CHAPTER III

MAIN RESULTS

3.1 Demiclosedness Principle

In this section we prove the demiclosedness principle for mappings of asymptotically nonexpansive type either in a Banach space with the locally uniform Opial condition or in a Banach space satisfying Opial's condition and whose norm is UKK.

Lemma 3.1.1 Suppose X is a Banach space satisfying the Opial's condition and C is nonempty weakly compact convex subset of X and $T : C \to C$ is a uniformly continuous mapping of asymptotically nonexpansive type. Suppose also $\{x_n\}$ is a sequence in C converges weakly to x, and for which the sequence $\{x_n - Tx_n\}$ converges strongly to 0. Then $\{T^n x\}$ converges weakly to x.

Proof. We shall show that $T^n x \rightharpoonup x$. For each $m \in I\!N$, set

$$A_m = \overline{co}\{T^i x : i \geq m\} \quad \text{and } A = \cap_{m=1}^{\infty} A_m,$$

where $\overline{co}\{T^i x : i \geq m\}$ is the smallest closed convex of $\{T^i x : i \geq m\}$. Thus $A \neq \emptyset$, by C is weakly compact, and it is readily seen that

$$A = \overline{co}\omega_w(x)$$

where $\omega_w(x) = \{y \in X : T^{n_j} x \rightharpoonup y \text{ for some } n_j \uparrow \infty\}$.

We must show that $\{x\} = A = \overline{co}\omega_w(x)$. Let $y_0 \in A$, $T^{n_j} x \rightharpoonup y_0$. Claim $y_0 = x$. Assume that $y_0 \neq x$. Define $f : X \to I\!R$ by

$$f(y) = \limsup_{n\to\infty} \|x_n - y\|, \; y \in X.$$

Clearly f is well-define. By X satisfying Opial's condition and we have $x_n \rightharpoonup x$ and $x \neq y_0$,

$$f(x) = \limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y_0\| = f(y_0).$$

Then

$$f(x) < f(y_0).$$

Write $R := f(y_0) - f(x) > 0$. Clearly $\frac{R}{2} > 0$. Since T is asymptotically nonexpansive type,

$$\limsup_{n\to\infty}(\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}) \le 0 < \frac{R}{2}$$

for each $x \in C$. Then there exists $n_0 \in I\!N$ such that

$$\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\} < \frac{R}{2}$$

for all $n \ge n_0$. Since for $m_0 \ge n_0$, $y_0 \in \overline{co}\{T^i x : i \ge m_0 + 1\} = A_{m_0+1}$ and A_{m_0+1} is also closed convex hull, there exists an integer $p \ge 1$ and nonnegative number $t_1, t_2, ..., t_p$ with $\sum_{j=1}^{p} t_j = 1$ such that

$$\|y_0 - \sum_{j=1}^{p} t_j T^{m_0+j} x\| < \frac{R}{2}.$$

It follows that

$$
\begin{aligned}
f(y_0) &= \limsup_{n\to\infty} \ \|x_n - y_0\| \\
&= \limsup_{n\to\infty} \ \|x_n - \textstyle\sum_{j=1}^{p} t_j T^{m_0+j} x + \sum_{j=1}^{p} t_j T^{m_0+j} x - y_0\| \\
&\le \limsup_{n\to\infty} \ (\|x_n - \textstyle\sum_{j=1}^{p} t_j T^{m_0+j} x\| + \|\sum_{j=1}^{p} t_j T^{m_0+j} x - y_0\|) \\
&< \tfrac{R}{2} + \limsup_{n\to\infty} \ \|x_n - \textstyle\sum_{j=1}^{p} t_j T^{m_0+j} x\| \\
&= \tfrac{R}{2} + \limsup_{n\to\infty} \ \|\textstyle\sum_{j=1}^{p} t_j x_n - \sum_{j=1}^{p} t_j T^{m_0+j} x\|, \ \sum_{j=1}^{p} t_j = 1 \\
&\le \tfrac{R}{2} + \textstyle\sum_{j=1}^{p} t_j \limsup_{n\to\infty} \ \|x_n - T^{m_0+j} x\| \\
&\le \tfrac{R}{2} + \textstyle\sum_{j=1}^{p} t_j (\limsup_{n\to\infty} \ \|x_n - T^{m_0+j} x_n\| + \|T^{m_0+j} x_n - T^{m_0+j} x\|) \\
&\le \tfrac{R}{2} + \textstyle\sum_{j=1}^{p} t_j [\limsup_{n\to\infty} \ (\|T^{m_0+j} x_n - T^{m_0+j} x\| - \|x_n - x\|) \\
&\quad + \limsup_{n\to\infty} \|x_n - x\|], \|x_n - T^{m_0+j} x_n\| \to 0, \ \text{by uniform continuous of } T \\
&< \tfrac{R}{2} + \tfrac{R}{2} + \limsup_{n\to\infty} \ \|x_n - x\|, \ \ m_0 + j \ge m_0 \text{ by } T \text{ type}. \\
&= R + f(x) = f(y_0) - f(x) + f(x) = f(y_0).
\end{aligned}
$$

Thus $f(y_0) < f(y_0)$ a contradiction. Then $y_0 = x$. Thus $A = \{x\}$. Hence $T^n x \rightharpoonup x$. □

Theorem 3.1.2 Suppose X is a Banach space satisfying the locally uniform Opial condition, C is nonempty weakly compact convex subset of X and $T : C \to C$ is uniformly continuous mapping of asymptotically nonexpansive type. Then $I - T$ is demiclosed at zero.

Proof. We shall show that $I - T$ is demiclosed at zero. Let $(x_n) \subseteq C$ with $x_n \rightharpoonup x$ and $(x_n - Tx_n) \to 0$. By lemma 3.1.1, we have $T^n x \rightharpoonup x$. Let $\epsilon > 0$, by the definition of asymptotically nonexpansive type,

$$\limsup_{n\to\infty}(\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}) \leq 0 \text{ for each } x \in C.$$

It follows that,

$$\limsup_{m\to\infty}\left\{\limsup_{n\to\infty}\|T^n x - T^m x\|\right\} = \limsup_{m\to\infty}\left\{\limsup_{n\to\infty}\|T^m x - T^m(T^{n-m}x)\|\right\}$$
$$= \limsup_{m\to\infty}\left\{\limsup_{n\to\infty}(\|T^m x - T^m(T^{n-m}x)\| - \|x - T^{n-m}x\| + \|x - T^{n-m}x\|)\right\}$$
$$\leq \limsup_{m\to\infty}\{(\limsup_{n\to\infty}(\sup\{\|T^m x - T^m u\| - \|x - u\| : u \in C\})$$
$$+ \limsup_{n\to\infty}\|x - T^{n-m}x\|\}$$
$$= \limsup_{m\to\infty}(\sup\{\|T^m x - T^m u\| - \|x - u\| : u \in C\} + \limsup_{n\to\infty}\|x - T^n x\|)$$
$$\leq \limsup_{m\to\infty}(\sup\{\|T^m x - T^m u\| - \|x - u\| : u \in C\})$$
$$+ \limsup_{m\to\infty}\limsup_{n\to\infty}\|x - T^n x\|$$
$$\leq 0 + \limsup_{n\to\infty}\|x - T^n x\|.$$

It implies that,

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\|T^n x - T^m x\| \leq \limsup_{n\to\infty}\|x - T^n x\|.$$

By proposition 2.2.4, we have $T^m x \to x$. Since T is uniformly continuous, $T(T^m x) \to Tx$. But $T^m x \to x$, implies $T^{m+1}x \to x$. By the uniqueness of limit, $Tx = x$. □

Corollary 3.1.3 [6, 1995, p.933] Suppose X is a Banach space satisfying the locally uniform Opial condition, C is nonempty weakly compact convex subset of X and $T : C \to C$ is an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.

Theorem 3.1.4 Suppose X is a Banach space satisfying Opial's condition and whose norm is UKK and C is nonempty weakly compact convex subset of X, and $T : C \to C$ is a uniformly continuous mapping of asymptotically nonexpansive type. Then $I - T$ is demiclosed at zero.

