CHAPTER II

PRELIMINARIES

This chapter includes some notations, definitions, and some useful results.

2.1 Metric spaces and Banach spaces

In this section, we recall the basic definitions and elementary properties of metric spaces and Banach spaces.

Definition 2.1.1. [56] A metric space is a pair (X, d), where X is a set and d is a metric on X(or distance function on X), that is, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(M_1) $d(x,y) \geq 0$;

(M_2) $d(x,y) = 0$ if and only if $x = y$;

(M_3) $d(x,y) = d(y,x)$ (symmetry);

(M_4) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality).

The element of X are called the point of the metric (X, d).

Definition 2.1.2. [56] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *convergent* if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

x is called the limit of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x \text{ or, simple, } x_n \to x. \tag{2.1.1}$$

In this case, we say that $\{x_n\}$ converges to x. If $\{x_n\}$ is not convergent, it is said to be divergent.

Definition 2.1.3. [56] A sequence $\{x_n\}$ in a metric space $X = (X,d)$ is said to be *Cauchy* if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n \geq N$.

Definition 2.1.4. [56] If every Cauchy sequence in a metric space (X,d) converges then the metric space (X,d) is said to be complete.

The concepts of open, closed and bounded subsets of normed spaces are given as follows.

Definition 2.1.5. [56] Let (X,d) be a metric space and A be a subset of E.

(i) Given a point $x_0 \in X$, the *ball centered at x_0 and with radius $r > 0$* is the set $B(x_0, r) := \{x \in E : d(x_0, x) < r\}$.

(ii) A is open if for each $x_0 \in A$ there exists a $\delta > 0$ such that $B(x_0, \delta) \subseteq A$.

(iii) A is closed if the complement A^c is open.

Theorem 2.1.6. [56] *For a subset A of a metric space (X,d). Then A is closed if and only if the situation $x_n \in A$, $x_n \to x$ implies that $x \in A$.*

Definition 2.1.7. [56] Let A be a nonempty subset of a metric space (X,d). Then A is said to be bounded if $diam(C) := \sup_{x,y\in A} d(x,y) < +\infty$.

Definition 2.1.8. [56] A metric space (X,d) is said to be *compact* if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact considered as a subspace of X, that is, every sequence in M has a convergent subsequence whose limit is an element in M.

Definition 2.1.9. [56] A *norm* on a (real or complex) vector space E is a real-valued function on E whose valued at an $x \in E$ is denoted by $\|x\|$ and which has the properties

(N_1) $\|x\| \geq 0$;

(N_2) $\|x\| = 0 \Leftrightarrow x = 0$;

(N_3) $\|\alpha x\| = |\alpha|\|x\|$;

(N_4) $\|x + y\| \geq \|x\| + \|y\|$,

where x and y are arbitrary vectors in E and α is any scalar. A normed space E is a vector space with a norm defined on it which is denoted by $(E, \|\cdot\|)$ or simply by E.

Convergence of sequences and related concepts in normed spaces follow from the corresponding definition 2.1.2 and 2.1.3 for metric spaces and the fact that now $d(x, y) = \|x - y\|$.

Definition 2.1.10. [56] A Banach space is a complete normed space.

Definition 2.1.11. [57] Let A be a subset of normed space E. Then A is said to be *convex* if $(1 - \lambda)x + \lambda y \in A$ for all $x, y \in A$ and all scalar $\lambda \in [0, 1]$.

Next, we discuss some properties of linear operators.

Definition 2.1.12. [56] Let X and Y be linear spaces over the field $\mathbb{K}$.

(i) A mapping $T : X \to Y$ is called *a linear operator* if for all $x, y \in X$ and $\alpha \in \mathbb{K}$,

$$T(x + y) = Tx + Ty \text{ and } T(\alpha x) = \alpha Tx,$$

(ii) A mapping $T : X \to \mathbb{K}$ is called *a linear functional on* X if T is a linear operator.

Definition 2.1.13. [56] Let X and Y be normed spaces over the field $\mathbb{K}$ and $T : X \to Y$ a linear operator. T is said to be *bounded* on X, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

Definition 2.1.14. [56] Let E and Y be normed spaces over the field $\mathbb{K}$, $T : E \to Y$ an operator and $x_0 \in E$. We say that T is *continuous at* x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(x_0)\| < \varepsilon$ whenever $\|x - x_0\| < \delta$ and $x \in E$. If T is continuous at each $x \in E$, then T is said to be *continuous on* E.

Definition 2.1.15. [56] Let E be a normed space. Then the set of all bounded linear functionals on E is called *a dual space* of E and is denoted by E^*.

Weak convergence is defined in terms of bounded linear functionals on E as follows.

Definition 2.1.16. [56] A sequence $\{x_n\}$ is a normed space E is said to be *weakly convergent* if there exists an $x \in E$ such that for every $f \in E^*$,

$$\lim_{n\to\infty} f(x_n) = f(x).$$

This is written $x_n \rightharpoonup x$. The element x is called the weak limit of $\{x_n\}$, and we say that $\{x_n\}$ converges weakly to x.

A subset C of E is *weakly closed* if it is closed in the weak topology, that is, if it contains the weak limit of all of its weakly convergent sequences. The *weakly open* sets are now taken as those sets whose complements are weakly closed. The resulting topology on E is called the *weak topology* on E. Sets which are compact in this topology are said to be *weakly compact*.

Remark 2.1.17. [57] The weak topology of a normed space is a Hausdorff topology, i.e., if x, y are two distinct points in E, there exist two open sets G and H such that $x \in G$, $y \in H$, and $G \cap H \neq \emptyset$.

Definition 2.1.18. [57] A normed space E is said to be *reflexive* if the *canonical mapping* $G : E \to E^{**}$ (i.e. $G(x) = g_x$ for all $x \in E$ where $g_x(f) = f(x)$ for all $f \in E^*$) is surjective.

Next, we present some useful properties of duality mappings and Banach spaces having geometric structures such as convexity and smoothness.

Definition 2.1.19. [57] A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$

Definition 2.1.20. [57] A Banach space E is said to be *uniformly convex* if for each $0 < \varepsilon \leqslant 2$, there is $\delta > 0$ such that $\forall x, y \in E$, the condition $\|x\| = \|y\| = 1$, and $\|x - y\| \geq \varepsilon$ imply $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Definition 2.1.21. [57] Let E be a Banach space and $S = \{x \in E : \|x\| = 1\}$. Then E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1.2}$$

exists for all $x, y \in S$. It is also said to be *uniformly smooth* if the limit (2.1.2) is attained uniformly for $x, y \in S$.

