CHAPTER III

EXISTENCE RESULTS FOR FIXED POINT PROBLEMS, EQUILIBRIUM PROBLEMS AND QUSI-EQUILIBRIUM PROBLEMS

3.1 On the existence results for system of generalized strong vector quasi-equilibrium problems

In this section, we introduce a different kind of systems of generalized strong vector quasi-equilibrium problem without assuming that the dual of the ordering cone has a weak* compact base. Let X, Y and Z be real locally convex Hausdorff topological vector spaces, $K \subset X$ and $D \subset Y$ be nonempty compact convex subsets, and $C \subset Z$ be a nonempty closed convex cone. We also suppose that $S_1, S_2 : K \to 2^K$, $T_1, T_2 : K \to 2^D$ and $F_1, F_2 : K \times D \times K \to 2^Z$ are set-valued mappings. We consider the following system of generalized strong vector quasi-equilibrium problem (in short, *SGSVQEP*): finding $(\bar{x}, \bar{u}) \in K \times K$ and $\bar{v} \in T_1(\bar{x}), \bar{y} \in T_2(\bar{u})$ such that $\bar{x} \in S_1(\bar{x}), \bar{u} \in S_2(\bar{u})$ satisfying

$$\begin{aligned} &F_1(\bar{x}, \bar{y}, z) \subset C \quad \forall z \in S_1(\bar{x}), \\ &F_2(\bar{u}, \bar{v}, z) \subset C \quad \forall z \in S_2(\bar{u}). \end{aligned} \tag{3.1.1}$$

We call this $(\bar{x}, \bar{u})$ a strong solution for the *SGSVQEP*. We apply Kakutani-Fan-Glicksberg fixed point theorem to prove an existence theorem of strong solutions for the system of generalized strong vector quasi-equilibrium problem. Moreover, we also prove the closedness of the strong solution set for the system of generalized strong vector quasi-equilibrium problem.

Theorem 3.1.1. *For each $i = \{1, 2\}$, let $S_i : K \to 2^K$ be continuous set-valued mappings such that for any $x \in K$, $S_i(x)$ are nonempty closed convex subsets of K. Let $T_i : K \to 2^D$ be upper semi continuous set-valued mappings such that for any*

$x \in K$, $T_i(x)$ is nonempty closed convex subsets of D and $F_i : K \times D \times K \to 2^Z$ is set-valued mappings satisfying the following conditions:

(i) for all $(x, y) \in K \times D$, $F_i(x, y, S_i(x)) \subset C$;

(ii) for all $(y, z) \in D \times K$, $F_i(\cdot, y, z)$ are properly C-quasiconvex;

(iii) $F_i(\cdot, \cdot, \cdot)$ are upper C-continuous;

(iv) for all $y \in D$, $F_i(\cdot, y, \cdot)$ are lower $(-C)$-continuous.

Then SGSVQEP has a solution. Moreover, the set of all strong solutions is closed.

Proof. For any $(x, y) \in K \times D$, define set-valued mappings $A, B : K \times D \to 2^K$ by

$$A(x, y) = \{a \in S_1(x) : F_1(a, y, z) \subset C, \forall z \in S_1(x)\}$$

and

$$B(x, y) = \{b \in S_2(x) : F_2(b, y, z) \subset C, \forall z \in S_2(x)\}.$$

Step 1. Show that $A(x, y)$ and $B(x, y)$ are nonempty.

For any $x \in K$, we note that $S_1(x)$ and $S_2(x)$ are nonempty. Thus, for any $(x, y) \in K \times D$, we have $A(x, y)$ and $B(x, y)$ are nonempty.

Step 2. Show that $A(x, y)$ and $B(x, y)$ are convex subsets of K .

Let $a_1, a_2 \in A(x, y)$ and $\lambda \in [0, 1]$. Put $a = \lambda a_1 + (1 - \lambda)a_2$. Since $a_1, a_2 \in S_1(x)$ and $S_1(x)$ is convex set, we have $a \in S_1(x)$. We claim that $a \in A(x, y)$. In fact, if $a \notin A(x, y)$, then there exists $z^* \in S_1(x)$ such that

$$F_1(a, y, z^*) \not\subset C. \tag{3.1.2}$$

But since $F_1(a, y, z^*) \in F_1(a, y, S_1(x))$ and by (i), $F_1(a, y, S_1(x)) \subset C$. We see that (3.1.2) possess a contradiction. Therefore $a \in A(x, y)$ and hence $A(x, y)$ is a convex subset of K. Similarly, we have $B(x, y)$ is a convex subset of K.

Step 3. Show that $A(x,y)$ and $B(x,y)$ are closed subsets of K.

Let $\{a_\alpha\}$ be a net in $A(x,y)$ such that $a_\alpha \to a^*$. Thus, we have $a_\alpha \in S_1(x)$. Since $S_1(x)$ is a closed subset of K, it follows that $a^* \in S_1(x)$. By the lower semicontinuity of S_1 and Lemma 2.4.8(iii) , for any $z^* \in S_1(x)$ and any net $\{x_\alpha\} \to x$, there exists a net $\{z_\alpha\}$ such that $z_\alpha \in S_1(x_\alpha)$ and $z_\alpha \to z^*$. This implies that

$$F_1(a_\alpha, y, z_\alpha) \subset C. \tag{3.1.3}$$

Since $F_1(\cdot, y, \cdot)$ is lower $(-C)$-continuous, for any neighborhood U of the origin in Z, there is a subnet $\{a_\beta, z_\beta\}$ of $\{a_\alpha, z_\alpha\}$ such that

$$F_1(a^*, y, z^*) \subset F_1(a_\beta, y, z_\beta) + U + C. \tag{3.1.4}$$

From (3.1.3) and (3.1.4), we have

$$F_1(a^*, y, z^*) \subset U + C.$$

We claim that $F_1(a^*, y, z^*) \subset C$. Assume that there exists $p \in F_1(a^*, y, z^*)$ and $p \notin C$. Thus, we note that $0 \notin (C-p)$ and $C-p$ is closed. Hence $Z \setminus (C-p)$ is open and $0 \in Z \setminus (C-p)$. Since Z is a locally convex space, there exists a neighborhood U_0 of the origin such that $U_0 \subset Z \setminus (C-p)$ is convex and $U_0 = -U_0$. This implies that $0 \notin U_0 + (C-p)$, i.e., $p \notin U_0 + C$, which is a contradiction. Therefore $F_1(a^*, y, z^*) \subset C$. This mean that $a^* \in A(x,y)$ and so $A(x,y)$ is a closed subset of K. Similarly, we have $B(x,y)$ is a closed subset of K.

Step 4. Show that A and B are upper semicontinuous.

Let $\{(x_\alpha, y_\alpha) : \alpha \in I\} \subset K \times D$ be given such that $(x_\alpha, y_\alpha) \to (x,y) \in K \times D$, and let $a_\alpha \in A(x_\alpha, y_\alpha)$ such that $a_\alpha \to a$. Since $a_\alpha \in S_1(x_\alpha)$ and S_1 is upper semicontinuous, it follows by Lemma 2.4.8 (ii) that $a \in S_1(x)$. We now claim that $a \in A(x,y)$. Assume that $a \notin A(x,y)$. Then, there exists $z^* \in S_1(x)$ such that

$$F_1(a, y, z^*) \not\subseteq C,$$

which implies that there is a neighborhood U_0 of the origin in Z such that

$$F_1(a, y, z^*) + U_0 \not\subseteq C.$$

Since F_1 is upper C-continuous, for any neighborhood U of the origin in Z, there exists a neighborhood U_1 of (a, y, z^*) such that

$$F_1(\hat{a}, \hat{y}, \hat{z}) \subset F_1(a, y, z^*) + U + C \ , \quad \forall(\hat{a}, \hat{y}, \hat{z}) \in U_1.$$

Without loss of generality, we can assume that $U_0 = U$. This implies that

$$F_1(\hat{a}, \hat{y}, \hat{z}) \subset F_1(a, y, z^*) + U_0 + C \not\subseteq C + C \subset C \ , \quad \forall(\hat{a}, \hat{y}, \hat{z}) \in U_1.$$

Thus there is $\alpha_0 \in I$ such that

$$F_1(a_\alpha, y_\alpha, z_\alpha) \not\subseteq C \ , \quad \forall \alpha \geq \alpha_0,$$

which contradicts to $a_\alpha \in A(x_\alpha, y_\alpha)$. Hence $a \in A(x, y)$ and therefore A is a closed mapping. Since K is a compact set and $A(x, y)$ is a closed subset of K, we note that $A(x, y)$ is compact. Then $\overline{A(x, y)}$ is also compact. Hence, by Lemma 2.4.8 (i), A is a upper semicontinuous mapping. Similarly, we note that B is a upper semicontinuous mapping.

Step 5. Show that $SGSVQEP$ has a solution.