Proof. We shall show that $I - T$ is demiclosed at zero. Let $\{x_n\}$ be a sequence in C with $x_n \rightharpoonup x$ and $(x_n - Tx_n) \to 0$. We must show that $Tx = x$. It follows by lemma 3.1.1, that $T^n x \rightharpoonup x$. Let

$$r = \limsup_{n\to\infty} \|T^n x - x\| \text{ and } r_m = \limsup_{n\to\infty} \|T^n x - T^m x\|$$

for all $m \geq 1$. By the Opial's condition of X, we have

$$r = \limsup_{n\to\infty} \|T^n x - x\| \leq \limsup_{n\to\infty} \|T^n x - T^m x\| = r_m.$$

Then $r \leq r_m$ for all $m \geq 1$. We now show that $\lim_{m\to\infty} r_m = r$. Let $\epsilon > 0$. By the definition of asymptotically nonexpansive type,

$$\limsup_{n\to\infty}(\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}) \leq 0 < \frac{\epsilon}{2}$$

for each $x \in C$. Then there exists $m_0 \in I\!N$ such that for each $n \geq m_0$,

$$\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\} < \frac{\epsilon}{2}.$$

Thus for all $m \geq m_0$, we obtain that

$$\begin{aligned} r_m &= \limsup_{n\to\infty} \|T^n x - T^m x\| \\ &= \limsup_{n\to\infty} (\|T^m(T^{n-m}x) - T^m x\| - \|T^{n-m}x - x\| + \|T^{n-m}x - x\|) \\ &\leq \limsup_{n\to\infty} (\|T^m(T^{n-m}x) - T^m x\| - \|T^{n-m}x - x\|) \\ &\quad + \limsup_{n\to\infty} \|T^{n-m}x - x\| \end{aligned}$$

$< \limsup_{n\to\infty} \ (\frac{\epsilon}{2}) + \limsup_{n\to\infty} \ \|T^n x - x\|$

$= \frac{\epsilon}{2} + r.$

Then for each $m \geq m_0$, $r_m < \epsilon + r$ for all $\epsilon > 0$. Hence $\lim_{m\to\infty} r_m = r$. Claim that $r = 0$. Assume that $r \neq 0 \ \Rightarrow r > 0$. Then $\{T^n x\}$ does not contain any strongly convergent subsequence and, therefore, $\{T^n x\}$ has a subsequence $\{T^{n_j} x\}$ such that $sep(T^{n_j} x) > 0$. Set $\epsilon_0 = \frac{sep(T^{n_j} x)}{2r}$. By the definition of UKK, there exists a $\delta_0 > 0$ such that $\|v\| \leq 1 - \delta_0$ for any sequence $\{v_n\}$ in B_X converging weakly to v and such that $sep(v_n) \geq \epsilon_0$. Choose $0 < \eta < 1$ such that

$$(1+\eta)(1-\delta_0) < 1.$$

Since $\frac{\eta r}{2} > 0$ and T is an asymptotically nonexpansive type,

$$\limsup_{n\to\infty}(\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}) \leq 0 < \frac{\eta r}{2}$$

for each $x \in C$. Then there exists $N_1 \geq m_0$ such that for all $n \geq N_1$,

$$\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\} < \frac{\eta r}{2}.$$

For $m \geq N_1$, we have

$r_m = \limsup_{n\to\infty} \ \|T^n x - T^m x\|$

$= \limsup_{n\to\infty} \ (\|T^m(T^{n-m}x) - T^m x\| - \|T^{n-m}x - x\| + \|T^{n-m}x - x\|)$

$\leq \limsup_{n\to\infty} \ (\|T^m(T^{n-m}x) - T^m x\| - \|T^{n-m}x - x\|) + \limsup_{n\to\infty} \ \|T^{n-m}x - x\|$

$< \limsup_{n\to\infty} \ (\frac{\eta r}{2}) + \limsup_{n\to\infty} \ \|T^n x - x\|$

$= \frac{\eta r}{2} + r < \eta r + r = (1+\eta)r.$

Then $\limsup_{n\to\infty} \ \|T^n x - T^m x\| < (1+\eta)r$. It follows that there exists $j_0 \geq N_1$ such that

$$\|T^{n_j} x - T^m x\| < (1+\eta)r$$

for all $j \geq j_0$. Let

$$y_j = \frac{T^{n_j} x - T^m x}{(1+\eta)r}, \quad j \geq j_0.$$

Consider, $\|y_j\| = \frac{\|T^{n_j}x - T^m x\|}{(1+\eta)r} < 1,\ y_j \rightharpoonup \frac{(x-T^m x)}{(1+\eta)r}$, and
$sep(y_j) = \inf \frac{\|T^{n_j}x - T^m x\|}{(1+\eta)r} = \frac{sep(T^{n_j}x)}{(1+\eta)r} > \frac{sep(T^{n_j}x)}{2r} = \epsilon_0$.
Then we have, $\|y_j\| \leq 1,\ y_j \rightharpoonup \frac{(x-T^m x)}{(1+\eta)r}$, and $sep(y_j) \geq \epsilon_0$, it follows from the definition of UKK that

$$\frac{\|x - T^m x\|}{(1+\eta)r} \leq 1 - \delta_0.$$

It implies that

$$\|x - T^m x\| \leq (1+\eta)(1-\delta_0)r < r$$

for all $m \geq N_1$. Taking $m \to \infty$,we get,

$$r = \limsup_{m\to\infty} \|x - T^m x\| \leq (1+\eta)(1-\delta_0)r < r.$$

This is a contradiction, since $(1+\eta)(1-\delta_0) < 1$. Therefore, we must have $r = 0$ and $Tx = x$ by the continuity of T. □

Corollary 3.1.5 [6, 1995, p.939] Suppose X is a Banach space satisfying Opial's condition and whose norm is UKK and C is nonempty weakly compact convex subset of X, and $T : C \to C$ is an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero.

3.2 Fixed point theorems for asymptotically nonexpansive type.

In this section we provide a fixed point theorems for asymptotically nonexpansive type mappings which connect with Maluta's constant $D(X) < 1$ for a Banach space, for Banach space with weakly continuous duality map J_φ as the same mapping T and we show that the uniform Opial condition implies the fixed point property for mappings of asymptotically nonexpansive type defined on weakly compact convex subsets.

Lemma 3.2.1 Suppose that X is a Banach space such that $D(X) < 1$, that K is closed bounded convex subset of X, and that $T : K \to K$ is an asymptotically nonexpansive type mapping and weakly asymptotically regular on C. If $\{T^{n_k}x\}$ is a subsequence of $\{T^n x\}$ converges weakly to $x \in K$, then $\limsup_{k\to\infty} \|T^{n_k}x - x\| = 0$.

Proof. Assume that

$$\limsup_{k\to\infty} \|T^{n_k}x - x\| > 0.$$

Take a real number $q > 0$ small enough so that

$$0 < q < \limsup_{k\to\infty} \|T^{n_k}x - x\| \quad \text{and} \quad (1+q)D(X) < 1.$$

It then follow from the definition of $D(X)$ that

$$\limsup_{k\to\infty} \|T^{n_k}x - x\| \leq D(X) diam(\{T^{n_k}x\}).$$

By the definition of asymptotically nonexpansive type, there exists a natural number N such that

$$\sup\{\|T^n x - T^n y\| - \|x-y\| : y \in C\} < q^2/2,$$

for all $n \geq N$. Clearly we may assume that

$$\sup\{\|T^{n_k}x - T^{n_k}y\| - \|x-y\| : y \in C\} < q^2/2 \text{ and } n_{k+1} - n_k \geq N,$$

for all $k \geq 1$. However, for any fixed $i > j$, noting the fact that $T^{n_k+(n_i-n_j)}x \to x$ weakly as $k \to \infty$ and the weakly lower semi-continuity of the norm $\|\cdot\|$, we have

$$\begin{aligned}
\|T^{n_i}x - T^{n_j}x\| &= \|T^{n_j}x - T^{n_j}(T^{n_i-n_j}x)\| \\
&\leq \sup\{\|T^{n_j}x - T^{n_j}y\| - \|x-y\| : y \in C\} + \|T^{n_i-n_j}x - x\| \\
&\leq q^2/2 + \limsup_{k\to\infty} \|T^{n_i-n_j}x - T^{n_k+(n_i-n_j)}x\| \\
&< q^2/2 + \limsup_{k\to\infty} (\|T^{n_i-n_j}x - T^{n_i-n_j}(T^{n_k}x)\| - \|T^{n_k}x - x\|) \\
&\quad + \limsup_{k\to\infty} \|T^{n_k}x - x\| \\
&< (1+q)\limsup_{k\to\infty} \|T^{n_k}x - x\|.
\end{aligned}$$

We thus obtain

$$\limsup_{k\to\infty} \|T^{n_k}x - x\| \le (1+q)D(X) \limsup_{k\to\infty} \|T^{n_k}x - x\|$$

which implies that $\limsup_{k\to\infty} \|T^{n_k}x - x\| = 0$ since $(1+q)D(X) < 1$. □

Theorem 3.2.2 Suppose that X is a Banach space such that $D(X) < 1$, that C is a nonempty closed bounded convex subset of X, and $T : C \to C$ is continuous of asymptotically nonexpansive type mapping and T is weakly asymptotically regular on C. Further, suppose that there exists a nonempty closed convex subset K of C with the following property (ω):

$$x \in K \text{ implies } \quad \omega_w(x) \subset K,$$

where $\omega_w(x)$ is the weak ω-limit set of T at x; that is, the set

$$\{y \in X : y = weak - \lim_i T^{n_i}x \text{ for some } n_i \uparrow \infty\}.$$

Then T has a fixed point in K.

