CHAPTER II

PRELIMINARIES

This chapter contains the basic definitions, notation and some known results needed for later chapter.

2.1 Basic results.

Definition 2.1.1 Let X be a linear space over the field $I\!K$, denote either $I\!R$ or C A function $\|\bullet\| : X \to I\!R$ is said to be a norm on X if it satisfies the following conditions:

(i) $\|x\| \geq 0$, $\forall x \in X$

(ii) $\|x\| = 0 \Leftrightarrow x = 0$

(iii) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$

(iv) $\|\alpha x\| = |\alpha|\|x\|$, $\forall x \in X$ and $\forall \alpha \in I\!K$.

Definition 2.1.2 Let X be a linear space over the field $I\!K$. A function $< \cdot, \cdot >$: $X \times X \to I\!K$ that assigns to each ordered pair (x, y) of vectors in X a scalar $< x, y >$ is said to be an *inner product* on X if it satisfies the following conditions:

(i) $< x, x > \geq 0$, $\forall\, x \in X$ and $< x, x > = 0 \Leftrightarrow x = 0$

(ii) $< x, y > = \overline{< y, x >}$, $\forall\, x, y \in X$

(iii) $< \alpha x, y > = \alpha < x, y >$, $\forall\, x, y \in X$ and $\forall \alpha \in I\!K$

(iv) $< x + y, z > = < x, z > + < y, z >$, $\forall\, x, y, z \in X$.

Definition 2.1.3 A norm space X is said to be a *complete norm space* if every Cauchy sequence in X is a convergent sequence in X.

Definition 2.1.4 A complete norm linear space over the field $I\!K$ is called a *Banach space* over $I\!K$.

Definition 2.1.5 A subset C of a linear space X over the field $\mathbb{K}$ is *convex* if for any $x, y \in C$ implies

$$M = \{z \in X : z = \alpha x + (1-\alpha)y, 0 < \alpha < 1\} \subset C.$$

(M is called *closed segment with boundary point* x, y) or a subset C of X is *convex* if every $x, y \in C$ the segment joining x and y is contained in C.

Definition 2.1.6 A sequence $\{x_n\}$ in a normed space X is said to be *strongly convergence* (or convergence in norm) if there exists $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\| = 0 \text{ denote by } x_n \to x.$$

Definition 2.1.7 A sequence $\{x_n\}$ in a normed space X is said to be *weak convergent* if there exists an element $x \in X$ such that

$$\lim_{n\to\infty} f(x_n) = f(x), \forall f \in X^*$$

where X^* is the dual space. Denote by $x_n \rightharpoonup x$ or $\omega - \lim_{n\to\infty} x_n = x$.
It is clear that every strong convergence sequence implies weakly convergent sequence. And in a finite dimension normed space, weak convergent implies strong convergence.

Definition 2.1.8 A subset M of X is said to be *weakly compact* if every sequence $\{x_n\}$ in M contain subsequence converging weakly to a point in M.

Theorem 2.1.9 Let $\{x_n\}$ be a sequence in extended real numbers and let $b = \limsup_{n\to\infty} x_n$. Then

$$(1) \quad r > b \Rightarrow x_n < r \quad \textit{ultimately};$$

$$(2) \quad r < b \Rightarrow x_n > r \quad \textit{frequently}.$$

Ultimately mean from some index onward ; *frequently* mean for infinitely many indices.

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Theorem 2.1.10 Let $\{x_n\}$ be a sequence in extended real numbers and let $c = \liminf_{n\to\infty} x_n$. Then

$$(1) \quad r < c \Rightarrow x_n > r \ \ \textit{ultimately};$$

$$(2) \quad r > c \Rightarrow x_n < r \ \ \textit{frequently}.$$

Definition 2.1.11 Let X be a Banach space and let C be a nonempty closed convex subset of X. A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$

Definition 2.1.12 Let X be a Banach space and let C be a nonempty closed convex subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if, for each $n \geq 1$, there exists a sequence of positive real number $\{k_n\}$ with $k_n \to 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x, y \in C.$$

Definition 2.1.13 Let X be a Banach space and let C be a nonempty closed convex subset of X. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive type* if T^N is continuous for some integer $N \geq 1$ and

$$\limsup_{n\to\infty} [\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}] \leq 0 \text{ for each } x \in C.$$

Definition 2.1.14 Let X be a Banach space. An element $x \in X$ is said to be a *fixed point* of a mapping $T : X \to X$ iff $Tx = x$.

