CHAPTER III

ON THE EXISTENCE THEOREMS OF GENERALIZED EQUILIBRIUM PROBLEMS

3.1 Existence theorems of the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term in Banach spaces

In this section, we present the existence of solutions for hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term when C is a bounded(unbounded) subset of E.

Let C be a nonempty, closed and convex subset of a real reflexive Banach space E with its dual E^*, $F : C \rightrightarrows 2^{E^*}$ a multi-valued mapping. Let Ω be a bounded open set in $\mathbb{R}^N$, $T : E \to L^p(\Omega; \mathbb{R}^k)$ a linear continuous mapping, where $1 < p < \infty$, $k \geq 1$ and $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ a function. We shall denote $\hat{u} := Tu$, $j^\circ(x, y; h)$ denotes the Clarke's generalized directional derivative of a locally Lipschitz mapping $j(x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to direction $h \in \mathbb{R}^k$, where $x \in \Omega$. For the bifunction $f : C \times C \to [-\infty, +\infty]$ imposed the condition that the set $\mathcal{D}_1(f) = \{u \in C : \ f(u, v) \neq -\infty, \ \forall v \in C\}$ is nonempty, in this research, we discuss the following *hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term(HVIMN)*:

$$\begin{cases} \text{Find } u \in \mathcal{D}_1(f) \text{ and } u^* \in F(u) \text{ such that} \\ \langle u^*, v - u \rangle + f(u, v) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \\ \forall v \in C. \end{cases} \tag{3.1.1}$$

For proving our results, we first list all conditions of a bifunction f.

A bifunction $f : C \times C \to [-\infty, +\infty]$ is said to satisfy the condition Δ if

(i) $\mathcal{D}_1(f) = \{u \in C : \ f(u,v) \neq -\infty, \ \forall v \in C\}$ is nonempty;

(ii) $f(x,x) = 0$ for all $x \in C$;

(iii) $f(x,y) + f(y,x) \geq 0$ for all $x, y \in C$;

(iv) for all $y \in C$, $f(\cdot, y)$ is weakly upper semicontinuous;

(v) for all $x \in C$, $f(x, \cdot)$ is convex.

Theorem 3.1.1. *Let C be a nonempty, closed, bounded and convex subset of a real reflexive Banach space E. Let $f : C \times C \to [-\infty, +\infty]$ be a bifunction satisfying the condition Δ. Let $J : L^p(\Omega; \mathbb{R}^k) \to \mathbb{R}$ be a function*

$$J(\varphi) = \int_\Omega j(x, \varphi(x))dx,$$

and $T : E \to L^p(\Omega; \mathbb{R}^k)$ be a linear compact operator, where $1 < p < \infty$, $k \geq 1$ and Ω is a bounded open set in $\mathbb{R}^N$. Let $F : C \rightrightarrows E^*$ be a lower hemicontinuous set-valued mapping and stably f-quasimonotone with respect to the set $U(J,T)$ defined in (2.3.1)*. Suppose further that j satisfies either* (j1) *and* (j2) *or* (j1) *and* (j3)–(j4)*. Then $HVIMN$ has a solution.*

Proof. For any $v \in C$ define a multi-valued mapping $G : C \rightrightarrows E$ as follows:

$$G(v) := \{u \in C : \inf_{v^* \in F(v)} \langle v^*, v - u\rangle + f(u,v) + J^\circ(\hat{u}; \hat{v} - \hat{u}) \geq 0\}. \tag{3.1.2}$$

Consider two cases of G: (a) G is not a KKM mapping, and (b) G is a KKM mapping.

Case (a) If G is not a KKM mapping, then there exist $u_i \in C$ and $\lambda_i \in [0,1]$, $i = 1, 2, \ldots, N$, with $\sum_{i=1}^N \lambda_i = 1$ such that $u_0 = \sum_{i=1}^N \lambda_i u_i \notin \bigcup_{i=1}^N G(u_i)$, that is,

$$\inf_{u_i^* \in F(u_i)} \langle u_i^*, u_i - u_0\rangle + f(u_0, u_i) + J^\circ(\hat{u_0}; \hat{u}_i - \hat{u_0}) < 0, \forall i \in \{1, 2, \ldots, N\}. \tag{3.1.3}$$

We claim that there exists a neighborhood U of u_0 such that for all $v \in U \cap C$,

$$\inf_{u_i^* \in F(u_i)} \langle u_i^*, u_i - v\rangle + f(v, u_i) + J^\circ(\hat{v}; \hat{u}_i - \hat{v}) < 0, \forall i \in \{1, 2, \ldots, N\}.$$

If not, for any neighborhood U of u_0, there exist $v_0 \in U \cap C$ and $i_0 \in \{1, 2, \dots, N\}$ such that

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, u_{i_0} - v_0 \rangle + f(v_0, u_{i_0}) + J^\circ(\hat{v}_0; \hat{u}_{i_0} - \hat{v}_0) \geq 0.$$

Putting $U = B(u_0, \frac{1}{n})$, so there exists $v_n \in B(u_0, \frac{1}{n}) \cap C$ such that

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, u_{i_0} - v_n \rangle + f(v_n, u_{i_0}) + J^\circ(\hat{v}_n; \hat{u}_{i_0} - \hat{v}_n) \geq 0.$$

By Lemma 2.3.5 (ii), $v_n \to u_0$ and Δ (iv), we obtain that

$$\inf_{u_{i_0}^* \in F(u_{i_0})} \langle u_{i_0}^*, u_{i_0} - u_0 \rangle + f(u_0, u_{i_0}) + J^\circ(\hat{v}_n; \hat{u}_{i_0} - \hat{u}_0) \geq 0,$$

which is a contradiction with (3.1.3), so we have the claim.

From Remark 2.3.7 and (3.1.3), there exists a neighborhood U of u_0 such that for all $v \in U \cap C$,

$$\begin{aligned} &\inf_{u_i^* \in F(u_i)} \langle u_i^*, u_i - v \rangle + f(v, u_i) + \langle z_v^*, u_i - v \rangle \\ &= \inf_{u_i^* \in F(u_i)} \langle u_i^*, u_i - v \rangle + f(v, u_i) + J^\circ(\hat{v}; \hat{u}_i - \hat{v}) \\ &< 0, \ \forall i \in \{1, 2, \dots, N\}, \end{aligned}$$

which can be rewritten as

$$\inf_{u_i^* \in F(u_i)} \langle u_i^* - (-z_v^*), u_i - v \rangle + f(v, u_i) < 0, \ \forall i \in \{1, 2, \dots, N\}.$$

By the stable f-quasimonotonicity of F with respect to the set $U(J, T)$, we get that

$$\sup_{v^* \in F(v)} \langle v^* - (-z_v^*), u_i - v \rangle + f(v, u_i) \leq 0, \ \forall i \in \{1, 2, \dots, N\},$$

which is equivalent to

$$\sup_{v^* \in F(v)} \langle v^*, u_i - v \rangle + f(v, u_i) + J^\circ(\hat{v}; \hat{u}_i - \hat{v}) \leq 0, \ \forall i \in \{1, 2, \dots, N\}.$$

From (i) of Lemma 2.3.5, Δ (v) and the linearity of T, we have that

$$\sup_{v^* \in F(v)} \langle v^*, u_0 - v \rangle + f(v, u_0) + J^\circ(\hat{v}; \hat{u}_0 - \hat{v})$$

$$
\begin{aligned}
&= \sup_{v^*\in F(v)} \left\langle v^*, \sum_{i=1}^{N}\lambda_i u_i - v \right\rangle + f\left(v, \sum_{i=1}^{N}\lambda_i u_i\right) + J^\circ\left(\hat{v}; \sum_{i=1}^{N}\lambda_i \hat{u}_i - \hat{v}\right) \\
&\leq \sum_{i=1}^{N}\lambda_i \left[\sup_{v^*\in F(v)} \langle v^*, u_i - v\rangle + f(v, u_i) + J^\circ(\hat{v}; \hat{u}_i - \hat{v})\right] \\
&\leq 0. \qquad (3.1.4)
\end{aligned}
$$

By subadditivity of $v \mapsto J^\circ(x; v)$, gives that

$$J^\circ(\hat{v}; \hat{v} - \hat{u}_0) + J^\circ(\hat{v}; \hat{u}_0 - \hat{v}) \geq J^\circ(\hat{v}; 0) = 0.$$

Combining the last inequality with (3.1.4), we obtain that

$$\inf_{v^*\in F(v)} \langle v^*, v - u_0\rangle - f(v, u_0) + J^\circ(\hat{v}; \hat{v} - \hat{u}_0) \geq 0, \ \forall v \in U \cap C. \qquad (3.1.5)$$

It follows from Δ (iii) that

$$\inf_{v^*\in F(v)} \langle v^*, v - u_0\rangle + f(u_0, v) + J^\circ(\hat{v}; \hat{v} - \hat{u}_0) \geq 0, \ \forall v \in U \cap C. \qquad (3.1.6)$$

Let $v' \in C$ be any element and define

$$u_m = \frac{1}{m}v' + (1 - \frac{1}{m})u_0 \text{ for all } m \in \mathbb{N}.$$

Thus, $u_m \to u_0$ as $m \to \infty$ and hence there exists $M \in \mathbb{N}$ such that

$$u_m \in U \cap C \text{ for all } m > M.$$

For any given $u_0^* \in F(u_0)$, since F is lower hemicontinuous, there exists a sequence $\{u_m^*\}$ in $F(u_m)$ converging weakly star to u_0^*. It follows from (3.1.6) that for any $m > M$,

$$\langle u_m^*, u_m - u_0\rangle + f(u_0, u_m) + J^\circ(\hat{u}_m; \hat{u}_m - \hat{u}_0) \geq 0.$$

By (i) of Lemma 2.3.5, linearity of T, (A1) and (A4), we have that

$$0 \ \leq \ \left\langle u_m^*, \frac{1}{m}(v' - u_0)\right\rangle + f\left(u_0, u_0 + \frac{1}{m}(v' - u_0)\right) + J^\circ\left(\hat{u}_m; \frac{1}{m}(\hat{v'} - \hat{u}_0)\right)$$

$$\leq \frac{1}{m}\left[\langle u_m^*, v' - u_0\rangle + f(u_0, v') + J^\circ\left(\hat{u}_m; \hat{v'} - \hat{u}_0\right)\right].$$

Multiplying the last inequality by m and letting $m \to \infty$, from (ii) of Lemma 2.3.5, we obtain that

$$\langle u_0^*, v' - u_0\rangle + f(u_0, v') + J^\circ\left(\hat{u}_0; \hat{v'} - \hat{u}_0\right) \geq 0, \ \forall v' \in C. \tag{3.1.7}$$

Since $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$ and j satisfies either (j1) and (j2) or (j1) and (j3)-(j4), by Lemma 2.3.6, we get that

$$\int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq J^\circ(\hat{u}; \hat{v} - \hat{u}), \ \forall u, v \in C.$$

It follows from (3.1.7) that

$$\langle u_0^*, v' - u_0\rangle + f(u_0, v') + \int_\Omega j^\circ(x, \hat{u}_0(x); \hat{v'}(x) - \hat{u}_0(x))dx \geq 0, \forall v' \in C.$$

Hence $HVIMN$ has a solution.