Define the set-valued mappings $H_a : K \times D \to 2^{K \times D}$ and $G_b : K \times D \to 2^{K \times D}$ by

$$H_a(x, y) = (A(x, y), T_1(a)) \ \forall(x, y) \in K \times D$$

and

$$G_b(x, y) = (B(x, y), T_2(b)) \ \forall(x, y) \in K \times D.$$

Then H_a and G_b are upper semicontinuous and, for all $(x, y) \in K \times D$, $H_a(x, y)$ and $G_b(x, y)$ are nonempty closed convex subsets of $K \times D$.

Define the set-valued mapping $M : (K \times D) \times (K \times D) \to 2^{(K \times D) \times (K \times D)}$ by

$$M((x, y), (u, v)) = (H_u(x, y), G_x(u, v)), \quad \forall((x, y), (u, v)) \in (K \times D) \times (K \times D).$$

Then M is also upper semicontinuous and, for all $((x, y), (u, v)) \in (K \times D) \times (K \times D)$, $M((x, y), (u, v))$ is a nonempty closed convex subset of $(K \times D) \times (K \times D)$.

By Theorem 2.5.1, there exists a point $((\bar{x},\bar{y}),(\bar{u},\bar{v})) \in (K \times D) \times (K \times D)$ such that $((\bar{x},\bar{y}),(\bar{u},\bar{v})) \in M((\bar{x},\bar{y}),(\bar{u},\bar{v}))$, that is

$$(\bar{x},\bar{y}) \in H_{\bar{u}}(\bar{x},\bar{y}) \text{ and } (\bar{u},\bar{v}) \in G_{\bar{x}}(\bar{u},\bar{v}).$$

This implies that $\bar{x} \in A(\bar{x},\bar{y})$, $\bar{y} \in T_1(\bar{u})$, $\bar{u} \in B(\bar{u},\bar{v})$ and $\bar{v} \in T_2(\bar{x})$. Then, there exists $(\bar{x},\bar{u}) \in K \times K$ and $\bar{y} \in T_1(\bar{u})$, $\bar{v} \in T_2(\bar{x})$ such that $\bar{x} \in S_1(\bar{x})$, $\bar{u} \in S_2(\bar{u})$,

$$F_1(\bar{x},\bar{y},z) \subset C, \quad \forall z \in S_1(\bar{x}) \text{ and } F_2(\bar{u},\bar{v},z) \subset C, \quad \forall z \in S_2(\bar{u}).$$

Hence $SGSVQEP$ has a solution.

Step 6. Show that the set of solutions of $SGSVQEP$ is closed.

Let $\{(x_\alpha,u_\alpha) : \alpha \in I\}$ be a net in the set of solutions of $SGSVQEP$ such that $(x_\alpha,u_\alpha) \to (x^*,u^*)$. By definition of the set of solutions of $SGSVQEP$, we note that there exist $v_\alpha \in T_1(x_\alpha), y_\alpha \in T_2(u_\alpha)$, $x_\alpha \in S_1(x_\alpha)$ and $u_\alpha \in S_2(u_\alpha)$ satisfying

$$F_1(x_\alpha,y_\alpha,z) \subset C, \quad \forall z \in S_1(x_\alpha) \text{ and } F_2(u_\alpha,v_\alpha,z) \subset C, \quad \forall z \in S_2(u_\alpha).$$

Since S_1 and S_2 are continuous closed valued mappings, we obtain $x^* \in S_1(x^*)$ and $u^* \in S_2(u^*)$. Let $v_\alpha \to v^*$ and $y_\alpha \to y^*$. Since T_1 and T_2 are upper semicontinuous closed valued mappings, it follows by Lemma 2.4.8 (ii) that T_1 and T_2 are closed. Thus, we note that $v^* \in T_1(x^*)$ and $y^* \in T_2(u^*)$. Since $F_1(\cdot,y^*,\cdot)$ and $F_2(\cdot,v^*,\cdot)$ are lower $(-C)$-continuous, we have

$$F_1(x^*,y^*,z) \subset C, \quad \forall z \in S_1(x^*) \text{ and } F_2(u^*,v^*,z) \subset C, \quad \forall z \in S_2(u^*).$$

This means that (x^*,u^*) belongs to the set of solutions of $SGSVQEP$. Hence the set of solutions of $SGSVQEP$ is a closed set. This completes the proof. □

If we take $S = S_1 = S_2$, $F = F_1 = F_2$ and $T = T_1 = T_2$. Then from Theorem 3.1.1, we derive the following result.

Corollary 3.1.2. *Let $S : K \to 2^K$ be a continuous set-valued mapping such that for any $x \in K, S(x)$ is nonempty closed convex subset of K. Let $T : K \to 2^D$*

be an upper semicontinuous set-valued mapping such that for any $x \in K$, $T(x)$ *is a nonempty closed convex subset of* D *and* $F : K \times D \times K \to 2^Z$ *be set-valued mapping satisfy the following conditions:*

(i) *for all* $(x, y) \in K \times D, F(x, y, S(x)) \subset C$;

(ii) *for all* $(y, z) \in D \times K, F(\cdot, y, z)$ *is properly* C*-quasiconvex;*

(iii) $F(\cdot, \cdot, \cdot)$ *is an upper* C*-continuous;*

(iv) *for all* $y \in D, F(\cdot, y, \cdot)$ *is a lower* $(-C)$*-continuous;*

(v) *if* $x \in S(x)$ *and* $u \in S(u)$, *then* $T(x) = T(u)$.

Then GSVQEP has a solution. Moreover, the set of all solutions of GSVQEP is closed.

Now we give an example to explain that Theorem 3.1.1 is applicable.

Example 3.1.3. *Let* $X = Y = Z = \mathbb{R}$, $C = [0, +\infty)$ *and* $K = D = [0, 1]$. *For each* $x \in K$, *let* $S_1(x) = [x, 1]$, $S_2(x) = [0, x]$ *and* $T_1(x) = [1 - x, 1]$, $T_2(x) = [x, 1]$. *We consider the set-valued mappings* $F_1, F_2 : K \times D \times K \to 2^Z$ *defined by*

$$F_1(x, y, z) = [x - y + z, +\infty) \text{ for all } (x, y, z) \in K \times D \times K,$$

$$F_2(x, y, z) = [y - x + z, +\infty) \text{ for all } (x, y, z) \in K \times D \times K.$$

Then it is easy to check that all of condition $(i)-(iv)$ *in Theorem 3.1.1 are satisfied. Hence, by Theorem 3.1.1, SGSVQEP has a solution. Let* E *be the set of all strong solutions for SGSVQEP. Then we note that*

$$\begin{aligned} E &= \{(\bar{x}, \bar{u}, \bar{y}, \bar{v}) \in K \times K \times T_2(\bar{u}) \times T_1(\bar{x}) : \bar{x} \in S_1(\bar{x}), \bar{u} \in S_2(\bar{u}) \text{ such that} \\ &\quad F_1(\bar{x}, \bar{y}, z) \subset C, \ \forall z \in S_1(\bar{x}) \text{ and } F_2(\bar{u}, \bar{v}, z) \subset C, \ \forall z \in S_2(\bar{u})\} \\ &= \bigcup_{\frac{1}{3} \leq a \leq 0.5} (\{a\} \times [1 - a, 2a] \times [0, 1 - a] \times [1 - a, 1]). \end{aligned}$$

3.2 Existence results of generalized vector quasi-equilibrium problems in locally G-convex spaces

Let X, Y and Z be real locally G-convex Hausdorff topological vector spaces, $K \subset X$ and $D \subset Y$ be nonempty compact subsets, and $C \subset Z$ be a nonempty closed convex cone. We also suppose that $F : K \times D \times K \to 2^Z$, $S : K \to 2^K$ and $T : K \to 2^D$ are set-valued mappings.

The generalized vector quasi-equilibrium problem of type (I) ($GSVQEP\ I$) is to find $x^* \in K$ *and* $y^* \in T(x^*)$ such that

$$x^* \in S(x^*) \text{ and } F(x^*, y^*, z) \subset C \quad \forall z \in S(x^*).$$

The generalized vector quasi-equilibrium problem of type (II) ($GSVQEP\ II$) is to find $x^* \in K$ *and* $y^* \in T(x^*)$ such that

$$x^* \in S(x^*) \text{ and } F(x^*, y^*, z) \not\subset C \quad \forall z \in S(x^*).$$

We denote the set of all solutions to the $GSVQEP\ I$ and $GSVQEP\ II$ by $V_s(F)$ and $V_w(F)$, respectively. We prove the existence theorems of the generalized strong vector quasi-equilibrium problems in locally G-convex spaces, by using Kakutani-Fan-Glicksberg fixed point theorem for upper semicontinuous set-valued mappings with nonempty closed acyclic values, and the closedness of $V_s(F)$ and $V_w(F)$.

