CHAPTER IV

VECTORIAL VERSION OF EKELAND VARIATIONAL PRINCIPLES AND SADDLE POINT PROBLEMS

4.1 Vectorial form of Ekeland-type variational principle

In this section, we will present the vectorial form of equilibrium version of vector Ekeland's principle in the setting of complete metric spaces and ω-distances.

Theorem 4.1.1. *Let X be a complete metric space, $\omega : X \times X \to [0, \infty)$ be a ω-distance on X, Y be a locally convex space, C be a closed and convex cone in Y and D be a closed convex and bounded subset of C such that $0 \notin cl(D + C)$. Let $F : X \times X \to Y$ be a function satisfying the following conditions:*

(i) $F(x, x) = 0$ for all $x \in X$;

(ii) $F(x, y) + F(y, z) \in F(x, z) + C$ for every $x, y, z \in X$;

(iii) for each $x \in X$ the function $F(x, \cdot) : X \mapsto Y$ is (D,C)-lower semicontinuous;

(iv) for each fixed $x \in X$, $F(x, \cdot) : X \to Y$ is C-bounded below.

Then for every $x_0 \in X$ there exists $\overline{x} \in X$ such that

(i) $F(x_0, \overline{x}) + w(x_0, \overline{x})D \subseteq -C$;

(ii) $F(\overline{x}, x) + w(\overline{x}, x)D \nsubseteq -C$ for all $x \neq \overline{x}$.

Proof. Let $r \subset X \times X$ be a relation defined as following: for any $x, y \in X$

$$x \; r \; y \;\; \Leftrightarrow \;\; F(x, y) + \omega(x, y)D \subseteq -C.$$

We will first show that r is transitive. Suppose that $u_1 \; r \; u_2$ and $u_2 \; r \; u_3$. Thus, we have

$$\begin{aligned} & F(u_1,u_2)+\omega(u_1,u_2)D \subseteq -C \\ \textit{and} \quad & F(u_2,u_3)+\omega(u_2,u_3)D \subseteq -C. \end{aligned}$$

This implies that

$$F(u_1,u_2)+F(u_2,u_3)+\omega(u_1,u_2)D+\omega(u_2,u_3)D \subseteq -C. \tag{4.1.1}$$

By the assumption (ii), we obtain

$$F(u_1,u_3) \in F(u_1,u_2)+F(u_2,u_3)-C. \tag{4.1.2}$$

Therefore, by the convexity of D, we have

$$(\omega(u_1,u_2)+\omega(u_2,u_3))D = \omega(u_1,u_2)D+\omega(u_2,u_3)D.$$

Indeed, if $\omega(u_1,u_2)+\omega(u_2,u_3)=0$, we are done. If $\omega(u_1,u_2)+\omega(u_2,u_3)>0$, for $d_1,d_2 \in D$, we have

$$\frac{\omega(u_1,u_2)}{\omega(u_1,u_2)+\omega(u_2,u_3)}d_1+\frac{\omega(u_2,u_3)}{\omega(u_1,u_2)+\omega(u_2,u_3)}d_2 \in D.$$

So, we have

$$\omega(u_1,u_2)d_1+\omega(u_2,u_3)d_2 \in (\omega(u_1,u_2)+\omega(u_2,u_3))D.$$

Hence $\omega(u_1,u_2)D+\omega(u_2,u_3)D \subseteq (\omega(u_1,u_2)+\omega(u_2,u_3))D$, which implies that $(\omega(u_1,u_2)+\omega(u_2,u_3))D = \omega(u_1,u_2)D+\omega(u_2,u_3)D$.

By the definition of ω-distance, $\omega(u_1,u_3) \leq \omega(u_1,u_2)+\omega(u_2,u_3)$. Therefore, there is a real number $\varepsilon > 0$ such that

$$\begin{aligned} \omega(u_1,u_3)D &= \omega(u_1,u_2)D+\omega(u_2,u_3)D-\varepsilon D \\ &\subseteq \omega(u_1,u_2)D+\omega(u_2,u_3)D-C. \end{aligned} \tag{4.1.3}$$

From (4.1.1), (4.1.2) and (4.1.3), we have

$$F(u_1,u_3)+\omega(u_1,u_3)D \subseteq -C.$$

This implies that $u_1 \; r \; u_3$.

We define $S : X \to 2^X$ by

$$S(x) = \{y \in X : F(x, y) + \omega(x, y)D \subseteq -C\} \; for \; all \; x \in X.$$

It is easy to see that $x \in S(x)$, and so $S(x)$ is nonempty for all $x \in X$. By assumption (iii), we note that $S(x)$ is closed set for all $x \in X$. We now show that $S(x)$ is a countably orderable set by a relation $r \subset X \times X$.

Let

$$V(x) := \inf_{y \in S(x)} \xi_{(D,C)}(F(x, y))$$

where $\xi_{(D,C)}(z) := \inf\{r \in \mathbb{R} : z \in rD - C\}$ for all $z \in Y$.

Let W be any nonempty subset of A ordered by a relation s satisfying

$$u \; s \; v \;\Rightarrow\; u \; r^* \; v \quad \text{for every} \;\; u, v \in W, \; u \neq v.$$

Then, for any $u, v \in W$ with $u \neq v$, we note that

$$u \; s \; v \Rightarrow u \; r^* \; v \Rightarrow u \; r \; v \;\; and \;\; S(u) \subseteq S(v)$$

because r is transitive. Since $u \; r \; v$, $u \neq v$, thus $F(u, v) + \omega(u, v)D \subseteq -C$, which implies that $\xi_{(D,C)}(F(u, v)) \leq -\omega(u, v) < 0$. Moreover,

$$\begin{aligned}
V(u) &= \inf_{y \in S(u)} \xi_{(D,C)}(F(u, y)) \\
&\leq \inf_{y \in S(v)} \xi_{(D,C)}(F(u, y)) \\
&\leq \inf_{y \in S(v)} (\xi_{(D,C)}(F(u, v)) + \xi_{(D,C)}(F(v, y))) \\
&= \xi_{(D,C)}(F(u, v)) + \inf_{y \in S(v)} \xi_{(D,C)}(F(v, y)) \\
&< \inf_{y \in S(v)} \xi_{(D,C)}(F(v, y)) \\
&= V(v)
\end{aligned}$$

Thus $V(W) \subset \mathbb{R}$ is well ordered by the relation " $<$ " and hence $V(W)$ is at most countable. Since V is one-to-one mapping on W, W is at most countable.

For any $x \in X$, we let $(y_n) \subset S(x)$ with $y_n \; r \; y_{n+1}$ for all $n \in \mathbb{N}$. We next show that there is an element y_0 such that $y_n \; r \; y_0$ for all $n \in \mathbb{N}$.

