CHAPTER III

MAIN RESULTS

3.1 Fixed point in metric spaces

In this section we establish a common fixed point theorems in a complete metric spaces which improve a generalization of the fixed point theorems by Rashwan and Sadeek [16].

Theorems 3.1.1 Let (X,d) be a complete convex metric space, K be a nonempty closed subset of X. Let $S,T : K \to X$ be a non-self mapping such that $T(\partial K) \cup S(\partial K) \subseteq K$ and $\phi : [0,\infty) \to [0,\infty)$ satisfy the following:
(i) ϕ is continuous and strictly increasing in $\mathbb{R}^+$;
(ii) $\phi(t) = 0$ if and only if $t = 0$, and if a,b and c are three decreasing functions from $\mathbb{R}^+ \cup \{0\}$ into $[0,1)$ such that $a(t) + 2b(t) + c(t) < 1$, for all $t > 0$. Suppose that S and T satisfies the following condition

$$\begin{aligned} \phi[d(Sx,Ty)] &\leq a(d(x,y))\phi(d(x,y)) + b(d(x,y))\left[\phi(d(x,Sx) + \phi(d(y,Ty))\right] \\ &\quad + c(d(x,y))\min\{\phi(d(x,Ty),\phi(d(y,Sx)\}. \end{aligned} \tag{1}$$

Then S and T have a unique common fixed point.

Proof. Let $x_0 \in K$ and $\{x_n\}$ be a sequence in K and $\{y_n\}$ be a sequence in X which satisfying
(I) for any $n \in \mathbb{N}$,

$$\begin{aligned} y_{2n+1} &= Tx_{2n} \quad \text{and} \\ y_{2n} &= Sx_{2n-1}. \end{aligned}$$

(II) Let $y_{2n} \in K$ and $y_{2n+1} \in K$ we can represent that $y_{2n} = x_{2n}$ and $y_{2n+1} = x_{2n+1}$, unless as it occurs, that is, $y_{2n} \in K$ and $y_{2n+1} \notin K$. It follows that there exists $x_{2n+1} \in \partial K$

such that $d(y_{2n}, x_{2n+1}) + d(x_{2n+1}, y_{2n+1}) = d(y_{2n}, y_{2n+1})$ and another case, if $y_{2n} \notin K$ and $y_{2n+1} \in K$ then there exists , $x_{2n} \in \partial K$ and $d(y_{2n-1}, x_{2n}) + d(x_{2n}, y_{2n}) = d(y_{2n-1}, y_{2n})$.

Put

$$\begin{aligned} P_0 &= \{y_{2n} \in \{y_n\} : y_{2n} = x_{2n}\}, \\ P_1 &= \{y_{2n} \in \{y_n\} : y_{2n} \neq x_{2n}\}, \\ Q_0 &= \{y_{2n+1} \in \{y_n\} : y_{2n+1} = x_{2n+1}\}, \\ Q_1 &= \{y_{2n+1} \in \{y_n\} : y_{2n+1} \neq x_{2n+1}\}. \end{aligned}$$

We prove the following three lemmas before using them in this theorems.

Lemma 3.1.2 Let $e_n = d(y_n, y_{n+1})$ for any $n \in \mathbb{N}$. Then $\{e_n\}$ is a strictly decreasing sequence.

Proof. Case 1 If $y_{2n} \in P_0$ and $y_{2n+1} \in Q_0$ then $y_{2n} = x_{2n}$ and $y_{2n+1} = x_{2n+1}$

Note that

$$\begin{aligned} \phi(e_{2n}) &= \phi(d(y_{2n}, y_{2n+1})) = \phi(d(Sx_{2n-1}, Tx_{2n})) \qquad (2) \\ &\leq a(d(x_{2n-1}, x_{2n}))\phi(d(x_{2n-1}, x_{2n})) \\ &\quad + b(d(x_{2n-1}, x_{2n}))\left[\phi(d(x_{2n-1}, Sx_{2n-1}) + \phi(d(x_{2n}, Tx_{2n})\right] \\ &\quad + c(d(x_{2n-1}, x_{2n})) \min\{\phi(d(x_{2n-1}, Tx_{2n}), \phi(d(x_{2n}, Sx_{2n-1})\} \\ &= a(d(x_{2n-1}, x_{2n}))\phi(d(x_{2n-1}, x_{2n})) \\ &\quad + b(d(x_{2n-1}, x_{2n}))\left[\ \phi(d(y_{2n-1}, y_{2n}) + \phi(d(y_{2n}, y_{2n+1})\ \right] \\ &\quad + c(d(x_{2n-1}, x_{2n})) \min\{\phi(d(x_{2n-1}, y_{2n+1}), \phi(d(y_{2n}, y_{2n})\} \\ &\leq a(d(y_{2n-1}, y_{2n}))\phi(d(y_{2n-1}, y_{2n})) \\ &\quad + b(d(y_{2n-1}, y_{2n}))\left[\phi(d(y_{2n-1}, y_{2n}) + \phi(d(y_{2n}, y_{2n+1})\right] \qquad (3) \end{aligned}$$

By (3), we have

$$1 - b(d(y_{2n-1}, y_{2n}))\phi\left(d\left(y_{2n}, y_{2n+1}\right)\right)$$

$$\leq [a(d(y_{2n-1}, y_{2n})) + b(d(y_{2n-1}, y_{2n}))]\,[\phi(d(y_{2n-1}, y_{2n})] \tag{4}$$

Then,

$$\phi\left(d\left(y_{2n}, y_{2n+1}\right)\right) \leq \left[\frac{a(d(y_{2n-1}, y_{2n})) + b(d(y_{2n-1}, y_{2n}))}{1 - b(d(y_{2n-1}, y_{2n}))}\right][\phi(d(y_{2n-1}, y_{2n})] \tag{5}$$

Since

$$a(d(y_{2n-1}, y_{2n})) + 2b(d(y_{2n-1}, y_{2n})) + c(d(y_{2n-1}, y_{2n})) < 1, \tag{6}$$

We have

$$a(d(y_{2n-1}, y_{2n})) + 2b(d(y_{2n-1}, y_{2n})) < 1.$$

So that $a(d(y_{2n-1}, y_{2n})) + b(d(y_{2n-1}, y_{2n})) < 1 - b(d(y_{2n-1}, y_{2n})) < 1$.

Therefore $\phi(e_{2n}) = \phi\left(d\left(y_{2n}, y_{2n+1}\right)\right) < \phi(d(y_{2n-1}, y_{2n}) = \phi(e_{2n-1})$.

Since ϕ is a strictly increasing function, we obtain $e_{2n} < e_{2n-1}$.

Case 2 If $y_{2n} \in P_0$ and $y_{2n+1} \in Q_1$ then $y_{2n} = x_{2n}$ and there exist $x_{2n+1} \in \partial K$ with $y_{2n+1} \neq x_{2n+1}$and $d\left(x_{2n}, x_{2n+1}\right) + d(x_{2n+1}, y_{2n+1}) = d\left(x_{2n}, y_{2n+1}\right)$.