Theorem 3.2.1. *Let X, Y and Z be real locally G-convex Hausdorff topological vector spaces, $K \subset X$ and $D \subset Y$ be nonempty compact subsets, and $C \subset Z$ be a nonempty closed convex cone. Let $S : K \to 2^K$ be a continuous set-valued mapping such that for any $x \in K$, the set $S(x)$ is a nonempty closed contractible subset of K. Let $T : K \to 2^D$ be an upper semicontinuous set-valued mapping with nonempty closed acyclic values and $F : K \times D \times K \to 2^Z$ be a set-valued mapping satisfying the following conditions:*

(i) for all $(x, y) \in K \times D$, $F(x, y, S(x)) \subset C$;

(ii) for all $(y, z) \in D \times K$, $F(\cdot, y, z)$ *is properly* C*-quasiconvex;*

(iii) $F(\cdot, \cdot, \cdot)$ *is upper* C*-continuous;*

(iv) for all $y \in D$, $F(\cdot, y, \cdot)$ *is lower* $(-C)$*-continuous.*

Then, the solution set $V_s(F)$ *is a nonempty and closed subset of* K.

Proof. For any $(x, y) \in K \times D$, we define a set-valued mapping $G : K \times D \to 2^K$ by

$$G(x, y) = \{u \in S(x) : F(u, y, z) \subset C, \forall z \in S(x)\}.$$

Since for any $(x, y) \in K \times D$, $S(x)$ is nonempty. So, by assumption(i), we have $G(x, y)$ is nonempty. Next, we divide the proof into five steps.

Step 1. To show that G is acyclic.

Since every contractible set is acyclic, it is enough to show that $G(x, y)$ is contractible. Let $u \in G(x, y)$, thus $u \in S(x)$ and $F(u, y, z) \subset C$ $\forall z \in S(x)$. Since $S(x)$ is contractible, there exists a continuous mapping $h : S(x) \times [0, 1] \longrightarrow S(x)$ such that $h(s, 0) = s$ $\forall s \in S(x)$ and $h(s, 1) = u$ $\forall s \in S(x)$. Now, we set $H(s, t) = tu + (1 - t)h(s, t)$ for all $(s, t) \in G(x, y) \times [0, 1]$. Then H is a continuous mapping and we see that $H(s, 0) = s$ $\forall s \in G(x, y)$ and $H(s, 1) = u$ $\forall s \in G(x, y)$. Let $(s, t) \in G(x, y) \times [0, 1]$. We claim that $H(s, t) \in G(x, y)$. In fact, if $H(s, t) \notin G(x, y)$, then there exists $z^* \in S(x)$ such that

$$F(H(s, t), y, z^*) \not\subset C.$$

Since $(\cdot, y, z^*)$ is properly C-quasiconvex, we can assume that

$$F(u, y, z^*) \subset F(tu + (1 - t)h(s, t), y, z^*) + C.$$

It follows that

$$F(u, y, z^*) \subset F(H(s,t), y, z^*) + C \not\subset C + C \subset C,$$

which contradicts to $u \in G(x,y)$. Therefore $H(s,t) \in G(x,y)$, and hence G is contractible.

Step 2. To show that $G(x,y)$ is a closed subset of K.

Let $\{a_\alpha\}$ be a net in $G(x,y)$ such that $a_\alpha \to a^*$. Then $a_\alpha \in S(x)$. Since $S(x)$ is a closed subset of K, $a^* \in S(x)$. Since S is a lower semicontinous, it follows by Lemma 2.4.8 (iii) that, for any $z^* \in S(x)$ and any net $x_\alpha \to x$, there exists a net $\{z_\alpha\}$ such that $z_\alpha \in S(x_\alpha)$ and $z_\alpha \to z^*$. This implies that

$$F(a_\alpha, y, z_\alpha) \subset C. \tag{3.2.1}$$

Since $F(\cdot, y, \cdot)$ is lower $(-C)$-continuous, we note that, for any neighborhood U of the origin in Z, there exists a subnet $\{a_\beta, z_\beta\}$ of $\{a_\alpha, z_\alpha\}$ such that

$$F(a^*, y, z^*) \subset F(a_\beta, y, z_\beta) + U + C. \tag{3.2.2}$$

From (3.2.1) and (3.2.2), we have

$$F(a^*, y, z^*) \subset U + C. \tag{3.2.3}$$

We claim that $F(a^*, y, z^*) \subset C$. Assume that there exists $p \in F(a^*, y, z^*)$ and $p \notin C$. Then we note that $0 \notin (C - p)$ and the set $C - p$ is closed. Thus, $Z \setminus (C - p)$ is open and $0 \in Z \setminus (C - p)$. Since Z is a locally G-convex space, there exists a neighborhood U_0 of the origin such that $U_0 \subset Z \setminus (C - p)$ and $U_0 = -U_0$. Thus, we note that $0 \notin U_0 + (C - p)$ and hence $p \notin U_0 + C$, which contradicts to (3.2.3). Hence $F(a^*, y, z^*) \subset C$ and therefore $a^* \in G(x,y)$. Then $G(x,y)$ is a closed subset of K.

Step 3. To show that G is upper semicontinuous.

Let $\{(x_\alpha, y_\alpha) : \alpha \in I\} \subset K \times D$ be given such that $(x_\alpha, y_\alpha) \to (x, y) \in K \times D$, and let $a_\alpha \in G(x_\alpha, y_\alpha)$ such that $a_\alpha \to a$. Since $a_\alpha \in S(x_\alpha)$ and S is upper semicontinuous, it follows by Lemma 2.4.8 (ii) that $a \in S(x)$. We claim that $a \in G(x, y)$. Assume that $a \notin G(x, y)$. Then, there exists $z^* \in S(x)$ such that

$$F(a, y, z^*) \not\subset C,$$

which implies that there is a neighborhood U_0 of the origin in Z such that

$$F(a, y, z^*) + U_0 \not\subset C.$$

Since F is upper C-continuous, it follows that, for any neighborhood U of the origin in Z, there exists a neighborhood U_1 of (a, y, z^*) such that

$$F(\hat{a}, \hat{y}, \hat{z}) \subset F(a, y, z^*) + U + C\ , \quad \forall (\hat{a}, \hat{y}, \hat{z}) \in U_1.$$

Without loss of generality, we can assume that $U_0 = U$. This implies that

$$F(\hat{a}, \hat{y}, \hat{z}) \subset F(a, y, z^*) + U_0 + C \not\subset C + C \subset C\ , \quad \forall (\hat{a}, \hat{y}, \hat{z}) \in U_1.$$

Thus there is $\alpha_0 \in I$ such that

$$F(a_\alpha, y_\alpha, z_\alpha) \not\subset C\ , \quad \forall \alpha \geq \alpha_0,$$

it is a contradiction to $a_\alpha \in G(x_\alpha, y_\alpha)$. Hence $a \in G(x, y)$ and therefore G is a closed mapping. Since K is a compact set and $G(x, y)$ is a closed subset of K, $G(x, y)$ is compact. This implies that $\overline{G(x, y)}$ is compact. Then, by Lemma 2.4.8 (i), we have G is upper semicontinuous.

Step 4. To show that the solution set $V_s(F)$ is nonempty.

Define the set-valued mapping $Q : K \times D \to 2^{K \times D}$ by

$$Q(x, y) = (G(x, y), T(x)) \ \ \forall (x, y) \in K \times D.$$

Then Q is an upper semicontinuous mapping. Moreover, we note that $Q(x, y)$ is a nonempty closed acyclic subset of $K \times D$ for all $(x, y) \in K \times D$. By Lemma 2.3.5, there exists a point $(\bar{x},\bar{y}) \in (K \times D)$ such that $(\bar{x},\bar{y}) \in Q(\bar{x},\bar{y})$. Thus we have $\bar{x} \in G(\bar{x},\bar{y})$, $\bar{y} \in T(\bar{x})$. It follows that, there exist $\bar{x} \in K$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$F(\bar{x},\bar{y},z) \subset C \quad \forall z \in S(\bar{x})$$

Hence, the solution set $V_s(F) \neq \emptyset$.

Step 5. To show that the solutions set $V_s(F)$ is closed.

Let $\{x_\alpha : \alpha \in I\}$ be a net in $V_s(F)$ such that $x_\alpha \to x^*$. By definition of the solution set $V_s(F)$, we note that $x_\alpha \in S(x_\alpha)$ and there exists $y_\alpha \in T(x_\alpha)$ satisfying

$$F(x_\alpha, y_\alpha, z) \subset C \quad \forall z \in S(x_\alpha).$$

Since S is a closed mapping, $x^* \in S(x^*)$. From the compactness of D, we can assume that $y_\alpha \to y^*$. Since T is an upper semicontinuous mapping, it follows by Lemma 2.4.8 (ii) that T is closed. Thus, we have $y^* \in T(x^*)$. Since $F(\cdot, y^*, \cdot)$ is lower $(-C)$-continuous, we have

$$F(x^*, y^*, z) \subset C \quad \forall z \in S(x^*).$$

This means that x^* belongs to $V_s(F)$. Therefore, the solution set $V_s(F)$ is closed. This completes the proof. □

Theorem 3.2.1 extends Theorem 3.1 of Long et al.[18] to locally G-convex Hausdorff topological vector spaces.