Case (b) If G is a KKM mapping. We consider the following mapping:

$$u \mapsto \inf_{v^* \in F(v)} \langle v^*, v - u\rangle + f(u, v) + J^\circ(\hat{u}; \hat{v} - \hat{u}). \tag{3.1.8}$$

We claim that the above mapping is weakly upper semicontinuous. For given a sequence $\{\mu_n\} \subset C$ such that $\mu_n \rightharpoonup \mu_0$, it follows from linearity and compactness of T that $T\mu_n \to T\mu_0$; that is $\hat{\mu_n} \to \hat{\mu_0}$ as $n \to \infty$. Since $f(\cdot, y)$ is weakly upper semicontinuous for all $y \in C$ and Lemma 2.3.5 (ii), we have that

$$\begin{aligned}
&\limsup_{n\to\infty}\left[\inf_{v^*\in F(v)}\langle v^*, v - \mu_n\rangle + f(\mu_n, v) + J^\circ(\hat{\mu_n}; \hat{v} - \hat{\mu_n})\right] \\
&\quad \leq \limsup_{n\to\infty}\left(\inf_{v^*\in F(v)}\langle v^*, v - \mu_n\rangle\right) + \limsup_{n\to\infty} f(\mu_n, v) + \limsup_{n\to\infty} J^\circ(\hat{\mu_n}; \hat{v} - \hat{\mu_n}) \\
&\quad \leq \inf_{v^*\in F(v)}\langle v^*, v - \mu_0\rangle + f(\mu_0, v) + J^\circ(\hat{\mu_0}; \hat{v} - \hat{\mu_0}).
\end{aligned}$$

Thus we have the claim. Then $G(v)$ is weakly closed. It follows from the convexity boundedness and closedness of a subset C in a reflexive Banach space E, we have that C is weakly compact. Since $G(v) \subset C$, we get that $G(v)$ is weakly compact for each $v \in C$. Thus, all conditions of Lemma 2.2.3 are satisfied in the weak topology and so we obtain that $\bigcap_{v\in C} G(v) \neq \emptyset$. Taking $u_0 \in \bigcap_{v\in C} G(v)$, we have

$$\inf_{v^*\in F(v)} \langle v^*, v - u_0\rangle + f(u_0, v) + J^\circ(\hat{u}_0; \hat{v} - \hat{u}_0) \geq 0, \ \forall v \in C. \tag{3.1.9}$$

Let $v_0 \in C$ be arbitrarily fixed and define $u_n = u_0 + \frac{1}{n}(v_0 - u_0)$. Then $u_n \in C$ for all $n \geq 1$. For any $u_0^* \in F(u_0)$, since F is lower hemicontinuous, there exists a sequence $\{u_n^*\}$ in $F(u_n)$ converging weakly star to u_0^*. From (3.1.9), for any $n \geq 1$, we have

$$\langle u_n^*, u_n - u_0\rangle + f(u_0, u_n) + J^\circ(\hat{u}_0; \hat{u}_n - \hat{u}_0) \geq 0.$$

In view of the linearity of T, Δ (v) and Lemma 2.3.5 (i), we have

$$\begin{aligned} 0 &\leq \langle u_n^*, \frac{1}{n}(v_0 - u_0)\rangle + f\left(u_0, u_0 + \frac{1}{n}(v_0 - u_0)\right) + J^\circ\left(\hat{u}_0; \frac{1}{n}(\hat{v}_0 - \hat{u}_0)\right) \\ &\leq \frac{1}{n}\left[\langle u_n^*, v_0 - u_0\rangle + f(u_0, v_0) + J^\circ(\hat{u}_0; \hat{v}_0 - \hat{u}_0)\right]. \end{aligned}$$

Multiplying the inequality above by n and letting $n \to \infty$, we obtain that

$$\langle u_0^*, v_0 - u_0\rangle + f(u_0, v_0) + J^\circ(\hat{u}_0; \hat{u}_0 - \hat{u}_0) \geq 0, \ \forall v_0 \in C. \tag{3.1.10}$$

Since $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$ and j satisfies either (j1) and (j2) or (j1) and (j3)–(j4), by Lemma 2.3.6, we get that

$$\int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq J^\circ(\hat{u}; \hat{v} - \hat{u}), \ \forall u, v \in C.$$

This together with (3.1.10), we get

$$\langle u_0^*, v_0 - u_0\rangle + f(u_0, v_0) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \ \forall v_0 \in C.$$

Hence u_0 is a solution of $HVIMN$. This completes the proof. □

Remark 3.1.2. Theorem 3.1.1 generalizes and improves some recent results. In fact,

(i) If $f(u,v) = \phi(v) - \phi(u)$, where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function such that $C_\phi = C \cap \mathrm{dom}\phi \neq \emptyset$, then $\mathcal{D}_1(f) = C_\phi$, then Theorem 3.1.1 reduces to Theorem 4.1 of Tang and Huang [38];

(ii) If $f \equiv 0$, then Theorem 3.1.1 reduces to Theorem 3.1 of Zhang and He [37];

(iii) If $f \equiv 0$ and the mapping F in Theorem 3.1.1 is single-valued, then Theorem 3.1.1 extends Corollary 1 of Costea and Rădulescu [25] by relaxing the stably pseudomonotonicity of F in Corollary 1 of [25] to F being stably quasimonotone;

(iv) Theorem 3.1.1 generalizes and improves Theorem 2 of Motreanu and Rădulescu [30] by extending F from the single-valued case to the set-valued one and by relaxing the monotonicity of F in Theorem 2 of [30] to F being stably f-quasimonotone;

(v) If $f \equiv 0$ and the mapping F is single-valued, then Theorem 3.1.1 generalizes and improves Theorem 2 of Panagiotopolos et al. [89] be relaxing the monotonicity of F in [89] to F being stably quasimonotone and by extending F from the single-valued case to the set-valued one.

Next, we omit the boundedness of C in Theorem 3.1.1, we need to introduce the concept of f-coercivity.

Proposition 3.1.3. *Consider the following f-coercivity conditions:*

(A) *There exists a nonempty subset V_0 contained in a weakly compact subset V_1 of C such that the set*

$$D = \{u \in C : \inf_{v^* \in F(v)} \langle v^*, v-u\rangle + f(u,v) + J^\circ(\hat{u}; \hat{v} - \hat{u}) \geq 0, \ \forall v \in V_0\}$$

is weakly compact or empty.

(B) *There exists* $n_0 \in \mathbb{N}$ *such that for every* $u \in C \backslash B_{n_0}$*, there exists some* $v \in C$ *with* $\|v\| < \|u\|$ *such that*

$$\sup_{u^* \in F(u)} \langle u^*, v-u \rangle + f(u,v) + J^\circ(\hat{u}; \hat{v} - \hat{u}) \leq 0.$$

(C) *There exists* $n_0 \in \mathbb{N}$ *such that for every* $u \in C \backslash B_{n_0}$*, there exists some* $v \in C$ *with* $\|v\| < \|u\|$ *such that*

$$\sup_{u^* \in F(u)} \langle u^*, v-u \rangle + f(u,v) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx < 0.$$

Then we have

(i) (A) $\Rightarrow$ (B), *if* F *is stably* f*-quasimonotone with respect to the set* $U(J,T)$.

(ii) (C) $\Rightarrow$ (B), *if* $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$, j *satisfies either* (j1) *and* (j2) *or* (j1) *and* (j3)-(j4).

Proof. (i) If $D = \emptyset$, since V_0 is nonempty and contained in a weakly compact subset V_1 of C, then there exists a natural number $M < \infty$ such that $\|z\| < M$ for all $z \in V_0$. Taking $n_0 = M$, we obtain that for every $u \in C \backslash B_{n_0}$, there exists $v \in V_0 \neq \emptyset$ such that $v \in B_{n_0}$ and

$$\inf_{v^* \in F(v)} \langle v^*, v-u \rangle + f(u,v) + J^\circ(\hat{u}; \hat{v} - \hat{u}) < 0. \tag{3.1.11}$$

If $D \neq \emptyset$, then D is weakly compact. Since $D \cup V_0 \subset D \cup V_1$, which is a weakly compact subset, we conclude that there exists a natural number $M < \infty$ such that $\|z\| < M$ for all $z \in D \cup V_0$. Taking $n_0 = M$, for every $u \in C \backslash B_{n_0}$ and (3.1.11) holds. Hence, from the proves of both case we can conclude that there exists $n_0 \in \mathbb{N}$ such that, for any $u \in C \backslash B_{n_0}$, there exists $v \in B_{n_0}$ such that (3.1.11) holds. Therefore,

$$\begin{aligned} 0 &> \inf_{v^* \in F(v)} \langle v^*, v-u \rangle + f(u,v) + \langle z_u^*, v-u \rangle \\ &= \inf_{v^* \in F(v)} \langle v^*, v-u \rangle + f(u,v) + J^\circ(\hat{u}; \hat{v} - \hat{u}), \end{aligned}$$

and hence

$$\sup_{v^* \in F(v)} \langle v^* - (-z_u^*), u - v\rangle + f(u,v) > 0.$$

Since F is stably f-quasimonotonicity with respect to $U(J,T)$, we have

$$\inf_{u^* \in F(u)} \langle u^* - (-z_u^*), u - v\rangle + f(u,v) \geq 0,$$

and then

$$\sup_{u^* \in F(u)} \langle u^*, v - u\rangle + f(u,v) + J^\circ(\hat{u}; \hat{v} - \hat{u}) \leq 0.$$

This complete the proof.