Note that,

$$\begin{aligned}
\phi(e_{2n}) &= \phi\left(d\left(y_{2n}, y_{2n+1}\right)\right) = \phi\left(d\left(Sx_{2n-1}, Tx_{2n}\right)\right) \\
&\leq a(d(x_{2n-1}, x_{2n}))\phi(d(x_{2n-1}, x_{2n})) \\
&\quad + b(d(x_{2n-1}, x_{2n}))\,[\phi(d(x_{2n-1}, Sx_{2n-1}) + \phi(d(x_{2n}, Tx_{2n})] \\
&\quad + c(d(x_{2n-1}, x_{2n}))\min\{\phi(d(x_{2n-1}, Tx_{2n}), \phi(d(x_{2n}, Sx_{2n-1})\} \\
&\leq a(d(y_{2n-1}, y_{2n}))\phi(d(y_{2n-1}, y_{2n})) \\
&\quad + b(d(y_{2n-1}, y_{2n}))\,[\phi(d(y_{2n-1}, y_{2n}) + \phi(d(y_{2n}, y_{2n+1})] \\
&\quad + c(d(y_{2n-1}, y_{2n}))\min\{\phi(d(y_{2n-1}, y_{2n+1}), \phi(d(y_{2n}, y_{2n})\}.
\end{aligned}$$

So,

$$1 - b(d(y_{2n-1}, y_{2n}))\phi\left(d\left(y_{2n}, y_{2n+1}\right)\right)$$

$$\leq \left[a(d(y_{2n-1}, y_{2n})) + b(d(y_{2n-1}, y_{2n}))\right]\left[\phi(d(y_{2n-1}, y_{2n})\right].$$

Then,

$$\phi\left(d\left(y_{2n}, y_{2n+1}\right)\right) \leq \left[\frac{a(d(y_{2n-1}, y_{2n})) + b(d(y_{2n-1}, y_{2n}))}{1 - b(d(y_{2n-1}, y_{2n}))}\right]\left[\phi(d(y_{2n-1}, y_{2n})\right]. \quad (7)$$

Since $a(d(y_{2n-1}, y_{2n})) + 2b(d(y_{2n-1}, y_{2n})) + c(d(y_{2n-1}, y_{2n})) < 1$, we have

$a(d(y_{2n-1}, y_{2n})) + 2b(d(y_{2n-1}, y_{2n})) < 1$

So $a(d(y_{2n-1}, y_{2n})) + b(d(y_{2n-1}, y_{2n}))< 1 - b(d(y_{2n-1}, y_{2n})) < 1$.

Therefore $\phi(e_{2n}) = \phi\left(d\left(y_{2n}, y_{2n+1}\right)\right) < \phi(d(y_{2n-1}, y_{2n}) = \phi(e_{2n-1})$.

Since ϕ is a strictly increasing function, we obtain $e_{2n} < e_{2n-1}$.

Therefore $\{e_n\}$ is a strictly decreasing.

Case 3 If $y_{2n} \in P_1$ and $y_{2n+1} \in Q_0$ then there is $x_n \in \partial K$, $y_{2n} \neq x_{2n}$, $y_{2n+1} = x_{2n+1}$, Note that $y_{2n-1} \in Q_0$ and there is a point, and $y_{2n} = Sx_{2n-1} \in T(\partial K) \cup S(\partial K) \subseteq K$ such that

$$d\left(y_{2n-1}, x_{2n}\right) + d(x_{2n}, y_{2n}) = d\left(y_{2n-1}, y_{2n}\right).$$

Consider,

$$\begin{aligned}
\phi(e_{2n}) &= \phi\left(d\left(y_{2n}, y_{2n+1}\right)\right) = \phi\left(d\left(Sx_{2n-1}, Tx_{2n}\right)\right) \\
&\leq a(d(x_{2n-1}, x_{2n}))\phi(d(x_{2n-1}, x_{2n})) \\
&\quad +b(d(x_{2n-1}, x_{2n}))\left[\phi(d(x_{2n-1}, Sx_{2n-1}) + \phi(d(x_{2n}, Tx_{2n})\right] \\
&\quad +c(d(x_{2n-1}, x_{2n}))\min\left\{\phi(d(x_{2n-1}, Tx_{2n}), \phi(d(x_{2n}, Sx_{2n-1})\right\} \\
&\leq a(d(x_{2n-1}, x_{2n}))\phi(d(y_{2n-1}, y_{2n})) \\
&\quad +b(d(x_{2n-1}, x_{2n}))\left[\phi(d(x_{2n-1}, y_{2n}) + \phi(d(x_{2n}, y_{2n+1})\right] \\
&\quad +c(d(x_{2n-1}, x_{2n}))\min\left\{\phi(d(x_{2n-1}, y_{2n+1}), \phi(d(x_{2n}, y_{2n})\right\} \\
&\leq a(d(x_{2n-1}, x_{2n}))\phi(d(y_{2n-1}, y_{2n})) \\
&\quad +b(d(x_{2n-1}, x_{2n}))\left[\phi(d(y_{2n-1}, y_{2n}) + \phi(d(y_{2n}, y_{2n+1})\right] \\
&\quad +c(d(x_{2n-1}, x_{2n}))\min\left\{\phi(d(x_{2n-1}, y_{2n+1}), \phi(d(x_{2n}, y_{2n})\right\} \qquad (8)
\end{aligned}$$

We now consider in case $\phi(d(x_{2n-1}, y_{2n+1}) \geq \phi(d(x_{2n}, y_{2n})$.

Then $\min\left\{\phi(d(x_{2n-1}, y_{2n+1}), \phi(d(x_{2n}, y_{2n})\right\} = \phi(d(x_{2n}, y_{2n})$ and from (8), we have

$$\begin{aligned}
\phi(e_{2n}) &= \phi\left(d\left(y_{2n}, y_{2n+1}\right)\right) \\
&\leq a(d(x_{2n-1}, x_{2n}))\phi(d(y_{2n-1}, y_{2n})) \\
&\quad +b(d(x_{2n-1}, x_{2n}))\left[\phi(d(y_{2n-1}, y_{2n}) + \phi(d(y_{2n}, y_{2n+1})\right] \\
&\quad +c(d(x_{2n-1}, x_{2n}))\phi(d(x_{2n}, y_{2n}) \\
&\leq a(d(x_{2n-1}, x_{2n}))\phi(d(y_{2n-1}, y_{2n})) \\
&\quad +b(d(x_{2n-1}, x_{2n}))\left[\phi(d(y_{2n-1}, y_{2n}) + \phi(d(y_{2n}, y_{2n+1})\right] \\
&\quad +c(d(x_{2n-1}, x_{2n}))\phi(d(y_{2n}, y_{2n+1}).
\end{aligned}$$

This implies that

$$\left[1 - b(d(x_{2n-1}, x_{2n})) - c(d(x_{2n-1}, x_{2n}))\right]\phi\left(d\left(y_{2n}, y_{2n+1}\right)\right)$$

$$\leq a(d(x_{2n-1}, x_{2n})) + b(d(x_{2n-1}, x_{2n}))\left[\phi(d(y_{2n-1}, y_{2n})\right].$$

Since $a(d(x_{2n-1},x_{2n}))+2b(d(x_{2n-1},x_{2n}))+c(d(x_{2n-1},x_{2n}))<1$, we have

$$\frac{a(d(x_{2n-1},x_{2n}))+b(d(x_{2n-1},x_{2n}))}{1-b(d(x_{2n-1},x_{2n}))-c(d(x_{2n-1},x_{2n}))}<1.$$

Therefore $\phi\left(d\left(y_{2n},y_{2n+1}\right)\right)<\phi\left(d(y_{2n-1},y_{2n}\right))$.

