# CHAPTER IV

# GREEN'S RELATIONS

In this chapter, we present Green's relations on $T_{RE}(X)$ and $T_{SE}(X)$. We investigate characterizations of left principal ideal, right principal ideal and principal ideal on $T_{RE}(X)$ and $T_{SE}(X)$. And then we determine when elements of $T_{RE}(X)$ and $T_{SE}(X)$ are equivalence respect to Green's relations.

## 4.1 Green's relations on full regressive transformation semigroups

We let $X$ be a partially ordered set. Green's relations on $T_{RE}(X)$ are studied in this section.

**Theorem 4.1.1.** *Let $\alpha, \beta \in T_{RE}(X)$. Then $\beta \in \alpha T_{RE}(X)$ if and only if for every $P \in \pi(\alpha)$, there exists $Q \in \pi(\beta)$ such that $P \subseteq Q$ and $Q\beta_* \leq P\alpha_*$.*

**Proof.** Suppose that $\beta \in \alpha T_{RE}(X)$. Then there exists $\gamma \in T_{RE}(X)$ such that $\beta = \alpha\gamma$. Let $P \in \pi(\alpha)$. Thus $P\alpha_* = y$ for some $y \in X\alpha$, it follows that $P\beta = P\alpha\gamma = \{y\gamma\}$. Hence $y\gamma \in X\beta$. Let $Q = y\gamma\beta^{-1}$. Then $Q \in \pi(\beta)$ and $Q\beta_* = y\gamma$. Since $P\beta = \{y\gamma\}$, we deduce $P \subseteq Q$. By the regressiveness of $\gamma$, we have that $y\gamma \leq y$. This implies that $Q\beta_* \leq P\alpha_*$ as required.

For the converse, suppose that for every $P \in \pi(\alpha)$, there exists $Q \in \pi(\beta)$ such that $P \subseteq Q$ and $Q\beta_* \leq P\alpha_*$. We construct $\gamma \in T_{RE}(X)$ such that $\beta = \alpha\gamma$ in the following. For each $x \in X\alpha$, there exists a unique $P_x \in \pi(\alpha)$ such that $x = P_x\alpha_*$. By assumption, $P_x \subseteq Q_x$ and $Q_x\beta_* \leq P_x\alpha_*$ for some $Q_x \in \pi(\beta)$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} Q_x\beta_*, & \text{if } x \in X\alpha; \\ x, & \text{otherwise.} \end{cases}$$

Since $\pi(\beta)$ is a partition of $X$, it can be shown that $\gamma$ is well-defined. Let $x \in X$. If $x \in X\alpha$, then $x\gamma = Q_x\beta_*$. Hence $x\gamma = Q_x\beta_* \leq P_x\alpha_* = x$. If $x \notin X\alpha$, then $x\gamma = x$.

These prove that $x\gamma \leq x$ for all $x \in X$, so $\gamma \in T_{RE}(X)$. Next, to show that $\beta = \alpha\gamma$, let $x \in X$. Since $x\alpha \in X\alpha$, there exists $P_{x\alpha} \in \pi(\alpha)$ such that $P_{x\alpha}\alpha_* = x\alpha$, hence $x \in P_{x\alpha}$. By the definition of $\gamma$, we have $x\alpha\gamma = Q_{x\alpha}\beta_*$ where $Q_{x\alpha} \in \pi(\beta)$, $P_{x\alpha} \subseteq Q_{x\alpha}$ and $Q_{x\alpha}\beta_* \leq P_{x\alpha}\alpha_*$. Since $x \in P_{x\alpha} \subseteq Q_{x\alpha}$, we have $x\alpha\gamma = Q_{x\alpha}\beta_* = x\beta$. This proves that $\beta = \alpha\gamma$, so $\beta \in \alpha T_{RE}(X)$.

The theorem is thereby proved. $\square$

**Theorem 4.1.2.** *Let $\beta, \alpha \in T_{RE}(X)$. Then $(\alpha, \beta) \in \mathcal{R}$ if and only if $\alpha = \beta$.*

**Proof.** Suppose that $\alpha\mathcal{R}\beta$. Thus $\beta \in \alpha T_{RE}(X)$ and $\alpha \in \beta T_{RE}(X)$. To show that $\alpha = \beta$, let $x \in X$. Then $x \in P$ for some $P \in \pi(\alpha)$. By Theorem 4.1.1, there exists $Q \in \pi(\beta)$ such that $P \subseteq Q$ and $Q\beta_* \leq P\alpha_*$. By Theorem 4.1.1, there exists $P' \in \pi(\alpha)$ such that $Q \subseteq P'$ and $P'\alpha_* \leq Q\beta_*$. Since $\pi(\alpha)$ is a partition of $X$, $P = P'$. It follows that $P\alpha_* = Q\beta_*$ and $x \in Q$. Hence $x\alpha = x\beta$. $\square$

**Example 12.** Define $\alpha, \beta : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$x\alpha = \begin{cases} 1, & \text{if } x \leq 4; \\ x, & \text{otherwise,} \end{cases}$$

$$x\beta = \begin{cases} 1, & \text{if } x \leq 2; \\ 3, & \text{if } x = 4; \\ x, & \text{otherwise,} \end{cases}$$

We note that $\alpha, \beta \in T_{RE}(\mathbb{Z}^+)$. It is clear that

$$\pi(\alpha) = \{\{1,2,3,4\}\} \cup \{\{x\} : x \in \mathbb{Z}^+ \text{ and } x > 4\},$$

$$\pi(\beta) = \{\{1,2\},\{3,4\}\} \cup \{\{x\} : x \in \mathbb{Z}^+ \text{ and } x > 4\}.$$

Hence we can verify that $\alpha \in \beta T_{RE}(X)$ via Theorem 4.1.1. Moreover, $\alpha \neq \beta$ which implies that $(\alpha, \beta) \notin \mathcal{R}$ by Theorem 4.1.2.