Proof. Let $\Im$ be a free Ultrafilter on the set of positive integer(maximal element of the family of all proper filters on the set of positive integer). We then define a mapping S on K by

$$S(x) = w - \lim_{\Im} T^n x, \quad x \in K.$$

Since K is weakly compact, $S(x)$ is well define for all $x \in K$. By the definition of asymptotically nonexpansive type, we obtain S is nonexpansive mapping on K. Hence, S has a fixed point $x \in K$, that is,

$$w - \lim_{\Im} T^n x = x. \tag{3.2.1}$$

This yields a subsequence $\{T^{n_i}x\}$ of $\{T^n x\}$ converge weakly to x. Now we show that x is a fixed point of T. Since T is weakly asymptotically regular at x, it follows that for all integers $m \ge 0$, $w - \lim_{i\to\infty} T^{n_i+m}x = x$. By lemma 3.2.1, we have $\limsup_{i\to\infty} \|T^{n_i+m}x - x\| = 0$ for all $m \ge 0$. Let $\epsilon > 0$, by the definition of asymptotically nonexpansive type, there exists $N \in I\!N$ such that

$$\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\} < \epsilon,$$

for all $n \geq N$. For $n \geq N$, it implies that

$$\|x - T^n x\| \leq \limsup_{i\to\infty} \|x - T^{n_i+n}x\| + \limsup_{i\to\infty} \|T^{n_i+n}x - T^n x\|$$
$$\leq \limsup_{i\to\infty} (\sup\{\|T^n u - T^n x\| - \|u - x\| : u \in C\}) + \limsup_{i\to\infty} \|T^{n_i}x - x\|$$
$$< \epsilon.$$

Then $T^n x \to x$. Hence $Tx = x$ by continuity of T. □

As a direct consequence of Theorem 3.2.2 we have the following:

Corollary 3.2.3 [10, 1994, p.1353] Let C and X be as in Theorem 3.2.2 and let $T : C \to C$ be an asymptotically nonexpansive mappings. Suppose there exists a nonempty bounded closed convex subset of K of C with the property(ω). Then T has a fixed point in K.

Proof. This follows since an asymptotically nonexpansive mapping is of asymptotically nonexpansive type. □

Corollary 3.2.4 Let X be a Banach space such that $D(X) < 1$, let C be a bounded closed convex subset of X, and suppose $T : C \to C$ is a continuous mappings of asymptotically nonexpansive type. Then T has a fixed point.

Corollary 3.2.5 Suppose that X is a Banach space which is uniformly convex in every direction and for which $D(X) < 1$ and that C is a closed bounded convex subset of X. Then, if $T : C \to C$ is a continuous mappings of asymptotically nonexpansive type, T has a fixed point.

Proof. Using the same argument presented in the proof of Theorem 5 in [10]. □

Theorem 3.2.6 Suppose X is a Banach space with a weakly continuous duality map J_φ, C is a weakly compact convex subset of X, and $T : C \to C$, is an asymptotically nonexpansive type. Further, suppose that there exists a nonempty closed convex subset K of C with the following property (ω):

$$x \in K \text{ implies } \omega_w(x) \subseteq K$$

where $\omega_w(x)$ is the weak $\omega - limit$ set of T at x; that is, the set

$$\{y \in K : y = weak - \lim_i T^{n_i}x \text{ for some } n_i \uparrow \infty\}.$$

Then T has a fixed point in K.

Proof. For each $x \in C$, define the functional r_x by

$$r_x(y) = \limsup_{n\to\infty} \|T^n x - y\|.$$

Then by lemma 2.1.20(ii), when x lies in K, r_x is a constant over $y \in K$ and this constant is independent of $x \in K$; that is

$$\limsup_{n\to\infty} \|T^n x - y\| = r \quad \text{for all} \quad x, y \in K.$$

Now fixed $x \in K$ and $\{T^{n_i}x\}$ be a subsequence of $\{T^n x\}$ converging weakly to some y that is in K by property (ω) and such that $r' := \lim_{i\to\infty} \|T^{n_i}x - y\|$ exists. Since J_φ is the Gâteaux derivative of the convex function $\Phi(\|x\|)$, it follow by [12] that

$$\Phi(\|x+y\|) = \Phi(\|x\|) + \int_0^1 < y, J_\varphi(x+ty) > dt$$

for all $x, y \in X$. For any integer $n, m \geq 1$, we have

$$\Phi(\|T^n x - T^m x\|) = \Phi(\|(T^n x - y) + (y - T^m x)\|)$$

$$= \Phi(\|T^n x - y\|) + \textstyle\int_0^1 < (y - T^m x), J_\varphi((T^n x - y) + t(y - T^m x)) > dt.$$

Substituting n_i for n and let i go to infinity, we get

$$\lim_{i\to\infty} \Phi(\|T^{n_i}x - T^m x\|) = \Phi(r') + \textstyle\int_0^1 < (y - T^m x), J_\varphi(t(y - T^m x)) > dt$$

$$= \Phi(r') + \textstyle\int_0^1 \|y - T^m x\| \varphi(t\|y - T^m x\|) dt$$

$$= \Phi(r') + \Phi(\|y - T^m x\|).$$

It follow that,

$$\Phi(r') + \Phi(r) = \limsup_{m\to\infty} (\lim_{i\to\infty} \Phi(\|T^{n_i}x - T^m x\|))$$

$$\leq \limsup_{m\to\infty} \ \Phi(\limsup_{n\to\infty} \ (\|T^n x - T^m x\|))$$

$$\leq \limsup_{m\to\infty} \ \Phi(\limsup_{m\to\infty} \ (\sup\{\|T^m x - T^m u\| - \|x - u\| : u \in C\})$$

$$+ \limsup_{n\to\infty} \|x - T^{n-m}x\|)$$

$$\leq \limsup_{m\to\infty} \Phi(\limsup_{m\to\infty} (\sup\{\|T^m x - T^m u\| - \|x - u\| : u \in C\})$$

$$+ \limsup_{n\to\infty} \|x - T^n x\|)$$

$$\leq \limsup_{m\to\infty} \Phi(\limsup_{n\to\infty} \|x - T^n x\|)$$

$$\leq \limsup_{m\to\infty} \Phi(r) = \Phi(r).$$

Which implies that $\Phi(r') = 0$. Hence $r' = 0$, i.e. $\{T^{n_i}x\}$ strongly converges to y. This proves that, for each $x \in K$, the strong ω -limit set $\omega(x) := \{y \in X : y - \text{strong} - \lim_i T^{n_i}x \text{ for some } n_i \uparrow \infty\}$ of T at x is nonempty. It is clearly closed. We further claim that $\omega(x)$ is norm-compact. In fact, given any sequence $\{u_j\}$ in $\omega(x)$. It is easy to construct a subsequence $\{T^{m_j}x\}$ of $\{T^n x\}$ such that $\|T^{m_j}x - u_j\| < \frac{1}{j}$ for all $j \geq 1$. Repeating the argument above, we get a subsequence $\{T^{m_{j'}}x\}$ of $\{T^{m_j}x\}$ converging strongly to some $z \in \omega(x)$. Hence, $u_{j'} \to z$ strongly indicating the norm-compactness of $\omega(x)$. Now by lemma 2.1.20, T has a fixed point and complete the proof. □

Let C be nonempty weakly compact convex subset of a Banach space X and let $T : C \to C$ be a mapping of asymptotically nonexpansive type. Denote by $\Im$ the family of all closed convex nonempty subset K of C with the following property (ω);

$$x \in K \text{ implies } \omega_w(x) \subset K$$

where $\omega_w(x) = \{y \in X : y = \lim_i T^{n_i}x \text{ weakly for some } n_i \uparrow \infty\}$ is the weak ω-set of T at x. Let $\Im$ be ordered by inclusion. Then one easily sees that Zorn's lemma can be used to obtain a minimal element K in $\Im$.

