CHAPTER III

REGULARITY FOR SOME SUBSEMIGROUPS OF FULL TRANSFORMATION SEMIGROUPS

In this chapter, we characterize the regular, left regular, right regular and completely regular elements of some subsemigroups of $T(X)$.

3.1 Regularity for full transformation semigroups

Firstly, we characterize the regularity, left regularity, right regularity and completely regularity for each element of $T(X)$.

Theorem 3.1.1. *Every element of $T(X)$ is regular. Hence, $T(X)$ is a regular semigroup.*

Example 2. Define $\alpha : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$x\alpha = \begin{cases} 1, & \text{if } x \leq 2; \\ 2, & \text{if } x = 3; \\ x, & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in T(\mathbb{Z}^+)$. To show that α is neither right regular nor left regular element of $T(X)$, suppose that $\alpha \in RReg(T(\mathbb{Z}^+))$. Thus $\alpha = \alpha^2\beta$ for some $\beta \in T(\mathbb{Z}^+)$. By the definition of α, we have $1\alpha = 2\alpha = 1$ and $3\alpha = 2$. Hence

$$1 = 1\alpha = 1\alpha^2\beta = (1\alpha)\alpha\beta = 1\alpha\beta = 2\alpha\beta = (3\alpha)\alpha\beta = 3\alpha^2\beta = 3\alpha = 2,$$

a contradiction. Thus we have shown that α is not a right regular element of $T(\mathbb{Z}^+)$.

Next, we suppose that $\alpha = \beta\alpha^2$ for some $\beta \in T(\mathbb{Z}^+)$. Since $(3\beta\alpha)\alpha = 3\beta\alpha^2 = 3\alpha = 2$, we deduce that $3\beta\alpha \in 2\alpha^{-1} = \{3\}$. Hence $3 = 3\beta\alpha \in \mathbb{Z}^+\alpha = \mathbb{Z}^+\backslash\{3\}$, it is impossible. Thus α is not a left regular element of $T(\mathbb{Z}^+)$.

A natural question is under what conditions each element of $T(X)$ is left regular, right regular or completely regular.

Theorem 3.1.2. *Let $\alpha \in T(X)$. Then $\alpha \in LReg(T(X))$ if and only if $P \cap X\alpha \neq \emptyset$ for all $P \in \pi(\alpha)$.*

Proof. Suppose that $\alpha \in LReg(T(X))$. Then $\alpha = \beta\alpha^2$ for some $\beta \in T(X)$. Let $P \in \pi(\alpha)$. Then $P = y\alpha^{-1}$ for some $y \in X\alpha$. There is an element $x \in X$ such that $x\alpha = y$, hence

$$y = x\alpha = x\beta\alpha^2 = (x\beta\alpha)\alpha.$$

Thus $x\beta\alpha \in y\alpha^{-1}$, from which it follows that $P \cap X\alpha \neq \emptyset$.

Conversely, suppose that $P \cap X\alpha \neq \emptyset$ for all $P \in \pi(\alpha)$. We construct $\beta \in T(X)$ such that $\alpha = \beta\alpha^2$. For every $x \in X$, there exists a unique $P_x \in \pi(\alpha)$ such that $x \in P_x$. By assumption, we have $P_x \cap X\alpha \neq \emptyset$. We choose and fix an element $x_{P_x} \in P_x \cap X\alpha$ and $x'_{P_x} \in X$ such that $x'_{P_x}\alpha = x_{P_x}$. Define $\beta : X \to X$ by

$$x\beta = x'_{P_x} \text{ for all } x \in X.$$

Clearly, $\beta \in T(X)$. To show that $\alpha = \beta\alpha^2$, let $x \in X$. Then

$$x\beta\alpha^2 = x'_{P_x}\alpha^2 = x_{P_x}\alpha = x\alpha,$$

it follows that $\alpha = \beta\alpha^2$. Therefore $\alpha \in LReg(T(X))$. □

Theorem 3.1.3. *Let $\alpha \in T(X)$. Then $\alpha \in RReg(T(X))$ if and only if $\alpha|_{X\alpha}$ is injective.*

Proof. Suppose that $\alpha \in RReg(T(X))$. Then $\alpha = \alpha^2\beta$ for some $\beta \in T(X)$. Let $x, y \in X\alpha$ be such that $x\alpha = y\alpha$. Since $x, y \in X\alpha$, $x = x'\alpha$ and $y = y'\alpha$ for some $x', y' \in X$. Therefore

$$x = x'\alpha = x'\alpha^2\beta = (x'\alpha)\alpha\beta = (x\alpha)\beta = y\alpha\beta = (y'\alpha)\alpha\beta = y'\alpha^2\beta = y'\alpha = y.$$

This shows that $\alpha|_{X\alpha}$ is injective.

Conversely, suppose that $\alpha|_{X\alpha}$ is injective. We construct $\beta \in T(X)$ such that

$\alpha = \alpha^2\beta$. For every $x \in X\alpha^2$, we choose and fix an element $x' \in X\alpha$ such that $x = x'\alpha$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x', & \text{if } x \in X\alpha^2; \\ x, & \text{otherwise.} \end{cases}$$

We note that $\beta \in T(X)$. To verify that $\alpha = \alpha^2\beta$, let $x \in X$. Since $x\alpha^2 \in X\alpha^2$, there exists $(x\alpha^2)' \in X\alpha$ such that $(x\alpha^2)'\alpha = (x\alpha)\alpha$. It follows from assumption that $(x\alpha^2)' = x\alpha$. Therefore $x\alpha^2\beta = (x\alpha^2)' = x\alpha$. Hence $\alpha \in RReg(T(X))$ as required. □

Theorem 3.1.4. *Let $\alpha \in T(X)$. Then $\alpha \in CReg(T(X))$ if and only if for every $P \in \pi(\alpha), |P \cap X\alpha| = 1$.*

Proof. Suppose that $\alpha \in CReg(T(X))$. Then $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$ for some $\beta \in T(X)$. Let $P \in \pi(\alpha)$ and $x \in P$. From

$$x\alpha = x\alpha\beta\alpha = (x\alpha\beta)\alpha \text{ and } x\alpha\beta = x\beta\alpha \in X\alpha,$$

we see that $x\alpha\beta \in P \cap X\alpha$. Thus $P \cap X\alpha \neq \emptyset$. Let $a, b \in P \cap X\alpha$, then $a\alpha = b\alpha$, $a = a'\alpha$ and $b = b'\alpha$ for some $a', b' \in X$. We observe that

$$a = a'\alpha = a'\alpha\beta\alpha = (a'\alpha)\beta\alpha = a\beta\alpha = (a\alpha)\beta = (b\alpha)\beta = b\beta\alpha = b'\alpha\beta\alpha = b'\alpha = b.$$

This means that $|P \cap X\alpha| = 1$.

Conversely, suppose that $|P \cap X\alpha| = 1$ for all $P \in \pi(\alpha)$. For each $P \in \pi(\alpha)$, let $P \cap X\alpha = \{x_P\}$. Since $x_P \in X\alpha$, there exists $P' \in \pi(\alpha)$ such that $P' = x_P\alpha^{-1}$. By assumption, there is a unique $x_{P'} \in P' \cap X\alpha$ and $x_P = x_{P'}\alpha$. Define $\beta : X \to X$ by

$$x\beta = x_{P'} \text{ for all } x \in P \text{ and for each } P \in \pi(\alpha).$$

Clearly, β is well-defined because $\pi(\alpha)$ is a partition of X. To show that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$, let $x \in X$. By $\pi(\alpha)$ is a partition of X, $x\alpha \in P$ for some $P \in \pi(\alpha)$. It follows from assumption that $x\alpha \in P \cap X\alpha = \{x_P\}$. Since $x_P \in P$, we conclude that $x_P\beta = x_{P'}$. Thus

$$x\alpha\beta\alpha = x_P\beta\alpha = x_{P'}\alpha = x_P = x\alpha.$$

Since $x\alpha = x_P$, we get $x \in P'$. By the definition of β, $x\beta = x_{P''}$ where $x_{P''}\alpha = x_{P'}$ and $P'' \in \pi(\alpha)$. We then have

$$x\alpha\beta = x_P\beta = x_{P'} = x_{P''}\alpha = x\beta\alpha.$$

These mean that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$, hence $\alpha \in CReg(T(X))$. □

Corollary 3.1.5. *$CReg(T(X)) = RReg(T(X)) \cap LReg(T(X))$.*

Proof. Assume that $\alpha \in RReg(T(X)) \cap LReg(T(X))$. To show that $\alpha \in CReg(T(X))$, let $P \in \pi(\alpha)$. By Theorem 3.1.2, $P \cap X\alpha \neq \emptyset$. Let $a, b \in P \cap X\alpha$. Then $a\alpha = b\alpha$ and $a, b \in X\alpha$. It follows from Theorem 3.1.3 that $a = b$. This proves that $|P \cap X\alpha| = 1$. By Theorem 3.1.4, α is completely regular as required. □

Example 3. Define $\alpha : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$x\alpha = \begin{cases} 1, & \text{if } x \in \{1, 3\}; \\ 2, & \text{if } x \in \{2, 4\}; \\ x - 2, & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in T(\mathbb{Z}^+)$. Note that $\pi(\alpha) = \{\{1, 3\}, \{2, 4\}\} \cup \{\{x\} : x \in \mathbb{Z}^+ \text{ and } x > 4\}$. For each $P \in \pi(\alpha)$, we have $P \cap \mathbb{Z}^+\alpha \neq \emptyset$. By Theorem 3.1.2, α is a left regular element of $T(\mathbb{Z}^+)$. Since $1, 3 \in \mathbb{Z}^+\alpha$ and $1\alpha = 3\alpha$, by Theorem 3.1.3, α is not right regular. Next, we define $\beta : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$x\beta = \begin{cases} 1, & \text{if } x \in \{1, 3\}; \\ 2, & \text{if } x \in \{2, 4\}; \\ x + 2, & \text{otherwise.} \end{cases}$$

We have $\beta \in T(\mathbb{Z}^+)$ and it is clear that $\beta|_{\mathbb{Z}^+\beta}$ is an injection. By Theorem 3.1.3, β is right regular. Since $\{5\} \in \pi(\beta)$ and $\{5\} \cap \mathbb{Z}^+\beta = \emptyset$, it follows from Theorem 3.1.2 that β is not left regular. By Corollary 3.1.5, we also get α and β are not completely regular elements of $T(\mathbb{Z}^+)$.

3.2 Regularity for full regressive transformation semigroups

Throughout this section, let $(X, \leq)$ be a partially ordered set. We give necessary and sufficient conditions for elements in $T_{RE}(X)$ to be regular, left regular, right regular or completely regular.

Theorem 3.2.1. *Let $\alpha \in T_{RE}(X)$. Then $\alpha \in Reg(T_{RE}(X))$ if and only if $\alpha^2 = \alpha$.*

Proof. Assume that $\alpha \in Reg(T_{RE}(X))$. Then there exists an element $\beta \in T_{RE}(X)$ such that $\alpha = \alpha\beta\alpha$. Suppose that $\alpha^2 \neq \alpha$. Then $x\alpha^2 \neq x\alpha$ for some $x \in X$. Since $\alpha, \beta \in T_{RE}(X)$, we deduce that $x\alpha^2 < x\alpha$ and $x\alpha\beta \leq x\alpha$. We consider two cases as follows :

Case 1. $x\alpha\beta = x\alpha$. Then $x\alpha = x\alpha\beta\alpha = x\alpha\alpha < x\alpha$.

Case 2. $x\alpha\beta < x\alpha$. Then $x\alpha = x\alpha\beta\alpha \leq x\alpha\beta < x\alpha$.

These lead to a contradiction. Therefore $\alpha^2 = \alpha$. □

It is known that for every $\alpha \in T(X)$, α is an idempotent if and only if $x\alpha = x$ for all $x \in X\alpha$. Then by Theorem 3.2.1, $x\alpha = x$ for all $x \in X\alpha$ where α belongs to $Reg(T_{RE}(X))$.

Theorem 3.2.2. *The following statements are equivalent.*

(1) *$Reg(T_{RE}(X))$ is a subsemigroup of $T_{RE}(X)$.*

(2) *For every $\alpha, \beta \in Reg(T_{RE}(X))$ and $x \in X$, if $x \in X\alpha \backslash X\beta$, then $x\beta \in X\alpha$.*

(3) *For every $\alpha, \beta \in Reg(T_{RE}(X))$, $X\alpha\beta = X\beta\alpha$.*

(4) *For every subchain C of X, $|C| \leq 2$.*

(5) *$Reg(T_{RE}(X)) = T_{RE}(X)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that there exist $\alpha, \beta \in Reg(T_{RE}(X))$ and $x \in X$ such that $x \in X\alpha \backslash X\beta$ but $x\beta \notin X\alpha$. As was mentioned above, we have that $x\alpha = x$ and

$(x\beta)\alpha \neq x\beta$ since $x \in X\alpha$ and $x\beta \notin X\alpha$, respectively. Since α is regressive, $(x\beta)\alpha < x\beta$. Then

$$x(\alpha\beta)^2 = (x\alpha)\beta\alpha\beta = (x\beta\alpha)\beta \leq x\beta\alpha < x\beta = x\alpha\beta.$$

This proves that $(\alpha\beta)^2 \neq \alpha\beta$. By Theorem 3.2.1, we have that $\alpha\beta \notin Reg(T_{RE}(X))$.

