CHAPTER I

INTRODUCTION

One of the most important problems in nonlinear analysis is the so called equilibrium problem (abbreviated (EP)), which can be formulated as follows. Let C be a nonempty set and $f : C \times C \to \mathbb{R}$ a given function. The problem consists on finding an element $\tilde{x} \in C$ such that

$$f(\tilde{x}, y) \geq 0, \text{ for all } y \in C. \tag{EP}$$

The element $\tilde{x}$ satisfying (EP) is called equilibrium point of f on C.

(EP) has been extensively studied in recent years (e.g. [1, 2, 3, 4] and the references therein). Apart from its theoretical interest, important problems arising from economics, mechanics, electricity and other practical sciences motivate the study of (EP). Equilibrium problems include, as particular cases, optimization problems, saddle point (minimax) problems, variational inequalities, Nash equilibria problems, complementarity problems, fixed point problems, etc. As far as we know the term "equilibrium problem"was attributed in Blum and Oettli [4], but the problem itself has been investigated more than twenty years before in a paper of Ky Fan [5] in connection with the so called "intersection theorems"(i.e., results stating the nonemptiness of a certain family of sets). Ky Fan considered (EP) when C is a compact convex subset of a Hausdorff topological vector space and termed it "minimax inequality". Since that time, the existence theorems for solution of general versions of the equilibrium problem have been wildly studied by many authors, for example, mixed equilibrium problem (MEP) [6, 7], generalized equilibrium problem (GEP) [8], generalized mixed equilibrium problem (GMEP) [9], and so on.

On the other hand, The study of inequality problems captured special attention in the last decades, one of the most recent and general type of inequalities being the hemivariational inequalities. The notion of hemivariational inequality was introduced by Panagiotopoulos [10, 11] at the beginning of the 1980s as a variational formulation for several classes of mechanical problems with nonsmooth and nonconvex energy super-potentials. In the case of convex super-potentials, hemivariational inequalities reduce to variational inequalities which were studied earlier by many authors (see e.g. Fichera [12] or Hartman and Stampacchia [13]). For example, Carl [14], Carl et al. [15, 16] and Xiao and Huang [17] studied the existence of solutions of some kinds of hemi-variational inequalities using the method of sub-super solutions. Migorski and Ochal [18], and Park and Ha [19, 20] studied the problem using the regularized approximating method. Goelevan et al. [21] and Liu [22] proved the existence of solutions using the method of the first eigenfunction. For more related works regarding the existence of solutions for hemivariational inequalities, we refer to [11, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36] and references therein. Recently, Zhang and He [37] established some existence results for hemivariational inequalities of the Hartman-Stampacchia type involving stably quasimonotone set-valued mappings on bounded, closed and convex subsets in reflexive Banach spaces. Very recently, Tang and Huang [38] studied the existence of solutions for the variational-hemivariational inequalities in reflexive Banach spaces. By using the concept of stable ϕ-quasimonotonicity and the properties of Clarke's generalized directional derivative and Clarke's generalized gradient, they obtained some existence theorems when the constrained set is nonempty, bounded (or unbounded), closed and convex.

Related to the variational inequality problems and the equilibrium problems, we have the problem of finding the fixed points of the nonlinear mappings, which is the current interest in functional analysis. Fixed-point iteration process for nonlinear operators in Hilbert spaces and Banach spaces including Mann, Halpern and Ishikawa iterations process have been studied extensively by many authors to approximate fixed point of various classes of operators and to solve variational inequalities in both Hilbert spaces and Banach spaces and the references therein. In 1952, the original Mann iteration was defined in a matrix formulation by Mann [39]. In 1967, Halpern [40] proposed the new innovation iteration process which resemble in Mann's iteration. In 1974, Ishikawa [41] introduced the iterative scheme which later, it is said to be Ishikawa iteration and studied its strong convergence theorem for Lipschitzian pseudo-contractive mappings in Hilbert spaces. In 2000, the viscosity approximation method for solving nonlinear operator equations has recently attracted much attention. One advantage of the viscosity approximation method is that one can obtain a solution which satisfies some particular properties, for example, it solves a variational inequality. The viscosity approximation method was introduced to nonexpansive mappings by Moudafi [42] in Hilbert spaces and further developed in the framework of Banach spaces by Xu [43]. Recently, [44] Marino and Xu considered a general iterative method which is more general than viscosity approximation method. They proved that the iteration converges strongly to a fixed point of a nonexpansive mapping which solves some variational inequalities in a Hilbert spaces. Another popular method for solving nonlinear operator equations is the hybrid projection method which is developed in Nakajo and Takahashi [45], Kamimura and Takahashi [46] and Martinez-Yanes and Xu [47]. Motivated by the hybrid projection method, Takahashi and Zembayashi [48] introduced the iterative scheme which is called the shrinking projection method.

