# CHAPTER II

# PRELIMINARIES

## 2.1 Normed and Banach spaces

**Definition 2.1.1** Let $X$ be a linear space over the field $\mathbb{R}$. A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is said to be a *normed* on $X$ if it satisfies the following conditions :

(i) $\|x\| \geq 0, \ \forall x \in X$

(ii) $\|x\| = 0 \Longleftrightarrow x = 0$

(iii) $\|x + y\| \leq \|x\| + \|y\|, \ \forall x, y \in X$ (the triangle inequality)

(iv) $\|\alpha x\| = |\alpha| \, \|x\|, \ \forall x \in X$ and $\forall \alpha \in \mathbb{R}$.

A linear space $X$ over $\mathbb{R}$ with norm $\|\cdot\|$ defined on $X$ is called a *normed linear space* or simply a *normed space* over $\mathbb{R}$, written $(X, \|\cdot\|)$ or simply $X$.

**Definition 2.1.2** A normed space $X$ is called *complete* if every Cauchy sequence in $X$ converges to a vector in $X$.

**Definition 2.1.3** A complete normed linear space over field $\mathbb{R}$ is called a *Banach space* over $\mathbb{R}$.

A normed linear space carries a natural metric, namely, the distance $d : X \times X \longrightarrow \mathbb{R}$ defined by taking

$$d(x, y) = \|x - y\|$$

Taking $\alpha = -1$ in (iv) and combining this with (iii) instantly gives

$$\|x - y\| \leq \|x\| + \|y\|$$

and the triangle inequality for $d$ follows from this.

**Definition 2.1.4** A Banach space is a normed linear space $(X, \|\cdot\|)$ which is complete relative to the metric $d$ defined above.

**Definition 2.1.5** Let $X$ be a linear space over the field $\mathbb{R}$. A function $\langle\cdot,\cdot\rangle : X \times X \longrightarrow \mathbb{R}$ that assigns to each ordered pair $(x, y)$ of vector in $X$ a scalar $\langle x, y\rangle$ is said to be *inner product* on $X$ if it satisfies the following conditions:

(i) $\langle x, x\rangle \geq 0,\ \forall x \in X$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$

(ii) $\langle x, y\rangle = \langle y, x\rangle,\ \forall x, y \in X$

(iii) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle,\ \forall x, y \in X$ and $\forall \alpha \in \mathbb{R}$.

(iv) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle\ \forall x, y, z \in X$.

The scalar $\langle x, y\rangle$ is called the inner product of $x$ and $y$. A linear space $X$ equipped with an inner product $\langle\cdot,\cdot\rangle$ defined on $X$ is called *inner product space*. We also see that an inner product on $X$ relative to the norm on $X$ given by $\|x\| = \sqrt{\langle x, y\rangle}$ and the metric $d$ on $X$ given by $d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y\rangle}$ for all $x, y \in X$

**Definition 2.1.6** A *Hilbert space* is a complete inner product space.

**Definition 2.1.7** Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a *contraction* on $X$ if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y).$$

**Theorem 2.1.8 (Banach contraction mapping principle)** Consider a metric space $X = (X, d)$. Suppose that $X$ is complete and let $T : X \to X$ be a contraction on $X$. Then $T$ has a unique fixed point.

**Proof.** See [13].

**Definition 2.1.9** A bounded convex subset of $K$ of a Banach space $X$ is said to have *normal structure* if every bounded, convex subset $H$ of $K$ that contains more than one point contains a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \quad diam(H).$$

**Definition 2.1.10** A Banach space $X$ is said to have *normal structure* if every bounded, convex subset of $X$ has normal structure.

**Definition 2.1.11** The Banach space $X$ is said to be *uniformly convex* whenever given $\varepsilon > 0$ there exist $\delta > 0$ such that if $x, y \in S(X)$ where $S(X) = \{x \in X : \|x\| = 1\}$ and $\|x - y\| \geq \varepsilon$ then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

**Definition 2.1.12** A normed linear space $E$ is *uniformly convex in every direction* (UCED) if and only if for every nonzero member $z$ of $E$ and $\varepsilon > 0$, there exist a $\delta > 0$ such that $|\lambda| < \varepsilon$ if $\|x\| = \|y\| = 1$, $x - y = \lambda z$, and $\|\frac{x+y}{2}\| > 1 - \delta$.