(2) $\Rightarrow$ (3) Let $\alpha, \beta \in Reg(T_{RE}(X))$. To show that $X\alpha\beta \subseteq X\beta\alpha$, let $x \in X$. By Theorem 3.2.1, we deduce that $\alpha^2 = \alpha$.

Case 1. $x\alpha \in X\beta$. Since $\beta \in Reg(T_{RE}(X))$, $x\alpha\beta = x\alpha$. Then

$$x\alpha\beta = x\alpha = x\alpha^2 = x\alpha\beta\alpha \in X\beta\alpha.$$

Case 2. $x\alpha \notin X\beta$. By assumption, $x\alpha\beta = y\alpha$ for some $y \in X$. Then

$$x\alpha\beta = y\alpha = y\alpha^2 = x\alpha\beta\alpha \in X\beta\alpha.$$

From two cases, we conclude that $X\alpha\beta \subseteq X\beta\alpha$. By symmetry, we have $X\beta\alpha \subseteq X\alpha\beta$. Hence $X\alpha\beta = X\beta\alpha$.

(3) $\Rightarrow$ (4) Suppose that there exists a subchain C of X such that $|C| > 2$. Choose elements $a, b, c \in C$ such that $a < b < c$. Define $\alpha, \beta : X \rightarrow X$ as follow :

$$x\alpha = \begin{cases} b, & \text{if } x = c; \\ x, & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} a, & \text{if } x = b; \\ x, & \text{otherwise.} \end{cases}$$

Consider

$$x\alpha^2 = \begin{cases} b\alpha = b = x\alpha, & \text{if } x = c; \\ x\alpha, & \text{otherwise} \end{cases}$$

and

$$x\beta^2 = \begin{cases} a\beta = a = x\beta, & \text{if } x = b; \\ x\beta, & \text{otherwise,} \end{cases}$$

hence $\alpha, \beta \in Reg(T_{RE}(X))$. But $b = c\alpha = c\beta\alpha \in X\beta\alpha$ and $X\alpha\beta = (X\backslash\{c\})\beta = X\backslash\{b, c\}$ which imply that $X\alpha\beta \neq X\beta\alpha$.

(4) $\Rightarrow$ (5) Assume that $Reg(T_{RE}(X)) \neq T_{RE}(X)$. Then there exists $\alpha \in T_{RE}(X)\backslash Reg(T_{RE}(X))$, hence $\alpha^2 \neq \alpha$ by Theorem 3.2.1. Then $x\alpha^2 \neq x\alpha$ for some $x \in X$. By regressiveness of α, $x\alpha^2 < x\alpha$. If $x\alpha = x$, then $x\alpha^2 = (x\alpha)\alpha = x\alpha$ which is a contradiction. Hence $x\alpha < x$. Now, there exists a subchain $\{x\alpha^2, x\alpha, x\}$ of X including more than two elements as desired. □

Theorem 3.2.3. *Let $\alpha \in T_{RE}(X)$. Then $\alpha \in LReg(T_{RE}(X))$ if and only if for every $P \in \pi(\alpha)$ and $x \in P$, there exists $Q \in \pi(\alpha)$ such that $Q\alpha_* \in P$ and $y \leq x$ for some $y \in Q$.*

Proof. Suppose that $\alpha \in LReg(T_{RE}(X))$. Then there exists $\beta \in T_{RE}(X)$ such that $\alpha = \beta\alpha^2$. Let $P \in \pi(\alpha)$ and $x \in P$. By $\pi(\alpha)$ is a partition of X, $x\beta \in Q$ for some $Q \in \pi(\alpha)$. Since $x\alpha = x\beta\alpha^2 = (x\beta\alpha)\alpha$, it follows that $x\beta\alpha \in P$. Thereby $Q\alpha_* = x\beta\alpha \in P$ and $x\beta \leq x$ as required.

For the converse, suppose that for every $P \in \pi(\alpha)$ and $x \in P$, there exists $Q \in \pi(\alpha)$ such that $Q\alpha_* \in P$ and $y \leq x$ for some $y \in Q$. For each $x \in X$, there is a unique $P \in \pi(\alpha)$ such that $x \in P$. By assumption, we choose and fix $P_x \in \pi(\alpha)$ and $x' \in P_x$ such that $x' \leq x$ and $P_x\alpha_* \in P$. Define $\beta : X \to X$ by

$$x\beta = x' \text{ for all } x \in X.$$

For any $x \in X$, we then have $x\beta = x' \leq x$ and

$$x\beta\alpha^2 = x'\alpha\alpha = P_x\alpha_*\alpha = P\alpha_* = x\alpha.$$

Therefore $\beta \in T_{RE}(X)$ and $\alpha = \beta\alpha^2$, respectively. This proves that α is a left regular element of $T_{RE}(X)$ as desired. □

Theorem 3.2.4. *Let $\alpha \in T_{RE}(X)$. Then $\alpha \in RReg(T_{RE}(X))$ if and only if $\alpha^2 = \alpha$.*

Proof. Suppose that $\alpha \in RReg(T_{RE}(X))$. Then $\alpha = \alpha^2\beta$ for some $\beta \in T_{RE}(X)$. Suppose that $\alpha^2 \neq \alpha$. Then there exists $x \in X$ such that $x\alpha^2 \neq x\alpha$, hence

$$x\alpha = x\alpha^2\beta \leq x\alpha^2 < x\alpha.$$

This is a contradiction. Therefore $\alpha^2 = \alpha$. □

Completely regularity is directly characterized from Theorem 3.2.1.

Corollary 3.2.5. *Let $\alpha \in T_{RE}(X)$. Then $\alpha \in CReg(T_{RE}(X))$ if and only if $\alpha^2 = \alpha$.*

Corollary 3.2.6. *$CReg(T_{RE}(X)) = RReg(T_{RE}(X)) \cap LReg(T_{RE}(X))$.*

3.3 Regularity for semigroups of full transformations that preserve an equivalence

In this section, we let E an equivalence relation on X, we investigate regularity, left regularity, right regularity and completely regularity for elements of $T_E(X)$.

Theorem 3.3.1. [1] *Let $\alpha \in T_E(X)$. Then $\alpha \in Reg(T_E(X))$ if and only if for every $A \in X/E$, there exists $B \in X/E$ such that $A \cap X\alpha \subseteq B\alpha$.*

In general, $T_E(X)$ is not a regular semigroup as we show in the below example.

Example 4. Let $A_1 = \{2n : n \in \mathbb{Z}^+\}$ and $A_2 = \{2n-1 : n \in \mathbb{Z}^+\}$. Define $E = (A_1 \times A_1) \cup (A_2 \times A_2)$. We note E is an equivalence relation on $\mathbb{Z}^+$ and $\mathbb{Z}^+/E = \{A_1, A_2\}$. Consider $\alpha : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by

$$x\alpha = 2x \text{ for all } x \in \mathbb{Z}^+.$$

Clearly, $(x\alpha, y\alpha) \in A_1 \times A_1 \subseteq E$ for all $x, y \in \mathbb{Z}^+$. Thus $\alpha \in T_E(\mathbb{Z}^+)$. We see that $A_1\alpha = \{4n : n \in \mathbb{Z}^+\}$ and $A_2\alpha = \{4n-2 : n \in \mathbb{Z}^+\}$. This implies that $A_1 \cap \mathbb{Z}^+\alpha = A_1 \not\subseteq B\alpha$ for all $B \in \mathbb{Z}^+/E$. By Theorem 3.3.1, $\alpha \notin Reg(T_E(\mathbb{Z}^+))$ and hence $T_E(\mathbb{Z}^+)$ is not a regular semigroup.

Theorem 3.3.2. [1] *$T_E(X)$ is a regular semigroup if and only if $E = X \times X$ or $E = I_X$.*

Corollary 3.3.3. *$Reg(T_E(X))$ is a subsemigroup of $T_E(X)$ if and only if $E = X \times X$ or $E = I_X$.*

Proof. Assume that $E \neq X \times X$ and $E \neq I_X$. Since $E \neq I_X$, there are distinct elements $a, c \in X$ such that $(a, c) \in E$. Then $a, c \in A$ for some $A \in X/E$. Since $E \neq X \times X$, $(a, b) \notin E$ for some $b \in X$. Hence $b \in B$ for some $B \in X/E$ and $B \neq A$. Define $\alpha, \beta : X \to X$ by

$$x\alpha = \begin{cases} a, & \text{if } x \in A; \\ x, & \text{otherwise.} \end{cases}$$

and

$$x\beta = \begin{cases} c, & \text{if } x \in B; \\ x, & \text{otherwise.} \end{cases}$$

Then $\alpha, \beta \in T_E(X)$. By the definitions of α and β, we have that $X\alpha = \{a\} \cup X\backslash A$ and $X\beta = X\backslash B$. It is not difficult to verify that $C \cap X\alpha \subseteq C\alpha$ and $C \cap X\beta \subseteq C\beta$ for all $C \in X/E$. It follows from Theorem 3.3.1 that $\alpha, \beta \in Reg(T_E(X))$. Since $A\alpha\beta = \{a\}\beta = \{a\}$, $B\alpha\beta = B\beta = \{c\}$ and $C\alpha\beta = C\beta = C$ for all $C \in X/E\backslash\{A, B\}$ and $A \cap X\alpha\beta = A \cap (X\backslash A \cup \{a\})\beta = \{a, c\}$, we conclude that $\alpha\beta$ is not a regular element by Theorem 3.3.1. Therefore $Reg(T_E(X))$ is not a subsemigroup of $T_E(X)$. □

The next corollary follows immediately from Theorem 3.3.2 and Corollary 3.3.3.

Corollary 3.3.4. *The following statements are equivalent.*

(1) *$T_E(X)$ is a regular semigroup.*

(2) *$Reg(T_E(X))$ is a subsemigroup of $T_E(X)$.*

(3) *$E = X \times X$ or $E = I_X$.*

Example 5. Let $A_1 = \{1\}, A_2 = \{2\}$ and $A_3 = \mathbb{Z}^+ \setminus \{1, 2\}$ and define

$$E = \bigcup_{i=1}^{3}(A_i \times A_i).$$

We observe that E is an equivalence relation on $\mathbb{Z}^+$. We define $\alpha : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$x\alpha = x + 1 \text{ for all } x \in \mathbb{Z}^+.$$

It is easy to verity that $\alpha \in T_E(\mathbb{Z}^+)$. Assume that $\alpha \in RReg(T_E(\mathbb{Z}^+))$. Then $\alpha = \alpha^2\beta$ for some $\beta \in T_E(\mathbb{Z}^+)$. We note that $2 = 1\alpha = 1\alpha^2\beta = 2\alpha\beta = 3\beta$ and $3 = 2\alpha = 2\alpha^2\beta = 3\alpha\beta = 4\beta$. Since $\beta \in T_E(\mathbb{Z}^+)$ and $(3, 4) \in E$, we conclude that $(2, 3) = (3\beta, 4\beta) \in E$ which is a contradiction. Hence α is not a right regular element of $T_E(\mathbb{Z}^+)$.

Next, assume that $\alpha = \beta\alpha^2$ for some $\beta \in T_E(\mathbb{Z}^+)$. Since $1\alpha = 1\beta\alpha^2 = (1\beta)\alpha$ and α is injective, we conclude that $1 = 1\beta\alpha \in \mathbb{Z}^+\alpha = \mathbb{Z}^+ \setminus \{1\}$. This is impossible, therefore α is not a left regular element of $T_E(\mathbb{Z}^+)$.

Theorem 3.3.5. *Let $\alpha \in T_E(X)$. Then $\alpha \in LReg(T_E(X))$ if and only if for every $A \in X/E$, there exists $B \in X/E$ such that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$.*

Proof. Assume that $\alpha \in LReg(T_E(X))$. Then $\alpha = \beta\alpha^2$ for some $\beta \in T_E(X)$. Let $A \in X/E$. By Lemma 2.2.4, there is $B \in X/E$ such that $A\beta \subseteq B$. Let $P \in \pi_A(\alpha)$ and $x \in P \cap A$. Since $A\beta \subseteq B$, we have that $x\beta \in B$. Hence $x\alpha = x\beta\alpha^2 = (x\beta\alpha)\alpha$ which implies that $x\beta\alpha \in P$ as we wish to show.