Definition 2.1.15 A mapping $f : C \to C$ is *demiclosed* at y if for each $\{x_n\} \subset C$ with $x_n \rightharpoonup x$ and $f(x_n) \to y$, then $f(x) = y$.

Definition 2.1.16 A nonexpansive mapping $T : C \to C$ is said to be *asymptotically regular* at $x \in C$ if for each $x \in X$,

$$\lim_{n\to\infty} (T^n x - T^{n+1} x) = 0.$$

Definition 2.1.17 A nonexpansive mapping $T : C \to C$ is said to be *weakly asymptotically regular* at $x \in C$ if for each $x \in X$,

$$\omega - \lim_{n\to\infty}(T^n x - T^{n+1}x) = 0.$$

Definition 2.1.18 Let M be the set a mapping $f : M \to I\!R$ is *weak lower semi − continuous* if and only if

$$f(x) \leq \liminf_{n\to\infty} f(x_n) \text{ whenever } x_n \rightharpoonup x \text{ in } M.$$

Definition 2.1.19 The norm of X is said to be $Uniform\ Kadec - Klee(UKK)$ if given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $\{x_n\}$ is a sequence in B_X converging weakly to x, and such that $sep(x_n) \geq \epsilon$, then $\|x\| \leq 1 - \delta(\epsilon)$,
where $sep(x_n) := \inf\{\|x_n - x_m\| : n \neq m\}$, and $B_X := \{x \in X : \|x\| \leq 1\}$.

Lemma 2.1.20[8] Let T be an asymptotically nonexpansive type on a nonempty weakly compact convex subset of a Banach space X. Then there is a closed convex nonempty subset K of C and $\rho \geq 0$ such that

(i) if $x \in K$, then every weak limit point of $\{T^n x\}$ is contained in K,

(ii) $\rho_x(y) = \rho$ for all $x, y \in K$, where ρ_x is the function define by

$$\rho_x(y) = \limsup_{n\to\infty} \|T^n x - y\|, \quad y \in X.$$

Then by [8, lemma 1] the function $\rho_x(\cdot)$ is a constant on K and this constant is independent of $x \in K$.

Lemma 2.1.21 [8, lemma 2] Let C be a nonempty weakly compact convex subset of a Banach space X and let $T : C \to C$ be a mapping of asymptotically nonexpansive type such that T^N is continuous for some integer $N \geq 1$. Suppose there exists a closed convex nonempty subset K of C which has properties:

(i) $x \in K \Rightarrow \omega_w(x) \subseteq K$,

(ii) for each x in K and each subsequence $\{n_i\}$ of the positive integers $\{n\}$, $\{T^{n_i}x\}$ admits a norm-convergent subsequence; and

(iii) for each $x \in K$, $\omega(x)$ is norm-compact,

where $\omega(x) = \{y \in X : y = \|\cdot\| - \lim_{i\to\infty} T^{n_i}x$ for some $n_i \uparrow \infty\}$ is the ω-set of T at x.

Then T has a fixed point.

Definition 2.1.22 Let X be a Banach space. A closed convex subset K of X has *normal structure* if any bounded convex subset H of K which contain more than one point contain a non-diametral point, i.e. there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - x\| : x \in H\} < diam(H) := \sup\{\|x - y\| : x, y \in H\}.$$

A Banach space X is said to have *normal structure* if for any closed convex subset of X has normal structure.

2.2 Opial's Property.

Definition 2.2.1 A Banach space X is said to satisfying *Opial's condition*, if a sequence $\{x_n\}$ in X is converges weakly to x and $x \neq y$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Definition 2.2.2 A Banach space X is said to satisfying *uniform Opial condition*, if for each $c > 0$, there exists an $r > 0$ such that

$$1 + r \leq \liminf_{n\to\infty} \|x + x_n\|,$$

for each $x \in X$ with $\|x\| \geq c$ and weak null sequence $\{x_n\}$ in X with $\liminf_{n\to\infty} \|x_n\| \geq 1$.