**Theorem 4.1.3.** *Let $\alpha, \beta \in T_{RE}(X)$. Then the following statements are equivalent.*

(1) $\alpha \in T_{RE}(X)\beta$.

(2) *For every $P \in \pi(\alpha)$, there exists $Q \in \pi(\beta)$ such that $P\alpha_* = Q\beta_*$ and for every $x \in P$, $y \leq x$ for some $y \in Q$.*

**Proof.** Suppose that (1) holds. Then there exists $\delta \in T_{RE}(X)$ such that $\alpha = \delta\beta$. Let $P \in \pi(\alpha)$. Then $P\alpha_* \in X\alpha = X\delta\beta \subseteq X\beta$ which implies that $P\alpha_* \in X\beta$. Then there exists $Q \in \pi(\beta)$ such that $Q\beta_* = P\alpha_*$. Let $x \in P$. Since $x\delta\beta = x\alpha = P\alpha_* = Q\beta_*$, it follows that $x\delta \in Q$. By $\delta \in T_{RE}(X)$, we deduce that $x\delta \leq x$ as desired.

Conversely, suppose that (2) holds. We construct a map $\delta \in T_{RE}(X)$ such that $\alpha = \delta\beta$ in the following. For each $x \in X$, there exists $P_x \in \pi(\alpha)$ such that $x \in P_x$. By assumption, we choose and fix $Q_x \in \pi(\beta)$ and $y_x \in Q_x$ such that $P_x\alpha_* = Q_x\beta_*$ and $y_x \leq x$. Define $\delta : X \to X$ by

$$x\delta = y_x \text{ for all } x \in X.$$

Let $x \in X$. Then $x\delta = y_x \leq x$ and

$$x\delta\beta = y_x\beta = Q_x\beta_* = P_x\alpha_* = x\alpha.$$

These prove $\delta \in T_{RE}(X)$ and $\alpha = \delta\beta$, respectively. Therefore $\alpha \in T_{RE}(X)\beta$. □

**Theorem 4.1.4.** *Let $\alpha, \beta \in T_{RE}(X)$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathcal{L}$.

(2) *$X\beta \subseteq X\alpha$ and for every $P \in \pi(\alpha)$, there exists $Q \in \pi(\beta)$ such that $P\alpha_* = Q\beta_*$ and for every $p \in P$ and $q \in Q$, there exist $a \in P$ and $b \in Q$ such that $b \leq p$ and $a \leq q$.*

**Proof.** Suppose that (1) holds. Then $\beta \in T_{RE}(X)\alpha$ and $\alpha \in T_{RE}(X)\beta$. It is clear that $X\beta \subseteq X\alpha$. Let $P \in \pi(\alpha)$. Since $\alpha \in T_{RE}(X)\beta$ and by Theorem 4.1.3, there exists $Q \in \pi(\beta)$ such that $P\alpha_* = Q\beta_*$ and for all $p \in P$, $b \leq p$ for some $b \in Q$. Since $\beta \in T_{RE}(X)\alpha$ and by Theorem 4.1.3, there exists $P' \in \pi(\alpha)$ such that $Q\beta_* = P'\alpha_*$ and for every $q \in Q$, $a \leq q$ for some $a \in P'$. Since $P\alpha_* = Q\beta_* = P'\alpha_*$, it follows that $P = P'$. Hence (2) holds.

Conversely, assume that (2) holds. By Theorem 4.1.3, we suddenly have $\alpha \in T_{RE}(X)\beta$. It suffices to show that $\beta \in T_{RE}(X)\alpha$. Let $Q \in \pi(\beta)$. Since $Q\beta_* \in X\beta$, by assumption, $Q\beta_* \in X\alpha$. Hence $Q\beta_* = P\alpha_*$ for some $P \in \pi(\alpha)$. By assumption, there exists $Q' \in \pi(\beta)$ such that $P\alpha_* = Q'\beta_*$ and for every $q \in Q'$, $a \leq q$ for some $a \in P$. We then have that $Q = Q'$. By Theorem 4.1.3, $\beta \in T_{RE}(X)\alpha$.

Hence the proof of theorem is complete. □

As an immediate consequence of Theorem 4.1.2 and $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{L} \circ \mathcal{I} = \mathcal{L}$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{R} = \mathcal{L} \cap \mathcal{I} = \mathcal{I}$ where $\mathcal{I}$ is the identity relation on $T_{RE}(X)$. We have the following result.

**Theorem 4.1.5.** *Let $\alpha, \beta \in T_{RE}(X)$. Then the following statements hold.*

(1) $(\alpha, \beta) \in \mathcal{D}$ *if and only if* $\alpha\mathcal{L}\beta$.