In case $y_m = y_{m+1} = y_{m+2} = ...$ for some $m \in \mathbb{N}$, we can put $y_0 := y_m$ and so we have done. Then, it is enough to consider in case $\sum_{i=1}^{\infty} \omega(y_i, y_{i+1}) > 0$. Since $y_n \; r \; y_{n+1}$ for each $n \in \mathbb{N}$, we obtain

$$F(y_n, y_{n+1}) + \omega(y_n, y_{n+1})D \subseteq -C. \tag{4.1.4}$$

From (4.1.4) and the assumption (ii), we observe that

$$\begin{aligned} F(y_1, y_2) &\in F(y_1, y_{m+1}) - F(y_2, y_3) - F(y_3, y_4) - ... - F(y_m, y_{m+1}) + C \\ &\subseteq F(y_1, y_{m+1}) + (\omega(y_2, y_3)D + C) + (\omega(y_3, y_4)D + C) + ... \\ &\quad +(\omega(y_m, y_{m+1})D + C) + C \\ &\subseteq F(y_1, y_{m+1}) + \textstyle\sum_{i=2}^{m}(\omega(y_i, y_{i+1})D) + C, \end{aligned}$$

for all $m \in \mathbb{N}$. Since F is C-bounded below, there exists $z \in Y$ such that

$$F(y_1, y_2) \in z + C + \sum_{i=2}^{m}(\omega(y_i, y_{i+1})D). \tag{4.1.5}$$

By the convexity of D, we have

$$\sum_{i=1}^{m}(\omega(y_i, y_{i+1})D) = (\sum_{i=1}^{m}\omega(y_i, y_{i+1}))D \tag{4.1.6}$$

for any $m \in \mathbb{N}$. Therefore, it follows from (4.1.5) and (4.1.6) that

$$F(y_1, y_2) \in z + C + (\sum_{i=2}^{m}\omega(y_i, y_{i+1}))D.$$

Since $0 \notin cl(D + C)$, by the Separation Theorem, there exists $f^* \in Y^*$ such that

$$\langle f^*, 0\rangle < inf\{\langle f^*, d + c\rangle, \forall d \in D, \forall c \in C\}.$$

This implies that $0 < \varepsilon < \langle f^*, d+c\rangle = \langle f^*, d\rangle + \langle f^*, c\rangle$ for some $\varepsilon > 0$, and for any $d \in D$, $c \in C$. Hence $\inf_{d \in D}\langle f^*, d\rangle > 0$ and $\langle f^*, c\rangle \geq 0$ for any $c \in C$. Hence, for each $m \in \mathbb{N}$, we have

$$\langle f^*, F(y_1, y_2)\rangle = \langle f^*, z\rangle + \langle f^*, c\rangle + \sum_{i=2}^{m}\omega(y_i, y_{i+1})\langle f^*, d\rangle$$

for some $c \in C$ and $d \in D$. Since $\langle f^*, c\rangle \geq 0$ for any $c \in C$, it follows that

$$\langle f^*, F(y_1, y_2)\rangle \geq \langle f^*, z\rangle + \sum_{i=2}^{m}\omega(y_i, y_{i+1}) \inf_{d \in D}\langle f^*, d\rangle.$$

Since $\inf_{d\in D}\langle f^*, d\rangle > 0$, we have that $\sum_{i=1}^{m}\omega(y_i, y_{i+1})$ is bounded above by $\frac{\langle f^*, F(y_1,y_2)\rangle - \langle f^*, z\rangle}{\inf_{d\in D}\langle f^*, d\rangle}$. Moreover, $(\sum_{i=1}^{m}\omega(y_i, y_{i+1}))$ is a monotone sequence then the series $\sum_{i=1}^{\infty}\omega(y_i, y_{i+1})$ converges. This implies that $\lim_{i\to\infty}\omega(y_i, y_{i+1}) = 0$. It is easy to see that (y_n) is a Cauchy sequence in $S(x)$. By the completeness of X and closedness of $S(x)$, (y_n) converges to a certain $y_0 \in S(x)$. Since r is transitive and $y_n\ r\ y_{n+1}$, then $y_n\ r\ y_m$ for all $m > n$, and so $y_n\ r\ y_0$. This entails that $S(x)$ satisfied the condition in Theorem 2.2.23. Now, all of the proof consists in applying Theorem 2.2.23 to show that $S(x)$ has an r-maximal element $\bar{x} \in S(x)$. Let us observe that for $x \in X$, any r-maximal element of $S(x)$ is an r-maximal element of X. Hence, (i) holds for $\bar{x}$. Finally, we show that $\bar{x}$ satisfies (ii). Assume that $\bar{x}\ r\ z$ for some $z \neq \bar{x}$. Since r is transitive and $\bar{x}$ is r-maximal, $z\ r\ \bar{x}$. Consequently, $V(\bar{x}) > V(z)$ and $V(z) > V(\bar{x})$, a contradiction. Hence $\bar{x}$ satisfies (ii). □

Remark 4.1.2. We see in the proof that we do not use the symmetry condition of the metric. So, the conclusion in Theorem 4.1.1 still holds if we replace the word "metric space" by "quasi-metric space".

By setting $D = \{\varepsilon e\}$ for all $\varepsilon > 0$ in Theorem 4.1.1, we obtain the following Corollary which is proven by Ansari [75].

Corollary 4.1.3. [[75], *Theorem* 3.1] *Let (X, d) be a complete quasi-metric space, $\omega : X \times X \to [0, \infty)$ be a ω-distance on X, Y be a locally convex Hausdorff topological vector space, C be a proper, closed and convex cone in Y with apex at origin and $intC \neq \emptyset$, and $e \in Y$ be a fixed vector such that $e \in intC$. Let $F : X \times X \to Y$ be a function satisfying the following:*

(i) $F(x, x) = 0$, for all $x \in X$;

(ii) $F(x, y) + F(y, z) \in F(x, z) + C$ for all $x, y, z \in X$;

(ii) for each fixed $x \in X$, the function $F(x, \cdot) : X \to Y$ is (e, C)-lower semicontinuous and C-bounded below.

Then for every $\varepsilon > 0$ and for every $x_0 \in X$, there exists $\bar{x} \in X$ such that *(a)* $F(x_0, \bar{x}) + \varepsilon\omega(x_0, \bar{x})e \in -C$

(b) $F(\bar{x}, x) + \varepsilon\omega(\bar{x}, x)e \notin -C$*, for all* $x \in X, x \neq \bar{x}$.

If $F(x, y) = f(y) - f(x)$, where $f : X \to R$ is lower semicontinuous and bounded below, then we have the following result.