Definition 2.2.3 A Banach space X is said to satisfying *locally uniform Opial condition*, if for any weak null sequence $\{x_n\}$ in X with $\liminf_{n\to\infty} \|x_n\| \geq 1$ and any

$c > 0$, there exists $r > 0$ such that

$$1 + r \leq \liminf_{n\to\infty} \|x + x_n\|, \quad \text{for all} \quad x \in X \quad \text{with} \quad \|x\| \geq c.$$

It is clear that

locally uniform Opial condition $\Rightarrow$ *uniform Opial condition*

$\Rightarrow$ *Opial's condition.*

Proposition 2.2.4 [6, Proposition 2.2, p.931] A Banach space X satisfies the locally uniform Opial condition if and only if for any sequence $\{x_n\}$ in X which converges weakly to $x \in X$ and for any sequence $\{y_m\}$ in X,

$$\limsup_{m\to\infty}(\limsup_{n\to\infty} \|x_n - y_m\|) \leq \limsup_{n\to\infty} \|x_n - x\|$$

implies $\{y_m\}$ converges to x in norm.

Definition 2.2.5 For a Banach space X we define Opial's modulus of X, denote by r_X, as follows

$$r_X(c) = \inf\left\{\liminf_{n\to\infty} \|x + x_n\| - 1\right\},$$

where $c \geq 0$ and the infimum is taken over all $x \in X$ with $\|x\| \geq c$ and sequences $\{x_n\}$ in X such that $\omega - \lim x_n = 0$ and $\liminf \|x_n\| \geq 1$.
It is easy to see that the function r_X is nondecreasing and that X satisfies the uniform Opial condition if and only if $r_X(c) > 0$ for all $c > 0$.

2.3 Maluta's constant.

Definition 2.3.1[9] Let X be a Banach space. Then the *Maluta's constant* $D(X)$ of X is defined by

$$D(X) = \sup\left\{\frac{\limsup_{n\to\infty} \; d(x_{n+1}, co(x_1, ..., x_n))}{diam(x_n)}\right\}$$

where the supremum taken over all bounded nonconstant sequence $\{x_n\}$ in X.

Let μ be mean on positive integers $\mathbb{N}$, i.e. a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \leq \mu(a) \leq \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, ...) \in l^\infty$. According to time and circumstance, we use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbb{N}$ is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, ...) \in l^\infty$. Using the Hahn-Banach theorem, or the Tychonoff fixed point theorem, we can prove the exists of a Banach limit. We known that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, ...) \in l^\infty$. So, if $a = (a_1, a_2, ...) \in l^\infty$ and $a_n \to c$, as $n \to \infty$ we have $\mu_n(a_n) = \mu(a) = c$.

Let $S(X) = \{x \in X : \|x\| = 1\}$. Then the norm of X is said to be Gâteaux differentiable (and X is said to be smooth) if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3.1}$$

exists for each x and y in $S(X)$. It is also said to be uniformly Gâteaux differentiable if for each $y \in S(X)$, the limit (2.3.1) attained uniformly for x in $S(X)$.

It has been proven that Opial's condition implies weakly normal structure and, hence, the fixed point property for nonexpansive mappings and Opial's condition implies the fixed point property for asymptotically nonexpansive mappings provide by [10].

By a gauge we mean a continuous strictly increasing function φ defined $\mathbb{R}^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. We associate with a gauge φ a (generally multivalued) duality map $J_\varphi : X \to X^*$ defined by

$$J_\varphi(x) = \{x^* \in X^* : < x, x^* >= \|x\|\varphi(\|x\|) \text{ and } \|x^*\| = \varphi(\|x\|)\}.$$

Clearly the (normalized) duality map J corresponds to the gauge $\varphi(t) = t$. Browder [11] initiated the study of certain classes of nonlinear operators by means of a duality map J_φ. Set for $t \geq 0$,

$$\Phi(t) = \int_0^t \varphi(r)dr.$$

Then it is known that $J_\varphi(x)$ is the subdifferential of convex function $\Phi(\|\cdot\|)$ at x. Now recall that X is said to have a *weakly continuous duality map* if there exists a gauge φ such that the duality map J_φ is single-valued and continuous from X with the weak topology to X^* with the weak* topology. A space with a weakly continuous duality map is easily seen to satisfy Opial's condition ($cf.$[11]). Every $l^p(1 < p < \infty)$ space has a weakly continuous duality map with the gauge $\varphi(t) = t^{p-1}$.