Conversely, for every $A \in X/E$, we choose $A' \in X/E$ such that for every $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in A'$. Let $x \in X$. Since X/E and $\pi(\alpha)$ are partitions of X, there exist $A \in X/E$ and $P \in \pi(\alpha)$ such that $x \in A$ and $x \in P$. Hence $P \in \pi_A(\alpha)$. By assumption, we choose $x' \in A'$ such that $x'\alpha \in P$ and $A' \in X/E$. We also have that $x'\alpha\alpha = x\alpha$. Define $\beta : X \to X$ by

$$x\beta = x' \text{ for all } x \in X.$$

Let $x, y \in E$ be such that $(x, y) \in E$. Then there exists $A \in X/E$ such that $x, y \in A$. By the definition of β, $x\beta, y\beta \in A'$ where $A' \in X/E$. Hence $\beta \in T_E(X)$. Let $x \in X$. We then deduce that $x\beta\alpha^2 = x'\alpha\alpha = x\alpha$. Therefore α is a left regular element of $T_E(X)$ as required. □

Theorem 3.3.6. *Let $\alpha \in T_E(X)$. Then $\alpha \in RReg(T_E(X))$ if and only if*

(1) *$\alpha|_{X\alpha}$ is an injection and*

(2) *for every $x, y \in X\alpha$, $(x\alpha, y\alpha) \in E$ implies that $(x, y) \in E$.*

Proof. Assume that $\alpha \in RReg(T_E(X))$. Then $\alpha = \alpha^2\beta$ for some $\beta \in T_E(X)$. We note that $\alpha \in RReg(T(X))$, it follows from Theorem 3.1.3 that $\alpha|_{X\alpha}$ is an injection. Let $x, y \in X\alpha$ be such that $(x\alpha, y\alpha) \in E$. Thus $x = x'\alpha$ and $y = y'\alpha$ for some $x', y' \in X$. Since $\beta \in T_E(X)$, $(x\alpha\beta, y\alpha\beta) \in E$. Hence

$$(x, y) = (x'\alpha, y'\alpha) = (x'\alpha^2\beta, y'\alpha^2\beta) = (x\alpha\beta, y\alpha\beta) \in E$$

which implies that (2) holds.

Conversely, assume that (1) and (2) hold. Let $A \in X/E$ be such that $A \cap X\alpha^2 \neq \emptyset$. We choose and fix an element $x_A \in A \cap X\alpha^2$. For each $x \in A \cap X\alpha^2$, there exists a unique $x' \in X\alpha$ such that $x = x'\alpha$ by $\alpha|_{X\alpha}$ is injective. We observe that $(x'\alpha, x'_A\alpha) = (x, x_A) \in E$. It follows from assumption that $(x', x'_A) \in E$. Define $\beta_A : A \to X$ by

$$x\beta_A = \begin{cases} x', & \text{if } x \in X\alpha^2; \\ x'_A, & \text{if } x \notin X\alpha^2. \end{cases}$$

Then we define the map $\beta : X \to X$ by

$$\beta|_A = \begin{cases} \beta_A, & \text{if } A \cap X\alpha^2 \neq \emptyset; \\ i_A, & \text{otherwise,} \end{cases}$$

for all $A \in X/E$ (i_A is the identity mapping on A). Since X/E is a partition of X, β is well-defined. Let $x, y \in X$ be such that $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$. By the definition of β, we have $(x\beta, y\beta) = (x\beta|_A, y\beta|_A)$. If $A \cap X\alpha^2 = \emptyset$, then $(x\beta, y\beta) = (x, y) \in E$. If $A \cap X\alpha^2 \neq \emptyset$, by the definition of β_A, we then have $(x\beta_A, x'_A), (y\beta_A, x'_A) \in E$. By transitivity of E, $(x\beta, y\beta) \in E$, hence $\beta \in T_E(X)$.

Finally, to show that $\alpha = \alpha^2\beta$, let $x \in X$, so $x\alpha^2 \in X\alpha^2$. Then there exists $A \in X/E$ such that $x\alpha^2 \in A$. By the definition of β_A, $x\alpha^2\beta_A = (x\alpha^2)'$ where $(x\alpha^2)'\alpha = x\alpha^2 = (x\alpha)\alpha$. Since $(x\alpha^2)'$ is unique, $(x\alpha^2)' = x\alpha$. Thus $x\alpha^2\beta = x\alpha^2\beta_A = x\alpha$. Therefore $\alpha \in RReg(T_E(X))$ as asserted. □

From Example 5, we observe that $2, 3 \in \mathbb{Z}^+\alpha$ such that $(2\alpha, 3\alpha) = (3, 4) \in E$ but $(2, 3) \notin E$. Hence α does not satisfy (2) in Theorem 3.3.6.

Theorem 3.3.7. *Let $\alpha \in T_E(X)$. Then $\alpha \in CReg(T_E(X))$ if and only if for every $A \in X/E$, there exists $B \in X/E$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A(\alpha)$.*

Proof. Assume that α is a completely regular element in $T_E(X)$. Then $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$ for some $\beta \in T_E(X)$. Let $A \in X/E$. By Lemma 2.2.4, there exists $A' \in X/E$ such that $A\beta \subseteq A'$. Let $P \in \pi_A(\alpha)$ and $x \in P \cap A$. Hence $x\beta \in A'$. Since $x\alpha\beta\alpha = x\alpha = P\alpha_*$, we deduce that $x\alpha\beta \in P$. We note that $x\alpha\beta = x\beta\alpha \in A'\alpha$. Hence $P \cap A'\alpha \neq \emptyset$ which implies that $P \cap X\alpha \neq \emptyset$. Since $\alpha \in CReg(T(X))$, by Theorem 3.1.4, we get $|P \cap X\alpha| = 1$. It follows from $P \cap A'\alpha \subseteq P \cap X\alpha$ that $|P \cap A'\alpha| = |P \cap X\alpha| = 1$.

Conversely, suppose that for every $A \in X/E$, there exists $B \in X/E$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A(\alpha)$. Let $A \in X/E$. By assumption, we choose $A' \in X/E$ such that $|P \cap A'\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A(\alpha)$. For each $P \in \pi_A(\alpha)$, let $x_P \in P \cap A'\alpha$. This means that $x_P = x\alpha$ for some $x \in A'$ and then $x \in P'$ for some $P' \in \pi_{A'}(\alpha)$. We let $A'' \in X/E$ be such that $|Q \cap A''\alpha| = 1$ for all $Q \in \pi_{A'}(\alpha)$ and $x_{P'} \in P' \cap A''\alpha$. Hence $x_{P'}\alpha = P'\alpha_* = x_P$. For each $P, Q \in \pi_A(\alpha)$, we note by Lemma 2.2.4 that $x_P, x_Q \in A'\alpha \subseteq B$ for some $B \in X/E$. And $x_{P'}, x_{Q'} \in A''\alpha \subseteq B'$ where $B' \in X/E$, thus $(x_{P'}, x_{Q'}) \in E$. This implies that for all $P, Q \in \pi_A(\alpha)$, $(x_{P'}, x_{Q'}) \in E$. Let $x \in X$. Since X/E and $\pi(\alpha)$ are partitions of X, $x \in A$ for some $A \in X/E$ and $x \in P_x$ for some $P_x \in \pi(\alpha)$. Then $P_x \in \pi_A(\alpha)$. Define $\beta : X \to X$ by

$$x\beta = x_{P'_x} \text{ for all } x \in X.$$

Let $x, y \in X$ be such that $(x, y) \in E$. Then $x, y \in A$ where $A \in X/E$. We note that $P_x, P_y \in \pi_A(\alpha)$. Hence $(x_{P'_x}, x_{P'_y}) \in E$ which implies that $\beta \in T_E(X)$.

To show that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$, let $x \in X$. $x\alpha \in A$ for some $A \in X/E$ and $x\alpha \in P_{x\alpha}$ where $P_{x\alpha} \in \pi_A(\alpha)$. By assumption, we note that $x\alpha \in P_{x\alpha} \cap X\alpha = \{x_{P_{x\alpha}}\}$. Hence $x\alpha = x_{P_{x\alpha}}$. By the definition of β, $x\alpha\beta = x_{P'_{x\alpha}}$ where $x_{P'_{x\alpha}}\alpha = x_{P_{x\alpha}}$. Thus $x\alpha\beta\alpha = x_{P'_{x\alpha}}\alpha = x\alpha$. Moreover, $x\alpha = x_{P_{x\alpha}} = P'_{x\alpha}\alpha_*$. Then we have $x \in P'_{x\alpha}$. By the definition of β, $x\beta = x_{P'_x}$ and $x_{P'_x}\alpha = x_{P_x}$. Since $x \in P'_{x\alpha} \cap P_x$ and $\pi(\alpha)$ is a partition of X, $P'_{x\alpha} = P_x$. Hence $x_{P'_{x\alpha}} = x_{P_x}$, so

$$x\alpha\beta = x_{P'_{x\alpha}} = x_{P_x} = x_{P'_x}\alpha = x\beta\alpha$$

which completes the proof. □

Corollary 3.3.8. *$CReg(T_E(X)) = RReg(T_E(X)) \cap LReg(T_E(X))$.*

Proof. Suppose that $\alpha \in RReg(T_E(X)) \cap LReg(T_E(X))$. Let $A \in X/E$. Since α is left regular, by Theorem 3.3.5, there exists $B \in X/E$ such that for every $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$. Let $P \in \pi_A(\alpha)$, then $P \cap B\alpha \neq \emptyset$. We note here that $P \cap B\alpha \subseteq P \cap X\alpha$. It is enough to show that $|P \cap X\alpha| = 1$. Let $a, b \in P \cap X\alpha$. Then $a\alpha = b\alpha$ and $a, b \in X\alpha$. It follows from α is right regular and Theorem 3.3.6 that $a = b$. We conclude that $\alpha \in CReg(T_E(X))$ by Theorem 3.3.7. □

Example 6. Define $\alpha, \beta : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$x\alpha = \begin{cases} x, & \text{if } x \leq 3; \\ x-1, & \text{otherwise,} \end{cases}$$

$$x\beta = \begin{cases} x, & \text{if } x < 3; \\ x+1, & \text{otherwise.} \end{cases}$$

Recall an equivalence relation E on $\mathbb{Z}^+$ from Example 5. We see that $A\alpha \subseteq A$ and $A\beta \subseteq A$ for all $A \in \mathbb{Z}^+/E$. Hence $\alpha, \beta \in T_E(\mathbb{Z}^+)$. Since $3\alpha = 4\alpha$ and $3, 4 \in \mathbb{Z}^+\alpha$, we have $\alpha|_{\mathbb{Z}^+\alpha}$ is not injective. Thus by Theorem 3.3.6, α is not right regular. For each

$A \in \mathbb{Z}^+/E$, we have $P \cap A\alpha \neq \emptyset$ for all $P \in \pi_A(\alpha)$. It follows from Theorem 3.3.5 that α is left regular in $T_E(\mathbb{Z}^+)$.

We note that $\{3\} = 4\beta^{-1} \in \pi_{A_3}(\beta)$ and $x\alpha \neq 3$ for all $x \in \mathbb{Z}^+$. By Theorem 3.3.5, we conclude that $\alpha \notin LReg(T_E(\mathbb{Z}^+))$. Clearly, $\beta|_{\mathbb{Z}^+\beta}$ is injective and we note that $(x\beta, y\beta) \in E$ implies $(x, y) \in E$ for all $x, y \in \mathbb{Z}^+\beta$. Hence β is right regular by Theorem 3.3.6

From this example, we notice that $\alpha \in LReg(T_E(\mathbb{Z}^+))\backslash RReg(T_E(\mathbb{Z}^+))$ and $\beta \in RReg(T_E(\mathbb{Z}^+))\backslash LReg(T_E(\mathbb{Z}^+))$. By Corollary 3.3.8, we deduce that α and β are not completely regular.

3.4 Regularity for semigroups of transformations that preserve double direction equivalence

Deng et al. [4] have given some characterizations of regularity on $T_{E^*}(X)$. In this section, we then determine the rest regularity of $T_{E^*}(X)$.

