CHAPTER VI

DECOMPOSITION OF COMMUTATIVE (ORDERED) Γ-SEMIGROUPS INTO ARCHIMEDEAN COMPONENTS

In this chapter, we divide into two sections, and many properties of the relation " $(\bar{\eta})\ \eta$ " on a (ordered) Γ-semigroup M are provided. We prove that for commutative (ordered) Γ-semigroups, we have the usual relation $(\mathcal{N})\ n$ is equal to the relation $(\bar{\eta})\ \eta$. It is shown that if M is commutative, then M is, uniquely, a (ordered) semilattice of archimedean sub-Γ-semigroups of M which mean that they are decomposable, in a unique way, into their archimedean components.

6.1 Decomposition of Commutative Γ-Semigroups into Archimedean Components

Before the characterizations of the semilattices of archimedean sub-Γ-semigroups for the main theorems, we give some auxiliary results which are necessary in what follows. We begin by recalling the following two lemmas which proof can be found in [13].

Lemma 6.1.1.([13]) *If $\rho \in SC(M)$, then the following statements hold:*

(a) *For each $x \in M$, the ρ-class $(x)_\rho$ is a sub-Γ-semigroup of M.*

(b) *The set M/ρ is a commutative Γ-semigroup with $(x)_\rho\gamma(y)_\rho = (x\gamma y)_\rho$ for all $x, y \in M$ and $\gamma \in \Gamma$.*

Lemma 6.1.2.([13]) *If M is a Γ-semigroup, then the relation n is the least semilattice congruence on M.*

We now characterize the relationship between the relation η and the usual relation n in Γ-semigroups, and we prove that for commutative Γ-semigroups, the relation η coincides with the usual relation n.

Proposition 6.1.3. *If M is a Γ-semigroup, then the following statements hold:*

(a) *$a \mid a$ for all $a \in M$.*

(b) *If $a, b, c \in M$ is such that $a \mid b$ and $b \mid c$, then $a \mid c$.*

(c) *If $a, b \in M$ is such that $a \mid b$, then $c\gamma a \mid c\gamma b$ for all $c \in M$ and $\gamma \in \Gamma$.*

(d) *If M is commutative, and $a, b \in M$ is such that $a \mid b$, then $a\gamma c \mid b\gamma c$ for all $c \in M$ and $\gamma \in \Gamma$.*

Proof. (a) Clearly, $a \mid a$ for all $a \in M$. Hence the statement (a) holds.

(b) If $a, b, c \in M$ is such that $a \mid b$ and $b \mid c$, then $b = a$ or $b = a\gamma x$ for some $x \in M$ and $\gamma \in \Gamma$, and $c = b$ or $c = b\beta y$ for some $y \in M$ and $\beta \in \Gamma$. If $b = a\gamma x$ and $c = b\beta y$, then $c = b\beta y = a\gamma x\beta y$. Hence $a \mid c$. In another case, we can prove that $a \mid c$. Therefore the statement (b) holds.

(c) If $a, b \in M$ is such that $a \mid b$, then $b = a$ or $b = a\alpha x$ for some $x \in M$ and $\alpha \in \Gamma$. Thus $c\gamma b = c\gamma a$ or $c\gamma b = c\gamma a\alpha x$ for all $c \in M$ and $\gamma \in \Gamma$. Hence $c\gamma a \mid c\gamma b$ for all $c \in M$ and $\gamma \in \Gamma$. Therefore the statement (c) holds.

(d) If M is commutative, then by (c), the statement (d) also holds.

Therefore we complete the proof of the proposition. □

Recall that for $x, y \in M$, we write $x\mu y$ if and only if $x \mid y$ or $x \mid y\gamma_1 y\gamma_2 y ... y\gamma_m y$ for some $m \in \mathbb{N}$ and $\gamma_1, \gamma_2, ..., \gamma_m \in \Gamma$, and $\eta = \mu \cap \mu^{-1}$.

The following proposition is easy to verify.

Proposition 6.1.4. *If M is a Γ-semigroup, then the following statements hold:*

(a) η *is reflexive.*

(b) η *is symmetric.*

(c) $(a\gamma a, a) \in \eta$ *for all* $a \in M$ *and* $\gamma \in \Gamma$.

Proposition 6.1.5. *If* M *is a commutative* Γ*-semigroup, then the following statements hold:*

(a) *If* $a, b \in M$ *is such that* $a \mid b$*, then for any* $n \in \mathbb{N}$ *and* $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$,

$$a\gamma_1 a\gamma_2 a \ldots a\gamma_n a \mid b\gamma_1 b\gamma_2 b \ldots b\gamma_n b.$$

(b) *If* $a, b \in M$ *and* $\beta \in \Gamma$*, then for any* $n \in \mathbb{N}$ *and* $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$,

$$a\beta(b\gamma_1 b\gamma_2 b \ldots b\gamma_n b) \mid (a\beta b)\gamma_1(a\beta b)\gamma_2(a\beta b) \ldots (a\beta b)\gamma_n(a\beta b).$$

Proof. (a) Let $a, b \in M$ be such that $a \mid b, n \in \mathbb{N}$ and $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$. Then $b = a$ or $b = a\beta x$ for some $x \in M$ and $\beta \in \Gamma$. If $b = a$, then $b\gamma_1 b\gamma_2 b \ldots b\gamma_n b = a\gamma_1 a\gamma_2 a \ldots a\gamma_n a$. Hence $a\gamma_1 a\gamma_2 a \ldots a\gamma_n a \mid b\gamma_1 b\gamma_2 b \ldots b\gamma_n b$. If $b = a\beta x$, then $b\gamma_1 b\gamma_2 b \ldots b\gamma_n b = (a\beta x)\gamma_1(a\beta x)\gamma_2(a\beta x) \ldots (a\beta x)\gamma_n(a\beta x) = (a\gamma_1 a\gamma_2 a \ldots a\gamma_n a)\beta (x\beta x\beta x \ldots x\beta x)$. Hence $a\gamma_1 a\gamma_2 a \ldots a\gamma_n a \mid b\gamma_1 b\gamma_2 b \ldots b\gamma_n b$.

