CHAPTER V

NATURAL PARTIAL ORDER

In chapter, we present the characterization of the natural partial order on $T_{SE}(X)$ and give a necessary and sufficient condition for elements in $T_{SE}(X)$ to be minimal, maximal and covering elements with respect to the order.

5.1 Characterizations

As was mention from Theorem 3.5.1, $T_{SE}(X)$ is a regular semigroup. Then we deduce the natural partial order on $T_{SE}(X)$ as follows : for $\alpha, \beta \in T_{SE}(X)$,

$$\alpha \leq \beta \text{ if and only if } \alpha = \delta\beta = \beta\gamma \text{ for some } \delta, \gamma \in E(T_{SE}(X)).$$

The following theorem investigates the conditions when $\alpha \leq \beta$ for all $\alpha, \beta \in T_{SE}(X)$.

Theorem 5.1.1. *Let $\alpha, \beta \in T_{SE}(X)$. Then $\alpha \leq \beta$ if and only if $\pi(\beta)$ refines $\pi(\alpha)$ and for every $P \in \pi(\alpha), P\alpha_* \in P\beta$.*

Proof. Suppose that $\alpha \leq \beta$. Then $\alpha = \delta\beta = \beta\gamma$ for some $\delta, \gamma \in E(T_{SE}(X))$. Let $P \in \pi(\beta)$. Then $P\beta_* = x$ for some $x \in X\beta$. Since $P\alpha = P\beta\gamma$, $P \subseteq (x\gamma)\alpha^{-1} \in \pi(\alpha)$. This proves that $\pi(\beta)$ is a refinement of $\pi(\alpha)$. Let $P \in \pi(\alpha)$ and $x \in P$. Then $x\alpha = P\alpha_*$. Since $\delta \in E(T_{SE}(X)), x\delta^2 = x\delta$. Therefore

$$x\alpha = x\delta\beta = (x\delta)\delta\beta = x\delta\alpha$$

which implies that $x\delta \in P$. Hence $P\alpha_* = x\delta\beta \in P\beta$.

Conversely, suppose that $\pi(\beta)$ refines $\pi(\alpha)$ and for every $P \in \pi(\alpha), P\alpha_* \in P\beta$. For each $x \in X\beta$, there exists a unique $Q_x \in \pi(\beta)$ such that $x = Q_x\beta_*$. By assumption, there exists a unique $P_x \in \pi(\alpha)$ such that $Q_x \subseteq P_x$. It follows from Proposition 2.2.9(1) that $P_x \subseteq A$ for some $A \in X/E$, hence $Q_x \subseteq A$. By Proposition 2.2.9(3), we have $Q_x\beta_* \in A\beta \subseteq A$ and $P_x\alpha_* \in A\alpha \subseteq A$, hence $(x, P_x\alpha_*) = (Q_x\beta_*, P_x\alpha_*) \in E$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} P_x\alpha_*, & \text{if } x \in X\beta; \\ x, & \text{otherwise.} \end{cases}$$

It is clear that $\gamma \in T_{SE}(X)$. To show that $\gamma \in E(T_{SE}(X))$, let $x \in X$. If $x \notin X\beta$, then $x\gamma^2 = x\gamma$. If $x \in X\beta$, then $x = Q_x\beta_*, x\gamma = P_x\alpha_*$ and $Q_x \subseteq P_x$ for some $Q_x \in \pi(\beta)$ and $P_x \in \pi(\alpha)$. By assumption, $P_x\alpha_* \in P_x\beta$, there exists $y \in P_x$ such that $P_x\alpha_* = y\beta$. Since $y\beta \in X\beta$, there exists $Q_{y\beta} \in \pi(\beta)$ such that $y\beta = Q_{y\beta}\beta_*$. Hence $Q_{y\beta} \cap P_x \neq \emptyset$ which implies that $Q_{y\beta} \subseteq P_x$ by assumption. From the definition of γ, we have $y\beta\gamma = P_x\alpha_*$. Thus $y\gamma^2 = (P_x\alpha_*)\gamma = y\beta\gamma = P_x\alpha_* = x\gamma$. This shows that $\gamma \in E(T_{SE}(X))$ as required.

