CHAPTER V

BANDS OF WEAKLY r-ARCHIMEDEAN (ORDERED) Γ-SEMIGROUPS

In this chapter, we divide into two sections. We give some characterizations of weakly r-archimedean (ordered) Γ-semigroups of a (ordered) Γ-semigroup M that are given by the relation $(\bar{\eta}_r \cap \bar{\eta}_r^{-1})$ $\eta_r \cap \eta_r^{-1}$ on M, and are also necessary for our considerations. Finally, we give some characterizations of bands of r-archimedean (ordered) Γ-semigroups, bands of weakly r-archimedean (ordered) Γ-semigroups and regular bands of weakly r-archimedean (ordered) Γ-semigroups of (ordered) Γ-semigroups.

5.1 Bands of Weakly r-Archimedean Γ-Semigroups

The following two lemmas are also necessary for the main results.

Lemma 5.1.1. *If ρ_1 and ρ_2 are left congruences on a Γ-semigroup M, then so are $(\rho_1)^n$ and $(\rho_1 \circ \rho_2)^n$ for all $n \in \mathbb{N}$.*

Proof. Similar to the proof of Lemma 5.8 [18], we obtain it. □

Similar result holds if we replace the word "left" by "right". Then we get Corollary 5.1.2.

Corollary 5.1.2. *If ρ_1 and ρ_2 are congruences on a Γ-semigroup M, then so are $(\rho_1)^n$ and $(\rho_1 \circ \rho_2)^n$ for all $n \in \mathbb{N}$.*

Lemma 5.1.3. *If E is an equivalence relation on a Γ-semigroup M, then*

$$E^\flat := \{(a,b) \mid (a,b), (x\alpha a, x\alpha b), (a\beta y, b\beta y), (x\alpha a\beta y, x\alpha b\beta y) \in E \text{ for all } x, y \in M \text{ and } \alpha, \beta \in \Gamma\}$$

is the largest congruence on M contained in E.

Proof. Similar to the proof of Proposition 5.13 [18], we obtain it. □

We give some characterizations of weakly r-archimedean Γ-semigroups that are given by the relation $\eta_r \cap \eta_r^{-1}$ on a Γ-semigroup M and some characterizations of bands of weakly r-archimedean sub-Γ-semigroups.

Lemma 5.1.4. *A sub-Γ-semigroup T of a Γ-semigroup M is weakly r-archimedean if and only if $(a,b) \in \eta_r \cap \eta_r^{-1}$ for all $a, b \in T$.*

Proof. Assume that a sub-Γ-semigroup T is weakly r-archimedean, and let $a, b \in T$. Then $b \in a\Gamma M$ or there exists an integer $m \geq 2$ such that $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in a\Gamma M$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma$, and $a \in b\Gamma M$ or there exists an integer $n \geq 2$ such that $a\beta_1 a\beta_2 a \ldots a\beta_{n-1} a \in b\Gamma M$ for some $\beta_1, \beta_2, \ldots, \beta_{n-1} \in \Gamma$. Thus $b \in a \cup a\Gamma M$ or $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in a \cup a\Gamma M$, and $a \in b \cup b\Gamma M$ or $a\beta_1 a\beta_2 a \ldots a\beta_{n-1} a \in b \cup b\Gamma M$. Hence $(a,b) \in \eta_r$ and $(b,a) \in \eta_r$, so $(a,b) \in \eta_r \cap \eta_r^{-1}$.

Conversely, assume that $(a,b) \in \eta_r \cap \eta_r^{-1}$ for all $a, b \in T$. Now, let $a, b \in T$. Then $(a,b) \in \eta_r \cap \eta_r^{-1}$, so $(a,b) \in \eta_r$. Thus $b \in a \cup a\Gamma M$ or there exists an integer $m \geq 2$ such that $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in a \cup a\Gamma M$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma$. Hence, let $\gamma \in \Gamma$. If $b \in a \cup a\Gamma M$, then $b\gamma b \in a\Gamma M \cup a\Gamma M\Gamma M \subseteq a\Gamma M$. If $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in a \cup a\Gamma M$, then $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\gamma b \in a\Gamma M \cup a\Gamma M\Gamma M \subseteq a\Gamma M$. Therefore T is weakly r-archimedean. □

As a consequence of this result, we obtain Theorem 5.1.5.

Theorem 5.1.5. *A Γ-semigroup M is a band of weakly r-archimedean sub-Γ-semigroups of M if and only if it satisfies the condition for all $a, x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$,*

$$(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \eta_r \cap \eta_r^{-1}. \quad (\star)$$

Proof. Assume that M is a band of weakly r-archimedean sub-Γ-semigroups of M. Then there exists a band congruence ρ on M such that the ρ-class $(x)_\rho$ of M containing x is a weakly r-archimedean sub-Γ-semigroup of M for all $x \in M$. Now, let $a, x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since ρ is a band congruence on M, we have $(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \rho$. Then there exist $b_1, b_2, b_3, b_4 \in M$ such that $a, a\gamma a \in (b_1)_\rho, x\alpha a, x\alpha a\gamma a \in (b_2)_\rho, a\beta y, a\gamma a\beta y \in (b_3)_\rho, x\alpha a\beta y, x\alpha a\gamma a\beta y \in (b_4)_\rho$. Since $(b_1)_\rho, (b_2)_\rho, (b_3)_\rho$ and $(b_4)_\rho$ are weakly r-archimedean, it follows from Lemma 5.1.4 that

$$(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \eta_r \cap \eta_r^{-1}.$$

Conversely, assume that M satisfies the condition $(\star)$.

(i) Clearly, $(a, a) \in \eta_r$ for all $a \in M$.

(ii) Let $a, b, c \in M$ be such that $(a, b) \in \eta_r$ and $(b, c) \in \eta_r$. Then $b \in a \cup a\Gamma M$ or there exists an integer $m \geq 2$ such that $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in a \cup a\Gamma M$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma$, and $c \in b \cup b\Gamma M$ or there exists an integer $n \geq 2$ such that $c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c \in b \cup b\Gamma M$ for some $\beta_1, \beta_2, \ldots, \beta_{n-1} \in \Gamma$. Thus $b = a$ or $b = a\alpha s_1$ for some $s_1 \in M$ and $\alpha \in \Gamma$ or $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b = a$ or $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b = a\alpha s_1$ for some $s_1 \in M$ and $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma$, and $c = b$ or $c = b\beta s_2$ for some $s_2 \in M$ and $\beta \in \Gamma$ or $c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c = b$ or $c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c = b\beta s_2$ for some $s_2 \in M$ and $\beta_1, \beta_2, \ldots, \beta_{n-1} \in \Gamma$.

Now, suppose that $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b = a\alpha s_1$ and $c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c = b\beta s_2$. Put $p = c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c = b\beta s_2$. By hypothesis, $(p, b\alpha_1 b\beta s_2) = (b\beta s_2, b\alpha_1 b\beta s_2) \in \eta_r \cap \eta_r^{-1}$ and so $(b\alpha_1 b\beta s_2, p) \in \eta_r$. Thus $p \in b\alpha_1 b\beta s_2 \cup b\alpha_1 b\beta s_2 \Gamma M$ or there exists an integer $m_1 \geq 2$ such that $p\gamma_1 p\gamma_2 p \ldots p\gamma_{m_1 - 1} p \in b\alpha_1 b\beta s_2 \cup b\alpha_1 b\beta s_2 \Gamma M$ for some $\gamma_1, \gamma_2, \ldots, \gamma_{m_1-1} \in \Gamma$. Thus $p = b\alpha_1 b\beta s_2$ or $p = b\alpha_1 b\beta s_2 \delta_1 s_3$ for some $s_3 \in M$ and $\delta_1 \in \Gamma$ or $p\gamma_1 p\gamma_2 p \ldots p\gamma_{m_1-1} p = b\alpha_1 b\beta s_2$ or $p\gamma_1 p\gamma_2 p \ldots p\gamma_{m_1-1} p = b\alpha_1 b\beta s_2 \delta_1 s_3$ for some $s_3 \in M$ and $\delta_1 \in \Gamma$. Hence there exists an integer $k \geq n$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k-1} c \in b\alpha_1 b\beta s_2 \cup b\alpha_1 b\beta s_2 \Gamma M$ for some $\lambda_1, \lambda_2, \ldots, \lambda_{k-1} \in \Gamma$.