(b) Let $a, b \in M, n \in \mathbb{N}$ and $\beta, \gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$. Put $\beta_1 = \beta_2 = \ldots = \beta_n = \beta_{n+1} = \beta$. Then $(a\beta b)\gamma_1(a\beta b)\gamma_2(a\beta b) \ldots (a\beta b)\gamma_n(a\beta b) = (a\beta_1 b)\gamma_1(a\beta_2 b) \gamma_2(a\beta_3 b) \ldots (a\beta_n b)\gamma_n(a\beta_{n+1} b) = (a\beta_1 a\beta_2 a \ldots a\beta_n a)\beta(b\gamma_1 b\gamma_2 b \ldots b\gamma_n b) = a\beta(b\gamma_1 b \gamma_2 b \ldots b\gamma_n b)\beta(a\beta_1 a\beta_2 a \ldots a\beta_{n-1} a)$. Hence

$$a\beta(b\gamma_1 b\gamma_2 b \ldots b\gamma_n b) \mid (a\beta b)\gamma_1(a\beta b)\gamma_2(a\beta b) \ldots (a\beta b)\gamma_n(a\beta b).$$

Therefore the proof is completed. □

Proposition 6.1.6. *If* M *is a commutative* Γ*-semigroup, then the following statements hold:*

(a) η *is transitive.*

(b) η *is left compatible.*

(c) η *is right compatible.*

(d) $(a\gamma b, b\gamma a) \in \eta$ *for all* $a, b \in M$ *and* $\gamma \in \Gamma$.

Proof. (a) Let $a, b, c \in M$ be such that $(a, b) \in \eta$ and $(b, c) \in \eta$. Since $(a, b) \in \eta$, we have

$$a \mid b \text{ or } a \mid b\gamma_1 b\gamma_2 b \dots b\gamma_m b \text{ for some } m \in \mathbb{N} \text{ and } \gamma_1, \gamma_2, \dots, \gamma_m \in \Gamma,$$

and

$$b \mid a \text{ or } b \mid a\beta_1 a\beta_2 a \dots a\beta_n a \text{ for some } n \in \mathbb{N} \text{ and } \beta_1, \beta_2, \dots, \beta_n \in \Gamma.$$

Since $(b, c) \in \eta$, we have

$$b \mid c \text{ or } b \mid c\alpha_1 c\alpha_2 c \dots c\alpha_t c \text{ for some } t \in \mathbb{N} \text{ and } \alpha_1, \alpha_2, \dots, \alpha_t \in \Gamma,$$

and

$$c \mid b \text{ or } c \mid b\lambda_1 b\lambda_2 b \dots b\lambda_h b \text{ for some } h \in \mathbb{N} \text{ and } \lambda_1, \lambda_2, \dots, \lambda_h \in \Gamma.$$

Assume that $a \mid b\gamma_1 b\gamma_2 b \dots b\gamma_m b$, $b \mid a\beta_1 a\beta_2 a \dots a\beta_n a$, $b \mid c\alpha_1 c\alpha_2 c \dots c\alpha_t c$ and $c \mid b\lambda_1 b\lambda_2 b \dots b\lambda_h b$. Since M is commutative and using Proposition 6.1.5(*a*), we have

$$b\gamma_1 b\gamma_2 b \dots b\gamma_m b \mid c'\gamma_1 c'\gamma_2 c' \dots c'\gamma_m c' \text{ where } c' = c\alpha_1 c\alpha_2 c \dots c\alpha_t c.$$

By Proposition 6.1.3(*b*), we have $a \mid c'\gamma_1 c'\gamma_2 c' \dots c'\gamma_m c'$. In a similar way, we prove that $c \mid a'\lambda_1 a'\lambda_2 a' \dots a'\lambda_h a'$ where $a' = a\beta_1 a\beta_2 a \dots a\beta_n a$. Hence $(a, c) \in \eta$. In another case, we can prove that $(a, c) \in \eta$. Therefore η is transitive.

(*b*) Let $a, b, c \in M$ and $\gamma \in \Gamma$ be such that $(a, b) \in \eta$. Then

$$a \mid b \text{ or } a \mid b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \text{ for some } m \in \mathbb{N} \text{ and } \gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma,$$

and

$$b \mid a \text{ or } b \mid a\beta_1 a\beta_2 a \ldots a\beta_n a \text{ for some } n \in \mathbb{N} \text{ and } \beta_1, \beta_2, \ldots, \beta_n \in \Gamma.$$

Assume that $a \mid b\gamma_1 b\gamma_2 b \ldots b\gamma_m b$ and $b \mid a\beta_1 a\beta_2 a \ldots a\beta_n a$. By Proposition 6.1.3($c$), we have $c\gamma a \mid c\gamma(b\gamma_1 b\gamma_2 b \ldots b\gamma_m b)$. Since M is commutative and using Proposition 6.1.5(b), we have

$$c\gamma(b\gamma_1 b\gamma_2 b \ldots b\gamma_n b) \mid (c\gamma b)\gamma_1(c\gamma b)\gamma_2(c\gamma b) \ldots (c\gamma b)\gamma_m(c\gamma b).$$

By Proposition 6.1.3(b), we have

$$c\gamma a \mid (c\gamma b)\gamma_1(c\gamma b)\gamma_2(c\gamma b) \ldots (c\gamma b)\gamma_m(c\gamma b).$$

In a similar way, we prove that

$$c\gamma b \mid (c\gamma a)\beta_1(c\gamma a)\beta_2(c\gamma a) \ldots (c\gamma a)\beta_n(c\gamma a).$$

Hence $(c\gamma a, c\gamma b) \in \eta$. In another case, we can prove that $(c\gamma a, c\gamma b) \in \eta$. Therefore η is left compatible.

(c) Since M is commutative, it follows from (b).

(d) Since M is commutative and using Proposition 6.1.4(a), $(a\gamma b, b\gamma a) \in \eta$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Hence we have the proposition. □

Immediately from Propositions 6.1.4 and 6.1.6, we have Theorem 6.1.7.