Theorem 3.4.1. [4] *Let $\alpha \in T_{E^*}(X)$. Then $\alpha \in Reg(T_{E^*}(X))$ if and only if for every $A \in X/E$, $A \cap X\alpha \neq \emptyset$.*

Theorem 3.4.2. [4] *$T_{E^*}(X)$ is regular if and only if X/E is finite.*

Theorem 3.4.3. *$Reg(T_{E^*}(X))$ is a subsemigroup of $T_{E^*}(X)$.*

Proof. Let $\alpha, \beta \in Reg(T_{E^*}(X))$. To show that $\alpha\beta \in Reg(T_{E^*}(X))$, let $A \in X/E$. By Theorem 3.4.1, we have $A \cap X\beta \neq \emptyset$. Choose $a \in A \cap X\beta$. Thus $a = a'\beta$ for some $a' \in X$ and then $a' \in a\beta^{-1}$. It follows from Proposition 2.2.5(1) that $a\beta^{-1} \subseteq B$ for some $B \in X/E$. We then have $B \cap X\alpha \neq \emptyset$ via Theorem 3.4.1. Let $b \in B \cap X\alpha$ and $b' \in X$ such that $b = b'\alpha$. Since $(a', b) \in E$ and $\beta \in T_{E^*}(X)$, we conclude that $(a, b'\alpha\beta) = (a'\beta, b\beta) \in E$. Therefore $b'\alpha\beta \in A$ and hence $A \cap X\alpha\beta \neq \emptyset$. By Theorem 3.4.1, we have $\alpha\beta \in Reg(T_{E^*}(X))$ as required. □

Theorem 3.4.4. *Let $\alpha \in T_{E^*}(X)$. Then $\alpha \in LReg(T_{E^*}(X))$ if and only if for every $P \in \pi(\alpha)$, $P \cap X\alpha \neq \emptyset$.*

Proof. From Theorem 3.1.2, we have $\alpha = \beta\alpha^2$ for some $\beta \in T(X)$ if and only if for every $P \in \pi(\alpha), P \cap X\alpha \neq \emptyset$. To complete the proof, we have to show that β which is defined in Theorem 3.1.2 belongs to $T_{E^*}(X)$. Let $x, y \in X$ be such that $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$. Since $\pi(\alpha)$ is a partition of X, we note that $x \in P_x$ and $y \in P_y$ for some $P_x, P_y \in \pi(\alpha)$. By Proposition 2.2.5(2), we get $P_x, P_y \subseteq A$. Since $x_{P_x} \in P_x$ and $x_{P_y} \in P_y$, we then have $(x_{P_x}, x_{P_y}) \in E$. Since $\alpha \in T_{E^*}(X)$ and $(x'_{P_x}\alpha, x'_{P_y}\alpha) = (x_{P_x}, x_{P_y}) \in E$, we conclude that $(x\beta, y\beta) = (x'_{P_x}, x'_{P_y}) \in E$. On the other hand, let $x, y \in X$ be such that $(x\beta, y\beta) \in E$. Hence by the definition of β, $x \in P_x$ and $y \in P_y$ for some $P_x, P_y \in \pi(\alpha)$ and satisfy $x\beta = x'_{P_x}, y\beta = x'_{P_y}$. We note by $(x'_{P_x}, x'_{P_y}) \in E$ and $\alpha \in T_{E^*}(X)$ that $(x_{P_x}, x_{P_y}) = (x'_{P_x}\alpha, x'_{P_y}\alpha) \in E$. That is $x_{P_x}, x_{P_y} \in A$ for some $A \in X/E$. Since $x_{P_x} \in P_x, x_{P_y} \in P_y$ and Proposition 2.2.5(2), we observe that $P_x, P_y \subseteq A$, thus $(x, y) \in E$ and therefore $\beta \in T_{E^*}(X)$. □

Theorem 3.4.5. *Let $\alpha \in T_{E^*}(X)$. Then $\alpha \in RReg(T_{E^*}(X))$ if and only if*

(1) *$\alpha|_{X\alpha}$ is an injection and*

(2) *if there exists $A \in X/E$ such that $A \cap X\alpha^2 = \emptyset$, then there exists an injection $\varphi : \{A \in X/E : A \cap X\alpha^2 = \emptyset\} \to \{A \in X/E : A \cap X\alpha = \emptyset\}$.*

Proof. Suppose that $\alpha = \alpha^2\beta$ for some $\beta \in T_{E^*}(X)$. It follows from Theorem 3.1.3 that $\alpha|_{X\alpha}$ is injection. Next, we prove that (2) is hold in the following. Suppose that $\{A \in X/E : A \cap X\alpha^2 = \emptyset\} \neq \emptyset$. Let $A \in \{A \in X/E : A \cap X\alpha^2 = \emptyset\}$. By Proposition 2.2.6(1), we let $A' \in X/E$ such that $A\beta \subseteq A'$. Claim that $A' \cap X\alpha = \emptyset$, suppose not. Let $x \in X$ be such that $x\alpha \in A'$ and choose $a \in A$. Then $a\beta \in A'$ which implies that $(x\alpha^2\beta, a\beta) = (x\alpha, a\beta) \in E$. Since $\beta \in T_{E^*}(X)$, we have $(x\alpha^2, a) \in E$. Hence $x\alpha^2 \in A$ which is a contradiction. Thus $A' \cap X\alpha = \emptyset$. Define $\varphi : \{A \in X/E : A \cap X\alpha^2 = \emptyset\} \to \{A \in X/E : A \cap X\alpha = \emptyset\}$ by

$$A\varphi = A' \text{ for all } A \in X/E \text{ and } A \cap X\alpha^2 = \emptyset.$$

To show that φ is an injection, let $A, B \in \{A \in X/E : A \cap X\alpha^2 = \emptyset\}$ be such that $A\varphi = B\varphi$. By definition of φ, $A\varphi = A'$ and $B\varphi = B'$ where $A\beta \subseteq A'$ and $B\beta \subseteq B'$ for some $A', B' \in X/E$, respectively. It follows by Proposition 2.2.6(2) that $A = A'\beta^{-1}$ and $B = B'\beta^{-1}$. Since $A' = B'$, we deduce that $A = B$. Thus φ is an injection, hence (2) holds.

Conversely, suppose that α satisfies (1) and (2). For any $x \in X\alpha^2$, we choose and fix an element $x' \in X\alpha$ such that $x = x'\alpha$. Let $A \in X/E$ be such that $A \cap X\alpha^2 \neq \emptyset$. Then we fix $a \in A \cap X\alpha$ and define $\beta_A : A \to X$ by

$$x\beta_A = \begin{cases} x', & \text{if } x \in X\alpha^2; \\ a', & \text{otherwise.} \end{cases}$$

Let $A \in X/E$ be such that $A \cap X\alpha = \emptyset$ and $x \in A$, by (2) we fix $\tilde{x} \in A\varphi$ and define $\beta_A : A \to X$ by

$$x\beta_A = \tilde{x} \text{ for all } x \in A.$$

For convenience, we may assume that there exists $A \in X/E$ such that $A \cap X\alpha^2 = \emptyset$. Define $\beta : X \to X$ by

$$\beta|_A = \beta_A \text{ for all } A \in X/E.$$

Since X/E is a partition of X, β is well-defined. Let $x, y \in X$ be such that $(x, y) \in E$. Then $x, y \in A$ for some $A \in X/E$. There are two cases to consider :

Case 1. $A \cap X\alpha^2 = \emptyset$. Then $(x\beta, y\beta) = (\tilde{x}, \tilde{y}) \in E$.

Case 2. $A \cap X\alpha^2 \neq \emptyset$. Without loss of generality, we assume that $x, y \in X\alpha^2$. Hence $x\beta = x'$ and $y\beta = y'$ where $x = x'\alpha$ and $y = y'\alpha$, respectively. Since $\alpha \in T_{E^*}(X)$ and $(x'\alpha, y'\alpha) \in E$, we conclude that $(x\beta, y\beta) = (x', y') \in E$.

Next, let $x, y \in X$ be such that $(x\beta, y\beta) \in E$. Thus $x\beta, y\beta \in B$ for some $B \in X/E$. If $B \cap X\alpha = \emptyset$, then by the definition of β, $x\beta, y\beta \in B = A\varphi$ where $A \in X/E$. Since φ is injective, $x, y \in A$. If $B \cap X\alpha \neq \emptyset$, then by the definition of β, we may assume that $x\beta = x', y\beta = y'$ for some $x', y' \in X\alpha$ and $x = x'\alpha, y = y'\alpha$. Since $(x', y') = (x\beta, y\beta) \in E$ and $\alpha \in T_{E^*}(X)$, we deduce that $(x, y) = (x'\alpha, y'\alpha) \in E$.

It follows that $\beta \in T_{E^*}(X)$. Let $x \in X$, then $x\alpha^2 \in X\alpha^2$ and there exists $(x\alpha^2)' \in X\alpha$ such that $(x\alpha^2)'\alpha = x\alpha^2 = (x\alpha)\alpha$. We note by (1) that $(x\alpha^2)' = x\alpha$. Therefore $x\alpha^2\beta = (x\alpha^2)' = x\alpha$. Hence α is right regular as required. □

Example 7. Let $A_1 = \{1\}, A_2 = \{2,3\}, A_3 = \{4,5,6\}$ and for $n \geq 4$

$$A_n = \left\{x \in \mathbb{Z}^+ : \tfrac{(n-1)n}{2} < x \leq \tfrac{n(n+1)}{2}\right\}.$$

Define $E = \bigcup_{i \in \mathbb{Z}^+} (A_i \times A_i)$. Clearly, E is an equivalence relation on $\mathbb{Z}^+$. Now, we define $\alpha : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$x\alpha = \min A_{n+1} \text{ for all } x \in A_n \text{ and for each } A_n \in \mathbb{Z}^+/E.$$

Since $\mathbb{Z}^+/E$ is a partition of $\mathbb{Z}^+$, α is well-defined. To show $\alpha \in T_{E^*}(\mathbb{Z}^+)$, let $x, y \in \mathbb{Z}^+$ be such that $(x,y) \in E$. Thus $x, y \in A_n$ for some $A_n \in \mathbb{Z}^+/E$. This implies that $(x\alpha, y\alpha) = (\min A_{n+1}, \min A_{n+1}) \in E$. Next, let $x, y \in \mathbb{Z}^+$ be such that $(x\alpha, y\alpha) \in E$. By the definition of α, we then have $x\alpha = y\alpha = \min A_n$ for some $A_n \in \mathbb{Z}^+/E$ and $n > 1$. Therefore $x, y \in A_{n-1}$ and hence $(x,y) \in E$. We deduce that $\alpha \in T_{E^*}(\mathbb{Z}^+)$ as required. We note that $\alpha|_{\mathbb{Z}^+\alpha}$ is injective but since $\{A \in \mathbb{Z}^+/E : A \cap X\alpha^2 = \emptyset\} = \{A_1, A_2\}$ and $\{A \in \mathbb{Z}^+/E : A \cap X\alpha = \emptyset\} = \{A_1\}$, there is no injection from

$$\{A \in \mathbb{Z}^+/E : A \cap X\alpha^2 = \emptyset\} \text{ to } \{A \in \mathbb{Z}^+/E : A \cap X\alpha = \emptyset\}.$$

Hence α does not satisfy condition (2) in Theorem 3.4.5. Hence $\alpha \notin RReg(T_{E^*}(\mathbb{Z}^+))$.

Theorem 3.4.6. *Let $\alpha \in T_{E^*}(X)$. Then $\alpha \in CReg(T_{E^*}(X))$ if and only if for every $P \in \pi(\alpha)$, $|P \cap X\alpha| = 1$.*

Proof. It follows from Theorem 3.1.4 that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$ for some $\beta \in T(X)$ if and only if for every $P \in \pi(\alpha), |P \cap X\alpha| = 1$. It is enough to show that β which defined in Theorem 3.1.4 is an element of $T_{E^*}(X)$. Let $x, y \in X$ be such that $(x,y) \in E$. Then $x, y \in A$ for some $A \in X/E$. Since $\pi(\alpha)$ is a partition of X, $x \in P$ and $y \in Q$ for some $P, Q \in \pi(\alpha)$. This implies that $P \cap A \neq \emptyset$ and $Q \cap A \neq \emptyset$, respectively.

From Proposition 2.2.5(2), $P, Q \subseteq A$. Hence $x_P, x_Q \in A$. Since $\alpha \in T_{E^*}(X)$ and $(x_{P'}\alpha, x_{Q'}\alpha) = (x_P, x_Q) \in E$, we conclude that $(x\beta, y\beta) = (x_{P'}, x_{Q'}) \in E$. Assume that $(x\beta, y\beta) \in E$ for some $x, y \in X$. It follows from the definition of β that $x\beta = x_{P'}$ and $y\beta = x_{Q'}$ where $x \in P$, $y \in Q$ and $P, Q \in \pi(\alpha)$. Since $(x_{P'}, x_{Q'}) = (x\beta, y\beta) \in E$ and $\alpha \in T_{E^*}(X)$, $(x_P, x_Q) = (x_{P'}\alpha, x_{Q'}\alpha) \in E$. This implies that $x_P, x_Q \in A$ for some $A \in X/E$. Since $x_P \in P$ and Proposition 2.2.5(2), we note that $x \in P \subseteq A$. Similarly, we have $y \in Q \subseteq A$. Hence $(x, y) \in E$ which implies that $\beta \in T_{E^*}(X)$. □

Corollary 3.4.7. *Let $\alpha \in T_{E^*}(X)$. Then α is completely regular if and only if α is both left and right regular.*

Proof. Assume that α is both left and right regular. Thus $\alpha \in RReg(T(X)) \cap LReg(T(X))$. By Corollary 3.1.5, we have $\alpha \in CReg(T(X))$. It follows from Theorem 3.1.4 and Theorem 3.4.6 that $\alpha \in CReg(T_{E^*}(X))$. □

3.5 Regularity for self-E-preserving transformation semigroups

Theorem 3.5.1. *Every element of $T_{SE}(X)$ is regular. Hence, $T_{SE}(X)$ is a regular semigroup.*

Proof. Let $\alpha \in T_{SE}(X)$. For each $x \in X\alpha$, choose $x' \in X$ such that $x = x'\alpha$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} x', & \text{if } x \in X\alpha; \\ x, & \text{otherwise.} \end{cases}$$

It is clear that $\beta \in T(X)$. Let $x \in X$. Then

$$(x, x\beta) = \begin{cases} (x'\alpha, x') \in E, & \text{if } x \in X\alpha; \\ (x, x) \in E, & \text{otherwise} \end{cases}$$

and

$$x\alpha\beta\alpha = (x\alpha)'\alpha = x\alpha.$$

This proves that $\beta \in T_{SE}(X)$ and $\alpha = \alpha\beta\alpha$, respectively. □

The following example shows that there is an element of $T_{SE}(X)$ is neither left regular nor right regular.

