CHAPTER VII

THE LEAST REGULAR ORDER WITH RESPECT TO A REGULAR CONGRUENCE ON ORDERED Γ-SEMIGROUPS

In this chapter, we characterize the concept of regular congruences on ordered Γ-semigroups, and prove that for an ordered Γ-semigroup M, the following statements hold:

(i) Every ordered semilattice congruence is a regular congruence.

(ii) There exists the least regular order on the Γ-semigroup M/ρ with respect to a regular congruence ρ on M.

(iii) The regular congruences are not ordered semilattice congruences in general.

7.1 The Least Regular Order with respect to a Regular Congruence on Ordered Γ-Semigroups

Before the characterizations of regular congruences on ordered Γ-semigroups for the main theorems, we give some auxiliary results which are necessary in what follows. We begin by recalling the following two lemmas which proof can be found in [13].

Lemma 7.1.1.([13]) *If $x \in M$ and $\rho \in SC(M)$, then the following statements hold:*

(a) $f(x)_\rho = \{a \in M \mid a \in (x)_\rho$ *or* $u\gamma a \in (x)_\rho$ *for some* $u \in f(x)_\rho$ *and* $\gamma \in \Gamma\}$.

(b) $f(x)_\rho = t$.

(c) *If* $b \in f(x)_\rho$, *then* $f(b)_\rho \subseteq f(x)_\rho$.

(d) $\rho = \{(x,y) \mid f(x)_\rho = f(y)_\rho\}$.

Lemma 7.1.2.([13]) *If $x \in M$ and $\rho \in OSC(M)$, then the following statements hold:*

(a) $F(x)_\rho = \{a \in M \mid a \in [(x)_\rho)$ *or* $u\gamma a \in [(x)_\rho)$ *for some* $u \in F(x)_\rho$ *and* $\gamma \in \Gamma\}$.

(b) $F(x)_\rho = T$.

(c) *If* $b \in F(x)_\rho$, *then* $F(b)_\rho \subseteq F(x)_\rho$.

(d) $\rho = \{(x,y) \mid F(x)_\rho = F(y)_\rho\}$.

Similar to the proof of Corollary 5.1.2, we have Lemma 7.1.3.

Lemma 7.1.3. *Let ρ_1 and ρ_2 be relations on a Γ-semigroup M. If ρ_1 and ρ_2 are compatible, then so are ρ_1^n and $(\rho_1 \circ \rho_2)^n$ for all $n \in \mathbb{N}$.*

We now characterize the regular congruences on ordered Γ-semigroups, and answer the question that does there exist the least regular order on the Γ-semigroup M/ρ with respect to a regular congruence ρ on an ordered Γ-semigroup M?

Theorem 7.1.4. *Let ρ be a semilattice congruence on a Γ-semigroup M. Define an order $\preceq$ on M/ρ as follows:*

$$(x)_\rho \preceq (y)_\rho \text{ if and only if } f(y)_\rho \subseteq f(x)_\rho \text{ for all } x, y \in M.$$

Then $(M/\rho; \preceq)$ is an ordered Γ-semigroup.

Proof. Let $(x)_\rho = (x')_\rho$ and $(y)_\rho = (y')_\rho$ be such that $(x)_\rho \preceq (y)_\rho$. Then $f(y)_\rho \subseteq f(x)_\rho$, so $f(y')_\rho = f(y)_\rho \subseteq f(x)_\rho = f(x')_\rho$. Hence $(x')_\rho \preceq (y')_\rho$, so