Corollary 3.2.2. *Let X, Y and Z be real locally convex Hausdorff topological vector spaces, $K \subset X$ and $D \subset Y$ be two nonempty compact convex subsets, and $C \subset Z$ be a nonempty closed convex cone. Let $S : K \to 2^K$ be a continuous set-valued mapping such that for any $x \in K$, $S(x)$ is a nonempty closed convex subset of K. Let $T : K \to 2^D$ be an upper semicontinuous set-valued mapping such that for any $x \in K$, $T(x)$ is a nonempty closed convex subset of D. Let $F : K \times D \times K \to 2^Z$ be a set-valued mapping satisfy the following conditions:*

(i) for all $(x,y) \in K \times D$, $F(x,y,S(x)) \subset C$;

(ii) for all $(y,z) \in D \times K$, $F(\cdot,y,z)$ *is properly* C*-quasiconvex;*

(iii) $F(\cdot,\cdot,\cdot)$ *is upper* C*-continuous;*

(iv) for all $y \in D$, $F(\cdot,y,\cdot)$ *is lower* $(-C)$*-continuous.*

Then, the solutions set $V_s(F)$ *is a nonempty and closed subset of* K.

Theorem 3.2.3. *Let* X, Y *and* Z *be real locally G-convex Hausdorff topological vector spaces,* $K \subset X$ *and* $D \subset Y$ *be nonempty compact subsets, and* $C \subset Z$ *be a nonempty closed convex cone. Let* $S : K \to 2^K$ *be a continuous set-valued mapping such that for any* $x \in K$, *the set* $S(x)$ *is a nonempty closed contractible subset of* K. *Let* $T : K \to 2^D$ *be an upper semicontinuous set-valued mapping with nonempty closed acyclic values and* $F : K \times D \times K \to 2^Z$ *be a set-valued mapping satisfying the following conditions:*

(i) for all $(x,y) \in K \times D$, $F(x,y,S(x)) \not\subset C$;

(ii) for all $(y,z) \in D \times K$, $F(\cdot,y,z)$ *is properly* C*-quasiconvex;*

(iii) $F(\cdot,\cdot,\cdot)$ *is upper* C*-continuous;*

(iv) for all $y \in D$, $F(\cdot,y,\cdot)$ *is lower* $(-C)$*-continuous.*

Then, the solutions set $V_w(F)$ *is a nonempty and closed subset of* K.

Proof. For any $(x,y) \in K \times D$, define a set-valued mapping $B : K \times D \to 2^K$ by

$$B(x,y) = \{u \in S(x) : F(u,y,z) \not\subset C, \forall z \in S(x)\}.$$

Proceeding as in the proof of Theorem 3.2.1, we need to prove that $B(x,y)$ is closed acyclic subset of $K \times D$ for all $(x,y) \in K \times D$. We divide the remainder of the proof into three steps.

Step 1. To show that $B(x, y)$ is a closed subset of K .

Let $\{a_\alpha\}$ be a net in $B(x,y)$ such that $a_\alpha \to a^*$. Then $a_\alpha \in S(x)$ and $F(a_\alpha, y, z) \not\subset C, \forall z \in S(x)$. Since $S(x)$ is a closed subset of K, we have $a^* \in S(x)$. By the lower semicontinuity of S and Lemma 2.4.8 (iii), we note that, for any $z \in S(x)$ and any net $x_\alpha \to x$, there exists a net $\{z_\alpha\}$ such that $z_\alpha \in S(x_\alpha)$ and $z_\alpha \to z$. Thus, we have

$$F(a_\alpha, y, z_\alpha) \not\subset C, \tag{3.2.4}$$

which implies that there exists a neighborhood U_0 of the origin in Z such that

$$F(a_\alpha, y, z_\alpha) + U_0 \not\subset C. \tag{3.2.5}$$

Since $F(\cdot, y, \cdot)$ are lower $(-C)$-continuous, it follows that, for any neighborhood U of the origin in Z, there exists a subnet $\{a_\beta, z_\beta\}$ of $\{a_\alpha, z_\alpha\}$ such that

$$F(a^*, y, z) \subset F(a_\beta, y, z_\beta) + U + C. \tag{3.2.6}$$

Without loss of generality, we can assume that $U = U_0$. Then, by (3.2.4), (3.2.5)and (3.2.6), we have

$$F(a^*, y, z) \subset F(a_\alpha, y, z_\alpha) + U_0 + C \not\subset C + C \subset C.$$

This means that $a^* \in B(x, y)$ and so $B(x, y)$ is a closed subset of K.

Step 2. To show that B is upper semicontinuous.

Let $\{(x_\alpha, y_\alpha) : \alpha \in I\} \subset K \times D$ be given such that $(x_\alpha, y_\alpha) \to (x, y) \in K \times D$, and let $a_\alpha \in B(x_\alpha, y_\alpha)$ such that $a_\alpha \to a$. Then $a_\alpha \in S(x_\alpha)$ and $F(a_\alpha, y, z) \not\subset C, \forall z \in S(x_\alpha)$. Since S is closed mapping, it follows by Lemma 2.4.8 (ii) that $a \in S(x)$. We claim that $a \in B(x, y)$. Indeed, if $a \notin B(x, y)$, then there exists a $z_0 \in S(x)$ such that

$$F(a, y, z_0) \subset C. \tag{3.2.7}$$

Since F is upper C-continuous, we note that, for any neighborhood U of the origin in Z, there exists a neighborhood U_0 of (a, y, z_0) such that

$$F(a^*, y^*, z^*) \subset F(a, y, z_0) + U + C\ ,\ \ \forall (a^*, y^*, z^*) \in U_0. \tag{3.2.8}$$

From (3.2.7) and (3.2.8), we obtain

$$F(a^*, y^*, z^*) \subset U + C\ ,\ \ \forall (a^*, y^*, z^*) \in U_0. \tag{3.2.9}$$

As in the proof of Step 2 in Theorem 3.2.1, we can show that $F(a^*, y^*, z^*) \subset C$ for all $(a^*, y^*, z^*) \in U_0$. Hence there is $\alpha_0 \in I$ such that

$$F(a_\alpha, y_\alpha, z_\alpha) \subset C\ ,\ \ \forall \alpha \geq \alpha_0,$$

it is a contradiction to $a_\alpha \in B(x_\alpha, y_\alpha)$. Hence $a \in B(x, y)$ and therefore B is a closed mapping. Since K is a compact set and $B(x, y)$ is a closed subset of K, $B(x, y)$ is compact. This implies that $\overline{B(x, y)}$ is compact. Then, by Lemma 2.4.8 (i), B is upper semicontinuous.

Step 3. To show that the solution set $V_w(F)$ is nonempty and closed.

Define the set-valued mapping $P : K \times D \to 2^{K \times D}$ by

$$P(x, y) = (B(x, y), T(x))\ \ \forall (x, y) \in K \times D.$$

Then P is an upper semicontinuous mapping. Moreover, we note that $P(x, y)$ is a nonempty closed acyclic subset of $K \times D$ for all $(x, y) \in K \times D$. Hence, by Lemma 2.3.5, there exists a point $(\bar{x}, \bar{y}) \in (K \times D)$ such that $(\bar{x}, \bar{y}) \in P(\bar{x}, \bar{y})$. Thus, we have $\bar{x} \in B(\bar{x}, \bar{y})$ and $\bar{y} \in T(\bar{x})$. This implies that there exists $\bar{x} \in K$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$F(\bar{x}, \bar{y}, z) \not\subset C \quad \forall z \in S(\bar{x})$$

Hence $V_W(F) \neq \emptyset$. Similarly, by the proof of step 5 in Theorem 3.2.1, $V_w(F)$ is closed. This completes the proof. □

Next, we discuss the stability of the solutions for the generalized strong vector quasi-equilibrium problem *GSVQEP II*. Let X, Y be Banach spaces and let Z be a real locally G-convex Hausdorff topological vector space. Let $K \subset X$ and $D \subset Y$ be nonempty compact subsets, and $C \subset Z$ be a nonempty closed convex cone. Let

$$E := \{(S,T) \mid S : K \to 2^K \text{ is a continuous set-valued mapping with nonempty closed contractible values and } T : K \to 2^D \text{ is an upper semicontinuous set-valued mapping with nonempty closed acyclic values } \}.$$

Let B_1, B_2 be compact sets in a normed space. Recall that the Hausdorff metric is defined by

$$H(B_1, B_2) := max\{\sup_{b \in B_1} d(b, B_2), \sup_{b \in B_2} d(b, B_1)\},$$

where $d(b, B_2) := \inf_{a \in B_2} \|b - a\|$.

For $(S_1, T_1), (S_2, T_2) \in E$, we define

$$\rho((S_1, T_1), (S_2, T_2)) := \sup_{x \in K} H_1(S_1(x), S_2(x)) + \sup_{x \in K} H_2(T_1(x), T_2(x)),$$

where H_1, H_2 are the appropriate Hausdorff metrics. Obviously, (E, ρ) is a metric space. Now we assume that F satisfies the assumptions of Theorem 3.2.3. Then, for each $(S,T) \in E$, *GSVQEP II* has a solution x^*. Let

$$\varphi(S,T) = \{x \in K : x \in S(x) \text{ and } \exists y \in T(x), F(x,y,z) \not\subset C \ \forall z \in S(x)\}.$$

Thus $\varphi(S,T) \neq \emptyset$, which conclude that φ defines a set-valued mapping from E into K.

We also need the following lemma in the sequel.