Example 8. We note that the relation E which is defined in Example 7 is an equivalence relation on $\mathbb{Z}^+$. Define $\alpha : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$x\alpha = \begin{cases} \min A_n, & \text{if } x \in A_n \text{ and } x \text{ is odd;} \\ \max A_n, & \text{if } x \in A_n \text{ and } x \text{ is even.} \end{cases}$$

To show $\alpha \in T_{SE}(\mathbb{Z}^+)$, let $x \in \mathbb{Z}^+$. Since $\{A_n : n \in \mathbb{Z}^+\}$ is a partition of $\mathbb{Z}^+$, there exists $n \in \mathbb{Z}^+$ such that $x \in A_n$. By the definition of α, $x\alpha \in \{\min A_n, \max A_n\}$. Hence $(x, x\alpha) \in A_n \times A_n \subseteq E$, so $\alpha \in T_{SE}(\mathbb{Z}^+)$. Next, we verify that α is neither left regular nor right regular of $T_{SE}(\mathbb{Z}^+)$. Suppose that α is a left regular element of $T_{SE}(\mathbb{Z}^+)$. Then $\alpha = \beta\alpha^2$ for some $\beta \in T_{SE}(\mathbb{Z}^+)$. Since $(5\beta\alpha)\alpha = 5\beta\alpha^2 = 5\alpha = 4$, we have $5\beta\alpha = 5$. Since $(5, 5\beta) \in E$ and the definition of E, we get $5\beta \in A_3$. This is a contradiction because $5 = (5\beta)\alpha \in A_3\alpha = \{4, 6\}$. Hence α is not left regular. Suppose that α is right regular element of $T_{SE}(\mathbb{Z}^+)$. Then $\alpha = \alpha^2\beta$ for some $\beta \in T_{SE}(\mathbb{Z}^+)$. We see that $\{4, 6\}\alpha = \{6\}$ and $5\alpha = 4$. Thus

$$6 = 6\alpha = 6\alpha^2\beta = 6\alpha\beta = 4\alpha\beta = 5\alpha^2\beta = 5\alpha = 4$$

which is a contradiction. Hence α is not a right regular.

Theorem 3.5.2. *Let $\alpha \in T_{SE}(X)$. Then $\alpha \in LReg(T_{SE}(X))$ if and only if for every $P \in \pi(\alpha), P \cap X\alpha \neq \emptyset$.*

Proof. By Theorem 3.1.2, we note that $\alpha = \beta\alpha^2$ for some $\beta \in T(X)$ if and only if $P \cap X\alpha \neq \emptyset$ for all $P \in \pi(\alpha)$. It is enough to show that β is defined in Theorem 3.1.2

belongs to $T_{SE}(X)$. Let $x \in X$. Since $\alpha \in T_{SE}(X)$, we have $(x, x\alpha), (x_{P_x}, x_{P_x}\alpha) \in E$. By the transitivity of E and $x_{P_x}\alpha = x\alpha$, we conclude that $(x, x_{P_x}) \in E$. Since $(x'_{P_x}, x_{P_x}) = (x'_{P_x}, x'_{P_x}\alpha) \in E$, we deduce $(x, x\beta) = (x, x'_{P_x}) \in E$. Hence $\beta \in T_{SE}(X)$. □

Theorem 3.5.3. *Let $\alpha \in T_{SE}(X)$. Then $\alpha \in RReg(T_{SE}(X))$ if and only if $\alpha|_{X\alpha}$ is injection.*

Proof. We note by Theorem 3.1.3 that $\alpha = \alpha^2\beta$ for some $\beta \in T(X)$ if and only if $\alpha|_{X\alpha}$ is injection. We will verify that β in Theorem 3.1.3 belongs to $T_{SE}(X)$. Let $x \in X$.

$$(x, x\beta) = \begin{cases} (x'\alpha, x') \in E, & \text{if } x \in X\alpha^2; \\ (x, x) \in E, & \text{otherwise.} \end{cases}$$

This implies that $\beta \in T_{SE}(X)$. □

Theorem 3.5.4. *Let $\alpha \in T_{SE}(X)$. Then $\alpha \in CReg(T_{SE}(X))$ if and only if for every $P \in \pi(\alpha), |P \cap X\alpha| = 1$.*

Proof. It follows from Theorem 3.1.4 that $\alpha \in CReg(T(X))$ if and only if for every $P \in \pi(\alpha), |P \cap X\alpha| = 1$. It is enough to show that β which is defined in Theorem 3.1.4 is in $T_{SE}(X)$. Let $x \in X$. By $\pi(\alpha)$ is a partition of X, we have $x \in P$ for some $P \in \pi(\alpha)$. Since $x\alpha = x_P\alpha$, $(x, x_P\alpha) = (x, x\alpha) \in E$. Since $(x, x_P\alpha) \in E$ and $(x_P, x_P\alpha) \in E$, we conclude that $(x, x_P) \in E$ by transitivity of E. Then $(x, x_{P'}\alpha) = (x, x_P) \in E$. Since $(x_{P'}, x_{P'}\alpha) \in E$ and by transitive of E, $(x, x_{P'}) \in E$. It follows that $(x, x\beta) = (x, x_{P'}) \in E$, hence $\beta \in T_{SE}(X)$. □

By using the proof as given for Theorem 3.1.5, we then have the following characterization.

Corollary 3.5.5. $CReg(T_{SE}(X)) = RReg(T_{SE}(X)) \cap LReg(T_{SE}(X))$.

Example 9. Let $A_1 = \{2n - 1 : n \in \mathbb{Z}^+\}, A_2 = \{2n : n \in \mathbb{Z}^+\}$ and

$$E = (A_1 \times A_1) \cup (A_2 \times A_2).$$

Clearly, E is an equivalence relation on $\mathbb{Z}^+$. Recall α and β are defined in Example 3. We observe that $A_1\alpha \subseteq A_1$ and $A_2\alpha \subseteq A_2$, hence $\alpha \in T_{SE}(\mathbb{Z}^+)$. From Example 3, we show that $P \cap \mathbb{Z}^+\alpha \neq \emptyset$ for all $P \in \pi(\alpha)$ and $\alpha|_{\mathbb{Z}^+\alpha}$ is not injective. By Theorem 3.5.2 and Theorem 3.5.3, we conclude that α is left regular but not right regular, respectively. Similarly, we note that $\beta \in T_{SE}(\mathbb{Z}^+)$ and β is right regular but not left regular. Hence from Corollary 3.5.5, α and β are not completely regular.

3.6 Regularity for order preserving transformation semigroups

For this section, we let $(X, \leq)$ be a totally ordered set. The following example shows that in general, $\mathcal{O}(X)$ is not a regular semigroup.

Example 10. Define $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$x\alpha = x + 1 \text{ for all } x \in \mathbb{R}^+.$$

Then $\alpha \in \mathcal{O}(\mathbb{R}^+)$. We claim that α is not regular in $O(\mathbb{R}^+)$.

Assume that $\alpha = \alpha\beta\alpha$ for some $\beta \in O(\mathbb{R}^+)$. Since α is injective, $x = x\beta\alpha$ for all $x \in \mathbb{R}^+$. Thus $\mathbb{R}^+ = \mathbb{R}^+\alpha\beta = (1, \infty)\beta$. So, $1\beta = a\beta$ for some $a \in (1, \infty)$. For fix $b \in (1, a)$, we note that $1\beta \leq b\beta \leq a\beta = 1\beta$. This implies that $b\beta = a\beta$. Consider,

$$a = (a-1)\alpha = (a-1)\alpha\beta\alpha = a\beta\alpha = b\beta\alpha = (b-1)\alpha\beta\alpha = (b-1)\alpha = b.$$

It is a contradiction.

Theorem 3.6.1. *Let $\alpha \in \mathcal{O}(X)$. Then $\alpha \in Reg(\mathcal{O}(X))$ if and only if there exists a partition π of X such that $|P \cap X\alpha| = 1$ for all $P \in \pi$ and $(\pi, \preceq)$ is a totally ordered set.*

Proof. Suppose that $\alpha = \alpha\beta\alpha$ for some $\beta \in \mathcal{O}(X)$. For each $P \in \pi(\alpha)$, we let

$$P' = \cup\{Q \in \pi(\beta) : Q\beta_* \in P\} \text{ and denote } \pi = \{P' : P \in \pi(\alpha)\}.$$

Let $P \in \pi(\alpha)$ and $x \in P$. By assumption, $x\alpha = x\alpha\beta\alpha$. Since $\pi(\beta)$ is a partition of X, $x\alpha \in Q$ for some $Q \in \pi(\beta)$. Hence $P\alpha_* = x\alpha\beta\alpha = Q\beta_*\alpha$. This implies that $Q\beta_* \in P$ and then $\emptyset \neq Q \subseteq P'$. Moreover, we note that $x\alpha \in P' \cap X\alpha$. This means that $P' \cap X\alpha \neq \emptyset$. Let $a, b \in P' \cap X\alpha$. Then there exist $Q, \tilde{Q} \in \pi(\beta)$ such that $Q\beta_*, \tilde{Q}\beta_* \in P$ and $a \in Q, b \in \tilde{Q}$. Since $a, b \in X\alpha$, $a = a'\alpha$ and $b = b'\alpha$ for some $a', b' \in X$. It follows that

$$a = a'\alpha = a'\alpha\beta\alpha = a\beta\alpha = Q\beta_*\alpha = P\alpha_* = \tilde{Q}\beta_*\alpha = b\beta\alpha = b'\alpha\beta\alpha = b'\alpha = b.$$

Therefore $|P' \cap X\alpha| = 1$ for each $P' \in \pi$. Claim that π is a partition of X, it is clear that $\cup\pi = \cup\pi(\beta) = X$. Let $P', Q' \in \pi$ be such that $P' \cap Q' \neq \emptyset$. Then there exist $P, Q \in \pi(\alpha)$ such that $P' = \cup\{\tilde{P} \in \pi(\beta) : \tilde{P}\beta_* \in P\}$ and $Q' = \cup\{\tilde{Q} \in \pi(\beta) : \tilde{Q}\beta_* \in Q\}$. Let $x \in P' \cap Q'$. Since $\pi(\beta)$ is a partition of X, $x \in \tilde{P}$ for some $\tilde{P} \in \pi(\beta)$. By the definition of P' and Q', $\tilde{P}\beta_* \in P \cap Q$. Since $\pi(\alpha)$ is a partition of X, $P = Q$. Thus $P' = Q'$. This shows that π is a partition of X. We need to prove that $(\pi, \preceq)$ is a totally ordered set. Since π is a partition of X, $(\pi, \preceq)$ is a partially ordered set. Let $P', Q' \in \pi$ be such that $Q' \not\preceq P'$. Then there exist $a \in P'$ and $b \in Q'$ such that $b \not\leq a$. Since $(x, \leq)$ is a totally ordered set, $a < b$. To verify that $P' \preceq Q'$, suppose that there exist $x \in P'$ and $y \in Q'$ such that $x \not\leq y$. By $(X, \leq)$ is a totally ordered set, we have $y < x$. Since $x, a \in P'$, we conclude that $x\beta, a\beta \in P$. Similarly, we then have $y\beta, b\beta \in Q$. Since $P' \neq Q'$, $P \neq Q$. Consider,

$$P\alpha_* = a\beta\alpha \leq b\beta\alpha = Q\alpha_*.$$

It follows from $P \neq Q$ that $P\alpha_* < Q\alpha_*$. This implies that

$$P\alpha_* < Q\alpha_* = y\beta\alpha \leq x\beta\alpha = P\alpha_*$$

which is a contradiction. Thus $x \leq y$ for all $x \in P$ and $y \in Q$. Hence $P \preceq Q$ and then $(\pi, \preceq)$ is a totally ordered set.