(ii) By Lemma 2.3.6, we get that

$$\int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq J^\circ(\hat{u}; \hat{v} - \hat{u}), \ \forall u, v \in E.$$

Combining with (C), we get (B). □

Remark 3.1.4. (1) If $f(u,v) = \phi(v) - \phi(u)$, where $\phi : C \to \mathbb{R}$, then the f-coercivity conditions of Proposition 3.1.3 reduce to ϕ-coercivity conditions of Proposition 4.1 in [38].

(2) If $f(u,v) \equiv 0$, then the f-coercivity conditions of Proposition 3.1.3 reduce to coercivity conditions of Proposition 3.1 in [37].

Next, we present the main theorem for the unbounded constrained set.

Theorem 3.1.5. *Let C be a nonempty, closed, unbounded and convex subset of a real reflexive Banach space E. Let $f : C \times C \to [-\infty, +\infty]$ be a bifunction satisfying the condition Δ. Let $J : L^p(\Omega; \mathbb{R}^k) \to \mathbb{R}$ be a function*

$$J(\varphi) = \int_\Omega j(x, \varphi(x))dx,$$

and $T : E \to L^p(\Omega; \mathbb{R}^k)$ be a linear compact operator, where $1 < p < \infty, k \geq 1$ and Ω is a bounded open set in $\mathbb{R}^N$. Let $F : C \rightrightarrows E^*$ be a lower hemicontinuous set-valued mapping and stably f-quasimonotone with respect to the set $U(J,T)$ defined in (2.3.1). *Suppose further that j satisfies either* (j1) *and* (j2) *or* (j1) *and* (j3)-(j4). *If the condition* (B) *holds, then $HVIMN$ has a solution.*

Proof. Take $m > n_0$. Since B_m is bounded and convex, from (3.1.7) or (3.1.10) in Theorem 3.1.1, we can conclude that there exists $u_m \in B_m \cap C$ and $u_m^* \in F(u_m)$ such that

$$\langle u_m^*, v - u_m \rangle + f(u_m, v) + J^\circ(\hat{u_m}; \hat{v} - \hat{u_m}) \geq 0, \ \forall v \in B_m \cap C. \tag{3.1.12}$$

We consider two cases.

(a) If $\|u_m\| = m$, then $\|u_m\| > n_0$. Since the condition (B) holds, there is some $v_0 \in K$ with $\|v_0\| < \|u_m\|$ such that

$$\langle u_m^*, v_0 - u_m \rangle + f(u_m, v_0) + J^\circ(\hat{u_m}; \hat{v}_0 - \hat{u_m}) \leq 0. \tag{3.1.13}$$

Let $v \in C$. Since $\|v_0\| < \|u_m\| = m$, there is $t \in (0,1)$ such that $v_t = v_0 + t(v - v_0) \in B_m \cap C$. Notice that T is a linear mapping and $f(x, \cdot)$ is a convex function. By (3.1.12), (3.1.13) and (i) of Lemma 2.3.5, It follows that

$$\begin{aligned}
0 &\leq \langle u_m^*, v_t - u_m \rangle + f(u_m, v_t) + J^\circ(\hat{u_m}; \hat{v}_t - \hat{u_m}) \\
&= \langle u_m^*, (1-t)v_0 + tv - u_m \rangle + f(u_m, (1-t)v_0 + tv) \\
&\quad + J^\circ(\hat{u_m}; (1-t)\hat{v}_0 + t\hat{v} - \hat{u_m}) \\
&\leq (1-t)\left[\langle u_m^*, v_0 - u_m \rangle + f(u_m, v_0) + J^\circ(\hat{u_m}; \hat{v}_0 - \hat{u_m})\right] \\
&\quad + t\left[\langle u_m^*, v - u_m \rangle + f(u_m, v) + J^\circ(\hat{u_m}; \hat{v} - \hat{u_m})\right] \\
&\leq t\left[\langle u_m^*, v - u_m \rangle + f(u_m, v) + J^\circ(\hat{u_m}; \hat{v} - \hat{u_m})\right], \ \forall v \in C.
\end{aligned} \tag{3.1.14}$$

Dividing by t, we have that

$$\langle u_m^*, v - u_m \rangle + f(u_m, v) + J^\circ(\hat{u_m}; \hat{v} - \hat{u_m}) \geq 0, \ \forall v \in C. \tag{3.1.15}$$

(b) If $\|u_m\| < m$, then for any $v \in C$, there is $t \in (0,1)$ such that $v_t = n_m + t(v - v_m) \in B_m \cap C$. Notice that T is a linear mapping and $f(x, \cdot)$ is a convex function. By (3.1.12) and (i) of Lemma 2.3.5, It follows that

$$0 \leq \langle u_m^*, v_t - u_m \rangle + f(u_m, v_t) + J^\circ(\hat{u_m}; \hat{v}_t - \hat{u_m})$$

$$\leq \; t\left[\langle u_m^*, v - u_m\rangle + f(u_m, v) + J^\circ(\hat{u_m}; \hat{v} - \hat{u_m})\right], \; \forall v \in C.$$

Dividing by t, we have that (3.1.15) holds.

Since $J(\varphi) = \int_\Omega j(x, \varphi(x))dx$ and j satisfies either (j1) and (j2) or (j1) and (j3)-(j4), by Lemma 2.3.6, we have that

$$\int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq J^\circ(\hat{u}; \hat{v} - \hat{u}), \; \forall u, v \in E,$$

and so

$$\begin{aligned} 0 \;&\leq\; \langle u_m^*, v - u_m\rangle + f(u_m, v) + J^\circ(\hat{u_m}; \hat{v} - \hat{u_m}) \\ &\leq\; \langle u_m^*, v - u_m\rangle + f(u_m, v) + \int_\Omega j^\circ(x, \hat{u_m}(x); \hat{v}(x) - \hat{u_m}(x))dx, \; \forall v \in C. \end{aligned}$$

This shows that $HVIMN$ has a solution. □

Remark 3.1.6. (1) If $f(u, v) = \phi(v) - \phi(u)$, where $\phi : C \to \mathbb{R}$, then Theorem 3.1.5 reduces to Theorem 4.2 of [38].

(2) If $f \equiv 0$, then Theorem 3.1.5 reduces to Theorem 3.2 of [37]. Theorem 3.1.5 also generalizes and improves Theorem 2 of [25] by extending F from single-valued case to set-valued one and relaxing the corresponding coercivity condition and stable pseudomonotonicity of the operator in [25].

If the constraint set C is bounded, then the solution set of $HVIMN$ is obviously bounded. In the case of constraint set C is unbounded, the solution set of $HVIMN$ may be unbounded. In the sequel, we provide a sufficient condition to the boundedness of the solution set of $HVIMN$, when C is unbounded. The following theorem also generalizes Theorem 4.3 of [38].

Theorem 3.1.7. *Let C be a nonempty, closed, unbounded and convex subset of a real reflexive Banach space E. Let $f : C \times C \to [-\infty, +\infty]$ be a bifunction satisfying the condition Δ. Let $J : L^p(\Omega; \mathbb{R}^k) \to \mathbb{R}$ be a function*

$$J(\varphi) = \int_\Omega j(x, \varphi(x))dx,$$

and $T : E \to L^p(\Omega;\mathbb{R}^k)$ be a linear compact operator, where $1 < p < \infty, k \geq 1$ and Ω is a bounded open set in $\mathbb{R}^N$. Let $F : C \rightrightarrows E^*$ be a lower hemicontinuous set-valued mapping and stably f-quasimonotone with respect to the set $U(J,T)$ defined in (2.3.1). *Suppose further that j satisfies either* (j1) *and* (j2) *or* (j1) *and* (j3)-(j4). *If the condition* (C) *holds, then HVIMN is nonempty and bounded.*

Proof. From Proprsition 3.1.3, we have (C) $\Rightarrow$ (B). By Theorem 3.1.5, we know that the solution set of the problem (3.1.1) is nonempty. If the solution set is unbounded, then there exist $u_0 \in C$ and $u_0^* \in F(u_0)$ such that $\|u_0\| > n_0$ and

$$\langle u_0^*, v - u_0\rangle + f(u_0, v) + \int_\Omega j^\circ(x, \hat{u_0}(x); \hat{v}(x) - \hat{u_0}(x))dx \geq 0, \ \forall v \in C. \quad (3.1.16)$$

Since $\|u_0\| > n_0$, it follows from condition (C) that, there exists $v_0 \in C$ with $\|v_0\| < \|u_0\|$ such that

$$\sup_{u^* \in F(u_0)} \langle u^*, v_0 - u_0\rangle + f(u_0, v_0) + \int_\Omega j^\circ(x, \hat{u_0}(x); \hat{v_0}(x) - \hat{u_0}(x))dx < 0,$$

which is a contradiction with (3.1.16). This complete the proof. □

Remark 3.1.8. (1) If $f(u,v) = \phi(v) - \phi(u)$, where $\phi : C \to \mathbb{R}$, then Theorem 3.1.7 reduces to Theorem 4.3 of [38].

(2) If $f \equiv 0$, then Theorem 3.1.7 reduces to Theorem 3.3 of [37].

3.2 New generalized mixed equilibrium problem with respect to relaxed semi-monotone mappings in Banach Spaces

In this section, we introduce the new generalized mixed equilibrium problem with respect to relaxed semi-monotone mappings. Using the KKM technique, we obtain the existence of solutions for the generalized mixed equilibrium problem in Banach spaces. Furthermore, we also introduce a hybrid projection algorithm for finding a common element in the solution set of a generalized mixed equilibrium problem and the fixed point set of an asymptotically nonexpansive mapping. The strong convergence theorem of the proposed sequence is obtained in a Banach space setting.

Let $f : C \times C \to \mathbb{R}$ be a bifunction, $\eta : C \times C \to E^{**}$ a mapping, $\xi : E^{**} \to \mathbb{R}$, $\varphi : C \to R$ two real-valued functions and let $A : C \times C \to E^*$ be a η-ξ semi-monotone mapping. We consider the problem of finding $u \in C$ such that

$$f(u,v) + \langle A(u,u), \eta(v,u)\rangle + \varphi(v) \geq \varphi(u), \quad \forall v \in C, \tag{3.2.1}$$

which is called *the generalized mixed equilibrium problem with respect to relaxed η-ξ semi-monotone mapping* (GMEP(f, A, η, φ)). The set of such $u \in C$ is denoted by $GMEP(f, A, \eta, \varphi)$, i.e.

$$GMEP(f, A, \eta, \varphi) = \{u \in C : f(u,v) + \langle A(u,u), \eta(v,u)\rangle + \varphi(v) \geq \varphi(u), \quad \forall v \in C\}.$$

3.2.1 Existence Theorem

In the first part, we obtain the existence theorem for (GMEP(f, A, η, φ)).