Case 1: $p\gamma_1 p\gamma_2 p \ldots p\gamma_{m_1-1}p = b\alpha_1 b\beta s_2\delta_1 s_3$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2\delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2\delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2\delta_1 s_3) \in \eta_r \cap \eta_r^{-1}$$

where $r_1 = b\beta s_2\delta_1 s_3$, there exists an integer $k_1 \geq nm_1 \geq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_1-1}c \in b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M$.

Case 2: $p\gamma_1 p\gamma_2 p \ldots p\gamma_{m_1-1}p = b\alpha_1 b\beta s_2$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2) \in \eta_r \cap \eta_r^{-1}$$

where $r_1 = b\beta s_2$, there exists an integer $k_1 \geq nm_1 \geq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_1-1}c \in b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M$.

Case 3: $p = b\alpha_1 b\beta s_2\delta_1 s_3$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2\delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2\delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2\delta_1 s_3) \in \eta_r \cap \eta_r^{-1}$$

where $r_1 = b\beta s_2\delta_1 s_3$, there exists an integer $k_1 \geq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_1-1}c \in b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M$.

Case 4: $p = b\alpha_1 b\beta s_2$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2) \in \eta_r \cap \eta_r^{-1}$$

where $r_1 = b\beta s_2$, there exists an integer $k_1 \geq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_1-1}c \in b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M$.

If we continue in this way, there exist $r_{m-2} \in M$, an integer $k_{m-2} \geq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k_{m-2}-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_{m-2}-1}c \in b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2}$ $\cup b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2}\Gamma M$. Therefore $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_{m-2}-1}c \in b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2} \cup b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2}\Gamma M = a\alpha s_1\alpha_1 r_{m-2} \cup a\alpha s_1 \alpha_1 r_{m-2}\Gamma M \subseteq a\Gamma M \subseteq a \cup a\Gamma M$. Hence $(a, c) \in \eta_r$. In another case, we can show that $(a, c) \in \eta_r$. By (i) and (ii), $\eta_r \cap \eta_r^{-1}$ is an equivalence relation on M.

(*iii*) Let

$$\rho := \{(a,b) \mid (a,b), (x\alpha a, x\alpha b), (a\beta y, b\beta y), (x\alpha a\beta y, x\alpha b\beta y) \in \eta_r \cap \eta_r^{-1} \text{ for all } x, y \in M \text{ and } \alpha, \beta \in \Gamma\}.$$

Since $\eta_r \cap \eta_r^{-1}$ is an equivalence relation on M, it follows from Lemma 5.1.3 that ρ is the largest congruence on M contained in $\eta_r \cap \eta_r^{-1}$. By condition $(\star)$, ρ is a band congruence on M.

(*iv*) For any $x \in M$, let $a, b \in (x)_\rho$. Then $(a,b) \in \rho$, so $(a,b) \in \eta_r \cap \eta_r^{-1}$. Since ρ is a band congruence on M, $(x)_\rho$ is a sub-Γ-semigroup of M. It follows from Lemma 5.1.4 that $(x)_\rho$ is a weakly r-archimedean sub-Γ-semigroup of M. Therefore M is a band of weakly r-archimedean sub-Γ-semigroups of M.

Hence the proof is completed. □

We briefly recall here the definition of ordered Γ-semigroup. A partially ordered Γ-semigroup M is called an ***ordered Γ-semigroup*** if for any $a, b, c \in M$ and $\gamma \in \Gamma$, $a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

Theorem 5.1.6. *Let*

$$\rho := \{(a,b) \mid (a,b), (x\alpha a, x\alpha b), (a\beta y, b\beta y), (x\alpha a\beta y, x\alpha b\beta y) \in \eta_r \cap \eta_r^{-1} \text{ for all } x, y \in M \text{ and } \alpha, \beta \in \Gamma\}$$

be a congruence on a Γ-semigroup M. Then M/ρ is an ordered Γ-semigroup.

Proof. Let $\preceq$ be a relation on M/ρ defined as following:

$$\preceq := \{((x)_\rho, (y)_\rho) \mid (x_1, y_1) \in \rho^m \text{ for some } x_1 \in (x)_\rho, y_1 \in (y)_\rho \text{ and } m \in \mathbb{N}\}.$$

We shall show that $(M/\rho; \preceq)$ is an ordered Γ-semigroup.

(*i*) For any $(x)_\rho \in M/\rho$, we have $(x)_\rho \preceq (x)_\rho$ because $(x,x) \in \rho$. Thus $\preceq$ is reflexive.

(ii) Let $(x)_\rho, (y)_\rho \in M/\rho$ be such that $(x)_\rho \preceq (y)_\rho$ and $(y)_\rho \preceq (x)_\rho$. Then there exist $x_1, x_2 \in (x)_\rho, y_1, y_2 \in (y)_\rho$ and $m, n \in \mathbb{N}$ such that $(x_1, y_1) \in \rho^m$ and $(y_2, x_2) \in \rho^n$. Thus there exist $w_1, w_2, \ldots, w_{m-1}, w_1^{'}, w_2^{'}, \ldots, w_{n-1}^{'} \in M$ such that

$$(x_1, w_1), (w_1, w_2), .., (w_{m-1}, y_1) \in \rho, \tag{5.1.1}$$

and

$$(y_2, w_1^{'}), (w_1^{'}, w_2^{'}), .., (w_{n-1}^{'}, x_2) \in \rho. \tag{5.1.2}$$

Since $(x_1, w_1) \in \rho$, we have $(x_1, w_1), (x\alpha x_1, x\alpha w_1), (x_1\beta y, w_1\beta y), (x\alpha x_1\beta y, x\alpha w_1\beta y) \in \eta_r \cap \eta_r^{-1}$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$.

Let $x, y \in M$ and $\alpha, \beta \in \Gamma$, and $p = x\alpha x_1\beta y$. Suppose that $(x\alpha x_1\beta y, x\alpha w_1\beta y) \in \eta_r \cap \eta_r^{-1}$. Then $(x\alpha w_1\beta y, x\alpha x_1\beta y) \in \eta_r$. Thus $x\alpha x_1\beta y \in x\alpha w_1\beta y \cup x\alpha w_1\beta y\Gamma M$ or there exists an integer $k_1 \geq 2$ such that $(x\alpha x_1\beta y)\alpha_1(x\alpha x_1\beta y)\alpha_2 (x\alpha x_1\beta y)\ldots(x\alpha x_1\beta y)\alpha_{k_1-1}(x\alpha x_1\beta y) \in x\alpha w_1\beta y \cup x\alpha w_1\beta y\Gamma M$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{k_1-1} \in \Gamma$. Then $x\alpha x_1\beta y = x\alpha w_1\beta y$ or $x\alpha x_1\beta y = x\alpha w_1\beta y\delta_1 s_1$ for some $s_1 \in M$ and $\delta_1 \in \Gamma$ or $(x\alpha x_1\beta y)\alpha_1(x\alpha x_1\beta y)\alpha_2(x\alpha x_1\beta y)\ldots(x\alpha x_1\beta y)\alpha_{k_1-1}(x\alpha x_1\beta y) = x\alpha w_1\beta y$ or $(x\alpha x_1\beta y)\alpha_1(x\alpha x_1\beta y)\alpha_2(x\alpha x_1\beta y)\ldots(x\alpha x_1\beta y)\alpha_{k_1-1}(x\alpha x_1\beta y) = x\alpha w_1 \beta y\delta_1 s_1$ for some $s_1 \in M$ and $\delta_1 \in \Gamma$.

