ภาคผนวก

ภาคผนวก ก ผลงานตีพิมพ์ที่เกี่ยวข้องกับวิทยานิพนธ์

1. Prompak, K., Kaewpoonsuk, A., Maneechukate, T., Maneejiraprakarn, N., Pengpad, S. and Wardkein P. (2012). A new oscillation frequency discovery of the driven spring-mass system predicted by the multi-time differential equation. European Journal of Scientific Research, 92(3), 397-410.
2. Prompak, K., Kaewpoonsuk, A., Maneechukate, T. and Wardkein P. (2012). An oscillation discovery of the forced vibrating system predicted by the multi-time differential equation. Scientific Research and Essays, 7(39), 3292-3301.

European Journal of Scientific Research
ISSN 1450-216X Vol. 92 No 3 December, 2012, pp.397-410

http://www.europeanjournalofscientificresearch.com

A New Oscillation Frequency Discovery of the Driven Spring-Mass System Predicted by the Multi-Time differential Equation

Kriangsak Prompak
Department of Physics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand
E-mail: mum_c51@hotmail.com

Anucha Kaewpoonsuk
Corresponding Author, Department of Physics, Faculty of Science
Naresuan University, Phitsanulok 65000, Thailand
E-mail: anuchak@nu.ac.th

Thongchai Maneechukate
Department of Physics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand

Narongrit Maneejiraprakarn
Department of Physics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand

Sangwan Pengpad
Department of Physics, Faculty of Science, Naresuan University
Phitsanulok 65000, Thailand

Paramote Wardkein
Department of Telecommunication Engineering
Faculty of Engineering, King
Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand

Abstract

In this article an experiment of the forced oscillator was set up. Here a vibrating mass on a spring is driven vertically by the loudspeaker that suspends it. From this experiment the spectrum of output signal of an infrared distance measuring sensor (GP2Y0A02YK) which is used to transform the oscillatory motion of the mass into an electrical signal, exhibits four sharp frequency peaks, and it is also found that two of these frequency peaks cannot be described by the solution of ordinary differential equation from conventional ordinary differential equation text books. However, it can be solved by multi-time variable technique, a mathematical tool. The latter solution consists of the sum of four terms: natural response, forced response and the two new terms being the result of multiplying between natural and forced responses. This analytical solution reveals the frequency components and their behaviors more precisely and corresponds well to the spectrum of the experimental result.

Keywords: Oscillation, Mass-Spring, Multi-Time, Differential Equation

1. Introduction

The dynamics of a spring-mass system is one of the most popular topics analyzed in textbooks and undergraduate physics courses (Keller et al, 1993; Zill, 2005). The spring-mass system is the common example of a mechanical oscillator. When perturbed, it begins to oscillate with a natural frequency. Yet, this free oscillation decays exponentially with time due to retarding forces such as a resisting force due to the surrounding medium, i.e., these forces cause the mechanical energy of the oscillator to decrease. Any system that behaves in this way is known as a damped harmonic oscillator.

The oscillation of the damped harmonic oscillator (spring-mass system) acted on by an external driving force is called driven oscillation or forced oscillation. The oscillations occur mainly in electric circuits and in machinery. Using the conventional method (Taylor, 2005; Arya, 1997; James et al, 1989) for finding the solution of a linear differential equation describing the oscillation of such system, the solution is given by the sum of two parts. The natural response, the solution for a damped harmonic oscillator discussed above, delays out eventually. Another one is the forced response, the solution due to the external driving force, persists after the natural response has died away. The natural response depends on the initial conditions at time $t=0$ and on the initial values of the forced response also at time $t=0$ (Dimarogonas, 1996).

Recently, the analysis of a second-order oscillator based on a multi-time variable technique was proposed (Maneechukate, 2008). The result demonstrates that the amplitude of the natural response $x_n(t)$ of the system depends on initial value x_0 and the forced response $x_f(\tau)$ at any arbitrary time τ according to the following equation

$$x(t,\tau)=\underbrace{\left[x_0-x_f(\tau)\right]\cos\omega t}_{x_n(t)}+x_f(\tau),$$

where $x(t,\tau)$ is the complete solution of the separated time scales.

The solution above is somehow different from one achieved by the conventional method:

$$x(t)=\underbrace{\left[x_0-x_f(0)\right]\cos\omega t}_{x_n(t)}+x_f(t),$$

It is remarked that amplitude of the natural response $x_n(t)$ depends on the initial condition x_0 and the forced response $x_f(t)$ at the initial time , i.e., $t=0$.

In the past, the various concepts of multi-time have been introduced, e.g., multitime wave functions were first considered by Dirac in 1932, the term Multitime Partial Differential Equation was proposed by Roychowdhury (2001); in the later concept a two time $t=(t_1,t_2)$ was used, then, a single-time wave front $y(t)$ is replaced by a new periodic function of two variables, $\hat{y}(t_1,t_2)$, motivated by the wide separated time scales.

In this paper, we apply the multi-time variable technique to solve a mathematical model in the form of second order differential equation describing the damped harmonic oscillation of the spring-mass system acted on by the sinusoidal external force. The obtained analytical result consists of the sum of four components: natural response, forced response and the new two terms coming from the product of natural and forced responses. These two new terms do not appear when we use conventional method.

To verify that such analytical result is valid, an experiment of the forced-oscillating system is set up. Here an oscillating mass on a spring acted by an sinusoidal external force is used as an example to illustrate behavior of such solution. A large loudspeaker is used to generate the sinusoidal external force to the spring-mass system. The oscillatory motion of the mass on the spring is transformed as an electrical signal by the popular GP2Y0A02YK infrared distance measuring sensor, which is placed under the vibrating mass. From such experiment, the spectrum of output signal of the sensor exhibits four sharp frequency peaks, which agree well with the solution of such mathematical model. Moreover, the solution can clearly describe the behavior of each frequency peak in the spectrum.

2. Materials and Methods

In this research the forced oscillation of the mass on spring is described by a linear second order differential equation system, with multi-time variable technique. First of all, we would therefore like to describe the multi-time variable technique briefly as follows.

2.1. Multi-time Variable for Separating Time Scale of System

Definition (I)

The concept of multi-time variable technique is that the time of system can be separated into two different variables, t and τ, which the natural response is a function of t, and the forced response is a function of τ. As a result, the complete response of system can be written as

$$x(t,\tau)=x_n(t)+x_f(\tau), \qquad (1)$$

where τ is time considered after the beginning of time of system, t, by the time Δt. Alternatively, we can say that t is shifted τ by Δt, namely

$$\tau = t+\Delta t. \qquad (2)$$

Definition (II)

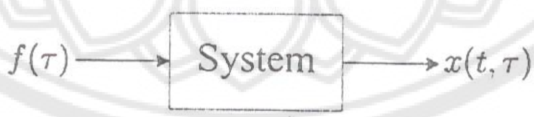
The linear second order differential equation that describes the oscillation of system acted on by an external input can be written as

$$c_2\frac{d^2x(t,\tau)}{dt^2}+c_1\frac{dx(t,\tau)}{dt}+c_0x(t,\tau)=f(\tau), \qquad (3)$$

where c_2, c_1 and c_0 are constant and $f(\tau)$ is the external input.

These definitions are applied to solve the linear system as shown in Figure 1.

Figure 1: Block diagram of this system.



Dividing (3) by c_2 gives the equation of motion,

$$\frac{d^2x(t,\tau)}{dt^2}+\frac{c_1}{c_2}\frac{dx(t,\tau)}{dt}+\frac{c_0}{c_2}x(t,\tau)=\frac{f(\tau)}{c_2}. \qquad (4)$$

When we consider the natural response, (4) has to be written as

$$\frac{d^2x(t,\tau)}{dt^2}+\frac{c_1}{c_2}\frac{dx(t,\tau)}{dt}+\frac{c_0}{c_2}x(t,\tau)=0, \qquad (5)$$

of which the solution reads

$$x_n(t) = e^{-\frac{c_1}{2c_2}t}\left(a\cos\omega_n t + b\cos\omega_n t\right), \tag{6}$$

where $\omega_n = \sqrt{\omega_0^2 - (c_1/2c_2)^2}$, $\omega_0 = \sqrt{c_0/c_2}$ and the constant a and b of the natural response depend on the initial conditions and on the forced response $x_f(\tau)$, which will be discussed later. In addition, (6) can usefully be rewritten as

$$x_n(t) = Ae^{-\frac{c_1}{2c_2}t}\cos(\omega_n t - \phi), \tag{7}$$

where $A = \sqrt{a^2 + b^2}$ is the amplitude of the oscillation and $\phi = \tan^{-1}(b/a)$ is the phase constant. For convenience we will set $b = 0$, then $A = a$ and $\phi = 0$. Substituting these values to (7), we get

$$x_n(t) = ae^{-\frac{c_1}{2c_2}t}\cos\omega_n t. \tag{8}$$

Next, to find the forced response , we can rewrite (4) as

$$\frac{d^2 x_f(\tau)}{d\tau} + \frac{c_1}{c_2}\frac{dx_f(\tau)}{d\tau} + \frac{c_0}{c_2}x_f(\tau) = \frac{f(\tau)}{c_2}. \tag{9}$$

Using the technique of Fourier Transform, we define the input of system $f(\tau) = F_0\cos(\omega_f \tau)$, where F_0 and ω_f are the amplitude and angular frequency of the input respectively, then we obtain the forced response output:

$$x_f(\tau) = F_0\left|H(j\omega_f)\right|\cos\left(\omega_f \tau + \angle H(j\omega_f)\right), \tag{10}$$

where $\left|H(j\omega_f)\right| = 1/\left[c_2\sqrt{\left(\omega_0^2 - \omega_f^2\right)^2 + (c_1/c_2)^2\omega_f^2}\right]$ is the system amplitude gain at frequency ω_f and $\angle H(j\omega_f) = -\tan^{-1}\left[\dfrac{(c_1/c_2)\omega_f}{\omega_0^2 - \omega_f^2}\right]$ is the phase shift of the system at frequency ω_f.

Finally, substituting (8) in (1), the complete response in the viewpoint of multi-time variable is

$$x(t,\tau) = ae^{-\frac{c_1}{2c_2}t}\cos(\omega_n t) + x_f(\tau), \tag{11}$$

where $x_f(\tau)$ was defined in (10).