Corollary 3.2.7 Suppose X is a Banach space with a weakly continuous duality map J_φ, C is a weakly compact convex subset of X, and $T : C \to C$, is an asymptotically nonexpansive type. Then T has a fixed point in C.

Corollary 3.2.8 [10, 1994, p.1350] Suppose X is a Banach space with a weakly continuous duality map J_φ, C is a weakly compact convex subset of X, and $T : C \to C$, is an asymptotically nonexpansive. Further, suppose that there exists a nonempty closed convex subset K of C with the following property (ω):

$$x \in K \text{ implies } \omega_w(x) \subseteq K$$

where $\omega_w(x)$ is the weak $\omega - limit$ set of T at x; that is, the set

$$\{y \in K : y = weak - \lim_i T^{n_i}x \text{ for some } n_i \uparrow \infty\}.$$

Then T has a fixed point in K.

Lemma 3.2.9 Let C be a nonempty weakly compact convex subset of a Banach space X satisfying Opial's condition and let T be a mapping of asymptotically nonexpansive type on C. Let $\{x_n\}$ be a sequence in C which satisfies the following condition

$$w - \lim_n T^m x_n = z_m, \quad \text{for all } m \geq 0.$$

Then $\lim_m \; b_m = \inf\{b_m : m \geq 0\}$, where $b_m = \limsup_{n\to\infty} \; \|T^m x_n - z_m\|$.

Proof. Let $\epsilon > 0$, and let $b = \inf\{b_m : m \geq 1\}$. Then $b + \frac{\epsilon}{2}$ is not lower bound, there exist natural number m_0 such that $b_{m_0} - b < \epsilon/2$. By the definition of asymptotically nonexpansive type, there exists a natural number N_1 such that

$$\sup\{\|T^n(z_{m_0}) - T^n y\| - \|z_{m_0} - y\| : y \in C\} < \epsilon/3, \forall\, n \geq N_1.$$

Using Opial's condition, we have for $j \geq N_1$,

$$\begin{aligned} b_{m_0+j} &= \limsup_{n\to\infty} \; \|T^{m_0+j}x_n - z_{m_0+j}\| \\ &\leq \limsup_{n\to\infty} \; \|T^{m_0+j}x_n - T^j z_{m_0}\| \\ &\leq \limsup_{n\to\infty} \; (\|T^{m_0+j}x_n - T^j z_{m_0}\| - \|T^{m_0}x_n - z_{m_0}\| + \|T^{m_0}x_n - z_{m_0}\|) \\ &\leq \limsup_{n\to\infty} \; (\|T^j(T^{m_0}x_n) - T^j z_{m_0}\| - \|T^{m_0}x_n - z_{m_0}\|) \end{aligned}$$

$$+\limsup_{n\to\infty} \|T^{m_0}x_n - z_{m_0}\|$$

$$< (\epsilon/3) + \limsup_{n\to\infty} \|T^{m_0}x_n - z_{m_0}\| < (\epsilon/2) + b_{m_0}.$$

This implies that $|b_m - b| < \epsilon$, for all $m \geq N$, for chosen $N = N_1 + m_0$.
Therefore $\lim_{m\to\infty} b_m = \inf\{b_m : m \geq 0\}$. □

Theorem 3.2.10 Suppose that X is a Banach space satisfying the uniform Opial condition, C is a nonempty weakly compact convex subset of X, and $T : C \to C$ is continuous of asymptotically nonexpansive type mapping. Then T has a fixed point.

Proof. Let K, ρ_x and ρ be as in lemma 2.1.20. Let $x \in K$ and let $\{T^{n_j}x\}$ be a weakly convergence subsequence of $\{T^n x\}$. Passing to subsequence and using diagonal method, we may assume that $\{T^{n_j+m}x\}$ converge weakly for every $m \geq 0$, say $w - \lim_n T^{n_j+m}x_n = z_m$. Let $b_m = \limsup_{j\to\infty} \|T^{n_j+m}x - z_m\|$. By lemma 3.2.9, $\{b_m\}$ converge to $b = \inf\{b_m : m \geq 0\} \geq 0$. We note by lemma 2.1.20,(i), that $z_m \in K$ for each m ≥ 0. By weak lower semi-continuity of the norm implies

$$\|z_m - z_{m'}\| \leq \limsup_{j\to\infty} \|z_m - T^{n_j+m'}x\| \leq \rho,$$

for all $m, m' \geq 0$. Hence $diam(\{z_m : m \geq 0\}) \leq \rho$. We claim that:

(*) for any $\epsilon > 0$ there exist $y \in K$, $m' > 0$ and $N > 0$ such that $\|T^n y - z_{n+m'}\| \leq \epsilon$ whenever $n > N$.

To prove our claim, we distinguish two cases.

Case I. $\lim_{m\to\infty} b_m = 0$. For any $\epsilon > 0$, there is $m' > 0$ such thus if $m > m'$, then $b_m < \epsilon/3$. By the definition of asymptotically nonexpansive type, there exists $m_0 \geq m'$ such that

$$\sup\{\|T^{n_j+m'+k}x - T^k u\| - \|T^{n_j+m'}x - u\| \ : u \in K\} < \epsilon/3,$$

for all $k \geq m_0$. Thus for $k \geq m_0$,

$$\|z_{m'+k} - T^k z_{m'}\| \leq \limsup_{j\to\infty} \|z_{m'+k} - T^{n_j+m'+k}x\| + \limsup_{j\to\infty} \|T^{n_j+m'+k}x - T^k z_{m'}\|$$

$$\leq b_{m'+k} + \limsup_{j\to\infty} (\|T^{n_j+m'+k}x - T^k z_{m'}\| - \|T^{n_j+m'}x - z_{m'}\|)$$

$$+\limsup_{j\to\infty} \|T^{n_j+m'}x - z_{m'}\|$$

$$\leq b_{m'+k} + \epsilon/3 + b_{m'} < \epsilon.$$

If we choose $y = z_{m'}$ and $N = m'$, then $(*)$ holds in this case.

Case II. $\lim_{m\to\infty} b_m = b > 0$. It follows by the uniform Opial property of X that for any $\epsilon > 0$ there is $\delta > 0$ and an integer $N > 1$ such that for all integer $m \geq N$ and $z \in X$,

$$\limsup_{j\to\infty} \|T^{n_j+m}x - z\| \leq b + \delta \Rightarrow \|z - z_m\| \leq \epsilon. \tag{3.2.2}$$

We may assume that N is chosen so large that for all $m \geq N$,

$$b_m < b + \delta/2.$$

By the definition of asymptotically nonexpansive type, there exists an integer $m_0 \geq N$ such that

$$\sup\{\|T^n x - T^n u\| - \|x - u\| : u \in C\} < \delta/2,$$

for all $n \geq m_0$. So if $k > m_0$, then

$$\limsup_{j\to\infty} \|T^{n_j+m_0+k}x - T^k z_{m_0}\| \leq \limsup_{j\to\infty} (\|T^{n_j+m_0+k}x - T^k z_{m_0}\| - \|T^{n_j+m_0}x - z_{m_0}\|)$$

$$+ \limsup_{j\to\infty} \|T^{n_j+m_0}x - z_{m_0}\|$$

$$< \delta/2 + b_{m_0} < b + \delta.$$

If we choose $m' = m_0$ and $y = z_{m_0}$, then for any $j \geq N$, we have $\|T^j y - z_{m'+j}\| \leq \epsilon$ by (3.2.2). This proves $(*)$.

Finally, we can show that $\rho = 0$, we distinguish two cases.