Remark 2.1.22. [57]

(i) E is uniformly convex if and only if E^* is uniformly smooth.

(ii) E is smooth if and only if E^* is strictly convex.

Definition 2.1.23. [57] Let $S(E) = \{x \in E : \|x\| = 1\}$ denote the unit sphere of a Banach space E. A Banach space E is said to have

(i) *a Gâteaux differentiable norm* (we also say that E is smooth), if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1.3}$$

exists for each $x, y \in S(E)$;

(ii) *a uniformly Gâteaux differentiable norm* , if for each y in $S(E)$, the limit (2.1.3) is uniformly attained for $x \in S(E)$;

(iii) *a Fréchet differentiable norm*, if for each $x \in S(E)$, the limit (2.1.3) is attained uniformly for $y \in S(E)$;

(iv) *a uniformly Fréchet differentiable norm* (we also say that E is uniformly smooth), if the limit (2.1.3) is attained uniformly for $(x, y) \in S(E) \times S(E)$.

Definition 2.1.24. [57] A Banach space E is said to have Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

It is known that if E is uniformly convex, then E has the Kadec-Klee property.

Lemma 2.1.25. [58] *Let E be a uniformly convex Banach space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < b \leq \alpha_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that $\limsup_{n\to\infty} \|x_n\| \leq d$, $\limsup_{n\to\infty} \|y_n\| \leq d$ and $\lim_{n\to\infty} \|\alpha_n x_n + (1-\alpha_n)y_n\| = d$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Definition 2.1.26. [57] Let E^* be dual space of a Banach space E. The mapping $J : E \to 2^{E^*}$ defined by

$$J(x) = \{j \in E^* : \langle j, x\rangle = \|x\|^2 = \|j\|_*^2\}, \quad \text{for all} x \in E,$$

is called the *duality mapping of* E.

Definition 2.1.27. [57] A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be gauge function if $\varphi(0) = 0$ and $\lim\limits_{t\to\infty} \varphi(t) = \infty$.

Definition 2.1.28. [57] Let E be a normed space and φ a gauge function. Then the mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{j \in E^* : \langle x, j\rangle = \|x\|\|j\|_*, \quad \|j\|_* = \varphi(\|x\|)\}, \quad \forall x \in E. \tag{2.1.4}$$

is called the *duality mapping with gauge function* φ.

In the particular case $\varphi(t) = t$, the duality mapping $J_\varphi = J$ is called the normalized duality mapping in Definition 2.1.26.

Definition 2.1.29. [59] A Banach space E is said to *has a weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\}$ with $x_n \rightharpoonup x$, the sequence $\{J_\varphi(x_n)\}$ converges weakly* to $J_\varphi(x)$.

Remark 2.1.30. For the gauge function φ, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Phi(t) = \int_0^t \varphi(s)ds \tag{2.1.5}$$

is continuous convex strictly increasing function on $\mathbb{R}^+$. Therefore, Φ has continuous inverse function Φ^{-1}.

The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [54].

Lemma 2.1.31. ([54]) *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

In particular, for all $x, y \in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$. Then the following identity holds:*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y-x\|), \quad \forall x, y \in E.$$

Definition 2.1.32. [60] Let E be a Banach space having a weakly continuous duality mapping J_φ with a gauge function φ, an operator A is said to be *strongly positive* if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J_\varphi(x)\rangle \geq \bar{\gamma}\|x\|\varphi(\|x\|) \tag{2.1.6}$$

and

$$\|\alpha I - \beta A\| = \sup_{\|x\|\leq 1} |\langle(\alpha I - \beta A)x, J_\varphi(x)\rangle|, \ \ \alpha \in [0,1], \beta \in [-1,1],$$

Lemma 2.1.33. [60, Lemma 3.1] *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ. Let A be a strong positive linear bounded operator on E with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \varphi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \leq \varphi(1)(1 - \rho\bar{\gamma})$.*

Now, we introduce the concept of best approximation in normed spaces. Let C be a nonempty subset of a normed space E and let $x \in E$. An element $y_0 \in C$ is said to be a *best approximation* to x if

$$\|x - y_0\| = d(x, C),$$

where $d(x,C) = \inf_{y\in C} \|x - y\|$. The number $d(x, C)$ is call *the distance from x to C*. The (possibly empty) set of all best approximations from x to C is denoted by

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}.$$

This defines a mapping P_C from E into 2^C and is called the *metric projection* onto C.

Existence and characterization of best approximations are shown in the following two theorems.

Theorem 2.1.34. [57] *Let C be a nonempty, closed and convex subset of a strictly convex reflexive (e.g, uniformly convex) Banach space E and let $x \in E$. Then there exists a unique element $y_0 \in C$ such that $\|x - y_0\| = d(x, C)$.*

Theorem 2.1.35. [57] *Let C be a nonempty closed convex subset of a smooth Banach space E and let $x \in E$ and $y \in C$. Then the following are equivalent:*

(a) *y is a best approximation to $x : y = P_C x$.*

(b) *y is a solution of the variational inequality :*

$$\langle y - z, J_\varphi(x - y)\rangle \geq 0 \text{ for all } z \in C,$$

where J_φ is a duality mapping with gauge function φ and P_C is the metric projection from E onto C.

Next, we present the concept of normal structure.

Let C be a nonempty, bounded, closed and convex subset of a Banach space E. The diameter of C be defined by $d(C) := \sup\{\|x - y\| : x, y \in C\}$. For each $x \in C$, denote $r(x, C) = \sup\{\|x - y\| : y \in C\}$ and denote by $r(C) := \inf\{r(x, C) : x \in C\}$ *the chebyshev radius of C relative to itself. The normal structure coefficient $N(E)$ of E* is defined by

$$N(E) := \inf\left\{\frac{d(C)}{r(C)} : C \text{ is a bounded, closed and convex of } E \text{ with } d(C) > 0\right\}.$$

A Banach space E is said to have *uniform normal structure* if $N(E) > 1$. It is known that every Banach space with a uniform normal structure is reflexive. Every uniformly convex and uniformly smooth Banach spaces have uniform normal structure.

In order to prove our main result, we need the following lemmas and definitions.

Let l^∞ be the Banach space of all bounded real-valued sequences. Let **LIM** be a continuous linear functional on l^∞ satisfying $\|\mathbf{LIM}\| = 1 = \mathbf{LIM}(1)$. Then we know that **LIM** is mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n \in \mathbb{N}\} \leq \mathbf{LIM}(a) \leq \sup\{a_n : n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. Occasionally, we shall use $\mathbf{LIM}_n(a_n)$ instead of $\mathbf{LIM}(a)$. A mean **LIM** on $\mathbb{N}$ is called a Banach limit if

$$\mathbf{LIM}_n(a_n) = \mathbf{LIM}_n(a_{n+1})$$

for every $a = (a_1, a_2, \dots) \in l^\infty$. Using the Hahn-Banach theorem, or the Tychonoff fixed point theorem, we can prove the existence of a Banach limit. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leq \mathbf{LIM}_n(a_n) \leq \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \dots) \in l^\infty$.
Subsequently, the following result was showed in [61].