On the other hand, if $\phi(d(x_{2n-1},y_{2n+1})<\phi(d(x_{2n},y_{2n})$, then from (8) we have

$$\begin{aligned}
\phi(e_{2n}) &= \phi\left(d\left(y_{2n},y_{2n+1}\right)\right)\\
&\leq a(d(x_{2n-1},x_{2n}))\phi(d(y_{2n-1},y_{2n}))\\
&\quad +b(d(x_{2n-1},x_{2n}))\left[\phi(d(y_{2n-1},y_{2n})+\phi(d(y_{2n},y_{2n+1})\right]\\
&\quad +c(d(x_{2n-1},x_{2n}))\phi(d(x_{2n-1},y_{2n+1})\\
&\leq a(d(x_{2n-1},x_{2n}))\phi(d(y_{2n-1},y_{2n}))\\
&\quad +b(d(x_{2n-1},x_{2n}))\left[\phi(d(y_{2n-1},y_{2n})+\phi(d(y_{2n},y_{2n+1})\right]\\
&\quad +c(d(x_{2n-1},x_{2n}))\phi(d(x_{2n},y_{2n})\\
&\leq a(d(x_{2n-1},x_{2n}))\phi(d(y_{2n-1},y_{2n}))\\
&\quad +b(d(x_{2n-1},x_{2n}))\left[\phi(d(y_{2n-1},y_{2n})+\phi(d(y_{2n},y_{2n+1})\right]\\
&\quad +c(d(x_{2n-1},x_{2n}))\phi(d(y_{2n},y_{2n+1}).
\end{aligned}$$

We implies that

$$\begin{aligned}
&\left[1-b(d(x_{2n-1},x_{2n}))-c(d(x_{2n-1},x_{2n}))\right]\phi\left(d\left(y_{2n},y_{2n+1}\right)\right)\\
&\qquad\leq a(d(x_{2n-1},x_{2n}))+b(d(x_{2n-1},x_{2n}))\left[\phi(d(y_{2n-1},y_{2n})\right].
\end{aligned}$$

Since $a(d(x_{2n-1},x_{2n}))+2b(d(x_{2n-1},x_{2n}))+c(d(x_{2n-1},x_{2n}))<1$, we have

$$\frac{a(d(x_{2n-1},x_{2n}))+b(d(x_{2n-1},x_{2n}))}{1-b(d(x_{2n-1},x_{2n}))-c(d(x_{2n-1},x_{2n}))}<1.$$

Therefore $\phi\left(d\left(y_{2n},y_{2n+1}\right)\right)<\phi\left(d(y_{2n-1},y_{2n}\right))$.

Since ϕ is increasing, we have $d\left(y_{2n},y_{2n+1}\right)<d(y_{2n-1},y_{2n})$, i.e $e_{2n}<e_{2n-1}$.

Hence $\{e_n\}$ is a strictly decreasing sequence.

Case 4 If $y_{2n} \in P_1$and $y_{2n+1} \in Q_1$then $y_{2n} \neq x_{2n}$ and $y_{2n+1} \neq x_{2n+1}$ such that $x_{2n} \in \partial K$ and $Tx_{2n} \in K$. Then $y_{2n+1} = x_{2n+1}$, a contradiction.

From **case 1-case 4,** we have $e_{2n} < e_{2n-1}$. Similarly $e_{2n+1} < e_{2n}$.

Hence $\{e_n\}$ strictly decreasing sequence.

Lemma 3.1.3 $\lim_{n\to\infty} e_n = 0$.

Proof. Assume that $\lim_{n\to\infty} e_n = e$ with $e > 0$. Since $\{e_n\}$ strictly decreasing sequence, we have $e_{2n} < e_{2n-1}$. It follows as in the proof of Lemma 3.1.2 that

$$\begin{aligned} \phi(e_{2n}) &= \phi\left[d\left(y_{2n}, y_{2n+1}\right)\right] \\ &\leq r\phi\left[d(y_{2n-1}, y_{2n})\right] \\ &= r\phi\left[d(e_{2n-1})\right], n = 0, 1, 2, ... \end{aligned} \tag{9}$$

where

$$r = \frac{a(e_{2n-1}) + b(e_{2n-1})}{1 - b(e_{2n-1})} < 1$$

By (8) and induction,

$$\begin{aligned} \phi(e_{2n}) &= \phi\left[d\left(y_{2n}, y_{2n+1}\right)\right] \\ &\leq r^n\phi\left[d(y_0, y_1)\right] \\ &= r^n\phi(e_0) \\ &< \phi(e_0) \end{aligned}$$

Moreover, $\phi(e_{2n+1}) \leq r\phi(e_n)$. Since ϕ is continuous, it follows by letting $n \to \infty$ $\phi(e) \leq r\phi(e) < \phi(e)$ which is contradict, and hence $e = 0$.

Lemma 3.1.4 .$\{y_n\}$ is a Cauchy sequence.

Proof. Clearly $e_n \neq 0$ for any n. By Lemma 3.1.3 , $\{e_n\}$ strictly decreasing to zero. We shall show that $\{y_n\}$ is a Cauchy sequence. Assume that $\{y_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and integer $m(k)$ and $n(k)$ such that $k \leq m(k) < n(k)$ and $b_k = d(y_{m(k)}, y_{n(k)}) \geq \varepsilon$ for $k \in \mathbb{N}$.

For each k we may assume that $n(k)$ is chosen that smallest number greater than $m(k)$ and $d(y_{m(k)}, y_{n(k)}) \geq \varepsilon$.

$$\begin{aligned} b_k &= d(y_{m(k)}, y_{n(k)}) \geq \varepsilon > 0 \\ b_k - d(y_{m(k)}, y_{n(k)-1}) &> 0 \end{aligned} \tag{10}$$

Note that

$$b_k - d(y_{m(k)}, y_{n(k)-1}) = d(y_{m(k)}, y_{n(k)}) - d(y_{m(k)}, y_{n(k)-1}) \tag{11}$$

$$\varepsilon \leq b_k \leq d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)}) \tag{12}$$

$$\leq \varepsilon + d(y_{n(k)-1}, y_{n(k)}) \tag{13}$$

By letting $k \to \infty$ we have $b_k \to \varepsilon$.