$\preceq$ is well-defined. For any ρ-class $(x)_\rho$, $f(x)_\rho \subseteq f(x)_\rho$. Hence $(x)_\rho \preceq (x)_\rho$, so $\preceq$ is reflexive. Let $(x)_\rho \preceq (y)_\rho$ and $(y)_\rho \preceq (x)_\rho$. Then $f(x)_\rho = f(y)_\rho$. By Lemma 7.1.1(d), we have $(x, y) \in \rho$. Hence $(x)_\rho = (y)_\rho$, so $\preceq$ is anti-symmetric. Let $(x)_\rho \preceq (y)_\rho$ and $(y)_\rho \preceq (z)_\rho$. Then $f(y)_\rho \subseteq f(x)_\rho$ and $f(z)_\rho \subseteq f(y)_\rho$, so $f(z)_\rho \subseteq f(x)_\rho$. Hence $(x)_\rho \preceq (z)_\rho$, so $\preceq$ is transitive. Therefore $\preceq$ is an equivalence relation. Let $(x)_\rho \preceq (y)_\rho, c \in M$ and $\gamma \in \Gamma$. Then $f(y)_\rho \subseteq f(x)_\rho$. By Lemma 7.1.1($c$), we have $y \in f(x)_\rho \subseteq f(x\gamma c)_\rho$ and $c \in f(x\gamma c)_\rho$. Thus $y\gamma c \in f(x\gamma c)_\rho$, so it follows from Lemma 7.1.1(c) that $f(y\gamma c)_\rho \subseteq f(x\gamma c)_\rho$. Hence $(x\gamma c)_\rho \preceq (y\gamma c)_\rho$. Similarly, $(c\gamma x)_\rho \preceq (c\gamma y)_\rho$. Therefore M/ρ is an ordered Γ-semigroup. □

Theorem 7.1.5. *If ρ is an ordered semilattice congruence on an ordered Γ-semigroup M, then ρ is a regular congruence on M.*

Proof. Assume that $\rho \in OSC(M)$. We define the order $\preceq$ on the Γ-semigroup M/ρ by $(x)_\rho \preceq (y)_\rho$ if and only if $F(y)_\rho \subseteq F(x)_\rho$ for all $x, y \in M$. By a similar proof of Theorem 7.1.4 and using Lemma 7.1.2, we get M/ρ is an ordered Γ-semigroup. If $x \leq y$, then $y \in F(x)_\rho$. By Lemma 7.1.2(c), we have $F(y)_\rho \subseteq F(x)_\rho$. Therefore $(x)_\rho \preceq (y)_\rho$, so $\varphi(x) \preceq \varphi(y)$. Hence ρ is a regular congruence on M. □

Immediately from Theorem 7.1.5, we have that $OSC(M) \subseteq RC(M)$, and the congruence $\mathcal{N}$ is the regular congruence on M.

The next theorem answer that we can find the least regular order on the Γ-semigroup M/ρ with respect to a regular congruence ρ on an ordered Γ-semigroup M.

Theorem 7.1.6. *Let ρ be a regular congruence on an ordered Γ-semigroup M. Define a relation $(\leq \circ\rho)/\rho$ on M/ρ as follows:*

$$(\leq \circ\rho)/\rho := \{((x)_\rho, (y)_\rho) \mid (x_1, y_1) \in (\leq \circ\rho) \text{ for some } x_1 \in (x)_\rho \text{ and } y_1 \in (y)_\rho\}.$$

If $\preceq := \{((x)_\rho, (y)_\rho) \mid ((x)_\rho, (y)_\rho) \in ((\leq \circ\rho)/\rho)^m$ *for some* $m \in \mathbb{N}\}$, *then* $\preceq$ *is the least regular order on* M/ρ *with respect to the regular congruence* ρ *on* M.

Proof. We shall show that $\preceq$ is an order on M/ρ.

(*i*) Reflexive: For any $x \in M$, since $x \leq x\rho x$, we have $(x, x) \in (\leq \circ\rho)$. Thus $((x)_\rho, (x)_\rho) \in\preceq$.