Corollary 4.1.4. *Let* X *be a complete metric space,* $\omega : X \times X \to [0, \infty)$ *be a* ω*-distance on* X*,* Y *be a locally convex space,* C *be a closed and convex cone in* Y *and* D *be a closed convex and bounded subset of* C *such that* $0 \notin cl(D + C)$*. Let* $f : X \to Y$ *be (D,C)-lower semicontinuous and C-bounded below. Then for every* $x_0 \in X$ *there exists* $\overline{x} \in X$ *such that*

(i) $f(\overline{x}) + w(x_0, \overline{x})D \subseteq f(x_0) - C$;

(ii) $f(x) + w(\overline{x}, x)D \nsubseteq f(\overline{x}) - C$ *for all* $x \neq \overline{x}$.

We obtain that Corollary 4.1.4 is extension of the following

Corollary 4.1.5. *Let* X *be a complete metric space,* $\omega : X \times X \to [0, \infty)$ *be a* ω*-distance on* X*,* Y *be a locally convex space,* C *be a closed and convex cone in* Y *and* D *be a closed convex and bounded subset of* C *such that* $0 \notin cl(D + C)$*. Let* $f : X \to Y$ *be (D,C)-lower semicontinuous and C-bounded below. Then for every* $x_0 \in X$ *there exists* $\overline{x} \in X$ *such that*

(i) $(f(x_0) - C) \cap (f(\overline{x}) + w(x_0, \overline{x})D) \neq \emptyset$;

(ii) $(f(\overline{x}) - C) \cap (f(x) + w(\overline{x}, x)D) = \emptyset$ *for all* $x \neq \overline{x}$.

Proof. By all conditions of Corollary 4.1.4, we have for every $x_0 \in X$ there exists $\overline{x} \in X$ such that

$$f(\overline{x}) + w(x_0, \overline{x})D \subseteq f(x_0) - C \tag{4.1.7}$$

$$f(x) + w(\overline{x}, x)D \nsubseteq f(\overline{x}) - C \quad for \quad all \quad x \neq \overline{x}. \tag{4.1.8}$$

From (4.1.7), we have (i) holds.

If (ii) was not satisfied, we would have $(f(\overline{x})-C)\cap(f(x)+w(\overline{x},x)D)\neq\emptyset$ for some $x\neq\overline{x}$. Then there are $c_1\in C$ and $d_1\in D$ such that

$$f(\overline{x}) = f(x) + w(\overline{x},x)d_1 + c_1. \tag{4.1.9}$$

Since $0\notin cl(D+C)$, by the Separation Theorem, there exists $y^*\in Y^*$ such that $0<\varepsilon<\langle y^*, d+c\rangle = \langle y^*, d\rangle + \langle y^*, c\rangle$ for some $\varepsilon>0$, $d\in D$ and $c\in C$. Hence $\inf_{d\in D}\langle y^*, d\rangle > 0$ and $\langle y^*, c\rangle \geq 0$ for any $c\in C$.
From (4.1.9), we obtain that

$$\langle y^*, f(\overline{x})\rangle = \langle y^*, f(x) + w(\overline{x},x)d_1 + c_1\rangle > \langle y^*, f(x)\rangle.$$

Using the same method of (4.1.9), we conclude that $\langle y^* f(\overline{x})\rangle < \langle y^* f(x)\rangle$, a contradiction. Consequently (ii) holds □

If we set $Y=\mathbb{R}$, $C=[0,\infty)$ and $D=\{\varepsilon\}$ for $\varepsilon>0$ in Theorem 4.1.1, we have the following result which is a well-known Ekeland's variational principle in a more general setting.

Corollary 4.1.6. *Let X be a complete metric space, $\omega : X\times X\to[0,\infty)$ be a ω-distance on X, $f : X\times X\to\mathbb{R}$ be a function satisfying the following conditions:*

(i) $F(x,x)=0$ for all $x\in X$;

(ii) $F(x,y)+F(y,z)\geq F(x,z)$ for every $x,y,z\in X$;

(iii) for each $x\in X$ the function $F(x,\cdot) : X\mapsto\mathbb{R}$ is lower semicontinuous and bounded below.

Then for every $x_0\in X$ and $\varepsilon>0$, there exists $\overline{x}\in X$ such that

(i) $F(x_0,\overline{x})+\varepsilon\omega(x_0,\overline{x})\leq 0$;

(ii) $F(\overline{x},x)+\varepsilon\omega(\overline{x},x)>0$ for all $x\neq\overline{x}$.

Remark 4.1.7. By setting $w=d$ and $F(x,y)=f(y)-f(x)$, where $f : X\to R$ is lower semicontinuous and bounded below in Corollary 4.1.6, we obtain Theorem 4.1.1 proven by Ekeland [24, 76].

The following theorem provides the equivalence between the equilibrium version of Ekeland-type variational principle, the equilibrium problem, Caristi-Kirk type fixed point theorem and Oettli and Théra type theorem

Theorem 4.1.8. *Let X be a complete metric space, $\omega : X \times X \to [0, \infty)$ be a ω-distance on X, Y be a locally convex space, C be a closed and convex cone in Y and D be a closed convex and bounded subset of C. Let $T : X \to 2^X$ and $F : X \times X \to Y$ be a function satisfying the following condition:*

(i) $F(x, x) = 0$ *for all* $x \in X$;

(ii) $F(x, y) + F(y, z) \in F(x, z) + C$ *for every* $x, y, z \in X$;

(iii) for each $x \in X$ *the function* $F(x, \cdot) : X \mapsto Y$ *is (D,C)-lower semicontinuous;*

(iv) for each fixed $x \in X$, $F(x, \cdot) : x \to Y$ *is* C*-bounded below;*

(v) for each $x \in X$ *there is* $y \in X$ *such that* $y \in Tx$ *and* $F(x, y) + \omega(x, y)D \subseteq -C$.

Then T has at least one fixed point, i.e., there exists $x \in X$ such that $x \in Tx$.

Proof. By assumption (i)-(iv) applying in Theorem 4.1.1, there exists $\overline{x} \in X$ such that

$$F(\overline{x}, z) + w(\overline{x}, z)D \nsubseteq -C \quad \text{for} \quad \text{all } z \neq \overline{x}.$$

On the other hand by assumption (v), there exists $y \in T(\overline{x})$ such that

$$F(\overline{x}, y) + w(\overline{x}, y)D \subseteq -C.$$

Then we see that $\overline{x} = y$ and so $\overline{x} \in T(\overline{x})$, that is T has at least one fixed point. □

Remark 4.1.9. If we set $F(x, y) = f(y) - f(x)$, $D = \{\varepsilon\}$, $\varepsilon > 0$ and replace ω-distance by d-distance in Theorem 4.1.8, we obtain Theorem 3.1 in [77] and Theorem 4.1 in [78] (vectorial Caristi-Kirk fixed point theorem).