Theorem 6.1.7. *If M is a commutative Γ-semigroup, then η is a semilattice congruence on M.*

Lemma 6.1.8. *If M is a Γ-semigroup, and $a, b \in M$ is such that $a \mid b$, then $n(a) \subseteq n(b)$.*

Proof. Assume that $a, b \in M$ is such that $a \mid b$. Then $b = a$ or $b = a\gamma x$ for some $x \in M$ and $\gamma \in \Gamma$. Since $b \in n(b)$, we have $a \in n(b)$ or $a\gamma x \in n(b)$. Hence $a \in n(b)$, so $n(a) \subseteq n(b)$. □

Theorem 6.1.9. *If M is a commutative Γ-semigroup, then $\eta = n$.*

Proof. Let $a, b \in M$ be such that $(a, b) \in \eta$. Then

$$a \mid b \text{ or } a \mid b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \text{ for some } m \in \mathbb{N} \text{ and } \gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma,$$

and

$$b \mid a \text{ or } b \mid a\beta_1 a\beta_2 a \ldots a\beta_k a \text{ for some } k \in \mathbb{N} \text{ and } \beta_1, \beta_2, \ldots, \beta_k \in \Gamma.$$

Assume that $a \mid b\gamma_1 b\gamma_2 b \ldots b\gamma_m b$ and $b \mid a\beta_1 a\beta_2 a \ldots a\beta_k a$. Then

$$b\gamma_1 b\gamma_2 b \ldots b\gamma_m b = a \text{ or } b\gamma_1 b\gamma_2 b \ldots b\gamma_m b = a\alpha x \text{ for some } x \in M \text{ and } \alpha \in \Gamma.$$

Since $b \in n(b)$, we have $b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \in n(b)$. This implies that $a \in n(b)$ or $a\alpha x \in n(b)$. Hence $a \in n(b)$, so $n(a) \subseteq n(b)$. Since $b \mid a\beta_1 a\beta_2 a \ldots a\beta_k a$, by symmetry, we get $n(b) \subseteq n(a)$. Therefore $n(a) = n(b)$, so $(a, b) \in n$. In another case, we can prove that $(a, b) \in n$. Hence $\eta \subseteq n$. On the other hand, by Theorem 6.1.7 and Lemma 6.1.2, we have $n \subseteq \eta$. Therefore $\eta = n$.

Therefore we complete the proof of the theorem. □

Proposition 6.1.10. *For a Γ-semigroup M, $\delta \cap \delta^{-1} \subseteq n$.*

Proof. Let $a, b \in M$ be such that $(a, b) \in \delta \cap \delta^{-1}$. Then $(a, b) \in \delta$, so $b \mid a$. By Lemma 6.1.8, $n(b) \subseteq n(a)$. Since $(a, b) \in \delta^{-1}$, $(b, a) \in \delta$. By symmetry, we have $n(a) \subseteq n(b)$. Thus $n(a) = n(b)$, so $(a, b) \in n$. Therefore $\delta \cap \delta^{-1} \subseteq n$. □

Proposition 6.1.11. *If M is a Γ-semigroup and $a, b \in M$, then the following statements are equivalent:*

(a) $a \mid b$ *or* $a \mid b\gamma_1 b\gamma_2 b \dots b\gamma_m b$ *for some* $m \in \mathbb{N}$ *and* $\gamma_1, \gamma_2, \dots, \gamma_m \in \Gamma$.

(b) $b = a\beta y$ *for some* $y \in M$ *and* $\beta \in \Gamma$ *or* $b\beta_1 b\beta_2 b \dots b\beta_n b = a\beta_{n+1} y$ *for some* $n \in \mathbb{N}, y \in M$ *and* $\beta_1, \beta_2, \dots, \beta_n, \beta_{n+1} \in \Gamma$.

Proof. If $a \mid b$, then $b = a$ or $b = a\gamma x$ for some $x \in M$ and $\gamma \in \Gamma$. Thus $b\gamma b = a\gamma b$ or $b\gamma b = a\gamma x\gamma b$. Hence there exist $y \in M$ and $\beta_1, \beta_2 \in \Gamma$ such that $b\beta_1 b = a\beta_2 y$. Assume that $a \mid b\gamma_1 b\gamma_2 b \dots b\gamma_m b$ for some $m \in \mathbb{N}$ and $\gamma_1, \gamma_2, \dots, \gamma_m \in \Gamma$. Then

$$b\gamma_1 b\gamma_2 b \dots b\gamma_m b = a \text{ or } b\gamma_1 b\gamma_2 b \dots b\gamma_m b = a\beta x \text{ for some } x \in M \text{ and } \beta \in \Gamma.$$

Put $\gamma_{m+1} = \beta$. Then

$$b\gamma_1 b\gamma_2 b \dots b\gamma_m b\gamma_{m+1} b = a\gamma_{m+1} b \text{ or } b\gamma_1 b\gamma_2 b \dots b\gamma_m b\gamma_{m+1} b = a\beta x\gamma_{m+1} b.$$

Hence there exist $n \in \mathbb{N}, y \in M$ and $\beta_1, \beta_2, \dots, \beta_n, \beta_{n+1} \in \Gamma$ such that $b\beta_1 b\beta_2 b \dots b\beta_n b = a\beta_{n+1} y$. Therefore (a) implies (b).

Conversely, if $b = a\beta y$ for some $y \in M$ and $\beta \in \Gamma$, then $a \mid b$. Assume that $b\beta_1 b\beta_2 b \dots b\beta_n b = a\beta_{n+1} y$ for some $n \in \mathbb{N}, y \in M$ and $\beta_1, \beta_2, \dots, \beta_n, \beta_{n+1} \in \Gamma$. Then $a \mid b\beta_1 b\beta_2 b \dots b\beta_n b$. Hence there exist $m \in \mathbb{N}$ and $\gamma_1, \gamma_2, \dots, \gamma_m \in \Gamma$ such that $a \mid b\gamma_1 b\gamma_2 b \dots b\gamma_m b$. Therefore (b) implies (a). □

Using the relation η defined above, we prove that the commutative Γ-semigroups are, uniquely, semilattices of archimedean sub-Γ-semigroups. That is, they are decomposable into archimedean sub-Γ-semigroups, and the decomposition is unique.