**Theorem 2.1.13** (cf[9]) Compact convex sets have normal structure.

**Theorem 2.1.14** (cf[20]) If $X$ is a UCED Banach space then $X$ has a normal structure.

## 2.2 Fixed point theorems for nonexpansive mappings

In this section we establish a fixed point theorems for nonexpansive mapping applying to a uniformly convex Banach space and normal structure.

**Definition 2.2.1** Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a *nonexpansive* on $X$ if

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$.

**Definition 2.2.2** A subset $C$ of a weakly topological space $X$ is called *weakly compact* if every open cover of $C$ has a finite subcover.

**Theorem 2.2.3** (cf[10]) Let $X$ be a Banach space, $C$ a weakly compact set in $X$, and $T : C \to C$ a nonexpansive mapping. If $C$ has normal structure then $T$ has a fixed point in $C$.

**Proposition 2.2.4** (cf[9]) Every uniformly convex Banach space is reflexive.

**Corollary 2.2.5** (cf[9]) If $C$ is a nonempty closed bounded convex subset of a uniformly convex Banach space, then every nonexpansive $T : C \to C$ has a fixed point.

**Theorem 2.2.6** (cf[12]) Let $X$ be a reflexive Banach space. Suppose $K$ is a bounded closed convex subset of $X$ which has normal structure. Then any nonexpansive mappings $T : K \to K$ has a fixed point.

**Definition 2.2.7** Let $X$ be a Banach spaces. A mapping $T : X \to X$ is said to be *h-non expansive* for $h > 0$ if $\|Tx - Ty\| \leq \max\{\|x - y\|, h\}$ for $x, y \in X$.

**Theorem 2.2.8** (cf[9]) If $K$ is a nonempty closed bounded convex subset of a Banach space $X$ and if $T : K \to K$ is h-nonexpansive mapping then $\inf\{\|x - Tx\| : x \in K\} \leq h$.

**Definition 2.2.9** Let $K$ be a nonempty closed bounded convex subset of a Banach space $X$ and let $T : K \to K$ be a mapping, for some $h > 0$ and $p > 0$, the condition $\|Tx - Ty\| \leq h\|x - y\|^p \ (x, y \in K)$ is said to be *Hölder condition.*

**Theorem 2.2.10** Suppose $K$ is a nonempty closed bounded convex subset of a Banach space $X$ and suppose $T : K \to K$ satisfies the Hölder condition for some $h, p \in (0,1)$. Then $\inf\{\|x - Tx\| : x \in K\} \leq h^{(\frac{1}{1-p})}$.

**Proof.** See[9].

As we have seen a nonexpansive mapping is $h$-nonexpansive for any $h > 0$,that is, if $T : K \to K$ is nonexpansive then $T$ is $\frac{1}{n}$-nonexpansive and moreover there exists $z \in K$ such that $\|z - Tz\| < \frac{1}{n}\ \forall\ n \in \mathbb{N}$. We have the following.

**Lemma 2.2.11** Let $K$ be a nonempty closed bounded convex subset of a Banach space $X$. If $T : K \to K$ is a nonexpansive mapping, then $\inf\{\|x - Tx\| : x \in K\} = 0$.

**Proof.** See [9].

**Lemma 2.2.12** ( Goebel-Karlovitz ) Let $K$ be a subset of a Banach space $X$ which is minimal with respect to being nonempty weakly compact convex and $T$ invariant for some nonexpansive mapping $T$, and suppose $x_n \subseteq K$ satisfies $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Then for each $x \in K$, $\lim_{n\to\infty} \|x - x_n\| = diam(K)$.

**Proof.** See [9, p.212].

**Theorem 2.2.13** (cf[1]) Let $C$ be a nonempty, closed, bounded, convex set in a (real) Hilbert space $H$. Then each nonexpansive map $T : C \to C$ has at least one fixed point.