Case I. There is $N_0 > 0$ such that $\operatorname{diam}(\overline{co}\{z_m : m \geq N_0\}) = \rho' < \rho$. By $(*)$, there are $y \in K, m', N \in I\!N$ such that $\|T^n y - z_{n+m'}\| \leq (\rho - \rho')/2$, for all $n > N$. So if $n > \max\{N, N_0\}$, then

$$\|z_{N_0} - T^n y\| \leq \|z_{N_0} - z_{n+m'}\| + \|z_{n+m'} - T^n y\| \leq (\rho + \rho')/2.$$

Case II. diam $(\overline{co}\{z_m : m \geq N\}) = \rho$ for all $N \in I\!N$. Since X satisfies the uniform Opial condition and hence the Opial's condition and C is weakly compact, C has normal structure. Hence, there exists $z_0 \in \overline{co}\{z_m : m \in I\!N\}$, such that

$$\rho' = \sup_{m \in I\!N} \|z_0 - z_m\| < diam(\overline{co}\{z_m : m \in I\!N\}) = \rho.$$

By (1), there are $y \in K, m', N \in I\!N$ such that $\|T^n y - z_{n+m'}\| \leq (\rho - \rho')/2$ whenever $n \geq N$. So if $n > N$, then

$$\|z_0 - T^n y\| \leq \|z_0 - z_{n+m'}\| + \|z_{n+m'} - T^n y\| \leq (\rho + \rho')/2.$$

This prove $\rho = 0$. Hence $\lim_{n\to\infty} \|T^n x - x\| = 0$. Therefore $Tx = x$ by continuity of T . □

Corollary 3.2.11 [6, 1995, p.942] Suppose that X is a Banach space satisfying the uniform Opial condition, C is a nonempty weakly compact convex subset of X, and $T : C \to C$ T is an asymptotically nonexpansive mapping. Then T has a fixed point.

3.3 Strong convergence and weak convergence.

In this section we investigate the asymptotic behavior of the iterates $\{T^n x\}$ for a mappings of asymptotically nonexpansive type. And we show the strong convergence of sequence of fixed points for an asymptotically nonexpansive mapping in a Banach space with a uniformly Gâteaux differentiable norm.

Theorem 3.3.1 Suppose X is a Banach space satisfying the uniform Opial condition, C is nonempty weakly compact convex subset of X and $T : C \to C$ is an asymptotically nonexpansive type. Then given an $x \in C$, $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.

Proof. Let $x \in C$,

$\Rightarrow)$ Suppose that $\{T^n x\}$ converges weakly to a fixed point of T.

Then $T^n x \rightharpoonup y$, for some $y \in F(T)$.

To show that T is weakly asymptotically regular at x. We must show that $(T^n x - T^{n+1}x) \rightharpoonup 0$.

Let $f \in X^*$. By $T^n x \rightharpoonup y$, $f(T^n x) \to f(y)$, implies that

$$f(T^n x - y) = f(T^n x) - f(y) \to 0.$$

Consider

$$\begin{aligned} f(T^n x - T^{n+1}x) &= f(T^n x - y + y - T^{n+1}x) \\ &= f(T^n x - y) - f(T^{n+1}x - y) \\ &\to 0 - 0 = f(0). \end{aligned}$$

We have for any $f \in X^*$, $f(T^n x - T^{n+1}x) \to f(0) \Rightarrow (T^n x - T^{n+1}x) \rightharpoonup 0$.

Hence T is weakly asymptotically regular at x.

$\Leftarrow$) Suppose that T is weakly asymptotically regular at x. Then $(T^n x - T^{n+1}x) \rightharpoonup 0$.

To show that $\{T^n x\}$ converges weakly to a fixed point of T. We must show that

$$\omega_w(x) \subseteq F(T).$$

Let $y \in \omega_w(x)$. Then we have a subsequence $\{T^{n_j}x\}$ of $\{T^n x\}$ such that $T^{n_j}x \rightharpoonup y$.

For all integers $m \geq 0$,

$$\begin{aligned} (T^{n_j}x - T^{n_j+m}x) &= (T^{n_j}x - T^{n_j+1}x + T^{n_j+1}x - T^{n_j+2}x + T^{n_j+2}x \\ &\quad + ... + T^{n_j+(m-1)}x + T^{n_j+(m-1)}x - T^{n_j+m}x) \\ &= (T^{n_j}x - T^{n_j+1}x) + (T^{n_j+1}x - T^{n_j+2}x) + ... \\ &\quad + (T^{n_j+(m-1)}x - T^{n_j+m}x) \\ &\rightharpoonup 0 + 0 + ... + 0 = 0. \end{aligned}$$

Thus $(T^{n_j}x - T^{n_j+m}x) \rightharpoonup 0$. Since $T^{n_j}x \rightharpoonup y$,

$$T^{n_j}x - (T^{n_j}x - T^{n_j+m}x) \rightharpoonup y - 0 = y.$$

Then $T^{n_j+m}x \rightharpoonup y$. Set $b_m = \limsup_{j\to\infty} \|T^{n_j+m}x - y\|$.

By the definition of Opial's condition, we have

$$\limsup_{j\to\infty} \|T^{n_j+m+k}x - y\| \leq \limsup_{j\to\infty} \|T^{n_j+m+k}x - T^k x\|$$

for all $m, k \geq 0$. Let $\epsilon > 0$, and $b := \inf\{b_m : m \geq 0\}$. Then $b + \frac{\epsilon}{2}$ is not lower bound of $\{b_m : m \geq 0\}$ there exists $m_0 \in I\!N$ such that $b_{m_0} < b + \frac{\epsilon}{2}$. Since T is an asymptotically nonexpansive type,

$$\limsup_{n\to\infty}(\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}) \leq 0 < \frac{\epsilon}{2}.$$

for each $x \in C$. Then there exists $N_1 \in I\!N$ such that for all $n \geq N_1$,

$$\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\} < \frac{\epsilon}{2}.$$

This implies that, for all integer $k \geq N_1$,

$$\begin{aligned}
b_{m_0+k} &= \limsup_{j\to\infty} \|T^{n_j+m_0+k}x - y\| \\
&\leq \limsup_{j\to\infty} \|T^{n_j+m_0+k}x - T^k y\| \\
&= \limsup_{j\to\infty} (\|T^k(T^{n_j+m_0}x) - T^k y\| - \|T^{n_j+m_0}x - y\| + \|T^{n_j+m_0}x - y\|) \\
&\leq \limsup_{j\to\infty} (\|T^k(T^{n_j+m_0}x) - T^k y\| - \|T^{n_j+m_0}x - y\|) + \limsup_{j\to\infty} \|T^{n_j+m_0}x - y\| \\
&< \tfrac{\epsilon}{2} + \limsup_{j\to\infty} \|T^{n_j+m_0}x - y\| \\
&= \tfrac{\epsilon}{2} + b_{m_0}.
\end{aligned}$$

Hence $\lim_{m\to\infty} b_m = \inf\{b_m : m \geq 0\}$. For all $m \geq N_1$, we note that for each fixed $y \in C$

$$\begin{aligned}
\|T^m y - y\| &= \limsup_{j\to\infty} \|T^m y - y\| \\
&= \limsup_{j\to\infty} \|T^m y - T^{n_j+2m}x + T^{n_j+2m}x - y\| \\
&\leq \limsup_{j\to\infty} \|T^m y - T^{n_j+2m}x\| + \limsup_{j\to\infty} \|T^{n_j+2m}x - y\| \\
&= \limsup_{j\to\infty} (\|T^m y - T^m(T^{n_j+m}x)\| - \|y - T^{n_j+m}x\| + \|y - T^{n_j+m}x\|) \\
&\quad + \limsup_{j\to\infty} \|T^{n_j+2m}x - y\| \\
&\leq \limsup_{j\to\infty} (\|T^m y - T^m(T^{n_j+m}x)\| - \|y - T^{n_j+m}x\|) + \limsup_{j\to\infty} \|y - T^{n_j+m}x\|
\end{aligned}$$

$$+\limsup_{j\to\infty} \|T^{n_j+2m}x - y\|$$

$$< \frac{\epsilon}{2} + b_m + b_{2m}$$

Then $\|T^m y - y\| < \frac{\epsilon}{2} + b_m + b_{2m},\ \forall\ m \geq N_1$.