Upon finding a from the initial condition $x(0,\tau) = x_0$, we set $t = 0$ in (11), getting

$$a = x_0 - x_f(\tau) \tag{12}$$

Thus, (11) can be written as

$$x(t,\tau) = \left[x_0 - x_f(\tau)\right]e^{-\frac{c_1}{2c_2}t}\cos(\omega_n t) + x_f(\tau). \tag{13}$$

In appendix of this paper, the solution of ordinary second order differential equation solved by conventional method , in case $\phi = 0$, is described. It is convenient to write such solution here:

$$x(t) = \left[x_0 - x_f(0)\right]e^{-\frac{c_1}{2c_2}t}\cos(\omega_n t) + x_f(t). \tag{14}$$

Comparing (13) with (14), it is easy to see that the natural response term in (13) depends on the values of the forced response for all time $t \geq 0$, whereas the natural response term in (14) depends on the initial values of the forced response at time $t = 0$ only. This is the obvious difference between the multi-time variable technique and the conventional method.

Next, substitution 10 in 13 gives

$$x(t,\tau) = \left[x_0 - F_0\left|H(j\omega_f)\right|\cos\left(\omega_f\tau + \angle H(j\omega_f)\right)\right]e^{-\frac{c_1}{2c_2}t}\cos(\omega_n t) + F_0\left|H(j\omega_f)\right|\cos\left(\omega_f\tau + \angle H(j\omega_f)\right). \tag{15}$$

From a trigonometric identity,

$$\cos u \cos v = \frac{1}{2}\left[\cos(u-v)+\cos(u+v)\right],$$

and the assumption of τ in (2), (15) can be rewritten as

$$x(t,\tau)=e^{-\frac{c_1}{2c_2}t}x_0\cos(\omega_n t)-e^{-\frac{c_1}{2c_2}t}\frac{D}{2}\cos\left[\left(\omega_f+\omega_n\right)t+\phi_f\right]-e^{-\frac{c_1}{2c_2}t}\frac{D}{2}\cos\left[\left(\omega_f-\omega_n\right)t+\phi_f\right] \\ +D\cos\left(\omega_f t+\phi_f\right), \tag{16}$$

where $D=F_0\left|H(j\omega_f)\right|$ is the magnitude of forced response and $\phi_f=\omega_f\Delta t+\angle H(j\omega_f)$ is the phase difference between the forced response and the external force.

From (16), the frequencies of the first, second, third, and fourth terms are ω_n, $\omega_f+\omega_n$, $\omega_f-\omega_n$, and ω_f, respectively, which can be easily illustrated by a simulated spectrum as shown in Figure 2, whereas, using conventional method, the frequency components contained in the solution (A6), which is discussed in the appendix, have only two frequencies: ω_n and ω_f, i.e., the frequency components $\omega_f+\omega_n$ and $\omega_f-\omega_n$ do not appear as illustrated in Figure 3.

Figure 2: The simulated spectrum of the solution obtained by using the multi-time variable technique.

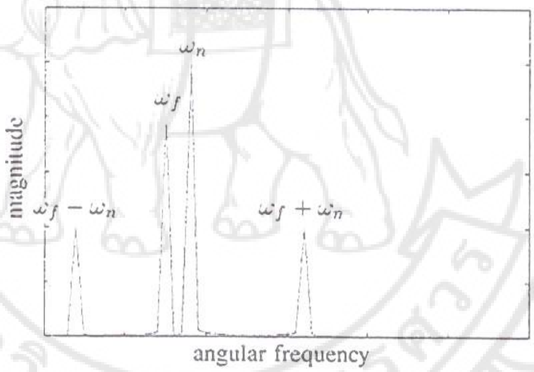
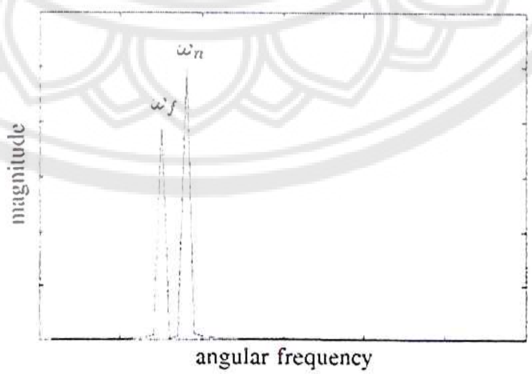


Figure 3: The simulated spectrum of the conventional solution.

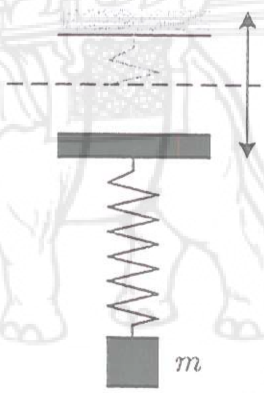


At $t=0$, the magnitudes of the second and third term, i.e., of two new terms, are equal to half the magnitude of the forced response (the fourth term). As t increases, the exponential factor $e^{-\frac{c_1}{2c_2}t}$ decreases continuously and approaches zero asymptotically. In effect, the magnitudes of the second and third terms, including the first term, will decrease continuously. In addition, the magnitude of forced response becomes large when ω_f gets close to ω_0 (by observing the value of $|H(j\omega_f)|$ as above). As a result, the magnitudes of the second and third terms also increase, i.e., half the magnitude of forced response at $t=0$.

2.2. The Driven Oscillation of a Mass-Spring System Described by the Multi-time Differential Equation

To explain the forced vibration of a spring-mass system we consider a vibrating mass on a spring that is acted on by the external force as shown in Figure 4.

Figure 4: The forced mass-spring system.



Without external forcing, using Newton's second and Hooke's laws we get

$$m\frac{d^2x(t)}{dt^2}=-kx(t)-\beta\frac{dx(t)}{dt}, \tag{17}$$

where m is the mass of an object, k is the spring constant, and β is the damping constant. Rearranging (17) we get

$$m\frac{d^2x(t)}{dt^2}+\beta\frac{dx(t)}{dt}+kx(t)=0. \tag{18}$$

When the support of the spring is acted on vertically by a driving force, using multi-time variable technique the equation describing the forced motion of the mass on the spring is in form

$$m\frac{d^2x(t,\tau)}{dt^2}+\beta\frac{dx(t,\tau)}{dt}+kx(t,\tau)=f(\tau). \tag{19}$$

Since (19) is similar to (3) where $m=c_2$, $\beta=c_1$ and $k=c_0$, hence, the natural response of this system, in case $\phi=0$, is

$$x_n(t)=ae^{-\frac{\beta}{2m}t}\cos(\omega_n t), \tag{20}$$

where the damped natural angular frequency $\omega_n = \sqrt{\omega_0^2 - (\beta/2m)^2}$ and the free natural angular frequency $\omega_0 = \sqrt{k/m}$. The forced response, when the external force $f(\tau) = F_0 \cos(\omega_f \tau)$, is

$$x_f(\tau) = F_0 \left|H(j\omega_f)\right| \cos\left(\omega_f \tau + \angle H(j\omega_f)\right), \tag{21}$$

where $\left|H(j\omega_f)\right| = 1 / \left[m\sqrt{\left(\omega_0^2 - \omega_f^2\right)^2 + (\beta/m)^2 \omega_f^2} \right]$ and $\angle H(j\omega_f) = -\tan^{-1}\left[\dfrac{(\beta/m)\omega_f}{\omega_0^2 - \omega_f^2}\right]$.

Substituting (20) in (1), the complete response of the forced spring-mass system in the viewpoint of multi-time variable is

$$x(t,\tau) = a e^{-\frac{\beta}{2m}t} \cos(\omega_n t) + x_f(\tau), \tag{22}$$

where $x_f(\tau)$ was defined in (21).

Setting $t = 0$ in (22) and giving $x(0,\tau) = x_0$ as an initial condition, namely the initial position or the displacement at $t = 0$, the parameter a is given by

$$a = x_0 - x_f(\tau) \tag{23}$$

Then, (22) can be written as

$$x(t,\tau) = \left[x_0 - x_f(\tau)\right] e^{-\frac{\beta}{2m}t} \cos(\omega_n t) + x_f(\tau), \tag{24}$$

which, of course, the natural response term depends on the values of the forced response for all time $t \geq 0$, whereas, using the conventional method, the natural response depends on the initial value of the forced response at time $t = 0$ only, as discussed previously.

Substituting (21) into (24), the complete response becomes

$$\begin{aligned} x(t,\tau) = {} & \left[x_0 - F_0 \left|H(j\omega_f)\right| \cos\left(\omega_f \tau + \angle H(j\omega_f)\right)\right] e^{-\frac{\beta}{2m}t} \cos(\omega_n t) \\ & + F_0 \left|H(j\omega_f)\right| \cos\left(\omega_f \tau + \angle H(j\omega_f)\right). \end{aligned} \tag{25}$$

From a trigonometric identity discussed above and the assumption of τ in (2), (25) can be rewritten as

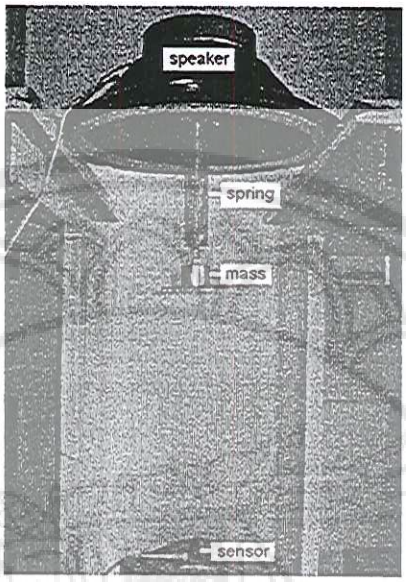
$$\begin{aligned} x(t,\tau) = {} & e^{-\frac{\beta}{2m}t} x_0 \cos(\omega_n t) - e^{-\frac{\beta}{2m}t} \frac{D}{2} \cos\left[\left(\omega_f + \omega_n\right)t + \phi_f\right] - e^{-\frac{\beta}{2m}t} \frac{D}{2} \cos\left[\left(\omega_f - \omega_n\right)t + \phi_f\right] \\ & + D\cos\left(\omega_f t + \phi_f\right), \end{aligned} \tag{26}$$

where $D = F_0 \left|H(j\omega_f)\right|$ and $\phi_f = \omega_f \Delta t + \angle H(j\omega_f)$.

Considering (26), the frequencies of the first, second, third, and fourth terms are ω_n, $\omega_f + \omega_n$, $\omega_f - \omega_n$, and ω_f, respectively, which the second and third terms do not appear if the conventional method is used to find the solution. At time $t = 0$, the magnitudes of both the second and third terms, i.e., of new terms, are equal to half the magnitude of the forced response (the fourth term); as t increases, the exponential factor $e^{-\frac{\beta}{2m}t}$ decreases continuously and approaches zero. From this effect, the magnitudes of the second and third terms, including the first term, decrease continuously. In addition, the magnitude of forced response becomes large when the oscillating frequency of the driving force, ω_f, is close to the free undamped natural frequency $\omega_0 = \sqrt{k/m}$. As a result, the magnitudes of the second and third terms are also large, i.e., half the magnitude of forced response at time $t = 0$.