To show that $\beta\gamma = \alpha$, let $x \in X$. Then $x\beta = Q_{x\beta}\beta_*$ and $x \in Q_{x\beta} \subseteq P_{x\beta}$ for some $Q_{x\beta} \in \pi(\beta)$ and $P_{x\beta} \in \pi(\alpha)$. Then $x\beta\gamma = P_{x\beta}\alpha_* = x\alpha$, so $\beta\gamma = \alpha$.

Next, for each $P \in \pi(\alpha)$, by assumption, we choose and fix $x_P \in P$ such that $P\alpha_* = x_P\beta$. Since $\pi(\alpha)$ is a partition of X, for each $x \in X$, we let $P_x \in \pi(\alpha)$ such that $x \in P_x$. Define $\delta : X \to X$ by

$$x\delta = x_{P_x} \text{ for all } x \in X.$$

Let $x \in X$. Then $x \in P_x$ for some $P_x \in \pi(\alpha)$. By Proposition 2.2.9(1), there exists $A \in X/E$ such that $P_x \subseteq A$. Since $x\delta = x_{P_x} \in P_x \subseteq A$, $(x, x\delta) \in E$. Thus $\delta \in T_{SE}(X)$. Consider,

$$x\delta\beta = x_{P_x}\beta = P_x\alpha_* = x\alpha.$$

Since $x_{P_x} \in P_x$, by the definition of δ we have $x_{P_x}\delta = x_{P_x}$ and $x\delta^2 = x_{P_x}\delta = x_{P_x} = x\delta$. Therefore $\alpha = \delta\beta$ and $\delta \in E(T_{SE}(X))$, respectively. Thus the theorem is completely proved. □

Corollary 5.1.2. *Let $\alpha, \beta \in T_{SE}(X)$. Then $\alpha \leq \beta$ if and only if for every $A \in X/E$, $\pi_A(\beta)$ is a refinement of $\pi_A(\alpha)$ and for every $P \in \pi_A(\alpha), P\alpha_* \in P\beta$.*

Proof. Suppose that $\alpha \leq \beta$. Let $A \in X/E$ and $P \in \pi_A(\beta)$. We then have $P \in \pi(\beta)$ and $P \cap A \neq \emptyset$. By Theorem 5.1.1, there exists $Q \in \pi(\alpha)$ such that $P \subseteq Q$. Thus $\emptyset \neq P \cap A \subseteq Q \cap A$ which implies that $Q \in \pi_A(\alpha)$ and $P \subseteq Q$. It is clear from

Proposition 2.2.9(2) that $\cup\pi_A(\beta) = \cup\pi_A(\alpha)$. Hence $\pi_A(\beta)$ is a refinement of $\pi_A(\alpha)$. Moreover, for any $P \in \pi_A(\alpha)$, we then have $P \in \pi(\alpha)$. By Theorem 5.1.1, $P\alpha_* \in P\beta$.

Conversely, suppose that for every $A \in X/E, \pi_A(\beta)$ is a refinement of $\pi_A(\alpha)$ and for all $P \in \pi_A(\alpha), P\alpha_* \in P\beta$. To show that $\alpha \leq \beta$, let $P \in \pi(\beta)$. From Proposition 2.2.9(1), there exists $A \in X/E$ such that $P \subseteq A$, so $P \in \pi_A(\beta)$. By assumption, $P \subseteq Q$ for some $Q \in \pi_A(\alpha)$. Since $\pi_A(\alpha) \subseteq \pi(\alpha)$, $\pi(\beta)$ refines $\pi(\alpha)$. Next, let $P \in \pi(\alpha)$. By Proposition 2.2.9(1), we have $P \subseteq A$ for some $A \in X/E$, hence $P \in \pi_A(\alpha)$. By assumption, $P\alpha_* \in P\beta$. It follows from Theorem 5.1.1 that $\alpha \leq \beta$ as desired. □

5.2 Compatibility

Recall that for any partial order ρ on a semigroup S, an element $c \in S$ is said to be *left compatible* with ρ if for every $(a, b) \in \rho$ implies $(ca, cb) \in \rho$. *Right compatible* with ρ is defined dually. Next, we describe the left and right compatible elements in $T_{SE}(X)$.