Case 1: $x\alpha x_1\beta y = x\alpha w_1\beta y$. Then $p = x\alpha w_1\beta r_1$ where $r_1 = y$. Next, since $(w_1, w_2) \in \rho, (x\alpha w_1\beta r_1, x\alpha w_2\beta r_1) \in \eta_r \cap \eta_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha w_1\beta r_1) \in \eta_r$. Then $x\alpha w_1\beta r_1 \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) \in x\alpha w_2\beta r_1 \cup x \alpha w_2\beta r_1\Gamma M$ for some $\beta_1, \beta_2, \ldots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$ or $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 1.1: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$. Then $p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 1.2: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 1.3: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$. Then $p\beta_1 p\beta_2 p\ldots p\beta_{k_2-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 1.4: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p\beta_1 p\beta_2 p\ldots p\beta_{k_2-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 2: $x\alpha x_1\beta y = x\alpha w_1\beta y\delta_1 s_1$. Then $p = x\alpha w_1\beta r_1$ where $r_1 = y\delta_1 s_1$. Next, since $(w_1, w_2) \in \rho$, $(x\alpha w_1\beta r_1, x\alpha w_2\beta r_1) \in \eta_r \cap \eta_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha w_1\beta r_1) \in \eta_r$. Then $x\alpha w_1\beta r_1 \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ for some $\beta_1, \beta_2, \ldots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$ or $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 2.1: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$. Then $p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 2.2: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 2.3: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$. Then $p\beta_1 p\beta_2 p\ldots p\beta_{k_2-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 2.4: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p\beta_1 p\beta_2 p\ldots p\beta_{k_2-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 3: $(x\alpha x_1\beta y)\alpha_1(x\alpha x_1\beta y)\alpha_2(x\alpha x_1\beta y)\ldots(x\alpha x_1\beta y)\alpha_{k_1-1}(x\alpha x_1\beta y) = x\alpha w_1\beta y$. Then $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p = x\alpha w_1\beta r_1$ where $r_1 = y$. Next, since $(w_1, w_2) \in \rho$, $(x\alpha w_1\beta r_1, x\alpha w_2\beta r_1) \in \eta_r \cap \eta_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha w_1\beta r_1) \in \eta_r$. Then $x\alpha w_1\beta r_1 \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha w_1\beta r_1)\beta_1 (x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ for some $\beta_1, \beta_2, \ldots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$ or $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1 \delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1$

$\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\dots(x\alpha w_1 \beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 3.1: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$. Then $p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 3.2: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 3.3: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\dots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$. Then $(p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1} p)\dots(p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p) = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 3.4: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\dots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$. Then $(p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\dots p \alpha_{k_1-1}p)\dots(p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p) = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 4: $(x\alpha x_1\beta y)\alpha_1(x\alpha x_1\beta y)\alpha_2(x\alpha x_1\beta y)\dots(x\alpha x_1\beta y)\alpha_{k_1-1}(x\alpha x_1\beta y) = x\alpha w_1\beta y \delta_1 s_1$. Then $p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p = x\alpha w_1\beta r_1$ where $r_1 = y\delta_1 s_1$. Next, since $(w_1, w_2) \in \rho$, $(x\alpha w_1\beta r_1, x\alpha w_2\beta r_1) \in \eta_r \cap \eta_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha w_1\beta r_1) \in \eta_r$. Then $x\alpha w_1\beta r_1 \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\dots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) \in x\alpha w_2\beta r_1 \cup x \alpha w_2\beta r_1\Gamma M$ for some $\beta_1, \beta_2, \dots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$ or $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\dots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\dots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 4.1: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$. Then $p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 4.2: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 4.3: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\dots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) =$

$x\alpha w_2\beta r_1$. Therefore $(p\alpha_1 p\alpha_2 p \ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p \ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p \ldots p\alpha_{k_1-1}p) \ldots (p\alpha_1 p\alpha_2 p \ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p \ldots p\alpha_{k_1-1}p) = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 4.4: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1) \ldots (x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$. Then $(p\alpha_1 p\alpha_2 p \ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p \ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p \ldots p\alpha_{k_1-1}p) \ldots (p\alpha_1 p\alpha_2 p \ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p \ldots p\alpha_{k_1-1}p) = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

If we continue in this way, we have $p = x\alpha y_1\beta r_m$ or there exists an integer $k \geq 2$ such that $p\gamma_1 p\gamma_2 p \ldots p\gamma_{k-1}p = x\alpha y_1\beta r_m$ for some $\lambda_1, \lambda_2, \ldots, \lambda_{k-1} \in \Gamma$. Since $(y_1, y_2) \in \rho$, $(x\alpha y_1\beta r_m, x\alpha y_2\beta r_m) \in \eta_r \cap \eta_r^{-1}$. Thus $(x\alpha y_2\beta r_m, x\alpha y_1\beta r_m) \in \eta_r$. Then $x\alpha y_1\beta r_m \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$ or there exists an integer $k' \geq 2$ such that $(x\alpha y_1\beta r_m)\gamma_1'(x\alpha y_1\beta r_m)\gamma_2'(x\alpha y_1\beta r_m) \ldots (x\alpha y_1\beta r_m)\gamma_{k'-1}'(x\alpha y_1\beta r_m) \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$ for some $\gamma_1', \gamma_2', \ldots, \gamma_{k'-1}' \in \Gamma$.

Put $q = x\alpha y_1\beta r_m$.

Case I: $p = q$ and $q \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$. Then $p \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M \subseteq x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \eta_r$.

Case II: $p = q$ and $q\gamma_1' q\gamma_2' q \ldots q\gamma_{k'-1}' q \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$. Then $p\gamma_1' p\gamma_2' p \ldots p\gamma_{k'-1}' p \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M \subseteq x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \eta_r$.

Case III: $p\gamma_1 p\gamma_2 p \ldots p\gamma_{k-1}p = q$ and $q \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$. Then $p\gamma_1 p\gamma_2 p \ldots p\gamma_{k-1}p \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M \subseteq x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \eta_r$.

Case IV: $p\gamma_1 p\gamma_2 p \ldots p\gamma_{k-1}p = q$ and $q\gamma_1' q\gamma_2' q \ldots q\gamma_{k'-1}' q \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$. Then $(p\gamma_1 p\gamma_2 p \ldots p\gamma_{k-1}p)\gamma_1'(p\gamma_1 p\gamma_2 p \ldots p\gamma_{k-1}p)\gamma_2'(p\gamma_1 p\gamma_2 p \ldots p\gamma_{k-1}p) \ldots (p\gamma_1 p\gamma_2 p \ldots p\gamma_{k-1}p)\gamma_{k'-1}'(p\gamma_1 p\gamma_2 p \ldots p\gamma_{k-1}p) \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M \subseteq x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \eta_r$.

Similar to the proof as above, by (5.1.2), we can prove that $(x\alpha x_1\beta y, x\alpha y_2\beta$

$y) \in \eta_r$. Therefore $(x\alpha x_1\beta y, x\alpha y_2\beta y) \in \eta_r \cap \eta_r^{-1}$.

In another case, we have the following statements:

1. If $(x_1, w_1) \in \eta_r \cap \eta_r^{-1}$, then $(x_1, y_2) \in \eta_r \cap \eta_r^{-1}$.

2. If $(x\alpha x_1, x\alpha w_1) \in \eta_r \cap \eta_r^{-1}$, then $(x\alpha x_1, x\alpha y_2) \in \eta_r \cap \eta_r^{-1}$.

3. If $(x_1\beta y, w_1\beta y) \in \eta_r \cap \eta_r^{-1}$, then $(x_1\beta y, y_2\beta y) \in \eta_r \cap \eta_r^{-1}$.

Therefore $(x_1, y_2) \in \rho$, so $(x)_\rho = (x_1)_\rho = (y_2)_\rho = (y)_\rho$. Hence $\preceq$ is anti-symmetric.