In order to confirm that the analytical result above is valid, an experiment that has a vibrating mass on a spring acted on by an external force is designed as shown in Figure 5.

Figure 5: The setup of the forced mass-spring system experiment



In this experiment a vibration system consists of a mass (m = 0.1 kg) attached to one end of a spring; here the natural frequency f_n of this system is about 1.70 Hz, i.e., $\omega_n = 2\pi f_n$ (the damping constant β can be roughly calculated from the equation $\omega_n = \sqrt{(k/m)-(\beta/2m)^2}$, where $\omega_n = 2\pi f_n$, f_n=1.70 Hz, k =14 N/m, and m =0.10 kg.). The other end of this spring is connected vertically to the center of a large loudspeaker. Hence, such the vibrating system is forced to undergo periodic oscillation of the loudspeaker in the direction of the mass motion ; the loudspeaker is connected with a power amplifier circuit that receives the sine signal from the function generator, so that we may easily adjust the frequency and amplitude of the external force.

To measure the oscillatory motion of the mass on spring, the infrared distance measuring sensor, GP2Y0A02YK, is placed under the mass. When the position of the mass varies, the output voltage of sensor follows this variation. The voltage signal and its spectrum are displayed on the oscilloscope.

To confirm linearity of the designed measurement system in the experiment above, the mass is put first into motion while the external force, applying by loudspeaker, is switched off. The output signal spectrum of sensor should have only one frequency, namely the frequency of mass's oscillation. Second, let the external force operate while the mass does not oscillate (the spring is wrapped with wire tape). Spectrum of output sensor signal in this case should also have only one frequency, the frequency of loudspeaker oscillation.

In the next process of experimentation, as the external force is switched on, initially we slightly lift the mass to one side of its equilibrium position and then release it; as a result, the oscillation of the mass on the spring is being driven by the periodic external force beginning at $t = 0$. In this case the obtained spectrum of sensor's output signal should correspond to (26), i.e, it have four sharp frequency peaks. The frequencies of the external force used in the experiment are as follows: 1.40, 1.50, 1.90 and 2.00 Hz. The experimental results of these above experiments are illustrated in the next section.

3. Results

The experimental result that the mass oscillates, while the external force is switched off, is shown in Figure 6. The upper trace signal is the output signal of sensor, GP2Y0A02YK, and the lower trace signal is its spectrum. Figure 7 shows the experimental result that the external force oscillates, while the mass does not oscillate; here the frequency of the external force is 2.00 Hz.

Next, the experimental result in case the vibrating mass on spring is forced by the driving force 1.40 Hz is illustrated in Figure 8. Similarly, the experimental results of the vibrating mass forced by the driving forces 1.50, 1.90 and 2.00Hz are shown in Figures 10, 11 and 12, respectively.

Figure 6: The experimental result due to spring-mass system oscillation in the absence of external force, the upper trace is the output signal of the GP2Y0A02YK sensor, and the lower trace is its spectrum

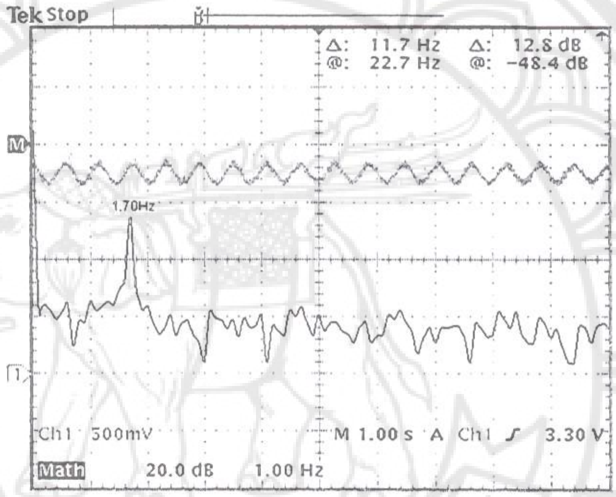


Figure 7: The experimental result in case the mass on spring does not oscillate, while the external force operates; the upper trace is the output signal of the GP2Y0A02YK sensor, and the lower trace is its spectrum.

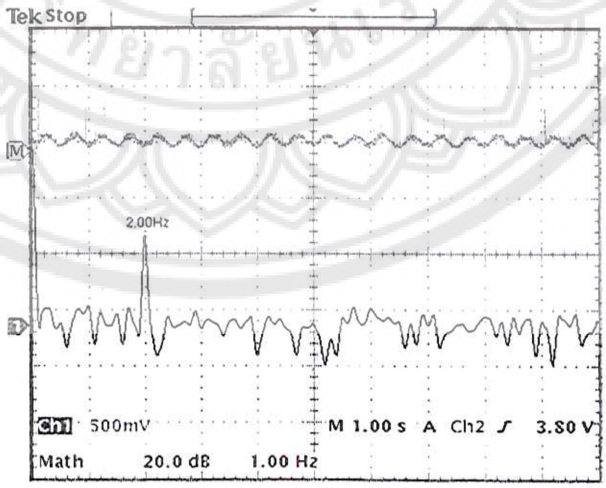


Figure 8: The experimental result of the forced spring-mass system, in case the frequency of the external force is equal to 1.40 Hz. The upper trace is the complete output response of the oscillating mass in time domain. The lower trace shows the spectrum of the upper trace signal.

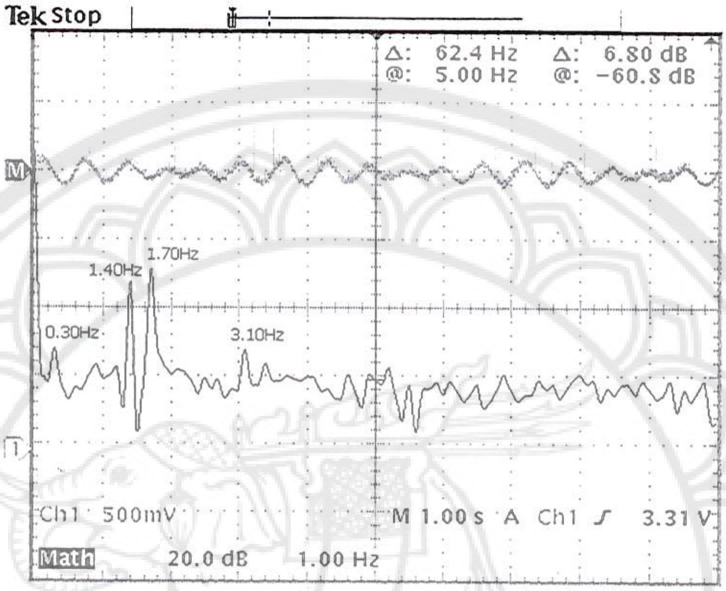


Figure 9: (Continued from Figure 8). After a short time, the frequency component 1.70 Hz slightly decreases, and the frequency components 0.30 and 3.10 Hz cannot be observed.

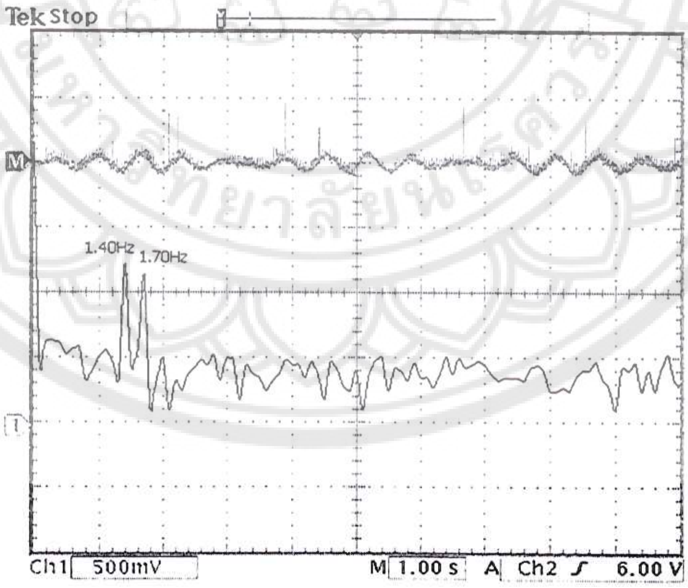


Figure 10: The experimental result of the forced spring-mass system with the frequency of the external force 1.50 Hz. The upper trace is the complete output response of the motion of the mass in time domain. The lower trace shows the spectrum of the upper trace signal.

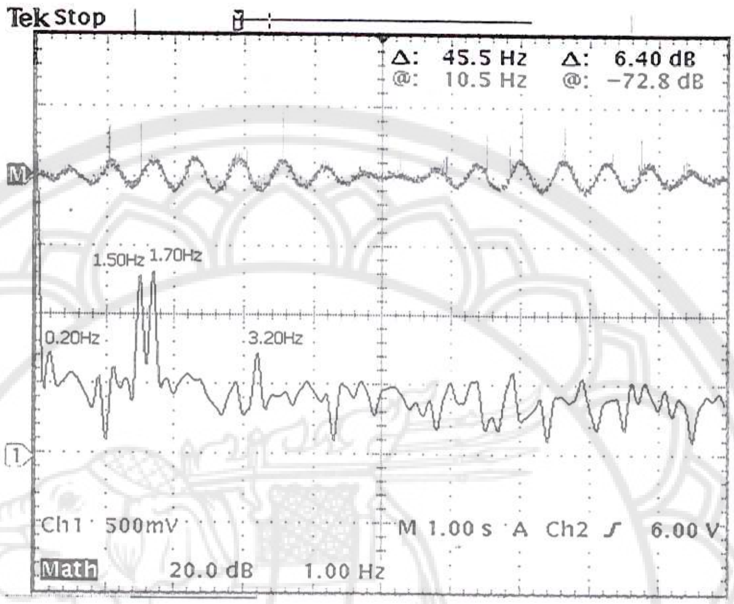


Figure 11: The experimental result of the forced spring-mass system with the frequency of the external force equal to 1.90 Hz; The upper trace is the complete output response of the motion of the mass on spring in time domain. The lower trace shows the spectrum of the upper trace signal.