4.2 Existence theorems for the n-vectorial saddle point problems

In this section, we introduce the n-vectorial saddle point problem and prove the existence of a saddle point for VSP_n under assuming compactness and uncompactness by using *Fan-KKM* Theorem.

Suppose that C is a closed convex cone in the topological vector space E such that $intC \neq \emptyset$ and $0 \notin intC$ where $intC$ denotes the interior of C. For each i=1, 2, ..., n, we let K_i be nonempty convex subsets of Hausdorff topological vector spaces X_i and let $f : \prod_{i=1}^{n} K_i \to E$ be a vector valued mapping. Considering the following n-vectorial saddle point problem is to find $\bar{z} := (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$ such that

$$VSP_n : \begin{cases} f(\bar{z}) - f(x_1, \bar{x}_2, \bar{x}_3, ..., \bar{x}_n) & \notin -intC \quad \forall x_1 \in K_1 \\ f(\bar{x}_1, x_2, \bar{x}_3, ..., \bar{x}_n) - f(\bar{z}) & \notin -intC \quad \forall x_2 \in K_2 \\ f(\bar{x}_1, \bar{x}_2, x_3, ..., \bar{x}_n) - f(\bar{z}) & \notin -intC \quad \forall x_3 \in K_3 \\ \vdots & \\ f(\bar{x}_1, \bar{x}_2, \bar{x}_3, ..., x_n) - f(\bar{z}) & \notin -intC \quad \forall x_n \in K_n \end{cases}$$

A point $\bar{z}$ is said to be a saddle point of f on $\prod_{i=1}^{n} K_i$, if it is a solution for VSP_n. Note that when $E = \mathbb{R}$ and $C = [0, +\infty)$, problem VSP_n is reduced to the saddle point problem of a real valued function, i.e., finding $\bar{z} := (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in \prod_{i=1}^{n} K_i$ such that

$$SP_n : \begin{cases} f(\bar{z}) - f(x_1, \bar{x}_2, \bar{x}_3, ..., \bar{x}_n) & \geq 0 \quad \forall x_1 \in K_1 \\ f(\bar{x}_1, x_2, \bar{x}_3, ..., \bar{x}_n) - f(\bar{z}) & \geq 0 \quad \forall x_2 \in K_2 \\ f(\bar{x}_1, \bar{x}_2, x_3, ..., \bar{x}_n) - f(\bar{z}) & \geq 0 \quad \forall x_3 \in K_3 \\ \vdots & \\ f(\bar{x}_1, \bar{x}_2, \bar{x}_3, ..., x_n) - f(\bar{z}) & \geq 0 \quad \forall x_n \in K_n \end{cases}$$

At the beginning of this section, we consider and show the existence theorems for VSP_3 that is the problem to find $(\bar{x}, \bar{y}, \bar{z}) \in K_1 \times K_2 \times K_3$ such that

$$VSP_3 : \begin{cases} f(\bar{x}, \bar{y}, \bar{z}) - f(x, \bar{y}, \bar{z}) & \notin -intC \quad \forall x \in K_1, \\ f(\bar{x}, y, \bar{z}) - f(\bar{x}, \bar{y}, \bar{z}) & \notin -intC \quad \forall y \in K_2, \\ f(\bar{x}, \bar{y}, z) - f(\bar{x}, \bar{y}, \bar{z}) & \notin -intC \quad \forall z \in K_3. \end{cases}$$

The following Lemma is useful for our main result.

Lemma 4.2.1. *Let K be a nonempty convex subset of a topological vector space X, let a vector valued map $\phi : K \to E$ be C-properly quasiconcave and let A be a nonempty finite subset of K. For any $e \in E$, if $e - \phi(\hat{x}) \notin -intC$ for some $\hat{x} \in co(A)$. Then, there exists $x \in A$ such that $e - \phi(x) \notin -intC$.*

Proof. Let $e \in E$ and $A = \{x_1, x_2, ..., x_n\}$ be a finite subset of K and $\hat{x} = \sum_{i=1}^{n} \alpha_i x_i$ where $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i \geq 0$ for all i = 1, 2, ...,n. We will prove it by the mathematical induction. Let $n = 2$. Since ϕ is C-properly quasiconcave on K, we have

$$\begin{aligned} e - \phi(\alpha_1 x_1 + \alpha_2 x_2) &\in e - \phi(x_1) - C \\ \text{or} \quad e - \phi(\alpha_1 x_1 + \alpha_2 x_2) &\in e - \phi(x_2) - C. \end{aligned}$$

If $e - \phi(x_1) \in -intC$ and $e - \phi(x_2) \in -intC$, then $e - \phi(\hat{x}) = e - \phi(\alpha_1 x_1 + \alpha_2 x_2) \in -intC$, which is a contradiction with assumption so $e - \phi(x_1) \notin -intC$ or $e - \phi(x_2) \notin -intC$. This completes the proof of case n=2. Assume that the statement is true for $n \in \mathbb{N}$ and for each $e \in E$,

$$e - \phi(\hat{x}) = e - \phi(\sum_{i=1}^{n+1} \alpha_i x_i) \notin -intC \tag{4.2.1}$$

where $\sum_{i=1}^{n+1} \alpha_i = 1$ and $\alpha_i \geq 0$ for all i = 1, 2, ..., n+1. Let $\alpha := \sum_{i=1}^{n} \alpha_i = 1 - \alpha_{n+1}$ and $x := \sum_{i=1}^{n} \frac{\alpha_i}{\alpha} x_i$, thus $\hat{x} := \sum_{i=1}^{n+1} \alpha_i x_i = \alpha x + \alpha_{n+1} x_{n+1}$. Since ϕ is C-properly quasiconcave and by (4.2.1), we have

$$\begin{aligned} e - \phi(\alpha x + \alpha_{n+1} x_{n+1}) &\in e - \phi(x) - C \\ \text{or} \quad e - \phi(\alpha x + \alpha_{n+1} x_{n+1}) &\in e - \phi(x_{n+1}) - C. \end{aligned}$$

By the same argument in case n=2, we have

$$e-\phi(x) \notin -intC \text{ or } e-\phi(x_{n+1})-C.$$

If $e-\phi(x) \notin -intC$, then by the induction hypothesis, there exists $x_i \in A$ such that $e-\phi(x_i) \notin -intC$, which completes the proof. □

Remark 4.2.2. If we replace the assumption of the map ϕ in Lemma 4.2.1 by C-properly quasiconvex then we have the result: for any $e \in E$, if $\phi(\hat{x})-e \notin -intC$ for some $\hat{x} \in co(A)$, then there exists $x \in A$ such that $\phi(x)-e \notin -intC$.