(2) $(\alpha, \beta) \in \mathcal{H}$ *if and only if* $\alpha = \beta$.

**Example 13.** Consider maps $\alpha, \beta : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by :

$$x\alpha = \begin{cases} 1, & \text{if } x = 2n-1 \text{ for some } n \in \mathbb{Z}^+; \\ 2, & \text{otherwise} \end{cases}$$

and

$$x\beta = \begin{cases} 1, & \text{if } x = 1 \text{ or } x = 2n+2 \text{ for some } n \in \mathbb{Z}^+; \\ 2, & \text{otherwise.} \end{cases}$$

It is not difficult to show that $\alpha, \beta \in T_{RE}(\mathbb{Z}^+)$. Note that

$$\pi(\alpha) = \{\{2n-1 : n \in \mathbb{Z}^+\}, \{2n : n \in \mathbb{Z}^+\}\},$$

$$\pi(\beta) = \{\{2n+1 : n \in \mathbb{Z}^+\} \cup \{2\}, \{2n+2 : n \in \mathbb{Z}^+\} \cup \{1\}\}.$$

From Theorem 4.1.4, we conclude that $(\alpha, \beta) \in \mathcal{L}$ and clearly, $\alpha \neq \beta$. Let $i_{\mathbb{Z}^+}$ be an identity map on $\mathbb{Z}^+$. We can show that $\alpha \in T_{RE}(\mathbb{Z}^+)i_{\mathbb{Z}^+}$ and $(\alpha, i_{\mathbb{Z}^+}) \notin \mathcal{L}$ via Theorem 4.1.3 and Theorem 4.1.4, respectively.

Next, Green's relation $\mathcal{J}$ is considered, we need the following lemma.

**Lemma 4.1.6.** *If $\alpha, \beta, \delta, \gamma \in T(X)$ are such that $\alpha = \delta\beta\gamma$, then*

$$\mathcal{A} = \{\cup\mathcal{A}_Q : Q \in \pi(\beta) \text{ and } Q \cap X\delta \neq \emptyset\}$$

*is a refinement of $\pi(\alpha)$ where $\mathcal{A}_Q = \{P \in \pi(\delta) : P\delta_* \in Q\}$.*

**Proof.** Let $\alpha, \beta, \delta, \gamma \in T(X)$ be such that $\alpha = \delta\beta\gamma$. By Theorem 2.2.8, $\pi(\delta)$ refines $\pi(\alpha)$. To show that $\cup\mathcal{A} = \cup\pi(\alpha) = X$, let $x \in X$. We then have $x \in P$ for some $P \in \pi(\delta)$. Since $x\delta\beta \in X\beta$, $x\delta\beta = Q\beta_*$ for some $Q \in \pi(\beta)$. Then $P\delta_* = x\delta \in Q$, hence $Q \cap X\delta \neq \emptyset$. Therefore $P \in \mathcal{A}_Q$ and $x \in P \subseteq \cup\mathcal{A}_Q \subseteq \cup\mathcal{A}$. Hence $X = \cup\mathcal{A}$. Let $Q \in \pi(\beta)$ be such that $Q \cap X\delta \neq \emptyset$. To show that there is $\tilde{P} \in \pi(\alpha)$ such that $\cup\mathcal{A}_Q \subseteq \tilde{P}$, let $x \in Q \cap X\delta$. Then there exists an element $x' \in X$ such that $x'\delta = x$. Since $\pi(\delta)$ is a partition of $X$, $x' \in P$ for some $P \in \pi(\delta)$ and $P\delta_* = x'\delta$. Since $\pi(\delta)$ refines $\pi(\alpha)$, $P \subseteq \tilde{P}$ for some $\tilde{P} \in \pi(\alpha)$. Let $y \in \cup\mathcal{A}_Q$. Then $y \in P'$ for some $P' \in \mathcal{A}_Q$. By the definition of $\mathcal{A}_Q$, $P'\delta_* \in Q$. Hence

$$y\delta\beta = P'\delta_*\beta = Q\beta_* = x\beta = x'\delta\beta.$$

Since $x' \in P \subseteq \tilde{P}$, $x'\alpha = \tilde{P}\alpha_*$. Thus

$$y\alpha = y\delta\beta\gamma = x'\delta\beta\gamma = x'\alpha = \tilde{P}\alpha_*$$

which implies that $y \in \tilde{P}$, hence $\cup\mathcal{A}_Q \subseteq \tilde{P}$. This proves that $\mathcal{A}$ refines $\pi(\alpha)$ as required.

□

**Theorem 4.1.7.** *Let $\alpha, \beta \in T_{RE}(X)$. Then the following statements are equivalent.*

(1) $\alpha \in T_{RE}(X)\beta T_{RE}(X)$.