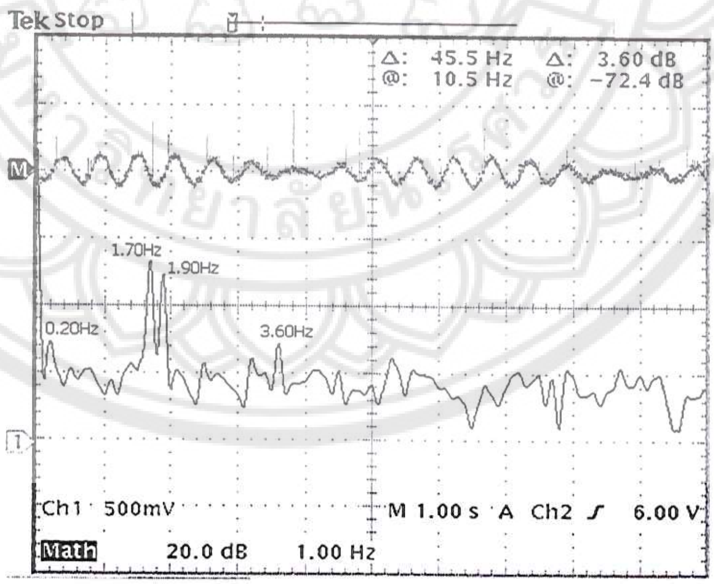
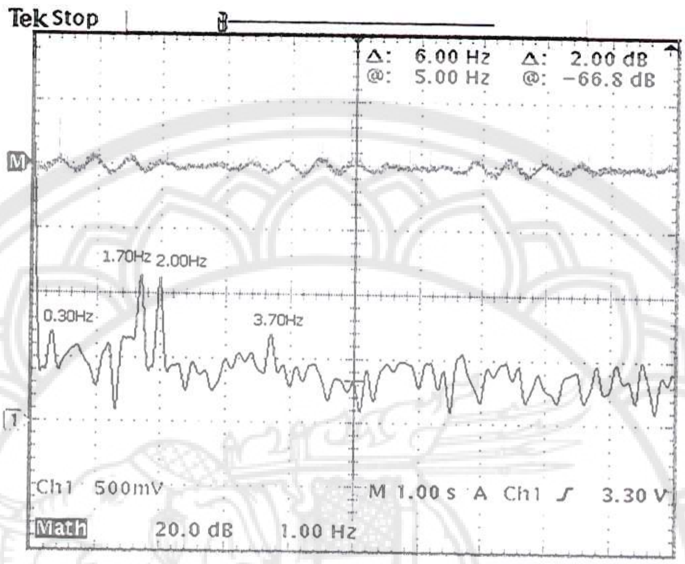


Figure 12: The experimental result of the forced spring-mass system with the frequency of the external force equal to 2.00 Hz; The upper trace is the complete output response of the motion of the mass in time domain. The lower trace shows the spectrum of the upper trace signal.



4. Discussion

The experimental results in Figure 6 and Figure 7 confirm that the designed measurement system is linear, since in each figure the spectrum has only one frequency, i.e., a sharp frequency peak in Figure 7 shows the frequency of oscillation of the mass on spring, namely 1.70 Hz. The sharp frequency peak in Figure 8 shows the frequency of the external force which is equal to 2.00 Hz.

Next, when the oscillating mass is acted on by the external force 1.40 Hz, the obtained spectrum, as shown in Figure 8, matches well with all the terms in (26), where the frequency components 1.70, 3.10, 0.30, and 1.40 Hz of spectrum are equivalent to the first, second, third, and fourth terms of (26), respectively. Consider Figure 8 in detail, a moment after the mass is pulled or pushed away from its equilibrium position and released, the magnitudes of the frequency components 3.10 and 0.30 Hz are about half the magnitude of the frequency component 1.40 Hz, corresponding to the amplitudes of the second and third terms in (26), or of new terms, which are equal to half the amplitude of the fourth term at time $t=0$. After a short time, the magnitude of the frequency component 1.70 Hz slightly decreases, and the frequency components 3.10 and 0.30 Hz cannot be observed, as shown in Figure 9, corresponding to the effect of the factor $e^{-\frac{\beta}{2m}t}$ in (26), whereas the magnitude of the frequency component 1.40 Hz has a steady or fixed amplitude.

Figure 10 is similar to Figure 8. It shows experimental result due to the external forcing frequency 1.50 Hz. In this figure, the frequency components 1.70, 3.20, 0.20, and 1.50 Hz of spectrum correspond to the first, second, third, and fourth terms of (26), respectively. After a short time, the magnitude of the frequency component 1.70 Hz slightly decreases, and the frequency components 3.20 and 0.20 Hz cannot be observed.

Analogous to Figure 8, Figure 11 shows experimental result due to the external forcing frequency equal to 1.90 Hz. In this figure, the frequency components 1.70, 3.60, 0.20, and 1.90 Hz of spectrum correspond to the first, second, third, and fourth terms of (26), respectively. After a short time, the magnitude of the frequency component 1.70 Hz slightly decreases, and the frequency components 3.60 and 0.20 Hz cannot be observed.

Similar to Figure 8, Figure 12 shows experimental result due to the external forcing frequency equal to 2.00 Hz. In this figure, the frequency components 1.70, 3.70, 0.30, and 2.00 Hz of spectrum correspond to the first, second, third, and fourth terms of (26), respectively. After a short time, the magnitude of the frequency component 1.70 Hz slightly decreases, and the frequency components 3.70 and 0.30 Hz cannot be observed.

5. Conclusion

From experimental and analytical results corresponding to each other we may conclude that the two occurring new frequency components are an outcome of an oscillating system driven by external force. The analytical result using the multi-time variable technique can predict four frequency components in the spectrum of the experimental result more precisely.

Acknowledgment

Financial support from the Thailand Research Fund through the Royal Golden Jubilee Ph.D. Program (Grant No. PHD/0014/2553) to Arum Kitipongwatana and Paramote Wardkein is acknowledged.

References

[1] Keller F.J., Gettys W.E., Skove M.J. (1993) Physics Classical and Modern. USA: MrGraw-Hill.
[2] Zill D.G. (2005) A First Course in Differential Equations with Modeling Applications. USA: Brooks/Cole Cengage Learning.
[3] Taylor J.R. (2005) Classical Mechanics. USA: University Science Books.
[4] Arya A.P. (1997) Introduction to Classical Mechanics. USA: Benjamin Cummings.
[5] James M.L., Smith G.M., Wolford J.C., Whaley P.W. (1989) Vibration of Mechanical and Structural Systems with Microcomputer Applications. USA: Harper and Row.
[6] Dimarogonas A (1996) Vibration for Engineers. USA: Prentice-Hall.
[7] Maneechukate, T., J., Koseeyaporn, P., Wardkein, and P., Koseeyaporn, 2008. "Wide-band amplitude control of the second-order oscillator circuit", *AEU - International Journal of Electronics and Communications*, 62, pp. 666-673.
[8] Roychowdhury, J.S., 2001. "Analyzing circuits with wildly separated timescales using numerical PDE methods", *IEEE Transactions on Circuits and Systems*, 48, pp. 578-594.

Appendix : The Solution of Second Order Differential Equation Solved Using the Conventional Method

In conventional method, the linear second order differential equation describing the oscillation output of damped oscillator, $x(t)$, forced by an external forcing $f(t)$ can be written as

$$c_2 \frac{d^2x(t)}{dt^2} + c_1 \frac{dx(t)}{dt} + c_0 x(t) = f(t).$$

The solution of the above equation consists of the natural response $x_n(t)$ and the forced response $x_f(t)$, i.e.,

$$x(t) = x_n(t) + x_f(t). \quad \text{(A1)}$$

The natural response of this system is

$$x_n(t) = e^{-\frac{c_1}{2c_2}t}\left(a\cos\omega_n t + b\cos\omega_n t\right),$$

or

$$x_n(t) = Ae^{-\frac{c_1}{2c_2}t}\cos\left(\omega_n t - \phi\right),$$

where $A = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}\left(b/a\right)$.

Defining $b = 0$, $\phi = 0$, thus

$$x_n(t) = ae^{-\frac{c_1}{2c_2}t}\cos\omega_n t. \qquad \text{(A2)}$$

Substituting (A2) into (A1) gives

$$x(t) = ae^{-\frac{c_1}{2c_2}t}\cos\omega_n t + x_f(t). \qquad \text{(A3)}$$

Upon finding a from the initial position $x(0) = x_0$. We set $t = 0$ in (A3) yielding $a = x_0 - x_f(0)$.

Thus, (A3) can be rewritten as

$$x(t) = \left[x_0 - x_f(0)\right]e^{-\frac{c_1}{2c_2}t}\cos\omega_n t + x_f(t). \qquad \text{(A4)}$$

If the external input $f(t) = F_0\cos(\omega_f t)$, then the forced response is

$$x_f(t) = F_0\left|H(j\omega_f)\right|\cos\left(\omega_f t + \angle H(j\omega_f)\right), \qquad \text{(A5)}$$

where $\left|H(j\omega_f)\right| = 1/\left[c_2\sqrt{\left(\omega_0^2 - \omega_f^2\right)^2 + \left(c_1/c_2\right)^2\omega_f^2}\right]$ and $\angle H(j\omega_f) = -\tan^{-1}\left[\frac{\left(c_1/c_2\right)\omega_f}{\omega_0^2 - \omega_f^2}\right]$.

Substituting (A5) to (A4) and determining $D = F_0\left|H(j\omega_f)\right|$ and $\phi_c = \angle H(j\omega_f)$, (A4) can be rewritten as

$$x(t) = \underbrace{\left[x_0 - D\cos(\phi_c)\right]}_{\text{constant value}}e^{-\frac{c_1}{2c_2}t}\cos\omega_n t + D\cos(\omega_f t + \phi_c). \qquad \text{(A6)}$$

Scientific Research and Essays Vol. 7(39), pp. 3292-3301, 4 October, 2012
Available online at http://www.academicjournals.org/SRE
DOI: 10.5897/SRE12. 249
ISSN 1992-2248 ©2012 Academic Journals

Full Length Research Paper

An oscillation discovery of the forced vibrating system predicted by the multi-time differential equation

Kriangsak Prompak[1], Anucha Kaewpoonsuk[1], Thongchai Maneechukate[1]* and Paramote Wardkein[2]

[1]Department of Physics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand.
[2]Department of Telecommunication Engineering, Faculty of Engineering, King Mongkut's Institute of Technology Ladkrabang, Bangkok 10520, Thailand.

Accepted 14 September, 2012

In this article, an experiment of the forced oscillating pendulum system was set up. Here, a pendulum is driven sinusoidally by the loudspeaker. From this experiment, the spectrum of output signal of a Hall-effect sensor (UGN3503) which is used to transform the oscillatory motion of the pendulum into an electrical signal, exhibits four sharp frequency peaks, and it is also found that two of these frequency peaks cannot be described by the solution of ordinary differential equation from conventional ordinary differential equation text books. However, it can be solved by multi-time variable technique, a mathematical tool. The latter solution consists of the sum of four terms: Natural response, forced response and the two new terms being the result of multiplying between natural and forced responses. This analytical solution reveals the frequency components and their behaviors more precisely and corresponds well to the spectrum of the experimental result.