Lemma 4.2.3. *For each i=1,2,3, let X_i be Hausdorff topological vector spaces, $K_i \subset X_i$ be nonempty convex subsets and $f : K_1 \times K_2 \times K_3 \to E$ be a vector valued function satisfying the conditions (i) and (ii).*

(i) f is C-properly quasiconcave and C-u.s.c. in the first argument on the convex hull of every nonempty finite subset of K_1

(ii) f is C-properly quasiconcave and C-l.s.c. in the second and third argument on the convex hull of every nonempty finite subset of K_2 and K_3 respectively.

Then, for each finite subset A_i of K_i where i=1,2,3, there exist $\hat{x} \in co(A_1)$, $\hat{y} \in co(A_2)$, and $\hat{z} \in co(A_3)$ such that

$$\begin{aligned} f(\hat{x},\hat{y},\hat{z})-f(u,\hat{y},\hat{z}) &\notin -intC \quad \forall u \in co(A_1), \\ f(\hat{x},v,\hat{z})-f(\hat{x},\hat{y},\hat{z}) &\notin -intC \quad \forall v \in co(A_2), \\ f(\hat{x},\hat{y},w)-f(\hat{x},\hat{y},\hat{z}) &\notin -intC \quad \forall w \in co(A_3). \end{aligned}$$

Proof. Take $K := K_1 \times K_2 \times K_3$ and for $(u,v,w) \in K$, we define the following subsets

$$\begin{aligned} L(u,v,w) &= \{x \in K_1 : f(x,v,w)-f(u,v,w) \notin -intC\}, \\ M(u,v,w) &= \{y \in K_2 : f(u,v,w)-f(u,y,w) \notin -intC\}, \\ N(u,v,w) &= \{z \in K_3 : f(u,v,w)-f(u,v,z) \notin -intC\}. \end{aligned}$$

By the definition of three sets, they are nonempty sets because $(u,v,w) \in P(u,v,w) := L(u,v,w) \times M(u,v,w) \times N(u,v,w)$. For each $i = 1,2,3$, we suppose A_i is the finite subset of K_i and set $A := A_1 \times A_2 \times A_3$. Define the set-valued mapping $Q : co(A) \to 2^{co(A)}$ by

$$Q(u,v,w) = \{(x,y,z) \in co(A) : (x,y,z) \in P(u,v,w)\} \quad \forall (u,v,w) \in co(A).$$

We will show that Q is a KKM mapping. Assume that there exists a finite set $(\{u_1,...,u_l\} \times \{v_1,...,v_m\} \times \{w_1,...,w_n\}) \subset co(A)$ such that

$$co(\{u_1,...,u_l\} \times \{v_1,...,v_m\} \times \{w_1,...,w_n\}) \not\subset \bigcup_{i=1,j=1,k=1}^{l,m,n} Q(u_i,v_j,w_k).$$

Then, there exists

$$\begin{aligned}(u_0,v_0,w_0) &= (\sum_{i=1}^{l} \alpha_i u_i, \sum_{j=1}^{m} \beta_j v_j, \sum_{k=1}^{n} \gamma_k w_k) \\ &\in co(\{u_1,...,u_l\} \times \{v_1,...,v_m\} \times \{w_1,...,w_n\})\end{aligned}$$

such that $u_0 \notin L(u_i,v_j,w_k)$ or $v_0 \notin M(u_i,v_j,w_k)$ or $w_0 \notin N(u_i,v_j,w_k)$ for $i = 1,...,l$, $j = 1,...,m$ and $k = 1,...,n$. We consider the case $u_0 \notin L(u_i,v_j,w_k)$ for $i = 1,...,l$, $j = 1,...,m$ and $k = 1,...,n$. Let $j \in \{1,...,m\}$ and $k \in \{1,...,n\}$ be fixed. Clearly,

$$f(u_0,v_j,w_k) - f(u_0,v_j,w_k) \notin -intC.$$

Since f is C-properly quasiconcave in the first argument and by Lemma 4.2.1, there exists $u_i \in \{u_1,...,u_l\}$ such that

$$f(u_0,v_j,w_k) - f(u_i,v_j,w_k) \notin -intC,$$

it is a contradiction with $u_0 \notin L(u_i,v_j,w_k)$. Similarly on other cases, we also obtain a contradiction and so we have Q is a KKM mapping.

Next, we will show that $Q(u,v,w)$ is closed for each $(u,v,w) \in co(A)$. Let $\{(u_\lambda,v_\lambda,w_\lambda)\}_{\lambda\in I} \subseteq Q(u,v,w)$ such that $(u_\lambda,v_\lambda,w_\lambda) \to (u',v',w') \in co(K)$. Assume that $(u',v',w') \notin Q(u,v,w)$, then we have $u' \notin L(u,v,w)$ or $v' \notin M(u,v,w)$ or

$w' \notin N(u,v,w)$. Consider the case $u' \notin L(u,v,w)$. Then, we have $f(u',v,w) - f(u,v,w) \in -intC$, it follows that there is $-c' \in intC$ such that

$$-c' = f(u',v,w) - f(u,v,w) \in -intC. \tag{4.2.2}$$

Since f is C-u.s.c. in the first argument, there exists an open neighborhood U of u' such that for any $c \in intC$ there is an $\lambda_0 \in I$ such that

$$f(u',v,w) \in f(u_\lambda,v,w) - C + intC \quad \forall u_\lambda \in U \quad where \quad \lambda \geq \lambda_0.$$

Set $c = c'$ and by (4.2.2), we obtain that

$$f(u_\lambda,v,w) - f(u,v,w) \in -intC.$$

Then $u_\lambda \notin L(u,v,w)$ which is a contradiction. For other cases, the proof is similar by using the C-lower semicontinility of f. This implies that $Q(u,v,w)$ is closed for each $(u,v,w) \in co(A)$. Since $X_1 \times X_2 \times X_3$ is a Hausdorff space, $co(A)$ is compact and also $Q(u,v,w)$ is compact. By the *Fan-KKM* Theorem, we obtain that

$$\bigcap_{(u,v,w)\in co(A)} Q(u,v,w) \neq \emptyset.$$

Hence there exist $(\hat{x},\hat{y},\hat{z}) \in co(A)$ such that $(\hat{x},\hat{y},\hat{z}) \in P(u,v,w)$ for all $(u,v,w) \in co(A)$. Then $\hat{x} \in L(u,v,w), \hat{y} \in M(u,v,w)$ and $\hat{z} \in N(u,v,w)$ for all $(u,v,w) \in co(A)$. Therefore $\hat{x} \in L(u,\hat{y},\hat{z}) \ \ \forall u \in co(A_1), \ \ \hat{y} \in M(\hat{x},v,\hat{z}) \ \ \forall v \in co(A_2)$ and $\hat{z} \in N(\hat{x},\hat{y},w) \ \ \forall w \in co(A_3)$. This completes the proof. □

Remark 4.2.4. Lemma 4.2.3 is the generalization of Lemma 3.1 in [38]. Moreover the idea of the proof in Lemma 4.2.3 similar to that obtained by Chadli and Mahdioui [38]. In the same way of the proof in Lemma 4.2.3, we can extend this result to n-tuples.