## 2.3 Fixed point theorems in metric spaces

**Definition 2.3.1** A metric space $(X, d)$ is *convex* if for each $x, y \in X$ with $x \neq y$ there exists $z \in X$, $x \neq z \neq y$, such that $d(x, z) + d(z, y) = d(x, y)$

**Lemma 2.3.2** Let $K$ be a closed subset of a complete and convex metric space $(X,d)$. If $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ ( the boundary of $K$) such that

$$d(x,z) + d(z,y) = d(x,y).$$

In this section we introduced two fixed point theorems for self mappings of complete metric spaces by Rashwan and Sadeek [16]. Let $\Phi$ denote the class of function $\phi : [0,\infty) \longrightarrow [0,\infty)$ which satisfy the condition $\phi(t) = t^n$, $n \in \mathbb{N}$ for every $t \geq 0$

**Theorem 2.3.3** Let $S$ and $T$ be the mappings of complete metric space $(X,d)$ into itself, and $\phi : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following statements:

(i) $\phi$ is continuous and strictly increasing in $\mathbb{R}^+$;

(ii) $\phi(t) = 0$ if and only if $t = 0$, and if $a, b$, and $c$ are decreasing functions from $\mathbb{R}^+ \cup \{0\}$ into $[0,1]$ such that $a(t) + 2b(t) + c(t) < 1$ for every $t > 0$ and suppose that $S$ and $T$ satisfy the following condition:

$$\begin{aligned}\phi(d(Sx,Ty)) \;\leq\; & a(d(x,y))\phi(d(x,Sx)) + b(d(y,Ty))\left[\phi(d(x,y)) + \phi(d(Sx,Ty))\right] \\ & + c(d(x,y)) \min\left\{\phi(d(x,Ty)), \phi(d(y,Sx))\right\}\end{aligned}$$

where $x, y \in X$ and $x \neq y$. Then $S$ and $T$ have a unique fixed point in $X$.

**Definition 2.3.4** Let $(X,d)$ be a metric space, and let $S$ and $f$ be self mappings of $X$. Then $S$ and $f$ are said to be *weakly commuting* at a point $x$ if

$$d(Sfx, fSx) \leq d(Sx, fx).$$

**Definition 2.3.5** Let $S$ and $f$ be self mappings of $X$. Then $S$ and $f$ are said to be *compatible* if $\lim_{n\to\infty} d(Sfx_n, fSx_n) = 0$ whenever $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$.

**Theorem 2.3.6** (cf[15]) Let $(X, d)$ be a complete convex metric space, $K$ a nonempty closed subset of $X$ and let $S$ and $T$ be self mappings of $X$ with $S$ or $T$ continuous and $\partial K \subseteq S(K) \cap T(K)$. Suppose that $f$ and $g$ are mappings from $K$ into $X$ which satisfy

(a) $f(K) \cap K \subseteq T(K)$ and $g(K) \cap K \subseteq S(K)$,

(b) $(S, f)$ and $(T, g)$ are two pairs of compatible mappings,

(c) there exists a lower semicontinuous function $\delta : (0, \infty) \to (0, \infty)$ such that for any $\varepsilon > 0$, $\delta(\varepsilon) > \varepsilon$ and for $x, y \in X$, $\varepsilon \leq M(x, y) < \delta(\varepsilon)$ imply $d(fx, gy) < \varepsilon$ where $M(x, y) = \max d(Sx, Ty), d(Sx, fx), d(Ty, gy)\}$,

(d) for every $x \in K$, $Tx \in \partial K \Rightarrow gx \in K$ and $Sx \in \partial K \Rightarrow fx \in K$,

(e) for every $x, y \in K$,

(i) $Tx \in \partial K$, $Sy \in K$ and $d(Sy, Tx) + d(Tx, fy) = d(Sy, fy) \Rightarrow d(Tx, gx) \leq d(Sy, fy)$ and

(ii) $Sx \in \partial K$, $Ty \in K$ and $d(Ty, Sx) + d(Sx, gy) = d(Sy, fy) \Rightarrow d(Sx, fx) \leq d(Ty, gy)$.

Then all of the $f, g, S$ and $T$ have a unique common fixed point.

## 2.4 Banach Limits

In this section we give Banach limits which useful for the next chapter.

