CHAPTER II

PRELIMINARIES

In this chapter, we give some precise definitions, notations and basic results which will be used in our study. Moreover, we will show some necessary propositions that we usually refer to and we show some relationships among subsemigroups of $T(X)$ in the rest of this chapter.

2.1 Elementary concepts

Definition 2.1.1. Let S be a semigroup and $x \in S$. Then

(1) x is a *regular element* if $x = xyx$ for some $y \in S$.

(2) x is a *left regular element* if $x = yx^2$ for some $y \in S$.

(3) x is a *right regular element* if $x = x^2y$ for some $y \in S$.

(4) x is a *completely regular element* if $x = xyx$ and $xy = yx$ for some $y \in S$.

And we denote

$$\begin{aligned} Reg(S) &= \{x \in S : x \text{ is a regular element of } S\}, \\ RReg(S) &= \{x \in S : x \text{ is a right regular element of } S\}, \\ LReg(S) &= \{x \in S : x \text{ is a left regular element of } S\}, \\ CReg(S) &= \{x \in S : x \text{ is a completely regular element of } S\}. \end{aligned}$$

We call S a *regular semigroup* if all its elements are regular, that is $Reg(S) = S$.

Theorem 2.1.2. *For a semigroup S, $CReg(S) \subseteq LReg(S) \cap RReg(S) \subseteq Reg(S)$.*

Proof. Let S be a semigroup. Clearly, $CReg(S) \subseteq LReg(S) \cap RReg(S)$. To show that $LReg(S) \cap RReg(S) \subseteq Reg(S)$, let $x \in LReg(S) \cap RReg(S)$. Then $x = yx^2$ and $x = x^2z$ for some $y, z \in S$. Since

$$x = yx^2 = (yx)x = y(x^2z)x = (yx^2)zx = xzx,$$

we have $x \in Reg(S)$ as required. □

In general, left regularity and right regularity are not a generalization of regularity. If S is a commutative semigroup, then $Reg(S), LReg(S), RReg(S)$ and $CReg(S)$ coincide.

Definition 2.1.3. Let X be a nonempty set and σ a relation on X. Then

(1) σ is *reflexive* if $(x, x) \in \sigma$ for all $x \in X$.

(2) σ is *symmetric* if $(x, y) \in \sigma$ implies $(y, x) \in \sigma$ for all $x, y \in X$.

(3) σ is *anti-symmetric* if $(x, y), (y, x) \in \sigma$ imply $x = y$ for all $x, y \in X$.

(4) σ is *transitive* if $(x, y), (y, z) \in \sigma$ imply $(x, z) \in \sigma$ for all $x, y, z \in X$.

If σ satisfies $(1), (2)$ and (3), then σ is a *partially order* on X. And we call σ an *equivalence relation* on X if σ satisfies $(1), (2)$ and (4).

For a nonempty set X and a partially order $\leq$ on X, we say that $(X, \leq)$ is a *partially ordered set.* For $x, y \in X$, we denote $(x, y) \in \leq$ by $x \leq y$ and write $x < y$ if $x \leq y$ and $x \neq y$.

Let X be a nonempty set and E an equivalence relation on X, we denote

$$X/E = \{\{x \in X : (x, a) \in E\} : a \in X\}$$

and call $A \in X/E$ an *equivalence class* or *E-class.*

Definition 2.1.4. Let $(X, \leq)$ be a partially ordered set. For $x, y \in X$, x and y are said to be *comparable* if $x \leq y$ or $y \leq x$. A nonempty subset C of X is called a *chain* if all elements $x, y \in C$ are comparable. In particular, if X is a chain, we say that $(X, \leq)$ is a *totally ordered set.*

Let $x, y \in X$ be such that $x < y$, if $x \leq z$ and $z \leq y$ imply $x = z$ or $z = y$, for all $z \in X$, then we call x an *upper cover for y* and y is a *lower cover for x.*

Definition 2.1.5. Let $(X, \leq)$ be a partially ordered set and $x \in X$. Then

(1) x is a *maximal element* of $(X, \leq)$ if for all $y \in X$, $x \leq y$ implies $x = y$.

(2) x is a *minimal element* of $(X, \leq)$ if for all $y \in X$, $y \leq x$ implies $y = x$.

(3) x is *isolated* if it is incomparable with every element in X except itself.

Definition 2.1.6. Let π be a collection of nonempty subsets of X. We say that π is a *partition* of X if π satisfies the following conditions.

(1) $\cup\pi = X$ and

(2) for every $A, B \in \pi$, $A \cap B \neq \emptyset$ implies $A = B$.

Theorem 2.1.7. *Let E be arbitrary equivalence relation on a nonempty set X. Then X/E is a partition of X.*

Theorem 2.1.8. *Let π be a partition of a nonempty set X. Then $E = \bigcup_{A \in \pi} (A \times A)$ is an equivalence relation on X and $X/E = \pi$.*

Let A, B be nonempty subsets of a semigroup S and $s \in S$, we denote $AB = \{ab : a \in A \text{ and } b \in B\}$ and $sA = \{sa : a \in A\}$. If S is a semigroup with identity, then we means $S^1 = S$. If S has no an identity element, we let $S^1 = S \cup \{1\}$ and define the binary operation on S^1 by

$$1 \cdot s = s \cdot 1 = s \text{ for all } s \in S^1 \text{ and } a \cdot b = ab \text{ for all } a, b \in S.$$

An element x in a semigroup S is called an *idempotent element* if $x = x^2$. We denote the set of all idempotent elements of S by $E(S)$.