(iii) Let $(x)_\rho, (y)_\rho, (z)_\rho \in M/\rho$ be such that $(x)_\rho \preceq (y)_\rho$ and $(y)_\rho \preceq (z)_\rho$. Then there exist $x_1 \in (x)_\rho, y_1, y_2 \in (y)_\rho, z_2 \in (z)_\rho$ and $m, n \in \mathbb{N}$ such that $(x_1, y_1) \in \rho^m$ and $(y_2, z_2) \in \rho^n$. Thus $x_1\rho^m y_1 \rho y_2 \rho^n z_2$, so $(x_1, z_2) \in \rho^{m+n+1}$. Hence $(x)_\rho \preceq (z)_\rho$. Therefore $\preceq$ is transitive.

(iv) Let $(x)_\rho, (y)_\rho \in M/\rho$ be such that $(x)_\rho \preceq (y)_\rho, (z)_\rho \in M/\rho$ and $\gamma \in \Gamma$. Then there exist $x_1 \in (x)_\rho, y_1 \in (y)_\rho$ and $m \in \mathbb{N}$ such that $(x_1, y_1) \in \rho^m$. It follows from Corollary 5.1.2 that $(z\gamma x_1, z\gamma y_1), (x_1\gamma z, y_1\gamma z) \in \rho^m$. Since $z\gamma x_1 \in (z\gamma x)_\rho, x_1\gamma z \in (x\gamma z)_\rho, z\gamma y_1 \in (z\gamma y)_\rho$ and $y_1\gamma z \in (y\gamma z)_\rho$, we have $(z\gamma x)_\rho \preceq (z\gamma y)_\rho$ and $(x\gamma z)_\rho \preceq (y\gamma z)_\rho$. Thus $(z)_\rho\gamma(x)_\rho = (z\gamma x)_\rho \preceq (z\gamma y)_\rho = (z)_\rho\gamma(y)_\rho$ and $(x)_\rho\gamma(z)_\rho = (x\gamma z)_\rho \preceq (y\gamma z)_\rho = (y)_\rho\gamma(z)_\rho$. Hence $\preceq$ is compatible, so $(M/\rho; \preceq)$ is an ordered Γ-semigroup.

This completes the proof. □

Immediately from Theorems 5.1.5 and 5.1.6, we have Corollary 5.1.7.

Corollary 5.1.7. *If a Γ-semigroup M is a band of weakly r-archimedean sub-Γ-semigroups of M, then there exists a congruence ρ on M such that M/ρ is an ordered Γ-semigroup.*

Lemma 5.1.8. *Let T be a left ideal of a Γ-semigroup M. Then the following statements are equivalent:*

(a) *T is a weakly r-archimedean sub-Γ-semigroup of M.*

(b) *T is an r-archimedean sub-Γ-semigroup of M.*

(c) *For any $a, b \in T, (a, b) \in \eta_r \cap \eta_r^{-1}$.*

Proof. By Lemma 5.1.4, (b) implies (c) and (c) implies (a). Now, we shall prove that (a) implies (b). Suppose that T is a weakly r-archimedean sub-Γ-semigroup of M, and let $a, b \in T$. It follows from Lemma 5.1.4 that $(a, b) \in \eta_r \cap \eta_r^{-1}$, so $(a, b) \in \eta_r$. Thus $b \in a \cup a\Gamma M$ or there exists an integer $m \geq 2$ such that $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in a \cup a\Gamma M$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma$. For any $\gamma \in \Gamma$, if $b = a$ or $b = a\alpha x$ for some $x \in M$ and $\alpha \in \Gamma$, then $b\gamma b = a\gamma b \in a\Gamma T$ or $b\gamma b = a\alpha x\gamma b \in a\Gamma T$ since T is a left ideal of M. If $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b = a$ or $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b = a\alpha x$ for some $x \in M$ and $\alpha \in \Gamma$, then $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\gamma b = a\gamma b \in a\Gamma T$ or $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\gamma b = a\alpha x\gamma b \in a\Gamma T$ since T is a left ideal of M. Therefore T is an r-archimedean sub-Γ-semigroup of M. □

Combining Theorem 5.1.5 and Lemma 5.1.8, we obtain Corollary 5.1.9.

Corollary 5.1.9. *Let M be a Γ-semigroup. Then following statements are equivalent:*

(a) *M is a band of weakly r-archimedean left ideals of M.*

(b) *M is a band of r-archimedean left ideals of M.*

(c) *M satisfies the condition for all $a, x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$,*

$$(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \eta_r \cap \eta_r^{-1}. \qquad (\star)$$

5.2 Bands of Weakly r-Archimedean Ordered Γ-Semigroups

We now give some characterizations of weakly r-archimedean sub-Γ-semigroups of an ordered Γ-semigroup M that are given by the relation $\bar{\eta}_r \cap \bar{\eta}_r^{-1}$ on M and some characterizations of bands of r-archimedean sub-Γ-semigroups, bands of weakly r-archimedean sub-Γ-semigroups and regular bands of weakly r-archimedean sub-Γ-semigroups of ordered Γ-semigroups.

Lemma 5.2.1. *A sub-Γ-semigroup T of an ordered Γ-semigroup M is weakly r-archimedean if and only if $(a,b) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$ for all $a, b \in T$.*

Proof. Suppose that T is weakly r-archimedean, and let $a, b \in T$. Then $b \in (a\Gamma M]$ or there exists an integer $m \geq 2$ such that $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in (a\Gamma M]$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma$, and $a \in (b\Gamma M]$ or there exists an integer $n \geq 2$ such that $a\beta_1 a\beta_2 a \ldots a\beta_{n-1} a \in (b\Gamma M]$ for some $\beta_1, \beta_2, \ldots, \beta_{n-1} \in \Gamma$. Thus $b \in (a \cup a\Gamma M]$ or $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in (a \cup a\Gamma M]$, and $a \in (b \cup b\Gamma M]$ or $a\beta_1 a\beta_2 a \ldots a\beta_{n-1} a \in (b \cup b\Gamma M]$. Hence $(a,b) \in \bar{\eta}_r$ and $(b,a) \in \bar{\eta}_r$, so $(a,b) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$.

Conversely, let $a, b \in T$. By hypothesis, $(a,b) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$, so $(a,b) \in \bar{\eta}_r$. Thus $b \in (a \cup a\Gamma M]$ or there exists an integer $m \geq 2$ such that $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in (a \cup a\Gamma M]$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma$. Hence, let $\gamma \in \Gamma$. If $b \in (a \cup a\Gamma M]$, then $b\gamma b \in (a \cup a\Gamma M]\Gamma M \subseteq (a\Gamma M]$. If $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in (a \cup a\Gamma M]$, then $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\gamma b \in (a \cup a\Gamma M]\Gamma M \subseteq (a\Gamma M]$. Therefore T is weakly r-archimedean. □

As a consequence of this result, we obtain Theorem 5.2.2.

Theorem 5.2.2. *An ordered Γ-semigroup M is a band of weakly r-archimedean sub-Γ-semigroups of M if and only if it satisfies the condition for all $a, x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$,*

$$(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}. \qquad (\star)$$

Proof. Assume that M is a band of weakly r-archimedean sub-Γ-semigroups of M. Then there exists a band congruence ρ on M such that the ρ-class $(x)_\rho$ of M containing x is a weakly r-archimedean sub-Γ-semigroup of M for all $x \in M$. Let $a, x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since ρ is a band congruence on M, we have $(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \rho$. Then there exist $b_1, b_2, b_3, b_4 \in M$ such that $a, a\gamma a \in (b_1)_\rho, x\alpha a, x\alpha a\gamma a \in (b_2)_\rho, a\beta y, a\gamma a\beta y \in (b_3)_\rho, x\alpha a\beta y, x\alpha a\gamma a\beta y \in (b_4)_\rho$. Since $(b_1)_\rho, (b_2)_\rho, (b_3)_\rho$ and $(b_4)_\rho$ are weakly r-archimedean, it follows from Lemma 5.2.1 that

$$(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}.$$

Conversely, assume that M satisfies the condition $(\star)$.