Proposition 2.1.36. [61, Proposition 3.2] *Let C be a nonexpty, closed and convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm and admits the duality mapping J_φ. Suppose that $\{x_n\}$ is a bounded sequence of C and let* LIM_n *be a Banach limit and $z \in E$. Then*

$$\mathrm{LIM}_n\Phi(\|x_n - z\|) = \inf_{y\in C} \mathrm{LIM}_n\Phi(\|x_n - y\|),$$

if and only if

$$\mathrm{LIM}_n\langle y - z, j_\varphi(x_n - z)\rangle \leq 0, \ \ \forall y \in C.$$

In the following, we also need the following lemma.

Lemma 2.1.37. [57] *Let C be a nonempty, closed and convex subset of a reflexive Banach space E and $f : C \to (-\infty, \infty]$ a proper lower semicontinuous convex function such that $f(x_n) \to \infty$ as $\|x_n\| \to \infty$. Then there exists $x_0 \in D(f)$ such that $f(x_0) = \inf_{x\in C} f(x)$.*

Lemma 2.1.38. [62] *Let $\{a_n\}$ be a sequence of real numbers. Then, $a_n \to 0$ if and only if for any subsequence $\{a_{n_i}\}$ of $\{a_n\}$, there exists a subsequence $\{a_{n_{i_j}}\}$ of $\{a_{n_i}\}$ such that $\{a_{n_{i_j}}\}$ converges to 0.*

Lemma 2.1.39. [63] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence such that

(a) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $\limsup_{n\to\infty} b_n/\alpha_n \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

2.2 Some nonlinear operators

In this section, we first recall some definitions related to the single-valued and multi-valued operators. Next, we introduce the notation of stable f monotonicity which will be used in Section 3.1 and give some examples. Throughout of this section, let E be a reflexive Banach space with the norm $\|\cdot\|$, E^* be its dual and let $\langle\cdot,\cdot\rangle$ denote the duality pairing of E^* and E.

Definition 2.2.1. [57] Let $f : E \to (-\infty, \infty]$ be a function and $\{x_n\} \subset E$. Then f is said to be

(i) *lower semicontinuous* on E if for any $x_0 \in E$, $f(x_0) \leq \liminf_{n\to\infty} f(x_n)$ whenever $x_n \to x_0$.

(ii) *upper semi (or hemi) continuous* on E if for any $x_0 \in E$, $\limsup_{n\to\infty} f(x_n) \leq f(x_0)$ whenever $x_n \to x_0$.

(iii) *weakly lower semicontinuous* on E if for any $x_0 \in E$, $f(x_0) \leq \liminf_{n\to\infty} f(x_n)$ whenever $x_n \rightharpoonup x_0$.

(iv) *weakly upper semicontinuous* on E if for any $x_0 \in E$, $\limsup_{n\to\infty} f(x_n) \leq f(x_0)$ whenever $x_n \rightharpoonup x_0$.

The following definition of continuity for multi-valued mappings can be founded in [64].

Definition 2.2.2. Let $F : C \rightrightarrows E^*$ be a multi-valued mapping. the F is said to be

(i) lower semicontinuous at x_0 if, for any $x_0^* \in F(x_0)$ and sequence $\{x_n\} \subset C$ with $x_n \to x_0$, there exists a sequence $x_n^* \in F(x_n)$ which converges to $x_n^* \to x_0^*$;

(ii) lower hemicontinuous if, the restriction of F to every line segment of C is lower semicontinuous with respect to the weak topology in E^*.

The following version of the KKM theorem is due to Ky Fan [65].

Lemma 2.2.3. [65] Let C be a nonempty subset of a Hausdorff topological vector space X and let $G : C \rightrightarrows X$ be a multi-valued mapping satisfying the following properties:

(i) G is a KKM mapping, i.e.,

$$co\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i), \text{ for } x_i \in K, i = 1, 2, \ldots, n;$$

(ii) $G(x)$ is closed in X for every $x \in C$;

(iii) $G(x_0)$ is compact in X for some $x_0 \in C$.

Then $\bigcap_{x \in C} G(x) \neq \emptyset$.

Theorem 2.2.4. [66]**(Kakutani-Fan-Glicksberg Fixed Point Theorem)** *Let E be a locally convex Hausdorff topological vector space, C be a nonempty, convex, compact subset of E. Suppose $T : C \rightrightarrows C$ is a upper semi-continuous mapping with nonempty, closed and convex values. Then T has a fixed point in C.*

Definition 2.2.5. [57] $T : E \to E$ be a mapping.

(i) T is said to be *Lipschitzian* if there exists a constant $L \geq 0$ such that for all $x, y \in E$,

$$\|Tx - Ty\| \leq L\|x - y\|.$$

(ii) T is said to be *contraction* if there exists a constant $0 \leq \alpha < 1$ such that for all $x, y \in E$,

$$\|Tx - Ty\| \leq \alpha \|x - y\|.$$

(iii) T is said to be *nonexpansive* if for all $x, y \in E$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

(iv) T is called an *asymptotically nonexpansive mapping* if there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $\lim_{n \to \infty} k_n = 1$ such that for all $x, y \in E$,

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \ \forall n \geq 1.$$

(v) T is called an *uniformly Lipschitz mapping* if there exists a constant $k \geq 0$ such that for all $x, y \in E$,

$$\|T^n x - T^n y\| \leq k\|x - y\|, \ \forall n \geq 1.$$

From the definition we have every nonexpansive mapping is asymptotically nonexpansive mapping with a sequence $k_n \equiv 1$ and every asymptotically nonexpansive mapping is a uniformly Lipschitz mapping with Lipschitz constant $k = \sup_{n \in \mathbb{N}} k_n$.

Definition 2.2.6. [57] An element $x \in E$ is said to be

(i) a *fixed point* of a mapping $T : E \to E$ provided $Tx = x$.

(ii) a *common fixed point* of two mappings $S, T : X \to X$ provided $Sx = x = Tx$.

The set of all fixed points of T is denoted by $F(T)$.

The following existence theorem of a contraction mapping and an asymptotically nonexpansive mapping are useful tools for our main results.