Proposition 6.1.12. *If* M *is a commutative* Γ*-semigroup, then the* η*-class* $(x)_\eta$ *is an archimedean sub-*Γ*-semigroup of* M *for all* $x \in M$.

Proof. Let $x \in M$. Since $\eta \in SC(M)$ and using Lemma 6.1.1(a), we have the η-class $(x)_\eta$ is a sub-Γ-semigroup of M. Let $a, b \in M$ be such that $a, b \in (x)_\eta$. Then $(a, b) \in \eta$. Thus

$$a \mid b \text{ or } a \mid b\gamma_1 b\gamma_2 b \ldots b\gamma_t b \text{ for some } t \in \mathbb{N} \text{ and } \gamma_1, \gamma_2, \ldots, \gamma_t \in \Gamma,$$

and

$$b \mid a \text{ or } b \mid a\beta_1 a\beta_2 a \ldots a\beta_h a \text{ for some } h \in \mathbb{N} \text{ and } \beta_1, \beta_2, \ldots, \beta_h \in \Gamma.$$

Since $a \mid b$ or $a \mid b\gamma_1 b\gamma_2 b \ldots b\gamma_t b$, it follows from Proposition 6.1.11 that there exist $u \in \mathbb{N}, s \in M$ and $\alpha_1, \alpha_2, \ldots, \alpha_u \in \Gamma$ such that $b\alpha_1 b\alpha_2 b \ldots b\alpha_u b = a\alpha_{u+1} s$. Similarly, there exist $v \in \mathbb{N}, k \in M$ and $\lambda_1, \lambda_2, \ldots, \lambda_v \in \Gamma$ such that $a\lambda_1 a\lambda_2 a \ldots a\lambda_v a = b\lambda_{v+1} k$. Thus $b\alpha_1 b\alpha_2 b \ldots b\alpha_u b\alpha_{u+1} b = a\alpha_{u+1} s\alpha_{u+1} b = b\alpha_{u+1} s\alpha_{u+1} a$, so $b\alpha_{u+1} s \mid b\alpha_1 b\alpha_2 b \ldots b\alpha_u b\alpha_{u+1} b$. Then $b \mid b\alpha_{u+1} s$. Hence $(b\alpha_{u+1} s, b) \in \eta$, so $b\alpha_{u+1} s \in (b)_\eta = (x)_\eta$. Thus $b\alpha_1 b\alpha_2 b \ldots b\alpha_u b\alpha_{u+1} b = a\alpha_{u+1} b\alpha_{u+1} s$ where $u + 1 \in \mathbb{N}$ and $b\alpha_{u+1} s \in (x)_\eta$, that is $a \mid_{(x)_\eta} b\alpha_1 b\alpha_2 b \ldots b\alpha_u b\alpha_{u+1} b$.

In a similar way, we can prove that there exist $n \in \mathbb{N}, z \in (x)_\eta$ and $\beta_1, \beta_2, \ldots, \beta_n$ such that $a\beta_1 a\beta_2 a \ldots a\beta_n a = b\beta_{n+1} z$. Hence $b \mid_{(x)_\eta} a\beta_1 a\beta_2 a \ldots a\beta_n a$. Therefore $(x)_\eta$ is an archimedean sub-Γ-semigroup of M. □

Immediately from Theorem 6.1.7 and Proposition 6.1.12, we have Theorem 6.1.13.

Theorem 6.1.13. *If M is a commutative Γ-semigroup, then M is a semilattice of archimedean sub-Γ-semigroups of M.*

Proposition 6.1.14. *If M is a commutative Γ-semigroup, and ρ is a semilattice congruence on M such that the ρ-class $(x)_\rho$ is an archimedean sub-Γ-semigroup of M for all $x \in M$, then $\rho = \eta$.*

Proof. Let $a, b \in M$ be such that $(a, b) \in \rho$. Then, since $a, b \in (b)_\rho$ and $(b)_\rho$ is archimedean, we get

$$a \mid_{(b)_\rho} b \text{ or } a \mid_{(b)_\rho} b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \text{ for some } m \in \mathbb{N} \text{ and } \gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma,$$

and

$$b \mid_{(b)_\rho} a \text{ or } b \mid_{(b)_\rho} a\beta_1 a\beta_2 a \ldots a\beta_n a \text{ for some } n \in \mathbb{N} \text{ and } \beta_1, \beta_2, \ldots, \beta_n \in \Gamma.$$

Hence $a \mid b$ or $a \mid b\gamma_1 b\gamma_2 b \ldots b\gamma_m b$, and $b \mid a$ or $b \mid a\beta_1 a\beta_2 a \ldots a\beta_n a$. Therefore $(a, b) \in \eta$, so $\rho \subseteq \eta$. By Lemma 6.1.2 and Theorem 6.1.9, we have η is the least semilattice congruence on M. Hence $\eta \subseteq \rho$, so $\eta = \rho$.

Hence the proof is completed. □

The following theorem is the main result of this section which is immediate from Theorem 6.1.7 and Propositions 6.1.12 and 6.1.14.

Theorem 6.1.15. *If M is a commutative Γ-semigroup, then M is, uniquely, a semilattice of archimedean sub-Γ-semigroups of M.*

6.2 Decomposition of Commutative Ordered Γ-Semigroups into Archimedean Components

Before the characterizations of the ordered semilattices of archimedean sub-Γ-semigroups for the main theorems, we give some auxiliary results which are necessary in what follows. We begin by recalling the following lemma, which proof can be found in [13].

Lemma 6.2.1. ([13]) *If M is an ordered Γ-semigroup, then the relation $\mathcal{N}$ is the least ordered semilattice congruence on M.*

The first purpose of this section is to characterize the relationship between the relation $\bar{\eta}$ and the usual relation $\mathcal{N}$ in ordered Γ-semigroups, and we prove that for commutative ordered Γ-semigroups, the relation $\bar{\eta}$ coincides with the usual relation $\mathcal{N}$.

Our first aim is to give some basic propositions for the main theorems.