(2) *There exist a refinement $\mathcal{B}$ of $\pi(\alpha)$ and an injection $\varphi : \mathcal{B} \to \pi(\beta)$ such that for every $P \in \mathcal{B}, \tilde{P}\alpha_* \leq (P\varphi)\beta_*$ where $P \subseteq \tilde{P}$ and $\tilde{P} \in \pi(\alpha)$ and for every $x \in P$, $y \leq x$ for some $y \in P\varphi$.*

**Proof.** Assume that (1) holds. Then $\alpha = \delta\beta\gamma$ for some $\delta, \gamma \in T_{RE}(X)$. Recall $\mathcal{A}$ from Lemma 4.1.6, we then have $\mathcal{A}$ is a refinement of $\pi(\alpha)$. Define $\varphi : \mathcal{A} \to \pi(\beta)$ by $\cup\mathcal{A}_Q\varphi = Q$. It is easy to verify that $\varphi$ is well-defined. Suppose that $\cup\mathcal{A}_Q\varphi = \cup\mathcal{A}_{Q'}\varphi$. By the definition of $\varphi$, we conclude that $Q = Q'$, hence $\cup\mathcal{A}_Q = \cup\mathcal{A}_{Q'}$. This proves that $\varphi$ is an injection.

Next, let $\cup\mathcal{A}_Q \in \mathcal{A}$ where $Q \in \pi(\beta)$ and $Q \cap X\delta \neq \emptyset$ and let $x \in \cup\mathcal{A}_Q$. Then $x \in P$ for some $P \in \pi(\delta)$ and $P\delta_* \in Q$. We deduce that $x\delta = P\delta_* \in Q = \cup\mathcal{A}_Q\varphi$. Since $\mathcal{A}$ is a refinement of $\pi(\alpha)$, $x \in \cup\mathcal{A}_Q \subseteq \tilde{P}$ for some $\tilde{P} \in \pi(\alpha)$. Hence

$$\tilde{P}\alpha_* = x\alpha = x\delta\beta\gamma = Q\beta_*\gamma \leq Q\beta_* = (\cup\mathcal{A}_Q\varphi)\beta_*.$$

Since $\delta$ is regressive, $x\delta \leq x$. Therefore (2) holds.

For the converse, suppose that (2) holds. We construct maps $\delta, \gamma \in T_{RE}(X)$ such that $\alpha = \delta\beta\gamma$ in the following. For each $x \in X$, there exists some $P_x \in \mathcal{B}$ such that $x \in P_x$. By assumption, we choose and fix an element $y_x \in P_x\varphi$ such that $y_x \leq x$. Define $\delta : X \to X$ by

$$x\delta = y_x \text{ for all } x \in X.$$

Clearly, $\delta \in T_{RE}(X)$. Since $\beta_* : \pi(\beta) \to X\beta$ and $\varphi$ are injective, we then have $\varphi\beta_* : \mathcal{B} \to X\beta$ is also injective. For each $x \in \mathcal{B}\varphi\beta_*$, there exists a unique $P'_x \in \mathcal{B}$ such that $x = P'_x\varphi\beta_*$ and $\tilde{P}'_x \in \pi(\alpha)$ such that $P'_x \subseteq \tilde{P}'_x$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} \tilde{P}'_x\alpha_*, & \text{if } x \in \mathcal{B}\varphi\beta_*; \\ x, & \text{otherwise.} \end{cases}$$

Since $\mathcal{B}$ is a refinement of $\pi(\alpha)$, we deduce that $\gamma$ is well-defined. It follows from assumption that $x\gamma = \tilde{P}'_x\alpha_* \leq (P'_x\varphi)\beta_* = x$ which implies that $\gamma \in T_{RE}(X)$. Finally, let $x \in X$. Then $x \in P_x$ for some $P_x \in \mathcal{B}$. Since $\mathcal{B}$ is a refinement of $\pi(\alpha)$, $\tilde{P}_x\alpha_* = x\alpha$ where $P_x \subseteq \tilde{P}_x$ for some $\tilde{P}_x \in \pi(\alpha)$. Hence $x\delta = y_x$ where $y_x \in P_x\varphi$. It follows that $y_x\beta = (P_x\varphi)\beta_* \in \mathcal{B}\varphi\beta_*$. By the definition of $\gamma$, $y_x\beta\gamma = \tilde{P}'_{y_x\beta}\alpha_*$ where $P'_{y_x\beta} \in \mathcal{B}, \tilde{P}'_{y_x\beta} \in \pi(\alpha)$ and $P'_{y_x\beta} \subseteq \tilde{P}'_{y_x\beta}$. Since $P_x\varphi\beta_* = y_x\beta = P'_{y_x\beta}\varphi\beta_*$ and $\varphi\beta_*$ is injective, $P_x = P'_{y_x\beta}$. Hence $\tilde{P}_x = \tilde{P}'_{y_x\beta}$. Therefore

$$x\delta\beta\gamma = y_x\beta\gamma = \tilde{P}_x\alpha_* = x\alpha.$$

Hence the theorem is completely proved. □

The next corollary is immediate from Theorem 4.1.7.