Key words: Vibration, pendulum, model, multi-time.

INTRODUCTION

The dynamics of a simple pendulum is one of the most popular topics analyzed in textbooks and undergraduate physics courses (Keller et al., 1993). The simple pendulum is also one of the examples presented when nonlinear oscillations are studied (the oscillation amplitude is not small) (Marion, 1970; Mickens, 1996). For small values of the oscillation amplitude, it is possible to linearize the equation of motion of the pendulum (Zill, 2005) and, in this regime, the oscillatory motion is a simple harmonic motion, that is, the restoring force is proportional to the angular displacement.

Pendulum is also the common example of a mechanical oscillator. When perturbed, it begins to oscillate with a natural frequency. Yet, this free oscillation decays exponentially with time due to friction forces, that is, these forces cause the mechanical energy of the oscillator to decrease. Any system that behaves in this way is known as a damped harmonic oscillator.

The oscillation of the damped harmonic oscillator (pendulum) acted on by an external driving force is called forced oscillation or driven oscillation. The oscillations occur mainly in machinery and in electric circuits. Using the conventional method (Taylor, 2005; Arya, 1997; James et al., 1989) for finding the solution of a linear differential equation describing the oscillation of such system, the solution is given by the sum of two parts. The natural response, the solution for a damped harmonic oscillator discussed earlier, delays out eventually. Another one is the forced response, the solution due to the external driving force, persists after the natural response has died away. The natural response depends on the initial conditions at time $t=0$ and on the initial values of the forced response also at time $t=0$ (Dimarogonas, 1996).

Recently, the analysis of a second-order oscillator based on a multi-time variable technique was proposed (Maneechukate et al., 2008). The result demonstrates that the amplitude of the natural response $x_n(t)$ of the system depends on initial value x_0 and the forced response $x_f(\tau)$ at any arbitrary time τ according to the following equation:

*Corresponding author. E-mail: thongchaim@nu.ac.th.

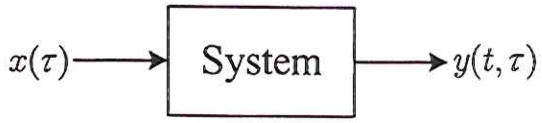


Figure 1. Block diagram of this system.

$$x(t,\tau)=\underbrace{\left[x_0-x_f(\tau)\right]\cos\omega t}_{x_n(t)}+x_f(\tau),$$

where $x(t,\tau)$ is the complete solution of the separated time scales.
This solution is somehow different from the one achieved by the conventional method:

$$x(t)=\underbrace{\left[x_0-x_f(0)\right]\cos\omega t}_{x_n(t)}+x_f(t).$$

It is remarked that amplitude of the natural response $x_n(t)$ depends on the initial condition x_0 and the forced response $x_f(t)$ at the initial time, that is, $t=0$.
In this paper, we apply the multi-time variable technique to solve a mathematical model in the form of second order differential equation describing the damped harmonic oscillation of the pendulum acted on by the sinusoidal external force. The obtained analytical result consists of the sum of four components: natural response, forced response and the new two terms coming from the product of natural and forced responses. These two new terms do not appear when we use conventional method.

To verify that such analytical result is valid, an experiment of the forced-oscillating system is set up. Here, a mechanical pendulum acted by a sinusoidal external force is used as an example to illustrate behavior of such solution. A large loudspeaker is used to generate the sinusoidal external force to the pendulum. The oscillatory motion of the pendulum is transformed as an electrical signal by the popular UGN3503 Hall Effect Sensor, which is placed near a magnet fixed next to pendulum's rotating point. From such experiment, the spectrum of output signal of the sensor exhibits four sharp frequency peaks, which agree well with the solution of such mathematical model. Moreover, the solution can clearly describe the behavior of each frequency peak in the spectrum.

MATERIALS AND METHODS

In this research, the forced vibration of the pendulum is described by a linear second order differential equation system, with multi-time variable technique. First of all, we would therefore like to describe the multi-time variable technique briefly as follows.

Multi-time variable for separating time scale of system

Definition (I)

The concept of multi-time variable technique is that the time of the system can be separated into two different variables, t and τ, which the natural response is a function of t, and the forced response is a function of τ. As a result, the complete response of system can be written as:

$$y(t,\tau)=y_n(t)+y_f(\tau), \tag{1}$$

where τ is time considered after the beginning of the time of the system, t, by the time Δt. Alternatively, we can say that t is shifted τ by Δt, namely:

$$\tau=t+\Delta t. \tag{2}$$

Definition (II)

The linear second order differential equation that describes the oscillation of system acted on by an external input can be written as:

$$c_2\frac{d^2y(t,\tau)}{dt^2}+c_1\frac{dy(t,\tau)}{dt}+c_0y(t,\tau)=x(\tau). \tag{3}$$

These definitions are applied to solve the linear system as shown in Figure 1. Dividing Equation 3 by c_2 gives the equation of motion:

$$\frac{d^2y(t,\tau)}{dt^2}+\frac{c_1}{c_2}\frac{dy(t,\tau)}{dt}+\frac{c_0}{c_2}y(t,\tau)=\frac{x(\tau)}{c_2}. \tag{4}$$

When we consider the natural response, Equation 4 has to be written as:

$$\frac{d^2y_n(t)}{dt^2}+\frac{c_1}{c_2}\frac{dy_n(t)}{dt}+\frac{c_0}{c_2}y_n(t)=0, \tag{5}$$

Of which the solution reads:

$$y_n(t)=e^{-\frac{c_1}{2c_2}t}\left(a\cos\omega_n t+b\sin\omega_n t\right) \tag{6}$$

According to Appendix A, where $\omega_n=\sqrt{\omega_0^2-(c_1/2c_2)^2}$, $\omega_0=c_0/c_2$ and the constants a and b of the natural response depend on the initial conditions and on the forced response $y_f(\tau)$, which will be discussed later. In addition, Equation 6 can usefully be rewritten as:

$$y_n(t)=Ae^{-\frac{c_1}{2c_2}t}\cos(\omega_n t-\phi), \tag{7}$$

where $A=\sqrt{a^2+b^2}$ is the amplitude of the motion and $\phi=\tan^{-1}(b/a)$ is the phase constant. For convenience, we will set $b=0$, then $A=a$ and $\phi=0$. Substituting these values to Equation 7, we get:

$$y_n(t)=ae^{-\frac{c_1}{2c_2}t}\cos(\omega_n t). \tag{8}$$

Next, to find the forced response $y_f(\tau)$, we can rewrite Equation 4 as:

$$\frac{d^2y_f(\tau)}{d\tau^2}+\frac{c_1}{c_2}\frac{dy_f(\tau)}{\partial\tau}+\frac{c_0}{c_2}y_f(\tau)=\frac{x(\tau)}{c_2}. \tag{9}$$

Using the technique of Fourier transform, we define the input of system $x(\tau)=X_0\cos(\omega_f\tau)$, where X_0 and ω_f are the amplitude and angular frequency of the input, respectively, then we obtain the forced response output:

$$y_f(\tau)=X_0\left|H(j\omega_f)\right|\cos\left(\omega_f\tau+\angle H(j\omega_f)\right), \tag{10}$$

where

$\left|H(j\omega_f)\right|=1/c_2\sqrt{\left((c_0/c_2)-\omega_f^2\right)^2+(c_1/c_2)^2\omega_f^2}$ is the system amplitude gain at frequency ω_f and $\angle H(j\omega_f)=-\tan^{-1}\left(\frac{(c_1/c_2)\omega_f}{(c_0/c_2)-\omega_f^2}\right)$ is the phase shift of the system at frequency ω_f (Appendix B). Finally, substituting Equations 8 and 10 in Equation 1, the complete response in the viewpoint of multi-time variable is

$$\begin{aligned}y(t,\tau)&=y_n(t)+y_f(\tau)\\&=ae^{-\frac{c_1}{2c_2}t}\cos(\omega_n t)+y_f(\tau),\end{aligned} \tag{11}$$

where $y_f(\tau)$ was defined in Equation 10.

Upon finding a from the initial position $y(0,\tau)=y_0$, we set $t=0$ in Equation 11, getting:

$$\begin{aligned}a&=y(0,\tau)-y_f(\tau)\\&=y_0-y_f(\tau).\end{aligned} \tag{12}$$

Thus, Equation 11 can be written as:

$$y(t,\tau)=\left[y_0-y_f(\tau)\right]e^{-\frac{c_1}{2c_2}t}\cos(\omega_n t)+y_f(\tau). \tag{13}$$

In Appendix C, the solution of ordinary second order differential equation solved by conventional method is described. It is convenient to write such solution here:

$$y(t)=\left[y_0-y_f(0)\right]e^{-\frac{c_1}{2c_2}t}\cos(\omega_n t)+y_f(t). \tag{14}$$

Comparing Equation 13 with Equation 14, it is easy to see that the natural response term in Equation 13 depends on the values of the forced response for all time $t\geq 0$, whereas the natural response term in Equation 14 depends on the initial values of the forced response at time $t=0$ only. This is the obvious difference between the multi-time variable technique and the conventional method.

Next, substituting Equation 10 in Equation 13 gives:

$$\begin{aligned}y(t,\tau)=&\left[y_0-X_0\left|H(j\omega_f)\right|\cos\left(\omega_f\tau+\angle H(j\omega_f)\right)\right]e^{-\frac{c_1}{2c_2}t}\cos(\omega_n t)\\&+X_0\left|H(j\omega_f)\right|\cos\left(\omega_f\tau+\angle H(j\omega_f)\right).\end{aligned} \tag{15}$$

From a trigonometric identity,

$\cos m\cos n=(1/2)\left[\cos(m-n)+\cos(m+n)\right]$,

and the assumption of τ in Equation 2, Equation 15 can be rewritten as:

$$\begin{aligned}y(t,t+\Delta t)=&e^{-\frac{c_1}{2c_2}t}y_0\cos(\omega_n t)-e^{-\frac{c_1}{2c_2}t}\frac{B}{2}\cos\left[(\omega_f+\omega_n)t+\phi_f\right]\\&-e^{-\frac{c_1}{2c_2}t}\frac{B}{2}\cos\left[(\omega_f-\omega_n)t+\phi_f\right]+B\cos(\omega_f t+\phi_f),\end{aligned} \tag{16}$$

where $B=X_0\left|H(j\omega_f)\right|$ is the magnitude of forced response and $\phi_f=\omega_f\Delta t+\angle H(j\omega_f)$ is the phase difference between the forced response and the external force.