(*ii*) Transitive: Let $((x)_\rho, (y)_\rho) \in\preceq$ and $((y)_\rho, (z)_\rho) \in\preceq$. Then there exist $m, n \in \mathbb{N}$ such that $((x)_\rho, (y)_\rho) \in ((\leq \circ\rho)/\rho)^m$ and $((y)_\rho, (z)_\rho) \in ((\leq \circ\rho)/\rho)^n$. Thus there exists a sequence of elements $(w_1)_\rho, (w_2)_\rho, \ldots, (w_{m-1})_\rho, (k_1)_\rho, (k_2)_\rho, \ldots, (k_{n-1})_\rho \in M/\rho$ such that $((w_{i-1})_\rho, (w_i)_\rho) \in (\leq \circ\rho)/\rho$ and $((k_{j-1})_\rho, (k_j)_\rho) \in (\leq \circ\rho)/\rho$ where $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, (w_0)_\rho = (x)_\rho, (w_m)_\rho = (y)_\rho = (k_0)_\rho$ and $(k_n)_\rho = (z)_\rho$. Since $((y)_\rho, (y)_\rho) \in (\leq \circ\rho)/\rho$, we have $((x)_\rho, (z)_\rho) \in ((\leq \circ\rho)/\rho)^{m+n+1}$. Hence $((x)_\rho, (z)_\rho) \in\preceq$.

(*iii*) Anti-symmetric: Let $((x)_\rho, (y)_\rho) \in\preceq$ and $((y)_\rho, (x)_\rho) \in\preceq$. Then there exist $m, n \in \mathbb{N}$ such that

$$((x)_\rho, (y)_\rho) \in ((\leq \circ\rho)/\rho)^m \text{ and } ((y)_\rho, (x)_\rho) \in ((\leq \circ\rho)/\rho)^n. \tag{7.1.1}$$

By (7.1.1), there exist a sequence of elements $(w_1)_\rho, (w_2)_\rho, \ldots, (w_{m-1})_\rho, (k_1)_\rho, (k_2)_\rho, \ldots, (k_{n-1})_\rho \in M/\rho$ such that

$$((w_{i-1})_\rho, (w_i)_\rho) \in (\leq \circ\rho)/\rho \text{ and } ((k_{j-1})_\rho, (k_j)_\rho) \in (\leq \circ\rho)/\rho \tag{7.1.2}$$

where $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, (w_0)_\rho = (x)_\rho = (k_n)_\rho$ and $(w_m)_\rho = (y)_\rho = (k_0)_\rho$. By (7.1.2), there exist $w'_{i-1} \in (w_{i-1})_\rho, w'_i \in (w_i)_\rho, k'_{j-1} \in (k_{j-1})_\rho$ and $k'_j \in (k_j)_\rho$ such that

$$(w'_{i-1}, w'_i) \in (\leq \circ\rho) \text{ and } (k'_{j-1}, k'_j) \in (\leq \circ\rho) \tag{7.1.3}$$

for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. By (7.1.3), there exist $w''_i, k''_j \in M$ such that

$$w'_{i-1} \leq w''_i \rho w'_i \text{ and } k'_{j-1} \leq k''_j \rho k'_j \tag{7.1.4}$$

for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Since ρ is regular, there exist a regular order $\preceq_1$ on M/ρ such that $(M/\rho; \preceq_1)$ is an ordered Γ-semigroup, and the mapping $\varphi : M \to M/\rho$ defined by $a \mapsto (a)_\rho$ is isotone. Consequently, by (7.1.4), we have for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$,

$$\begin{cases} (w_{i-1})_\rho = (w'_{i-1})_\rho \preceq_1 (w''_i)_\rho = (w'_i)_\rho = (w_i)_\rho \\ \\ (k_{j-1})_\rho = (k'_{j-1})_\rho \preceq_1 (k''_j)_\rho = (k'_j)_\rho = (k_j)_\rho. \end{cases} \tag{7.1.5}$$

Hence

$$\begin{cases} (x)_\rho = (w_0)_\rho \preceq_1 (w_1)_\rho \preceq_1 \cdots \preceq_1 (w_m)_\rho = (y)_\rho \\ \\ (y)_\rho = (k_0)_\rho \preceq_1 (k_1)_\rho \preceq_1 \cdots \preceq_1 (k_n)_\rho = (x)_\rho. \end{cases} \tag{7.1.6}$$