Proposition 6.2.2. *If M is an ordered Γ-semigroup, then the following statements hold:*

(a) $a \parallel a$ *for all* $a \in M$.

(b) *If* $a, b, c \in M$ *is such that* $a \parallel b$ *and* $b \parallel c$, *then* $a \parallel c$.

(c) *If* $a, b \in M$ *is such that* $a \parallel b$, *then* $c\gamma a \parallel c\gamma b$ *for all* $c \in M$ *and* $\gamma \in \Gamma$.

(d) *If* M *is commutative, and* $a, b \in M$ *is such that* $a \parallel b$, *then* $a\gamma c \parallel b\gamma c$ *for all* $c \in M$ *and* $\gamma \in \Gamma$.

Proof. (a) Clearly, $a \parallel a$ for all $a \in M$. Hence the statement (a) holds.

(b) If $a, b, c \in M$ is such that $a \parallel b$ and $b \parallel c$, then $b \leq a$ or $b \leq a\gamma x$ for some $x \in M$ and $\gamma \in \Gamma$, and $c \leq b$ or $c \leq b\beta y$ for some $y \in M$ and $\beta \in \Gamma$. If $b \leq a\gamma x$ and $c \leq b\beta y$, then $c \leq b\beta y \leq a\gamma x\beta y$. Hence $a \parallel c$. In another case, we can prove that $a \parallel c$. Therefore the statement (b) holds.

(c) If $a, b \in M$ is such that $a \parallel b$, then $b \leq a$ or $b \leq a\alpha x$ for some $x \in M$ and $\alpha \in \Gamma$. Thus $c\gamma b \leq c\gamma a$ or $c\gamma b \leq c\gamma a\alpha x$ for all $c \in M$ and $\gamma \in \Gamma$. Hence $c\gamma a \parallel c\gamma b$ for all $c \in M$ and $\gamma \in \Gamma$. Therefore the statement (c) holds.

(d) If M is commutative, then by (c), the statement (d) also holds. □

Recall that for $x, y \in M$, we write $x\bar{\mu}y$ if and only if $x \parallel y$ or $x \parallel y\gamma_1 y\gamma_2 y \ldots y\gamma_m y$ for some $m \in \mathbb{N}$ and $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma$, and $\bar{\eta} = \bar{\mu} \cap \bar{\mu}^{-1}$.

Proposition 6.2.3. *If M is an ordered Γ-semigroup, then the following statements hold:*

(a) $\bar{\eta}$ *is reflexive.*

(b) $\bar{\eta}$ *is symmetric.*

(c) *If* $a, b \in M$ *is such that* $a \leq b$, *then* $(a, a\gamma b) \in \bar{\eta}$ *for all* $\gamma \in \Gamma$.

Proof. (a) If $a \in M$, then it follows from Proposition 6.2.2(a) that $a \parallel a$. Thus $(a, a) \in \bar{\eta}$. Hence $\bar{\eta}$ is reflexive.

(b) It is obvious.

(c) Let $a, b \in M$ and $\gamma \in \Gamma$ be such that $a \leq b$. Then $a\gamma a \leq a\gamma b$, so $a\gamma b \parallel a\gamma a$. Clearly, $a \parallel a\gamma b$. Hence $(a, a\gamma b) \in \bar{\eta}$. □

Proposition 6.2.4. *If M is a commutative ordered Γ-semigroup, then the following statements hold:*

(a) *If $a, b \in M$ is such that $a \parallel b$, then for any $n \in \mathbb{N}$ and $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$,*

$$a\gamma_1 a\gamma_2 a \ldots a\gamma_n a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_n b\ .$$

(b) *If $a, b \in M$ and $\beta \in \Gamma$, then for any $n \in \mathbb{N}$ and $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$,*

$$a\beta(b\gamma_1 b\gamma_2 b \ldots b\gamma_n b) \parallel (a\beta b)\gamma_1(a\beta b)\gamma_2(a\beta b)\ldots(a\beta b)\gamma_n(a\beta b).$$

Proof. (a) Let $a, b \in M$ be such that $a \parallel b, n \in \mathbb{N}$ and $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$. Then $b \leq a$ or $b \leq a\beta x$ for some $x \in M$ and $\beta \in \Gamma$. If $b \leq a$, then $b\gamma_1 b\gamma_2 b \ldots b\gamma_n b \leq a\gamma_1 a\gamma_2 a \ldots a\gamma_n a$. Hence $a\gamma_1 a\gamma_2 a \ldots a\gamma_n a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_n b$. If $b \leq a\beta x$, then $b\gamma_1 b\gamma_2 b \ldots b\gamma_n b \leq (a\beta x)\gamma_1(a\beta x)\gamma_2(a\beta x)\ldots(a\beta x)\gamma_n(a\beta x) = (a\gamma_1 a\gamma_2 a \ldots a\gamma_n a)\beta$ $(x\beta x\beta x \ldots x\beta x)$. Hence $a\gamma_1 a\gamma_2 a \ldots a\gamma_n a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_n b$.

(b) Let $a, b \in M, n \in \mathbb{N}$ and $\beta, \gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$. Put $\beta_1 = \beta_2 = \ldots = \beta_n = \beta_{n+1} = \beta$. Then $(a\beta b)\gamma_1(a\beta b)\gamma_2(a\beta b)\ldots(a\beta b)\gamma_n(a\beta b) = (a\beta_1 b)\gamma_1(a\beta_2 b)\gamma_2$ $(a\beta_3 b)\ldots(a\beta_n b)\gamma_n(a\beta_{n+1} b) = (a\beta_1 a\beta_2 a \ldots a\beta_n a)\beta(b\gamma_1 b\gamma_2 b \ldots b\gamma_n b) = a\beta(b\gamma_1 b\gamma_2 b$ $\ldots b\gamma_n b)\beta(a\beta_1 a\beta_2 a \ldots a\beta_{n-1} a)$. Hence

$$a\beta(b\gamma_1 b\gamma_2 b \ldots b\gamma_n b) \parallel (a\beta b)\gamma_1(a\beta b)\gamma_2(a\beta b)\ldots(a\beta b)\gamma_n(a\beta b).$$