Theorem 5.2.1. *Let $\alpha \in T_{SE}(X)$. Then α is left compatible with $\leq$ on $T_{SE}(X)$ if and only if α is surjective.*

Proof. Suppose that α is not a surjection. Let $a' \in X \backslash X\alpha$. Then there exists $A \in X/E$ such that $a' \in A$. We choose and fix $a \in A\alpha$, hence $a \neq a'$. By Proposition 2.2.9(3), we have that $a, a' \in A$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} a', & \text{if } x = a; \\ x, & \text{otherwise.} \end{cases}$$

Since $a, a' \in A$, we get $\beta \in T_{SE}(X)$. We note that

$$\pi(\beta) = \{\{a, a'\}\} \cup \{\{x\} : x \in X\backslash\{a, a'\}\}.$$

It is easy to see that $\pi(i_X)$ refines $\pi(\beta)$ and $P\beta_* \in Pi_X$ for all $P \in \pi(\beta)$ where i_X is the identity map on X. By Theorem 5.1.1, we deduce that $\beta \leq i_X$. Since $a' \in X\alpha\beta$, we have $Q = a'(\alpha\beta)^{-1} \in \pi(\alpha\beta)$. Then

$$Q = a'(\alpha\beta)^{-1} = a'\beta^{-1}\alpha^{-1} = \{a, a'\}\alpha^{-1} = a\alpha^{-1}.$$

Since $Q\alpha i_X = (a\alpha^{-1})\alpha i_X = \{ai_X\} = \{a\}$, $Q(\alpha\beta)_* = a' \notin Q\alpha i_X$. By Theorem 5.1.1, we conclude that $\alpha\beta \not\leq \alpha i_X$. This proves that α is not left compatible with $\leq$ on $T_{SE}(X)$.

Conversely, assume that α is surjective. Then $Y\alpha^{-1}\alpha = Y$ for all $Y \subseteq X$. Let $\beta, \gamma \in T_{SE}(X)$ be such that $\beta \leq \gamma$. To show that $\alpha\beta \leq \alpha\gamma$ via Theorem 5.1.1, let $P \in \pi(\alpha\gamma)$. Then $P(\alpha\gamma)_* = y$ for some $y \in X\alpha\gamma$. Since $X\alpha\gamma \subseteq X\gamma$, $y \in X\gamma$. Let $Q \in \pi(\gamma)$ be such that $Q\gamma_* = y$. Since $\beta \leq \gamma$ and by Theorem 5.1.1, $Q \subseteq \tilde{P}$ for some $\tilde{P} \in \pi(\beta)$. Note here that

$$P\alpha\beta = y(\alpha\gamma)^{-1}\alpha\beta = (y\gamma^{-1})\alpha^{-1}\alpha\beta = (y\gamma^{-1})\beta = Q\beta \subseteq \tilde{P}\beta = \{\tilde{P}\beta_*\}.$$

Hence $P \subseteq \tilde{P}\beta_*(\alpha\beta)^{-1} \in \pi(\alpha\beta)$. That is $\pi(\alpha\gamma)$ refines $\pi(\alpha\beta)$. Next, let $P \in \pi(\alpha\beta)$. Then $P = y(\alpha\beta)^{-1}$ for some $y \in X\alpha\beta$. We then have $y \in X\beta$, so $Q\beta_* = y$ for some $Q \in \pi(\beta)$. Since $\beta \leq \gamma$, by Theorem 5.1.1 we have $Q\beta_* \in Q\gamma$. Consider

$$P(\alpha\beta)_* = y\beta^{-1}\alpha^{-1}\alpha\beta_* = Q\alpha^{-1}\alpha\beta_* = Q\beta_* \in Q\gamma = Q\alpha^{-1}\alpha\gamma = y\beta^{-1}\alpha^{-1}\alpha\gamma = P\alpha\gamma.$$

It follows from Theorem 5.1.1 that $\alpha\beta \leq \alpha\gamma$. Therefore α is a left compatible with $\leq$ on $T_{SE}(X)$. □

Theorem 5.2.2. *Let $\alpha \in T_{SE}(X)$. Then α is right compatible with $\leq$ on $T_{SE}(X)$ if and only if for every $A \in X/E$, $A \in \pi(\alpha)$ or $|P| = 1$ for all $P \in \pi_A(\alpha)$.*