From Equation 16, the frequencies of the first, second, third, and fourth terms are ω_n, $\omega_f+\omega_n$, $\omega_f-\omega_n$, and ω_f, respectively. $\omega_f+\omega_n$ and $\omega_f-\omega_n$ are the frequencies of two new terms, as discussed previously. At $t=0$, the magnitudes of the second and third term, that is, of two new terms, are equal to half the magnitude of the forced response (the fourth term). As t increases, the exponential factor $e^{-\frac{c_1}{2c_2}t}$ decreases continuously and approaches zero asymptotically. In effect, the magnitudes of the second and third terms, including the first term, will decrease continuously. In addition, the magnitude of forced response becomes large when ω_f gets close to $\sqrt{c_0/c_2}$ (by observing the value of $\left|H(j\omega_f)\right|$ as mentioned earlier). As a result, the magnitudes of the second and third terms also increase, that is, half the magnitude of forced response at $t=0$.

The forced vibration of a pendulum described by the multi-time second order differential equation

To explain the forced vibration of a pendulum, we consider the pendulum acted on by the external force as shown in Figure 2. Without external forcing, when the bob of the pendulum is pulled from its equilibrium position, the restoring force magnitude acting on the bob is given by:

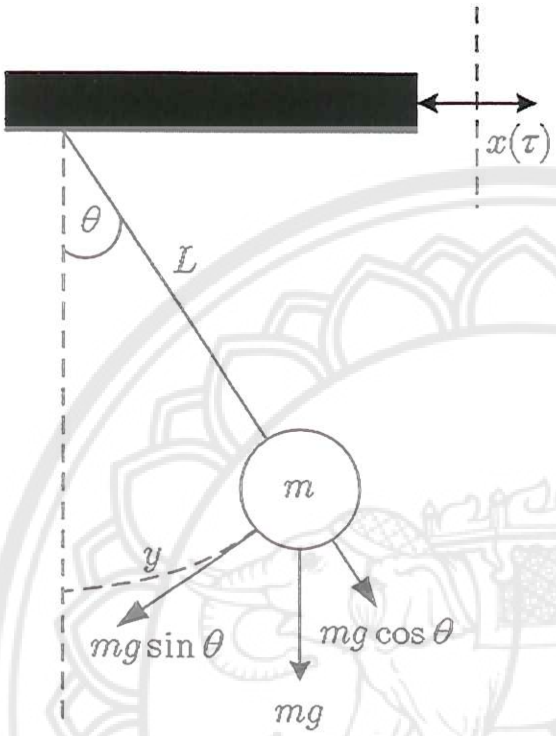


Figure 2. The forced oscillating pendulum.

$$F = -mg\sin\theta \tag{17}$$

where m is the mass of the bob and g is the acceleration due to gravity. Suppose the pendulum is set in motion, if the angle θ is small then $\sin\theta \approx \theta$; the motion of the pendulum is the damped harmonic oscillation, and the arc y can be approximately considered to be a straight line. From the relation $y = L\theta$, Equation 17 can be rewritten as:

$$F = -mg\theta = mg\frac{y}{L}. \tag{18}$$

The motion of the pendulum will eventually be damped out, because there is a damping force proportional to the velocity of the bob acting in the direction opposite the motion: $-\beta\frac{dy}{dt}$, where β is the damping constant. Including these forces in Newton's second law for the bob, we obtain:

$$-mg\frac{y}{L} - \beta\frac{dy}{dt} = m\frac{d^2y}{dt^2}. \tag{19}$$

Rearranging Equation 19, we get:

$$m\frac{d^2y(t)}{dt^2} + \beta\frac{dy(t)}{dt} + mg\frac{y(t)}{L} = 0. \tag{20}$$

When the support of the pendulum is acted on horizontally by a driving force $x(\tau)$, using the multi-time variable technique, the equation describing the forced motion of the pendulum is in the form:

$$m\frac{d^2y(t,\tau)}{dt^2} + \beta\frac{dy(t,\tau)}{\partial t} + mg\frac{y(t,\tau)}{L} = x(\tau). \tag{21}$$

If a periodic external force is given by $x(\tau) = X_0\cos(\omega_f\tau)$, where $\tau = t + \Delta t$, Equation 21 becomes:

$$m\frac{d^2y(t,\tau)}{dt^2} + \beta\frac{dy(t,\tau)}{dt} + mg\frac{y(t,\tau)}{L} = X_0\cos(\omega_f\tau). \tag{22}$$

Dividing Equation 22 by the mass gives the equation of motion,

$$\frac{d^2y(t,\tau)}{dt^2} + \frac{\beta}{m}\frac{dy(t,\tau)}{dt} + \frac{g}{L}y(t,\tau) = \frac{X_0}{m}\cos(\omega_f\tau). \tag{23}$$

When we want to find the natural response, Equation 23 can be rewritten as:

$$\frac{d^2y_n(t)}{dt^2} + \frac{\beta}{m}\frac{dy_n(t)}{dt} + \frac{g}{L}y_n(t) = 0. \tag{24}$$

Of which solution reads:

$$y_n(t) = e^{-\frac{\beta}{2m}t}\left(a\cos\omega_n t + b\sin\omega_n t\right), \tag{25}$$

where $\omega_n = \sqrt{\omega_0^2 - (\beta/2m)^2}$ and $\omega_0 = \sqrt{g/L}$, are the damped and undamped natural angular frequency of the system, respectively. The constants a and b of the natural response depend on the initial conditions, namely, the initial position y_0 and the initial velocity v_0, and on the forced response $y_f(\tau)$, which will be discussed later. Of course, Equation 25 can usefully be rewritten as:

$$y_n(t) = Ae^{-\frac{\beta}{2m}t}\cos\left(\omega_n t - \phi\right), \tag{26}$$

where, as before, $A = \sqrt{a^2 + b^2}$ is the amplitude of the motion and $\phi = \tan^{-1}\left(b/a\right)$ is the phase constant. For simplicity, we will set $b = 0$, then $A = a$ and $\phi = 0$. Substituting these values to Equation 26, we get

$$y_n(t) = ae^{-\frac{\beta}{2m}t}\cos\left(\omega_n t\right). \tag{27}$$

Next, when the forced response is determined, Equation 23 is rewritten as:

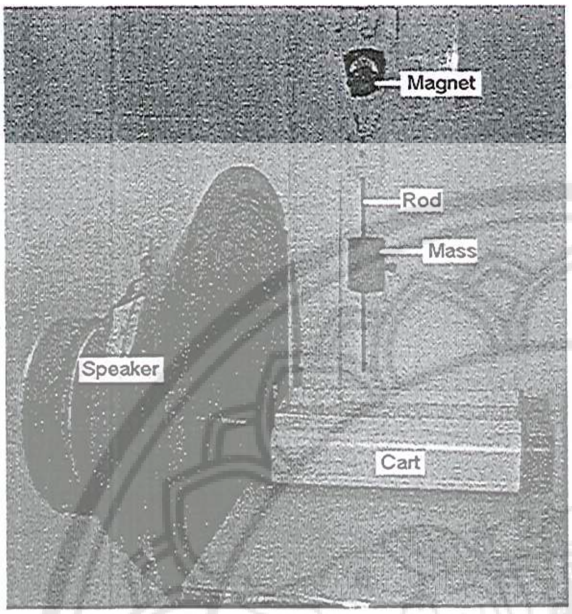


Figure 3. The setup of the forced pendulum experiment.

$$\frac{d^2 y_f(\tau)}{d\tau^2} + \frac{\beta}{m}\frac{dy_f(\tau)}{d\tau} + \frac{g}{L}y_f(\tau) = \frac{X_0}{m}\cos(\omega_f \tau). \quad (28)$$

Similar to the multi-time variable for separating time scale of system, the forced response of Equation 28 is

$$y_f(\tau) = X_0 \left|H(j\omega_f)\right| \cos\left(\omega_f \tau + \angle H(j\omega_f)\right), \quad (29)$$

where $\left|H(j\omega_f)\right| = 1/m\sqrt{\left(\omega_0^2-\omega_f^2\right)^2 + (\beta/m)^2\omega_f^2}$ and $\angle H(j\omega_f) = -\tan^{-1}\left(\frac{(\beta/m)\omega_f}{\omega_0^2-\omega_f^2}\right)$.

The sum of Equations 27 and 29 represent the complete solution, that is,

$$\begin{aligned} y(t,\tau) &= y_n(t) + y_f(\tau) \\ &= ae^{-\frac{\beta}{2m}t}\cos(\omega_n t) \\ &\quad + X_0\left|H(j\omega_f)\right|\cos\left(\omega_f\tau + \angle H(j\omega_f)\right). \end{aligned} \quad (30)$$

Setting $t = 0$ in Equation 30 and giving $y(0,\tau) = y_0$ as an initial condition, namely the initial position or the displacement at $t = 0$, the parameter a is given by:

$$\begin{aligned} a &= y(0,\tau) - y_f(\tau) \\ &= y_0 - X_0\left|H(j\omega_f)\right|\cos\left(\omega_f\tau + \angle H(j\omega_f)\right). \end{aligned} \quad (31)$$

Substituting Equation 31 into Equation 30, the complete response becomes:

$$\begin{aligned} y(t,\tau) &= e^{-\frac{\beta}{2m}t}\left[y_0 - X_0\left|H(j\omega_f)\right|\cos\left(\omega_f\tau + \angle H(j\omega_f)\right)\right]\cos(\omega_n t) \\ &\quad + X_0\left|H(j\omega_f)\right|\cos\left(\omega_f\tau + \angle H(j\omega_f)\right). \end{aligned} \quad (32)$$

Similar to Equation 16 in the multi-time variable for separating time scale of system, Equation 32 can be rewritten as:

$$\begin{aligned} y(t,t+\Delta t) &= e^{-\frac{\beta}{2m}t}y_0\cos(\omega_n t) - e^{-\frac{\beta}{2m}t}\frac{B}{2}\cos\left[\left(\omega_f+\omega_n\right)t + \phi_f\right] \\ &\quad - e^{-\frac{\beta}{2m}t}\frac{B}{2}\cos\left[\left(\omega_f-\omega_n\right)t + \phi_f\right] + B\cos\left(\omega_f t + \phi_f\right), \end{aligned} \quad (33)$$

where $B = X_0\left|H(j\omega_f)\right|$ and $\phi_f = \omega_f\Delta t + \angle H(j\omega_f)$.

Considering Equation 33, the frequencies of the first, second, third, and fourth terms are ω_n, $\omega_f+\omega_n$, $\omega_f-\omega_n$, and ω_f, respectively. At time $t = 0$, the magnitudes of both the second and third terms, that is, of new terms, are equal to half the magnitude of the forced response (the fourth term); as t increases, the exponential factor $e^{-\frac{\beta}{2m}t}$ decreases continuously and approaches zero. From this effect, the magnitudes of the second and third terms, including the first term, decrease continuously. In addition, the magnitude of forced response becomes large when the oscillating frequency of the driving force, ω_f, is close to the free undamped natural frequency $\omega_0 = \sqrt{g/L}$. As a result, the magnitudes of the second and third terms are also large, that is, half the magnitude of forced response at time $t = 0$.