Definition 2.1.9. [15] Let S be a semigroup and $a, b \in S$. We say that

(1) $(a, b) \in \mathcal{L}$ if $S^1a = S^1b$.

(2) $(a, b) \in \mathcal{R}$ if $aS^1 = bS^1$.

(3) $(a,b) \in \mathcal{J}$ if $S^1aS^1 = S^1bS^1$.

We then have $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ are equivalence relations on S. Let $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. Since $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$, we have $\mathcal{D}$ and $\mathcal{H}$ are also equivalence relations on S. Five equivalence relations on S are called *Green's relations.*

Mitsch [11] defined the *natural partial order* on any semigroup S as follows : for $a, b \in S$,

$$a \leq b \text{ if and only if } a = xb = by, a = ay \text{ for some } x, y \in S^1.$$

This order coincides with the natural partial order for a regular semigroup which is the following : for $a, b \in S$,

$$a \leq b \text{ if and only if } a = eb = bf \text{ for some } e, f \in E(S)$$

where $E(S)$ is the set of all idempotents of S.

2.2 Subsemigroups of full transformation semigroups

Let S be a semigroup. A nonempty subset T of S is called a *subsemigroup* of S if $xy \in T$ for all $x, y \in T$. Let X be a nonempty set and let $T(X)$ denote the semigroup of full transformations from X into itself under composition of mappings.

For a partially ordered set $(X, \leq)$, let E be an equivalence relation on X. The following subsemigroups of $T(X)$ are considered as in [14, 7, 1, 4, 3] respectively, defined by

$$\begin{aligned}
\mathcal{O}(X) &= \{\alpha \in T(X) : \forall x, y \in X, x \leq y \text{ implies } x\alpha \leq y\alpha\}, \\
T_{RE}(X) &= \{\alpha \in T(X) : \forall x \in X, x\alpha \leq x\}, \\
T_E(X) &= \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \text{ implies } (x\alpha, y\alpha) \in E\}, \\
T_{E^*}(X) &= \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \text{ if and only if } (x\alpha, y\alpha) \in E\},
\end{aligned}$$

$$EOP(X) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in E \text{ and } x \leq y \text{ imply } (x\alpha, y\alpha) \in E \text{ and } x\alpha \leq y\alpha\}.$$

All subsemigroups clearly contain i_X where i_X is the identity map on X. Now, we define a new subsemigroup of $T(X)$ by

$$T_{SE}(X) = \{\alpha \in T(X) : \forall x \in X, (x, x\alpha) \in E\}.$$

We call $T_{SE}(X)$ the *self-E-preserving transformation semigroup on* X.

Next, we will briefly recall some characterizations for above semigroups and introduce some notations that will be used in the sequel.

Theorem 2.2.1. *Let E be an equivalence relation on a nonempty set X. Then*

$$T_{SE}(X) \subseteq T_{E^*}(X) \subseteq T_E(X).$$

Proof. Clearly, $T_{E^*}(X)$ is a subsemigroup of $T_E(X)$. Let $\alpha \in T_{SE}(X)$. To show $\alpha \in T_{E^*}(X)$, let $x, y \in X$ be such that $(x, y) \in E$. Since $\alpha \in T_{SE}(X)$, $(x, x\alpha), (y, y\alpha) \in E$. It follows from E is symmetric that $(x\alpha, x) \in E$. We conclude that $(x\alpha, y\alpha) \in E$ by the transitivity of E. Conversely, suppose that $(x\alpha, y\alpha) \in E$ for some $x, y \in X$. We note that $(x, x\alpha), (y, y\alpha) \in E$. Since E is symmetric, $(y\alpha, y) \in E$. By the transitivity of E, we then have $(x, y) \in E$. Therefore $\alpha \in T_{E^*}(X)$. □

Definition 2.2.2. [1] For a nonempty set X and $\alpha \in T(X)$, $\pi(\alpha)$ denotes the decomposition of X induced by α, namely

$$\pi(\alpha) = \{y\alpha^{-1} : y \in X\alpha\}$$

and define $\alpha_* : \pi(\alpha) \to X\alpha$ by

$$P\alpha_* = x\alpha \text{ for each } P \in \pi(\alpha) \text{ and } x \in P.$$

Then $\pi(\alpha)$ is a partition of X and α_* is a bijection.

For a nonempty subset A of X and $\alpha \in T(X)$, we write

$$\pi_A(\alpha) = \{P \in \pi(\alpha) : P \cap A \neq \emptyset\}.$$

Theorem 2.2.3. *Let A be a nonempty subset of X and $\alpha \in T(X)$, we denote*

$$\pi(A, \alpha) = \{P \cap A : P \in \pi_A(\alpha)\}.$$

Then $\pi(A, \alpha)$ is a partition of A induced by α.