Since $\preceq_1$ is an order on M/ρ, we have $(x)_\rho = (y)_\rho$.
Hence $\preceq$ is an equivalence relation. We shall show that $\preceq$ is compatible. Since $\leq$ and ρ are compatible, then by Lemma 7.1.3, $\leq \circ\rho$ is compatible. Let $((x)_\rho, (y)_\rho) \in (\leq \circ\rho)/\rho, c \in M$ and $\gamma \in \Gamma$. Then there exist $x' \in (x)_\rho$ and $y' \in (y)_\rho$ such that $(x', y') \in (\leq \circ\rho)$. Since $\leq \circ\rho$ is compatible, we have $(c\gamma x', c\gamma y'), (x'\gamma c, y'\gamma c) \in (\leq \circ\rho)$. Since $c\gamma x' \in (c)_\rho\gamma(x')_\rho$ and $c\gamma y' \in (c)_\rho\gamma(y')_\rho$, we have $((c)_\rho\gamma(x')_\rho, (c)_\rho\gamma(y')_\rho) \in (\leq \circ\rho)/\rho$. Similarly, $((x')_\rho\gamma(c)_\rho, (y')_\rho\gamma(c)_\rho) \in (\leq \circ\rho)/\rho$. Hence $(\leq \circ\rho)/\rho$ is compatible. By Lemma 7.1.3, $((\leq \circ\rho)/\rho)^n$ is compatible for every $n \in \mathbb{N}$. Therefore $\preceq$ is compatible, so $\preceq$ is an order on M/ρ. If $x \leq y$, then $x \leq y\rho y$. Thus $(x, y) \in (\leq \circ\rho)$. Hence $((x)_\rho, (y)_\rho) \in (\leq \circ\rho)/\rho \subseteq \preceq$. Therefore $\preceq$ is a regular order on M/ρ. Let $((x)_\rho, (y)_\rho) \in \preceq$. Then there exists $m \in \mathbb{N}$ such that $((x)_\rho, (y)_\rho) \in ((\leq \circ\rho)/\rho)^m$. For any regular order $\preceq_\rho$ on M/ρ with respect to the

regular congruence ρ on M, by using a similar proof in (*iii*), we know that there exist $z_0, z_1, \ldots, z_m \in M$ such that

$$(x)_\rho = (z_0)_\rho \preceq_\rho (z_1)_\rho \preceq_\rho \cdots \preceq_\rho (z_m)_\rho = (y)_\rho.$$

Hence $((x)_\rho, (y)_\rho) \in \preceq_\rho$. Therefore $\preceq$ is the least regular order on M/ρ with respect to the regular congruence ρ on M.

Hence the proof of the theorem is completed. □

We shall give an example of an ordered Γ-semigroup M with there exists a regular congruence on M which is not an ordered semilattice congruence.

Example 7.1.1. Let $\Gamma = \{\gamma\}$ and $M = \{a, b, c, d\}$ be an ordered Γ-semigroup with

$$x\gamma y = \begin{cases} b & \text{if } x, y \in \{a, b\}, \\ c & \text{otherwise} \end{cases}$$

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (b,c), (b,d), (c,d)\}.$$

By Example 3.5 [13], we have $(M; \leq)$ is an ordered Γ-semigroup and

$$\begin{aligned} \mathcal{N} &= M \times M, \\ n &= \{(a,a), (b,b), (c,c), (d,d), (a,b), (b,a), (c,d), (d,c)\}. \end{aligned}$$

Since $b \leq c$ and $b\gamma c = c$, we get $(b, b\gamma c) = (b, c) \notin n$. Hence $n \notin OSC(M)$. We observe here that $M/n = \{(a)_n, (c)_n\}$. Define a relation $\preceq$ on M/n as follows:

$$\preceq := \{((a)_n, (a)_n), ((c)_n, (c)_n), ((a)_n, (c)_n)\}.$$

We can easily show that $(M/n; \preceq)$ is an ordered Γ-semigroup. We note that

$$\begin{aligned} b \leq c &\quad \text{implies} \quad (b)_n = (a)_n \preceq (c)_n, \\ b \leq d &\quad \text{implies} \quad (b)_n = (a)_n \preceq (c)_n = (d)_n, \\ c \leq d &\quad \text{implies} \quad (c)_n \preceq (c)_n = (d)_n. \end{aligned}$$

Therefore n is a regular congruence on M. Hence $OSC(M) \neq RC(M)$.