Proof. Assume that there exists $A \in X/E$ such that $A \notin \pi(\alpha)$ and $|P'| > 1$ for some $P' \in \pi_A(\alpha)$. By Proposition 2.2.9(2), we have $P' \subseteq A$. Since $A \notin \pi(\alpha)$, it follows that $P' \neq A$. We choose $p' \in P'$ and $a \in A \setminus P'$. Then $p'\alpha = P'\alpha_*$ and $a\alpha \neq P'\alpha_*$. Now, define $\beta : X \to X$ by

$$x\beta = \begin{cases} a, & \text{if } x = p'; \\ x, & \text{otherwise.} \end{cases}$$

Let $x \in X$.

$$(x, x\beta) = \begin{cases} (p', a) \in E, & \text{if } x = p'; \\ (x, x) \in E, & \text{otherwise,} \end{cases}$$

thus $\beta \in T_{SE}(X)$. It is easy to see that $\pi(i_X) = \{\{x\} : x \in X\}$ is a refinement of $\pi(\beta)$ and $P\beta_* \in Pi_X$ for all $P \in \pi(\beta)$. By Theorem 5.1.1, $\beta \leq i_X$. Note that

$$(P'\alpha_*)(i_X\alpha)^{-1} = (P'\alpha_*)\alpha^{-1}i_X^{-1} = P'i_X^{-1} = P'.$$

Hence we deduce $P' \in \pi(i_X\alpha)$. By the definition of β and $P'\backslash\{p'\} \neq \emptyset$, we have that $P'\beta\alpha = (\{a\} \cup P'\backslash\{p'\})\,\alpha = \{a\alpha, P'\alpha_*\}$. Claim that $P' \nsubseteq Q$ for all $Q \in \pi(\beta\alpha)$. Suppose not, there exists $Q \in \pi(\beta\alpha)$ such that $P' \subseteq Q$. Since $\{a\alpha, P'\alpha_*\} = P'\beta\alpha$, we observe that $\{a\alpha, P\alpha_*\} \subseteq Q\beta\alpha = \{Q(\beta\alpha)_*\}$ which is a contradiction. So we have the claim. This proves that $\pi(i_X\alpha)$ does not refine $\pi(\beta\alpha)$. By Theorem 5.1.1, we conclude that $\beta\alpha \nleq i_X\alpha$. Therefore α is not a right compatible.

Conversely, suppose that for all $A \in X/E$, $A \in \pi(\alpha)$ or $|P| = 1$ for all $P \in \pi_A(\alpha)$. Let $\beta, \gamma \in T_{SE}(X)$ be such that $\beta \leq \gamma$. To show that $\beta\alpha \leq \gamma\alpha$ via Corollary 5.1.2, let $A \in X/E$. We consider two cases as follow :

Case 1. $A \in \pi(\alpha)$. Then $A\alpha_* = y$ for some $y \in X\alpha$. By Proposition 2.2.9(3), $A\beta \subseteq A$. Since $A\beta\alpha \subseteq A\alpha = \{y\}$, we note that $A \subseteq y(\beta\alpha)^{-1} \in \pi(\beta\alpha)$. By Proposition 2.2.9(1), there exists $B \in X/E$ such that $y(\beta\alpha)^{-1} \subseteq B$. Then $A = B$ since X/E is a partition of X. Hence $A = y(\beta\alpha)^{-1}$ which implies that $\pi_A(\beta\alpha) = \{A\}$. Similarly, we have that $\pi_A(\gamma\alpha) = \{A\}$. Hence $\pi_A(\gamma\alpha)$ refines $\pi_A(\beta\alpha)$. Moreover, let $P \in \pi_A(\beta\alpha) = \{A\}$. Then

$$P(\beta\alpha)_* = A(\beta\alpha)_* = y \in \{y\} = A\gamma\alpha = P\gamma\alpha.$$

Case 2. $|P| = 1$ for all $P \in \pi_A(\alpha)$. Let $P \in \pi_A(\gamma\alpha)$. $P(\gamma\alpha)_* = y$ for some $y \in X\gamma\alpha$. Then $P\gamma \subseteq y\alpha^{-1}$. Since $y\alpha^{-1} \in \pi_A(\alpha)$, by assumption, $|y\alpha^{-1}| = 1$. Let $y\alpha^{-1} = \{x\}$ for some $x \in X$. We then have $P\gamma = \{x\}$ and $P \cap A \neq \emptyset$, hence $P = x\gamma^{-1} \in \pi_A(\gamma)$. Since $\beta \leq \gamma$, by Corollary 5.1.2, $\pi_A(\gamma)$ refines $\pi_A(\beta)$. Hence $P \subseteq Q$ for some $Q \in \pi_A(\beta)$. This means that $P\beta \subseteq Q\beta = \{Q\beta_*\}$. Now, we consider $P\beta\alpha \subseteq Q\beta\alpha = \{Q\beta_*\alpha\}$, thus $P \subseteq (Q\beta_*\alpha)(\beta\alpha)^{-1}$. Note that