We can choose some p large enough such that

$$\phi^p(2\varepsilon) < \frac{\varepsilon}{3}.$$

Then choose some k large enough such that

$$b_k < 2\varepsilon \text{ and } \sup_{j\geq k} e_j < \frac{\varepsilon}{3p}$$

Now, we obtain that

$$\begin{aligned} \varepsilon &\leq d(y_{m(k)}, y_{n(k)}) \\ &\leq \sum_{j=m(k)}^{m(k)+p-1} d(y_j, y_{j+1}) + d(y_{m(k)+p}, y_{n(k)+p}) + \sum_{j=n(k)}^{n(k)+p-1} d(y_j, y_{j+1}) \\ &\leq p \sup_{j\geq m(k)} e_j + \phi^p(b_k) + p \sup_{j\geq n(k)} e_j \\ &< p\frac{\varepsilon}{3p} + \phi^p(2\varepsilon) + p\frac{\varepsilon}{3p} \\ &= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ , a contradiction.} \end{aligned} \tag{14}$$

Hence $\{y_n\}$ is a Cauchy sequence in a complete convex metric space.

Proof of theorem 3.1.1. Since X is complete and K is closed, there exists a point $z \in K \subseteq X$ such that $\lim_{n\to\infty} y_n = z$ and so $\lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} y_{2n+1} = z = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Sx_{2n-1}$. We shall show that $Tz = z = Sz$.

Using condition (1) we obtain

$$\begin{aligned} \phi\left[d(y_{2n+2}, Tz)\right] &= \phi\left[d(Sx_{2n+1}, Tz)\right] \\ &\leq a(d(x_{2n+1}, z))\phi(d(x_{2n+1}, z)) \\ &\quad + b(d(x_{2n+1}, z))\left[\phi(d(x_{2n+1}, Sx_{2n+1}) + \phi(d(z, Tz)\right] \\ &\quad + c(d(x_{2n+1}, z))\min\left\{\phi(d(x_{2n+1}, Tz), \phi(d(z, Sx_{2n+1})\right\} \end{aligned} \tag{15}$$

Letting $n \to \infty$ and use $b < \frac{1}{2}$ we have

$$\phi\left[d(z, Tz)\right] < \frac{1}{2}\phi\left[d(z, Tz)\right] \tag{16}$$

Then $\phi\left[d(z, Tz)\right] = 0$ and we have $d(z, Tz) = 0$. That is $Tz = z$.

$$\begin{aligned} \phi\left[d(Sz, y_{2n+1})\right] &= \phi\left[d(Sz, Tx_{2n})\right] \\ &\leq a(d(z, x_{2n}))\phi(d(z, x_{2n})) \\ &\quad + b(d(z, x_{2n}))\left[\phi(d(z, Sz) + \phi(d(x_{2n}, Tx_{2n})\right] \\ &\quad + c(d(z, x_{2n}))\min\left\{\phi(d(z, Tx_{2n}), \phi(d(x_{2n}, Sz)\right\} \end{aligned} \tag{17}$$

Letting $n \to \infty$ and use $b < \frac{1}{2}$ we have

$$\phi\left[d(Sz, z)\right] < \frac{1}{2}\phi\left[d(Sz, z)\right] \tag{18}$$

Then $\phi\left[d(Sz, z)\right] = 0$ and we have $d(Sz, z) = 0$. That is $Sz = z$.

Therefore S and T have a common fixed point $z \in K \subseteq X$.

For uniqueness, suppose that w and z (with $w \neq z$) are two fixed points of S and T.

$$\begin{aligned}
\phi\left[d(w,z)\right] &= \phi\left[d(Sw,Tz)\right] \\
&\leq a(d(w,z))\phi(d(w,z)) + b(d(w,z)) \\
&\quad [\phi(d(w,Sw) + \phi(d(z,Tz)] \\
&\quad + c(d(w,z)) \min\{\phi(d(w,Tz), \phi(d(z,Sw)\} \\
&\leq [a(d(w,z)) + c(d(w,z))]\, \phi(d(z,w) \\
&< \phi(d(z,w), \text{ a contradiction.}
\end{aligned}$$

This complete the proof of theorem. □

Corollary 3.1.5 (cf[16]) Let $T : K \to X$ be a non-self mapping satisfying $T(\partial K) \subseteq K$ of a complete convex metric space (X,d) and $\phi \in \Phi$ such that for every $x, y \in \partial K$,

$$\begin{aligned}
\phi\left[d(Tx,Ty)\right] &\leq a\phi(d(x,y)) + b\phi(d(x,Tx) \\
&\quad + c\phi(d(y,Ty)
\end{aligned} \tag{19}$$

where a, b and c are three nonnegative constants satisfying $a + b + c < 1$.

Then T has a unique fixed point.

3.2 Fixed point theorems in Banach spaces

We now determine the behavior of nonexpansive mappings defined on $\overline{B_r} = \{x \in X : \|x\| \leq r\}$ (i.e. the closed ball of radius r center at 0) and prove an existence fixed point theorems of Leray-Schauder condition for nonexpansive mappings. It is worth resulting that a more general case will be presented in this section.

Theorem 3.2.1 Let X be a uniformly convex Banach space and $\overline{B_r} = \{x \in X : \|x\| \leq r\}$ with $r > 0$. Suppose $F : \overline{B_r} \to X$ is nonexpansive such that $x = \lambda F(x)$ for all $x \in \partial\overline{B_r}$ and for all $\lambda \in (0,1)$. Then F has a fixed point in $\overline{B_r}$

Proof. Suppose that $F : \overline{B_r} \to X$ is nonexpansive and define a mapping $S : X \to \overline{B_r}$ by

$$S(x) = \begin{cases} x, & \|x\| \le r, \\ r\frac{x}{\|x\|}, & \|x\| > r \end{cases} \tag{20}$$

We now show that $S : X \to \overline{B_r}$ is nonexpansive.

Let $x, y \in X$ and if $x, y \in \overline{B_r}$, then

$$\|Sx - Sy\| = \|x - y\|. \tag{21}$$

If $x \in \overline{B_r}$ but $y \notin \overline{B_r}$, then

$$\begin{aligned} \|Sx - Sy\| &= \left\| x - r\frac{y}{\|y\|} \right\| \\ &\le \|x - y\|. \end{aligned} \tag{22}$$

Similarly if $y \in \overline{B_r}$ and $x \notin \overline{B_r}$, then we have S is nonexpansive.

Therefore $S \circ F : \overline{B_r} \to \overline{B_r}$ is nonexpansive mapping. By Corollary 2.2.5, there exists $z \in \overline{B_r}$ such that

$$S(F(z)) = z. \tag{23}$$

If $F(z) \in \overline{B_r}$, then

$$z = S(F(z)) = F(z). \tag{24}$$

Hence F has a fixed point.