(i) Clearly, $(a, a) \in \bar{\eta}_r$ for all $a \in M$.

(ii) Let $a, b, c \in M$ be such that $(a, b) \in \bar{\eta}_r$ and $(b, c) \in \bar{\eta}_r$. Then $b \in (a \cup a\Gamma M]$ or there exists an integer $m \geq 2$ such that

$$b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in (a \cup a\Gamma M] \text{ for some } \alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma,$$

and $c \in (b \cup b\Gamma M]$ or there exists an integer $n \geq 2$ such that

$$c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c \in (b \cup b\Gamma M] \text{ for some } \beta_1, \beta_2, \ldots, \beta_{n-1} \in \Gamma.$$

Thus $b \leq a$ or $b \leq a\alpha s_1$ for some $s_1 \in M$ and $\alpha \in \Gamma$ or $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \leq a$ or $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \leq a\alpha s_1$ for some $s_1 \in M$ and $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma$, and $c \leq b$ or $c \leq b\beta s_2$ for some $s_2 \in M$ and $\beta \in \Gamma$ or $c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c \leq b$ or $c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c \leq b\beta s_2$ for some $s_2 \in M$ and $\beta_1, \beta_2, \ldots, \beta_{n-1} \in \Gamma$.

Now, suppose that $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \leq a\alpha s_1$ and $c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c \leq b\beta s_2$. Let $p = c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c$. By hypothesis, $(b\beta s_2, b\alpha_1 b\beta s_2) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$ and so

$(b\alpha_1 b\beta s_2, b\beta s_2) \in \bar{\eta}_r$. Thus $b\beta s_2 \in (b\alpha_1 b\beta s_2 \cup b\alpha_1 b\beta s_2 \Gamma M]$ or there exists an integer $m_1 \geq 2$ such that $(b\beta s_2)\gamma_1(b\beta s_2)\gamma_2(b\beta s_2)\dots(b\beta s_2)\gamma_{m_1-1}(b\beta s_2) \in (b\alpha_1 b\beta s_2 \cup b\alpha_1 b\beta s_2 \Gamma M]$ for some $\gamma_1, \gamma_2, \dots, \gamma_{m_1-1} \in \Gamma$. Since $p \leq b\beta s_2, p \leq b\beta s_2 \in (b\alpha_1 b\beta s_2 \cup b\alpha_1 b\beta s_2 \Gamma M]$ or $p\gamma_1 p\gamma_2 p \dots p\gamma_{m_1-1} p \leq b\beta s_2 \gamma_1 b\beta s_2 \gamma_2 b\beta s_2 \dots b\beta s_2 \gamma_{m_1-1} b\beta s_2 \in (b\alpha_1 b \beta s_2 \cup b\alpha_1 b\beta s_2 \Gamma M]$. Thus $p \leq b\alpha_1 b\beta s_2$ or $p \leq b\alpha_1 b\beta s_2 \delta_1 s_3$ for some $s_3 \in M$ and $\delta_1 \in \Gamma$ or $p\gamma_1 p\gamma_2 p \dots p\gamma_{m_1-1} p \leq b\alpha_1 b\beta s_2$ or $p\gamma_1 p\gamma_2 p \dots p\gamma_{m_1-1} p \leq b\alpha_1 b\beta s_2 \delta_1 s_3$ for some $s_3 \in M$ and $\delta_1 \in \Gamma$. Hence there exists an integer $k \geq n$ such that $c\lambda_1 c\lambda_2 c \dots c\lambda_{k-1} c \in (b\alpha_1 b\beta s_2 \cup b\alpha_1 b\beta s_2 \Gamma M]$ for some $\lambda_1, \lambda_2, \dots, \lambda_{k-1} \in \Gamma$.

Case 1: $p\gamma_1 p\gamma_2 p \dots p\gamma_{m_1-1} p \leq b\alpha_1 b\beta s_2 \delta_1 s_3$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2 \delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2 \delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2 \delta_1 s_3) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$$

where $r_1 = b\beta s_2 \delta_1 s_3$, there exists an integer $k_1 \geq nm_1 \geq n$ and $\lambda_1, \lambda_2, \dots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \dots c\lambda_{k_1-1} c \in (b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M]$.

Case 2: $p\gamma_1 p\gamma_2 p \dots p\gamma_{m_1-1} p \leq b\alpha_1 b\beta s_2$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$$

where $r_1 = b\beta s_2$, there exists an integer $k_1 \geq nm_1 \geq n$ and $\lambda_1, \lambda_2, \dots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \dots c\lambda_{k_1-1} c \in (b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M]$.

Case 3: $p \leq b\alpha_1 b\beta s_2 \delta_1 s_3$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2 \delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2 \delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2 \delta_1 s_3) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$$

where $r_1 = b\beta s_2 \delta_1 s_3$, there exists an integer $k_1 \geq n$ and $\lambda_1, \lambda_2, \dots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \dots c\lambda_{k_1-1} c \in (b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M]$.

Case 4: $p \leq b\alpha_1 b\beta s_2$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$$

where $r_1 = b\beta s_2$, there exists an integer $k_1 \geq n$ and $\lambda_1, \lambda_2, \dots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \dots c\lambda_{k_1-1} c \in (b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M]$.

If we continue in this way, there exist $r_{m-2} \in M$, an integer $k_{m-2} \geq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k_{m-2}-1} \in \Gamma$ such that

$$\begin{aligned} c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_{m-2}-1} c &\in (b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2} \cup \\ &\quad b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2}\Gamma M]. \end{aligned}$$

Hence

$$\begin{aligned} c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_{m-2}-1} c &\in (b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2} \cup \\ &\quad b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2}\Gamma M] \\ &\subseteq (a\alpha s_1 \alpha_1 r_{m-2} \cup a\alpha s_1 \alpha_1 r_{m-2}\Gamma M] \\ &\subseteq (a\Gamma M] \\ &\subseteq (a \cup a\Gamma M]. \end{aligned}$$

Therefore $(a, c) \in \bar{\eta}_r$. In another case, we can show that $(a, c) \in \bar{\eta}_r$. By (i) and (ii), $\bar{\eta}_r \cap \bar{\eta}_r^{-1}$ is an equivalence relation on M.

(iii) Let

$$\begin{aligned} \rho := \{(a, b) \mid\ & (a, b), (x\alpha a, x\alpha b), (a\beta y, b\beta y), (x\alpha a\beta y, x\alpha b\beta y) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1} \\ & \text{for all } x, y \in M \text{ and } \alpha, \beta \in \Gamma\}. \end{aligned}$$

Since $\bar{\eta}_r \cap \bar{\eta}_r^{-1}$ is an equivalence relation on M, it follows from Lemma 5.1.3 that ρ is the largest congruence on M contained in $\bar{\eta}_r \cap \bar{\eta}_r^{-1}$. By condition $(\star)$, ρ is a band congruence on M.

(iv) For any $x \in M$, let $a, b \in (x)_\rho$. Then $(a, b) \in \rho$, so $(a, b) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$. Since ρ is a band congruence on M, $(x)_\rho$ is a sub-Γ-semigroup of M. It follows from Lemma 5.2.1 that $(x)_\rho$ is a weakly r-archimedean sub-Γ-semigroup of M. Therefore M is a band of weakly r-archimedean sub-Γ-semigroups of M.

Hence the proof is completed. □

Theorem 5.2.3. *Let*

$$\rho \ := \ \{(a,b) \mid (a,b),(x\alpha a, x\alpha b),(a\beta y, b\beta y),(x\alpha a\beta y, x\alpha b\beta y) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$$
$$\text{for all } x,y \in M \text{ and } \alpha,\beta \in \Gamma\}$$

be a congruence on an ordered Γ-semigroup M. Then ρ is a regular congruence on M.