Proof. Clearly, $\cup\pi(A, \alpha) \subseteq A$. Let $x \in A$, we note by $\pi(\alpha)$ is a partition of X that $x \in P$ for some $P \in \pi(\alpha)$. Thus $P \cap A \neq \emptyset$. Hence $P \in \pi_A(\alpha)$ and $x \in \cup\pi(A, \alpha)$. Next, let $P', Q' \in \pi(A, \alpha)$ be such that $P' \cap Q' \neq \emptyset$. Then there exists $x \in P' \cap Q'$ for some $x \in X$. Since $P', Q' \in \pi(A, \alpha)$, we conclude that $P' = P \cap A$ and $Q' = Q \cap A$ for some $P, Q \in \pi_A(\alpha)$. This implies that $x \in P \cap Q$. It follows from $\pi(\alpha)$ is a partition of X that $P = Q$ and hence $P' = Q'$. Therefore $\pi(A, \alpha)$ is a partition of A as required. □

Lemma 2.2.4. [1] *Let E be an equivalence relation on X and $\alpha \in T_E(X)$. Then for each $B \in X/E$, there exists $B' \in X/E$ such that $B\alpha \subseteq B'$. Consequently, for each $A \in X/E$, the set $A\alpha^{-1}$ is either $\emptyset$ or a union of some E-classes.*

Proposition 2.2.5. *Let E be an equivalence relation on X and $\alpha \in T_{E^*}(X)$. Then the following statements hold.*

(1) *If $P \in \pi(\alpha)$, then there exists $A \in X/E$ such that $P \subseteq A$.*

(2) *For every $P \in \pi(\alpha)$ and $A \in X/E$, if $P \cap A \neq \emptyset$, then $P \subseteq A$.*

(3) *For every $A \in X/E$, $\pi_A(\alpha)$ is a partition of A.*

Proof. (1) Let $P \in \pi(\alpha)$ and $x \in P$. Since X/E is a partition of X, there exists $A \in X/E$ such that $x \in A$. For each $p \in P$, we have $x\alpha = p\alpha$ which implies that $(x\alpha, p\alpha) \in E$. Since $\alpha \in T_{E^*}(X)$, we deduce that $(x, p) \in E$. Hence $p \in A$. This shows that $P \subseteq A$.

(2) Let $P \in \pi(\alpha)$ and $A \in X/E$ be such that $P \cap A \neq \emptyset$. By (1), there exists $B \in X/E$ such that $P \subseteq B$. This implies that $A \cap B \neq \emptyset$. Since X/E is a partition of

X, we have that $A = B$.

(3) Let $A \in X/E$. It follows from (2) that $\cup\pi_A(\alpha) \subseteq A$. Let $x \in A$. Since $\pi(\alpha)$ is a partition of X, there exists $P \in \pi(\alpha)$ such that $x \in P$, so $P \cap A \neq \emptyset$. Hence $P \in \pi_A(\alpha)$ and therefore $A \subseteq \cup\pi_A(\alpha)$. □

Proposition 2.2.6. *Let E be an equivalence relation on X and $\alpha \in T_{E^*}(X)$. Then the following statements hold.*

(1) *For every $A \in X/E$, there exists $B \in X/E$ such that $A\alpha \subseteq B$.*

(2) *For every $A, B \in X/E$, if $A\alpha \subseteq B$, then $B\alpha^{-1} = A$.*

Proof. (1) Since $T_{E^*}(X)$ is a subsemigroup of $T_E(X)$, by Lemma 2.2.4 we have (1) is true.

(2) Suppose that $A\alpha \subseteq B$ where $A, B \in X/E$. Clearly, $A \subseteq B\alpha^{-1}$. Let $b \in B\alpha^{-1}$. Then $b\alpha \in B$. Let $a \in A$. By assumption, $a\alpha \in B$, hence $(a\alpha, b\alpha) \in E$. Since $\alpha \in T_{E^*}(X)$, we get $(a, b) \in E$. That is, $b \in A$ and then $B\alpha^{-1} \subseteq A$. □

Definition 2.2.7. For a nonempty set X, let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of X. We say that $\mathcal{B}$ is a *refinement* of $\mathcal{A}$ or $\mathcal{B}$ *refines* $\mathcal{A}$ if $\cup\mathcal{B} = \cup\mathcal{A}$ and for each $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$ such that $B \subseteq A$.

Theorem 2.2.8. *Let $\alpha, \beta \in T(X)$. Then $\alpha \in \beta T(X)$ if and only if $\pi(\beta)$ refines $\pi(\alpha)$.*

Proof. Assume that $\alpha \in \beta T(X)$. Then $\alpha = \beta\delta$ for some $\delta \in T(X)$. We note $\cup\pi(\alpha) = \cup\pi(\beta)$. Let $P \in \pi(\beta)$. Hence $P = y\beta^{-1}$ where $y \in X\beta$. Thus $P\alpha = P\beta\delta = \{y\delta\}$ which implies that $P \subseteq y\delta\alpha^{-1}$. Since $y\delta\alpha^{-1} \in \pi(\alpha)$, we conclude that $\pi(\beta)$ refines $\pi(\alpha)$.

Conversely, assume that $\pi(\beta)$ refines $\pi(\alpha)$. For each $x \in X\beta$, there exists a unique $P_x \in \pi(\beta)$ such that $P_x = x\beta^{-1}$. By assumption, there exists a unique $Q_x \in \pi(\alpha)$ such that $P_x \subseteq Q_x$. Define $\delta : X \to X$ by

$$x\delta = \begin{cases} Q_x\alpha_*, & \text{if } x \in X\beta; \\ x, & \text{otherwise.} \end{cases}$$

By $\pi(\beta)$ is a partition of X, we have δ is well-defined. Let $x \in X$, hence $x\beta \in X\beta$. By the definition of δ,

$$x\beta\delta = Q_{x\beta}\alpha_* = x\alpha$$

since $x \in P_{x\beta} \subseteq Q_{x\beta}$ where $P_{x\beta} \in \pi(\beta)$ and $Q_{x\beta} \in \pi(\alpha)$. Therefore $\alpha = \beta\delta$, so the theorem is thereby proved. □

From Theorem 2.1.7, Definition 2.2.2 and Proposition 2.2.5(1), we conclude that $\pi(\alpha)$ refines X/E for all $\alpha \in T_{E^*}(X)$. Since $T_{SE}(X)$ is a subsemigroup of $T_{E^*}(X)$, we have the following result immediately.