For solving the mixed equilibrium problem, let us assume the following conditions for a bifunction $f : C \times C \to \mathbb{R}$:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e. $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) for all $y \in C$, $f(\cdot, y)$ is weakly upper semicontinuous;

(A4) for all $x \in C$, $f(x, \cdot)$ is convex.

The following lemmas can be found in [9].

Lemma 3.2.1. [9] *Let C be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to E^*$ be an η-hemicontinuous and relaxed η-ξ monotone mapping. Let f be a bifunction from*

$C \times C$ to $\mathbb{R}$ satisfying (A1) *and* (A4) *and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. Let $r > 0$ and $z \in C$. Assume that*

(i) $\eta(x, x) = 0, \quad \forall x \in C.$

(ii) *for any fixed $u, v \in C$, the mapping $x \mapsto \langle Tv, \eta(x, u)\rangle$ is convex.*

Then the following problems (3.2.2) and (3.2.3) are equivalent:

Find $x \in C$ such that

$$f(x,y) + \varphi(y) + \langle Tx, \eta(y,x)\rangle + \frac{1}{r}\langle y - x, J(x - z)\rangle \geq \varphi(x), \forall y \in C; \tag{3.2.2}$$

and

Find $x \in C$ such that

$$f(x,y) + \varphi(y) + \langle Ty, \eta(y,x)\rangle + \frac{1}{r}\langle y - x, J(x - z)\rangle \geq \varphi(x) + \xi(y - x), \forall y \in C. \tag{3.2.3}$$

Lemma 3.2.2. [9] *Let C be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to E^*$ be an η-hemicontinuous and relaxed η-ξ monotone mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1), (A3) *and* (A4) *and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. Let $r > 0$ and $z \in C$. Assume that*

(i) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$;*

(ii) *for any fixed $u, v \in C$, the mapping $x \mapsto \langle Tv, \eta(x, u)\rangle$ is convex and lower semicontinuous;*

(iii) *$\xi : E \to \mathbb{R}$ is weakly lower semicontinuous; that is, for any net $\{x_\beta\}$, $\{x_\beta\}$ converges to x in $\sigma(E, E^*)$ implies that $\xi(x) \leq \liminf_\beta \xi(x_\beta)$.*

Then, the solution set of the problem (3.2.2) is nonempty; that is, there exists $x_0 \in C$ *such that*

$$f(x_0, y) + \langle Tx_0, \eta(y, x_0)\rangle + \varphi(y) + \frac{1}{r}\langle y - x_0, J(x_0 - z)\rangle \geq \varphi(x_0), \quad \forall y \in C.$$

we prove the following crucial lemma concerning the generalized mixed equilibrium problem with respect to relaxed η–ξ semi-monotone mapping (GMEP(f, A, η, φ)) in a real Banach space with the smooth and strictly convex second dual space.

Lemma 3.2.3. *Let E be a real Banach space with the smooth and strictly convex second dual space E^{**}, let C be a nonempty bounded closed convex subset of E^{**}, let $A : C \times C \to E^*$ be a relaxed η–ξ semi-monotone mapping. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1), (A3) *and* (A4) *and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $r > 0$ and $z \in C$. Assume that*

(i) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$;*

(ii) *for any fixed $u, v, w \in C$, the mapping $x \mapsto \langle A(v, w), \eta(x, u)\rangle$ is convex and lower semicontinuous;*

(iii) *for each $x \in C, A(x, \cdot) : C \to E^*$ is finite-dimensional continuous: that is, for any finite-dimensional subspace $F \subset E^{**}, A(x, \cdot) : C \cap F \to E^*$ is continuous;*

(iv) *$\xi : E^{**} \to \mathbb{R}$ is convex lower semicontinuous.*

Then there exists $u_0 \in C$ such that

$$f(u_0, v) + \langle A(u_0, u_0), \eta(v, u_0)\rangle + \varphi(v) + \frac{1}{r}\langle v - u_0, J(u_0 - z)\rangle \geq \varphi(u_0), \quad \forall v \in C. \tag{3.2.4}$$

Proof. Let $F \subseteq E^{**}$ be a finite-dimensional subspace with $C_F := F \cap C \neq \emptyset$. For each $w \in C$, consider the following problem: Find $u_0 \in C_F$ such that

$$f(u_0, v) + \langle A(w, u_0), \eta(v, u_0)\rangle + \varphi(v) + \frac{1}{r}\langle v - u_0, J(u_0 - z)\rangle - \varphi(u_0) \geq 0, \ \forall v \in C_F. \tag{3.2.5}$$

Since $C_F \subseteq F$ is bounded closed and convex, $A(w, \cdot)$ is continuous on C_F and relaxed η–ξ monotone for each fixed $w \in C$, from Lemma 3.2.2, we know that problem (3.2.5) has a solution $u_0 \in C_F$.

Now, define a set-valued mapping $G : C_F \to 2^{C_F}$ as follows:

$$\begin{aligned} Gw = \{u \in C_F : & f(u, v) + \langle A(w, u), \eta(v, u)\rangle + \varphi(v) \\ & + \frac{1}{r}\langle v - u, J(u - z)\rangle - \varphi(u) \geq 0, \quad \forall v \in C_F\}. \end{aligned}$$

It follows from Lemma 3.2.1 that, for each fixed $w \in C_F$,

$$\begin{aligned} &\{u \in C_F : f(u, v) + \langle A(w, u), \eta(v, u)\rangle + \varphi(v) \\ &\qquad + \frac{1}{r}\langle v - u, J(u - z)\rangle - \varphi(u) \geq 0, \forall v \in C_F\} \\ &= \{u \in C_F : f(u, v) + \langle A(w, v), \eta(v, u)\rangle + \varphi(v) \\ &\qquad + \frac{1}{r}\langle v - u, J(u - z)\rangle - \varphi(u) \geq \xi(v - u), \forall v \in C_F\}. \end{aligned}$$

Since every convex lower semicontinuous function in Banach spaces is weakly lower semicontinuous, the proper convex lower semicontinuity of φ and ξ, condition (ii), (A3) and (A4) imply that $G : C_F \to 2^{C_F}$ has nonempty bounded closed and convex values. Using (A3) and the complete continuity of $A(\cdot, u)$, we can conclude that G is upper semicontinuous. It follows from Theorem 2.2.4 that G has a fixed point $w^* \in C_F$, i.e.,

$$\begin{aligned} & f(w^*, v) + \langle A(w^*, w^*), \eta(v, w^*)\rangle + \varphi(v) \\ & \quad + \frac{1}{r}\langle v - w^*, J(w^* - z)\rangle - \varphi(w^*) \geq 0, \quad \forall v \in C_F. \end{aligned} \tag{3.2.6}$$

Let

$$\mathcal{F} = \{F \subset E^{**} : F \text{ is finite dimensional with } F \cap C \neq \emptyset\}$$

and let

$$W_F = \{u \in C : f(u,v) + \langle A(u,v), \eta(v,u)\rangle + \varphi(v) + \frac{1}{r}\langle v-u, J(u-z)\rangle - \varphi(u) \geq \xi(v-u), \forall v \in C_F\}, \quad \forall F \in \mathcal{F}.$$

By (3.2.6) and Lemma 3.2.1, we know that W_F is nonempty and bounded. Denote by $\overline{W}_F^*$ the weak*-closure of W_F in E^{**}. Then, $\overline{W}_F^*$ is weak* compact in E^{**}.

For any $F_i \in \mathcal{F}$, $i = 1, 2, \ldots, N$, we know that $W_{\bigcap_{i=1}^N F_i} \subset \bigcap_{i=1}^N W_{F_i}$, so $\{\overline{W}_F^* : F \in \mathcal{F}\}$ has the finite intersection property. Therefore, it follows that

$$\bigcap_{F\in\mathcal{F}} \overline{W}_F^* \neq \emptyset.$$

Let $u_0 \in \bigcap_{F\in\mathcal{F}} \overline{W}_F^*$. We claim that

$$f(u_0,v) + \langle A(u_0,u_0), \eta(v,u_0)\rangle + \varphi(v) + \frac{1}{r}\langle v-u_0, J(u_0-z)\rangle - \varphi(u_0) \geq 0, \quad \forall v \in C.$$

Indeed, for each $v \in C$, let $F \in \mathcal{F}$ be such that $v \in C_F$ and $u_0 \in C_F$. Then, there exist $u_j \in W_F$ such that $u_j \rightharpoonup u_0$. The definition of W_F implies that

$$f(u_j,v) + \langle A(u_j,v), \eta(v,u_j)\rangle + \varphi(v) + \frac{1}{r}\langle v-u_j, J(u_j-z)\rangle - \varphi(u_j) \geq \xi(v-u_j),$$

that is

$$f(u_j,v) + \langle A(u_j,v), \eta(v,u_j)\rangle + \varphi(v) + \frac{1}{r}\langle v-z, J(u_j-z)\rangle - \frac{1}{r}\|z-u_j\|^2 - \varphi(u_j) \geq \xi(v-u_j),$$

for all $j = 1, 2, \ldots$. Using the complete continuity of $A(\cdot,u)$, (A3), (ii), the continuity of J, the convex lower semicontinuity of φ, ξ and $\|\cdot\|^2$ and letting $j \to \infty$, we get

$$f(u_0,v) + \langle A(u_0,v), \eta(v,u_0)\rangle + \varphi(v) + \frac{1}{r}\langle v-u_0, J(u_0-z)\rangle - \varphi(u_0) \geq \xi(v-u_0), \ \forall v \in C.$$

From Lemma 3.2.1, we have

$$f(u_0,v) + \langle A(u_0,u_0), \eta(v,u_0)\rangle + \varphi(v) + \frac{1}{r}\langle v-u_0, J(u_0-z)\rangle - \varphi(u_0) \geq 0, \ \forall v \in C.$$

Hence, we complete the proof. □

Setting $A \equiv 0$ and $\varphi \equiv 0$ in Lemma 3.2.3, we have the following result.