Conversely, assume there exists a partition π of X such that $|P \cap X\alpha| = 1$ for each $P \in \pi$ and $(\pi, \preceq)$ is a totally ordered set. For each $P \in \pi$, we choose $x_P \in X$ be such that $P \cap X\alpha = \{x_P\alpha\}$. Let $x \in X$. Since π is a partition of X, $x \in P_x$ for some $P_x \in \pi$. Define $\beta : X \to X$ by

$$x\beta = x_{P_x} \text{ for all } x \in X.$$

By π is a partition of X, β is well-defined. Let $x, y \in X$ be such that $x \leq y$. Then there exist $P_x, P_y \in \pi$ be such that $x \in P_x$ and $y \in P_y$. If $P_x = P_y$, then $x_{P_x} = x_{P_y}$. Suppose that $P_x \neq P_y$. Since $x \leq y$ and by assumption, $P_x \preceq P_y$. To show that $x_{P_x} \leq x_{P_y}$, assume that $x_{P_x} \not\leq x_{P_y}$. Since $(X, \leq)$ is a totally ordered set, we have $x_{P_y} < x_{P_x}$. It is clear by $\alpha \in \mathcal{O}(X)$ that $x_{P_x}\alpha \leq x_{P_y}\alpha$. By assumption, $x_{P_x}\alpha \in P_x$ and $x_{P_y}\alpha \in P_y$. From P_x and P_y are distinct elements in π and π is a partition of X, we conclude that $x_{P_y}\alpha < x_{P_x}\alpha$. We note by $P_x \preceq P_y$ that $x_{P_x}\alpha \leq x_{P_y}\alpha$. This is a contradiction. Thus $x_{P_x} \leq x_{P_y}$ which implies that $\beta \in \mathcal{O}(X)$. Finally, let $x \in X$. We note by assumption that $x\alpha \in P_{x\alpha} \cap X\alpha = \{x_{P_{x\alpha}}\alpha\}$ where $x\alpha \in P_{x\alpha}$ and $P_{x\alpha} \in \pi$. Thus $x\alpha\beta\alpha = x_{P_{x\alpha}}\alpha = x\alpha$. □

Theorem 3.6.2. *Let $\alpha \in \mathcal{O}(X)$. Then $\alpha \in LReg(\mathcal{O}(X))$ if and only if for every $P \in \pi(\alpha), P \cap X\alpha \neq \emptyset$.*

Proof. By Theorem 3.1.2, we note that $\alpha = \beta\alpha^2$ for some $\beta \in T(X)$ if and only if $P \cap X\alpha \neq \emptyset$ for all $P \in \pi(\alpha)$. It is enough to show that β defined in Theorem 3.1.2 belong to $\mathcal{O}(X)$. Let $x, y \in X$ be such that $x \leq y$. By the definition of β, $x\beta = x_{P_x}$ and $y\beta = x_{P_y}$ where $P_x, P_y \in \pi(\alpha)$ and $x \in P_x, y \in P_y$. If $P_x = P_y$, then $x\beta = x_{P_x} = x_{P_y} = y\beta$. Assume that $P_x \neq P_y$. It follows from $x \leq y$ and Proposition 2.2.11 that $P_x \preceq P_y$. To show that $x_{P_x} \leq x_{P_y}$, assume that $x_{P_x} \not\leq x_{P_y}$. Since $(X, \leq)$ is a totally ordered set, $x_{P_y} < x_{P_x}$. It follows from $\alpha \in \mathcal{O}(X)$ that $x_{P_y}\alpha \leq x_{P_x}\alpha$. By assumption, $x_{P_y}\alpha \in P_y$ and $x_{P_x}\alpha \in P_x$. We note that $x_{P_y}\alpha < x_{P_x}\alpha$ from $P_x \cap P_y = \emptyset$. Since $P_x \preceq P_y$, we conclude that $x_{P_x}\alpha \leq x_{P_y}\alpha < x_{P_x}\alpha$. This is a contradiction. Hence $x\beta = x_{P_x} \leq x_{P_y} = y\beta$ and thus $\beta \in \mathcal{O}(X)$ as we wished to show. □

Theorem 3.6.3. *Let $\alpha \in \mathcal{O}(X)$. Then $\alpha \in RReg(\mathcal{O}(X))$ if and only if*

(1) *$\alpha|_{X\alpha}$ is an injection and*

(2) *there exist* $x_P \in X$ *corresponding to* P *for all* $P \in \pi(\alpha)$ *such that* $P \preceq Q$ *implies* $x_P \leq x_Q$ *for all* $P, Q \in \pi(\alpha)$ *and for* $P \in \pi(\alpha)$ *such that* $P \cap X\alpha^2 \neq \emptyset$ *implies* $x_P \in X\alpha$ *and* $x_P\alpha \in P$.

Proof. Suppose that $\alpha = \alpha^2\beta$ for some $\beta \in \mathcal{O}(X)$. From Theorem 3.1.3, we note that $\alpha|_{X\alpha}$ is an injection. Let $P \in \pi(\alpha)$ be such that $P \cap X\alpha^2 \neq \emptyset$ and $a, b \in P \cap X\alpha^2$. Then $a\alpha = b\alpha$ and $a, b \in X\alpha$. By $\alpha|_{X\alpha}$ is injective, we have $a = b$. For each $P \in \pi(\alpha)$ such that $P \cap X\alpha^2 \neq \emptyset$, we let $x_P = x\beta$ where $x \in P \cap X\alpha^2$. For each $P \in \pi(\alpha)$ such that $P \cap X\alpha^2 = \emptyset$, we choose $x \in P$ and let $x_P = x\beta$. Let $P, Q \in \pi(\alpha)$ be such that $P \preceq Q$. We note that $x\beta = x_P$ and $y\beta = x_Q$ for some $x \in P$ and $y \in Q$. If $P = Q$, then $x_P = x_Q$. Assume that $P \neq Q$. It follows from $P \preceq Q$ that $x \leq y$. Since $\beta \in \mathcal{O}(X)$, we get $x_P = x\beta \leq y\beta = x_Q$. Let $P \in \pi(\alpha)$ be such that $P \cap X\alpha^2 \neq \emptyset$. To show that $x_P \in X\alpha$ and $x_P\alpha \in P$, let $x \in P \cap X\alpha^2$. Then there exists $x' \in X$ such that $x'\alpha^2 = x$. Note that $x_P = x\beta = x'\alpha^2\beta = x'\alpha \in X\alpha$ and $x_P\alpha = x\beta\alpha = x'\alpha^2\beta\alpha = (x'\alpha)\alpha = x'\alpha^2 = x \in P$. Thus (2) is true.

To prove the converse, assume that (1) and (2) hold. Let $x \in X$. By $\pi(\alpha)$ is a partition of X, $x \in P_x$ for some $P_x \in \pi(\alpha)$. Define $\beta : X \to X$ by

$$x\beta = x_{P_x} \text{ for all } x \in X.$$

Clearly, β is well-defined. Let $x, y \in X$ be such that $x \leq y$. Then there exist $P_x, P_y \in \pi(\alpha)$ such that $x \in P_x$ and $y \in P_y$. If $P_x = P_y$, then $x\beta = y\beta$. Suppose that $P_x \neq P_y$. Since $x \leq y$, by Proposition 2.2.11, we have $P_x \preceq P_y$. It follows from (2) that $x\beta = x_{P_x} \leq x_{P_y} = y\beta$. Hence $\beta \in \mathcal{O}(X)$. Finally, let $x \in X$. Then $x\alpha^2 \in X\alpha^2$. By the definition of β, we have $(x\alpha^2)\beta = x_{P_{x\alpha^2}}$ where $x\alpha^2 \in P_{x\alpha^2}$ and $P_{x\alpha^2} \in \pi(\alpha)$. This means that $P_{x\alpha^2} \cap X\alpha^2 \neq \emptyset$. We note from (2) that $x_{P_{x\alpha^2}} \in X\alpha$ and $x_{P_{x\alpha^2}}\alpha \in P_{x\alpha^2}$. That is $(x_{P_{x\alpha^2}}\alpha)\alpha = (x\alpha^2)\alpha$. It follows from (1) that $x_{P_{x\alpha^2}}\alpha = x\alpha^2 = (x\alpha)\alpha$. Since $x_{P_{x\alpha^2}} \in X\alpha$ and by (1), we then have $x_{P_{x\alpha^2}} = x\alpha$. Thus $x\alpha^2\beta = x\alpha$ and therefore $\alpha \in RReg(\mathcal{O}(X))$ as required. □

Theorem 3.6.4. *Let* $\alpha \in \mathcal{O}(X)$. *Then* $\alpha \in CReg(\mathcal{O}(X))$ *if and only if for every* $P \in \pi(\alpha)$, $|P \cap X\alpha| = 1$.

Proof. It follows from Theorem 3.1.4 that $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$ for some $\beta \in T(X)$ if and only if for every $P \in \pi(\alpha), |P \cap X\alpha| = 1$. It is enough to show that β which defined in Theorem 3.1.4 belongs to $\mathcal{O}(X)$. Let $x, y \in X$ be such that $x \leq y$. We note that $x \in P$ and $y \in Q$ for some $P, Q \in \pi(\alpha)$. If $P = Q$, then by assumption we have $x_P = x_Q$ and hence $x\beta = x_{P'} = x_{Q'} = y\beta$. Suppose that $P \neq Q$. It follows from Proposition 2.2.11 that $P \preceq Q$. This implies that $x_P \leq x_Q$. Then there exist $P', Q' \in \pi(\alpha)$ and $x_{P'} \in P' \cap X\alpha, x_{Q'} \in Q' \cap X\alpha$ such that $P'\alpha_* = x_{p'}\alpha = x_P$ and $Q'\alpha_* = x_{Q'}\alpha = x_Q$. Since $P'\alpha_* = x_P \leq x_Q = Q'\alpha_*$ and by Proposition 2.2.11, we conclude that $P' \preceq Q'$. Hence $x\beta = x_{P'} \leq x_{Q'} = y\beta$. Therefore $\beta \in \mathcal{O}(X)$. □

Corollary 3.6.5. *Let $\alpha \in \mathcal{O}(X)$. Then α is completely regular if and only if α is both left and right regular.*

3.7 Regularity for E-order-preserving transformation semigroups

Throughout of this section, we assume that $(X, \leq)$ is a totally ordered set and E is an equivalence on X. The following example shows that $EOP(X)$ need not to be regular and there exists an element of $EOP(X)$ which is neither left regular nor right regular.

Example 11. Let $A_1 = \{3(k-1)+1 : k \in \mathbb{Z}^+\}, A_2 = \{3(k-1)+2 : k \in \mathbb{Z}^+\}$ and $A_3 = \{3k : k \in \mathbb{Z}^+\}$. Define $E = \bigcup_{i=1}^{3} A_i \times A_i$. It is clearly that E is an equivalence relation on $\mathbb{Z}^+$ and $\mathbb{Z}^+/E = \{A_1, A_2, A_3\}$. Define $\alpha : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$x\alpha = \begin{cases} 6k-5+r, & \text{if } r = 1,2; \\ 6k, & \text{if } r = 3, \end{cases}$$

where $x = 3(k-1)+r$ for some $k, r \in \mathbb{Z}^+$ and $r \leq 3$. It is easy to verify that $\alpha \in EOP(\mathbb{Z}^+)$. Assume that $\alpha = \alpha\beta\alpha$ for some $\beta \in EOP(\mathbb{Z}^+)$. Since

$$2\alpha = 2\alpha\beta\alpha = 3\beta\alpha \text{ and } 3\alpha = 3\alpha\beta\alpha = 6\beta\alpha$$

and α is an injective, we deduce that $2 = 3\beta$ and $3 = 6\beta$. Because of $3 \leq 6$ and $(3, 6) \in E$, we then have $(2, 3) = (3\beta, 6\beta) \in E$ which is a contradiction. Therefore α is not regular of $EOP(\mathbb{Z}^+)$.

Suppose that $\alpha = \alpha^2\beta$ for some $\beta \in EOP(\mathbb{Z}^+)$. Since $\beta \in EOP(\mathbb{Z}^+)$ and $(3, 6) \in E$ and $3 \leq 6$, we have that $(3\beta, 6\beta) \in E$. We note that

$$2 = 1\alpha = 1\alpha^2\beta = 2\alpha\beta = 3\beta \text{ and } 3 = 2\alpha = 2\alpha^2\beta = 3\alpha\beta = 6\beta.$$

It would follow that $(3\beta, 6\beta) = (2, 3) \notin E$ which is a contradiction. This proves that α is not right regular of $EOP(\mathbb{Z}^+)$.