Lemma 3.2.4. *[15, 73] Let W be a metric space and let A, $A_n (n = 1, 2, ...)$ be compact sets in W. Suppose that for any open set $O \supset A$, there exists n_0 such* *that $A_n \supset O$ for all $n \geq n_0$. Then, any sequence $\{x_n\}$ satisfying $x_n \in A_n$ has a convergent subsequence with a limit in A.*

In the following theorem, we replace the convex set by the contractible set and acyclic set in Theorem 4.1 in [18]. All of this reason, we referred to the above Lemma 3.2.4, we could acquire the as same as result that it appeared in the following theorem. Now, we need only to present stability theorem for the solution set mapping φ for *GSVQEP II*.

Theorem 3.2.5. $\varphi : E \to 2^K$ *is an upper semicontinuous mapping with compact values.*

Proof. Since K is compact, we need only to show that φ is a closed mapping. In fact, let $((S_n, T_n), x_n) \in Graph(\varphi)$ be such that $((S_n, T_n), x_n) \to ((S, T), x^*)$. Since $x_n \in \varphi(S_n, T_n)$, we have $x_n \in S_n(x_n)$ and there exists $y_n \in T_n(x_n)$ such that

$$F(x_n, y_n, z) \not\subset C \quad \forall z \in S_n(x_n). \tag{3.2.10}$$

By the same argument as in the proof of Theorem 4.1 in [18], we can show that $x^* \in S(x^*)$ and $y^* \in T(x^*)$.

Since S is lower semicontinuous at x^* and $x_n \to x^*$, it follows by Lemma 2.4.8 (iii) that, for any $z \in S(x^*)$, there exists $z_n \in S(x_n)$ such that $z_n \to z$. To finish the proof of the theorem, we need to show that $F(x^*, y^*, z) \not\subset C$ for all $z \in S(x^*)$. Since $\rho((S_n, T_n), (S, T)) \to 0$, it follows by the same argument as in the proof of Theorem 4.1 in [18] that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in S_{n_k}(x_{n_k})$, $y_{n_k} \in T_{n_k}(x_{n_k})$, $z_{n_k} \in S_{n_k}(x_{n_k})$ and

$$F(x_{n_k}, y_{n_k}, z_{n_k}) \not\subset C.$$

From the upper C-continuous of F, we have

$$F(x^*, y^*, z) \not\subset C \qquad \forall z \in S(x^*).$$

Then $((S,T),x^*) \in Graph(\varphi)$, and so $Graph(\varphi)$ is closed. The theorem is proved. □

3.3 Existence solutions of vector equilibrium problems and fixed point of multi-valued mappings

In this section, we assume that X and Y are Hausdorff topological vector spaces, K is a nonempty convex subset of X and C is a pointed closed convex cone in Y with $intC \neq \emptyset$, $T : K \to 2^K$ is a set-valued mapping and for a given vector valued mapping $F : K \times K \to Y$ such that $F(x,x) = 0$ for each $x \in K$, let us present the fixed point problem of a multi-valued mapping together with the vector equilibrium problem, in particular, it is to find $x \in K$ such that

$$x \in T(x), \quad F(x,y) \notin -C \setminus \{0\} \text{ for all } y \in K. \tag{3.3.1}$$

This problem shows the relationship in sense of intersection between fixed points of the multi-valued mapping and the vector equilibrium problem so the set of all solutions of the problem (3.3.1) is denoted by $F(T) \cap VEP(F)$. This problem includes vector quasi-equilibrium problems (in short, $VQEP$) and vector quasi-variational inequalities (in short, $VQVI$) as special cases.

For the main purpose of this section, we provide sufficient conditions and prove the existence solutions of intersection between the set of all fixed points of the multi-valued mapping and the set of all solutions for vector equilibrium problem by using the generalization of the Fan-Browder fixed point theorem . We also study the existence solutions of intersection between the set of all fixed points of the multi-valued mapping and the set of all solutions for vector variational inequality. Consequently, our results extend the existence theorems of vector quasi-equilibrium problems and vector quasi-variational inequalities. To do this, the following lemma is necessary.

Lemma 3.3.1. *Let K be a nonempty and convex subset of X. Let $T : K \to 2^K$*

be set-valued mapping such that for any $x \in K$, $T(x)$ is a nonempty convex subset of K. Assume that $F : K \times K \to Y$ is a hemicontinuous in the first argument, C-convex in the second argument and C-strong pseudomonotone. Then the following statements are equivalent:

(i) Find $x \in K$ such that $x \in T(x)$ and $F(x,y) \notin -C \setminus \{0\} \quad \forall y \in K$

(ii) Find $x \in K$ such that $x \in T(x)$ and $F(y,x) \in -C \quad \forall y \in K$.

Proof. $(i) \to (ii)$ It is clear by the C-strong pseudomonotone.
$(ii) \to (i)$ Let $x \in K$ such that

$$x \in T(x) \text{ and } F(y,x) \in -C \text{ for all } y \in K. \tag{3.3.2}$$

For any $y \in K$ and $\alpha \in (0,1)$, we set $z_\alpha = \alpha y + (1-\alpha)x$ and so we have $z_\alpha \in K$ because K is convex. By the assumption, we conclude that

$$x \in T(x) \text{ and } F(z_\alpha, x) \in -C. \tag{3.3.3}$$

Since F is C-convex in the second argument and by (3.3.3), we get

$$\begin{aligned} 0 &= F(z_\alpha, \alpha y + (1-\alpha)x) \\ &\in \alpha F(z_\alpha, y) + (1-\alpha)F(z_\alpha, x) - C \\ &\subseteq \alpha F(z_\alpha, y) + (-C) + (-C) \\ &\subseteq \alpha F(z_\alpha, y) - C. \end{aligned}$$

This implies that $\alpha F(z_\alpha, y) \in C$ and since C is a convex cone then we have $F(z_\alpha, y) \in C$. Since F is a hemicontinuous in the first argument and $z_\alpha \to x$ as $\alpha \to 0^+$, we have $F(x,y) \in C$ for all $y \in K$. Therefore we obtain that

$$x \in T(x) \text{ and } F(x,y) \notin -C \setminus \{0\} \; \forall y \in K.$$

This completes the proof. □

Theorem 3.3.2. *Let K be a nonempty compact convex subset of X and $F : K \times K \to Y$ be a C-strong pseudomonotone, hemicontinuous in the first argument, C-convex and l.s.c. in the second argument such that $0 = F(x,x)$ for all $x \in K$. Let $T : K \to 2^K$ be a set-valued mapping such that for any $x \in K$, $T(x)$ is nonempty convex subset of K and for any $y \in K$, $T^{-1}(y)$ is open in K. Assume the set $P := \{x \in X \mid x \in T(x)\}$ is open in K and for any $x \in K$, $T(x) \cap \{y \in K \mid F(y,x) \notin -C\} \neq \emptyset$. Then $F(T) \cap VEP(F) \neq \emptyset$.*

Proof. For any $x \in K$, we define the set-valued mappings $A, B : K \to 2^K$ by

$$\begin{aligned} A(x) &= \{y \in K | \; F(y,x) \notin -C\} \\ \text{and} \quad B(x) &= \{y \in K | \; F(x,y) \in -C \setminus \{0\}\}. \end{aligned}$$

Since $F(x,x) = 0$ for all $x \in K$, we get that $A(x)$ and $B(x)$ are nonempty sets. We define the set-valued mapping $H : K \to 2^K$ by

$$H(x) = \begin{cases} B(x) & \text{if } \; x \in P \\ T(x) & \text{if } \; x \in K \setminus P. \end{cases}$$

Clearly, $H(x)$ is nonempty for each $x \in K$ and we have $H(x)$ is convex. Indeed, let $y_1,\ y_2 \in B(x)$ and $\alpha \in (0,1)$. Since F is C-convex in the second argument, we have

$$\begin{aligned} F(x, \alpha y_1 + (1-\alpha)y_2) &\in \alpha F(x,y_1) + (1-\alpha)F(x,y_2) - C \\ &\subseteq (-C \setminus \{0\}) - C \\ &= -C \setminus \{0\}. \end{aligned}$$

Then $\alpha y_1 + (1-\alpha)y_2 \in B(x)$ and hence $B(x)$ is convex. Since $T(x)$ is convex, then $H(x)$ is also convex.