Theorem 2.2.7. (Banach contraction principle, [67]) *Let (X, d) be a complete metric space and $T : X \to X$ be a contraction mapping. Then we have the following:*

(i) *There exists a unique fixed point $x^* \in X$.*

(ii) *For an arbitrary $x_0 \in X$, the Picard iteration process defined by*

$$x_{n+1} = Tx_n, \ n \geq 0$$

converges to x^*.

Theorem 2.2.8. [68] *Let C be a bounded closed convex subset of a Banach space E and let $T : C \to C$ be a nonexpansive mapping. If E is a reflexive Banach space with normal structure, then T has a fixed point.*

Theorem 2.2.9. [54, Theorem 1] *Suppose E is a Banach space with uniformly normal structure, C is a nonempty bounded subset of E, and $T : C \to C$ is a uniformly k-Lipschitzian mapping with $k < \sqrt{N(E)}$. Suppose also there exists a nonempty bounded closed convex subset C^* of C with the following property (P):*

$$x \in C^* \text{ implies } \omega_w(x) \subset C^*,$$

where $\omega_w(x)$ is the weak ω-limit set of T at x, i.e. the set

$$\{y \in E : y = weak - \lim_{j\to\infty} T^{n_j}x \text{ for some } n_j \to \infty\}.$$

Then T has a fixed point in C^*.

Theorem 2.2.10. [69] *Let C be a bounded, closed and convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing, convex and continuous function $\gamma : [0, \infty) \to [0, \infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\Big(\frac{1}{k_m}\Big\|S^m\Big(\sum_{i=1}^{n}\lambda_i x_i\Big) - \sum_{i=1}^{n}\lambda_i S^m x_i\Big\|\Big) \leq \max_{1\leq j\leq k\leq n}(\|x_j - x_k\| - \frac{1}{k_m}\|S^m x_j - S^m x_k\|)$$

for all $n \in \mathcal{N}$, $\{x_1, x_2, \ldots, x_n\} \subset C$, $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subset [0, 1]$ with $\sum_{i=1}^{n}\lambda_i = 1$ and an asymptotically nonexpansive mapping S of C into E with the sequence $\{k_m\}$.

Lemma 2.2.11. [70, Lemma 1.6] *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $S : C \to C$ be an asymptotically nonexpansive mapping. Then $(I - S)$ is demiclosed at 0, i.e., if $x_n \rightharpoonup x$ and $(I - S)x_n \to 0$, then $x \in F(S)$.*

We next present the concept of Nonlinear semigroup.

Definition 2.2.12. A family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ of mappings of C into itself is called an *asymptotically nonexpansive semigroup* on C if it satisfies the following conditions:

(S1) $T(0)x = x$ for all $x \in C$;

(S2) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(S3) there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x, y \in C$
$\|T^n(t)x - T^n(t)y\| \leq k_n\|x - y\|, \ \forall t \geq 0, \ \forall n \geq 1;$

(S4) for all $x \in C$, the mapping $t \mapsto T(t)x$ is continuous.

An asymptotically nonexpansive semigroup $\mathcal{S}$ is called *nonexpansive semigroup* if $k_n = 1$ for all $n \geq 1$. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is,

$$F(\mathcal{S}) := \{x \in C : T(t)x = x, 0 \leq t < \infty\} = \bigcap_{t\geq 0} F(T(t)).$$

A family $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ of mappings of C into itself is called a *strongly continuous semigroup of Lipchitszian mappings* on C if it satisfies the following conditions (S1), (S2), (S4) in previous and (S3′):

(S3′) for each $t > 0$, there exists a bounded measurable function $L_t : (0, \infty) \to [0, \infty)$ such that $\|T(t)x - T(t)y\| \leq L_t\|x - y\|, \quad \forall x, y \in C.$

A strongly continuous semigroup of Lipchitszian mappings $\mathcal{S}$ is called *strongly continuous semigroup of nonexpansive mappings* if $L_t = 1$ for all $t > 0$, and *strongly continuous semigroup of asymptotically nonexpansive* if $\limsup_{t\to\infty} L_t \leq 1$. Note that for asymptotically nonexpansive semigroup $\mathcal{S}$, we can always assume that the Lipchitszian constant $\{L_t\}_{t>0}$ are such that $L_t \geq 1$ for each $t > 0$, L_t is non-increasing in t, and $\lim_{t\to\infty} L_t = 1$; otherwise we replace L_t, for each $t > 0$, with $L_t := \max\{\sup_{s\geq t} L_s, 1\}$. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is,

$$F(\mathcal{S}) := \{x \in C : T(t)x = x, 0 \leq t < \infty\} = \bigcap_{t\geq 0} F(T(t)).$$

Theorem 2.2.13. [71] *Suppose C is a weakly compact, convex subset of a Banach space E, and suppose that C has complete normal structure. Let $\mathcal{S}$ be a commutative nonexpansive semigroup of C into itself. Then, there is a point $x \in C$ such that $T(x) = x$ for each $T \in \mathcal{S}$.*

Now, we present the concept of uniformly asymptotically regular semigroup. $\mathcal{S}$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) on C if for all $h \geq 0$ and any bounded subset B of C,

$$\lim_{s\to\infty} \sup_{x\in B} \|T(h)(T(s)x) - T(s)x\| = 0.$$

The nonexpansive semigroup $\{\sigma_t : t > 0\}$ defined by the following lemma is an example of u.a.r. nonexpansive semigroup. Other examples of u.a.r. operator semigroup can be found in [72, Examples 17, 18].

Lemma 2.2.14. (see [73, Lemma 2.7]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, B a bounded closed convex subset of C, and $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. For each $h > 0$, set $\sigma_t(x) = \frac{1}{t}\int_0^t T(s)xds$, then*

$$\lim_{t\to\infty} \sup_{x\in B} \|\sigma_t(x) - T(h)\sigma_t(x)\| = 0. \tag{2.2.1}$$

Example 2.2.15. *The set $\{\sigma_t : t > 0\}$ defined by Lemma 2.2.14 is u.a.r. nonexpansive semigroup. In fact, it is obvious that $\{\sigma_t : t > 0\}$ is a nonexpansive semigroup. For each $h > 0$, we have*

$$\begin{aligned}\|\sigma_t(x) - \sigma_h\sigma_t(x)\| &= \left\|\sigma_t(x) - \frac{1}{h}\int_0^h T(s)\sigma_t(x)ds\right\| \\ &= \left\|\frac{1}{h}\int_0^h (\sigma_t(x) - T(s)\sigma_t(x))ds\right\| \\ &\le \frac{1}{h}\int_0^h \|\sigma_t(x) - T(s)\sigma_t(x)\|ds.\end{aligned}$$

Applying Lemma 2.2.14, we have

$$\limsup_{t\to\infty}\sup_{x\in B}\|\sigma_t(x) - \sigma_h\sigma_t(x)\| \le \frac{1}{h}\int_0^h \limsup_{t\to\infty}\sup_{x\in B}\|\sigma_t(x) - \sigma_h\sigma_t(x)\|ds = 0.$$

Now, we collect some notations of monotonicity. The following definitions (i),(ii) and (iii) are well-known definitions, (iv) can be found in [74], (v) and (vi) can be found in [37], and (vii)-(x) can be found in [38].