$$\emptyset \neq A \cap P \subseteq A \cap (Q\beta_*\alpha)(\beta\alpha)^{-1},$$

hence $(Q\beta_*\alpha)(\beta\alpha)^{-1} \in \pi_A(\beta\alpha)$. This proves that $\pi_A(\gamma\alpha)$ refines $\pi_A(\beta\alpha)$. Next, let $P \in \pi_A(\beta\alpha)$. Then $P(\beta\alpha)_* = y$ for some $y \in X$ which implies that $P\beta \subseteq y\alpha^{-1}$. By assumption, $y\alpha^{-1} = \{x\}$ for some $x \in X$, hence $P\beta = \{x\}$. Therefore $P = x\beta^{-1} \in \pi_A(\beta)$. It follows from $\beta \leq \gamma$ and Corollary 5.1.2, we have $P\beta_* \in P\gamma$. Hence $P(\beta\alpha)_* \in P\beta\alpha \subseteq (P\gamma)\alpha$.

From two cases, we conclude that $\beta\alpha \leq \gamma\alpha$ by Corollary 5.1.2. This shows that α is right compatible with $\leq$ on $T_{SE}(X)$. □

5.3 Minimal, maximal and covering elements

To characterize maximality respect to the natural order, the following lemma is needed.

Lemma 5.3.1. *Let $\alpha, \beta \in T_{SE}(X)$ be such that $\alpha \leq \beta$ and $A \in X/E$. If $A\alpha = A$, then $x\alpha = x\beta$ for all $x \in A$.*

Proof. Assume that $A\alpha = A$. Let $x \in A$. By $\pi(\beta)$ is a partition of X, $x \in Q$ for some $Q \in \pi(\beta)$. Since $\alpha \leq \beta$ and Theorem 5.1.1, there exists $P \in \pi(\alpha)$ such that $Q \subseteq P$. Hence $x \in P$. It follows that $x\beta = Q\beta_*$ and $x\alpha = P\alpha_*$. By assumption and Proposition 2.2.9(3), $x\beta \in A\beta \subseteq A = A\alpha$. Since $\alpha_* : \pi(\alpha) \to X\alpha$ is surjective, there exists $P' \in \pi(\alpha)$ such that $x\beta = P'\alpha_*$. We note by Theorem 5.1.1 that $P'\alpha_* \in P'\beta$. Then $x\beta = y\beta$ for some $y \in P'$ which implies that $y \in Q$. This means that $P \cap P' \neq \emptyset$. Since $\pi(\alpha)$ is a partition of X, $P = P'$. Therefore

$$x\beta = P'\alpha_* = P\alpha_* = x\alpha.$$

The proof is complete. □

Theorem 5.3.2. *Let $\alpha \in T_{SE}(X)$. If for every $A \in X/E$, $A\backslash A\alpha = \emptyset$ or $|P| = 1$ for all $P \in \pi_A(\alpha)$, then α is a maximal element.*

Proof. Assume that for every $A \in X/E, A\backslash A\alpha = \emptyset$ or $|P| = 1$ for all $P \in \pi_A(\alpha)$. Suppose that $\alpha \leq \beta$ for some $\beta \in T_{SE}(X)$. To verify that $\alpha = \beta$, let $x \in X$. Then

$x \in A$ for some $A \in X/E$. It follows from assumption that $A\backslash A\alpha = \emptyset$ or $|P| = 1$ for all $P \in \pi_A(\alpha)$. If $A\backslash A\alpha = \emptyset$, then by Lemma 5.3.1, we have $x\alpha = x\beta$. Suppose that $|P| = 1$ for all $P \in \pi_A(\alpha)$. Since $\pi(\beta)$ is a partition of X, there exists $Q \in \pi(\beta)$ such that $x \in Q$. By assumption and Theorem 5.1.1, $Q \subseteq P$ for some $P \in \pi(\alpha)$ and $P\alpha_* \in P\beta$. Since $x \in A \cap P$, we get $P \in \pi_A(\alpha)$. It follows from assumption that $|P| = 1$, then $Q = P = \{x\}$. Hence $x\alpha = P\alpha_* \in P\beta = \{x\beta\}$ and so $\alpha = \beta$. Thus α is a maximal element. □