In order to confirm that the aforementioned analytical result is valid, an experiment that has a vibrating pendulum acted on by an external force is designed as shown in Figure 3.

In this experiment, a vibration system, the pendulum consists of a light rod of length 0.10 m (pendulum's arm) and a mass 0.10 kg attached at one end. The other end of this rod is suspended to a support fixed on an experimental cart, here, the natural frequency f_n of the pendulum's oscillation is about 1.50 Hz (the damping constant β can be roughly calculated from the equation, $\omega_n = \sqrt{(g/L)-(\beta/2m)^2}$, where $\omega_n = 2\pi f_n$, $f_n = 1.50$ Hz, $g = 9.8$ m/s^2, $L = 0.10$ m and $m = 0.10$ kg.). The body of the cart is connected horizontally to the center of a large loudspeaker. Hence, such that the vibrating system is forced to undergo periodic oscillation of the loudspeaker in the direction of the pendulum motion; the loudspeaker is connected with a power amplifier circuit that receives the sine signal from the function generator, so that we may easily adjust the frequency and amplitude of the external force

To measure the oscillatory motion of the pendulum, the Hall Effect Sensor, UGN3503, is placed close to a circular flat magnet fixed on pendulum arm close to the rotating point of it as shown

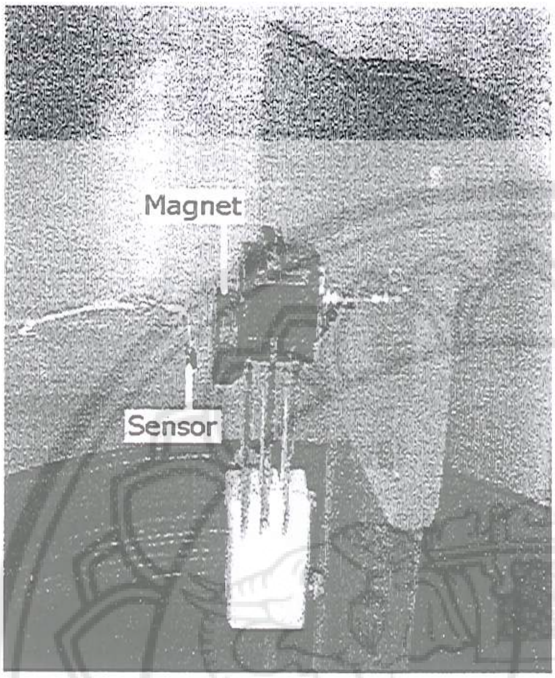


Figure 4. Hall Effect Sensor placed nearby the flat circular magnet.

in Figure 4. When the position of the mass varies, the output voltage of the sensor follows this variation. The voltage signal and its spectrum are displayed on the oscilloscope.

To confirm linearity of the designed measurement system in the aforementioned experiment, the pendulum is put first into motion while the external force, applied by loudspeaker, is switched off. The output signal spectrum of sensor should have only one frequency, namely, the frequency of pendulum's oscillation. Second, let the external force operate while the pendulum does not oscillate (the pendulum is fixed at rest on the experimental cart). Spectrum of output sensor signal in this case should also have only one frequency, the frequency of loudspeaker oscillation.

In the next process of experimentation, as the external force is switched on, initially, we slightly pull the mass to one side of its equilibrium position, which the angle of the pendulum with respect to the vertical is small (less than 5°) and then release it. This is equivalent to giving energy to the pendulum, that is, the oscillation of the pendulum is maintained, and it is being driven by the periodic external force. In this case, the obtained spectrum of sensor's output signal should correspond to Equation 33. The frequencies of the force used in the experiment are as follows: 1.10, 1.20 and 1.90 Hz. The experimental results of these experiments are illustrated subsequently.

RESULTS

The experimental result which the pendulum oscillates, while the external force is switched off, is shown in Figure 5. The upper trace signal is the output signal of sensor,

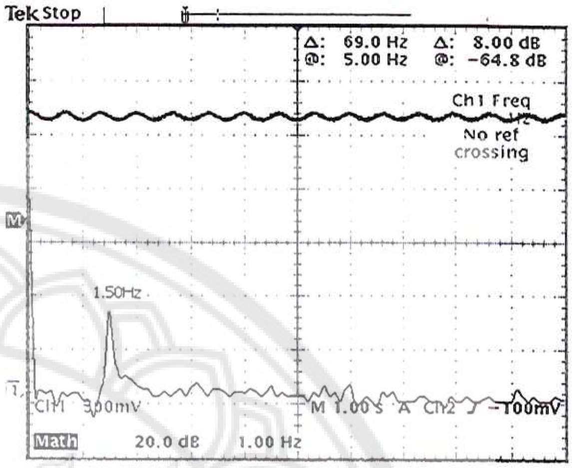


Figure 5. The experimental result due to pendulum oscillation in the absence of external force. The upper trace is the output signal of the UGN3503 sensor, and the lower trace is its spectrum.

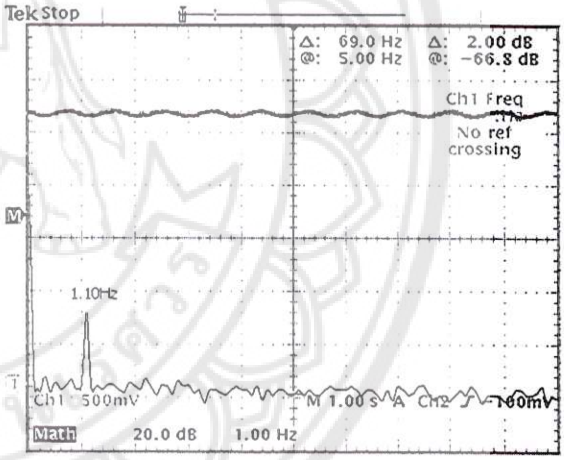


Figure 6. The experimental result in case pendulum does not oscillate, while the external force operates. The upper trace is the output signal of the UGN3503 sensor, and the lower trace is its spectrum.

UGN3503, and the lower trace signal is its spectrum. Figure 6 shows the experimental result that the external force oscillates, while the pendulum does not oscillate; here, the frequency of the external force is 1.10 Hz. Next, the experimental result in case the pendulum is forced by the driving force 1.10 Hz is illustrated in Figure 7. Similarly, the experimental results of the pendulum forced by the driving forces 1.20 and 1.90 Hz are as shown in Figures 9 and 10, respectively.

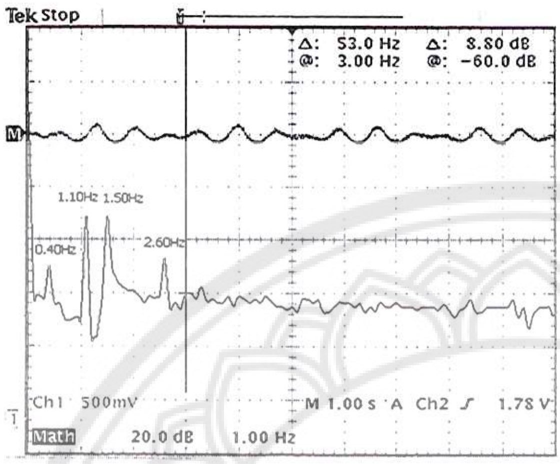


Figure 7. The experimental result of the forced pendulum, in case the frequency of the external force is equal to 1.10 Hz. The upper trace is the complete output response of the pendulum in time domain. The lower trace shows the spectrum of the upper trace signal.

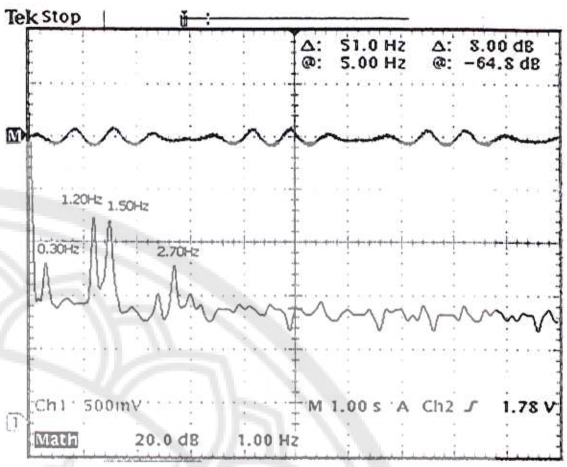


Figure 9. The experimental result of the forced pendulum with the frequency of the external force 1.20 Hz. The upper trace is the complete output response of the motion of the pendulum in time domain. The lower trace shows the spectrum of the upper trace signal.

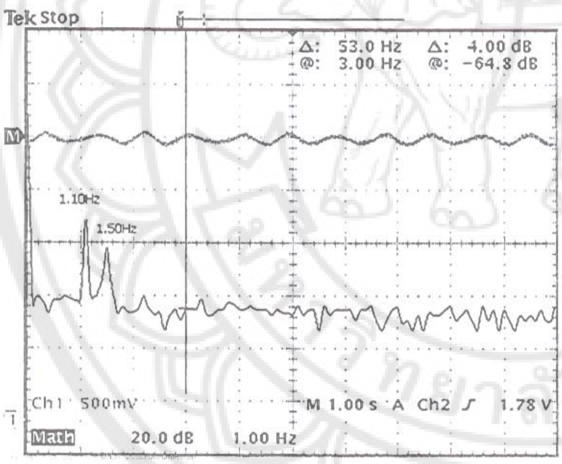


Figure 8. After a short time, the frequency component 1.50 Hz slightly decreases, and the frequency components 0.40 and 2.60 Hz cannot be observed (continued from Figure 7).

DISCUSSION

The experimental results in Figures 5 and 6 confirm that the designed measurement system is linear, since in each figure the spectrum has only one frequency, that is, a sharp frequency peak in Figure 5 showing the frequency of oscillation of the pendulum, namely 1.50 Hz. The sharp frequency peak in Figure 6 shows the frequency of the external force equal to 1.10 Hz.

Next, when the oscillating pendulum is acted on by the external force 1.10 Hz, the obtained spectrum, as shown in Figure 7, matches well with all the terms in Equation 33, where the frequency components 1.50, 2.60, 0.40, and 1.10 Hz of the spectrum are equivalent to the first, second, third, and fourth terms of Equation 33, respectively. Consider Figure 7 in detail, a moment after the mass is pulled away from its equilibrium position and released, the magnitudes of the frequency components 2.60 and 0.40 Hz are about half the magnitude of the frequency component 1.10 Hz, corresponding to the amplitudes of the second and third terms in Equation 33, or of new terms, which are equal to half the amplitude of the fourth term at time $t = 0$. After a short time, the magnitude of the frequency component 1.50 Hz slightly decreases, and the frequency components 2.60 and 0.40 Hz cannot be observed, as shown in Figure 8, corresponding to the effect of the factor $e^{-\frac{\beta}{2m}t}$ in Equation 33, whereas the magnitude of the frequency component 1.10 Hz has steady or fixed amplitude.