**Theorem 4.1.8.** *Let $\alpha, \beta \in T_{RE}(X)$. Then $(\alpha, \beta) \in \mathcal{J}$ if and only if*

(1) *there exist a refinement $\mathcal{B}$ of $\pi(\alpha)$ and an injection $\varphi : \mathcal{B} \to \pi(\beta)$ such that for every $P \in \mathcal{B}, \tilde{P}\alpha_* \leq P\varphi\beta_*$ where $P \subseteq \tilde{P}$ and $\tilde{P} \in \pi(\alpha)$ and for every $x \in P$, $y \leq x$ for some $y \in P\varphi$ and*

(2) *there exist a refinement $\mathcal{B}'$ of $\pi(\beta)$ and an injection $\varphi' : \mathcal{B}' \to \pi(\alpha)$ such that for every $Q \in \mathcal{B}', \tilde{Q}\beta_* \leq Q\varphi'\alpha_*$ where $Q \subseteq \tilde{Q}$ and $\tilde{Q} \in \pi(\beta)$ and for every $x' \in Q$, $y' \leq x'$ for some $y' \in Q\varphi'$.*

From Example 12, we can verify that $\alpha \in T_{RE}(\mathbb{Z}^+)\beta T_{RE}(\mathbb{Z}^+)$ by Theorem 4.1.7 and $(\alpha, \beta) \notin \mathcal{J}$ by Theorem 4.1.8. Recall $\alpha, \beta$ defined in Example 13, we have $(\alpha, \beta) \in \mathcal{J}$ via Theorem 4.1.8.

## 4.2 Green's relations on self-$E$-preserving transformation semigroups

Denote $E$ an equivalence relation on $X$. We discuss Green's relations of $T_{SE}(X)$.

**Theorem 4.2.1.** *For any $\alpha, \beta \in T_{SE}(X)$, $\alpha \in T_{SE}(X)\beta$ if and only if for every $A \in X/E$, $A\alpha \subseteq A\beta$.*

**Proof.** Suppose that $\alpha \in T_{SE}(X)\beta$. Then $\alpha = \delta\beta$ for some $\delta \in T_{SE}(X)$. Let $A \in X/E$. By Proposition 2.2.9(3), we then have $A\delta \subseteq A$, hence $A\alpha = A\delta\beta \subseteq A\beta$.

Conversely, assume that for every $A \in X/E$, $A\alpha \subseteq A\beta$. For each $x \in X$, there exists a unique $A \in X/E$ such that $x \in A$. By assumption, choose an element $x' \in A$ such that $x\alpha = x'\beta$. Define $\delta : X \to X$ by

$$x\delta = x' \text{ for all } x \in X.$$

Let $x \in X$. Since $x, x' \in A$, $(x, x\delta) = (x, x') \in E$. Hence $\delta \in T_{SE}(X)$ and $x\delta\beta = x'\beta = x\alpha$ for all $x \in X$. This implies that $\alpha = \delta\beta$, therefore $\alpha \in T_{SE}(X)\beta$ as required. □

**Theorem 4.2.2.** *For $\alpha, \beta \in T_{SE}(X)$, $(\alpha, \beta) \in \mathcal{L}$ if and only if $A\alpha = A\beta$ for all $A \in X/E$.*

**Theorem 4.2.3.** *Let $\alpha, \beta \in T_{SE}(X)$. Then $\alpha \in \beta T_{SE}(X)$ if and only if $\pi(\beta)$ refines $\pi(\alpha)$.*

**Proof.** We note by Theorem 2.2.8 that $\alpha = \beta\delta$ for some $\delta \in T(X)$ if and only if $\pi(\beta)$ refines $\pi(\alpha)$. It suffices to show that $\delta$ which is defined in Theorem 2.2.8 is in $T_{SE}(X)$. Let $x \in X$. If $x \notin X\beta$, then $(x, x\delta) = (x, x) \in E$. Assume that $x \in X\beta$. Then by the definition of $\delta$, $x\delta = Q_x\alpha_*$ where $P_x = x\beta^{-1}$ and $P_x \subseteq Q_x$ for some $P_x \in \pi(\beta)$ and $Q_x \in \pi(\alpha)$. By Proposition 2.2.9(1), $\pi(\alpha)$ and $\pi(\beta)$ refine $X/E$, we then have $P_x \subseteq A$ and $Q_x \subseteq B$ for some $A, B \in X/E$. Since $\beta \in T_{SE}(X)$, we conclude that $x \in A$. Since $P_x \subseteq Q_x$ and $X/E$ is a partition of $X$, we have that $A = B$, so $Q_x \subseteq A$. By Proposition 2.2.9(3), we note that $Q_x\alpha_* \in A\alpha \subseteq A$. It follows that $(x, x\delta) = (x, Q_x\alpha_*) \in E$. Therefore $\delta \in T_{SE}(X)$. □

**Theorem 4.2.4.** *For $\alpha, \beta \in T_{SE}(X)$, $(\alpha, \beta) \in \mathcal{R}$ if and only if $\pi(\alpha) = \pi(\beta)$.*

Since $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, the following theorem follows immediately from Theorem 4.2.2 and Theorem 4.2.4.

**Theorem 4.2.5.** *For $\alpha, \beta \in T_{SE}(X)$, $(\alpha, \beta) \in \mathcal{H}$ if and only if $\pi(\alpha) = \pi(\beta)$ and for every $A \in X/E$, $A\alpha = A\beta$.*

**Example 14.** Let $A_1 = \{1, 2, 3, 4\}$ and $A_2 = \mathbb{Z}^+ \backslash A_1$. Define

$$E = (A_1 \times A_1) \cup (A_2 \times A_2).$$

We then have $E$ is an equivalence relation on $\mathbb{Z}^+$. Define $\alpha, \beta, \gamma \in T(\mathbb{Z}^+)$ by

$$x\alpha = \begin{cases} x - 1, & \text{if } x \in A_1 \backslash \{1\}; \\ x, & \text{otherwise}, \end{cases}$$

$$x\beta = \begin{cases} x - 1, & \text{if } x = 3, 4; \\ x, & \text{otherwise} \end{cases}$$

and

$$x\gamma = \begin{cases} 2, & \text{if } x = 1; \\ x, & \text{otherwise.} \end{cases}$$