Proposition 2.2.9. *Let E be an equivalence relation on X and $\alpha \in T_{SE}(X)$. Then the following statements hold.*

(1) *$\pi(\alpha)$ is a refinement of X/E.*

(2) *$A = \cup\pi_A(\alpha)$ for all $A \in X/E$.*

(3) *$A\alpha \subseteq A$ for all $A \in X/E$.*

The next theorem is easy to verify.

Theorem 2.2.10. *For a partially ordered set $(X, \leq)$, let π be a collection of nonempty subsets of X. We define a relation $\preceq$ on π by*

$$P \preceq Q \text{ if and only if } P = Q \text{ or } x \leq y \text{ for all } x \in P, y \in Q.$$

Then $(\pi, \preceq)$ is a partially ordered set.

Proposition 2.2.11. *Let $(X, \leq)$ be a totally ordered set and $\alpha \in \mathcal{O}(X)$. Then $(\pi(\alpha), \preceq)$ is a totally ordered set and (for any $P, Q \in \pi(\alpha), P \preceq Q$ if and only if $P\alpha_* \leq Q\alpha_*$).*

Proof. Since $\pi(\alpha)$ is a partition of X, $(\pi(\alpha), \preceq)$ is a partially ordered set. Let $P, Q \in \pi(\alpha)$ be distinct. Fix $p \in P$ and $q \in Q$. Since $(X, \leq)$ is a totally ordered set, we assume that $p \leq q$. Claim that $P \preceq Q$, suppose not. Then there exist $x \in P$ and $y \in Q$ such that $x \not\leq y$. We note by $(X, \leq)$ is a totally ordered set that $y < x$. It follows from $\alpha \in \mathcal{O}(X)$ that

$$y\alpha \leq x\alpha = P\alpha_* = p\alpha \leq q\alpha = Q\alpha_* = y\alpha.$$

Hence $P\alpha_* = Q\alpha_*$ which is a contradiction with P and Q are distinct. Thus $x \leq y$ for all $x \in P$ and $y \in Q$. This means that $P \preceq Q$ and then $(\pi(\alpha), \preceq)$ is a totally ordered set. Let $P, Q \in \pi(\alpha)$ be such that $P \preceq Q$. Suppose that $P \neq Q$. Let $x \in P$ and $y \in Q$, then by $P \preceq Q$ we get $x \leq y$. Thus $P\alpha_* = x\alpha \leq y\alpha = Q\alpha_*$. On the other hand, let $P, Q \in \pi(\alpha)$ be such that $P\alpha_* \leq Q\alpha_*$. If $P = Q$, then the proposition is already proved. Suppose that $P \neq Q$. Since P and Q are distinct, $P\alpha_* < Q\alpha_*$. Assume that $Q \preceq P$. Let $x \in P$ and $y \in Q$. We note by assumption that $y \leq x$. It follows that $Q\alpha_* = y\alpha \leq x\alpha = P\alpha_* < Q\alpha_*$ which is a contradiction. Hence $Q \not\preceq P$. By $(\pi(\alpha), \preceq)$ is a totally ordered set, we have $P \preceq Q$. □

Proposition 2.2.12. *Let $(X, \leq)$ be a totally ordered set and E an equivalence relation on X. For $A \in X/E$ and $\alpha \in EOP(X)$,*

(1) *there exists a unique $B \in X/E$ such that $A\alpha \subseteq B$ and*

(2) *$(\pi(A, \alpha), \preceq)$ is a totally ordered set.*

Proof. Let $A \in X/E$ and $x \in A$. Since X/E is a partition of X, $x\alpha \in B$ for a unique $B \in X/E$. Claim that $A\alpha \subseteq B$, let $y \in A$. We note by $(X, \leq)$ is a totally ordered set that $x \leq y$ or $y \leq x$. It follows from $\alpha \in EOP(X)$ that $(x\alpha, y\alpha) \in E$ which implies that $y\alpha \in B$. Thus (1) holds.

It is clearly seen that $\pi(A, \alpha)$ is a partition of A and $(\pi(A, \alpha), \preceq)$ is a partially ordered set. To prove $(\pi(A, \alpha), \preceq)$ is a totally ordered set, let $P, Q \in \pi(A, \alpha)$ be such that $Q \not\preceq P$. Then there exist $p \in P$ and $q \in Q$ such that $q \not\leq p$. Since $(X, \leq)$ is a totally ordered set, $p < q$. By the definition of $\pi(A, \alpha)$, we have that $P = P' \cap A$ and $Q = Q' \cap A$ for some $P', Q' \in \pi_A(\alpha)$. Since $P \neq Q$, $P' \neq Q'$ and hence $P'\alpha_* \neq Q'\alpha_*$. Claim that $P \preceq Q$. Suppose that $y < x$ for some $x \in P$ and $y \in Q$. We note here that $p, q, x, y \in A$.