Therefore the proof is completed. □

Proposition 6.2.5. *If M is a commutative ordered Γ-semigroup, then the following statements hold:*

(a) *$\bar{\eta}$ is transitive.*

(b) *$\bar{\eta}$ is left compatible.*

(c) *$\bar{\eta}$ is right compatible.*

(d) *$(a\gamma b, b\gamma a) \in \bar{\eta}$ for all $a, b \in M$ and $\gamma \in \Gamma$.*

Proof. (a) Let $a, b, c \in M$ be such that $(a, b) \in \bar{\eta}$ and $(b, c) \in \bar{\eta}$. Since $(a, b) \in \bar{\eta}$, we have

$$a \parallel b \text{ or } a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \text{ for some } m \in \mathbb{N} \text{ and } \gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma,$$

and

$$b \parallel a \text{ or } b \parallel a\beta_1 a\beta_2 a \ldots a\beta_n a \text{ for some } n \in \mathbb{N} \text{ and } \beta_1, \beta_2, \ldots, \beta_n \in \Gamma.$$

Since $(b, c) \in \bar{\eta}$, we have

$$b \parallel c \text{ or } b \parallel c\alpha_1 c\alpha_2 c \ldots c\alpha_t c \text{ for some } t \in \mathbb{N} \text{ and } \alpha_1, \alpha_2, \ldots, \alpha_t \in \Gamma,$$

and

$$c \parallel b \text{ or } c \parallel b\lambda_1 b\lambda_2 b \ldots b\lambda_h b \text{ for some } h \in \mathbb{N} \text{ and } \lambda_1, \lambda_2, \ldots, \lambda_h \in \Gamma.$$

Assume that $a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_m b$, $b \parallel a\beta_1 a\beta_2 a \ldots a\beta_n a$, $b \parallel c\alpha_1 c\alpha_2 c \ldots c\alpha_t c$ and $c \parallel b\lambda_1 b\lambda_2 b \ldots b\lambda_h b$. Since M is commutative and using Proposition 6.2.4(a), we have

$$b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \parallel c'\gamma_1 c'\gamma_2 c' \ldots c'\gamma_m c' \text{ where } c' = c\alpha_1 c\alpha_2 c \ldots c\alpha_t c.$$

By Proposition 6.2.2(b), we have

$$a \parallel c'\gamma_1 c'\gamma_2 c' \dots c'\gamma_m c'.$$

In a similar way, we prove that

$$c \parallel a'\lambda_1 a'\lambda_2 a' \dots a'\lambda_h a' \text{ where } a' = a\beta_1 a\beta_2 a \dots a\beta_n a.$$

Hence $(a, c) \in \bar{\eta}$. In another case, we can prove that $(a, c) \in \bar{\eta}$. Therefore $\bar{\eta}$ is transitive.

(b) Let $a, b, c \in M$ and $\gamma \in \Gamma$ be such that $(a, b) \in \bar{\eta}$. Then

$$a \parallel b \text{ or } a \parallel b\gamma_1 b\gamma_2 b \dots b\gamma_m b \text{ for some } m \in \mathbb{N} \text{ and } \gamma_1, \gamma_2, \dots, \gamma_m \in \Gamma,$$

and

$$b \parallel a \text{ or } b \parallel a\beta_1 a\beta_2 a \dots a\beta_n a \text{ for some } n \in \mathbb{N} \text{ and } \beta_1, \beta_2, \dots, \beta_n \in \Gamma.$$

Assume that $a \parallel b\gamma_1 b\gamma_2 b \dots b\gamma_m b$ and $b \parallel a\beta_1 a\beta_2 a \dots a\beta_n a$. By Proposition 6.2.2($c$), we have $c\gamma a \parallel c\gamma b\gamma_1 b\gamma_2 b \dots b\gamma_m b$. Since M is commutative and using Proposition 6.2.4(b), we have

$$c\gamma(b\gamma_1 b\gamma_2 b \dots b\gamma_n b) \parallel (c\gamma b)\gamma_1(c\gamma b)\gamma_2(c\gamma b) \dots (c\gamma b)\gamma_m(c\gamma b).$$

By Proposition 6.2.2(b), we have

$$c\gamma a \parallel (c\gamma b)\gamma_1(c\gamma b)\gamma_2(c\gamma b) \dots (c\gamma b)\gamma_m(c\gamma b).$$

In a similar way, we prove that

$$c\gamma b \parallel (c\gamma a)\beta_1(c\gamma a)\beta_2(c\gamma a) \dots (c\gamma a)\beta_n(c\gamma a).$$

Hence $(c\gamma a, c\gamma b) \in \bar{\eta}$. In another case, we can prove that $(c\gamma a, c\gamma b) \in \bar{\eta}$. Therefore $\bar{\eta}$ is left compatible.

(c) Since M is commutative, it follows from (b).

(d) Since M is commutative and using Proposition 6.2.3(a), $(a\gamma b, b\gamma a) \in \bar{\eta}$ for all $a, b \in M$ and $\gamma \in \Gamma$. □

Immediately from Propositions 6.2.3 and 6.2.5, we have Theorem 6.2.6.

Theorem 6.2.6. *If M is a commutative ordered Γ-semigroup, then $\bar{\eta}$ is an ordered semilattice congruence on M.*

Lemma 6.2.7. *If M is an ordered Γ-semigroup, and $a, b \in M$ is such that $a \parallel b$, then $N(a) \subseteq N(b)$.*

Proof. Assume that $a, b \in M$ is such that $a \parallel b$. Then $b \leq a$ or $b \leq a\gamma x$ for some $x \in M$ and $\gamma \in \Gamma$. Since $b \in N(b)$, we have $a \in N(b)$ or $a\gamma x \in N(b)$. Hence $a \in N(b)$, so $N(a) \subseteq N(b)$. □

Theorem 6.2.8. *If M is a commutative ordered Γ-semigroup, then $\bar{\eta} = \mathcal{N}$.*