Corollary 3.2.4. *Let E be a real Banach space with the smooth and strictly convex second dual space E^{**}, let C be a nonempty bounded closed convex subset of E^{**}. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1), (A3) *and* (A4). *Let $r > 0$ and $z \in C$. Then there exists $u_0 \in C$ such that*

$$f(u_0, v) + \frac{1}{r}\langle v - u_0, J(u_0 - z)\rangle \geq 0, \quad \forall v \in C.$$

If E is reflexive (i.e. $E \equiv E^{**}$) smooth and strictly convex real Banach space, then we have the following result.

Corollary 3.2.5. *Let E be a reflexive smooth and strictly convex Banach space, let C be a nonempty bounded closed convex subset of E, let $A : C \times C \to E^*$ be a relaxed η-ξ semi-monotone mapping. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1), (A3) *and* (A4) *and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let $r > 0$ and $z \in C$. Assume that*

(i) *$\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in C$;*

(ii) *for any fixed $u, v, w \in C$, the mapping $x \mapsto \langle A(v, w), \eta(x, u)\rangle$ is convex and lower semicontinuous;*

(iii) *for each $x \in C, A(x, \cdot) : C \to E^*$ is finite-dimensional continuous.*

(iv) *$\xi : E \to \mathbb{R}$ is convex lower semicontinuous.*

Then there exists $u_0 \in C$ such that

$$f(u_0, v) + \langle A(u_0, u_0), \eta(v, u_0)\rangle + \varphi(v) + \frac{1}{r}\langle v - u_0, J(u_0 - z)\rangle \geq \varphi(u_0), \quad \forall v \in C.$$

If E is reflexive (i.e. $E \equiv E^{**}$) smooth and strictly convex, A is semi-monotone, then we obtain the following result.

Corollary 3.2.6. *Let E be a reflexive smooth and strictly convex Banach space, let C be a nonempty bounded closed convex subset of E, let $A : C \times C \to E^*$ be a semi-monotone mapping. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1), (A3) *and* (A4) *and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that, for any $r > 0$ and $z \in C$,*

(i) *for any fixed $u, v, w \in C$, the mapping $x \mapsto \langle A(v,w), x - u\rangle$ is convex and lower semicontinuous;*

(ii) *for each $x \in C, A(x,\cdot) : C \to E^*$ is finite-dimensional continuous.*

Then there exists $u_0 \in C$ such that

$$f(u_0, v) + \langle A(u_0, u_0), v - u_0\rangle + \varphi(v) + \frac{1}{r}\langle v - u_0, J(u_0 - z)\rangle \geq \varphi(u_0), \quad \forall v \in C.$$

Theorem 3.2.7. *Let E be a real Banach space with the smooth and strictly convex second dual space E^{**}, let C be a nonempty, bounded, closed and convex subset of E^{**}, let $A : C \times C \to E^*$ be a relaxed η–ξ semi-monotone mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. For any $r > 0$, define a mapping $\Phi_r : E^{**} \to C$ as follows:*

$$\Phi_r(x) = \Big\{u \in C : f(u,v) + \langle A(u,u), \eta(v,u)\rangle + \varphi(v) + \frac{1}{r}\langle v - u, J(u - x)\rangle \geq \varphi(u), \forall v \in C\Big\} \qquad (3.2.7)$$

for all $x \in E$. Assume that

(i) *$\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in C$;*

(ii) *for any fixed $u, v, w \in C$, the mapping $x \mapsto \langle A(v,w), \eta(x,u)\rangle$ is convex and lower semicontinuous;*

(iii) *for each* $x \in C, A(x,\cdot) : C \to E^*$ *is finite-dimensional continuous: that is, for any finite-dimensional subspace* $F \subset E^{**}, A(x,\cdot) : C \cap F \to E^*$ *is continuous;*

(iv) $\xi : E^{**} \to R$ *is convex lower semicontinuous;*

(v) *for any* $x, y \in C$, $\xi(x-y) + \xi(y-x) \geq 0$;

(vi) *for any* $x, y \in C$, $A(x,y) = A(y,x)$.

Then, the following holds:

(1) Φ_r *is single-valued;*

(2) $\langle \Phi_r x - \Phi_r y, J(\Phi_r x - x)\rangle \leq \langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle$ *for all* $x, y \in E$;

(3) $F(\Phi_r) = GMEP(f, A, \eta, \varphi)$;

(4) $GMEP(f, A, \eta, \varphi)$ *is closed and convex.*

Proof. For each $x \in E^{**}$, by Lemma 3.2.3, we conclude that $\Phi_r(x)$ is nonempty.

(1) We prove that Φ_r is single-valued. Indeed, for $x \in E^{**}$ and $r > 0$, let z_1, $z_2 \in \Phi_r(x)$. Then,

$$f(z_1, v) + \langle A(z_1, z_1), \eta(v, z_1)\rangle + \varphi(v) + \frac{1}{r}\langle v - z_1, J(z_1 - x)\rangle \geq \varphi(z_1), \ \forall v \in C$$

and

$$f(z_2, v) + \langle A(z_2, z_2), \eta(v, z_2)\rangle + \varphi(v) + \frac{1}{r}\langle v - z_2, J(z_2 - x)\rangle \geq \varphi(z_2), \ \forall v \in C.$$

Hence

$$f(z_1, z_2) + \langle A(z_1, z_1), \eta(z_2, z_1)\rangle + \varphi(z_2) + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x)\rangle \geq \varphi(z_1)$$

and

$$f(z_2, z_1) + \langle A(z_2, z_2), \eta(z_1, z_2)\rangle + \varphi(z_1) + \frac{1}{r}\langle z_1 - z_2, J(z_2 - x)\rangle \geq \varphi(z_2).$$

Adding the two inequalities, from (i) we have

$$f(z_2, z_1) + f(z_1, z_2) + \langle A(z_1, z_1) - A(z_2, z_2), \eta(z_2, z_1)\rangle + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq 0. \quad (3.2.8)$$

From (A2), we have

$$\langle A(z_1, z_1) - A(z_2, z_2), \eta(z_2, z_1)\rangle + \frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq 0. \quad (3.2.9)$$

That is,

$$\frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq \langle A(z_2, z_2) - A(z_1, z_1), \eta(z_2, z_1)\rangle. \quad (3.2.10)$$

Calculating the righthand side of (3.2.10), we have

$$\begin{aligned}
&\langle A(z_2, z_2) - A(z_1, z_1), \eta(z_2, z_1)\rangle \\
&= \langle A(z_2, z_2) - A(z_2, z_1) + A(z_2, z_1) - A(z_1, z_2) + A(z_1, z_2) - A(z_1, z_1), \eta(z_2, z_1)\rangle \\
&= \langle A(z_2, z_2) - A(z_2, z_1), \eta(z_2, z_1)\rangle + \langle A(z_2, z_1) - A(z_1, z_2), \eta(z_2, z_1)\rangle \\
&\quad + \langle A(z_1, z_2) - A(z_1, z_1), \eta(z_2, z_1)\rangle \\
&\geq 2\xi(z_2 - z_1) + \langle A(z_2, z_1) - A(z_1, z_2), \eta(z_2, z_1)\rangle,
\end{aligned}$$

and so,

$$\frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq 2\xi(z_2 - z_1) + \langle A(z_2, z_1) - A(z_1, z_2), \eta(z_2, z_1)\rangle. \quad (3.2.11)$$

In (3.2.11) exchanging the position of z_1 and z_2, we get

$$\frac{1}{r}\langle z_1 - z_2, J(z_2 - x) - J(z_1 - x)\rangle \geq 2\xi(z_1 - z_2) + \langle A(z_1, z_2) - A(z_2, z_1), \eta(z_1, z_2)\rangle. \quad (3.2.12)$$

Adding the inequalities (3.2.11) and (3.2.12) and using (v) and (vi), we have

$$\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq r\left(\xi(z_2 - z_1) + \xi(z_1 - z_2)\right) \geq 0. \quad (3.2.13)$$

Hence,

$$0 \le \langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle = \langle (z_2 - x) - (z_1 - x), J(z_1 - x) - J(z_2 - x)\rangle.$$

Since J is monotone and E^{**} is strictly convex, we obtain that $z_1 - x = z_2 - x$ and hence $z_1 = z_2$. Therefore Φ_r is single-valued.

(2) For $x, y \in C$, we have

$$f(\Phi_r x, \Phi_r y) + \langle A(\Phi_r x, \Phi_r x), \eta(\Phi_r y, \Phi_r x)\rangle + \varphi(\Phi_r y) - \varphi(\Phi_r x) + \frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x)\rangle \ge 0$$

and

$$f(\Phi_r y, \Phi_r x) + \langle A(\Phi_r y, \Phi_r y), \eta(\Phi_r x, \Phi_r y)\rangle + \varphi(\Phi_r x) - \varphi(\Phi_r y) + \frac{1}{r}\langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle \ge 0.$$

Adding the above two inequalities and by (i) and (A2), we get

$$\langle A(\Phi_r x, \Phi_r x) - A(\Phi_r y, \Phi_r y), \eta(\Phi_r y, \Phi_r x)\rangle + \frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \ge 0, \quad (3.2.14)$$

that is

$$\frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \ge \langle A(\Phi_r y, \Phi_r y) - A(\Phi_r x, \Phi_r x), \eta(\Phi_r y, \Phi_r x)\rangle. \quad (3.2.15)$$

After calculating (3.2.15), we have

$$\frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \ge 2\xi(\Phi_r y, \Phi_r x) + \langle A(\Phi_r y, \Phi_r x) - A(\Phi_r x, \Phi_r y), \eta(\Phi_r y, \Phi_r x)\rangle. \quad (3.2.16)$$

In (3.2.15) exchanging the position of $\Phi_r x$ and $\Phi_r y$, we get

$$\frac{1}{r}\langle \Phi_r x - \Phi_r y, J(\Phi_r y - y) - J(\Phi_r x - x)\rangle$$

$$\geq 2\xi(\Phi_r x, \Phi_r y) + \langle A(\Phi_r x, \Phi_r y) - A(\Phi_r y, \Phi_r x), \eta(\Phi_r x, \Phi_r y)\rangle. \quad (3.2.17)$$

Adding the inequalities (3.2.16), (3.2.17), use (i) and (vi), we have

$$\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq r(\xi(\Phi_r x, \Phi_r y) + \xi(\Phi_r y, \Phi_r x)). \quad (3.2.18)$$

It follows from (iv) that

$$\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq 0. \quad (3.2.19)$$

Hence

$$\langle \Phi_r x - \Phi_r y, J(\Phi_r x - x)\rangle \leq \langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle. \quad (3.2.20)$$

(3) Next, we show that $F(\Phi_r) = GMEP(f, A, \eta, \varphi)$. Indeed, we have the following:

$$\begin{aligned} u \in F(\Phi_r) \quad &\Leftrightarrow \quad u = \Phi_r u \\ &\Leftrightarrow \quad f(u,v) + \langle A(u,u), \eta(v,u)\rangle + \varphi(v) \\ &\qquad + \frac{1}{r}\langle v - u, J(u-u)\rangle \geq \varphi(u), \quad \forall v \in C \\ &\Leftrightarrow \quad f(u,v) + \langle A(u,u), \eta(v,u)\rangle + \varphi(v) \geq \varphi(u), \quad \forall v \in C \\ &\Leftrightarrow \quad u \in GMEP(f, A, \eta, \varphi). \end{aligned} \quad (3.2.21)$$

Hence, $F(\Phi_r) = GMEP(f, A, \eta, \varphi)$.