Definition 2.2.16. Let $\phi : C \to \mathbb{R}$ be a function, and $F : C \rightrightarrows E^*$ a multi-valued mapping. Then F is said to be

(i) monotone if, for each $u, v \in C$,

$$\langle v^* - u^*, v - u\rangle \ge 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(ii) pseudomonotone if, for each $u, v \in C$,

$$\langle u^*, v - u\rangle \ge 0 \Longrightarrow \langle v^*, v - u\rangle \ge 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(iii) quasimonotone if, for each $u, v \in C$,

$$\langle u^*, v - u\rangle > 0 \Longrightarrow \langle v^*, v - u\rangle \ge 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(iv) relaxed η–ξ monotone if there exist a mapping $\eta : C \times C \to E$ and a function $\xi : E \to \mathbb{R}$ positively homogeneous of degree p, that is, $\xi(tz) = t^p\xi(z)$ for all $t > 0$ and $z \in E$ such that

$$\langle Tx - Ty, \eta(x, y)\rangle \ge \xi(x - y), \quad \forall x, y \in C,$$

(v) stably pseudomonotone with respect to the set $U \subset E^*$ if, F and $F(\cdot) - \xi$ are pseudomonotone for every $\xi \in U$;

(vi) stably quasimonotone with respect to the set $U \subset E^*$ if, F and $F(\cdot) - \xi$ are quasimonotone for every $\xi \in U$;

(vii) ϕ-pseudomonotone if, for each $u, v \in K$,

$$\langle u^*, v-u\rangle + \phi(v) - \phi(u) \geq 0 \Longrightarrow \langle v^*, v-u\rangle + \phi(v) - \phi(u) \geq 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(viii) ϕ-quasimonotone if, for each $u, v \in K$,

$$\langle u^*, v-u\rangle + \phi(v) - \phi(u) > 0 \Longrightarrow \langle v^*, v-u\rangle + \phi(v) - \phi(u) \geq 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(ix) stably ϕ-pseudomonotone with respect to the set $U \subset E^*$ if, F and $F(\cdot) - \xi$ are ϕ-pseudomonotone for every $\xi \in U$;

(x) stably ϕ-quasimonotone with respect to the set $U \subset E^*$ if, F and $F(\cdot) - \xi$ are ϕ-quasimonotone for every $\xi \in U$.

We introduce the concept of stable f-quasimonotonicity with respect to the set $U \subset E^*$ which is useful for establishing existence theorems for the main results.

Definition 2.2.17. Let $f : C \times C \to \mathbb{R}$ be a bifunction, and $F : C \rightrightarrows E^*$ a multi-valued mapping. Then F is said to be

(i) f-pseudomonotone if, for each $u, v \in C$,

$$\langle u^*, v-u\rangle + f(u,v) \geq 0 \Longrightarrow \langle v^*, v-u\rangle + f(u,v) \geq 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(ii) f-quasimonotone if, for each $u, v \in C$,

$$\langle u^*, v-u\rangle + f(u,v) > 0 \Longrightarrow \langle v^*, v-u\rangle + f(u,v) \geq 0, \ \forall u^* \in F(u) \text{ and } v^* \in F(v);$$

(iii) stably f-pseudomonotone with respect to the set $U \subset E^*$ if, F and $F(\cdot) - \xi$ are f-pseudomonotone for every $\xi \in U$;

(iv) stably f-quasimonotone with respect to the set $U \subset E^*$ if, F and $F(\cdot) - \xi$ are f-quasimonotone for every $\xi \in U$.

Remark 2.2.18. 1. If $f(u,v) = \phi(v) - \phi(u)$, where $\phi : C \to \mathbb{R}$, then the notations of f-pseudomonotonicity, f-quasimonotonicity, stable f-pseudomonotonicity and stable f-quasimonotonicity of the operator F reduce to ones of ϕ-pseudomonotonicity, ϕ-quasimonotonicity, stable ϕ-pseudomonotonicity and stable ϕ-quasimonotonicity of the operator F, respectively. Those concepts are introduced in [38].

2. If $f \equiv 0$, then the notations of f-pseudomonotonicity, f-quasimonotonicity, stable f-pseudomonotonicity and stable f-quasimonotonicity of the operator F reduce to ones of pseudomonotonicity, quasimonotonicity, stable pseudomonotonicity and stable quasimonotonicity of the operator F, respectively. Those concepts are introduced in [37].

Remark 2.2.19. We represent the implications between monotonicity and some kinds of generalized monotonicity through the following diagrams:

(D1) : monotonicity $\Rightarrow$ stably psuedomonotonicity w.r.t. $U \Rightarrow$ psuedomonotonicity

$\Downarrow$ $\Downarrow$

stably quasimonotonicity w.r.t. $U \Rightarrow$ quasimonotonicity

(D2) : monotonicity $\Rightarrow$ stably ϕ-psuedomonotonicity w.r.t. $U \Rightarrow$ ϕ-psuedomonotonicity

$\Downarrow$ $\Downarrow$

stably ϕ-quasimonotonicity w.r.t. $U \Rightarrow$ ϕ-quasimonotonicity

(D3) : monotonicity $\Rightarrow$ stably f-psuedomonotonicity w.r.t. $U \Rightarrow$ f-psuedomonotonicity

$\Downarrow$ $\Downarrow$

stably f-quasimonotonicity w.r.t. $U \Rightarrow$ f-quasimonotonicity.

The inverse direction of each implication relationship mentioned by three diagrams above does not hold in general. Example 2 in [75] had shown that a stably pseudomonotone mapping with respect to a closed line segment is not necessarily monotone; Example 3.1 in [76] had shown that a quasimonotone mapping may not be pseudomonotone; Example 4.1 in [37] had shown that a stably quasimonotone mapping is not necessarily stable pseudomonotone; Example 3.1 in [38] had shown that stably ϕ-pseudomonotone mapping may not be monotone, while Example 4.1 in the same paper had shown that a stably ϕ-quasimonotone mapping is not necessarily stably ϕ-pseudomonotone and ϕ-quasimonotone mapping may not be ϕ-pseudomonotone.