Proof. Let $a, b \in M$ be such that $(a, b) \in \bar{\eta}$. Then

$$a \parallel b \text{ or } a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \text{ for some } m \in \mathbb{N} \text{ and } \gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma,$$

and

$$b \parallel a \text{ or } b \parallel a\beta_1 a\beta_2 a \ldots a\beta_n a \text{ for some } n \in \mathbb{N} \text{ and } \beta_1, \beta_2, \ldots, \beta_n \in \Gamma.$$

Assume that $a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_m b$ and $b \parallel a\beta_1 a\beta_2 a \ldots a\beta_n a$. Then

$$b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \leq a \text{ or } b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \leq a\alpha x \text{ for some } x \in M \text{ and } \alpha \in \Gamma.$$

Since $b \in N(b)$, we have $b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \in N(b)$. This implies that $a \in N(b)$ or $a\alpha x \in N(b)$. Hence $a \in N(b)$, so $N(a) \subseteq N(b)$. Since $b \parallel a\beta_1 a\beta_2 a \ldots a\beta_n a$, by symmetry, we get $N(b) \subseteq N(a)$. Therefore $N(a) = N(b)$, so $(a, b) \in \mathcal{N}$. In another case, we can prove that $(a, b) \in \mathcal{N}$. Hence $\bar{\eta} \subseteq \mathcal{N}$. On the other hand, by Theorem 6.2.6 and Lemma 6.2.1, we have $\mathcal{N} \subseteq \bar{\eta}$. Therefore $\bar{\eta} = \mathcal{N}$. □

We define a relation $\bar{\delta}$ on an ordered Γ-semigroup M as follows:

$$\bar{\delta} := \{(x,y) \mid y \parallel x\}.$$

Proposition 6.2.9. *If M is a commutative ordered Γ-semigroup, then the relation $\bar{\delta}$ is pseudoorder on M.*

Proof. Let $a, b \in M$ be such that $(a,b) \in \leq$. Then $a \leq b$, so $b \parallel a$. Thus $(a,b) \in \bar{\delta}$, so $\leq \subseteq \bar{\delta}$. Let $a, b, c \in M$ be such that $(a,b) \in \bar{\delta}$ and $(b,c) \in \bar{\delta}$. Then $b \parallel a$ and $c \parallel b$. By Proposition 6.2.2(b), we have $c \parallel a$. Thus $(a,c) \in \bar{\delta}$. Let $a, b, c \in M$ and $\gamma \in \Gamma$ be such that $(a,b) \in \bar{\delta}$. Then $b \parallel a$. By Proposition 6.2.2(c) and (d), we have $b\gamma c \parallel a\gamma c$ and $c\gamma b \parallel c\gamma a$. Thus $(a\gamma c, b\gamma c) \in \bar{\delta}$ and $(c\gamma a, c\gamma b) \in \bar{\delta}$. Therefore $\bar{\delta}$ is a pseudoorder on M. □

Proposition 6.2.10. *For an ordered Γ-semigroup M, $\bar{\delta} \cap \bar{\delta}^{-1} \subseteq \mathcal{N}$.*

Proof. Let $a, b \in M$ be such that $(a,b) \in \bar{\delta} \cap \bar{\delta}^{-1}$. Then $(a,b) \in \bar{\delta}$, so $b \parallel a$. By Lemma 6.2.7, $N(b) \subseteq N(a)$. Since $(a,b) \in \bar{\delta}^{-1}$, $(b,a) \in \bar{\delta}$. By symmetry, we have $N(a) \subseteq N(b)$. Thus $N(a) = N(b)$, so $(a,b) \in \mathcal{N}$. Therefore $\bar{\delta} \cap \bar{\delta}^{-1} \subseteq \mathcal{N}$. □

Proposition 6.2.11. *If M is an ordered Γ-semigroup and $a, b \in M$, then the following statements are equivalent:*

(a) *$a \parallel b$ or $a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_m b$ for some $m \in \mathbb{N}$ and $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma$.*

(b) *$b \leq a\beta y$ for some $y \in M$ and $\beta \in \Gamma$ or $b\beta_1 b\beta_2 b \ldots b\beta_n b \leq a\beta_{n+1} y$ for some $n \in \mathbb{N}, y \in M$ and $\beta_1, \beta_2, \ldots, \beta_n, \beta_{n+1} \in \Gamma$.*

Proof. If $a \parallel b$, then $b \leq a$ or $b \leq a\gamma x$ for some $x \in M$ and $\gamma \in \Gamma$. Thus $b\gamma b \leq a\gamma b$ or $b\gamma b \leq a\gamma x\gamma b$. Hence there exist $y \in M$ and $\beta_1, \beta_2 \in \Gamma$ such that $b\beta_1 b \leq a\beta_2 y$. Assume that $a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_m b$ for some $m \in \mathbb{N}$ and $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma$. Then

$$b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \leq a \text{ or } b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \leq a\beta x \text{ for some } x \in M \text{ and } \beta \in \Gamma.$$

Put $\gamma_{m+1} = \beta$. Then

$$b\gamma_1 b\gamma_2 b \ldots b\gamma_m b\gamma_{m+1} b \leq a\gamma_{m+1} b \text{ or } b\gamma_1 b\gamma_2 b \ldots b\gamma_m b\gamma_{m+1} b \leq a\beta x\gamma_{m+1} b.$$

Hence there exist $n \in \mathbb{N}, y \in M$ and $\beta_1, \beta_2, \ldots, \beta_n, \beta_{n+1} \in \Gamma$ such that

$$b\beta_1 b\beta_2 b \ldots b\beta_n b \leq a\beta_{n+1} y.$$

Therefore (a) implies (b).

Conversely, if $b \leq a\beta y$ for some $y \in M$ and $\beta \in \Gamma$, then $a \parallel b$. Assume that $b\beta_1 b\beta_2 b \ldots b\beta_n b \leq a\beta_{n+1} y$ for some $n \in \mathbb{N}, y \in M$ and $\beta_1, \beta_2, \ldots, \beta_n, \beta_{n+1} \in \Gamma$. Then $a \parallel b\beta_1 b\beta_2 b \ldots b\beta_n b$. Hence there exist $m \in \mathbb{N}$ and $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma$ such that

$$a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_m b.$$

Therefore (b) implies (a). □

Using the relation $\bar{\eta}$ defined above, we prove that the commutative ordered Γ-semigroups are, uniquely, ordered semilattices of archimedean sub-Γ-semigroups. That is, they are decomposable into archimedean sub-Γ-semigroups, and the decomposition is unique.