(4) Finally, we prove that $GMEP(f, A, \eta, \varphi)$ is nonempty closed and convex. For each $v \in C$, we define the multi-valued mapping $G : C \to 2^{E^{**}}$ by

$$G(v) = \{u \in C : f(u,v) + \langle A(u,u), \eta(v,u)\rangle + \varphi(v) \geq \varphi(u)\}.$$

Since $v \in G(v)$, we have $G(v) \neq \emptyset$.

First, we prove that $G(y)$ is closed for each $y \in C$. For any $y \in C$, let $\{x_n\}$ be any sequence in $G(y)$ such that $x_n \to x_0$. We claim that $x_0 \in G(y)$. Then, for each $y \in C$, we have

$$f(x_n, y) + \langle A(x_n, x_n), \eta(y, x_n)\rangle + \varphi(y) \geq \varphi(x_n).$$

By monotonicity of A, we obtain that

$$f(x_n, y) + \langle A(x_n, y), \eta(y, x_n)\rangle + \varphi(y) \geq \varphi(x_n) + \xi(y - x_n).$$

By (A3), (i), (ii), (iv), lower semicontinuity of φ and the completely continuity of A, we obtain the following

$$\begin{aligned} \varphi(x_0) + \langle A(x_0, y), \eta(x_0, y)\rangle &\leq \liminf_{n\to\infty} \varphi(x_n) + \liminf_{n\to\infty} \langle A(x_n, y), \eta(x_n, y)\rangle \\ &\leq \liminf_{n\to\infty} (\varphi(x_n) + \langle A(x_n, y), \eta(x_n, y)\rangle) \\ &= \liminf_{n\to\infty} (\varphi(x_n) - \langle A(x_n, y), \eta(y, x_n)\rangle) \\ &\leq \limsup_{n\to\infty} (\varphi(x_n) - \langle A(x_n, y), \eta(y, x_n)\rangle) \\ &\leq \limsup_{n\to\infty} (f(x_n, y) + \varphi(y) - \xi(y - x_n)) \\ &\leq f(x_0, y) + \varphi(y) - \xi(y - x_0). \end{aligned}$$

Hence,

$$f(x_0, y) + \langle A(x_0, y), \eta(y, x_0)\rangle + \varphi(y) \geq \varphi(x_0) + \xi(y - x_0), \;\; \forall y \in C.$$

From Lemma 3.2.1, we have

$$f(x_0, y) + \langle A(x_0, x_0), \eta(y, x_0)\rangle + \varphi(y) \geq \varphi(x_0), \;\; \forall y \in C.$$

This shows that $x_0 \in G(y)$ and hence $G(y)$ is closed for each $y \in C$. Thus $GMEP(f, A, \eta, \varphi) = \bigcap_{y\in C} G(y)$ is also closed.

Finally, we prove that $GMEP(f, A, \eta, \varphi)$ is convex. In fact, let $u, v \in F(\Phi_r)$ and $z_t = tu + (1-t)v$ for $t \in (0, 1)$. From (2), we know that

$$\langle \Phi_r u - \Phi_r z_t, J(\Phi_r z_t - z_t) - J(\Phi_r u - u)\rangle \geq 0.$$

This yields that

$$\langle u - \Phi_r z_t, J(\Phi_r z_t - z_t)\rangle \;\; \geq \;\; 0. \tag{3.2.22}$$

Similarly, we also have

$$\langle v - \Phi_r z_t, J(\Phi_r z_t - z_t)\rangle \geq 0. \tag{3.2.23}$$

It follows from (3.2.22) and (3.2.23) that

$$\begin{aligned}\|z_t - \Phi_r z_t\|^2 &= \langle z_t - \Phi_r z_t, J(z_t - \Phi_r z_t)\rangle \\ &= t\langle u - \Phi_r z_t, J(z_t - \Phi_r z_t)\rangle + (1-t)\langle v - \Phi_r z_t, J(z_t - \Phi_r z_t)\rangle \\ &\leq 0.\end{aligned}$$

Hence $z_t \in F(\Phi_r) = GMEP(f, A, \eta, \varphi)$ and hence $GMEP(f, A, \eta, \varphi)$ is convex. This completes the proof. □

If E is reflexive (i.e. $E \equiv E^{**}$) smooth and strictly convex, then the following result can be derived as a corollary of Theorem 3.2.7.

Corollary 3.2.8. *Let E be a reflexive smooth and strictly convex Banach space, let C be a nonempty, bounded, closed and convex subset of E, let $A : C \times C \to E^*$ be a relaxed η-ξ semi-monotone mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. Let $r > 0$ and $z \in C$ and define a mapping $\Phi_r : E \to C$ as follows:*

$$\Phi_r(x) = \Big\{u \in C : f(u,v) + \langle A(u,u), \eta(v,u)\rangle + \varphi(v) + \frac{1}{r}\langle v - u, J(u - x)\rangle \geq \varphi(u), \forall v \in C\Big\}$$

for all $x \in E$. Assume that

(i) *$\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in C$;*

(ii) *for any fixed $u, v, w \in C$, the mapping $x \mapsto \langle A(v,w), \eta(x,u)\rangle$ is convex and lower semicontinuous;*

(iii) *for each $x \in C, A(x,\cdot) : C \to E^*$ is finite-dimensional continuous;*

(iv) $\xi : E \to \mathbb{R}$ *is convex lower semicontinuous;*

(v) *for any* $x, y \in C$, $\xi(x-y) + \xi(y-x) \geq 0$;

(vi) *for any* $x, y \in C$, $A(x,y) = A(y,x)$.

Then, the following holds:

(1) Φ_r *is single-valued;*

(2) $\langle \Phi_r x - \Phi_r y, J(\Phi_r x - x)\rangle \leq \langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle$ *for all* $x, y \in E$;

(3) $F(\Phi_r) = GMEP(f, A, \eta, \varphi)$;

(4) $GMEP(f, A, \eta, \varphi)$ *is closed and convex.*

3.2.2 Strong Convergenve Theorems

In this section, we prove a strong convergence theorem by using a hybrid projection algorithm for an asymptotically nonexpansive mapping in a uniformly convex and smooth Banach space.

Theorem 3.2.9. *Let E be a real Banach space with the smooth and uniformly convex second dual space E^{**}, let C be a nonempty, bounded, closed and convex subset of E^{**}. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. Let $A : C \times C \to E^*$ be a relaxed η–ξ semi-monotone and let $S : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $k_n \to 1$ as $n \to \infty$. Assume that*

$\Omega := F(S) \cap GMEP(f, A, \eta, \varphi) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence in* C *generated by*

$$\begin{cases} x_0 \in C, D_0 = C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - S^n z\| \leq t_n \|x_n - S^n x_n\|\}, \ \ n \geq 0, \\ u_n \in C \text{ such that} \\ f(u_n, y) + \varphi(y) + \langle A(u_n, u_n), \eta(y, u_n)\rangle \\ \qquad\qquad + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq \varphi(u_n), \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle\}, \ \ n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \ \ n \geq 0, \end{cases} \tag{3.2.24}$$

where $\{t_n\}$ *and* $\{r_n\}$ *are real sequences in* $(0, 1)$ *such that* $\lim_{n\to\infty} t_n = 0$, *and* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly, as* $n \to \infty$, *to* $P_\Omega x_0$.

Proof. Firstly, we rewrite the algorithm (3.2.24) as the following:

$$\begin{cases} x_0 \in C, D_0 = C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - S^n z\| \leq t_n \|x_n - S^n x_n\|\}, \ \ n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle \Phi_{r_n} x_n - z, J(x_n - \Phi_{r_n} x_n)\rangle \geq 0\}, \ \ n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \ \ n \geq 0, \end{cases} \tag{3.2.25}$$

where Φ_r is the mapping defined by

$$\Phi_r(x) = \{z \in C : f(z, y) + \langle A(z, z), \eta(y, z)\rangle + \varphi(y) \\ + \frac{1}{r}\langle y - z, J(z - x)\rangle \geq \varphi(z), \ \forall y \in C\}. \tag{3.2.26}$$

We first show that the sequence $\{x_n\}$ is well defined. It is easy to verify that $C_n \cap D_n$ is closed and convex and $\Omega \subset C_n$ for all $n \geq 0$. Next, we prove that $\Omega \subset C_n \cap D_n$. Since $D_0 = C$, we also have $\Omega \subset C_0 \cap D_0$. Suppose that $\Omega \subset C_{k-1} \cap D_{k-1}$ for $k \geq 2$. It follows from Theorem 3.2.7 (2) that

$$\langle \Phi_{r_k} x_k - \Phi_{r_k} u, J(\Phi_{r_k} u - u) - J(\Phi_{r_k} x_k - x_k)\rangle \geq 0,$$

for all $u \in \Omega$. This implies that

$$\langle \Phi_{r_k} x_k - u, J(x_k - \Phi_{r_k} x_k)\rangle \geq 0,$$

for all $u \in \Omega$. Hence $\Omega \subset D_k$. By the mathematical induction, we get that $\Omega \subset C_n \cap D_n$ for each $n \geq 0$ and hence $\{x_n\}$ is well-defined. Put $w = P_\Omega x_0$. Since $\Omega \subset C_n \cap D_n$ and $x_{n+1} = P_{C_n \cap D_n}$, we have

$$\|x_{n+1} - x_0\| \leq \|w - x_0\|, \ n \geq 0. \tag{3.2.27}$$

Since $x_{n+2} \in D_{n+1} \subset D_n$ and $x_{n+1} = P_{C_n \cap D_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|x_{n+2} - x_0\|.$$