Since $p \leq q$ and $(p,q) \in E$, we deduce that $p\alpha \leq q\alpha$. Since $P'\alpha_* \neq Q'\alpha_*$, $p \in P'$ and $q \in Q'$, $P'\alpha_* = p\alpha < q\alpha = Q'\alpha_*$. Similarly, we conclude that $Q'\alpha_* = y\alpha \leq x\alpha = P'\alpha_*$. This implies that

$$P'\alpha_* < Q'\alpha_* \leq P'\alpha_*$$

which is a contradiction. Hence $x \leq y$ for all $x \in P$ and $y \in Q$. So we have the claim. Therefore $(\pi(A,\alpha), \preceq)$ is a totally ordered set as desired. □

2.3 Relationships between subsemigroups of full transformation semigroups

In this section, the set X under consideration is a totally ordered set with E an arbitrary equivalence relation on X. We define subsets of $T(X)$ by

$$\begin{aligned} T_{ER}(X) &= T_E(X) \cap T_{RE}(X), \\ \mathcal{O}_E(X) &= T_E(X) \cap \mathcal{O}(X), \\ T_{SER}(X) &= T_{SE}(X) \cap T_{RE}(X), \\ T_{SEO}(X) &= T_{SE}(X) \cap \mathcal{O}(X), \\ T_{OR}(X) &= \mathcal{O}(X) \cap T_{RE}(X). \end{aligned}$$

It is known that the intersection of subsemigroups of a semigroup S is either an empty set or itself a subsemigroup of S. Then $T_{ER}(X), \mathcal{O}_E(X), T_{SER}(X), T_{SEO}(X)$ and $T_{OR}(X)$ are subsemigroups of $T(X)$ containing i_X.

We characterize the conditions under which some of above subsemigroups of $T(X)$ are equal.

Theorem 2.3.1. *$T_{ER}(X) = T_{RE}(X)$ if and only if for every $A, B \in X/E$ such that $A \neq B$, if there exist $a \in A, b \in B$ such that $a < b$, then $|B| = 1$.*

Proof. Assume that there exist $A, B \in X/E$ such that $A \neq B$, $a < b$ for some $a \in A, b \in B$ and $|B| > 1$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a, & \text{if } x = b; \\ x, & \text{otherwise.} \end{cases}$$

Clearly, $x\alpha \leq x$ for all $x \in X$ and then $\alpha \in T_{RE}(X)$. Let $c \in B \backslash \{b\}$. Thus $(b, c) \in E$. Since $(b\alpha, c\alpha) = (a, c) \notin E$, $\alpha \notin T_E(X)$. Therefore $T_{ER}(X) \neq T_{RE}(X)$.

Conversely, suppose that for every $A, B \in X/E$ such that $A \neq B$, if there exist $a \in A, b \in B$ such that $a < b$, then $|B| = 1$. To show that $T_{ER}(X) = T_{RE}(X)$, let $\alpha \in T_{RE}(X)$ and let $x, y \in X$ be such that $(x, y) \in E$. Hence $x, y \in B$ for some $B \in X/E$. If $x = y$, then $(x\alpha, y\alpha) \in E$. Suppose that $x \neq y$. Then $|B| > 1$. Since $\alpha \in T_{RE}(X)$, $x\alpha \leq x$. It follows by assumption that $x\alpha \in B$. Similarly, we have that $y\alpha \in B$. This means that $(x\alpha, y\alpha) \in E$. Therefore, $\alpha \in T_E(X)$ and hence $T_{ER}(X) = T_{RE}(X)$. □

Theorem 2.3.2. *$T_{ER}(X) = T_E(X)$ if and only if $|X| = 1$.*

Proof. Suppose that $|X| > 1$. Let $a, b \in X$ be such that $a \neq b$. Then there exist $A, B \in X/E$ such that $a \in A$ and $b \in B$. Suppose that $a < b$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x \in A; \\ x, & \text{otherwise.} \end{cases}$$

Let $x, y \in X$ be such that $(x, y) \in E$. Then

$$(x\alpha, y\alpha) = \begin{cases} (b, b) \in E, & \text{if } x, y \in A; \\ (x, y) \in E, & \text{otherwise} \end{cases}$$

which implies that $\alpha \in T_E(X)$. Since $a\alpha = b \not\leq a$, we deduce that $\alpha \notin T_{ER}(X)$. □

Theorem 2.3.3. *$T_{SER}(X) = T_{RE}(X)$ if and only if $E = X \times X$.*

Proof. Suppose that $E \neq X \times X$. Then there exist $a, b \in X$ such that $(a, b) \notin E$. Suppose that $a < b$ and define $\alpha : X \to X$ as given in Theorem 2.3.1. Then $\alpha \in T_{RE}(X)$. Since $(b, b\alpha) = (b, a) \notin E$, $\alpha \notin T_{SE}(X)$. Hence $T_{SER}(X) \neq T_{RE}(X)$.

Assume that $E = X \times X$. We have that $T_{SE}(X) = T(X)$, hence $T_{SER}(X) = T_{SE}(X) \cap T_{RE}(X) = T(X) \cap T_{RE}(X) = T_{RE}(X)$. □

Theorem 2.3.4. *$T_{SER}(X) = T_{SE}(X)$ if and only if $E = I_X$ where I_X is the identity relation on X.*

Proof. Suppose that $E \neq I_X$. Then there exist $a, b \in X$ such that $a \neq b$ and $(a, b) \in E$. We assume that $a < b$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x = a; \\ x, & \text{otherwise.} \end{cases}$$

For each $x \in X$,

$$(x, x\alpha) = \begin{cases} (a, b) \in E, & \text{if } x = a; \\ (x, x) \in E, & \text{otherwise,} \end{cases}$$

hence $\alpha \in T_{SE}(X)$. Since $a\alpha = b \not\leq a$, we conclude that $\alpha \notin T_{RE}(X)$. Therefore $T_{SER}(X) \neq T_{SE}(X)$.