Proposition 6.2.12. *If M is a commutative ordered Γ-semigroup, then the $\bar{\eta}$-class $(x)_{\bar{\eta}}$ is an archimedean sub-Γ-semigroup of M for all $x \in M$.*

Proof. Let $x \in M$. Since $\bar{\eta} \in SC(M)$ and using Lemma 6.1.1(a), we have the $\bar{\eta}$-class $(x)_{\bar{\eta}}$ is a sub-Γ-semigroup of M. Let $a, b \in M$ be such that $a, b \in (x)_{\bar{\eta}}$. Then $(a, b) \in \bar{\eta}$. Thus

$$a \parallel b \text{ or } a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_t b \text{ for some } t \in \mathbb{N} \text{ and } \gamma_1, \gamma_2, \ldots, \gamma_t \in \Gamma,$$

and

$$b \parallel a \text{ or } b \parallel a\beta_1 a\beta_2 a \ldots a\beta_h a \text{ for some } h \in \mathbb{N} \text{ and } \beta_1, \beta_2, \ldots, \beta_h \in \Gamma.$$

Since $a \parallel b$ or $a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_t b$, it follows from Proposition 6.2.11 that there exist $u \in \mathbb{N}, s \in M$ and $\alpha_1, \alpha_2, \ldots, \alpha_u \in \Gamma$ such that $b\alpha_1 b\alpha_2 b \ldots b\alpha_u b \leq a\alpha_{u+1}s$. Similarly, there exist $v \in \mathbb{N}, k \in M$ and $\lambda_1, \lambda_2, \ldots, \lambda_v \in \Gamma$ such that $a\lambda_1 a\lambda_2 a \ldots a\lambda_v a \leq b\lambda_{v+1}k$. Thus $b\alpha_1 b\alpha_2 b \ldots b\alpha_u b\alpha_{u+1} b \leq a\alpha_{u+1}s\alpha_{u+1}b = b\alpha_{u+1}s\alpha_{u+1}a$, so $b\alpha_{u+1}s \parallel b\alpha_1 b\alpha_2 b \ldots b\alpha_u b\alpha_{u+1} b$. Clearly, $b \parallel b\alpha_{u+1}s$. Hence $(b\alpha_{u+1}s, b) \in \bar{\eta}$, so $b\alpha_{u+1}s \in (b)_{\bar{\eta}} = (x)_{\bar{\eta}}$. Thus we have $b\alpha_1 b\alpha_2 b \ldots b\alpha_u b\alpha_{u+1} b \leq a\alpha_{u+1}b\alpha_{u+1}s$ where $u+1 \in \mathbb{N}$ and $b\alpha_{u+1}s \in (x)_{\bar{\eta}}$, that is $a \parallel_{(x)_{\bar{\eta}}} b\alpha_1 b\alpha_2 b \ldots b\alpha_u b\alpha_{u+1} b$. In a similar way, we can prove that there exist $n \in \mathbb{N}, z \in (x)_{\bar{\eta}}$ and $\beta_1, \beta_2, \ldots, \beta_n$ such that $a\beta_1 a\beta_2 a \ldots a\beta_n a \leq b\beta_{n+1}z$. Hence $b \parallel_{(x)_{\bar{\eta}}} a\beta_1 a\beta_2 a \ldots a\beta_n a$. Therefore $(x)_{\bar{\eta}}$ is an archimedean sub-Γ-semigroup of M. □

Immediately from Theorem 6.2.6 and Proposition 6.2.12, we have Theorem 6.2.13.

Theorem 6.2.13. *If M is a commutative ordered Γ-semigroup, then M is an ordered semilattice of archimedean sub-Γ-semigroups of M.*

Proposition 6.2.14. *If M is a commutative ordered Γ-semigroup, and ρ is an ordered semilattice congruence on M such that the ρ-class $(x)_\rho$ is an archimedean sub-Γ-semigroup of M for all $x \in M$, then $\rho = \bar{\eta}$.*

Proof. Let $a, b \in M$ be such that $(a, b) \in \rho$. Then, since $a, b \in (b)_\rho$ and $(b)_\rho$ is archimedean, we get

$$a \parallel_{(b)_\rho} b \text{ or } a \parallel_{(b)_\rho} b\gamma_1 b\gamma_2 b \ldots b\gamma_m b \text{ for some } m \in \mathbb{N} \text{ and } \gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma,$$

and

$$b \parallel_{(b)_\rho} a \text{ or } b \parallel_{(b)_\rho} a\beta_1 a\beta_2 a \ldots a\beta_n a \text{ for some } n \in \mathbb{N} \text{ and } \beta_1, \beta_2, \ldots, \beta_n \in \Gamma.$$

Hence

$$a \parallel b \text{ or } a \parallel b\gamma_1 b\gamma_2 b \ldots b\gamma_m b,$$

and

$$b \parallel a \text{ or } b \parallel a\beta_1 a\beta_2 a \ldots a\beta_n a.$$

Therefore $(a, b) \in \bar{\eta}$, so $\rho \subseteq \bar{\eta}$. By Lemma 6.2.1 and Theorem 6.2.8, we have $\bar{\eta}$ is the least ordered semilattice congruence on M. Hence $\bar{\eta} \subseteq \rho$, so $\bar{\eta} = \rho$. □

Immediately from Theorem 6.2.6 and Propositions 6.2.12 and 6.2.14, we have Theorem 6.2.15.

Theorem 6.2.15. *If M is a commutative ordered Γ-semigroup, then M is, uniquely, an ordered semilattice of archimedean sub-Γ-semigroups of M.*

In comparison our above results with results of ordered semigroups, we see that for commutative ordered Γ-semigroups, we have the usual relation $\mathcal{N}$ is equal to the relation $\bar{\eta}$, and every commutative ordered Γ-semigroup is, uniquely, ordered semilattice of archimedean sub-Γ-semigroups which is an analogous result of ordered semigroups.