Let A, A_ϕ and A_f denote the sets consisting of generalized monotonicity in diagrams (D1), (D2) and (D3), respectively. Example 3.1 [38] had shown that each of generalized monotonicity taken from the set A is independent of any one taken from A_ϕ if $\phi = I_C$. Also, the following two examples illustrate that each of generalized monotonicity taken from the set A_ϕ is independent of any one taken from A_f if $f(u,v) \neq \phi(v) - \phi(u)$, where $\phi : C \to \mathbb{R}$.

Example 2.2.20. Let $E = \mathbb{R}^2$ and $C = [3,5] \times \{0\}$. Let $f : C \times C \to [-\infty, +\infty]$, $F : C \to E^* = \mathbb{R}^2$ and $\phi : C \to \mathbb{R}$ by

$$f(x,y) = y_1^2 - x_1 y_1 \quad \text{and} \quad F(x) = [-5,1] \times \{0\} \quad \text{and} \quad \phi(x) = x_1^2,$$

where $x = (x_1, 0), y = (y_1, 0)$. Then F is stably ϕ-pseudomonotone with respect to the set $V := \{(0,m) : m \in \mathbb{R}\}$ but not f-quasimonotone. Indeed, we first show that F is ϕ-pseudomonotone on C. If

$$\begin{aligned} 0 \leq \langle x^*, y - x \rangle + \phi(y) - \phi(x) &= x_1^*(y_1 - x_1) + y_1^2 - x_1^2 \\ &= (x_1^* + y_1 + x_1)(y_1 - x_1). \end{aligned}$$

Since $x_1^* \in [-5,1]$ and $x_1, y_1 \in [3,5]$, we get that $x_1^* + y_1 + x_1 > 0$. It implies that $y_1 - x_1 \geq 0$. Thus, we have

$$\langle y^*, y - x \rangle + \phi(y) - \phi(x) = (y_1^* + y_1 + x_1)(y_1 - x_1) \geq 0.$$

Hence F is ϕ-pseudomonotone on C.

Next, we show that $F(\cdot) - \xi$ is ϕ-pseudomonotone for each $\xi = (0, m) \in V$. If

$$0 \leq \langle x^* - \xi, y - x \rangle + \phi(y) - \phi(x) = x_1^*(y_1 - x_1) + y_1^2 - x_1^2 = (x_1^* + y_1 + x_1)(y_1 - x_1).$$

Since $x_1^* \in [-5, 1]$ and $x_1, y_1 \in [3, 5]$, we get that $x_1^* + y_1 + x_1 > 0$. It implies that $y_1 - x_1 \geq 0$. Thus, we have

$$\langle y^* - \xi, y - x \rangle + \phi(y) - \phi(x) = (y_1^* + y_1 + x_1)(y_1 - x_1) \geq 0.$$

Hence F is stably ϕ-pseudomonotone with respect to the set V.

Finally, we show that F is not f-quasimonotone on C. Take $x = (3, 0), y = (4, 0), x^* = (1, 0)$ and $y^* = (-5, 0)$. Then we have

$$\begin{aligned} \langle x^*, y - x \rangle + f(x, y) &= x_1^*(y_1 - x_1) + y_1(y_1 - x_1) \\ &= (x_1^* + y_1)(y_1 - x_1) \\ &= (1 + (4))(4 - 3) > 0, \end{aligned}$$

but

$$\begin{aligned} \langle y^*, y - x \rangle + f(x, y) &= y_1^*(y_1 - x_1) + y_1(y_1 - x_1) \\ &= (y_1^* + y_1)(y_1 - x_1) \\ &= (-5 + 4)(4 - 3) < 0, \end{aligned}$$

Hence F is not f-quasimonotone on C. □

Example 2.2.21. Let $E = \mathbb{R}^2$ and $C = [-1, 2] \times \{0\}$. Let $f : C \times C \to [-\infty, +\infty]$, $F : C \to E^* = \mathbb{R}^2$ and $\phi : C \to \mathbb{R}$ by

$$f(x, y) = y_1^2 - x_1 y_1 \text{ and } F(x) = [\frac{4}{3}, 3] \times \{0\} \text{ and } \phi(x) = x_1^2,$$

where $x=(x_1,0), y=(y_1,0)$. Then F is f-pseudomonotone with respect to the set $V:=\{(0,m): m\in\mathbb{R}\}$ but not ϕ-quasimonotone. Indeed, we first show that F is f-pseudomonotone on C. If

$$\begin{aligned} 0\leq \langle x^*, y-x\rangle + f(x,y) &= x_1^*(y_1-x_1)+y_1(y_1-x_1) \\ &= (x_1^*+y_1)(y_1-x_1). \end{aligned}$$

Since $x_1^*+y_1>0$, we have $y_1-x_1\geq 0$.
Thus

$$\begin{aligned} 0\leq \langle y^*, y-x\rangle + f(x,y) &= y_1^*(y_1-x_1)+y_1(y_1-x_1) \\ &= (y_1^*+y_1)(y_1-x_1). \end{aligned}$$

Hence F is f-pseudomonotone on C.

Next, we show that $F(\cdot)-\xi$ is pseudomonotone for all $\xi\in V$. If

$$\begin{aligned} 0\leq \langle x^*-\xi, y-x\rangle + f(x,y) &= x_1^*(y_1-x_1)+y_1(y_1-x_1) \\ &= (x_1^*+y_1)(y_1-x_1). \end{aligned}$$

Since $x_1^*+y_1>0$, we have $y_1-x_1\geq 0$.
Thus

$$\begin{aligned} 0\leq \langle y^*, y-x\rangle + f(x,y) &= y_1^*(y_1-x_1)+y_1(y_1-x_1) \\ &= (y_1^*+y_1)(y_1-x_1). \end{aligned}$$

Hence F is stably f-pseudomonotone with respect to the set V.

Finally, we show that F is not ϕ-quasimonotone on C. If we take $x=(-1,0), y=(-\frac{1}{2},0), x^*=(3,0)$ and $y^*=(\frac{4}{3},0)$, then we have that

$$\langle x^*, y-x\rangle + \phi(y)-\phi(x) = x_1^*(y_1-x_1)+y_1^2-x_1^2$$

$$
\begin{aligned}
&= x_1^*(y_1 - x_1) + y_1^2 - x_1^2 - x_1y_1 + x_1y_1 \\
&= (x_1^* + x_1 + y_1)(y_1 - x_1) \\
&= (3 - 1 - \frac{1}{2})(-\frac{1}{2} - (-1)) = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} > 0,
\end{aligned}
$$

but

$$
\begin{aligned}
\langle y^*, y - x\rangle + \phi(y) - \phi(x) &= y_1^*(y_1 - x_1) - y_1^2 + x_1^2 \\
&= y_1^*(y_1 - x_1) - y_1^2 + x_1^2 - x_1y_1 + x_1y_1 \\
&= (y_1^* - x_1 - y_1)(y_1 - x_1) < 0 \\
&= (\frac{4}{3} - 1 - \frac{1}{2})(-\frac{1}{2} - (-1)) = -\frac{1}{6} \cdot \frac{1}{2} = -\frac{1}{12} < 0.
\end{aligned}
$$

Hence F is not ϕ-quasimonotone on C. □

2.3 Clarke's generalized derivative

The purpose of this section is to present the basic facts of the theory of generalized differentiation for a locally Lipschitz function.