Next, suppose that $\alpha = \beta\alpha^2$ for some $\beta \in EOP(\mathbb{Z}^+)$. Since α is injective and $1\alpha = 1\beta\alpha^2 = (1\beta\alpha)\alpha$, $1 = 1\beta\alpha$, we conclude that $1 \in \mathbb{Z}^+\alpha$. This contradiction shows that α is not left regular of $EOP(\mathbb{Z}^+)$.

Example 11 inspires us to find necessary and sufficient conditions under which an element of $EOP(X)$ is regular, right regular, left regular or completely regular, respectively.

Theorem 3.7.1. *Let $\alpha \in EOP(X)$. Then $\alpha \in Reg(EOP(X))$ if and only if for every $A \in X/E$, there exists a partition π_A of A such that $(\pi_A, \preceq)$ is a totally ordered set and for every $P \in \pi_A$, there exists $x_P \in X$ corresponding to P such that $P \cap X\alpha \subseteq \{x_P\alpha\}$ and $P \preceq Q$ implies $x_P \leq x_Q$ and $(x_P, x_Q) \in E$ for $P, Q \in \pi_A$.*

Proof. Suppose that $\alpha = \alpha\beta\alpha$ for some $\beta \in EOP(X)$. Let $A \in X/E$. We note by Proposition 2.2.12(2) that

$$\pi(A, \beta) = \{P' \cap A : P' \in \pi_A(\beta)\}$$

is a totally ordered set. By the definition of $\pi(A, \beta)$, we have $\pi(A, \beta)$ is a partition of A. For every $P \in \pi(A, \beta)$, there exists $P' \in \pi_A(\beta)$ such that $P = P' \cap A$. We denote $x_P = P'\beta_*$. Let $P \in \pi(A, \beta)$ be such that $P \cap X\alpha \neq \emptyset$. We have that $P = P' \cap A$ for some $P' \in \pi_A(\beta)$. For arbitrary $x \in P \cap X\alpha$, $x = x'\alpha$ for some $x' \in X$. Hence

$$x = x'\alpha = x'\alpha\beta\alpha = x\beta\alpha = P'\beta_*\alpha = x_P\alpha.$$

This means that $P \cap X\alpha = \{x_P\alpha\}$. Let $P, Q \in \pi(A, \beta)$ be such that $P \preceq Q$. Then $P = P' \cap A$ and $Q = Q' \cap A$ for some $P', Q' \in \pi_A(\beta)$. Choose $x \in P$ and $y \in Q$. If $P = Q$, then $x_P = x_Q$. Assume that $P \neq Q$. By $P \preceq Q$, we have $x \leq y$. Since $(x, y) \in E$ and $x \leq y$, $(x_P, x_Q) = (x\beta, y\beta) \in E$ and $x_P = x\beta \leq y\beta = x_Q$.

For the converse, suppose that for every $A \in X/E$, there exists a partition π_A of A such that $(\pi_A, \preceq)$ is a totally ordered set and for every $P \in \pi_A$, there exists $x_P \in X$ corresponding to P such that $P \cap X\alpha \subseteq \{x_P\alpha\}$ and $P \preceq Q$ implies $x_P \leq x_Q$ and $(x_P, x_Q) \in E$ for $P, Q \in \pi_A$. We will construct $\beta \in EOP(X)$ in the following, let $x \in X$. Since X/E is a partition of X, $x \in A$ for some $A \in X/E$. We note by assumption that $x \in P_x$ for some $P_x \in \pi_A$. Define $\beta : X \to X$ by

$$x\beta = x_{P_x} \text{ for all } x \in X.$$

Clearly, β is well-defined. Let $x, y \in X$ be such that $(x, y) \in E$ and $x \leq y$. Then there exists $A \in X/E$ such that $x, y \in A$. Thus $x \in P_x$ and $y \in P_y$ for some $P_x, P_y \in \pi_A$. Since $x \leq y$ and by assumption, $P_x \preceq P_y$. It follows that $(x\beta, y\beta) = (x_{P_x}, x_{P_y}) \in E$ and $x\beta = x_{P_x} \leq x_{P_y} = y\beta$. Therefore $\beta \in EOP(X)$. Finally, let $x \in X$. Then $x\alpha \in A$ for some $A \in X/E$. It follows from the definition of β that $x\alpha\beta = x_{P_{x\alpha}}$ where $x\alpha \in P_{x\alpha}$ and $P_{x\alpha} \in \pi_A$. It is clear from assumption that $x\alpha \in P_{x\alpha} \cap X\alpha = \{x_{P_{x\alpha}}\alpha\}$. Thus $x\alpha\beta\alpha = x_{P_{x\alpha}}\alpha = x\alpha$. Hence the theorem is thereby proved. □

Corollary 3.7.2. *Let $\alpha \in EOP(X)$. Then $\alpha \in Reg(EOP(X))$ if and only if for every $A \in X/E$ such that $A \cap X\alpha \neq \emptyset$, there exist a partition π_A of A such that $(\pi_A, \preceq)$ is a totally ordered set and $B \in X/E$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A$.*

Proof. Assume that $\alpha \in Reg(EOP(X))$. Then there exists $\beta \in EOP(X)$ such that $\alpha = \alpha\beta\alpha$. Let $A \in X/E$ be such that $A \cap X\alpha \neq \emptyset$. By Proposition 2.2.12(1), we have $A\beta \subseteq B$ for some $B \in X/E$. Claim that $B\alpha \subseteq A$, let $b \in B$. Since $A \cap X\alpha \neq \emptyset$, we choose $a \in A \cap X\alpha$, that is $a = a'\alpha$ for some $a' \in X$. We note that $(a\beta, b) \in E$. Since $(X, \leq)$ is a totally ordered set, we conclude that $a\beta \leq b$ or $b \leq a\beta$. It follows from $\alpha \in EOP(X)$

that $(a\beta\alpha, b\alpha) \in E$. Note that $a\beta\alpha = a'\alpha\beta\alpha = a'\alpha = a \in A$, hence $b\alpha \in A$. So we have the claim. For $P \in \pi_B(\alpha)$, we denote

$$P' = \cup\{Q \in \pi(A,\beta) : Q\beta \subseteq P\} \text{ and define } \pi_A = \{P' : P \in \pi_B(\alpha)\}.$$

Let $P \in \pi_B(\alpha)$ and $x \in P \cap B$. We note that $x\alpha \in A$ and $x\alpha = x\alpha\beta\alpha$. From $\pi(A,\beta)$ is a partition of A, $x\alpha \in Q$ for some $Q \in \pi(A,\beta)$. By the definition of $\pi(A,\beta)$, we have that $Q = \tilde{Q} \cap A$ for some $\tilde{Q} \in \pi_A(\beta)$. This means that $x\alpha\beta \in Q\beta = \{\tilde{Q}\beta_*\}$, thus $\tilde{Q}\beta_*\alpha = x\alpha = P\alpha_*$. Hence $Q\beta = \{\tilde{Q}\beta_*\} \subseteq P$. This shows that $\emptyset \neq Q \subseteq P'$ and $x\alpha \in P' \cap B\alpha \subseteq P' \cap X\alpha$. Hence $\emptyset \neq P' \cap B\alpha \subseteq P' \cap X\alpha$. To verify that $|P' \cap X\alpha| = 1$, let $a, b \in P' \cap X\alpha$. That is $a = a'\alpha$ and $b = b'\alpha$ for some $a', b' \in X$. By $(X, \leq)$ is a totally ordered set, we assume that $a \leq b$. Since $a, b \in P' \subseteq \cup\pi(A,\beta) \subseteq A$, we have $(a,b) \in E$. It follows from $\beta \in EOP(X)$ that $(a\beta, b\beta) \in E$. By the definition of P', $a\beta, b\beta \in P$ for some $P \in \pi_B(\alpha)$. Hence $a\beta\alpha = b\beta\alpha$. We conclude that

$$a = a'\alpha = a'\alpha\beta\alpha = a\beta\alpha = b\beta\alpha = b'\alpha\beta\alpha = b'\alpha = b.$$

This implies that $|P' \cap X\alpha| = 1$ and hence $|P' \cap B\alpha| = |P' \cap X\alpha| = 1$ for all $P' \in \pi_A$. To show π_A is a partition of A. It is clear that $\cup\pi_A \subseteq \cup\pi(\beta, A) = A$, let $x \in A$. By $\pi(A,\beta)$ is a partition of A, we have $x \in Q$ for some $Q \in \pi(A,\beta)$. We note by $A\beta \subseteq B$ that $x\beta \in P$ for some $P \in \pi_B(\alpha)$. Hence $x \in Q \subseteq P' \subseteq \cup\pi_A$, then $\cup\pi_A = A$. Let $P', Q' \in \pi_A$ be such that $P' \cap Q' \neq \emptyset$. Then there exist $P, Q \in \pi_B(\alpha)$ such that $P' = \cup\{\tilde{P} \in \pi(A,\beta) : \tilde{P}\beta \subseteq P\}$ and $Q' = \cup\{\tilde{Q} \in \pi(A,\beta) : \tilde{Q}\beta \subseteq Q\}$. Let $x \in P' \cap Q'$. Since $\pi(A,\beta)$ is a partition of A, $x \in \tilde{P}$ for some $\tilde{P} \in \pi(A,\beta)$. By the definition of P' and Q', $\tilde{P}\beta \subseteq P \cap Q$. Since $\pi(\alpha)$ is a partition of X, we conclude that $P = Q$, hence $P' = Q'$. Therefore π_A is a partition of A. Next, we will verify that $(\pi_A, \preceq)$ is a totally ordered set. Since π_A is a partition of A, we get $(\pi_A, \preceq)$ is a partially ordered set. Let $P', Q' \in \pi_A$ be such that $Q' \npreceq P'$. Since $Q' \npreceq P'$, we conclude that $Q' \neq P'$ and $q \nleq p$ for some $q \in Q'$ and $p \in P'$. It follows from $(X, \leq)$ is a totally ordered set that $p < q$. Claim that $P' \preceq Q'$, suppose that there exist $x \in P'$ and $y \in Q'$ such that $y < x$. Since $x, p \in P'$, we have $x\beta, p\beta \in P \cap B$. Similarly, we deduce that $y\beta, q\beta \in Q \cap B$. Since $P' \neq Q'$, $P \neq Q$. We note here that

$$P\alpha_* = p\beta\alpha \leq q\beta\alpha = Q\alpha_*.$$

From $P \neq Q$, we have $P\alpha_* < Q\alpha_*$. Then

$$P\alpha_* < Q\alpha_* = y\beta\alpha \leq x\beta\alpha = P\alpha_*$$

which is a contradiction. Thus $x \leq y$ for all $x \in P'$ and $y \in Q'$. Hence $P' \preceq Q'$ and then $(\pi_A, \preceq)$ is a totally ordered set.

Conversely, suppose that for every $A \in X/E$ such that $A \cap X\alpha \neq \emptyset$, there exist a partition π_A of A such that $(\pi_A, \preceq)$ is a totally ordered set and $B \in X/E$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A$. Let $A \in X/E$. If $A \cap X\alpha = \emptyset$, then we let $\pi_A = \{A\}$ and fix $x_A \in A$. Clearly, $A \cap X\alpha \subseteq \{x_A\alpha\}$ and for $P, Q \in \pi_A$ such that $P \preceq Q$ implies $x_P \leq x_Q$ and $(x_P, x_Q) \in E$. Assume that $A \cap X\alpha \neq \emptyset$. It follows that there exist a partition π_A of A such that $(\pi_A, \preceq)$ is a totally ordered set and $B \in X/E$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A$. Then for each $P \in \pi_A$, we fix $x_P \in B$ such that $x_P\alpha \in P \cap B\alpha = P \cap X\alpha$. Let $P, Q \in \pi_A$ be such that $P \preceq Q$. If $P = Q$, then $x_P = x_Q$. Suppose that $P \neq Q$ and hence $x \leq y$ for all $x \in P$, $y \in Q$. We conclude that $x_P\alpha \leq x_Q\alpha$. Since π_A is a partition of A, $x_P\alpha < x_Q\alpha$. To show that $x_P \leq x_Q$, suppose that $x_P \nleq x_Q$. By $(X, \leq)$ is a totally ordered set, we have $x_Q < x_P$. By $x_P, x_Q \in B$ and $\alpha \in EOP(X)$, we have $x_Q\alpha \leq x_P\alpha < x_Q\alpha$. It is a contradiction. Hence $x_P \leq x_Q$ as required. By Theorem 3.7.1, we conclude that α is a regular element of $EOP(X)$. □

This leads directly to the following corollary when X is finite.