Theorem 4.2.5. *For each $i = 1,2,3$, let X_i be Hausdorff topological vector spaces, $K_i \subseteq X_i$ be nonempty compact convex subsets and $f : K_1 \times K_2 \times K_3 \to E$ be a vector valued mapping satisfying the conditions (i) and (ii) in Lemma 4.2.3. Then, VSP_3 has a saddle point.*

Proof. Let $\mathscr{K}$ be the family of all nonempty finite subsets of $K := K_1 \times K_2 \times K_3$ and for each $A := A_1 \times A_2 \times A_3 \in \mathscr{K}$, we suppose the following set

$$\mathscr{L}_A = \{(x, y, z) : x \in L(u, v, w), y \in M(u, v, w), z \in N(u, v, w) \ \forall (u, v, w) \in co(A)\}$$

By Lemma 4.2.3, we have $\mathscr{L}_A$ is nonempty for each A $\in \mathscr{K}$. Next, we will show that the family $\{\overline{\mathscr{L}_A}\}_{A\in\mathscr{K}}$ has the finite intersection property. Suppose that $A' := A'_1 \times A'_2 \times A'_3$ and $A'' := A''_1 \times A''_2 \times A''_3$ are two finite subsets of K. Setting $A := A' \cup A''$, by the definition of the set $\mathscr{L}_A$, we obtain that $\mathscr{L}_A \subset \mathscr{L}_{A'} \cap \mathscr{L}_{A''}$ and so we have

$$\emptyset \neq \overline{\mathscr{L}_A} \subset \overline{\mathscr{L}_{A'}} \cap \overline{\mathscr{L}_{A''}}$$

This leads to $\{\overline{\mathscr{L}_A}\}_{A\in\mathscr{K}}$ has finite intersection property. Since K is compact, $\bigcap_{A\in\mathscr{K}} \overline{\mathscr{L}_A} \neq \emptyset$. Let $(x, y, z) \in K$ be an arbitrary and $(\tilde{x}, \tilde{y}, \tilde{z}) \in \bigcap_{A\in\mathscr{K}} \overline{\mathscr{L}_A}$ be fixed. Set $D = \{(x, y, z), (\tilde{x}, \tilde{y}, \tilde{z})\}$, then we have $D \in \mathscr{K}$. Since $(\tilde{x}, \tilde{y}, \tilde{z}) \in \overline{\mathscr{L}_D}$, there exists a generalized sequence $\{(x_\alpha, y_\alpha, z_\alpha)\}_{\alpha\in I} \subset \overline{\mathscr{L}_D}$ such that $\{x_\alpha, y_\alpha, z_\alpha\} \to (\tilde{x}, \tilde{y}, \tilde{z})$. Since $(x_\lambda, y_\lambda, z_\lambda) := (\lambda x + (1-\lambda)\tilde{x}, \lambda y + (1-\lambda)\tilde{y}, \lambda z + (1-\lambda)\tilde{z}) \in co(D)$ and by the definition of $\mathscr{L}_D$, for $\alpha \in I$ and $\lambda \in [0, 1]$, we note that

$$x_\alpha \in L(x_\lambda, y_\lambda, z_\lambda), \quad y_\alpha \in M(x_\lambda, y_\lambda, z_\lambda), \quad z_\alpha \in N(x_\lambda, y_\lambda, z_\lambda).$$

Then, for all $\alpha \in I$ and $\lambda \in [0, 1]$,

$$\begin{aligned} f(x_\alpha, y_\lambda, z_\lambda) - f(x_\lambda, y_\lambda, z_\lambda) &\notin -intC, \\ f(x_\lambda, y_\lambda, z_\lambda) - f(x_\lambda, y_\alpha, z_\lambda) &\notin -intC, \\ f(x_\lambda, y_\lambda, z_\lambda) - f(x_\lambda, y_\lambda, z_\alpha) &\notin -intC. \end{aligned}$$

By Proposition 2.2.40, we conclude that for all $\lambda \in [0, 1]$,

$$\begin{aligned} f(\tilde{x}, y_\lambda, z_\lambda) - f(x, y_\lambda, z_\lambda) &\notin -intC, \\ f(x_\lambda, y, z_\lambda) - f(x_\lambda, \tilde{y}, z_\lambda) &\notin -intC, \\ f(x_\lambda, y_\lambda, z) - f(x_\lambda, y_\lambda, \tilde{z}) &\notin -intC. \end{aligned}$$

Therefore, we have

$$\begin{aligned} f(\tilde{x},\tilde{y},\tilde{z}) - f(x,\tilde{y},\tilde{z}) &\notin -intC, \\ f(\tilde{x},y,\tilde{z}) - f(\tilde{x},\tilde{y},\tilde{z}) &\notin -intC, \\ f(\tilde{x},\tilde{y},z) - f(\tilde{x},\tilde{y},\tilde{z}) &\notin -intC. \end{aligned}$$

Since (x, y, z) is an arbitrary element in $K_1 \times K_2 \times K_3$, we complete the proof. □

In Theorem 4.2.5, we set for each $z \in K_3$, $f(x, y, z) = g(x, y)$ for all $(x, y) \in K_1 \times K_2$, where $g : K_1 \times K_2 \to E$. Then we have the following corollary.

Corollary 4.2.6. *For each $i = 1, 2$, let X_i be Hausdorff topological vector spaces, $K_i \subseteq X_i$ be nonempty compact convex subsets and the vector valued mapping $f : K_1 \times K_2 \to E$ be C-properly quasiconcave and C-u.s.c. in the first argument on the convex hull of every nonempty finite subset of K_1 and C-properly quasiconcave and C-l.s.c. in the second argument on the convex hull of every nonempty finite subset of K_2. Then, there exists $(\bar{x}, \bar{y}) \in K_1 \times K_2$ such that*

$$VSP : \begin{cases} f(\bar{x},\bar{y}) - f(x,\bar{y}) \notin -intC & \forall x \in K_1, \\ f(\bar{x},y) - f(\bar{x},\bar{y}) \notin -intC & \forall y \in K_2. \end{cases}$$

Setting $E = \mathbb{R}$ and $C = [0, +\infty)$. Then Corollary 4.2.6 can be reduced to the following corollary.