By Theorems 7.1.3 and 7.1.4, we obtain Theorems 7.1.7 and 7.1.8.

Theorem 7.1.7. *If $\rho \in SC(M)$, then $(M/f(\rho); \preceq')$ is an ordered Γ-semigroup.*

Proof. Assume that $\rho \in SC(M)$. Then, by Theorem 7.1.4, $(M/\rho; \preceq)$ is an ordered Γ-semigroup. Now, define a mapping by $f(x)_\rho \gamma f(y)_\rho = f(x\gamma y)_\rho$ for all $x, y \in M$ and $\gamma \in \Gamma$. Let $f(x)_\rho = f(x')_\rho$ and $f(y)_\rho = f(y')_\rho$, and $\gamma \in \Gamma$. Then, by Lemma 7.1.1(d), we have $(x, x') \in \rho$ and $(y, y') \in \rho$. Thus $(x\gamma y, x'\gamma y') \in \rho$. Hence $(x\gamma y)_\rho = (x'\gamma y')_\rho$, so $f(x\gamma y)_\rho = f(x'\gamma y')_\rho$. Therefore the mapping is well-defined. For any $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

$$\begin{aligned}
(f(x)_\rho \alpha f(y)_\rho)\beta f(z)_\rho &= f(x\alpha y)_\rho \beta f(z)_\rho \\
&= f((x\alpha y)\beta z)_\rho \\
&= f(x\alpha(y\beta z))_\rho \\
&= f(x)_\rho \alpha f(y\beta z)_\rho \\
&= f(x)_\rho \alpha (f(y)_\rho \beta f(z)_\rho).
\end{aligned}$$

Therefore $M/f(\rho)$ is a Γ-semigroup. We define an order $\preceq'$ on $M/f(\rho)$ by $f(x)_\rho \preceq' f(y)_\rho$ if and only if $(x)_\rho \preceq (y)_\rho$ for all $x, y \in M$. We can easily show that $(M/f(\rho); \preceq')$ is an ordered Γ-semigroup. □

Theorem 7.1.8. *If $\rho \in OSC(M)$, then $(M/F(\rho); \preceq')$ is an ordered Γ-semigroup. Moreover, if $x \leq y$, then $F(x)_\rho \preceq' F(y)_\rho$.*

Proof. Assume that $\rho \in OSC(M)$. Then, by Theorem 7.1.5, $(M/\rho; \preceq)$ is an ordered Γ-semigroup, and there exists a mapping $\varphi : M \to M/\rho$ such that φ is isotone. Define a mapping and an order $\preceq'$ on $M/F(\rho)$ by $F(x)_\rho \gamma F(y)_\rho = F(x\gamma y)_\rho$, and $F(x)_\rho \preceq' F(y)_\rho$ if and only if $(x)_\rho \preceq (y)_\rho$ for all $x, y \in M$ and $\gamma \in \Gamma$. By a similar proof of Theorem 7.1.7 and Lemma 7.1.2, we get $(M/F(\rho); \preceq')$ is an ordered Γ-semigroup. If $x \leq y$, then $\varphi(x) \preceq \varphi(y)$. Hence $(x)_\rho \preceq (y)_\rho$, so $F(x)_\rho \preceq' F(y)_\rho$. □

By Theorems 7.1.7 and 7.1.8, we can easily prove Corollaries 7.1.9 and 7.1.10.

Corollary 7.1.9. *If $\rho \in SC(M)$, then $M/\rho \cong M/f(\rho)$.*

Corollary 7.1.10. *If $\rho \in OSC(M)$, then $M/\rho \cong M/F(\rho)$. Moreover, if $x \leq y$, then $F(x)_\rho \preceq' F(y)_\rho$.*

In comparison our above results with results of ordered semigroups, we see that $\mathcal{N}$ is the regular congruence on M which is an analogous result of ordered semigroups.