Clearly, $\alpha, \beta, \gamma \in T_{SE}(\mathbb{Z}^+)$. Note here that

$$\pi(\alpha) = \{\{1, 2\}\} \cup \{\{x\} : x \in \mathbb{Z}^+ \backslash \{1, 2\}\},$$

$$\pi(\beta) = \{\{2, 3\}\} \cup \{\{x\} : x \in \mathbb{Z}^+ \backslash \{2, 3\}\},$$

$$\pi(\gamma) = \{\{1, 2\}\} \cup \{\{x\} : x \in \mathbb{Z}^+ \backslash \{1, 2\}\}.$$

Hence by Theorem 4.2.4, we conclude that $(\alpha, \beta) \notin \mathcal{R}$. Since $A\alpha = A\beta$ for all $A \in \mathbb{Z}^+/E$, by Theorem 4.2.2, we have $(\alpha, \beta) \in \mathcal{L}$. We see that $\pi(\alpha) = \pi(\gamma)$ and $A_1\alpha = \{1, 2, 3\} \neq \{2, 3, 4\} = A_1\gamma$ where $A_1 \in \mathbb{Z}^+/E$. Thus by Theorem 4.2.4 and Theorem 4.2.2, we deduce that $(\alpha, \gamma) \in \mathcal{R}$ and $(\alpha, \gamma) \notin \mathcal{L}$, respectively.

**Theorem 4.2.6.** *For $\alpha, \beta \in T_{SE}(X)$, $\alpha \in T_{SE}(X)\beta T_{SE}(X)$ if and only if there exist a refinement $\mathcal{A}$ of $\pi(\alpha)$ and $\varphi : \mathcal{A} \to \pi(\beta)$ such that $\varphi$ is an injection and for every $P \in \mathcal{A}$, $P, P\varphi \subseteq A$ for some $A \in X/E$.*

**Proof.** Assume that $\alpha \in T_{SE}(X)\beta T_{SE}(X)$. Then $\alpha = \delta\beta\gamma$ for some $\delta, \gamma \in T_{SE}(X)$. Let $\mathcal{A} = \{\cup\mathcal{A}_Q : Q \in \pi(\beta) \text{ and } Q \cap X\delta \neq \emptyset\}$ where $\mathcal{A}_Q = \{P \in \pi(\alpha) : P\delta_* \in Q\}$. Then by Lemma 4.1.6, $\mathcal{A}$ is a refinement of $\pi(\alpha)$. Define $\varphi : \mathcal{A} \to \pi(\beta)$ by $(\cup\mathcal{A}_Q)\varphi = Q$. It is clear that $\varphi$ is well-defined. Suppose that $(\cup\mathcal{A}_Q)\varphi = (\cup\mathcal{A}_{Q'})\varphi$. By the definition of $\varphi$, $Q = Q'$. Thus $\mathcal{A}_Q = \mathcal{A}_{Q'}$ and so $\varphi$ is an injection. Let $\cup\mathcal{A}_Q \in \mathcal{A}$ where $Q \in \pi(\beta)$. We note that $Q \subseteq A$ for some $A \in X/E$ by Proposition 2.2.9(1). For every $P \in \mathcal{A}_Q$, we have $P\delta_* \in Q$ and then $P\delta_* \in A$. Hence $P \subseteq A$ by $\delta \in T_{SE}(X)$. Thus $\cup\mathcal{A}_Q \subseteq A$ which implies $\cup\mathcal{A}_Q \subseteq A$ and $(\cup\mathcal{A}_Q)\varphi = Q \subseteq A$.

Conversely, suppose that $\varphi : \mathcal{A} \to \pi(\beta)$ is an injection where $\mathcal{A}$ is a refinement of $\pi(\alpha)$ and $P, P\varphi \subseteq A$ for some $A \in X/E$ for each $P \in \mathcal{A}$. Let $x \in X$, so $x \in P$ for some $P \in \mathcal{A}$. Choose $\tilde{x} \in P\varphi$. We define $\delta : X \to X$ by

$$x\delta = \tilde{x} \text{ for all } x \in X.$$

Clearly, $P\varphi$ is unique and $P, P\varphi \subseteq A$ for some $A \in X/E$ then $\delta \in T_{SE}(X)$. Since $\beta_* : \pi(\beta) \to X\beta$ is an injection and by assumption, $\varphi\beta_* : \mathcal{A} \to X\beta$ is an injection. For each $x \in \mathcal{A}\varphi\beta_*$, there exists a unique $P_x \in \mathcal{A}$ such that $x = P_x\varphi\beta_*$, hence we fix $x' \in P_x$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} x'\alpha, & \text{if } x \in \mathcal{A}\varphi\beta_*; \\ x, & \text{otherwise.} \end{cases}$$