In addition to study iteration process, existence studies occupy a significant and important part in any theory and any class of problems. In nonlinear analysis theorems about coincidence points (and fixed points, as special cases), maximal elements, nonempty intersections, KKM mappings are among the most powerful tools for existence studies and hence have been intensively investigated. In 1961, using generalization of the classical Knaster-KuratowskiMazurkievicz theorem, Ky Fan [49] established an elementary but very basic geometric lemma for multi-valued maps. In 1968 Browder [50] obtained a fixed point theorem which is the more convenient form of Fan's lemma. Since then this result has been known as the FanBrowder fixed point theorem, and numerous generalizations of this have appeared in the literature, first in Hausdorff topological vector spaces and, later, in generalized convex spaces. Many of these generalizations have major applications in nonlinear analysis, game theory and abstract economies.

In CAT(0) spaces, fixed point theory was first studied by Kirk (see [51, 52]) in 2003-2004. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. In 2008, Kirk and Panyanak [53] specialized Lims concept [54] of Δ-convergence in a general metric space to CAT(0) spaces and showed that many Banach space results which involve weak convergence have precise analogs in this setting; for instance, the Opial property, the Kadec-Klee property and the demiclosedness principle for LANE mappings.

Motivated and inspired by the above works, the purposes of this thesis are to extend, to generalize and to improve existence theorems and the iteration schemes of some nonlinear operators for finding a common element of the solutions of equilibrium problems, variational inequality problems and fixed point problems in Banach spaces and CAT(0) spaces.

This thesis is divided into 6 chapters. Chapter 1 is an introduction to the research problem. Chapter 2 is dealing with some preliminaries and give some useful results that will be deplicated in later Chapter.

Chapter 3, 4 and 5 are the main results of this research. Precisely, in section 3.1 we establish some existence results for the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term in reflexive Banach spaces. Using the concept of the stable f-quasimonotonicity, the properties of Clarkes generalized directional derivative, Clarke's generalized gradient and KKM technique, some existence theorems of solutions are proved when the constrained set is nonempty, bounded (or unbounded), closed and convex. In section 3.2, we introduce the new generalized mixed equilibrium problem with respect to relaxed semi-monotone mappings. Using the KKM technique, we obtain the existence of solutions for the generalized mixed equilibrium problem in Banach spaces. Furthermore, we also introduce a hybrid projection algorithm for finding a common element in the solution set of a generalized mixed equilibrium problem and the fixed point set of an asymptotically nonexpansive mapping. The strong convergence theorem of the proposed sequence is obtained in a Banach space setting. In section 3.3, we prove the existence of solutions for a generalized mixed equilibrium problem under the new conditions imposed on the given bifunction and introduce the algorithm for solving a common element in the solution set of a generalized mixed equilibrium problem and the common fixed point set of finite family of asymptotically nonexpansive mappings. Also, the strong convergence theorems are obtained, under some appropriate conditions, in uniformly convex and smooth Banach spaces.

Section 4.1, we introduce the modified general iterative approximation methods for finding a common fixed point of nonexpansive semigroups which is a unique solution of some variational inequalities. The strong convergence theorems are established in the framework of a reflexive Banach space which admits a weakly continuous duality mapping. In section 4.2, we introduce the general iterative methods for finding a common fixed point of asymptotically nonexpansive semigroups which is a unique solution of some variational inequalities. We prove the strong convergence theorems of such iterative scheme in a reflexive Banach space which admits a weakly continuous duality mapping.

Section 5.1, we first give some properties for generalized hybrid mappings in complete CAT(0) spaces. Then, we prove Δ-convergence and strong convergence theorems of the proposed Picard, Mann and Ishikawa iterative schemes for such mappings in complete CAT(0) spaces. In section 5.2, we study the strong convergence theorems of Moudafis viscosity approximation methods for a nonexpansive mapping T in CAT(0) spaces. By using the concept of quasilinearization, we remark that the proof is different from that of Shi and Chen [55]. In fact, strong convergence theorems for two given iterative schemes are established in CAT(0) spaces without the property $\mathcal{P}$. In section 5.3, we study the strong convergence of Moudafis viscosity approximation methods for approximating a common fixed point of a one-parameter continuous semigroup of nonexpansive mappings in CAT(0) spaces. We prove that the proposed iterative scheme converges strongly to a common fixed point of a one-parameter continuous semigroup of nonexpansive mappings which is also a unique solution of the variational inequality.

The conclusion of research is in Chapter 6.