Conversely, suppose that E is the identity relation on X. Let $\alpha \in T_{SE}(X)$ and $x \in X$. Then we have that $(x, x\alpha) \in E$. By assumption, $x\alpha = x$ which implies that $\alpha \in T_{RE}(X)$. This proves that $T_{SE}(X) \subseteq T_{RE}(X)$, hence $T_{SE}(X) = T_{SE}(X) \cap T_{RE}(X) = T_{SER}(X)$. □

Theorem 2.3.5. *$\mathcal{O}_E(X) = \mathcal{O}(X)$ if and only if $E = X \times X$ or $E = I_X$.*

Proof. Assume that $E \neq X \times X$ and $E \neq I_X$. Then there exist $A, B \in X/E$ such that $|A| > 1$ and $B \neq A$. Let $a, c \in A$ be such that $a < c$ and $b \in B$. Since E is an equivalence relation on X, we have that $b \in B \backslash A$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} \max\{b, c\}, & \text{if } c \leq x; \\ \min\{b, c\}, & \text{otherwise.} \end{cases}$$

To show that $\alpha \in \mathcal{O}(X)$, let $x, y \in X$ be such that $x \leq y$.

Case 1. $c \leq x \leq y$ or $x \leq y < c$. Then we get that $x\alpha = y\alpha$.

Case 2. $x < c \leq y$. Then we have that $x\alpha = \min\{b, c\} < \max\{b, c\} = y\alpha$.

From two cases, we deduce that $\alpha \in \mathcal{O}(X)$. We note here that $(a, c) \in E$ and $(a\alpha, c\alpha) = (\min\{b, c\}, \max\{b, c\}) \notin E$. Hence $\alpha \notin T_E(X)$. This proves that $\mathcal{O}_E(X) \neq \mathcal{O}(X)$ as required.

Conversely, suppose that $E = X \times X$ or $E = I_X$. We then have that $T_E(X) = T(X)$. Therefore $\mathcal{O}_E(X) = T_E(X) \cap \mathcal{O}(X) = T(X) \cap \mathcal{O}(X) = \mathcal{O}(X)$. □

Theorem 2.3.6. *$\mathcal{O}_E(X) = T_E(X)$ if and only if $|X| = 1$.*

Proof. Assume that $|X| > 1$. Let $a, b \in X$ be such that $a < b$. Then $a \in A$ for some $A \in X/E$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x \in A \text{ and } x \neq b; \\ a, & \text{otherwise.} \end{cases}$$

To show that $\alpha \in T_E(X)$, let $x, y \in X$ be such that $(x, y) \in E$. Then $x, y \in B$ for some $B \in X/E$.

Case 1. $B \neq A$. This implies that $x, y \notin A$. Hence $(x\alpha, y\alpha) = (a, a) \in E$.

Case 2. $B = A$. If $b \in A$, then $x\alpha, y\alpha \in \{a, b\} \subseteq A$. Thus $(x\alpha, y\alpha) \in E$. If $b \notin A$, then $x \neq b$ and $y \neq b$ which implies that $(x\alpha, y\alpha) = (b, b) \in E$.

From two cases, we then have $\alpha \in T_E(X)$. Since $a < b$ and $a\alpha = b \not\leq a = b\alpha$, we get $\alpha \notin \mathcal{O}(X)$. Hence $\mathcal{O}_E(X) \neq T_E(X)$. □

Theorem 2.3.7. *$T_{SEO}(X) = T_{SE}(X)$ if and only if $E = I_X$.*

Proof. Assume that $E \neq I_X$. Then there exist $a, b \in X$ such that $(a, b) \in E$ and $a \neq b$. We may assume that $a < b$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x = a; \\ a, & \text{if } x = b; \\ x, & \text{otherwise.} \end{cases}$$

It is easy to verify that $\alpha \in T_{SE}(X)$. Since $a < b$ and $a\alpha > b\alpha$, $\alpha \notin \mathcal{O}(X)$. Therefore $T_{SEO}(X) \neq T_{SE}(X)$.

Conversely, suppose that $E = I_X$. Then we get $T_{SE}(X) = \{i_X\}$, it follows that $T_{SEO}(X) = T_{SE}(X) \cap \mathcal{O}(X) = \{i_X\} \cap \mathcal{O}(X) = \{i_X\} = T_{SE}(X)$. □

Theorem 2.3.8. *$T_{SEO}(X) = \mathcal{O}(X)$ if and only if $E = X \times X$.*

Proof. Suppose that $E \neq X \times X$. Then there exist $a, b \in X$ such that $(a, b) \notin E$. We may assume that $a < b$ and define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x \geq a; \\ a, & \text{otherwise.} \end{cases}$$

To show that $\alpha \in \mathcal{O}(X)$, let $x, y \in X$ be such that $x \leq y$.

Case 1. $a \leq x \leq y$ or $x \leq y < a$. Then we get $x\alpha = y\alpha$.

Case 2. $x < a \leq y$. Then we have that $x\alpha = a < b = y\alpha$.

From two cases, we deduce that $\alpha \in \mathcal{O}(X)$. Since $(a, a\alpha) = (a, b) \notin E$, $\alpha \notin T_{SE}(X)$. Hence $T_{SEO}(X) \neq \mathcal{O}(X)$.