Definition 2.3.1. [77] A function $g : E \to \mathbb{R}$ is called locally Lipschitz on E if for every $x \in E$ there exists a neighborhood U of x and a constant $L_x \geq 0$, so called Lipschitz constant, such that

$$|g(v) - g(w)| \leq L_x\|v - w\|, \quad \forall v, w \in U.$$

Definition 2.3.2. [77] The Clarke's generalized directional derivative of the locally Lipschitz mapping $g : E \to \mathbb{R}$ at the point $x \in E$ with respect to the direction $v \in E$, denoted by $g^\circ(x; v)$, is defined by

$$g^\circ(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{g(y + tv) - g(y)}{t}$$

We observe that in contrast to the usual directional derivative, the generalized directional derivative g° is always defined.

Definition 2.3.3. [77] Let $\varphi : E \to \mathbb{R}$ be a locally Lipschitz function. The Clarke's generalized gradient of φ at $x \in E$, denoted by $\partial g(x)$, is a subset of a dual space E^* defined as follows

$$\partial g(x) = \{\xi \in E^* : \langle \xi, v\rangle \leq g^\circ(x; v), \forall v \in E\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E^* and E.

Example 2.3.4. *In this case, $E = \mathbb{R}$ and $f(x) = |x|$ (f is Lipschitz by the triangle inequality). If $x > 0$, we calculate*

$$f^\circ(x; v) = \lim_{y \to x,\ t \downarrow 0} \frac{y + tv - y}{t} = v$$

so that $\partial f(x)$, the set of numbers f satisfying $v \geq \xi v$ for all v, reduces to the singleton $\{1\}$. Similarly, $\partial f(x) = \{-1\}$ if $x < 0$. The remaining case is $x = 0$. We find

$$f^\circ(0; v) = \begin{cases} v, & \text{if } v \geq 0 \\ -v, & \text{if } v < 0, \end{cases}$$

that is, $f^\circ(0; v) = |v|$. Thus $\partial f(0)$ consists of those ζ satisfying $|v| \geq \zeta v$ for all v, that is, $\partial f(0) = [-1, 1]$.

The following lemmas provide basic properties of the generalized directional derivative and the generalized gradient.

Lemma 2.3.5. [77, Proposition 2.1.1] Let $g : C \to \mathbb{R}$ be Lipschitz of rank L_x near x. Then

(i) the function $v \mapsto g^\circ(x; v)$ is finite, positively homogeneous, and subadditive on E, and satisfies

$$|g^\circ(x; v)| \leq L_x \|v\|;$$

(ii) $g^\circ(x; v)$ is upper semicontinuous as a function of (x, v) and, as a function of v alone, is Lipschitz of rank L_x on E;

(iii) $g^\circ(x; -v) = (-g)^\circ(x; v)$;

(iv) for every $v \in E$, we have $g^\circ(x; v) = \max\{\langle z, v\rangle,\ z \in \partial g(x)\}$;

(v) for every $x \in E$ the gradient $\partial g(x)$ is nonempty, convex and weakly*-compact subset of E^* which is bounded by the Lipschitz constant L_x of g near x.

Let Ω be a bounded open subset of $\mathbb{R}^N$ and $T : E \to L^p(\Omega; \mathbb{R}^k)$ be a linear compact operator, where $1 < p < \infty$ and $k \geq 1$. We shall denote $\hat{u} = T(u)$ and q by the conjugated exponent of p, i.e., $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Let $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ be a function such that the mapping

$$j(\cdot, y) : \Omega \to \mathbb{R} \text{ is measurable}, \quad \forall y \in \mathbb{R}^k. \tag{j1}$$

We assume that the following conditions holds: either there exists $l \in L^q(\Omega; \mathbb{R})$ such that

$$|j(x, y_1) - j(x, y_2)| \leq l(x)|y_1 - y_2|, \quad \forall x \in \Omega,\ \forall y_1, y_2 \in \mathbb{R}^k, \tag{j2}$$

or

$$\text{the mapping } j(x, \cdot) \text{ is locally Lipschitz}, \ \forall x \in \Omega, \tag{j3}$$

and there exists $C > 0$ such that

$$|z| \leq C(1 + |y|^{p-1}), \quad \forall x \in \Omega, \forall z \in \partial j(x, y). \tag{j4}$$

Lemma 2.3.6. [77, Proposition 2.7.5] If $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$, and j satisfies either (j1) and (j2) or (j1) and (j3)-(j4), then J is uniformly Lipschitz on bounded subsets, and one has

$$\partial J(\varphi) \subset \int_\Omega \partial j(x, \varphi(x))dx.$$

Let $J : L^p(\Omega; \mathbb{R}^k) \to \mathbb{R}$ be an arbitrary locally Lipschitz functional. For each $u \in E$ there exists (see, for example [77]) $z_u \in \partial J(\hat{u})$ such that

$$J^\circ(\hat{u}; \xi) = \langle z_u, \xi\rangle = \max\{\langle w, \xi\rangle, w \in \partial J(\hat{u})\}.$$

Denoting by $T^* : L^q(\Omega;\mathbb{R}^k) \to E^*$ the adjoint operator of T, i.e., $\langle x, T^*y^*\rangle = \langle Tx, y^*\rangle$ for all $x \in E$ and $y^* \in L^q(\Omega;\mathbb{R}^k)$. We define the subset $U(J,T)$ of E^* as follows:

$$U(J,T) = \{-z_u^* : u \in K, z_u^* = T^* z_u\}. \tag{2.3.1}$$

Remark 2.3.7. It is easy to obtain that $J^\circ(\hat{u};\hat{v}-\hat{u}) = \langle z_u^*, v-u\rangle$ for all $u, v \in E$. Indeed, Since T^* is an adjoint operator of T, for any $u, v \in E$, we have

$$\begin{aligned}
J^\circ(\hat{u};\hat{v}-\hat{u}) &= \langle z_u, \hat{v}-\hat{u}\rangle \\
&= \langle z_u, T(v) - T(u)\rangle \\
&= \langle z_u, T(v-u)\rangle \\
&= \langle T^* z_u, v-u\rangle \\
&= \langle z_u^*, v-u\rangle.
\end{aligned}$$

2.4 CAT(0) spaces

In this section, we present the special metric space which has the geometry defined on it. We also introduce the concept of several types of convergence on it.