Theorem 5.3.3. *Let $\alpha \in T_{SE}(X)$. If there exists $A \in X/E$ such that $A\backslash A\alpha \neq \emptyset$ and $|P| > 1$ for some $P \in \pi_A(\alpha)$, then α has an upper cover.*

Proof. Suppose that there exists $A \in X/E$ such that $A\backslash A\alpha \neq \emptyset$ and $|P| > 1$ for some $P \in \pi_A(\alpha)$. Choose $a \in A\backslash A\alpha$ and $a' \in P$. Clearly, $a, a' \in A$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} a, & \text{if } x = a'; \\ x\alpha, & \text{otherwise.} \end{cases}$$

We see that

$$(x, x\beta) = \begin{cases} (a', a) \in E, & \text{if } x = a'; \\ (x, x\alpha) \in E, & \text{otherwise} \end{cases}$$

for all $x \in X$, thus $\beta \in T_{SE}(X)$. Since $|P| > 1$, $P\backslash\{a'\} \neq \emptyset$. It is easy to verify that $\pi(\beta) = \pi(\alpha)\backslash\{P\} \cup \{P\backslash\{a'\}, \{a'\}\}$ and then $\pi(\beta)$ refines $\pi(\alpha)$. For $Q \in \pi(\alpha)$,

$$Q\alpha_* = \begin{cases} P\alpha_* \in \{P\alpha_*, a'\} = P\beta, & \text{if } Q = P; \\ Q\beta_* \in Q\beta, & \text{otherwise.} \end{cases}$$

By Theorem 5.1.1, we conclude that $\alpha \leq \beta$. Clearly, $\alpha \neq \beta$.

Suppose that $\alpha \leq \delta \leq \beta$ for some $\delta \in T_{SE}(X)$ and $\delta \neq \alpha$. To show that $\delta = \beta$, let $x \in X$. Since $\pi(\beta)$ is a partition of X, there exists $Q \in \pi(\beta)$ such that $x \in Q$. By $\delta \leq \beta$ and Theorem 5.1.1, we have $Q \subseteq P'$ for some $P' \in \pi(\delta)$. Similarly, we conclude that $P' \subseteq Q'$ for some $Q' \in \pi(\alpha)$. Thus $Q \subseteq P' \subseteq Q'$. If $Q \in \pi(\alpha)\backslash\{P\}$, then we have $Q = Q'$ and hence $Q = P' = Q'$. It follows from $\alpha \leq \delta \leq \beta$ and Theorem 5.1.1 that

$$Q'\alpha_* \in Q'\delta = P'\delta = \{P'\delta_*\} \subseteq P'\beta = \{Q\beta_*\}.$$

Hence $x\alpha = x\delta = x\beta$. This implies that $x'\alpha = x'\delta = x'\beta$ for all $x' \in X \backslash P$.

Assume that $Q \notin \pi(\alpha) \backslash \{P\}$. Therefore $Q = \{a'\}$ or $Q = P \backslash \{a'\}$. We note that $Q \subseteq P$. Since $\pi(\alpha)$ is a partition of X, we have $P' \subseteq Q' = P$. If $P' = P$, then by $\alpha \leq \delta$ and Theorem 5.1.1, we have

$$x\alpha = P\alpha_* \in P\delta = P'\delta = \{x\delta\}$$

which implies that $x'\alpha = x'\delta$ for all $x' \in X$. This is a contradiction, thus $P' \neq P$. There are two cases to consider :

Case 1. $Q = \{a'\}$. Claim that $P' = Q$, assume that $P' \backslash \{a'\} \neq \emptyset$. Let $p \in P' \backslash \{a'\}$, then $p \in P \backslash \{a'\}$. We note by $P \backslash \{a'\} \in \pi(\beta)$ and Theorem 5.1.1 that $P \backslash \{a'\} \subseteq P''$ for some $P'' \in \pi(\delta)$. This means that $P' \cap P'' \neq \emptyset$. By $\pi(\delta)$ is a partition of X, $P' = P''$. Consider,

$$P = \{a'\} \cup P \backslash \{a'\} \subseteq P' \subseteq P.$$

We observe that $P' = P$ which is a contradiction. So, we have the claim.