By $\mathcal{A}$ is a refinement of $\pi(\alpha)$, $\gamma$ is well-defined. Let $x \in X$, if $x \notin \mathcal{A}\varphi\beta_*$, then $(x, x\gamma) = (x, x) \in E$. Assume that $x \in \mathcal{A}\varphi\beta_*$. So, $x = P_x\varphi\beta_*$ for some $P_x \in \mathcal{A}$. By Proposition 2.2.9(1), there is $A \in X/E$ such that $P_x\varphi \subseteq A$. We conclude that $x = P_x\varphi\beta_* \in A\beta \subseteq A$ by Proposition 2.2.9(3). It follows from assumption that $P_x \subseteq A$. From $(x', x'\alpha) \in E$ and $x' \in A$, we then have $x'\alpha \in A$. Thus $\gamma \in T_{SE}(X)$. Now, for $x \in X$, $x \in P$ for some $P \in \mathcal{A}$. Since $\tilde{x}\beta = P\varphi\beta_* \subseteq \mathcal{A}\varphi\beta_*$, we conclude that $P\varphi\beta_* = \tilde{x}\beta = P_{\tilde{x}\beta}\varphi\beta_*$. It follows from $\varphi\beta_*$ is injective that $P = P_{\tilde{x}\beta}$. Hence $x, (\tilde{x}\beta)' \in P$. Since $\mathcal{A}$ is a refinement of $\pi(\alpha)$, there exists $P' \in \pi(\alpha)$ such that $P \subseteq P'$. Thus $x, (\tilde{x}\beta)' \in P'$ which implies that $x\alpha = (\tilde{x}\beta)'\alpha$. Consider

$$x\delta\beta\gamma = \tilde{x}\beta\gamma = (\tilde{x}\beta)'\alpha = x\alpha.$$

We get $\alpha \in T_{SE}(X)\beta T_{SE}(X)$ as required. □

**Theorem 4.2.7.** *For $\alpha, \beta \in T_{SE}(X)$, $\alpha \in T_{SE}(X)\beta T_{SE}(X)$ if and only if there exists an injection $\varphi' : \pi(\alpha) \to \pi(\beta)$ such that for every $P \in \pi(\alpha)$, $P, P\varphi' \subseteq A$ for some $A \in X/E$.*

**Proof.** Assume that $\alpha \in T_{SE}(X)\beta T_{SE}(X)$. We note by Theorem 4.2.6 that there exist a refinement $\mathcal{A}$ of $\pi(\alpha)$ and $\varphi : \mathcal{A} \to \pi(\beta)$ such that $\varphi$ is an injection and for all $P \in \mathcal{A}$, $P, P\varphi \subseteq A$ for some $A \in X/E$. For each $P \in \pi(\alpha)$, choose and fix $P' \in \mathcal{A}$ such that $P' \subseteq P$. Define $\varphi' : \pi(\alpha) \to \pi(\beta)$ by

$$P\varphi' = P'\varphi \text{ for all } P \in \pi(\alpha).$$

We then have $\varphi$ is well-defined. Let $P, Q \in \pi(\alpha)$ be such that $P\varphi' = Q\varphi'$. Hence $P'\varphi = Q'\varphi$. By $\varphi$ is injective, $P' = Q'$. Since $P \cap Q \neq \emptyset$ and $\pi(\alpha)$ is a partition of $X$, $P = Q$. Thus $\varphi'$ is injective. Let $P \in \pi(\alpha)$. Then $P', P'\varphi \subseteq A$ for some $A \in X/E$. We have $P \cap A \neq \emptyset$. By Proposition 2.2.9(1), we conclude that $P \subseteq A$. Therefore the proof is complete. □

**Theorem 4.2.8.** *For $\alpha, \beta \in T_{SE}(X)$, $(\alpha, \beta) \in \mathcal{J}$ if and only if there exist injections $\psi : \pi(\alpha) \to \pi(\beta)$ and $\psi' : \pi(\beta) \to \pi(\alpha)$ such that for every $P \in \pi(\alpha)$ and $Q \in \pi(\beta)$, $P, P\psi \subseteq A$ and $Q, Q\psi' \subseteq A'$ for some $A, A' \in X/E$.*

From Example 14, we can verify that $(\alpha, \beta), (\alpha, \gamma) \in \mathcal{J}$ via Theorem 4.2.8.

To obtain $\mathcal{D}$ relation, the following lemma is needed.

**Lemma 4.2.9.** *For $\alpha, \beta \in T_{SE}(X)$ and $A \in X/E$. If $\varphi : \pi_A(\beta) \to \pi_A(\alpha)$ is a bijection, then there exists $\delta_A : A \to X$ satisfies $\pi(\delta_A) = \pi_A(\beta)$ and $A\delta_A = A\alpha$.*

**Proof.** Assume that $\varphi : \pi_A(\beta) \to \pi_A(\alpha)$ is bijective. Let $x \in A$. By Proposition 2.2.9(2), there exists $P_x \in \pi_A(\beta)$ such that $x \in P_x$. We define $\delta_A : A \to X$ by