Proof. Let $\preceq$ be a relation on M/ρ defined as following:

$$\preceq := \{((x)_\rho,(y)_\rho) \mid (x_1,y_1) \in (\leq \circ\rho)^m \text{ for some } x_1 \in (x)_\rho, y_1 \in (y)_\rho \text{ and } m \in \mathbb{N}\}.$$

We firstly show that $\preceq$ is well-defined. Let $(x)_\rho = (x')_\rho$ and $(y)_\rho = (y')_\rho$ be such that $(x)_\rho \preceq (y)_\rho$. Then there exist $x_1 \in (x)_\rho, y_1 \in (y)_\rho$ and $m \in \mathbb{N}$ such that $(x_1,y_1) \in (\leq \circ\rho)^m$, so $x_1 \in (x')_\rho$ and $y_1 \in (y')_\rho$ such that $(x_1,y_1) \in (\leq \circ\rho)^m$. Thus $(x')_\rho \preceq (y')_\rho$. Hence $\preceq$ is well-defined.

(*i*) For any $(x)_\rho \in M/\rho$, we have $(x,x) \in (\leq \circ\rho)$ because $x \leq x\rho x$. Hence $(x)_\rho \preceq (x)_\rho$, so $\preceq$ is reflexive.

(*ii*) Let $(x)_\rho,(y)_\rho \in M/\rho$ be such that $(x)_\rho \preceq (y)_\rho$ and $(y)_\rho \preceq (x)_\rho$. Then there exist $x_1,x_2 \in (x)_\rho, y_1,y_2 \in (y)_\rho$ and $m,n \in \mathbb{N}$ such that $(x_1,y_1) \in (\leq \circ\rho)^m$ and $(y_2,x_2) \in (\leq \circ\rho)^n$. Thus there exist $w_1,w_2,\ldots,w_{m-1},w_1',w_2',\ldots,w_{n-1}' \in M$ such that

$$(x_1,w_1),(w_1,w_2),..,(w_{m-1},y_1) \in (\leq \circ\rho),$$

and

$$(y_2,w_1'),(w_1',w_2'),..,(w_{n-1}',x_2) \in (\leq \circ\rho).$$

Thus there exist $z_1,z_2,\ldots,z_m,z_1',z_2',\ldots,z_n' \in M$ such that

$$x_1 \leq z_1\rho w_1 \leq z_2\rho w_2 \leq \ldots \leq z_i\rho w_i \leq \ldots \leq z_m\rho y_1, \qquad (\star)$$

and

$$y_2 \leq z_1'\rho w_1' \leq z_2'\rho w_2' \leq \ldots \leq z_j'\rho w_j' \leq \ldots \leq z_n'\rho x_2. \qquad (\star\star)$$

Since $(z_1, w_1) \in \rho$, we have $(z_1, w_1), (x\alpha z_1, x\alpha w_1), (z_1\beta y, w_1\beta y), (x\alpha z_1\beta y, x\alpha w_1\beta y) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$.

Let $x, y \in M$ and $\alpha, \beta \in \Gamma$, and $p = x\alpha x_1\beta y$. Suppose that $(x\alpha z_1\beta y, x\alpha w_1 \beta y) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$. Then $(x\alpha w_1\beta y, x\alpha z_1\beta y) \in \bar{\eta}_r$. Thus $x\alpha z_1\beta y \in (x\alpha w_1\beta y \cup x\alpha w_1\beta y\Gamma M]$ or there exists an integer $k_1 \geq 2$ such that $(x\alpha z_1\beta y)\alpha_1(x\alpha z_1\beta y)\alpha_2 (x\alpha z_1\beta y)\ldots(x\alpha z_1\beta y)\alpha_{k_1-1}(x\alpha z_1\beta y) \in (x\alpha w_1\beta y \cup x\alpha w_1\beta y\Gamma M]$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{k_1-1} \in \Gamma$. Then $x\alpha z_1\beta y \leq x\alpha w_1\beta y$ or $x\alpha z_1\beta y \leq x\alpha w_1\beta y\delta_1 s_1$ for some $s_1 \in M$ and $\delta_1 \in \Gamma$ or $(x\alpha z_1\beta y)\alpha_1(x\alpha z_1\beta y)\alpha_2(x\alpha z_1\beta y)\ldots(x\alpha z_1\beta y)\alpha_{k_1-1}(x\alpha z_1\beta y) \leq x\alpha w_1\beta y$ or $(x\alpha z_1\beta y)\alpha_1(x\alpha z_1\beta y)\alpha_2(x\alpha z_1\beta y)\ldots(x\alpha z_1\beta y)\alpha_{k_1-1}(x\alpha z_1\beta y) \leq x\alpha w_1\beta y\delta_1 s_1$ for some $s_1 \in M$ and $\delta_1 \in \Gamma$.

Case 1: $x\alpha z_1\beta y \leq x\alpha w_1\beta y$. Since $x_1 \leq z_1$, $p \leq x\alpha z_1\beta y \leq x\alpha w_1\beta y$. Thus $p \leq x\alpha w_1\beta r_1$ where $r_1 = y$. Next, since $(z_2, w_2) \in \rho$, $(x\alpha z_2\beta r_1, x\alpha w_2\beta r_1) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha z_2\beta r_1) \in \bar{\eta}_r$. Then $x\alpha z_2\beta r_1 \in (x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M]$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \in (x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M]$ for some $\beta_1, \beta_2, \ldots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$ or $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1$ or $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 1.1: $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$. Since $w_1 \leq z_2$, $p \leq x\alpha w_1\beta r_1 \leq x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$. Thus $p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 1.2: $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$. Since $w_1 \leq z_2$, $p \leq x\alpha w_1\beta r_1 \leq x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$. Thus $p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 1.3: $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha$

$w_2\beta r_1$. Since $w_1 \leq z_2$,

$$
\begin{aligned}
p\beta_1 p\beta_2 p \ldots p\beta_{k_2-1}p &\leq (x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1} \\
&\quad (x\alpha w_1\beta r_1) \\
&\leq (x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1} \\
&\quad (x\alpha z_2\beta r_1) \\
&\leq x\alpha w_2\beta r_1.
\end{aligned}
$$

Thus $p\beta_1 p\beta_2 \ldots \beta_{k_2-1}p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 1.4: $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1\delta_2 s_2$. Since $w_1 \leq z_2$,

$$
\begin{aligned}
p\beta_1 p\beta_2 p \ldots p\beta_{k_2-1}p &\leq (x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1} \\
&\quad (x\alpha w_1\beta r_1) \\
&\leq (x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1} \\
&\quad (x\alpha z_2\beta r_1) \\
&\leq x\alpha w_2\beta r_1\delta_2 s_2.
\end{aligned}
$$

Thus $p\beta_1 p\beta_2 p \ldots p\beta_{k_2-1}p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 2: $x\alpha z_1\beta y \leq x\alpha w_1\beta y\delta_1 s_1$. Since $x_1 \leq z_1$, $p \leq x\alpha z_1\beta y \leq x\alpha w_1\beta y\delta_1 s_1$. Thus $p \leq x\alpha w_1\beta r_1$ where $r_1 = y\delta_1 s_1$. Since $(z_2, w_2) \in \rho$, $(x\alpha z_2\beta r_1, x\alpha w_2\beta r_1) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha z_2\beta r_1) \in \bar{\eta}_r$. Then $x\alpha z_2\beta r_1 \in (x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M]$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \in (x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M]$ for some $\beta_1, \beta_2, \ldots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$ or $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1$ or $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 2.1: $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$. Since $w_1 \leq z_2$, $p \leq x\alpha w_1\beta r_1 \leq x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$. Thus $p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 2.2: $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$. Since $w_1 \leq z_2$, $p \leq x\alpha w_1\beta r_1 \leq x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$. Thus $p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 2.3: $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\dots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1$. Since $w_1 \leq z_2$,

$$\begin{aligned} p\beta_1 p\beta_2 p\dots p\beta_{k_2-1}p &\leq (x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\dots(x\alpha w_1\beta r_1)\beta_{k_2-1} \\ &\quad (x\alpha w_1\beta r_1) \\ &\leq (x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\dots(x\alpha z_2\beta r_1)\beta_{k_2-1} \\ &\quad (x\alpha z_2\beta r_1) \\ &\leq x\alpha w_2\beta r_1. \end{aligned}$$