If $b = 0,\ b_m \to 0,$ and $b_{2m} \to 0 \ \Rightarrow \|T^m y - y\| \to 0,$ as $m \to \infty$. Hence $Ty = y$ by continuity of T for some $N \geq 1$. Then $y \in F(T)$. Suppose now $b > 0$. For $m \geq N_1$, let

$$z_j^{(m)} = \frac{(T^{n_j+m}x - y)}{b_m}.$$

Then, $z_j^{(m)} \rightharpoonup 0$ and $\limsup_{j\to\infty} \|z_j^{(m)}\| = 1$. By the definition of Opial's modulus r_X of X, *liminf* can be replace by *limsup*, that is for each $c > 0$

$$r_X(c) = \inf\{\limsup_{n\to\infty} \|x_n + x\| - 1 : \|x\| \geq c,\ x_n \rightharpoonup 0,\ \limsup_{n\to\infty} \|x_n\| \geq 1\}.$$

We obtain that,

$$\limsup_{j\to\infty} \|z_j^{(2m)} + z\| - 1 \geq r_X(c).$$

It implies that

$$\limsup_{j\to\infty} \|z_j^{(2m)} + z\| \geq 1 + r_X(c)$$

for each $z \in X$ with $\|z\| \geq c$. Taking $z = \frac{(y - T^m y)}{b_{2m}}$, it follows that

$$\limsup_{j\to\infty} \|z_j^{(2m)} + z\| = \limsup_{j\to\infty} \left\| \frac{(T^{n_j+2m}x - y)}{b_{2m}} + \frac{(y - T^m y)}{b_{2m}} \right\|$$

$$= \frac{1}{b_{2m}} \limsup_{j\to\infty} \|T^{n_j+2m}x - T^m y\|$$

$$= \frac{1}{b_{2m}} \limsup_{j\to\infty} (\|T^m(T^{n_j+m}x) - T^m y\| - \|T^{n_j+m}x - y\|$$

$$+ \|T^{n_j+m}x - y\|)$$

$$\leq \frac{1}{b_{2m}} (\limsup_{j\to\infty} (\|T^m(T^{n_j+m}x) - T^m y\| - \|T^{n_j+m}x - y\|)$$

$$+ \limsup_{j\to\infty} \|T^{n_j+m}x - y\|)$$

$$< \frac{1}{b_{2m}} (\frac{\epsilon}{2} + b_m).$$

It follows by the definition of Opial's modulus that,

$$r_X\left(\frac{\|y - T^m y\|}{b_{2m}}\right) + 1 < \frac{1}{b_{2m}} \left(\frac{\epsilon}{2} + b_m\right).$$

It implies that

$$b_{2m}\left(r_X\left(\frac{\|y-T^m y\|}{b_{2m}}\right)+1\right)<\frac{\epsilon}{2}+b_m.$$

Taking the limit as $m \to \infty$ we get,

$$b\left(r_X\left(\frac{\limsup_{m\to\infty}\ \|y-T^m y\|}{b}\right)+1\right)<\frac{\epsilon}{2}+b$$

$$\Rightarrow\left(r_X\left(\frac{\limsup_{m\to\infty}\ \|y-T^m y\|}{b}\right)+1\right)<\frac{\epsilon}{2b}+1$$

$$\Rightarrow r_X\left(\frac{\limsup_{m\to\infty}\ \|y-T^m y\|}{b}\right)<\frac{\epsilon}{2b}.$$

Since r_X is nondecreasing and continuous,

$$\limsup_{m\to\infty}\|y-T^m y\|=0$$

Which in turn implies $Ty=y$ by the continuity of T^N for some $N \geq 1$. Thus we have verified $\omega_w(x) \subseteq F(T)$. The complete of proof, we have to show that $\omega_w(x)$ is a singleton. This can be achieved by using the fact $\lim_{n\to\infty}\|T^n x-y\|=0$ exist for every $y \in \omega_w(x)$ and by a standard argument involving Opial's condition. □

Corollary 3.3.2 [6, 1995, p.940] Suppose X is a Banach space satisfying the uniform Opial condition, C is nonempty weakly compact convex subset of X and $T : C \to C$ is an asymptotically nonexpansive mapping. Then given an $x \in C$, $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.

Theorem 3.3.3 Suppose X is a Banach space with a weakly continuous duality map J_φ, C is a weakly compact convex subset of X, and $T : C \to C$, is an asymptotically nonexpansive type. Then if T is weakly asymptotically regular at $x \in C$, then $\{T^n x\}$ converges weakly to a fixed point of T.

Proof. First observe that for any $p \in F(T)$, the $\lim_{n\to\infty}\|T^n x-p\|$ exists. In fact, for all integers $n, m \geq 1$, we have

$\|T^{n+m}x-p\| \leq \limsup_{n\to\infty}\ \|T^{n+m}x-p\|$

$$= \limsup_{n\to\infty} \left(\|T^n(T^m x) - T^n p\| - \|T^m x - p\| + \|T^m x - p\|\right)$$

$$\leq \limsup_{n\to\infty} \left(\sup\{\|T^n u - T^n p\| - \|u - p\| : u \in C\}\right)$$

$$+ \limsup_{n\to\infty} \|T^m x - p\|$$

$$\leq \limsup_{n\to\infty} \|T^m x - p\| = \|T^m x - p\|.$$

It follows that for all integers $m \geq 1$,

$$\limsup_{n\to\infty} \|T^n x - p\| = \limsup_{n\to\infty} \|T^{n+m} x - p\| \leq \|T^m x - p\|.$$

This implies that $\limsup_{n\to\infty} \|T^n x - p\| \leq \liminf_{m\to\infty} \|T^m x - p\|$. Hence $\lim_{n\to\infty} \|T^n x - p\|$ exists. To show that $\{T^n x\}$ converges weakly to a fixed point of T, it suffices to show that

$$\omega_w(x) \subset F(T). \qquad (**)$$

As a matter of fact if (**) is proven and if $p_1 = \omega - \lim_j T^{n_j} x$ and $p_2 = \omega - \lim_i T^{n_i} x$ belong to $\omega_w(x)$ and $p_1 \neq p_2$, then Opial's condition of X, it implies that

$$\begin{aligned} \lim_{n\to\infty} \|T^n x - p_1\| &= \lim_{i\to\infty} \|T^{n_i} x - p_1\| \\ &< \lim_{i\to\infty} \|T^{n_i} x - p_2\| \\ &= \lim_{j\to\infty} \|T^{n_j} x - p_2\| \\ &< \lim_{j\to\infty} \|T^{n_j} x - p_1\| \\ &= \lim_{n\to\infty} \|T^n x - p_1\|, \end{aligned}$$

a contradiction. Hence $\omega_w(x)$ must be a singleton set. This implies that $T^n x \rightharpoonup y$, where $\{y\} = \omega_w(x)$.

We now show that $\omega_w(x) \subset F(T)$. Let $y = \omega - \lim_{i\to\infty} T^{n_j}$ be an arbitrary element of $\omega_w(x)$. By weakly asymptotically regular of T at x, we have for all integer $m \geq 0$,

$$\omega - \lim_j T^{n_j+m} x = y.$$

Let $r_m = \limsup_{j\to\infty} \|T^{n_j+m} x - y\|$ and $r = \inf\{r_m : m > 0\}$. Claim that $\lim_{m\to\infty} r_m = r$. Let $\epsilon > 0$, by the definition of asymptotically nonexpansive type, there exists

$N_1 \in \mathbb{N}$ such that,

$$\sup\{\|T^n x - T^n y\| - \|x - y\| : y \in C\} < \frac{\epsilon}{2}$$

for all $n \geq N_1$. Since $r < r + \frac{\epsilon}{2}$, there exists $m_0 \in \mathbb{N}$ such that $r < r_{m_0} < r + \frac{\epsilon}{2}$. Then for all $l \geq N_1$ and by Opial's condition of X, we have

$$\begin{aligned}
r_{m_0+l} &= \limsup_{j\to\infty} \|T^{n_j+m_0+l}x - y\| \leq \limsup_{j\to\infty} \|T^{n_j+m_0+l}x - T^l y\| \\
&= \limsup_{j\to\infty} (\|T^{n_j+m_0+l}x - T^l y\| - \|T^{n_j+m_0}x - y\| + \|T^{n_j+m_0}x - y\|) \\
&\leq \limsup_{j\to\infty} (\sup\{\|T^l u - T^l y\| - \|u - y\| : u \in C\}) \\
&\quad + \limsup_{j\to\infty} \|T^{n_j+m_0}x - y\| \\
&< \limsup_{j\to\infty} (\tfrac{\epsilon}{2}) + \limsup_{j\to\infty} \|T^{n_j+m_0}x - y\| \\
&= \tfrac{\epsilon}{2} + r_{m_0}.
\end{aligned}$$

It follows that $\lim_{m\to\infty} r_m = r$ exists. Now for all integers $m, j \geq 0$, we have

$$\begin{aligned}
\Phi(\|T^{n_j+2m}x - y\|) &= \Phi(\|(T^{n_j+2m}x - (T^m y) + (T^m y - y)\|) \\
&= \Phi(\|T^{n_j+2m}x - T^m y\|) \\
&\quad + \textstyle\int_0^1 < (T^m y - y), J_\varphi((T^{n_j+2m}x - T^m y) + t(T^m y - y)) > dt.
\end{aligned}$$