Corollary 4.2.7. *For each $i = 1, 2$, let X_i be Hausdorff topological vector spaces, $K_i \subseteq X_i$ be nonempty compact convex subsets and the vector valued mapping $f : K_1 \times K_2 \to \mathbb{R}$ is quasiconcave and u.s.c. in the first argument on the convex hull of every nonempty finite subset of K_1 and quasiconcave and l.s.c. in the second argument on the convex hull of every nonempty finite subset of K_2. Then, there exists $(\bar{x}, \bar{y}) \in K_1 \times K_2$ such that*

$$f(\bar{x},y) \geq f(\bar{x},\bar{y}) \geq f(x,\bar{y}) \quad for\ all\ (x,y) \in K_1 \times K_2.$$

The next theorem presents the existence solution for VSP_3 without assuming compactness of the subsets.

Theorem 4.2.8. *For each $i = 1, 2, 3$, let X_i be Hausdorff topological vector spaces, $K_i \subseteq X_i$ be nonempty convex subsets and $f : K_1 \times K_2 \times K_3 \to E$ be a vector value mapping satisfying the conditions $(i)-(ii)$ and if it satisfies the following condition:*

(iii) (The coercivity) there is a nonempty compact set $B := B_1 \times B_2 \times B_3 \subseteq K := K_1 \times K_2 \times K_3$ and there is a nonempty compact convex set $\tilde{B} := \tilde{B}_1 \times \tilde{B}_2 \times \tilde{B}_3 \subseteq K$ such that if $(x, y, z) \in K \cap B^C$, then

$$\begin{aligned} f(\tilde{x}, \tilde{y}, \tilde{z}) - f(x, \tilde{y}, \tilde{z}) &\in -intC, \\ f(\tilde{x}, y, \tilde{z}) - f(\tilde{x}, \tilde{y}, \tilde{z}) &\in -intC, \\ f(\tilde{x}, \tilde{y}, z) - f(\tilde{x}, \tilde{y}, \tilde{z}) &\in -intC. \end{aligned}$$

for some $(\tilde{x}, \tilde{y}, \tilde{z}) \in \tilde{B}$. Then VSP_3 has a saddle point.

Proof. Let $\mathscr{K}$ be the family of all nonempty finite subsets of $K := K_1 \times K_2 \times K_3$ and for each $A := A_1 \times A_2 \times A_3 \in \mathscr{K}$, we suppose the following set

$$\mathscr{L}_A = \{(x, y, z) \in B : x \in L(u, v, w), y \in M(u, v, w), z \in N(u, v, w)\ \forall (u, v, w) \in co(A \cup \tilde{B})\}$$

It is easy to see that $co(A \cup \tilde{B})$ is compact for every $A \in \mathscr{K}$. By Theorem 4.2.5, there exists $(\tilde{x}, \tilde{y}, \tilde{z}) \in co(A \cup \tilde{B})$ such that

$$\begin{aligned} f(\tilde{x}, \tilde{y}, \tilde{z}) - f(x, \tilde{y}, \tilde{z}) &\notin -intC \quad \text{for all } x \in co(A_1 \cup \tilde{B}_1), \\ f(\tilde{x}, y, \tilde{z}) - f(\tilde{x}, \tilde{y}, \tilde{z}) &\notin -intC \quad \text{for all } y \in co(A_2 \cup \tilde{B}_2), \\ f(\tilde{x}, \tilde{y}, z) - f(\tilde{x}, \tilde{y}, \tilde{z}) &\notin -intC \quad \text{for all } z \in co(A_3 \cup \tilde{B}_3). \end{aligned}$$

By the contrary of coercivity condition (iii) and since $\tilde{B} \subset co(AU\tilde{B})$, we deduce that $(\tilde{x}, \tilde{y}, \tilde{z}) \in B$. This means that $\mathscr{L}_A$ is nonempty for all $A \in \mathscr{K}$. Similarly proved in Theorem 4.2.5, it implies that the family $\{\overline{\mathscr{L}_A}\}_{A \in \mathscr{K}}$ has the finite intersection property and hence $\bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A}$ is also nonempty by the compactness of B. Let $(x, y, z) \in K$ be an arbitrary and $(\bar{x}, \bar{y}, \bar{z}) \in \bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A}$ be fixed. Setting $D = \{(x, y, z), (\bar{x}, \bar{y}, \bar{z})\}$, then we have $D \in \mathscr{K}$. Since $(\bar{x}, \bar{y}, \bar{z}) \in \overline{\mathscr{L}_D}$, there exists a generalized sequence $\{(x_\alpha, y_\alpha, z_\alpha)\}_{\alpha\in I} \subset \overline{\mathscr{L}_D}$ such that $(x_\alpha, y_\alpha, z_\alpha) \to (\bar{x}, \bar{y}, \bar{z})$. By the same argument of Theorem 4.2.5, we conclude that

$$\begin{aligned} f(\bar{x}, \bar{y}, \bar{z}) - f(x, \bar{y}, \bar{z}) &\notin -intC, \\ f(\bar{x}, y, \bar{z}) - f(\bar{x}, \bar{y}, \bar{z}) &\notin -intC, \\ f(\bar{x}, \bar{y}, z) - f(\bar{x}, \bar{y}, \bar{z}) &\notin -intC \end{aligned}$$

for all $(x, y, z) \in K$, which implies that VSP_3 has a saddle point and completes the proof. □

Remark 4.2.9. In Theorem 4.2.8, if we set for each $z \in K_3$, $f(x, y, z) = g(x, y)$ for all $(x, y) \in K_1 \times K_2$, where $g : K_1 \times K_2 \to E$ then we have Theorem 3.2 in [38]. In addition to this, if we let $E = \mathbb{R}$ and $C = [0, +\infty)$ then we also have Corollary 3.1 in [38].

The following theorem presents the existence solution for VSP_n which generalizes Theorem 4.2.5.

Theorem 4.2.10. *For each i =1, 2, ..., n, let $K_i \subseteq X_i$ be a nonempty compact convex subset and $f : \prod_{i=1}^{n} K_i \to E$ be a vector valued mapping satisfying the following conditions:*

(I) f is C-properly quasiconcave, C-u.s.c. in the first argument and C-properly quasiconvex, C-l.s.c. in the other arguments on the convex hull of every nonempty finite subset of $\prod_{i=1}^{n} K_i$.

Then, VSP_n has a saddle point.