Corollary 3.7.3. [3] *Let X be a finite set and $\alpha \in EOP(X)$. Then α is a regular element if and only if for every $A \in X/E$, there exists $B \in X/E$ such that $X\alpha \cap A \subseteq B\alpha$.*

Proof. Suppose that α is a regular element. Let $A \in X/E$. If $A \cap X\alpha \neq \emptyset$, then $A \cap X\alpha \subseteq A\alpha$. Assume that $A \cap X\alpha \neq \emptyset$. By Corollary 3.7.2, there exists a partition π_A of A such that $(\pi_A, \preceq)$ is a totally ordered set and $B \in X/E$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A$. Thus $P \cap X\alpha = P \cap B\alpha$ for all $P \in \pi_A$. Since π_A is a partition of A, we have $\bigcup_{P \in \pi_A} P = A$. Hence

$$A \cap X\alpha = \left(\bigcup_{P\in\pi_A} P\right) \cap X\alpha = \bigcup_{P\in\pi_A}(P \cap X\alpha) = \bigcup_{P\in\pi_A}(P \cap B\alpha) \subseteq B\alpha.$$

Conversely, assume that for every $A \in X/E$, there exists $B \in X/E$ such that $X\alpha \cap A \subseteq B\alpha$. We need to show that α is regular via Corollary 3.7.2. Let $A \in X/E$ be such that $A \cap X\alpha \neq \emptyset$. Since X is a finite set, we order $A \cap X\alpha = \{a_1, a_2, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$ for some $n \in \mathbb{N}$. Let $P_1 = \{x \in A : x \leq a_1\}$, $P_i = \{x \in A : a_{i-1} < x \leq a_i\}$ for all $i = 2, 3, \ldots, n-1$ and $P_n = \{x \in A : a_{n-1} < x\}$. It is easy to see that $\pi_A = \{P_i : i = 1, 2, \ldots, n\}$ is a partition of A. Moreover, $P_i \cap X\alpha = \{a_i\}$ for all $i = 1, 2, \ldots, n$. By assumption, we have $A \cap X\alpha \subseteq B\alpha$ for some $B \in X/E$. Then choose $x_i \in B$ such that $x_i\alpha = a_i$ for each $i = 1, 2, \ldots, n$. Hence $P_i \cap X\alpha = \{a_i\} = \{x_i\alpha\} = P_i \cap B\alpha$ for all $i = 1, 2, \ldots, n$. Thus $|P_i \cap X\alpha| = |P_i \cap B\alpha| = 1$ for all $i = 1, 2, \ldots, n$. To verify $(\pi_A, \preceq)$ is a totally ordered set, let $P_i, P_j \in \pi_A$ be distinct. We assume that $a_i < a_j$ from X is a totally ordered set. This implies that $i < j$. Claim that $P_i \preceq P_j$, let $x \in P_i$ and $y \in P_j$. It follows from the definition of P_i and P_j that $x \leq a_i \leq a_{j-1} < y$. Hence $(\pi_A, \preceq)$ is a totally ordered set. By Corollary 3.7.2, we observe that α is regular. □

Theorem 3.7.4. *Let $\alpha \in EOP(X)$. Then $\alpha \in LReg(EOP(X))$ if and only if for every $A \in X/E$, there exists $B \in X/E$ such that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$.*

Proof. Suppose that $\alpha = \beta\alpha^2$ for some $\beta \in EOP(X)$. Let $A \in X/E$ and $a \in A$. Since X/E is a partition of X, $a\beta \in B$ for some $B \in X/E$. We claim that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$. Let $P \in \pi_A(\alpha)$ and $x \in P \cap A$. Since X is a totally ordered set, we assume that $a \leq x$. From $(a, x) \in E$ and $a \leq x$, we then have $(a\beta, x\beta) \in E$. Since $a\beta \in B$, we conclude that $x\beta \in B$. Consider, $P\alpha_* = x\alpha = x\beta\alpha^2 = (x\beta\alpha)\alpha$. Therefore $x\beta\alpha \in P$ and $x\beta \in B$.

Conversely, suppose that for every $A \in X/E$, there exists $B \in X/E$ such that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$. Hence each $A \in X/E$, we fix $A' \in X/E$ and $x_P \in A'$ corresponding to $P \in \pi_A(\alpha)$ such that $x_P\alpha \in P$. We will construct $\beta \in EOP(X)$ in the following, let $x \in X$. Since X/E is a partition of X, $x \in A$ for some $A \in X/E$. Then there exists $P_x \in \pi_A(\alpha)$ such that $x \in P_x$. Define $\beta : X \to X$ by

$$x\beta = x_{P_x} \text{ for all } x \in X.$$

To show that $\beta \in EOP(X)$, let $x, y \in X$ be such that $(x, y) \in E$ and $x \leq y$. We then have $x, y \in A$ for some $A \in X/E$ and $x \in P_x, y \in P_y$ where $P_x, P_y \in \pi_A(\alpha)$. Clearly, $(x\beta, y\beta) = (x_{P_x}, x_{P_y}) \in E$. If $P_x = P_y$, then $x_{P_x} = x_{P_y}$. Suppose that $P_x \neq P_y$. Claim that $x_{P_x} \leq x_{P_y}$, suppose not. Since X is a totally ordered set, we have $x_{P_y} < x_{P_x}$. Since $(x_{P_x}, x_{P_y}), (x, y) \in E$, $x_{P_y} < x_{P_x}$ and $x \leq y$, we conclude that $x_{P_y}\alpha \leq x_{P_x}\alpha, x\alpha \leq y\alpha$ and $(x_{P_y}\alpha, x_{P_x}\alpha) \in E$. Then $P_x\alpha_* = x\alpha \leq y\alpha = P_y\alpha_*$. We note by $P_x \neq P_y$ that $P_x\alpha_* < P_y\alpha_*$. Similarly, we see that $(x_{P_y}\alpha, x_{P_x}\alpha) \in E$ and $x_{P_y}\alpha \leq x_{P_x}\alpha$. Hence $x_{P_y}\alpha\alpha \leq x_{P_x}\alpha\alpha$. By assumption, we have $x_{P_y}\alpha \in P_y$ and $x_{P_x}\alpha \in P_x$. It follows that

$$x_{P_x}\alpha\alpha = P_x\alpha_* < P_y\alpha_* = x_{P_y}\alpha\alpha \leq x_{P_x}\alpha\alpha.$$

This is a contradiction. Thus $x_{P_x} \leq x_{P_y}$ and then $x\beta \leq y\beta$. Therefore $\beta \in EOP(X)$. We need to verify that $\alpha = \beta\alpha^2$, let $x \in X$. Hence $x\beta\alpha^2 = x_{P_x}\alpha\alpha = P_x\alpha_* = x\alpha$, so α is a left regular element of $EOP(X)$ as required. □

Theorem 3.7.5. *Let $\alpha \in EOP(X)$. Then $\alpha \in RReg(EOP(X))$ if and only if*

(1) *$\alpha|_{X\alpha}$ is an injection and*

(2) *for every $A \in X/E$, there exist $x_P \in X$ corresponding to P for all $P \in \pi(A, \alpha)$ such that $P \preceq Q$ implies $x_P \leq x_Q$ and $(x_P, x_Q) \in E$ for all $P, Q \in \pi(A, \alpha)$ and if $P \cap X\alpha^2 \neq \emptyset$, then $x_P \in X\alpha$ and $x_P\alpha \in P$.*

Proof. Suppose that $\alpha = \alpha^2\beta$ for some $\beta \in EOP(X)$. We note from Theorem 3.1.3 that $\alpha|_{X\alpha}$ is an injection. Let $A \in X/E$ and $P \in \pi(A, \alpha)$ be such that $P \cap X\alpha^2 \neq \emptyset$. Claim that $|P \cap X\alpha^2| = 1$, let $a, b \in X\alpha^2$, then $a\alpha = b\alpha$ and $a, b \in X\alpha$. It follows from $\alpha|_{X\alpha}$ is an injection that $a = b$. For each $P \in \pi(A, \alpha)$ such that $P \cap X\alpha^2 \neq \emptyset$, we let $x_P = x\beta$ where $x \in P \cap X\alpha^2$. For each $P \cap X\alpha^2 = \emptyset$, we choose $x \in P$ and let $x_P = x\beta$. Let $P, Q \in \pi(A, \alpha)$ be such that $P \preceq Q$. We see that $x\beta = x_P$ and $y\beta = x_Q$ for some $x \in P$ and $y \in Q$. if $P = Q$, then $x_P = x_Q$. Assume that $P \neq Q$. Thus by assumption, $x < y$. Since $\pi(A, \alpha)$ is a partition of A, $(x, y) \in E$. This implies that $x_P = x\beta \leq y\beta = x_Q$ and $(x_P, x_Q) = (x\beta, y\beta) \in E$. Let $P \in \pi(A, \alpha)$ be such that $P \cap X\alpha^2 \neq \emptyset$. We note that $x_P\alpha = x\beta$ for some $x \in P \cap X\alpha^2$. There exists $x' \in X$ such that $x'\alpha^2 = x$. Then

$x_P = x\beta = x'\alpha^2\beta = x'\alpha$ and $x_P\alpha = x'\alpha\alpha = x \in P$ which imply that $x_P \in X\alpha$ and $x_P\alpha \in P$, respectively.

Conversely, assume that (1) and (2) are true. Let $x \in X$. Then $x \in A$ for some $A \in X/E$. By $\pi(A, \alpha)$ is a partition of A, we have $x \in P_x$ for some $P_x \in \pi(A, \alpha)$. Define $\beta : X \to X$ by

$$x\beta = x_{P_x} \text{ for all } x \in X.$$

It is clear that β is well-defined. Let $x, y \in X$ be such that $x \leq y$ and $(x, y) \in E$. Then there exists $A \in X/E$ such that $x, y \in A$. Thus $x \in P_x$ and $y \in P_y$ for some $P_x, P_y \in \pi(A, \alpha)$. By $x \leq y$ and Proposition 2.2.12(2), we conclude that $P_x \preceq P_y$. It follows that $x\beta \leq y\beta$ and $(x\beta, y\beta) \in E$, hence $\beta \in EOP(X)$. Let $x \in X$, then $x\alpha^2 \in X\alpha^2$. Thus $x\alpha^2\beta = x_{P_{x\alpha^2}}$ where $x\alpha^2 \in P$ for some $P \in \pi(A, \alpha)$. We note that $P \cap X\alpha^2 \neq \emptyset$. Hence $x_{P_{x\alpha^2}}\alpha \in P$ and $x_{P_{x\alpha^2}} \in X\alpha$. Thus $(x_{P_{x\alpha^2}}\alpha)\alpha = (x\alpha^2)\alpha$. By (1), we conclude that $(x_{P_{x\alpha^2}})\alpha = x\alpha^2$. Since $x_{P_{x\alpha^2}} \in X\alpha$ and by (1), we deduce again that $x_{P_{x\alpha^2}} = x\alpha$. Therefore $x\alpha^2\beta = x\alpha$ and hence α is right regular. □

Theorem 3.7.6. *Let $\alpha \in EOP(X)$. Then $\alpha \in CReg(EOP(X))$ if and only if for every $A \in X/E$, there exists $B \in X/E$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A(\alpha)$.*

Proof. Since $EOP(X)$ is a subsemigroup of $T_E(X)$ for a totally ordered set X, by Theorem 3.3.7, we have $\alpha = \alpha\beta\alpha$ and $\alpha\beta = \beta\alpha$ for some $\beta \in T_E(X)$ if and only if for every $A \in X/E$, there exists $B \in X/E$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A(\alpha)$. It enough to show that β which is defined in Theorem 3.3.7 is in $EOP(X)$. Let $x, y \in X$ be such that $(x, y) \in E$ and $x \leq y$. Then there exist $P_x, P_y \in \pi(\alpha)$ such that $x \in P_x$ and $y \in P_y$. If $P_x = P_y$, then we have $x\beta = y\beta$. Assume that $P_x \neq P_y$. Since $(x, y) \in E$, $x, y \in A$ for some $A \in X/E$ which implies that $P_x, P_y \in \pi_A(\alpha)$. It follows from Proposition 2.2.12(2) that $P_x \cap A \preceq P_y \cap A$. Hence $x_{P_x} \leq x_{P_y}$ and by $P_x \neq P_y$, we conclude that $x_{P_x} < x_{P_y}$. From Theorem 3.3.7, we have shown that $(x_{P'_x}, x_{P'_y}) \in E$ where $x_{P'_x}\alpha = x_{P_x}$ and $x_{P'_y}\alpha = x_{P_y}$. To prove that $x_{P'_x} \leq x_{P'_y}$, suppose not. Since $(X, \leq)$ is a totally ordered set, $x_{P'_y} \leq x_{P'_x}$. By $\alpha \in EOP(X)$, we note that

$$x_{P_y} = x_{P'_y}\alpha \leq x_{P'_x}\alpha = x_{P_x} < x_{P_y}.$$

It is a contradiction. Hence $x_{P'_x} \leq x_{P'_y}$ and so $x\beta = x_{P'_x} \leq x_{P'_y} = y\beta$. This implies that $\beta \in EOP(X)$ and theorem has been proved. □

Corollary 3.7.7. *$CReg(EOP(X)) = RReg(EOP(X)) \cap LReg(EOP(X))$.*