$$x\delta_A = (P_x\varphi)\alpha_* \text{ for all } x \in A.$$

$\delta_A$ is well-defined. By Proposition 2.2.9(2), we then have $\cup\pi_A(\beta) = A = \cup\pi(\delta_A)$. Claim that $\pi_A(\beta) = \pi(\delta_A)$. From the definition of $\delta_A$, we note that for all $P \in \pi_A(\beta)$, $P\delta_A = \{P\varphi\alpha_*\}$ so, $P \subseteq Q$ for some $Q \in \pi(\delta_A)$. For each $Q \in \pi(\delta_A)$, $Q\delta_{A*} = (P\varphi)\alpha_*$ for some $P \in \pi_A(\beta)$. To show $Q \subseteq P$, let $x \in Q$. Then there exists $P_x \in \pi_A(\beta)$ such that $x \in P_x$. So $Q\delta_{A*} = x\delta_A = (P_x\varphi)\alpha_*$, then we have $P\varphi\alpha_* = P_x\varphi\alpha_*$. Since $\varphi\alpha_*$ is a bijection, $P = P_x$. Hence $x \in P$, so $Q \subseteq P$. Therefore we conclude that $\pi_A(\beta) = \pi(\delta_A)$. By Proposition 2.2.9(2), $A = \cup\pi_A(\beta) = \cup\pi(\delta_A)$. So,

$$A\delta_A = (\cup\pi_A(\beta))\delta_A = (\cup\pi_A(\beta))\varphi\alpha = (\cup\pi_A(\beta)\varphi)\alpha = (\cup\pi_A(\alpha))\alpha = A\alpha.$$

Therefore the lemma is proved. □

**Theorem 4.2.10.** *For $\alpha, \beta \in T_{SE}(X)$, $(\alpha, \beta) \in \mathcal{D}$ if and only if for every $A \in X/E$, there exists a bijection $\varphi_A : \pi_A(\beta) \to \pi_A(\alpha)$.*

**Proof.** Suppose that $(\alpha, \beta) \in \mathcal{D}$. Then there exists $\delta \in T_{SE}(X)$ such that $(\alpha, \delta) \in \mathcal{L}$ and $(\delta, \beta) \in \mathcal{R}$. Let $A \in X/E$. For each $P \in \pi_A(\beta)$, we then have $P \in \pi(\beta)$ and $P \cap A \neq \emptyset$. By Theorem 4.2.4, we have $\pi(\delta) = \pi(\beta)$, so $P \in \pi(\delta)$. Since $P \cap A \neq \emptyset$ and by Proposition 2.2.9(1), we deduce that $P \subseteq A$, hence $P\delta_* \in A\delta$. From Theorem 4.2.2, we obtain that $A\alpha = A\delta$, that is $P\delta_* \in A\alpha \subseteq X\alpha$. Then there exists $Q_P \in \pi(\alpha)$ such that $Q_P\alpha_* = P\delta_*$. Since $Q_P\alpha_* \in A$ and $\alpha \in T_{SE}(X)$, we observe that $Q_P \subseteq A$ and then $Q_P \in \pi_A(\alpha)$. Define $\varphi_A : \pi_A(\beta) \to \pi_A(\alpha)$ by

$$P\varphi_A = Q_P \text{ for all } P \in \pi_A(\beta).$$

To show that $\varphi_A$ is well-defined, if $Q' \in \pi_A(\alpha)$ such that $Q'\alpha_* = P\delta_*$ then $Q = Q'$ by $Q\alpha_* = Q'\alpha_*$. Assume that $P\varphi_A = P'\varphi_A$. Then $Q_P = Q_{P'}$, hence $P\delta_* = Q_P\alpha_* = Q_{P'}\alpha_* = P'\delta_*$ which implies that $P = P'$. This shows that $\varphi$ is an injection. Claim that

$\varphi_A$ is a surjection, let $Q \in \pi_A(\alpha)$. Then $Q\alpha_* \in A\alpha = A\delta$. Thus there exists $P \in \pi(\delta)$ such that $P\delta_* = Q\alpha_*$. Since $\delta \in T_{SE}(X)$ and Proposition 2.2.9(3), we conclude that $P \subseteq A$. By $\pi(\delta) = \pi(\beta)$ then $P \in \pi(\beta)$ and so $P \in \pi_A(\beta)$. Therefore $P\varphi_A = Q$, thus we have the claim. Hence $\varphi_A$ is bijective as required.

Conversely, for every $A \in X/E$, there exists a bijection $\varphi_A : \pi_A(\beta) \to \pi_A(\alpha)$. It follows from Lemma 4.2.9 that there exists $\delta_A : A \to X$ corresponding to $A \in X/E$ such that $\pi_A(\beta) = \pi(\delta_A)$ and $A\delta_A = A\alpha$. Thus we define $\delta : X \to X$ by

$$\delta|_A = \delta_A \text{ for each } A \in X/E.$$

Since $X/E$ is a partition of $X$, $\delta$ is well-defined. We note that for each $A \in X/E$, $A\delta = A\delta_A = A\alpha \subseteq A$ by $\alpha \in T_{SE}(X)$. We then have $\delta \in T_{SE}(X)$. Finally, we can see that

$$\pi(\beta) = \bigcup_{A \in X/E} \pi_A(\beta) = \bigcup_{A \in X/E} \pi(\delta_A) = \bigcup_{A \in X/E} \pi_A(\delta) = \pi(\delta).$$

It follows from Theorem 4.2.4 that $(\delta, \beta) \in \mathcal{R}$. Consider,

$$A\delta = A\delta|_A = A\delta_A = A\alpha.$$

We then have $(\alpha, \delta) \in \mathcal{L}$ by Theorem 4.2.2. Therefore, $(\alpha, \beta) \in \mathcal{D}$.

Hence the theorem is completely proved. □