Conversely, assume that $E = X \times X$. Thus $T_{SE}(X) = T(X)$. Hence $T_{SEO}(X) = T_{SE}(X) \cap \mathcal{O}(X) = T(X) \cap \mathcal{O}(X) = \mathcal{O}(X)$. □

Theorem 2.3.9. *$T_{OR}(X) = \mathcal{O}(X)$ if and only if $|X| = 1$.*

Proof. Assume that $|X| > 1$. Then there exist $a, b \in X$ such that $a < b$. Define $\alpha : X \to X$ as given in Theorem 2.3.8. Then $\alpha \in \mathcal{O}(X)$. Since $a\alpha = b > a$, $\alpha \notin T_{RE}(X)$. Hence $T_{OR}(X) \neq \mathcal{O}(X)$. □

Theorem 2.3.10. *$T_{OR}(X) = T_{RE}(X)$ if and only if $|X| \leq 2$.*

Proof. Suppose that $|X| > 2$. Let $a, b, c \in X$ be such that $a < b < c$. We define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a, & \text{if } x = c; \\ x, & \text{otherwise.} \end{cases}$$

Then $\alpha \in T_{RE}(X)$. We note that $b < c$ and $b\alpha = b \nleq a = c\alpha$. Thus $\alpha \notin \mathcal{O}(X)$ and so $T_{OR}(X) \neq T_{RE}(X)$.

Conversely, assume that $|X| \leq 2$. Let $\alpha \in T_{RE}(X)$. To show that $\alpha \in T_{OR}(X)$, let $x, y \in X$ be such that $x < y$. Since $\alpha \in T_{RE}(X)$ and by assumption, we then have $x\alpha = x$ (since $x\alpha \leq x$). It follows that $x\alpha \leq y\alpha$ (since $y\alpha = x$ or $y\alpha = y$), which implies that $\alpha \in \mathcal{O}(X)$. Therefore $T_{OR}(X) = T_{RE}(X)$. □

Finally, we study relationships among $EOP(X), T_E(X)$ and $\mathcal{O}(X)$ where $(X, \leq)$ is a partially ordered set and E is an equivalence relation on X. Clearly, $T_E(X) \cap \mathcal{O}(X) \subseteq EOP(X)$.

Proposition 2.3.11. *Let X be a partially ordered set and E an arbitrary equivalence relation on X. Then $EOP(X) = \mathcal{O}(X)$ if and only if $\cup\mathcal{K} \subseteq E$ where $\mathcal{K} = \{C \times C : C \text{ is a subchain of } X\}$.*

Proof. Suppose that there exists $(a, b) \in \cup\mathcal{K}$ such that $(a, b) \notin E$. Then $a \in A$ and $b \in B$ for some $A, B \in X/E$. Since $(a, b) \in \cup\mathcal{K}$, $a, b \in C$ for some subchain C of X. Define $\alpha \in T(X)$ by

$$x\alpha = \begin{cases} a, & \text{if } x \in B; \\ b, & \text{otherwise.} \end{cases}$$

Let $x, y \in X$ be such that $x \leq y$ and $(x, y) \in E$. By the definition of α, we deduce that

$$(x\alpha, y\alpha) = \begin{cases} (a, a) \in E, & \text{if } x, y \in B; \\ (b, b) \in E, & \text{otherwise.} \end{cases}$$

It follows that $\alpha \in EOP(X)$. Since a and b are comparable, we may assume that $a < b$. Then we have $a\alpha = b \nleq a = b\alpha$. Hence $\alpha \notin \mathcal{O}(X)$.

Conversely, assume that $\cup\mathcal{K} \subseteq E$ where $\mathcal{K} = \{C \times C : C \text{ is a subchain of } X\}$.

To show that $EOP(X) = \mathcal{O}(X)$, let $\alpha \in EOP(X)$ and $a, b \in X$ be such that $a \leq b$. Thus $a, b \in C$ for some subchain C of X. By assumption, we have $(a, b) \in E$. Since $\alpha \in EOP(X)$, $a\alpha \leq b\alpha$. Hence $\alpha \in \mathcal{O}(X)$. Next, let $\alpha \in \mathcal{O}(X)$ and $x, y \in X$ be such that $(x, y) \in E$ and $x \leq y$. Since $\alpha \in \mathcal{O}(X)$, $x\alpha \leq y\alpha$ which implies that $x\alpha, y\alpha \in C$ for some subchain C of X. It follows from assumption that $(x\alpha, y\alpha) \in E$. Hence $\alpha \in EOP(X)$. □

Proposition 2.3.12. *Let X be a partially ordered set and E an arbitrary equivalence relation on X. Then $EOP(X) = T(X)$ if and only if for every two distinct a, b in X, $(a, b) \in E$ implies that a and b are incomparable.*

Proof. Suppose that there exist distinct elements a, b in X such that $(a, b) \in E$ and a and b are comparable. We may assume that $a < b$. Define $\beta \in T(X)$ by

$$x\beta = \begin{cases} a, & \text{if } x = b; \\ b, & \text{otherwise.} \end{cases}$$

By the definition of β, we then have $a\beta = b \not\leq a = b\beta$. This means that $\beta \notin EOP(X)$.