If $F(z) \notin \overline{B_r}$, then

$$\begin{aligned} z &= S(F(z)) = r\frac{F(z)}{\|F(z)\|} \\ &= \beta F(z), \text{ where } \beta = \frac{r}{\|F(z)\|} < 1. \end{aligned} \tag{25}$$

This is contradict to our assumption. □

Theorem 3.2.1 has the following immediate corollary.

Corollary 3.2.2 (cf[1]) Let C be a nonempty, closed, bounded, convex set in a (real)

Hilbert space H. Then each nonexpansive map $T : C \to C$ has at least one fixed point.

Proof. Since every Hilbert space is uniformly convex Banach space, we obtain corollary as require. □

It is natural to ask whether we can extend theorems to non expansive mappings as theorem 3.2.1 In this section we had proved fixed point theory in Banach space for non-self maps which satisfies boundary conditions as the follows:

Theorem 3.2.3 Let K be a nonempty closed bounded convex subset of a uniformly convex in every direction (UCED) Banach space X and let $T : K \to X$ be nonexpansive satisfying one of the following holds:

(i) $T(\partial K) \subseteq K$,

(ii) T is weakly inward condition,

(iii) $0 \in IntK$ and $Tx \neq mx$ for all $x \in \partial K$ and $m > 1$.

Then T has a fixed point in K.

Proof. (i) Suppose $T : K \to X$ is nonexpansive satisfying $T(\partial K) \subseteq K$, and let $x_0 \in K$. For each $n \geq 1$ define $T_n : K \to X$ by

$$T_n(x) = \frac{1}{n}x_0 + (1 - \frac{1}{n})Tx \text{ for } x \in K. \tag{26}$$

For each $x, y \in K$, we note that

$$\begin{aligned} \|T_nx - T_ny\| &= \left\| \left[\frac{1}{n}x_0 + (1 - \frac{1}{n})Tx\right] - \left[\frac{1}{n}x_0 + (1 - \frac{1}{n})Ty\right] \right\| \\ &= \left\| (1 - \frac{1}{n})Tx - (1 - \frac{1}{n})Ty \right\| \\ &= \left| (1 - \frac{1}{n}) \right| \|Tx - Ty\| \qquad (27) \\ &\leq \left| (1 - \frac{1}{n}) \right| \|x - y\| \qquad (28) \end{aligned}$$

Hence T_n is a contraction. Further $T_n(\partial K) \subseteq K$ since $T(\partial K) \subseteq K$, for $x \in \partial K$ and $x_0 \in K$.

Therefore, by assumption, T_n has a unique fixed point $x_n \in K$.

Then there exists a bounded sequence $\{x_n\} \subseteq K$ such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, by Theorem 2.5.8 [14]. The sequence $\{x_n\}$ is said to be regular (relative to K) if $r_K(\{x_n\}) = r_K(\{x_{n_k}\})$ for all subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{x_n\}$ is said to be asymptotically uniform if $A_K(\{x_n\}) = A_K(\{x_{n_k}\})$ for all subsequence $\{x_{n_k}\}$ of $\{x_n\}$ where $r_K(\{x_n\}) = \inf\left\{\limsup_{n\to\infty} \|y - x_n\| : y \in K\right\}$ and

$$A_K(\{x_n\}) = \left\{y \in K : \limsup_{n\to\infty} \|y - x_n\| = r_K(\{x_n\})\right\}.$$

Since $\{x_n\} \subseteq K$ is bounded, then $\{x_n\}$ has a subsequence which is regular with respect to K. By passing to a subsequence we may assume that $\{x_n\}$ is regular.

Then $A_K(\{x_n\})$ consists of exactly one point in such a space is UCED and x_n is asymptotically uniform with respect to K, i.e. $A_K(\{x_n\}) = A_K(\{x_{n_k}\})$ is singleton.

Take any $v \in A = A_K(\{x_n\})$ and $r = r_K(\{x_n\})$. By Banach contraction principle, T_n has a unique fixed point $x_n \in K$, that is, we have

$$x_n = \frac{1}{n}x_0 + (1 - \frac{1}{n})Tx_n \tag{29}$$

Let $\{x_{n_\lambda}\}$ be an subsequence of $\{x_n\}$. Then we can solve the asymptotic center A as above just by replacing the sequence $\{x_n\}$ by the subsequence $\{x_{n_\lambda}\}$.

We now show that $x_{n_\lambda} \to v$. Observe that

$$\begin{aligned} \|x_{n_\lambda} - x_{n_{\lambda+1}}\| &= \left\| \left[\frac{1}{n_\lambda}x_0 + (1 - \frac{1}{n_\lambda})Tx_{n_\lambda}\right] - \left[\frac{1}{n_{\lambda+1}}x_0 + (1 - \frac{1}{n_{\lambda+1}})Tx_{n_{\lambda+1}}\right] \right\| \\ &= \left\| \left[\frac{1}{n_\lambda} - \frac{1}{n_{\lambda+1}}\right] x_0 + (1 - \frac{1}{n_\lambda})Tx_{n_\lambda} - (1 - \frac{1}{n_{\lambda+1}})Tx_{n_{\lambda+1}} \right\|. \end{aligned} \tag{30}$$

Given $x_{n_{\lambda+1}} = Tx_{n_\lambda}$ we also have

$$\begin{aligned} \|x_{n_\lambda} - x_{n_{\lambda+1}}\| &= \left\| \left(\frac{1}{n_\lambda} - \frac{1}{n_{\lambda+1}}\right) x_0 + (1 - \frac{1}{n_\lambda})Tx_{n_\lambda} - (1 - \frac{1}{n_{\lambda+1}})T(Tx_{n_\lambda}) \right\| \\ &\leq \left\| (\frac{1}{n_\lambda})x_0 \right\| + \|Tx_{n_\lambda} - T(Tx_{n_\lambda})\| \\ &\leq \left\| (\frac{1}{n_\lambda})x_0 \right\| + \|x_{n_\lambda} - Tx_{n_\lambda}\|. \end{aligned} \tag{31}$$

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By letting $n_\lambda \longrightarrow \infty$ we have $\{x_{n_\lambda}\}$ is Cauchy sequence (since K is bounded). Therefore x_{n_λ} converges to a point, say $x \in K$. Claim that $x = v$.

For $n_\lambda \in \mathbb{N}$, we can find $y_{n_\lambda} \in K$ such that $y_{n_\lambda} = Tx_{n_\lambda}$,

and $\lim\limits_{n_\lambda \to \infty} \|y_{n_\lambda} - x_{n_\lambda}\| = 0$. Note that

$$\begin{aligned} \|x_{n_\lambda} - x\| &= \|x_{n_\lambda} - Tx_{n_\lambda} + Tx_{n_\lambda} - x\| \\ &\leq \|x_{n_\lambda} - Tx_{n_\lambda}\| + \|Tx_{n_\lambda} - x\|. \end{aligned} \tag{32}$$

By letting $n_\lambda \longrightarrow \infty$, $\lim\limits_{n_\lambda \to \infty} \|x_{n_\lambda} - x\| \leq \lim\limits_{n_\lambda \to \infty} \|Tx_{n_\lambda} - x\|$ or,

$\limsup\limits_{n_\lambda \to \infty} \|x_{n_\lambda} - x\| \leq \limsup\limits_{n_\lambda \to \infty} \|Tx_{n_\lambda} - x\|$.