Since $\{\|x_n - x_0\|\}$ is bounded, we have $\lim_{n\to\infty} \|x_n - x_0\| = d$ for some a constant d. Moreover, by the convexity of D_n, we also have $\frac{1}{2}(x_{n+1} + x_{n+2}) \in D_n$ and hence

$$\|x_0 - x_{n+1}\| \leq \left\|x_0 - \frac{x_{n+1} + x_{n+2}}{2}\right\| \leq \frac{1}{2}\Big(\|x_0 - x_{n+1}\| + \|x_0 - x_{n+2}\|\Big).$$

This implies that

$$\lim_{n\to\infty} \left\|\frac{1}{2}(x_0 - x_{n+1}) + \frac{1}{2}(x_0 - x_{n+2})\right\| = \lim_{n\to\infty} \left\|x_0 - \frac{x_{n+1} + x_{n+2}}{2}\right\| = d.$$

By Lemma 2.1.25, we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.2.28}$$

Next, we show that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.2.29}$$

To obtain (3.2.29), we need to show that $\lim_{n\to\infty} \|x_n - S^{n-k}x_n\| = 0, \ \forall k \in \mathcal{N}$. Fix $k \in \mathcal{N}$ and put $m = n - k$. Since $x_n = P_{C_{n-1} \cap D_{n-1}} x$, we have $x_n \in C_{n-1} \subseteq \cdots \subseteq C_m$. Since $t_m > 0$, there exist $y_1, \ldots, y_N \in C$ and nonnegative numbers $\lambda_1, \ldots, \lambda_N$ with $\lambda_1 + \cdots + \lambda_N = 1$ such that

$$\left\|x_n - \sum_{i=1}^{N} \lambda_i y_i\right\| < t_m, \tag{3.2.30}$$

and $\|y_i - S^m y_i\| \leq t_m \|x_m - S^m x_m\|$ for all $i \in \{1, \ldots, N\}$. Put $M = \sup_{x \in C} \|x\|, u = P_{F(S)}x$ and $r_0 = \sup_{n \geq 1} (1 + k_n)\|x_n - u\|$. Since C and $\{k_m\}$ are bounded, (3.2.30) implies

$$\left\|x_n - \frac{1}{k_m} \sum_{i=1}^{N} \lambda_i y_i\right\| \leq \left(1 - \frac{1}{k_m}\right) \|x\| + \frac{1}{k_m} \left\|x_n - \sum_{i=1}^{N} \lambda_i y_i\right\| \leq \left(1 - \frac{1}{k_m}\right) M + t_m, \tag{3.2.31}$$

and $\|y_i - S^m y_i\| \leq t_m \|x_m - S^m x_m\| \leq t_m (1 + k_m) \|x_m - u\| \leq r_0 t_m$ for all $i \in \{1, \ldots, N\}$. Therefore

$$\left\|y_i - \frac{1}{k_m} S^m y_i\right\| \leq \left(1 - \frac{1}{k_m}\right) M + r_0 t_m \tag{3.2.32}$$

for all $i \in \{1, \ldots, N\}$. Moreover, asymptotically nonexpansiveness of S and (3.2.27) give that

$$\left\|\frac{1}{k_m} S^m \left(\sum_{i=1}^{N} \lambda_i y_i\right) - S^m x_n\right\| \leq \left(1 - \frac{1}{k_m}\right) M + t_m. \tag{3.2.33}$$

It follows from Theorem 2.2.10, (3.2.31)–(3.2.33) that

$$\begin{aligned}
\|x_n - S^m x_n\| \leq\ & \left\|x_n - \frac{1}{k_m} \sum_{i=1}^{N} \lambda_i y_i\right\| + \frac{1}{k_m} \left\|\sum_{i=1}^{N} \lambda_i (y_i - S^m y_i)\right\| \\
& + \frac{1}{k_m} \left\|\sum_{i=1}^{N} \lambda_i S^m y_i - S^m \left(\sum_{i=1}^{N} \lambda_i y_i\right)\right\| \\
& + \left\|\frac{1}{k_m} S^m \left(\sum_{i=1}^{N} \lambda_i y_i\right) - S^m x_n\right\| \\
\leq\ & 2\left(1 - \frac{1}{k_m}\right) M + 2t_m + \frac{r_0 t_m}{k_m} \\
& + \gamma^{-1} \left(\max_{1 \leq i \leq j \leq N} \left(\|y_i - y_j\| - \frac{1}{k_m} \|S^m y_i - S^m y_j\|\right)\right) \\
\leq\ & 2\left(1 - \frac{1}{k_m}\right) M + 2t_m + \frac{r_0 t_m}{k_m} \\
& + \gamma^{-1} \left(\max_{1 \leq i \leq j \leq N} \left(\|y_i - \frac{1}{k_m} S^m y_i\| + \|y_j - \frac{1}{k_m} S^m y_j\|\right)\right) \\
\leq\ & 2\left(1 - \frac{1}{k_m}\right) M + 2t_m + \frac{r_0 t_m}{k_m} + \gamma^{-1} \left(2\left(1 - \frac{1}{k_m}\right) M + 2r_0 t_m\right).
\end{aligned}$$

Since $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} t_n = 0$, it follows from the last inequality that $\lim_{n\to\infty} \|x_n - S^m x_n\| = 0$. We have that

$$\begin{aligned} \|x_n - Sx_n\| &= \|x_n - S^{n-1}x_n\| + \|S^{n-1}x_n - Sx_n\| \\ &\leq \|x_n - S^{n-1}x_n\| + k_1\|S^{n-2}x_n - x_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x} \in C$. Therefore, we obtain $\tilde{x} \in F(S)$. Next, we show that $\tilde{x} \in GMEP(f, A, \eta, \varphi)$. By the construction of D_n, we see from Theorem 2.1.35 that $\Phi_{r_n} x_n = P_{D_n} x_n$. Since $x_{n+1} \in D_n$, we get

$$\|x_n - \Phi_{r_n} x_n\| \leq \|x_n - x_{n+1}\| \to 0. \tag{3.2.34}$$

From (C2), we also have

$$\frac{1}{r_n}\left\|J(x_n - \Phi_{r_n} x_n)\right\| = \frac{1}{r_n}\|x_n - \Phi_{r_n} x_n\| \to 0, \tag{3.2.35}$$

as $n \to \infty$. By (3.2.35), we also have $\Phi_{r_{n_i}} x_{n_i} \rightharpoonup \tilde{x}$. By the definition of $\Phi_{r_{n_i}}$, for each $y \in C$, we obtain

$$\begin{aligned} &f\left(\Phi_{r_{n_i}} x_{n_i}, y\right) + \left\langle A\left(\Phi_{r_{n_i}} x_{n_i}, \Phi_{r_{n_i}} x_{n_i}\right), \eta\left(y, \Phi_{r_{n_i}} x_{n_i}\right)\right\rangle + \varphi(y) \\ &+ \frac{1}{r_{n_i}}\left\langle y - \Phi_{r_{n_i}} x_{n_i}, J\left(\Phi_{r_{n_i}} x_{n_i} - x_{n_i}\right)\right\rangle \geq \varphi\left(\Phi_{r_{n_i}} x_{n_i}\right). \end{aligned}$$

By (A3), (3.2.35), (ii), the weakly lower semicontinuity of φ and complete continuity of A we have

$$\begin{aligned} \varphi(\tilde{x}) &\leq \liminf_{i\to\infty} \varphi\left(\Phi_{r_{n_i}} x_{n_i}\right) \\ &\leq \liminf_{i\to\infty} f\left(\Phi_{r_{n_i}} x_{n_i}, y\right) + \liminf_{i\to\infty} \left\langle A\left(\Phi_{r_{n_i}} x_{n_i}, \Phi_{r_{n_i}} x_{n_i}\right), \eta(y, \Phi_{r_{n_i}} x_{n_i})\right\rangle \\ &\quad + \varphi(y) + \liminf_{i\to\infty} \frac{1}{r_{n_i}}\left\langle y - \Phi_{r_{n_i}} x_{n_i}, J(\Phi_{r_{n_i}} x_{n_i} - x_{n_i})\right\rangle \\ &\leq f(\tilde{x}, y) + \varphi(y) + \langle A(\tilde{x}, \tilde{x}), \eta(y, \tilde{x})\rangle. \end{aligned}$$

Hence,

$$f(\tilde{x}, y) + \varphi(y) + \langle A(\tilde{x}, \tilde{x}), \eta(y, \tilde{x}) \rangle \geq \varphi(\tilde{x}).$$

This shows that $\tilde{x} \in GMEP(f, A, \eta, \varphi)$ and hence $\tilde{x} \in \Omega := F(S) \cap GMEP(f, A, \eta, \varphi)$.

Finally, we show that $x_n \to w$ as $n \to \infty$, where $w := P_\Omega x_0$. By the weakly lower semicontinuity of the norm, it follows from (3.2.27) that

$$\|x_0 - w\| \leq \|x_0 - \tilde{x}\| \leq \liminf_{i \to \infty} \|x_0 - x_{n_i}\| \leq \limsup_{i \to \infty} \|x_0 - x_{n_i}\| \leq \|x_0 - w\|.$$

This shows that

$$\lim_{i \to \infty} \|x_0 - x_{n_i}\| = \|x_0 - w\| = \|x_0 - \tilde{x}\|$$

and $\tilde{x} = w$. Since E^{**} is uniformly convex, we obtain that $x_0 - x_{n_i} \to x_0 - w$. It follows that $x_{n_i} \to w$. By Lemma 2.1.38, we have $x_n \to w$ as $n \to \infty$. This completes the proof. □

If S is a nonexpansive mapping in Theorem 3.2.9, then we obtain the following result concerning the problem of finding a common element of GMEP(f, A, η, φ) and the fixed point set of a nonexpansive mapping in a Banach space setting.