Taking the limit superior as j approaches the infinity, we get

$$\begin{aligned}
\Phi(r_{2m}) &= \limsup_{j\to\infty} \Phi(\|T^{n_j+2m}x - T^m y\|) \\
&\quad + \textstyle\int_0^1 < (T^m y - y), J_\varphi((y - T^m y) + t(T^m y - y)) > dt \\
&= \limsup_{j\to\infty} \Phi(\|T^{n_j+2m}x - T^m y\|) - \textstyle\int_0^1 \|T^m y - y\|\varphi(t\|T^m y - y\|)dt \\
&= \limsup_{j\to\infty} \Phi(\|T^{n_j+2m}x - T^m y\|) - \Phi(\|T^m y - y\|) \\
&\leq \limsup_{j\to\infty} \Phi((\|T^m(T^{n_j+m}x) - T^m y\| - \|T^{n_j+m}x - y\|) \\
&\quad + \limsup_{j\to\infty} \|T^{n_j+m}x - y\|) - \Phi(\|T^m y - y\|) \\
&\leq \limsup_{j\to\infty} \Phi\left(\limsup_{m\to\infty} (\sup\{\|T^m u - T^m y\| - \|u - y\| : u \in C\}) + r_m\right) \\
&\quad - \Phi(\|T^m y - y\|)
\end{aligned}$$

$$\leq \limsup_{j\to\infty} \Phi(0 + r_m) - \Phi(\|T^m y - y\|) = \Phi(r_m) - \Phi(\|T^m y - y\|).$$

It implies that,

$$\Phi(\|T^m y - y\|) \leq \Phi(r_m) - \Phi(r_{2m}).$$

It follow that

$$\Phi(\|T^m y - y\|) \leq \Phi(r_m) - \Phi(r_{2m}) \to 0, \text{ as } m \to \infty.$$

This implies that $T^m y \to y$ strongly and, hence $Ty = y$ by continuity of T^N for some $N \geq 1$. Hence $\omega_w(x) \subset F(T)$. □

Corollary 3.3.4 [10, 1994, p.1350] Suppose X is a Banach space with a weakly continuous duality map J_φ, C is a weakly compact convex subset of X, and $T : C \to C$, is an asymptotically nonexpansive mapping. Then if T is weakly asymptotically regular at $x \in C$, then $\{T^n x\}$ converges weakly to a fixed point of T.

Suppose now C is a weakly compact convex subset of a Banach space X and $T : C \to C$ is an asymptotically nonexpansive completely continuous mapping (we may always asumme $k_n \geq 1$ for all $n \geq 1$). For any $n \geq 1$, we take $t_n = \min\{1 - (k_n - 1)^{\frac{1}{2}}, 1 - \frac{1}{n}\}$. Fix a u in C and define for each integers $n \geq 1$ the contraction $S_n : C \to C$ by

$$S_n(x) = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x. \tag{3.3.1}$$

Then the Banach Contraction Principle yields a unique point $x_n \in C$ that is fixed by S_n, that is, we have

$$x_n = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n}T^n x_n. \tag{3.3.2}$$

Theorem 3.3.5 Suppose that X is a reflexive Banach space with weakly continuous duality map and uniformly Gâteaux differentiable norm. Suppose in addition T are weakly asymptotically regular. Then $\{x_n\}$ converge strongly to a fixed point of T.

Proof. Suppose that the sequence $\{x_n\}$ defined by (3.3.2). From corollary 3.2.7, the fixed point set $F(T)$ of T is nonempty. We now shows that $\{x_n\}$ converge strongly to a fixed point of T.

Now let μ be a Banach limit and define $f : C \to [0, \infty)$ by

$$f(z) = \mu_n \|x_n - z\| \quad \text{for every } z \in C.$$

Then, since the function f on C is convex and continuous, $f(z) \to \infty$ as $\|z\| \to \infty$, and X is reflexive it follows from [13, p. 79] that there exists $u \in C$ with $f(u) = \inf_{z \in C} f(z)$. Define the set

$$M = \{v \in C : f(v) = \inf_{z \in C} f(z)\}.$$

Then M is a nonempty, closed and convex. We further claim that M has the property (ω). If x is in M and $y = w - \lim_j T^{m_j} x$ belong to the weak ω-limit set $\omega_w(x)$ of T at x. By Theorem 3.3.3, we have $\{T^n x\}$ weak converge to fixed point of T, that is $y \in F(T)$. Since T is completely continuous, $\{T^n x\}$ strong convergence to y. It implies that

$$\begin{aligned} f(y) = \mu_n \|x_n - y\| &\le \mu_n \|x_n - T^n x_n\| + \mu_n \|T^n x_n - T^n x\| + \mu_n \|T^n x - y\| \\ &\le k_n \mu_n \|x_n - x\| = k_n f(x) = \inf_{z \in C} f(z) \\ &\le f(y). \end{aligned}$$

Then $f(y) = \inf_{z \in C} f(z)$ and hence $y \in M$. This show that y belongs to M and hence M satisfies the property (ω). It follows from corollary 3.2.7, that T has a fixed point $z_0 \in M$. Next, to show that (x_n) converges strongly to a fixed point of T. We note that, for any $w \in F(T)$,

$$\begin{aligned} \langle x_n - T^n x_n, J(x_n - w) \rangle &= \langle x_n - w, J(x_n - w) \rangle + \langle w - T^n x_n, J(x_n - w) \rangle \\ &\ge \|x_n - w\|^2 - \|w - T^n x_n\| \|x_n - w\| \\ &\ge -(k_n - 1) \|x_n - w\|^2 \\ &\ge -(k_n - 1) d^2 \end{aligned}$$

where $d = \text{diam } C$. Since x_n is a fixed point of S_n, it follows that

$$x_n - T^n x_n = \frac{k_n - t_n}{t_n}(u - x_n)$$

and from last inequality above,we get

$$\langle x_n - u, J(x_n - w)\rangle \leq s_n d^2, \tag{3.3.3}$$

where $s_n = \frac{t_n(k_n-1)}{(k_n-t_n)} \to 0$ as $n \to \infty$. So, putting $w = z_o$, we have

$$\langle x_n - u, J(x_n - z_0)\rangle \leq s_n d^2. \tag{3.3.4}$$

On the other hand, since z_0 is the minimizer of the function f on C, by [14, Lemma 3], we have

$$\mu_n\langle z - z_0, J(x_n - z_0)\rangle \leq 0$$

for all $z \in C$. In particular, we have

$$\mu_n\langle u - z_0, J(x_n - z_0)\rangle \leq 0. \tag{3.3.5}$$

Combing (3.3.4) and (3.3.5), we get

$$\mu_n\langle x_n - z_0, J(x_n - z_0)\rangle = \mu_n\|x_n - z_0\|^2 \leq 0.$$

Therefore, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to z_0. To show that $\{x_n\}$ converges strongly to a fixed point of T, let $\{x_{n_j}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_j} \to z$ and $x_{m_j} \to z'$. Thus $\mu_{n_j}\|x_{n_j} - z_0\|^2 \leq 0$ and hence there is a subsequence $\{x_{n_{j'}}\}$ of $\{x_{n_j}\}$ which converges strongly to a fixed point of T. Hence $z \in F(T)$. Similarly, we have $z' \in F(T)$. It follows from (3.3.3) that

$$\langle z - u, J(z - z')\rangle \leq 0$$

and

$$\langle z' - u, J(z' - z)\rangle \leq 0$$

Adding these two inequalities yields

$$\langle z - z' J(z - z')\rangle = \|z - z'\|^2 = 0.$$

So we have $z = z'$. Therefore $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 3.3.6 Suppose that X is Banach space with a uniformly Gâteaux differentiable norm such that $D(X) < 1$, that C is a closed bounded convex subset of X, and that $T : C \to C$ is an asymptotically nonexpansive mapping. Then, a mapping S_n on C given by (3.3.1) has a unique fixed point x_n in C. Further, if T is weakly asymptotically regular and completely continuous, then $\{x_n\}$ define by (3.3.2) converges strongly to a fixed point of T.