Conversely, assume that for every two distinct a, b in X, $(a, b) \in E$ implies that a and b are incomparable. Let $\alpha \in T(X)$ and $x, y \in X$ be such that $(x, y) \in E$ and $x \leq y$. We deduce that $x = y$ which implies that $(x\alpha, y\alpha) \in E$ and $x\alpha \leq y\alpha$. Therefore $\alpha \in EOP(X)$. □

Corollary 2.3.13. *Let X be a partially ordered set and E an arbitrary equivalence relation on X.*

(1) *If $E = X \times X$, then $EOP(X) = \mathcal{O}(X)$ and $T_E(X) = T(X)$.*

(2) *If $E = I_X$, then $T_E(X) = EOP(X) = T(X)$.*

Theorem 2.3.14. *Let X be a partially ordered set and E an arbitrary equivalence relation on X. If $EOP(X) \subseteq T_E(X)$, then*

(1) *$E = X \times X$ or*

(2) *for every $A \in X/E$ and arbitrary partition $\{P, Q\}$ of A, there exist $x \in P, y \in Q$ such that x and y are comparable.*

Proof. Suppose that $E \neq X \times X$ and (2) is not true. Then there exist $A \in X/E$ and a partition $\{P, Q\}$ of A such that a and b are incomparable for all $a \in P, b \in Q$. Since $E \neq X \times X$, choose $B \in X/E$ such that $B \neq A$ and fix $b \in B$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} b, & \text{if } x \in P; \\ x, & \text{otherwise.} \end{cases}$$

To show that $\alpha \in EOP(X)$, let $x, y \in X$ be such that $(x, y) \in E$ and $x \leq y$. Hence $x, y \in D$ for some $D \in X/E$.

Case 1. $D \neq A$. Then $x, y \notin P$. By the definition of α, $x\alpha = x$ and $y\alpha = y$. Hence $(x\alpha, y\alpha) \in E$ and $x\alpha \leq y\alpha$.

Case 2. $D = A$. Since $x \leq y$ and $\{P, Q\}$ is a partition of A, either $x, y \in P$ or $x, y \in Q$. This implies that $(x\alpha, y\alpha) \in E$ and $x\alpha \leq y\alpha$.

It follows by two cases that $\alpha \in EOP(X)$. Notice that for any $x \in P$ and $y \in Q$, $(x, y) \in E$ but $(x\alpha, y\alpha) = (b, y) \notin E$. Therefore $\alpha \notin T_E(X)$. □

Theorem 2.3.15. *Let X be a partially ordered set and E an arbitrary equivalence relation on X. Suppose that for every $A \in X/E$ and $x, y \in A$, there exist subchains $C_1, C_2, C_3, \ldots, C_n$ of A for some positive integer n such that $x \in C_1, y \in C_n$ and $C_i \cap C_{i+1} \neq \emptyset$ for all $i = 1, 2, \ldots, n-1$. Then $EOP(X) \subseteq T_E(X)$.*

Proof. Suppose that for every $A \in X/E$ and $x, y \in A$, there exist subchains $C_1, C_2, C_3, \ldots, C_n$ of A for some positive integer n such that $x \in C_1, y \in C_n$ and $C_i \cap C_{i+1} \neq \emptyset$ for all $i = 1, 2, \ldots, n-1$. Let $\alpha \in EOP(X)$ and $(x, y) \in E$. Hence $x, y \in A$ for some $A \in X/E$. It follows from assumption that there exist subchains $C_1, C_2, C_3, \ldots, C_n$ of A for some positive integer n such that $x \in C_1, y \in C_n$ and $C_i \cap C_{i+1} \neq \emptyset$ for all $i = 1, 2, \ldots, n-1$. Choose $c_i \in C_i \cap C_{i+1}$ for all $i = 1, 2, \ldots, n-1$. Since $x, c_1 \in C_1$, x and c_1 are comparable. Assume that $x \leq c_1$. By $\alpha \in EOP(X)$, we deduce that $(x\alpha, c_1\alpha) \in E$. For each

$i = 1, 2, \ldots, n-1$, we have $c_i, c_{i+1} \in C_{i+1}$. We conclude that $c_i \leq c_{i+1}$ or $c_{i+1} \leq c_i$. Since $(c_i, c_{i+1}) \in E$ and $\alpha \in EOP(X)$, $(c_i\alpha, c_{i+1}\alpha) \in E$. Similarly, we have that $(c_n\alpha, y\alpha) \in E$. It follows from transitivity of E that $(x\alpha, y\alpha) \in E$. This proves that $\alpha \in T_E(X)$. $\square$

Example 1. Let $X = \{a_1, a_2, a_3, b\}$ and $E = (\{a_1, a_2\} \times \{a_1, a_2\}) \cup (\{a_3, b\} \times \{a_3, b\})$. Define

$$\leq = \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_1, a_2), (a_1, a_3), (a_2, a_3), (b, b)\}.$$

Then $(X, \leq)$ is a partially ordered set and E is an equivalence relation on X. Define $\alpha, \beta, \delta \in T(X)$ by

$$x\alpha = \begin{cases} a_3, & \text{if } x = a_1; \\ b, & \text{if } x = a_2; \\ x, & \text{otherwise,} \end{cases}$$

$$x\beta = \begin{cases} a_2, & \text{if } x = a_1; \\ a_3, & \text{if } x = a_2; \\ x, & \text{otherwise} \end{cases}$$

and

$$x\delta = \begin{cases} a_1, & \text{if } x = a_3; \\ x, & \text{otherwise.} \end{cases}$$

It is easy to verify that $\alpha \in T_E(X) \setminus (\mathcal{O}(X) \cup EOP(X))$, $\beta \in \mathcal{O}(X) \setminus (T_E(X) \cup EOP(X))$ and $\delta \in EOP(X) \setminus (\mathcal{O}(X) \cup T_E(X))$.