**Definition 2.4.1** Let $l^\infty$ be a space of all bounded over filed $\mathbb{R}$. A linear continuous functional $\mu : l^\infty \to \mathbb{R}$ which we denote $\mu(x)$ by $\mu(x_n)$ is called a *Banach limit* if

(1) $\mu(x) \geq 0$ if $(x_n) \geq 0$ far all $n \in \mathbb{N}$

(2) $\mu(e) = 1 = \|\mu\|$ where $e = \{1, 1, ...\}$

(3) $\mu(x_n) = \mu(x_{n+1})$ for each $x = \{x_1, x_2, ...\} \in l^\infty$

**Theorem 2.4.2** (cf[19]) There exists a linear continuous functional $f$ on $l^\infty$ such that $\|\mu\| = \mu(1)$ and $\mu(x_n) = \mu(x_{n+1})$ for each $x = (x_1, x_2, ...) \in l^\infty$

**Theorem 2.4.3** (cf[19]) Let $\mu$ be a Banach limit. Then

$$\inf\{x_n : n \in \mathbb{N}\} \leq \mu(x) \leq \sup\{x_n : n \in \mathbb{N}\}$$

for every $x = (x_1, x_2, ...) \in l^{\infty}$and $x_n \longrightarrow c$ which $c \in \mathbb{R}$. It follows that $\mu(x_n) = \mu(x) = c$.

## 2.5 Boundary conditions

The following boundary conditions have been particularly useful in extending fixed point theory for non-self mappings:

**Definition 2.5.1** Let $K$ be subset of a Banach space $X$. A mapping $T : K \to X$ satisfies *Rothe's condition* if $T(\partial K) \subseteq K$ .

**Definition 2.5.2** Let $K$ be subset of a Banach space $X$. A mapping $T : K \to X$ is said to be *inward condition* if $Tx \in I_K(x)$, for all $x \in K$
where $I_K(x) = \{z \in X : z = x + \lambda(y - x)$, for some $y \in K$ and $\lambda \geq 0\}$ .

**Definition 2.5.3** Let $K$ be subset of a Banach space $X$. A mapping $T : K \to X$ is said to be *weakly inward condition* if $Tx \in \overline{I_K(x)}$ for all $x \in K$ where $\overline{I_K(x)}$ is a closure of $I_K(x)$.

**Definition 2.5.4** Let $K$ be subset of a Banach space $X$. A mapping $T : K \to X$ satisfies *Leray-Schauder's condition* if the $intK \neq \emptyset$ then there exists a $z \in intK$ such that $Tx - z \neq m(x - z)$ for $x \in \partial K$ and $m > 1$.

**Remark 2.5.5** It is well known that there hold the following:
Rothe's condition$\Longrightarrow$inward condition$\Longrightarrow$weakly inward condition$\Longrightarrow$Leray-Schauder's condition(in case $intK \neq \emptyset$).

**Theorem 2.5.6** (cf[14]) Let $K$ be a closed subset of a Banach space $X$ and $T : K \to X$ a contraction satisfying one of the following:

(i) $T(\partial K) \subseteq K$ .

(ii) $T$ is weakly inward.

(iii) $0 \in IntK$ and $Tx \neq mx$ for all $x \in \partial K$ and $m > 1$.

Then $T$ has a unique fixed point.

**Definition 2.5.7** A subset $K$ of a Banach space $X$ is said to be *star shape* if there exist a point $x_0 \in K$ such that $tx_0 + (1-t)x \in K$ for any $t \in (0,1)$ and $x \in K$, where $x_0$ is called a *center* of $K$.

**Theorem 2.5.8** (cf[14]) Let $X$ be a Banach space, $K$ a closed subset of $X$, and $T : K \to X$ a nonexpansive mapping such that $K$ is bounded and one of the following hold:

(i) $K$ is star shape and $T(\partial K) \subseteq K$ .

(ii) $K$ is star shape and $T$ is weakly inward.

(iii) $0 \in IntK$ and $Tx \neq mx$ for all $x \in \partial K$ and $m > 1$. Then there exists a bounded sequence $\{x_n\}$ in $K$ such that $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$

**Theorem 2.5.9** (cf[14]) Let $X$ be a uniformly convex Banach space, $K$ a closed bounded convex subset of $X$, and $T : K \to X$ a nonexpansive mapping satisfying one of (i)-(iii) of Theorem 2.5.6. Then $T$ has a fixed point.

**Definition 2.5.10** A sequence $\{x_n\}$ satisfying $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$ is called an *approximate fixed point sequence.*