Figure 9 is similar to Figure 7. It shows experimental result due to the external force frequency 1.20 Hz. In this figure, the frequency components 1.50, 2.70, 0.30, and 1.20 Hz of the spectrum correspond to the first, second, third, and fourth terms of Equation 33, respectively.

Analogous to Figure 7, Figure 10 shows experimental result due to the external forcing frequency equal to 1.90 Hz. In this figure, the frequency components 1.50, 3.40, 0.40, and 1.90 Hz of the spectrum correspond to the first, second, third, and fourth terms of Equation 33, respectively.

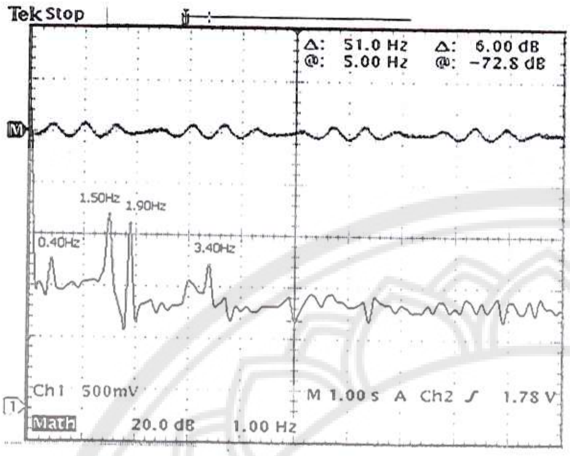


Figure 10. The experimental result of the forced pendulum with the frequency of the external force equal to 1.90 Hz. The upper trace is the complete output response of the motion of the pendulum in time domain. The lower trace shows the spectrum of the upper trace signal.

Conclusions

From the experimental and analytical results corresponding to each other, we may conclude that the two occurring new frequency components are an outcome of an oscillating system driven by external force. The analytical result using the multi-time variable technique can predict four frequency components in the spectrum of the experimental result more precisely.

REFERENCES

Arya AP (1997). Introduction to Classical Mechanics. Benjamin Cummings, United States of America. p. 86.

Dimarogonas A (1996). Vibration for Engineers. Prentice-Hall, United States of America. p. 131.

James, ML, Smith, GM, Wolford JC, Whaley PW (1989). Vibration of Mechanical and Structural Systems with Microcomputer Applications. Harper and Row, United States of America. p. 137.

Keller FJ, Gettys WE, Skove MJ (1993). Physics Classical and Modern. MrGraw-Hill, United States of America. p. 384.

Maneechukate T, Koseeyaporn J, Wardkein P, Koseeyaporn P (2008). Wide-band amplitude control of the second-order oscillator circuit. Int. J. Electron. Commun. (AEÜ) 62:666-673.

Marion JB (1970). Classical Dynamics of Particles and systems. Harcourt Brace Jovanovich, San Diego.

Mickens RE (1996). Oscillations in Planar Dynamics Systems. World Scientific, Singapore.

Taylor JR (2005). Classical mechanics. University Science Books, United States of America. P. 184.

Zill DG (2005). A First Course in Differential Equations with Modeling Applications. Brooks/Cole Cengage Learning, United States of America. P. 190.

APPENDIX

Appendix A: The general solution of homogeneous equation

The characteristic equation of Equation 5 is

$$r^2 + \frac{c_1}{c_2} r + \frac{c_0}{c_2} = 0.$$

Then, we get the characteristic roots:

$$r_1, r_2 = -\frac{c_1}{2c_2} \pm \sqrt{\left(\frac{c_1}{2c_2}\right)^2 - \frac{c_0}{c_2}}.$$

Here, we consider the solution for the damped harmonic oscillator, that is, $\left(\frac{c_1}{2c_2}\right)^2 - \frac{c_0}{c_2} < 0$; for this case, the roots will be complex:

$$r_1, r_2 = -\frac{c_1}{2c_2} \pm j\sqrt{\frac{c_0}{c_2} - \left(\frac{c_1}{2c_2}\right)^2}.$$

and we can write the solution, natural response, as:

$$y_n(t) = D_1 e^{\left(-\frac{c_1}{2c_2} + j\sqrt{\frac{c_0}{c_2} - \left(\frac{c_1}{2c_2}\right)^2}\right)t} + D_2 e^{\left(-\frac{c_1}{2c_2} - j\sqrt{\frac{c_0}{c_2} - \left(\frac{c_1}{2c_2}\right)^2}\right)t}. \quad \text{(A1)}$$

If we define $\omega_n = \sqrt{\frac{c_0}{c_2} - \left(\frac{c_1}{2c_2}\right)^2}$ as the natural angular frequency of the damped oscillator and $\omega_0 = \frac{c_0}{c_2}$ as that of the undamped oscillator, then Equation A1 can be written as:

$$y_n(t) = D_1 e^{\left(-\frac{c_1}{2c_2} + j\omega_n\right)t} + D_2 e^{\left(-\frac{c_1}{2c_2} - j\omega_n\right)t}. \quad \text{(A2)}$$

From Euler's equation, $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$, we can rewritten Equation A2 as:

$$y_n(t) = e^{-\frac{c_1}{2c_2}t}\left[D_1(\cos\omega_n t + j\sin\omega_n t) + D_2(\cos\omega_n t - j\sin\omega_n t)\right]. \quad \text{(A3)}$$

We know, of course, that $y_n(t)$ is real, whereas the two exponentials in Equation A3 are complex. Therefore, the coefficient D_2 must be the complex conjugate of D_1. Let us assume that

$$D_1 = \frac{a - jb}{2}$$

then

$$D_2 = \frac{a + jb}{2}$$

where a and b are real constants. We then substitute the complex constant D_1 and D_2 into Equation A3 obtaining the solution as Equation 6. That is,

$$y_n(t) = e^{-\frac{c_1}{2c_2}t}\left(a\cos\omega_n t + b\sin\omega_n t\right).$$

Appendix B: The frequency response of the system

From Equation 9, we can find the frequency response of the system via Fourier transform pair as follows:

$$F\left[\frac{d^n y_f(\tau)}{d\tau}\right] = (j\omega)^n Y_f(j\omega),$$
$$F\left[y_f(\tau)\right] = Y_f(j\omega)$$

and

$$F[x(\tau)] = X(j\omega).$$

Here, we have defined $x(\tau)$ as input of the system. Substituting these Fourier transform pair into Equation 9, we get:

$$(j\omega)^2 Y_f(j\omega) + \frac{c_1}{c_2}(j\omega)Y_f(j\omega) + \frac{c_0}{c_2}Y_f(j\omega) = \frac{X(j\omega)}{c_2},$$

or simply

$$\left[(j\omega)^2 + \frac{c_1}{c_2}(j\omega) + \frac{c_0}{c_2}\right]Y_f(j\omega) = \frac{X(j\omega)}{c_2},$$

Then, the ratio between $Y_f(j\omega)$ and $X(j\omega)$, the frequency response of system, is:

$$H(j\omega) = \frac{1}{c_2\left[\left(\frac{c_0}{c_2} - \omega^2\right) + \frac{c_1}{c_2}\omega j\right]},$$

where the system's magnitude response is

$$|H(j\omega)| = \frac{1}{c_2\sqrt{\left(\frac{c_0}{c_2} - \omega^2\right)^2 + \left(\frac{c_1}{c_2}\omega\right)^2}},$$

and phase response is

$$\angle H(j\omega) = -\tan^{-1}\left(\frac{(c_1/c_2)\omega}{\frac{c_0}{c_2} - \omega^2}\right).$$

Since the frequency of $x(\tau)$ is equal to ω_f, the system amplitude gain at frequency ω_f, $|H(j\omega_f)|$, is:

$$|H(j\omega_f)| = \frac{1}{c_2\sqrt{\left(\frac{c_0}{c_2} - \omega_f^2\right)^2 + \left(\frac{c_1}{c_2}\omega_f^2\right)^2}},$$

and the phase shift of the system at frequency ω_f is:

$$\angle H(j\omega_f) = -\tan^{-1}\left(\frac{\frac{c_1}{c_2}\omega_f}{\frac{c_0}{c_2} - \omega_f^2}\right).$$

Here, the input of the system is $x(\tau) = X_0\cos(\omega_f\tau)$, the sinusoidal input, where X_0 is its amplitude. The steady-state response is indicated by:

$$y_f(\tau) = X_0|H(j\omega_f)|\cos(\omega_f\tau + \angle H(j\omega_f)).$$

Therefore, it is clear that the sinusoidal steady-state response, or the forced response, has the same frequency as the input, whereas its amplitude and phase angle are determined by the system's magnitude response $|H(j\omega)|$ and phase response $\angle H(j\omega)$ at any given frequency ω_f.

Appendix C: The solution of second order differential equation solved using the conventional method

In conventional method, the linear second order differential equation describing the oscillation output of damped oscillator, $y(t)$, forced by an external forcing $x(t)$ can be written as:

$$c_2\frac{d^2y(t)}{dt} + c_1\frac{dy(t)}{dt} + c_0y(t) = x(t).$$

The solution of the aforementiond equation consists of the natural response $y_n(t)$ and the forced response $y_f(t)$, that is,

$$y(t) = y_n(t) + y_f(t). \quad \text{(C1)}$$

The natural response of this system is:

$$y_n(t) = e^{-\frac{c_1}{2c_2}t}(a\cos\omega_n t + b\sin\omega_n t),$$

or

$$y_n(t) = Ae^{-\frac{c_1}{2c_2}t}(\cos\omega_n t - \phi),$$

where $A = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$.

If $b = 0$ then $\phi = 0$, thus:

$$y_n(t) = ae^{-\frac{c_1}{2c_2}t}\cos\omega_n t. \quad \text{(C2)}$$

Substituting Equation C2 into Equation C1 gives:

$$y(t) = ae^{-\frac{c_1}{2c_2}t}\cos\omega_n t + y_f(t). \quad \text{(C3)}$$

Upon finding a from the initial position $y(0) = y_0$. We set $t = 0$ in Equation C3 yielding:

$$a = y_0 - y_f(0).$$

Thus, Equation C3 can be rewritten as:

$$y(t) = [y_0 - y_f(0)]e^{-\frac{c_1}{2c_2}t}\cos\omega_n t + y_f(t).$$