By the defining of H, we see that H has no fixed point, Indeed, suppose that there is $x \in K$ such that $x \in H(x)$. It is impossible for $x \in K \setminus P$, then $x \in P$ and so $x \in B(x)$. Thus $F(x,x) \in -C \setminus \{0\}$, a contradiction with $0 = F(x,x)$. Using the contrapositive of Theorem 2.5.3, we obtain that H has no the local intersection property. Define the set-valued mapping $G : K \to 2^K$ by

$$G(x) = \begin{cases} A(x) & \text{if } x \in P \\ A(x) \cap T(x) & \text{if } x \in K \setminus P. \end{cases}$$

From the C-strong pseudimonotonicity of F, we have $G(x) \subseteq H(x)$ for any $x \in K$. Next, we will show that for each $y \in K, G^{-1}(y)$ is open in K. For any $y \in K$, we denote the complement of $A^{-1}(y)$ by $[A^{-1}(y)]^C = \{x \in K | F(y,x) \in -C\}$. Since C is closed and F is $l.s.c.$ in the second argument, we have $[A^{-1}(y)]^C$ is closed in K and so $A^{-1}(y)$ is open in K. We note that

$$\begin{aligned} G^{-1}(y) &= (A^{-1}(y) \cap P) \cup (A^{-1}(y) \cap T^{-1}(y) \cap (K \setminus P)) \\ &= [A^{-1}(y) \cup (A^{-1}(y) \cap T^{-1}(y) \cap (K \setminus P))] \cap [P \cup (A^{-1}(y) \cap T^{-1}(y) \cap (K \setminus P))] \\ &= \{A^{-1}(y) \cap [A^{-1}(y) \cup (K \setminus P)]\} \cap \{[P \cup (A^{-1}(y) \cap T^{-1}(y))] \cap K\} \\ &= A^{-1}(y) \cap [P \cup (A^{-1}(y) \cap T^{-1}(y))]. \end{aligned}$$

Since for any $y \in K$, $T^{-1}(y)$, $A^{-1}(y)$ and P are open, we have $G^{-1}(y)$ is open in K. Thus, by the contrapositive of Lemma 2.4.10, we have

$$K \not\subseteq \bigcup_{y \in K} G^{-1}(y).$$

Hence, there exists $\bar{x} \in K$ such that $\bar{x} \notin G^{-1}(y)$ for all $y \in K$. That is $G(\bar{x}) = \emptyset$. If $\bar{x} \in K \setminus P$ then $A(x) \cap T(x) = \emptyset$, which contradicts with the assumption. Therefore $\bar{X} \in P$ and $A(x) = \emptyset$. This implies that $\bar{x} \in A(\bar{x})$ and $F(y,x) \in -C$ for all $y \in K$. This completes the proof by Lemma 3.3.1. ☐

The following example guarantees the assumption that the set $T(x) \cap A(x) \neq \emptyset$ where $A(x) = \{y \in K | \ F(y,x) \notin -C\}$

Example 3.3.3. *Let* $X, Y = \mathbb{R}$, $K = [-1,1]$ *and* $C = [0, \infty)$. *For any* $x, y \in K$, *we define two mappings* $F : K \times K \to 2^Y$ *and* $T : K \to 2^K$ *by*

$$F(x,y) = x - y \quad \forall x, y \in [-1,1]$$

and

$$T(x) = \begin{cases} (-1-x, 1] & \text{if } -1 \le x \le 0 \\ (x-1, 1] & \text{if } 0 \le x \le 1. \end{cases}$$

Clearly, $T(x)$ *is a nonempty convex subset of* K *and* $T^{-1}(y)$ *is open in* K. *If* $F(x,y) \notin -C \setminus \{0\}$ $\forall x, y \in K$, *then* $x \ge y$ *and it implies that for* $x \ge y$, $F(x,y) \in -C$. *This shows that* F *is* C*-strong pesudomonotone. Let* $x, y_1, y_2 \in K$ *and* $\alpha \in [0,1]$ *and since* $0 \in K$, *we obtain that*

$$\begin{aligned} F(x, \alpha y_1 + (1-\alpha)y_2) &= x - (\alpha y_1 + (1-\alpha)y_2) \\ &= \alpha(x - y_1) + (1-\alpha)y_2 - 0 \\ &\in \alpha F(x,y_1) + (1-\alpha)F(x,y_2) - K. \end{aligned}$$

Then F *is* C*-convex in the second argument and it is easy to see that* F *is a hemicontinuous in the first argument and l.s.c. in the second argument. Note that*

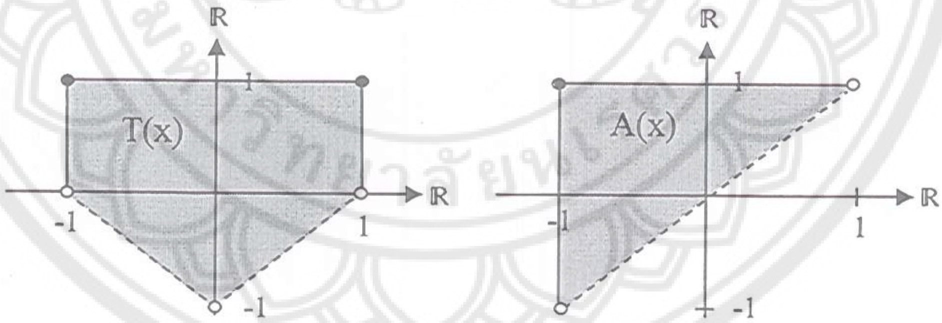

Figure 2: Example 3.3.3

$$\begin{aligned} A(x) &= \{y \in K \mid F(y,x) \notin -C\} \\ &= \{y \in [-1,1] \mid y > x\} \\ &= (x, 1] \quad \textit{where} \quad x < 1. \end{aligned}$$

If $-1 \leq x \leq 0$, then $T(x) = (-1-x, 1]$ which including $(0,1]$. Also $(0,1]$ is contained in $A(x)$ for all $-1 \leq x \leq 0$. Otherwise, $(x,1] \subseteq (x-1,1]$ for any $0 \leq x < 1$. This is to confirm that the set $T(x) \cap A(x) \neq \emptyset$ for each $x \in K$. Moreover, this example asserts that the set $P = \{x \in X \mid x \in T(x)\}$ is open in K because it equal to the set $(-0.5, 1]$ which is open in K.

Taking $T(x) \equiv K \ \ \forall x \in K$ in Theorem 3.3.2, we have the following results.

Corollary 3.3.4. *Let K be a nonempty compact convex subset of X and $F : K \times K \rightarrow Y$ be C-strong pseudomonotone, hemicontinuous in the first argument, $C-$convex and l.s.c. in the second argument such that $0 = F(x,x)$ for all $x \in K$. Then, VEP has a solution.*

If we set the vector-valued mapping $F \equiv 0$ then Theorem 3.3.2 is reduced to the following Corollary introduced by Browder (see Theorem 1, in [2]).

Corollary 3.3.5. *Let K be a nonempty compact convex subset of X. Let $T : K \rightarrow 2^K$ be a set-valued mapping such that for any $x \in K$, $T(x)$ is a nonempty convex subset of K and for any $y \in K$, $T^{-1}(y)$ is open in K. Then there exists $\bar{x}$ in K such that $\bar{x} \in T(\bar{x})$.*

If we set $Y = \mathbb{R}$ and $C = [0, \infty)$ in Theorem 3.3.2 together with Remark 2.2.30, we have the following result.

Corollary 3.3.6. *Let K be a nonempty compact convex subset of X and $F : K \times K \rightarrow \mathbb{R}$ be a monotone, hemicontinuous in the first argument, convex and l.s.c. in the second argument such that $0 = F(x,x)$ for all $x \in K$. Let $T : K \rightarrow 2^K$ be a set-valued mapping such that for any $x \in K$, $T(x)$ is nonempty convex subset of K and for any $y \in K$, $T^{-1}(y)$ is open in K. Assume the set $P := \{x \in X \mid x \in T(x)\}$ is open in K and for any $x \in K$, $T(x) \cap \{y \in K \mid F(y,x) > 0\} \neq \emptyset$. Then $F(T) \cap EP(F) \neq \emptyset$.*

Now, we let $L(X, Y)$ be a space of all linear continuous operators from X to Y. A mapping $A : K \to L(X, Y)$ is said to be *C-strong pseudomonotone* if it satisfies

$$\forall x, y \in K, \quad \langle A(x), y - x\rangle \notin -C \setminus \{0\} \Rightarrow \langle A(y), x - y\rangle \in -C$$

and it is called *hemicontinuous* if, for all $x, y \in K$ and for all $\lambda \in [0, 1]$, the mapping $\lambda \mapsto \langle T(x + \lambda(y - x), z)\rangle$ is continuous at 0^+.

As a direct consequence of Theorem 3.3.2, we obtain the following result.

Theorem 3.3.7. *Let K be a nonempty compact convex subset of X and $A : K \to L(X, Y)$ be C-strong pseudomonotone and hemicontinuous. Let $T : K \to 2^K$ be a set-valued mapping such that for any $x \in K$, $T(x)$ is a nonempty convex subset of K and for any $y \in K$, $T^{-1}(y)$ is open in K. Assume the set $P := \{x \in X \mid x \in T(x)\}$ is open in K and for any $x \in K$, $T(x) \cap \{y \in K \mid \langle A(y), x - y\rangle \notin -C\} \neq \emptyset$. Then there exists $\bar{x} \in K$ such that*

$$\bar{x} \in T(\bar{x}) \quad \text{and} \quad \langle A(\bar{x}), y - \bar{x}\rangle \notin -C\setminus\{0\} \;\; \forall y \in K.$$

Proof. We define the vector value mapping $F : K \times K \to Y$ by

$$F(x, y) = \langle Ax, y - x\rangle .$$

We will show that F satisfies all conditions in Theorem 3.3.2. Clearly $F(x, x) = 0$ and by the assumptions of A, we have F is C-strong pseudomonotone and hemicontinuous in the first argument. Let $x \in K$ by fixed. For any $y, z \in K$ and $\theta \in [0, 1]$, we obtain that

$$\begin{aligned} F(x, \theta y + (1 - \theta)z) &= \langle Ax, (\theta y + (1 - \theta)z) - x\rangle \\ &= \theta \langle Ax, y - x\rangle + (1 - \theta) \langle Ax, z - x\rangle \\ &\in \theta \langle Ax, y - x\rangle + (1 - \theta) \rangle Ax, z - x\rangle - C \\ &= \theta F(x, y) + (1 - \theta)F(x, z) - C. \end{aligned}$$

Then F is C- convex in the second argument.