Thus $p\beta_1 p\beta_2 p\dots p\beta_{k_2-1}p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 2.4: $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\dots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1\delta_2 s_2$. Since $w_1 \leq z_2$,

$$\begin{aligned} p\beta_1 p\beta_2 p\dots p\beta_{k_2-1}p &\leq (x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\dots(x\alpha w_1\beta r_1)\beta_{k_2-1} \\ &\quad (x\alpha w_1\beta r_1) \\ &\leq (x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\dots(x\alpha z_2\beta r_1)\beta_{k_2-1} \\ &\quad (x\alpha z_2\beta r_1) \\ &\leq x\alpha w_2\beta r_1\delta_2 s_2. \end{aligned}$$

Thus $p\beta_1 p\beta_2 p\dots p\beta_{k_2-1}p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 3: $(x\alpha z_1\beta y)\alpha_1(x\alpha z_1\beta y)\alpha_2(x\alpha z_1\beta y)\dots(x\alpha z_1\beta y)\alpha_{k_1-1}(x\alpha z_1\beta y) \leq x\alpha w_1\beta y$. Since $x_1 \leq z_1$, $p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p \leq (x\alpha z_1\beta y)\alpha_1(x\alpha z_1\beta y)\alpha_2(x\alpha z_1\beta y)\dots(x\alpha z_1\beta y)\alpha_{k_1-1}(x\alpha z_1\beta y) \leq x\alpha w_1\beta y$. Thus $p\alpha_1 p\alpha_2 p\dots p\alpha_{k_1-1}p \leq x\alpha w_1\beta r_1$ where $r_1 = y$. Next, since $(z_2, w_2) \in \rho$, $(x\alpha z_2\beta r_1, x\alpha w_2\beta r_1) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha z_2\beta r_1) \in \bar{\eta}_r$. Then $x\alpha z_2\beta r_1 \in (x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M]$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\dots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \in (x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M]$ for some $\beta_1, \beta_2, \dots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$ or

$x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2$ $(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1$ or $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 3.1: $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$. Since $w_1 \leq z_2$, $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p \leq x\alpha w_1\beta r_1 \leq x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$. Thus $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 3.2: $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$. Since $w_1 \leq z_2$, $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p \leq x\alpha w_1\beta r_1 \leq x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$. Thus $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 3.3: $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1$. Since $w_1 \leq z_2$,

$$\begin{aligned}
&(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\ldots\\
&(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\\
\leq\ & (x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1)\\
\leq\ & (x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1)\\
\leq\ & x\alpha w_2\beta r_1.
\end{aligned}$$

Therefore $(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\ldots(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p) \leq x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 3.4: $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1\delta_2 s_2$. Since $w_1 \leq z_2$,

$$\begin{aligned}
&(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\ldots\\
&(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\\
\leq\ & (x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1)\\
\leq\ & (x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1)\\
\leq\ & x\alpha w_2\beta r_1\delta_2 s_2.
\end{aligned}$$

Therefore $(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\ldots(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p) \leq x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 4: $(x\alpha z_1\beta y)\alpha_1(x\alpha z_1\beta y)\alpha_2(x\alpha z_1\beta y)\ldots(x\alpha z_1\beta y)\alpha_{k_1-1}(x\alpha z_1\beta y) \leq x\alpha w_1\beta y \delta_1 s_1$. Since $x_1 \leq z_1$, $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p \leq (x\alpha z_1\beta y)\alpha_1(x\alpha z_1\beta y)\alpha_2(x\alpha z_1\beta y)\ldots (x\alpha z_1\beta y)\alpha_{k_1-1}(x\alpha z_1\beta y) \leq x\alpha w_1\beta y\delta_1 s_1$. Hence $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p \leq x\alpha w_1\beta r_1$ where $r_1 = y\delta_1 s_1$. Next, since $(z_2, w_2) \in \rho$, $(x\alpha z_2\beta r_1, x\alpha w_2\beta r_1) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha z_2\beta r_1) \in \bar{\eta}_r$. Then $x\alpha z_2\beta r_1 \in (x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M]$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1) \beta_{k_2-1}(x\alpha z_2\beta r_1) \in (x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M]$ for some $\beta_1, \beta_2, \ldots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$ or $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1$ or $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 4.1: $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$. Since $w_1 \leq z_2$, $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p \leq x\alpha w_1\beta r_1 \leq x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1$. Thus $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 4.2: $x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$. Since $w_1 \leq z_2$, $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p \leq x\alpha w_1\beta r_1 \leq x\alpha z_2\beta r_1 \leq x\alpha w_2\beta r_1\delta_2 s_2$. Thus $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p \leq x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 4.3: $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha w_2\beta r_1$. Since $w_1 \leq z_2$,

$$
\begin{aligned}
&(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\ldots\\
&(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\\
\leq\ &(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1)\\
\leq\ &(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1)\\
\leq\ &x\alpha w_2\beta r_1.
\end{aligned}
$$

Therefore $(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\ldots (p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p) \leq x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 4.4: $(x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\ldots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \leq x\alpha$

$w_2\beta r_1\delta_2 s_2$. Since $w_1 \leq z_2$,

$$\begin{aligned}
&(p\alpha_1 p\alpha_2 p \dots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p \dots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p \dots p\alpha_{k_1-1}p)\dots \\
&(p\alpha_1 p\alpha_2 p \dots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p \dots p\alpha_{k_1-1}p) \\
\leq\ & (x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\dots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) \\
\leq\ & (x\alpha z_2\beta r_1)\beta_1(x\alpha z_2\beta r_1)\beta_2(x\alpha z_2\beta r_1)\dots(x\alpha z_2\beta r_1)\beta_{k_2-1}(x\alpha z_2\beta r_1) \\
\leq\ & x\alpha w_2\beta r_1\delta_2 s_2.
\end{aligned}$$

Therefore $(p\alpha_1 p\alpha_2 p \dots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p \dots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p \dots p\alpha_{k_1-1}p)\dots$ $(p\alpha_1 p\alpha_2 p \dots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p \dots p\alpha_{k_1-1}p) \leq x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

If we continue in this way, we have $p \leq x\alpha y_1\beta r_m$ or there exists an integer $k \geq 2$ such that $p\gamma_1 p\gamma_2 p \dots p\gamma_{k-1}p \leq x\alpha y_1\beta r_m$ for some $\lambda_1, \lambda_2, \dots, \lambda_{k-1} \in \Gamma$. Since $(y_1, y_2) \in \rho$, $(x\alpha y_1\beta r_m, x\alpha y_2\beta r_m) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$. Thus $(x\alpha y_2\beta r_m, x\alpha y_1\beta r_m) \in \bar{\eta}_r$. Then $x\alpha y_1\beta r_m \in (x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M]$ or there exists an integer $k' \geq 2$ such that $(x\alpha y_1\beta r_m)\gamma_1'(x\alpha y_1\beta r_m)\gamma_2'(x\alpha y_1\beta r_m)\dots(x\alpha y_1\beta r_m)\gamma_{k'-1}'(x\alpha y_1\beta r_m) \in (x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M]$ for some $\gamma_1', \gamma_2', \dots, \gamma_{k'-1}' \in \Gamma$.

Put $q = x\alpha y_1\beta r_m$.

Case I: $p \leq q$ and $q \in (x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M]$. Then $p \leq q \in (x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M] \subseteq (x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M]$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \bar{\eta}_r$.

