on M.*

Proof. By Proposition 3.2.9, Φ_I is a congruence on M. Since M is commutative, $(a\gamma b, b\gamma a) \in \Phi_I$ for all $a, b \in M$ and $\gamma \in \Gamma$. Let $x \in M$ and $\gamma \in \Gamma$. Then for $a \in M$,

$$\begin{aligned} a \in \langle\!\langle x\gamma x, I\rangle\!\rangle &\Rightarrow (x\gamma x)\Gamma a \subseteq I \\ &\Rightarrow (x\gamma x\Gamma a)\Gamma a \subseteq I\Gamma M \subseteq I \\ &\Rightarrow (x\Gamma a)\gamma(x\Gamma a) \subseteq I \\ &\Rightarrow x\Gamma a \subseteq I \\ &\Rightarrow a \in \langle\!\langle x, I\rangle\!\rangle. \end{aligned}$$

By Proposition 3.2.3, $\langle\!\langle x, I\rangle\!\rangle \subseteq \langle\!\langle x\gamma x, I\rangle\!\rangle$. Therefore we have $\langle\!\langle x\gamma x, I\rangle\!\rangle = \langle\!\langle x, I\rangle\!\rangle$, so $(x\gamma x, x) \in \Phi_I$. Let $x, y \in M$ be such that $x \leq y$ and $\gamma \in \Gamma$. Then for $a \in M$,

$$\begin{aligned} a \in \langle\langle x, I\rangle\rangle &\Rightarrow x\Gamma a \subseteq I \\ &\Rightarrow (x\Gamma a)\gamma y \subseteq I\Gamma M \subseteq I \\ &\Rightarrow (x\gamma y)\Gamma a \subseteq I \\ &\Rightarrow a \in \langle\langle x\gamma y, I\rangle\rangle, \end{aligned}$$

and

$$\begin{aligned} a \in \langle\langle x\gamma y, I\rangle\rangle &\Rightarrow (x\gamma y)\Gamma a \subseteq I \\ &\Rightarrow (x\gamma x\Gamma a)\Gamma a \subseteq ((x\gamma y\Gamma a)\Gamma a] \subseteq (I\Gamma M] \subseteq (I] \subseteq I \\ &\Rightarrow (x\Gamma a)\gamma(x\Gamma a) \subseteq I \\ &\Rightarrow x\Gamma a \subseteq I \\ &\Rightarrow a \in \langle\langle x, I\rangle\rangle. \end{aligned}$$

Thus $\langle\langle x, I\rangle\rangle = \langle\langle x\gamma y, I\rangle\rangle$, so $(x, x\gamma y) \in \Phi_I$. Hence Φ_I is an ordered semilattice congruence on M. □

Proposition 3.2.11. *If I is an ordered prime ideal of an ordered Γ-semigroup M, then*

$$\Phi_I = (I \times I) \cup (M \setminus I \times M \setminus I).$$

Proof. This follows directly from Lemma 3.2.1(b) and Proposition 3.2.5. □