Proof. For each $(u_1, u_2, ..., u_n) \in \prod_{i=1}^{n} K_i$, we define the following subsets

$$\begin{aligned}
L_1(u_1, u_2, ..., u_n) &= \{x_1 \in K_1 : f(x_1, u_2, ..., u_n) - f(u_1, u_2, u_3, ..., u_n) \notin -intC\} \\
L_2(u_1, u_2, ..., u_n) &= \{x_2 \in K_2 : f(u_1, u_2, ..., u_n) - f(u_1, x_2, u_3, ..., u_n) \notin -intC\} \\
L_3(u_1, u_2, ..., u_n) &= \{x_3 \in K_3 : f(u_1, u_2, ..., u_n) - f(u_1, u_2, x_3, ..., u_n) \notin -intC\} \\
&\vdots \\
L_n(u_1, u_2, ..., u_n) &= \{x_n \in K_n : f(u_1, u_2, ..., u_n) - f(u_1, u_2, u_3, ..., x_n) \notin -intC\}.
\end{aligned}$$

By the definition of these sets, they are nonempty sets because $(u_1, u_2, ..., u_n) \in \prod_{i=1}^{n} L_i(u_1, u_2, ..., u_n)$. Let $\mathscr{K}$ be the family of all nonempty finite subsets of $\prod_{i=1}^{n} K_i$ and for each $A = \prod_{i=1}^{n} A_i \in \mathscr{K}$, we suppose the following set

$$\mathscr{L}_A = \{(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} K_i : x_i \in L_i(w_1, w_2, ..., w_n) \ \ \forall (w_1, w_2, ..., w_n) \in co(A)\}.$$

By Remark 4.2.4, we have $\mathscr{L}_A$ is nonempty for each $A \in \mathscr{K}$. Using the similar idea of the proof in the Theorem 4.2.5, we have $\bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A} \neq \emptyset$. Let $(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} K_i$ be an arbitrary and $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in \bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A}$ be fixed. In the same way as the proof in the Theorem 4.2.5 once more, we conclude that

$$VSP_n : \begin{cases}
f(\bar{z}) - f(x_1, \bar{x}_2, \bar{x}_3, ..., \bar{x}_n) \notin -intC & \forall x_1 \in K_1 \\
f(\bar{x}_1, x_2, \bar{x}_3, ..., \bar{x}_n) - f(\bar{z}) \notin -intC & \forall x_2 \in K_2 \\
f(\bar{x}_1, \bar{x}_2, x_3, ..., \bar{x}_n) - f(\bar{z}) \notin -intC & \forall x_3 \in K_3 \\
\vdots \\
f(\bar{x}_1, \bar{x}_2, \bar{x}_3, ..., x_n) - f(\bar{z}) \notin -intC & \forall x_n \in K_n
\end{cases}$$

□

If we set $E = \mathbb{R}$ and $C = [0, +\infty)$, then Theorem 4.2.10 is reduced to the following corollary.

Corollary 4.2.11. *For each i =1, 2, ..., n, let $K_i \subseteq X_i$ be a nonempty compact convex subsets and $f : \prod_{i=1}^{n} K_i \to \mathbb{R}$ is quasiconcave, u.s.c. in the first argument and* *quasiconvex, l.s.c. in the other arguments on the convex hull of every nonempty finite subset of* $\prod_{i=1}^{n} K_i$. *Then, there exists a saddle point* $\bar{z} \in \prod_{i=1}^{n} K_i$ *for* SP_n.

The following theorem presents the existence solution for VSP_n which generalizes Theorem 4.2.8.

Theorem 4.2.12. *For each i=1, 2, ..., n, let* $K_i \subseteq X_i$ *be a nonempty convex subsets and* $f : \prod_{i=1}^{n} K_i \to E$ *be a vector valued mapping satisfying the condition (I) and if it satisfies the following condition:*

(II) (The coercivity) there is a nonempty compact set $B = \prod_{i=1}^{n} B_i \subseteq \prod_{i=1}^{n} K_i$ *and there is a nonempty compact convex set* $\widetilde{B} = \prod_{i=1}^{n} \widetilde{B}_i \subseteq \prod_{i=1}^{n} K_i$ *such that if* $(x_1, ..., x_n) \in \prod_{i=1}^{n} K_i \cap B^C$, *then*

$$\begin{array}{lcl} f(\tilde{z}) - f(x_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n) & \in & -intC \\ f(\tilde{x}_1, x_2, \tilde{x}_3, ..., \tilde{x}_n) - f(\tilde{z}) & \in & -intC \\ f(\tilde{x}_1, \tilde{x}_2, x_3, ..., \tilde{x}_n) - f(\tilde{z}) & \in & -intC \\ & \vdots & \\ f(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_{n-1}, x_n) - f(\tilde{z}) & \in & -intC \end{array}$$

for some $\tilde{z} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n) \in \widetilde{B}$.

Then VSP_n *has a saddle point.*

Proof. For each $(u_1, u_2, ..., u_n) \in \prod_{i=1}^{n} K_i$, we define $L_i(u_1, u_2, ..., u_n)$ same as Theorem 4.2.10. Let $\mathscr{K}$ be the family of all nonempty finite subsets of $\prod_{i=1}^{n} K_i$ and for each $A = \prod_{i=1}^{n} A_i \in \mathscr{K}$, we consider the following set

$$\mathscr{L}_A = \{(x_1, x_2, ..., x_n) \in B : x_i \in L_i(w_1, w_2, ..., w_n) \in co(A \cup \tilde{B})\}.$$

it is easy to see that $co(A \cup \tilde{B})$ is compact for every $A \in \mathscr{K}$. By Theorem 4.2.10, there exists $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n) \in co(A \cup \tilde{B})$ such that

$$\begin{array}{lll} f(\tilde{z}) - f(x_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n) & \notin -intC & \forall x_1 \in co(A_1 \cup \tilde{B}_1) \\ f(\tilde{x}_1, x_2, \tilde{x}_3, ..., \tilde{x}_n) - f(\tilde{z}) & \notin -intC & \forall x_2 \in co(A_2 \cup \tilde{B}_2) \\ f(\tilde{x}_1, \tilde{x}_2, x_3, ..., \tilde{x}_n) - f(\tilde{z}) & \notin -intC & \forall x_3 \in co(A_3 \cup \tilde{B}_3) \\ & \vdots & \\ f(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_{n-1}, x_n) - f(\tilde{z}) & \notin -intC & \forall x_n \in co(A_n \cup \tilde{B}_n) \end{array}$$

Since $\widetilde{B} \subseteq co(A \cup \widetilde{B})$ and by the contrapositive coercivity condition(II), we conclude that $(\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n) \in B$. This implies that $\mathscr{L}_A \neq \emptyset$ for all $A \in \mathscr{K}$. By the compactness of B, we now follow an idea similar to that in Theorem 4.2.5 which implies that $\bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A} \neq \emptyset$. Let $(x_1, x_2, ..., x_n) \in \prod_{i=1}^{n} K_i$ be an arbitrary and $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in \bigcap_{A \in \mathscr{K}} \overline{\mathscr{L}_A}$ be fixed. Setting $D = \{(x_1, x_2, ..., x_n), (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)\}$, then we have $D \in \mathscr{K}$. By the same argument of Theorem 4.2.8 applying to n-tuples, it implies that $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ is a saddle point for VSP_n. □