Case 2. $Q = P \backslash \{a'\}$. Similarly, we conclude that $P' = Q$.

It follows from two cases that $P' = Q$. Since $\delta \leq \beta$ and Theorem 5.1.1,

$$x\delta = P'\delta_* \in P'\beta = Q\beta = \{x\beta\}.$$

Therefore $\delta = \beta$ and hence β is an upper cover of α. □

As a direct consequence of Theorem 5.3.2 and Theorem 5.3.7, we have the following corollaries.

Corollary 5.3.4. *Let $\alpha \in T_{SE}(X)$. Then the following statements are equivalent.*

(1) *α has an upper cover.*

(2) *α is not a maximal element.*

(3) *There exists $A \in X/E$ such that $A \backslash A\alpha \neq \emptyset$ and $|P| > 1$ for some $P \in \pi_A(\alpha)$.*

Corollary 5.3.5. *Let $\alpha \in T_{SE}(X)$. Then α is a maximal element of $T_{SE}(X)$ if and only if for every $A \in X/E, \alpha|_A$ is injective or surjective.*

Theorem 5.3.6. *Let $\alpha \in T_{SE}(X)$. If α is not a minimal element, then there exists $A \in X/E$ such that $|\pi_A(\alpha)| > 1$.*

Proof. Assume that $\beta \leq \alpha$ for some $\beta \in T_{SE}(X)$ and $\beta \neq \alpha$. Then there exists $x \in X$ such that $x\beta \neq x\alpha$. We note by X/E is a partition of X that $x \in A$ for some $A \in X/E$. It follows from Proposition 2.2.9(2) that $x \in P$ for some $P \in \pi_A(\alpha)$. By assumption and Theorem 5.1.1, there exists $Q \in \pi(\beta)$ such that $P \subseteq Q$ and $Q\beta_* \in Q\alpha$. Since $\alpha_* : \pi(\alpha) \to X\alpha$ is surjective, $Q\beta_* = P'\alpha_*$ for some $P' \in \pi(\alpha)$. We note that $x\alpha \neq x\beta = Q\beta_* = P'\alpha_*$. Then $x \notin P'$ which implies that $P \neq P'$. From $x \in A$ and Proposition 2.2.9(3), we then have $x\beta \in A\beta \subseteq A$. Choose $x' \in P'$, we note that $(x', x\beta) = (x', x'\alpha) \in E$ by $\alpha \in T_{SE}(X)$. This means that $x' \in A$ and then $P' \in \pi_A(\alpha)$. Therefore $|\pi_A(\alpha)| > 1$ as desired. □

Theorem 5.3.7. *Let $\alpha \in T_{SE}(X)$. If there exists $A \in X/E$ such that $|\pi_A(\alpha)| > 1$, then α has a lower cover.*

Proof. Suppose that there exists $A \in X/E$ such that $|\pi_A(\alpha)| > 1$. Then we choose two distinct sets $P, P' \in \pi_A(\alpha)$. This means that $P\alpha_* \neq P'\alpha_*$. By Proposition 2.2.9(2) that $P, P' \subseteq A$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} P\alpha_*, & \text{if } x \in P \cup P'; \\ x\alpha, & \text{otherwise.} \end{cases}$$

By Proposition 2.2.9(3), we conclude that $P\alpha_* \in A\alpha \subseteq A$. Hence $(x, x\beta) \in E$ for all $x \in X$ which implies that $\beta \in T_{SE}(X)$. Note that

$$\pi(\beta) = \pi(\alpha) \backslash \{P, P'\} \cup \{P \cup P'\}.$$

This means that $\pi(\alpha)$ refines $\pi(\beta)$. Moreover, by the definition of β, we see that

$$Q\beta_* = \begin{cases} Q\alpha_* \in Q\alpha, & \text{if } Q \in \pi(\alpha)\backslash\{P, P'\}; \\ P\alpha_* \in Q\alpha, & \text{if } Q = P \cup P', \end{cases}$$

for all $Q \in \pi(\beta)$. It follows from Theorem 5.1.1 that $\beta \leq \alpha$ and clearly, $\beta \neq \alpha$.