Case II: $p \leq q$ and $q\gamma_1' q\gamma_2' q \dots q\gamma_{k'-1}' q \in (x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M]$. Then $p\gamma_1' p\gamma_2' p \dots p\gamma_{k'-1}' p \leq q\gamma_1' q\gamma_2' q \dots q\gamma_{k'-1}' q \in (x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M] \subseteq (x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M]$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \bar{\eta}_r$.

Case III: $p\gamma_1 p\gamma_2 p \dots p\gamma_{k-1}p \leq q$ and $q \in (x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M]$. Then $p\gamma_1 p\gamma_2 p \dots p\gamma_{k-1}p \leq q \in (x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M] \subseteq (x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M]$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \bar{\eta}_r$.

Case IV: $p\gamma_1 p\gamma_2 p \dots p\gamma_{k-1}p \leq q$ and $q\gamma_1' q\gamma_2' q \dots q\gamma_{k'-1}' q \in (x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M]$. Then $(p\gamma_1 p\gamma_2 p \dots p\gamma_{k-1}p)\gamma_1'(p\gamma_1 p\gamma_2 p \dots p\gamma_{k-1}p)\gamma_2'(p\gamma_1 p\gamma_2 p \dots p\gamma_{k-1}p)\dots(p\gamma_1 p\gamma_2 p \dots p\gamma_{k-1}p)\gamma_{k'-1}'(p\gamma_1 p\gamma_2 p \dots p\gamma_{k-1}p) \leq q\gamma_1' q\gamma_2' q \dots q\gamma_{k'-1}' q \in (x\alpha y_2\beta r_m \cup x\alpha y_2\beta$

$r_m\Gamma M] \subseteq (x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M]$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \bar{\eta}_r$.

Similar to the proof as above, by $(\star\star)$, we can prove that $(x\alpha x_1\beta y, x\alpha y_2\beta y) \in \bar{\eta}_r$. Therefore $(x\alpha x_1\beta y, x\alpha y_2\beta y) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$.

In another case, we have the following statements:

1. If $(z_1, w_1) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$, then $(x_1, y_2) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$.

2. If $(x\alpha z_1, x\alpha w_1) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$, then $(x\alpha x_1, x\alpha y_2) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$.

3. If $(z_1\beta y, w_1\beta y) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$, then $(x_1\beta y, y_2\beta y) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$.

Therefore $(x_1, y_2) \in \rho$, so $(x)_\rho = (x_1)_\rho = (y_2)_\rho = (y)_\rho$. Hence $\preceq$ is anti-symmetric.

(iii) Let $(x)_\rho, (y)_\rho, (z)_\rho \in M/\rho$ be such that $(x)_\rho \preceq (y)_\rho$ and $(y)_\rho \preceq (z)_\rho$. Then there exist $x_1 \in (x)_\rho, y_1, y_2 \in (y)_\rho, z_2 \in (z)_\rho$ and $m, n \in \mathbb{N}$ such that $(x_1, y_1) \in (\leq \circ\rho)^m$ and $(y_2, z_2) \in (\leq \circ\rho)^n$. Thus $x_1(\leq \circ\rho)^m y_1 \rho y_2 (\leq \circ\rho)^n z_2$, so $(x_1, z_2) \in (\leq \circ\rho)^m \circ \rho \circ (\leq \circ\rho)^n$. Since $\rho \circ \rho \subseteq \rho$, $(x_1, z_2) \in (\leq \circ\rho)^{m+n}$. Hence $(x)_\rho \preceq (z)_\rho$. Therefore $\preceq$ is transitive.

(iv) Let $(x)_\rho, (y)_\rho \in M/\rho$ be such that $(x)_\rho \preceq (y)_\rho, (z)_\rho \in M/\rho$ and $\gamma \in \Gamma$. Then there exist $x_1 \in (x)_\rho, y_1 \in (y)_\rho$ and $m \in \mathbb{N}$ such that $(x_1, y_1) \in (\leq \circ\rho)^m$. It follows from Corollary 5.1.2 that $(z\gamma x_1, z\gamma y_1), (x_1\gamma z, y_1\gamma z) \in (\leq \circ\rho)^m$. Since $z\gamma x_1 \in (z\gamma x)_\rho, x_1\gamma z \in (x\gamma z)_\rho, z\gamma y_1 \in (z\gamma y)_\rho$ and $y_1\gamma z \in (y\gamma z)_\rho$, we have $(z\gamma x)_\rho \preceq (z\gamma y)_\rho$ and $(x\gamma z)_\rho \preceq (y\gamma z)_\rho$. Thus $(z)_\rho\gamma(x)_\rho = (z\gamma x)_\rho \preceq (z\gamma y)_\rho = (z)_\rho\gamma(y)_\rho$ and $(x)_\rho\gamma(z)_\rho = (x\gamma z)_\rho \preceq (y\gamma z)_\rho = (y)_\rho\gamma(z)_\rho$. Hence $\preceq$ is compatible, so $(M/\rho; \preceq)$ is an ordered Γ-semigroup.

(v) Let $x, y \in M$ be such that $x \leq y$. Then $x \leq y\rho y$, so $(x, y) \in (\leq \circ\rho)$. Hence $(x)_\rho \preceq (y)_\rho$, so $\varphi(x) \preceq \varphi(y)$. Therefore ρ is a regular congruence on M.

This completes the proof. □

Immediately from Theorems 5.2.2 and 5.2.3, we have Corollary 5.2.4.

Corollary 5.2.4. *An ordered Γ-semigroup M is a band of weakly r-archimedean sub-Γ-semigroups of M if and only if it is a regular band of weakly r-archimedean sub-Γ-semigroups of M.*

Lemma 5.2.5. *Let M be a negative ordered Γ-semigroup, and T be a sub-Γ-semigroup of M. Then the following statements are equivalent:*

(a) *T is a weakly r-archimedean sub-Γ-semigroup of M.*

(b) *T is an r-archimedean sub-Γ-semigroup of M.*

(c) *For any $a, b \in T, (a, b) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$.*

Proof. By Lemma 5.2.1, (b) implies (c) and (c) implies (a). Now, we shall prove that (a) implies (b). Suppose that T is a weakly r-archimedean sub-Γ-semigroup of M, and let $a, b \in T$. It follows from Lemma 5.2.1 that $(a, b) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}$, so $(a, b) \in \bar{\eta}_r$. Thus $b \in (a \cup a\Gamma M]$ or there exists an integer $m \geq 2$ such that $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1} b \in (a \cup a\Gamma M]$ for some $\alpha_1, \alpha_2, \dots, \alpha_{m-1} \in \Gamma$. For any $\gamma \in \Gamma$, if $b \leq a$ or $b \leq a\alpha x$ for some $x \in M$ and $\alpha \in \Gamma$, then $a\alpha x \leq a$ since M is negative. Hence $b \leq a$, so $b\gamma b \leq a\gamma b \in a\Gamma T \subseteq (a\Gamma T]$. If $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1} b \leq a$ or $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1} b \leq a\alpha x$ for some $x \in M$ and $\alpha \in \Gamma$, then $a\alpha x \leq a$ since M is negative. Hence $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1} b \leq a$, so $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1} b\gamma b \leq a\gamma b \in a\Gamma T \subseteq (a\Gamma T]$. Therefore T is an r-archimedean sub-Γ-semigroup of M. □

Combining Theorem 5.2.2 and Lemma 5.2.5, we obtain Corollary 5.2.6.

Corollary 5.2.6. *Let M be a negative ordered Γ-semigroup. Then the following statements are equivalent:*

(a) *M is a band of weakly r-archimedean sub-Γ-semigroups of M.*

(b) *M is a band of r-archimedean sub-Γ-semigroups of M.*

(c) *M satisfies the condition for all* $a, x, y \in M$ *and* $\alpha, \beta, \gamma \in \Gamma$,

$$(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \bar{\eta}_r \cap \bar{\eta}_r^{-1}. \qquad (\star)$$