Then $\limsup\limits_{n_\lambda \to \infty} \|x_{n_\lambda} - x\|$ is a lower bounded of $\left\{ \limsup\limits_{n \to \infty} \|y - x_n\| : y \in K \right\}$.

Therefore

$$\limsup_{n_\lambda \to \infty} \|x_{n_\lambda} - x\| \leq r \tag{33}$$

Note that

$$\limsup_{n_\lambda \to \infty} \|x_{n_\lambda} - x\| \geq r. \tag{34}$$

From (33) and (34) we get $\limsup\limits_{n_\lambda \to \infty} \|x_{n_\lambda} - x\| = r$, by asymptotically uniform with uniqueness of $\{x_{n_\lambda}\}$ we have $x = v$. It follows that $Tx_{n_\lambda} \longrightarrow Tv$ and

$$\begin{aligned} \|v - Tv\| &= \|v - x_{n_\lambda} + x_{n_\lambda} - Tx_{n_\lambda} + Tx_{n_\lambda} - Tv\| \\ &\leq \|v - x_{n_\lambda}\| + \|x_{n_\lambda} - Tx_{n_\lambda}\| + \|Tx_{n_\lambda} - Tv\|. \end{aligned} \tag{35}$$

Letting $n_\lambda \longrightarrow \infty$, we have $\lim\limits_{n_\lambda \to \infty} \|v - Tv\| = \|v - Tv\| = 0$. Therefore $Tv = v$ and hence T has a fixed point.

(ii) Suppose T is weakly inward. Fix $x_0 \in K$ define for each $n \geq 1$ the mapping $T_n : K \to X$ by

$$T_n(x) = \frac{1}{n}x_0 + (1 - \frac{1}{n})Tx$$

It is easy to see that T is contraction. Moreover T_n is weakly inward, since $Tx \in \overline{I_K(x)}$ and $x_0 = x_0 + \beta(x_0 - x_0)$ and $\beta \geq 0$. By Banach contraction principle T_n has a unique fixed point $x_n \in K$. Then there exists a bounded sequence $\{x_n\} \subseteq K$ such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.

Similarly to (i), T has a fixed point.

(iii) Suppose $0 \in IntK$ and $Tx \neq mx$ for all $x \in \partial K$ and $m > 1$. Define $T_n : K \to X$ as above. Fix $0 \in IntK \subseteq K$. Assume that $T_n(x) = tx$ for some $x \in \partial K$ and $t > 1$.

Then

$$\begin{aligned} (1-\frac{1}{n})Tx &= tx \\ Tx &= \left(\frac{t}{1-\frac{1}{n}}\right)x, \ \frac{t}{1-\frac{1}{n}} > 1 \end{aligned} \tag{36}$$

which is contradict. Therefore $T_n(x) \neq tx$ for all $x \in \partial K$ and $t > 1$. By Banach contraction principle, T_n has a unique fixed point $x_n \in K$. Similarly to (i), T has a fixed point. □

Corollary 3.2.4 (cf[14]) Let X be a uniformly convex Banach space, K a closed bounded convex subset, and $T : K \to X$ a nonexpansive mapping satisfying one of (i)-(iii) of Theorem 3.2.3 holds. Then T has a fixed point.

Theorem 3.2.5 Let X be a reflexive Banach space and let C be a nonempty bounded closed convex subset of X which has normal structure. Let $T : C \to X$ be nonexpansive mapping satisfying $0 \in IntC$ and $Tx \neq mx$ for all $x \in \partial C$ and $m > 1$ or Leray-Schauder's condition . Then T has a fixed point.

Proof. Let $u \in C$. For each $n \geq 1$, define $T_n : C \to X$ by

$$T_n x = (1-\frac{1}{n})u + \frac{1}{n}Tx \tag{37}$$

Then for each $x, y \in C$,

$$\begin{aligned}
\|T_n x - T_n y\| &= \left\| \left[(1-\frac{1}{n})u + \frac{1}{n}Tx\right] - \left[(1-\frac{1}{n})u + \frac{1}{n}Ty\right] \right\| \\
&= \left\| \frac{1}{n}Tx - \frac{1}{n}Ty \right\| \\
&= \left| \frac{1}{n} \right| \|Tx - Ty\| \\
&\leq \left| \frac{1}{n} \right| \|x - y\| \qquad (38)
\end{aligned}$$

Therefore T_n is a contraction. Moreover T_n satisfies Leray-Schauder's condition, Assume $0 \in IntC$ and $Tx = tx$ for some $x \in \partial C$ and $t > 1$. We have

$$\left(\frac{1}{n}\right) Tx = tx. \qquad (39)$$

Then

$$Tx = \left(\frac{t}{\frac{1}{n}}\right) = tn > 1,$$

which is contradicted. Therefore $T_n(x) \neq tx$ for all $x \in \partial C$ and $t > 1$. By Banach contraction mapping principle, T_n has a unique fixed point x_n such that

$$x_n = (1 - \frac{1}{n})u + \frac{1}{n}Tx.$$

Define $g : C \to \mathbb{R}$ by $g(z) = \mu_n \|x_n - z\| \; \forall z \in C$ where μ_n is a Banach limit for all $n \in \mathbb{N}$. We can choose $\{x_n\} \subseteq C$ such that $g(x_n) \to d = \inf_{z \in C} g(z)$. Let $z_0 \in C$ and $\varepsilon > 0$ we can choose $\delta = \varepsilon$ for each $z \in C$.

if $\|z - z_0\| < \delta$ then

$$\begin{aligned}
\|g(z) - g(z_0)\| &= | \, \mu_n \|x_n - z\| - \mu_n \|x_n - z_0\| \, | \\
&= \mu_n (| \|x_n - z\| - \|x_n - z_0\| |) \qquad (40)
\end{aligned}$$

Note that

$$\begin{aligned}
| \|z\| - \|z_0\| | &\leq \|z - z_0\| \text{ and} \\
-(\|z - z_0\|) &\leq \|z\| - \|z_0\| \leq \|z - z_0\|.
\end{aligned}$$

Therefore

$$\begin{aligned} \|x_n - z\| - \|x_n - z_0\| &\leq \|x_n - z - (x_n - z_0)\| \\ &= \|z - z_0\|. \end{aligned}$$

From (40),

$$\|g(z) - g(z_0)\| \leq \mu_n \|z - z_0\|. \tag{41}$$

By existence of Banach limit $\mu(a_n) = \mu(a) = c$. From (41), we have

$$\|g(z) - g(z_0)\| \leq c_n \|z - z_0\| < c\delta = c\varepsilon = \varepsilon'.$$