Assume that $\beta \leq \delta \leq \alpha$ for some $\delta \in T_{SE}(X)$. We note by Theorem 5.1.1 that $\pi(\alpha)$ refines $\pi(\delta)$ and $\pi(\delta)$ refines $\pi(\beta)$. We will verify that $\pi(\alpha) = \pi(\delta)$ or $\pi(\delta) = \pi(\beta)$. Suppose that $\pi(\delta) \neq \pi(\beta)$. To show that $\pi(\delta) = \pi(\alpha)$, let $Q \in \pi(\delta)$ and $x \in Q$. Since $\pi(\delta)$ refines $\pi(\beta)$, there exists $Q' \in \pi(\beta)$ such that $Q \subseteq Q'$.

Case 1. $Q' \in \pi(\alpha)\backslash\{P, P'\}$. By $\pi(\alpha)$ is a partition of X, $x \in Q''$ for some $Q'' \in \pi(\alpha)$. Since $\pi(\alpha)$ refines $\pi(\delta)$, there exists $\tilde{Q} \in \pi(\delta)$ such that $Q'' \subseteq \tilde{Q}$. This implies that $x \in Q \cap \tilde{Q}$. It follows from $\pi(\delta)$ is a partition of X that $Q = \tilde{Q}$. Thus $Q'' \subseteq Q \subseteq Q'$. Since $Q', Q'' \in \pi(\alpha)$ and $\pi(\alpha)$ is a partition of X, we conclude that $Q'' = Q'$ and hence $Q = Q'' \in \pi(\alpha)$.

Case 2. $Q' = P \cup P'$. If $Q = Q'$, then it is easy to see that $\pi(\beta) = \pi(\delta)$ which lead to a contradiction. Thus $Q \neq Q'$. Since $x \in Q' = P \cup P'$, we have $x \in P$ or $x \in P'$.

Subcase 2.1. $x \in P$. We observe that $P \subseteq Q \subseteq P \cup P'$. To show that $Q = P$, assume that $P \neq Q$. Let $a \in Q\backslash P$. From $Q \subseteq P \cup P'$, we get $a \in P'$. Since $\pi(\alpha)$ refines $\pi(\delta)$, we conclude that $P' \subseteq Q$. Thus $P \cup P' \subseteq Q \subseteq P \cup P'$ which implies that $Q = Q'$. It is impossible. Hence $Q = P \in \pi(\alpha)$.

Subcase 2.2. $x \in P'$. By symmetry, we then have $Q = P' \in \pi(\alpha)$.

It follows from two cases that $\pi(\delta) \subseteq \pi(\alpha)$. Let $Q \in \pi(\alpha)$. Since $\pi(\alpha)$ refines $\pi(\delta)$, there exists $Q' \in \pi(\delta)$ such that $Q \subseteq Q'$. Hence we get from $\pi(\delta) \subseteq \pi(\alpha)$ that $Q' \in \pi(\alpha)$. By $\pi(\alpha)$ is a partition of X, we have $Q = Q'$. Thus $\pi(\alpha) \subseteq \pi(\delta)$. Therefore $\pi(\alpha) = \pi(\delta)$ as required. Next, we will show that $\alpha = \delta$ or $\delta = \beta$. There are two cases to consider :

Case 1. $\pi(\delta) = \pi(\alpha)$. To verify that $\delta = \alpha$, let $x \in X$. By $\pi(\alpha)$ is a partition of X, $x \in Q$ for some $Q \in \pi(\alpha)$. Since $\delta \leq \alpha$ and Theorem 5.1.1, $Q \subseteq R$ and $R\delta_* \in R\alpha$ for some $R \in \pi(\delta)$. Since $\pi(\delta) = \pi(\alpha)$, $Q = R$ and then $x\delta = x\alpha$. Hence $\delta = \alpha$.

Case 2. $\pi(\delta) = \pi(\beta)$. Similarly, we have $\delta = \beta$.

This means that β is a lower cover of α. Therefore theorem is proved. □

Corollary 5.3.8. *Let $\alpha \in T_{SE}(X)$. Then the following statements are equivalent.*

(1) *α has a lower cover.*

(2) *α is not a minimal element.*

(3) *There exists $A \in X/E$ such that $|\pi_A(\alpha)| > 1$.*

Corollary 5.3.9. *For $\alpha \in T_{SE}(X)$. Then α is a minimal element if and only if for every $A \in X/E$, $\alpha|_A$ is a constant mapping.*