Next, we will show that F is $l.s.c.$ in the second argument. Let $x \in K$ be fixed. Let $y_0 \in K$ and N be a neighborhood of $F(x, y_0)$. Since the linear operator Ax is continuous, there exists an open neighborhood M of y_0 such that $\forall y \in M, \langle Ax, y - x\rangle \in N$ because N is a neighborhood of $\langle Ax, y_0 - x\rangle$. Thus for all $y \in M, F(x, y) \in N$. Hence F is continuous in the second argument and so it is $l.s.c.$ in the second argument. Then all hypotheses of the Theorem 3.3.2 hold and hence there exists $\bar{x} \in K$ such that

$$\bar{x} \in T(\bar{x}) \text{ and } F(\bar{x}, y) = \langle A\bar{x}, y - x\rangle \notin -C \setminus \{0\} \ \ \forall y \in K.$$

This completes the proof. □

If we take $T(x) \equiv K \ \ \forall x \in K$ in previous Theorem 3.3.7, we have the following corollary.

Corollary 3.3.8. *Let K be a nonempty compact convex subset of X and $A : K \to L(X, Y)$ be C-strong pseudomonotone and hemicontinuous. Then, VVI has a solution.*

If we set $Y = \mathbb{R}$ and $C = [0, \infty)$ in Theorem 3.3.7, we have the following result.

Corollary 3.3.9. *Let K be a nonempty compact convex subset of X and $A : K \to L(X, \mathbb{R})$ be monotone and hemicontinuous in the first argument. Let $T : K \to 2^K$ be a set-valued mapping such that for any $x \in K$, $T(x)$ is a nonempty convex subset of K and for any $y \in K$, $T^{-1}(y)$ is open in K. Assume the set $P := \{x \in X \mid x \in T(x)\}$ is open in K and for any $x \in K$, $T(x) \cap \{y \in K \mid \langle A(y), x - y\rangle > 0\} \neq \emptyset$. Then $F(T) \cap VI(K, A) \neq \emptyset$.*

Remark 3.3.10. (1) Theorem 3.3.2 and Theorem 3.3.7 are the extensions of vector quasi-equilibrium problems and vector quasi-variational inequalities, respectively.

(2) If X is a real Banach space, then the Corollary 3.3.4 come to be Theorem 2.3 in [49].

3.4 The existence results for the new type of generalized strong vector quasi-equilibrium problems

In this section, we assume that X and Y are Hausdorff topological vector spaces, K is a nonempty convex subset of X and C is a pointed closed convex cone in Y with $\text{int}C \neq \varnothing$. For a given multi-valued bi-operator $F : K \times K \to 2^Y$ such that $\{0\} \subseteq F(x,x)$ for each $x \in K$, where 2^Y denotes the family of subsets of Y, the new type of generalized strong vector quasi-equilibrium problem (for short, $GSVQEP$) is the problem to find $x \in K$ such that

$$x \in A(x), \quad F(x,y) \nsubseteq -C \setminus \{0\} \text{ for all } y \in A(x) \tag{3.4.1}$$

where $A : K \to 2^K$ is a multi-valued map with nonempty values. If we set $F(x,y) = \langle Tx, \eta(y-x)\rangle \ \ \forall x, y \in K$ then the $GSVQEP$ reduces to the following generalized quasi-variational like inequality problem (for short, $GQVLIP$) which is the problem to find $x \in K$ such that

$$x \in A(x), \ \langle Tx, \eta(y-x)\rangle \nsubseteq -C \setminus \{0\} \text{ for all } y \in A(x), \tag{3.4.2}$$

where $T : K \to 2^{L(X,Y)}$ is a multi-valued mapping, $\eta : K \times K \to X$ is a nonlinear mapping and $L(X,Y)$ is denoted by the space of all continuous linear operators for X to Y.

We shall investigate the existence results for $GSVQEP$ and $GQVLIP$ with monotonicity and without monotonicity. First, we present the following lemma which is the Minty's type for $GSVQEP$.

Lemma 3.4.1. *Let K be a nonempty and convex subset of X, let $A : K \to 2^K$ be set-valued mapping such that for any $x \in K$, $A(x)$ is a nonempty convex subset of K and let $F : K \times K \to 2^y$ be g.h.c in the first argument, C-convex in the second argument and C-strongly pseudomonotone. Then the following problems are equivalent:*

(i) Find $x \in K$ such that $x \in A(x)$, $F(x,y) \nsubseteq -C \setminus \{0\}$, $\forall y \in A(x)$.

(ii) Find $x \in K$ such that $x \in A(x),\ F(y,x) \subseteq -C,\ \forall y \in A(x)$.

Proof. $(i) \rightarrow (ii)$ It is clear by the C-strong pseudomonotonicity.
$(ii) \rightarrow (i)$ Let $x \in K$. For any $y \in A(x)$ and $\theta \in (0,1)$, we set $z_\theta = \theta y + (1-\theta)X$ By the assumption (ii) and the convexity of A(x), we conclude that

$$x \in A(x),\ F(z_\theta, x) \subseteq -C.$$

Since F is C-convex in the second argument, we have

$$\begin{aligned} 0 &\in F(z_\theta, z_\theta) \\ &\subseteq \theta F(z_\theta, y) + (1-\theta)F(z_\theta, x) - C \\ &\subseteq \theta F(z_\theta, y) - C \end{aligned}$$

Then, we have $F(z_\theta, y) \cap C \neq \varnothing$, because C is a convex cone. Since F is g.h.c in the first argument, we have $x \in A(x),\ F(x,y) \cap C \neq \varnothing,\ \forall y \in A(x)$. It implies that $x \in A(x),\ F(x,y) \nsubseteq -C \setminus \{0\},\ \forall y \in A(x)$. This completes the proof. □

In the following theorem, we present the existence result for $GSVQEP$ by assuming the monotonicity of the function.

Theorem 3.4.2. *Let K be a nonempty compact convex subset of X. Let $A : K \rightarrow 2^K$ be a set valued mapping such that for any $x \in K,\ A(x)$ is a nonempty convex subset of K and for each $y \in K,\ A^{-1}(y)$ is open in K. Let the set $P := \{x \in X |\ x \in A(x)\}$ be closed. Assume that $F : K \times K \rightarrow 2^Y$ is C-strongly pseudomonotone, g.h.c. in the first argument, C-convex and l.s.c in the second argument. Then $GSVQEP$ is solvable.*

Proof. For any $x \in K$, we define the set-valued mappings $S,\ T : K \rightarrow 2^K$ by

$$\begin{aligned} S(x) &= \{y \in K | F(y,x) \nsubseteq -C\}, \\ T(x) &= \{y \in K | F(x,y) \subseteq -C \setminus \{0\}\}, \end{aligned}$$

and for any $y \in K$, we denoted the complement of $S^{-1}(y)$ by $(S^{-1}(y))^c = \{x \in K|\ F(y,x) \subseteq -C\}$. For each $x \in K$, we define multi-valued maps $G,\ H : K \to 2^K$ by

$$G(x) = \begin{cases} S(x) \cap A(x) & \text{if x} \in \text{P} \\ A(x) & \text{if } x \in K \setminus \text{P} \end{cases}$$

and

$$H(x) = \begin{cases} T(x) \cap A(x) & \text{if x} \in \text{P} \\ A(x) & \text{if } x \in K \setminus \text{P} \end{cases}$$

Clearly, $G(x)$ and $H(x)$ are nonempty for all $x \in K$, and by the C-strong pseudomonotonicity of F, we have $G(x) \subseteq H(x)$ for all $x \in K$. We claim that $H(x)$ is convex. Let $y_1, y_2 \in T(x)$ and $\theta \in (0,1)$. Since F is C-convex in the second argument, we have

$$\begin{aligned} F(x, \theta y_1 + (1-\theta) y_2) &\subseteq \theta F(x, y_1) + (1-\theta) F(x, y_2) - C \\ &\subseteq (-C \setminus \{0\}) - C \\ &\subseteq -C \setminus \{0\}. \end{aligned}$$