Proof. It follows by the Banach Contraction Principle that S_n has a unique fixed point x_n in C, that is we have

$$x_n = (1 - \frac{t_n}{k_n})u + \frac{t_n}{k_n} T^n x_n \tag{3.3.6}$$

for all $n \geq 1$. Applying Zorn's lemma we can get a subset K of C which is minimal with respect to being nonempty, closed, convex, and satisfying the property (ω). From corollary 3.2.3, the fixed point set $F(T)$ of T is nonempty. We now shows that $\{x_n\}$ converges strongly to a fixed point of T. Now let μ be a Banach limit and define $f : K \to [0, \infty)$ by

$$f(z) = \mu_n \|x_n - z\| \quad \text{for every } z \in K.$$

Then, since the function f on K is convex and continuous, $f(z) \to \infty$ as $\|z\| \to \infty$, and X is reflexive it follows from [13, p.79] that there exists $u \in K$ with $f(u) = \inf_{z \in K} f(z)$. Define the set

$$M = \{v \in K : f(v) = \inf_{z \in K} f(z)\}.$$

Then M is a nonempty, closed and convex. We further claim that M has the property (ω). We must show that $\{T^n x\}$ weak converges to fixed point of T. Let

$y \in \omega_w(x)$ then $y = w - \lim_j T^{m_j}x$. It follows by the weakly asymptotically regular and completely continuous of T, that $Ty = y$. Hence $\omega_w(x) \subseteq F(T)$. For any $n \in I\!N$, we have $\lim_{n\to\infty} \|T^n y - y\| = 0$. By lemma 2.1.20($i$), we have $\rho = 0$. It implies that $T^n x \to y$ and hence $\{T^n x\}$ converges weakly to fixed point of T. In fact, if x is in M, then from $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$, we have

$$\begin{aligned} f(y) = \mu_n\|x_n - y\| &\leq \mu_n\|x_n - T^n x_n\| + \mu_n\|T^n x_n - T^n x\| + \mu_n\|T^n x - y\| \\ &\leq k_n\mu_n\|x_n - x\| = k_n f(x) = \inf_{z\in K} f(z) \\ &\leq f(y). \end{aligned}$$

This show that y belongs to M and hence M satisfies the property (ω). It follows from corollary 3.2.3, that T has a fixed point $z_0 \in M$. Next, to show that (x_n) converges strongly to a fixed point of T. We note that, for any $w \in F(T)$,

$$\begin{aligned} \langle x_n - T^n x_n, J(x_n - w)\rangle &= \langle x_n - w, J(x_n - w)\rangle + \langle w - T^n x_n, J(x_n - w)\rangle \\ &\geq \|x_n - w\|^2 - \|w - T^n x_n\|\|x_n - w\| \\ &\geq -(k_n - 1)\|x_n - w\|^2 \\ &\geq -(k_n - 1)d^2 \end{aligned}$$

where $d =$ diam K. Since x_n is a fixed point of S_n, it follows that

$$x_n - T^n x_n = \frac{k_n - t_n}{t_n}(u - x_n) \tag{3.3.7}$$

and from last inequality above,we get

$$\langle x_n - u, J(x_n - w)\rangle \leq s_n d^2, \tag{3.3.8}$$

where $s_n = \frac{t_n(k_n-1)}{(k_n-t_n)} \to 0$ as $n \to \infty$. So, putting $w = z_o$, we have

$$\langle x_n - u, J(x_n - z_0)\rangle \leq s_n d^2. \tag{3.3.9}$$

On the other hand, since z_0 is the minimizer of the function f on K, by [14, Lemma 3], we have

$$\mu_n\langle z - z_0, J(x_n - z_0)\rangle \leq 0$$

for all $z \in K$. In particular, we have

$$\mu_n\langle u - z_0, J(x_n - z_0)\rangle \leq 0. \qquad (3.3.10)$$

Combining (3.3.9) and (3.3.10), we get

$$\mu_n\langle x_n - z_0, J(x_n - z_0)\rangle = \mu_n\|x_n - z_0\|^2 \leq 0.$$

Therefore, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to z_0. To show that $\{x_n\}$ converges strongly to a fixed point of T, let $\{x_{n_j}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_j} \to z$ and $x_{m_j} \to z'$. Thus $\mu_{n_j}\|x_{n_j} - z_0\|^2 \leq 0$ and hence there is a subsequence $\{x_{n_j'}\}$ of $\{x_{n_j}\}$ which converges strongly to a fixed point of T. Hence $z \in F(T)$. Similarly, we have $z' \in F(T)$. It follows from (3.3.8) that

$$\langle z - u, J(z - z')\rangle \leq 0$$

and

$$\langle z' - u, J(z' - z)\rangle \leq 0.$$

Adding these two inequalities yields

$$\langle z - z', J(z - z')\rangle = \|z - z'\|^2 = 0.$$

So we have $z = z'$. Therefore $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 3.3.7 Let X be a Banach space satisfying Opial's condition and whose norm is UKK. Let C be weakly compact convex subset of X and let $T : C \to C$ be a uniformly continuous of asymptotically nonexpansive type and which is asymptotically regular at the point $x \in C$. Then the iterates $\{T^n x\}$ converges weakly to a fixed point of T.

Proof. We shall show that $\{T^n x\}$ converge weakly to a fixed point of T.

Let $x_n := T^n x$ for $n = 1, 2, 3, \ldots$. Since T is asymptotically regular,

$$\lim_{n\to\infty}(T^n x - T^{n+1}x) = 0, \quad \forall x \in C$$

Then $(x_n - T^m x_n) = (T^n x - T^m(T^n x)) = (T^n x - T^{m+n} x) \to 0.$
Thus $(x_n - T^m x_n) \to 0$

Since $\{x_n\}$ is a sequence in weakly compact C, there exist subsequence$\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to y \in C$, the set $\omega_w(x) \neq \emptyset$, by Theorem 3.1.4, we have $\omega_w(x) \subset F(T)$. To show that for each $y \in \omega_w(x)$ the $\lim_{n\to\infty} \|T^n x - y\|$ exists. Let $y \in \omega_w(x)$ and any $m \in I\!N$ we have $y \in F(T)$,

$$\begin{aligned}
\limsup_{n\to\infty} \|T^n x - y\| &= \limsup_{n\to\infty} \|T^{n+m} x - y\| \\
&= \limsup_{n\to\infty} \|T^{n+m} x - T^n y\|, \quad \text{by } y \in F(T). \\
&\leq \limsup_{n\to\infty} (\|T^n(T^m x) - T^n y\| - \|T^m x - y\|) \\
&\quad + \limsup_{n\to\infty} \|T^m x - y\| \\
&\leq \limsup_{n\to\infty} (\sup\{\|T^n u - T^n y\| - \|u - y\| : u \in C\}) \\
&\quad + \limsup_{n\to\infty} \|T^m x - y\| \\
&\leq \|T^m x - y\|.
\end{aligned}$$

Thus $\limsup_{n\to\infty} \|T^n x - y\| \leq \liminf_{m\to\infty} \|T^m x - y\|$

Hence $\lim_{n\to\infty} \|T^n x - y\|$ exists.

Claim that $\omega_w(x)$ is singleton set. Let $p_1, p_2 \in \omega_w(x)$ and $p_1 \neq p_2$, there exists subsequence $\{T^{n_i} x\}$ and $\{T^{n_j} x\}$ of $\{T^n x\}$ such that $T^{n_i} x \rightharpoonup p_1, T^{n_j} x \rightharpoonup p_2$, then

$$\begin{aligned}
\lim_{n\to\infty} \|T^n x - p_1\| &= \lim_{i\to\infty} \|T^{n_i} x - p_1\| \\
&< \lim_{i\to\infty} \|T^{n_i} x - p_2\|, \text{ by Opial's condition} \\
&= \lim_{j\to\infty} \|T^{n_j} x - p_2\| \\
&< \lim_{j\to\infty} \|T^{n_j} x - p_1\|, \text{ by Opial's condition} \\
&= \lim_{n\to\infty} \|T^n x - p_1\|
\end{aligned}$$

Then $\lim_{n\to\infty} \|T^n x - p_1\| < \lim_{n\to\infty} \|T^n x - p_1\|$ a contradiction.

Since every subsequence $\{T^{n_j} x\}$ of $\{T^n x\}$ converges weakly to a fixed point of T and $\lim_{n\to\infty} \|T^n x - y\|$ exists. Hence $\{T^n x\}$ is weakly convergence to a fixed point of T. □

Corollary 3.3.8 Let X be a Banach space satisfying Opial's condition and whose norm is UKK. Let C be weakly compact convex subset of X and let $T : C \to C$ be an asymptotically nonexpansive mapping such that T^N continuous for some $N \geq 1$ and which is asymptotically regular at the point $x \in C$. Then the iterates $\{T^n x\}$ converges weakly to a fixed point of T.