Hence g is continuous. Furthermore, g is convex, let $z_1, z_2 \in C$ and $\lambda \in [0,1]$. For any $i \in \mathbb{N}$, we have

$$\begin{aligned} \mu_i \|x_i - [\lambda z_1 + (1-\lambda)z_2]\| &= \mu_i \|\lambda x_i + (1-\lambda)x_i - \lambda z_1 - (1-\lambda)z_2\| \\ &= \mu_i \|\lambda x_i - \lambda z_1 + (1-\lambda)x_i - (1-\lambda)z_2\| \\ &\leq \mu_i \{\|\lambda x_i - \lambda z_1\| + \|(1-\lambda)x_i - (1-\lambda)z_2\|\} \\ &= \mu_i \|\lambda x_i - \lambda z_1\| + \mu_i \|(1-\lambda)x_i - (1-\lambda)z_2\| \\ &= \mu_i \lambda \|x_i - z_1\| + \mu_i(1-\lambda)\|x_i - z_2\|. \end{aligned}$$

This implies

$$g[\lambda z_1 + (1-\lambda)z_2] \leq g(\lambda z_1) + g((1-\lambda)z_2). \tag{42}$$

Hence g is convex.

Define set

$$M = \left\{ v \in C : g(v) = \inf_{z \in C} g(z) \right\}.$$

Since X is reflexive, it follows by [3] that $M \neq \emptyset$. Moreover M we can verifies that M is bounded, closed and convex.

To show M is bounded. Let $M' = \{v' \in C : \mu_n \|x_n - v'\| \leq r + 1\}$ where $r = \inf_{y \in C} \mu_n \|x_n - y\|$. We see that $M \subseteq M'$, we need show that M' is bounded.

Let $z \in M'$ and $r = \inf_{y \in C} \mu_n \|x_n - y\|$, we have $\mu_n \|x_n - z\| \leq r + 1$ and there exist $x_m \in \{x_n\}$ such that

$$\begin{aligned} \mu_n \|x_m - z\| &= \mu_n \{\|x_m - x_n + x_n - z\|\} \\ &\leq \mu_n \|x_m - x_n\| + \mu_n \|x_n - z\| \\ &= 1 + r + 1 = r + 2. \end{aligned}$$

Since $\{x_m\} \subseteq \{x_n\}$ is bounded, we have M' is bounded.

We shall show M is closed. Let $\{x_n\} \subseteq M$ and $\lim_{n\to\infty} x_n = x$. We may assume $x \notin M$. Thus $x \in C \backslash M$, there is an open ball $B(x;r)$ for some $r > 0$ such that $B(x;r) \subseteq C \backslash M$. Since $\lim_{n\to\infty} x_n = x$, we have there is $N \in \mathbb{N}$ such that $x_n \in B(x;r)$ $\forall n \geq N$. It implies that $x_n \notin M$ for sufficiently large n, which is a contradicted. Therefore $x \in M$ and complete the proof that M is closed. Moreover M is convex. In particular, we need to show that $Tx \in M$ if $x \in M$

$$\begin{aligned} \mu_n \|x_n - Tx\| &\leq \mu_n \{\|x_n - Tx_n\| + \|Tx_n - Tx\|\} \\ &= \mu_n \|x_n - Tx_n\| + \mu_n \|Tx_n - Tx\| \end{aligned} \tag{43}$$

Note that

$$\begin{aligned} \|x_n - Tx_n\| &= \left\|(1 - \frac{1}{n})u + \frac{1}{n}Tx - Tx_n\right\| \\ &\leq (1 - \frac{1}{n}) \|u - Tx_n\|. \end{aligned} \tag{44}$$

From (43) and (44) we have

$$\mu_n \|x_n - Tx\| \leq \mu_n(1 - \frac{1}{n}) \|u - Tx_n\| + \mu_n \|Tx_n - Tx\| \tag{45}$$

By the existence of Banach limit,

$$\begin{aligned} \mu_n \|x_n - Tx\| &\leq \mu_n \|Tx_n - Tx\| \\ &\leq \mu_n \|x_n - x\|. \end{aligned}$$

Therefore $g(Tx) = \inf_{z \in C} g(z)$, $Tx \in M$ which implies T is invariant under M.
Hence T has a fixed point in M by Theorem 2.2.6. □

Corollary 3.2.6 Let X be a reflexive Banach space and let C be a nonempty bounded closed convex subset of X which has normal structure. Let $T : C \to X$ be nonexpansive mapping satisfying weakly inward condition . Then T has a fixed point.

Corollary 3.2.7 Let X be a reflexive Banach space and let C be a nonempty bounded closed convex subset of X which has normal structure. Let $T : C \to X$ be nonexpansive mapping satisfying inward condition . Then T has a fixed point.

Corollary 3.2.8 Let X be a reflexive Banach space and let C be a nonempty bounded closed convex subset of X which has normal structure. Let $T : C \to X$ be a nonexpansive mapping satisfying Rothe's condition . Then T has a fixed point.

Corollary 3.2.9(cf[12]) Let X be a reflexive Banach space. Suppose K is a bounded closed convex subset of X which has normal structure. Then any nonexpansive mappings $T : K \to K$ has a fixed point.

Proof. If $C = X$, then we have this corollary as required. □

3.3 Remarks for h_λ-contractive and h-nonexpansive

Definition 3.3.1 A mapping $T : K \to K$ where K is a subset of a Banach space X is said to be h_λ-*contractive* if for $h > 0$ and $\lambda \in (0, 1)$,

$$\|Tx - Ty\| \leq \lambda \max\{h,\ \|x - y\|\},\ x, y \in K.$$

Proposition 3.3.2 Let K be an open subset of a Banach space X and let $T : K \to X$

be a h_λ-contractive for $h > 0$ and $\lambda \in (0,1)$. Then $(I-T)(K)$ is an open subset of X.

Proof. Let $h > 0$ and $x_0 \in K$. Set $f = I - T$, there exists $y \in K$ and $\rho > 0$ such that $\|y - x_0\| \leq \rho$. Thus $B(\, x_0; \rho) \subseteq K$. We choose $r = (1-\lambda)\rho$ where $\lambda \in (0,1)$. Now fix $z \in B(\, f(x_0); r)$. We shall show that there is $w \in B(\, x_0; \rho)$ such that $f(w) = w - T(w) = z$, implies that $B(\, f(x_0); r) \subseteq f(K)$. Define the mappings $T_z : K \to X$ by setting

$$T_z(x) = T(x) + z, \quad x \in K.$$

We consider,

$$\begin{aligned}
\|T_z(x) - x_0\| &= \|T(x) + z - x_0\| && (46)\\
&= \|T(x) + T(x_0) - T(x_0) + z - x_0\| \\
&\leq \|T(x) - T(x_0)\| + \|z - (x_0 - T(x_0))\| \\
&\leq \lambda \max\{h,\ \|x - x_0\|\} + \|z - f(x_0)\| && (47)
\end{aligned}$$