Definition 2.4.1. [78] A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that

(i) $c(0) = x, c(l) = y$;

(ii) $d(c(t), c(t')) = |t-t'|$ for all $t, t' \in [0,l]$.

In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or metric) *segment* joining x and y.

When it is unique this geodesic segment is denoted by $[x,y]$.

Definition 2.4.2. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$.

Definition 2.4.3. A subset C of a CAT(0) space is convex if $[x,y] \subseteq C$ for all $x, y \in C$.

Definition 2.4.4. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X,d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\overline{\triangle}(x_1, x_2, x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for all $i, j \in 1, 2, 3$ (see Figure 1).

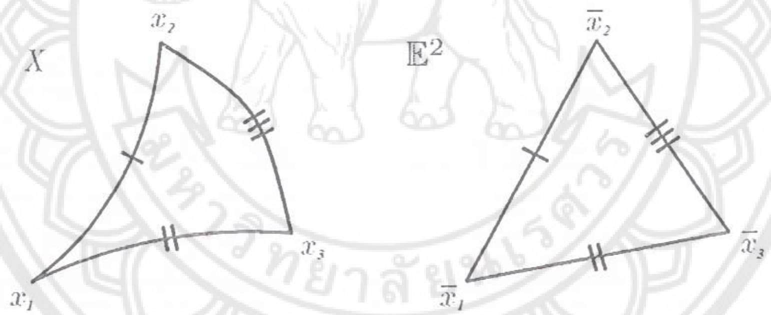

Figure 1 Comparison triangle

Definition 2.4.5. A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.
CAT(0) : Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \triangle$,

$$d(x,y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})$$

(see Figure 2).

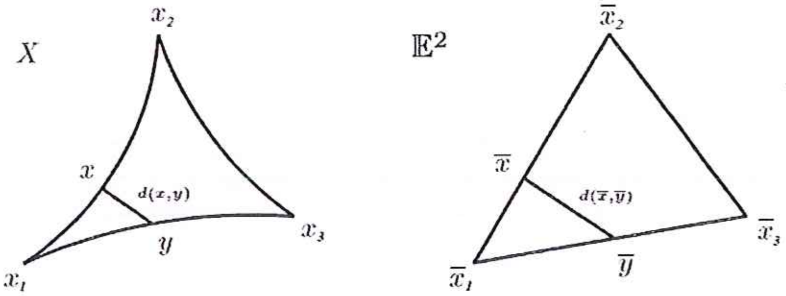

Figure 2 CAT(0) inequality

Definition 2.4.6. Let x, y_1, y_2 be the points in a CAT(0) space and y_0 be the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [79]. In fact (cf. [78], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, R-trees (see [78]), Euclidean buildings (see [80]), the complex Hilbert ball with a hyperbolic metric (see [81]), and many others. Complete CAT(0) spaces are often called Hadamard spaces.

Next, we collect some useful lemmas in CAT(0) spaces.

Lemma 2.4.7. [78, Proposition 2.2] *Let X be a CAT(0) space, $p, q, r, s \in X$ and $\lambda \in [0, 1]$. Then*

$$d(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s) \leq \lambda d(p, r) + (1-\lambda)d(q, s).$$

Lemma 2.4.8. [82, Lemma 2.4] *Let X be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0, 1]$. Then*

$$d(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d(x, z) + (1-\lambda)d(y, z).$$

Lemma 2.4.9. [82, Lemma 2.5] *Let X be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0,1]$. Then*

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x,z) + (1-\lambda)d^2(y,z) - \lambda(1-\lambda)d^2(x,y).$$

We give the concept of Δ-convergence and collect some basic properties. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [83] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

Definition 2.4.10. A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Uniqueness of asymptotic center implies that CAT(0) space X satisfies Opial's property, i.e., for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Since it is not possible to formulate the concept of demiclosedness in a CAT(0) setting, as stated in linear spaces, let us formally say that "$I - T$ is demiclosed at zero" if the conditions, $\{x_n\} \subseteq C$ Δ- converges to x and $d(x_n, Tx_n) \to 0$ imply $x \in F(T)$.

Lemma 2.4.11. [53] *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Lemma 2.4.12. [84] *If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.4.13. [84] *If C is a closed convex subset of X and $T : C \to X$ is a nonexpansive mapping, then the conditions $\{x_n\}$ Δ-convergence to x and $d(x_n, Tx_n) \to 0$, and imply $x \in C$ and $Tx = x$.*

Definition 2.4.14. [85] Let X be a CAT(0) space and $a, b, c, d \in X$. Then *quasilinearization* is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right). \tag{2.4.1}$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d) \tag{2.4.2}$$

for all $a, b, c, d \in X$.

Having the notion of quasilinearization, Kakavandi and Amini [86] introduced the following notion of convergence.

Definition 2.4.15. A sequence $\{x_n\}$ in the complete CAT(0) space (X, d) w-converges to $x \in X$ if $\lim_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0$, *i.e.*, $\lim_{n\to\infty}(d^2(x_n, x) - d^2(x_n, y) + d^2(x, y)) = 0$ for all $y \in X$.

It is obvious that convergence in the metric implies w-convergence, and it is easy to check that w-convergence implies Δ-convergence [86, Proposition 2.5], but it is showed in ([87, Example 4.7]) that the converse is not valid. However the following lemma shows another characterization of Δ-convergence as well as, more explicitly, a relation between w-convergence and Δ-convergence.

Lemma 2.4.16. [87, Theorem 2.6] *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if*

$$\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \le 0 \text{ for all } y \in X.$$

Theorem 2.4.17. [78] *Let X be a complete CAT(0) space, and let C be a nonempty closed convex subset of X. Then,*

(i) *for every $x \in X$, there exists a unique point $y_0 \in C$, denoted by $P_C x$, such that $d(x, y_0) = d(x, C) := \inf_{y\in C} d(x, y)$;*

(ii) *if x' belongs to the geodesic segment $[x, y_0]$, then $P_C x' = P_C x$.*

Theorem 2.4.18. [88] *Let C be a nonempty convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then*

$$u = P_C x \quad \text{if and only if} \quad \langle \overrightarrow{yu}, \overrightarrow{ux}\rangle \ge 0, \quad \text{for all } y \in C.$$