Then we have $T(x)$ is convex and so $H(x)$ is convex by convexity of $A(x)$. Next, we will show that $G^{-1}(y)$ is open in K for each $y \in K$. Since F is l.s.c. in the second argument and by the definition of $(S^{-1}(y))^c$, we have $(S^{-1}(y))^c$ closed and so $S^{-1}(y)$ is open in K. By assumption, we obtain that

$$G^{-1}(y) = (S^{-1}(y) \cap A^{-1}(y)) \cup (A^{-1}(y) \cap K \setminus P)$$

is open in K. It is easy to see that the mapping H has no fixed point because $0 \in F(x,x), \forall x \in K$ From the contrapositive of Generalization of the Fan-Browder fixed point Theorem and Lemma (2.4.10), we have

$$K \not\subseteq \bigcup_{y \in K} G^{-1}(y).$$

Hence, there exist $\bar{x} \in K$ such that $G(\bar{x}) = \varnothing$. If $\bar{x} \in K \setminus P$, we have $A(\bar{x}) = \varnothing$, which contradicts with the assumptions. Then $\bar{x} \in P$ and hence $S(\bar{x}) \cap A(\bar{x}) = \varnothing$. This means that $\bar{x} \in A(\bar{x})$ and $F(y, \bar{x}) \subseteq -C\ \ \forall y \in A(\bar{x})$. This completes the proof by Lemma 3.4.1 □

The following example shows that $GSVQEP$ has a solution under the condition of Theorem 3.4.2

Example 3.4.3. *Let $Y = \mathbb{R}$, $C = [0,\infty)$ and $K = [-1,1]$. Define the mappings $A : K \to 2^K$ and $F : K \times K \to 2^Y$ by*

$$A(x) = \begin{cases} [-0.5, x+0.5) & if \quad -1 \le x < 0 \\ (-0.5, 0.5) & if \quad x = 0 \\ (x-0.5, 0.5] & if \quad 0 < x \le 1 \end{cases}$$

and

$$F(x,y) = \begin{cases} [0, y-x] & if \quad x \le y \\ [y-x, 0] & if \quad x > y, \end{cases}$$

respectively. By the definition of A, we have the set $P = \{x \in X \mid x \in A(x)\} = [-0.5, 0.5]$ which is closed and for each $y \in K, A^{-1}(y)$ is open in K

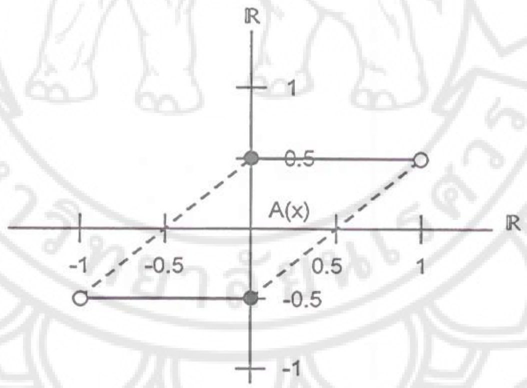

Figure 3: Example 3.4.3

We see that F is C-strongly pesudomonotone. Indeed, if $F(x,y) \not\subseteq C\backslash\{0\}$ then we only consider in the case $x < y$, so $F(x,y) = [0, y-x]$. That is

$$F(y,x) = [x-y, 0] \subseteq -C \quad for\ all\ x < y.$$

Let $x, y, z \in K$ and $\lambda \in [0,1]$. If $x < \lambda y + (1-\lambda)z$, then

$$\begin{aligned} F(x, \lambda y + (1-\lambda)z) &= [0, \lambda y + (1-\lambda)z - x] \\ &= [0, \lambda(y-x) + (1-\lambda)(z-x)] \\ &\subseteq [0, \lambda(y-x) + (1-\lambda)(z-x)] - C \\ &= \lambda[0, y-x] + (1-\lambda)[0, z-x] - C \\ &= \lambda F(x,y) + (1-\lambda)F(x,y) - C. \end{aligned}$$

Similarly in another case, we have F is C-convex in the second argument. Clearly, F is g.h.c. in the first argument and l.s.c. in the second argument.

Moreover, this example asserts that -0.5 is one of the solutions because if x = -0.5 then $A(x) = [-0.5, 0)$. Note that for all $y \in A(x)$, $y \geq x$. Therefore $F(-0.5, y) - [0, y+0.5] \not\subseteq C\backslash\{0\}$ $\forall y \in [-0.5, 0)$.

When F is not necessarily monotonicity, we have the following result.

Theorem 3.4.4. *Let K be a nonempty compact convex subset of X, let $A : K \to 2^K$ be a set-valued mapping such that for each $x \in K$, $A(x)$ is a nonempty convex subset of K and let the set $P := \{x \in X | \; x \in A(x)\}$ be closed. Assume that $F : K \times K \to 2^y$ is C-convex in the second argument and for each $y \in K$, the set $\{x \in K \mid F(x,y) \subseteq -C \setminus \{0\}\}$ is open. Then GSVQEP has a solution.*

Proof. We proceed by the contrary statements, that is, for each $x \in X$, $x \notin A(x)$ or there exists $y \in A(x)$ such that

$$F(x,y) \subseteq -C \setminus \{0\} \qquad (3.4.3)$$

For every $y \in K$, we define the set N_y as follows:

$$N_y = \{x \in K : F(x,y) \subseteq -C \setminus \{0\}\},$$

and define

$$M_y := N_y \cup P^C;$$

By the assumption, we have the set M_y is open in K and we see that $\{M_y\}_{y\in K}$ is an open cover of K. Since K is compact, there exists a finite subcover $\{M_{y_i}\}_{i=1}^n$ such that $K = \bigcup_{i=1}^{n} M_{y_i}$. By a partition of unity, there exists a family $\{\beta_i\}_{i=1}^n$ of real valued continuous functions subordinate to $\{M_{y_i}\}_{i=1}^n$ such that for all $x \in K$, $0 \le \beta_i(x) \le 1$ and $\sum_{i=1}^{n} \beta_i(x) = 1$ and for each $x \notin M_{y_i}$, $\beta_i(x) = 0$. Let $C := co\{y_1, y_2, ..., y_n\} \subseteq K$. Then C is a simplex of a finite dimensional space. Define a mapping $S : C \to C$ by

$$S(x) = \sum_{i=1}^{n} \beta_i(x) y_i \quad \forall x \in C. \tag{3.4.4}$$

Hence, we have S is continuous since β_i is continuous for each i. From Brouwer's fixed point theorem, there exists $x_0 \in C$ such that $x_0 = S(x_0)$. We define a set-valued mapping $T : K \to 2^Y$ by

$$T(x) = F(x, S(x)) \quad \textit{for all } x \in K. \tag{3.4.5}$$

Now, we note that for any $x \in K$, $\{i|\ x \in M_{y_i}\} \neq \varnothing$. Since F is C-convex in the second argument, it follows from (3.4.3), (3.4.4) and (3.4.5), we have

$$\begin{aligned} T(x) &= F(x, \sum_{i=1}^{n} \beta_i(x) y_i) \\ &\subseteq \sum_{i=1}^{n} \beta_i(x) F(x, y_i) - C \\ &\subseteq -C \setminus \{0\} - C \\ &= -C \setminus \{0\}, \end{aligned}$$

for all $x \in K$. Since $x_0 \in K$ and it is a fixed point of S, $0 = F(x, x) = F(x, S(x)) = T(x) \subseteq -C \setminus \{0\}$, which is a contradiction. This completes the proof. □

If we set $A \equiv I$, then Theorem 3.4.2 and Theorem 3.4.4 are reduced to Theorem 1 and Theorem 3 in Kom and Wong [74], respectively and Theorem 3.4.2 is also a multi-valued version of Theorem 2.3 in Kazmi and Khan [49].

Let $F(x,y) = \langle Tx, \eta(y,x)\rangle$ for all $x, y \in K$, where $\eta : K \times K \to X$ and $T : K \to 2^{L(X,Y)}$. As a consequence of Theorem 3.4.2 and using the same argument in Kum and Wang ([74],Theorem 2), we have the following existence result for $GQVLIP$.

Corollary 3.4.5. *Let K be a nonempty compact convex subset of X, Let $A : K \to 2^K$ be a set-valued mapping such that for any $x \in K, A(x)$ is a nonempty convex subset of K and for each $y \in K, A^{-1}(y)$ is open in K. Let the set $P := \{x \in X|\ x \in A(x)\}$ be closed, let $\eta : K \times K \to X$ be affine and continuous in first argument and hemicontinuous in second argument and let $T : K \to 2^{L(X,Y)}$ be a C-strong pseudomonotone and g.h.c. with nonempty compact values where $L(X,Y)$ is equipped with topology of bounded convergence. Then $GQVLIP$ has a solution.*

As a consequence of Theorem 3.4.4, we obtain the following existence result for $GQVLIP$

Corollary 3.4.6. *Let K be a nonempty compact convex subset of X. Let $A : K \to 2^K$ be a set-valued mapping such that for each $x \in X$, $A(x)$ is a nonempty convex subset of K and let the set $P := \{x \in X | x \in A(x)\}$ be closed. Assume that $\eta : K \times K \to X$ is affine in the first argument and $T : K \to 2^{L(X,Y)}$ is a nonlinear mapping such that, for every $y \in K$, the set $\{x \in K | \langle T(x), \eta(y,x)\rangle \subseteq -C \setminus \{0\}\}$ is open. Then $GQVLIP$ has a solution.*