Case 1 : If $\|x - x_0\| \geq h$, then we have from (47)

$$\begin{aligned}
\|T_z(x) - x_0\| &\leq \lambda\|x - x_0\| + \|z - f(x_0)\| \\
&\leq \lambda\rho + r \\
&= \rho.
\end{aligned}$$

Case2 : If $\|x - x_0\| < h$, then

$$\begin{aligned}
\|T_z(x) - x_0\| &\leq \lambda h + \|z - f(x_0)\| \\
&\leq \lambda h + r.
\end{aligned}$$

Case2.1 If $h \leq \rho$, then

$$\begin{aligned}
\|T_z(x) - x_0\| &\leq \lambda h + r \\
&\leq \lambda\rho + r \\
&= \lambda\rho + (1-\lambda)\rho \\
&= \rho.
\end{aligned}$$

Case 2.2 If $h > \rho$, then

$$\begin{aligned} \|T_z(x) - x_0\| &\leq \lambda h + r \\ &= \lambda h + (1-\lambda)\rho \\ &\leq \lambda h + (1-\lambda)h \\ &= h. \end{aligned}$$

By using the results in case 1 and case 2.1, it follows that $T_z : B(x_0;\rho) \to B(x_0;\rho)$, Banach contraction mapping principle, T_z has a unique fixed point, says $w \in B(x_0;\rho)$, $T_z(w) = w$.

Therefore $w = T_z(w) = T(w) + z$, which $f(w) = w - T(w) = z$ and the proof is complete. Another (case 2.2) we have $T_z : B(x_0;h) \to B(x_0;h)$, the proof is also complete. □

Proposition 3.3.3 Let K be a nonempty closed bounded convex subset of a Banach space X with $intK \neq \emptyset$ and let $T : K \to X$ be h-nonexpansive which satisfies Rothe's condition. Then for $\lambda \in (0,\infty)$ sufficiently small, the mapping $T_\lambda : K \to X$ defined by

$$T_\lambda(x) = (1-\lambda)x + \lambda Tx, \quad x \in K,$$

then there exists $z \in K$ such that $\|z - Tz\| \leq h$ for $h > 0$

(i.e., $\inf\{\|x - Tx\| : x \in K\} \leq h$).

Proof. Let a function $p : X \to (0,\infty]$ satisfies the number $p(x) := \inf\{k \in (0,\infty] : k^{-1}x \in K\}$

Let $x \in K$ and there is $r > 0$ such that $B(x,r) \subseteq K$. Thus $x \in intK$.

We may assume that $0 \in intK$. Since K is bounded, there is $t > 0$ such that $K \subseteq B(0,tr)$.

Consider $r \leq \|x\| \leq tr$. Then $p(x)^{-1}r \leq p(x)^{-1}\|x\| \leq p(x)^{-1}tr$.

Put $R = p(x)^{-1}r$.

$$R \leq p(x)^{-1}\|x\| \leq tR \tag{48}$$

and $p(x)^{-1}x \in \partial K$, $\forall x \neq 0$.

Thus there exist

$$c_1 = \frac{1}{tr} > 0 \text{ and } c_2 = \frac{1}{R} \text{ such that for all } x \in X$$

$$c_1 \|x\| \leq p(x) \leq c_2 \|x\| \qquad (49)$$

Now let $h > 0$ and $x \in K$ and suppose that $y = p(x)^{-1}x$.

Thus $y \in \partial K$ and $y = p(x)^{-1}x + (1 - p(x)^{-1}x)$. Therefore

$$\begin{aligned} p(y-x) &= p(p(x)^{-1}x - x) \\ &= p((p(x)^{-1} - 1)x) \\ &= (p(x)^{-1} - 1)p(x) \\ &= \left(\frac{1}{p(x)} - 1\right)p(x) \\ &= \left(\frac{1-p(x)}{p(x)}\right)p(x) = 1 - p(x). \end{aligned}$$

Now let $\lambda \in (0,1)$ and $T_\lambda : K \to K$ defined by $T_\lambda(x) = (1-\lambda)x + \lambda Tx$, $\forall x \in K$.

Hence

$$\begin{aligned} p(T_\lambda(x)) &= p((1-\lambda)x + \lambda Tx) \\ &\leq (1-\lambda)p(x) + \lambda p(Tx). \end{aligned}$$

On the other hand we consider the unit ball of K. Since $y \in \partial K$ then $Ty \in K$, it follows that

$$p(Ty) \leq 1. \qquad (50)$$

From(49) and (50), we have

$$\begin{aligned} p(Tx) &= p(Ty + Tx - Ty) \\ &\leq p(Ty) + p(Tx - Ty) \\ &\leq 1 + c_2 \|Tx - Ty)\|. \end{aligned}$$

Case1 If $\|x - y\| \leq h$, then

$$\begin{aligned} p(Tx) &\leq 1 + c_2 h \\ &\leq 1 + k_1 \text{ where } k_1 = c_2 h. \end{aligned}$$

From this, we have

$$p(T_\lambda(x)) \le (1-\lambda)p(x) + \lambda(1+k_1).$$

We can chose $\lambda \in (0,1)$ such that $\lambda \le 1-k_1$. Then

$$\begin{aligned} p(T_\lambda(x)) &\le (1-\lambda)p(x) + (1-k_1)(1+k_1) \\ &= (1-\lambda)p(x) + (1-(k_1)^2) \\ &\le (1-\lambda)p(x) + 1. \end{aligned}$$

By letting $\lambda \longrightarrow 1$, we have $p(T_\lambda(x)) \le 1$.

Case 2 If $\|x-y\| > h$, then

$$\begin{aligned} p(Tx) &\le 1 + c_2\|x-y\| \\ &\le 1 + c_2c_1^{-1}p(y-x) \\ &= 1 + k_2p(y-x) \quad \text{where } k_2 = c_2c_1^{-1} \\ &\le 1 + k_2(1-p(x)) \quad (\because\ p(y) \le 1). \end{aligned}$$

Then

$$\begin{aligned} p(T_\lambda(x)) &\le (1-\lambda)p(x) + \lambda(1+k_2(1-p(x)) \\ &= p(x) - \lambda p(x) + \lambda + \lambda k_2 - \lambda k_2 p(x) \\ &= (1-\lambda(1+k_2))\,p(x) + \lambda(1+k_2). \end{aligned}$$

We observe that if $p(x) \le 1$(i.e. if $x \in K$) and if λ is chosen so small that $(1-\lambda(1+k_2))\,p(x)$ $\le 1-\lambda(1+k_2)$ then $p(T_\lambda(x)) \le 1$. From case1 and case2 , we conclude that $T_\lambda : K \to K$ has a unique fixed point, by Banach contraction, which is a completion of the proof. □

Corollary 3.3.4 Let K be a nonempty closed bounded convex subset of a Banach space X with $intK \neq \emptyset$ and let $T : K \to X$ be nonexpansive which satisfies Rothe's condition. Then there exists an approximate fixed point sequence $\{x_n\}$ in K.