Theorem 3.2.10. *Let E be a real Banach space with the smooth and uniformly convex second dual space E^{**}, let C be a nonempty, bounded, closed and convex subset of E^{**}. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. Let $A : C \times C \to E^*$ be a relaxed η-ξ semi-monotone and let $S : C \to C$ be a nonexpansive mapping such that $\Omega := F(S) \cap GMEP(f, A, \eta, \varphi) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated* *by*

$$\begin{cases} x_0 \in C, D_0 = C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - Sz\| \leq t_n\|x_n - Sx_n\|\}, \ \ n \geq 0, \\ u_n \in C \text{ such that} \\ f(u_n, y) + \varphi(y) + \langle A(u_n, u_n), \eta(y, u_n)\rangle \\ \qquad\qquad + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq \varphi(u_n), \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle\}, \ \ n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \ \ n \geq 0, \end{cases}$$

where $\{t_n\}$ *and* $\{r_n\}$ *are real sequences in* $(0, 1)$ *such that* $\lim_{n\to\infty} t_n = 0$, *and* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly, as* $n \to \infty$, *to* $P_\Omega x_0$.

Putting $A \equiv 0$ and $\varphi \equiv 0$ in Theorem 3.2.9, then we have the following result in a Banach space.

Theorem 3.2.11. *Let* E *be a real Banach space with the smooth and uniformly convex second dual space* E^{**}, *let* C *be a nonempty, bounded, closed and convex subset of* E^{**}. *Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1)-(A4). *Let* $S : C \to C$ *be an asymptotically nonexpansive mapping with a sequence* $\{k_n\} \subset [1, \infty)$ *such that* $\Omega := F(S) \cap EP(f) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence in* C *generated by*

$$\begin{cases} x_0 \in C, D_0 = C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - S^n z\| \leq t_n\|x_n - S^n x_n\|\}, \ \ n \geq 0, \\ u_n \in C \text{ such that} \\ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \geq 0, \forall y \in C, n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle\}, \ \ n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \ \ n \geq 0, \end{cases}$$

where $\{t_n\}$ *and* $\{r_n\}$ *are real sequences in* $(0, 1)$ *such that* $\lim_{n\to\infty} t_n = 0$, *and* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly, as* $n \to \infty$, *to* $P_\Omega x_0$.

Putting $f \equiv 0, A \equiv 0$, $\varphi \equiv 0$ and $r_n \equiv 1$ in Theorem 3.2.9 and applying Lemma 2.1.35, we get $x_n = u_n$. Then we have the following new approximation method concerning the problem of finding a fixed of an asymptotically nonexpansive mapping in a Banach space.

Theorem 3.2.12. *Let E be a real Banach space with the smooth and uniformly convex second dual space E^{**}, let C be a nonempty, bounded, closed and convex subset of E^{**}. Let $S : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_0 \in C, C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - S^n z\| \leq t_n \|x_n - S^n x_n\|\}, \ n \geq 0, \\ x_{n+1} = P_{C_n} x_0, \ n \geq 0, \end{cases}$$

where $\{t_n\}$ and $\{r_n\}$ is a real sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 0$. Then $\{x_n\}$ converges strongly, as $n \to \infty$, to $P_{F(S)} x_0$.

If E is reflexive (i.e. $E \equiv E^{**}$) smooth and uniformly convex, then the following results can be derived as a corollary of Theorem 3.2.12.

Corollary 3.2.13. *Let E be a reflexive smooth and uniformly convex real Banach space, let C be a nonempty, bounded, closed and convex subset of E. Let $S : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty]$ such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_0 \in C, C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - S^n z\| \leq t_n \|x_n - S^n x_n\|\}, \ n \geq 0, \\ x_{n+1} = P_{C_n} x_0, \ n \geq 0, \end{cases}$$

where $\{t_n\}$ and $\{r_n\}$ is a real sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 0$. Then $\{x_n\}$ converges strongly, as $n \to \infty$, to $P_{F(S)} x_0$.

3.3 Existence and strong convergence theorems for generalized mixed equilibrium problems of a finite family of asymptotically nonexpansive mappings in Banach spaces

In this section, we first prove the existence results of solutions for GMEP under the new conditions imposed on the bifunction f. For a real Banach space E with the dual E^* and for C a nonempty closed convex subset of E, let $f : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ a real-valued function and $T : C \to E^*$ be a relaxed η-ξ monotone mapping. We consider the following generalized mixed equilibrium problem (GMEP) :

$$\text{Find } x \in C \text{ such that } f(x,y) + \langle Tx, \eta(y,x)\rangle + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \quad (3.3.1)$$

The set of such $x \in C$ is denoted by $GMEP(f,T)$, i.e.,

$$GMEP(f,T) = \{x \in C : f(x,y) + \langle Tx, \eta(y,x)\rangle + \varphi(y) \geq \varphi(x),\ \forall y \in C\}.$$

Next, we introduce the following iterative algorithm for finding a common element in the solution set of the GMEP and the common fixed point set of a finite family of asymptotically nonexpansive mappings $\{S_1, S_2, \ldots, S_N\}$ in a uniformly convex and smooth Banach space : $x_0 \in C, D_0 = C_0 = C$, and

$$\left\{ x_1 = P_{C_0 \cap D_0} x_0 = P_C x_0, \right.$$

$$\begin{cases} C_1 = \overline{co}\{z \in C : \|z - S_1 z\| \leq t_1 \|x_1 - S_1 x_1\|\}, \\ u_1 \in C \text{ such that} \\ f(u_1, y) + \varphi(y) + \langle Tu_1, \eta(y, u_1)\rangle + \frac{1}{r_1}\langle y - u_1, J(u_1 - x_1)\rangle, \forall y \in C, \\ D_1 = \{z \in C : \langle u_1 - z, J(x_1 - u_1)\rangle \geq 0\}, \\ x_2 = P_{C_1 \cap D_1} x_0, \end{cases}$$

$$\cdots$$

$$\begin{cases} C_N = \overline{co}\{z \in C_{N-1} : \|z - S_N z\| \le t_1 \|x_N - S_N x_N\|\}, \\ u_N \in C \text{ such that} \\ f(u_N, y) + \varphi(y) + \langle T u_N, \eta(y, u_N)\rangle + \frac{1}{r_N}\langle y - u_N, J(u_N - x_N)\rangle, \forall y \in C, \\ D_N = \{z \in D_{N-1} : \langle u_N - z, J(x_N - u_N)\rangle \ge 0\}, \\ x_{N+1} = P_{C_N \cap D_N} x_0, \end{cases}$$

$$\begin{cases} C_{N+1} = \overline{co}\{z \in C_N : \|z - S_1^2 z\| \le t_1 \|x_{N+1} - S_1^2 x_{N+1}\|\}, \\ u_{N+1} \in C \text{ such that} \\ f(u_{N+1}, y) + \varphi(y) + \langle T u_{N+1}, \eta(y, u_{N+1})\rangle \\ + \frac{1}{r_{N+1}}\langle y - u_{N+1}, J(u_{N+1} - x_{N+1})\rangle, \forall y \in C, \\ D_{N+1} = \{z \in D_N : \langle u_{N+1} - z, J(x_{N+1} - u_{N+1})\rangle \ge 0\}, \\ x_{N+2} = P_{C_{N+1} \cap D_{N+1}} x_0, \end{cases}$$

$\cdots$

$$\begin{cases} C_{2N} = \overline{co}\{z \in C_{2N-1} : \|z - S_N^2 z\| \le t_1 \|x_{2N} - S_N^2 x_{2N}\|\}, \\ u_{2N} \in C \text{ such that} \\ f(u_{2N}, y) + \varphi(y) + \langle T u_{2N}, \eta(y, u_{2N})\rangle + \frac{1}{r_{2N}}\langle y - u_{2N}, J(u_{2N} - x_{2N})\rangle, \forall y \in C, \\ D_{2N} = \{z \in D_{2N-1} : \langle u_{2N} - z, J(x_{2N} - u_{2N})\rangle \ge 0\}, \\ x_{2N+1} = P_{C_{2N} \cap D_{2N}} x_0, \end{cases}$$

$$\begin{cases} C_{2N+1} = \overline{co}\{z \in C_{2N} : \|z - S_1^3 z\| \le t_1 \|x_{2N+1} - S_1^3 x_{2N+1}\|\}, \\ u_{2N+1} \in C \text{ such that} \\ f(u_{2N+1}, y) + \varphi(y) + \langle T u_{2N+1}, \eta(y, u_{2N+1})\rangle \\ + \frac{1}{r_{2N+1}}\langle y - u_{2N+1}, J(u_{2N+1} - x_{2N+1})\rangle, \forall y \in C, \\ D_{2N+1} = \{z \in D_{2N} : \langle u_{2N+1} - z, J(x_{2N+1} - u_{2N+1})\rangle \ge 0\}, \\ x_{2N+2} = P_{C_{2N+1} \cap D_{2N+1}} x_0, \end{cases}$$

$\cdots.$

The above algorithm is called the hybrid iterative algorithm for a finite family of asymptotically nonexpansive mappings from C into itself. Since, for each $n \ge 1$, it can be written as $n = (h-1)N + i$, where $i = i(n) \in \{1, 2, \ldots, N\}$, $h = h(n) \ge 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$. Hence the above table can be written in the following form:

$$\begin{cases} x_0 \in C, D_0 = C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - S_{i(n)}^{h(n)} z\| \leq t_n \|x_n - S_{i(n)}^{h(n)} x_n\|\}, \ n \geq 1, \\ u_n \in C \text{ such that} \\ f(u_n, y) + \varphi(y) + \langle Tu_n, \eta(y, u_n)\rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle, \forall y \in C, n \geq 1, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \geq 0\}, \ n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \ n \geq 0. \end{cases} \tag{3.3.2}$$

3.3.1 Existence theorems for generalized mixed equilibrium problems

In this section, we prove the existence results of solutions for $GMEP(f, T)$ under the new conditions imposed on the bifunction f. We first assume that a bifunction f satisfied conditions (A1)–(A4) in Section 3.2. The following lemma is obtained from Kamraksa and Wangkeeree [9].

Lemma 3.3.1. [9, Lemma 3.3] *Let C be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to E^*$ be an η-hemicontinuous and relaxed η-ξ monotone mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let φ be a lower semicontinuous and convex function from C to $\mathbb{R}$. Let $r > 0$ and and define a mapping $\Phi_r : E \to C$ as follows:*

$$\Phi_r(x) = \Big\{z \in C : f(z, y) + \langle Tz, \eta(y, z)\rangle + \varphi(y) + \frac{1}{r}\langle y - z, J(z - x)\rangle \geq \varphi(z), \forall y \in C\Big\}$$

for all $x \in E$. Assume that

(i) $\eta(x, y) + \eta(y, x) = 0$, *for all $x, y \in C$;*

(ii) *for any fixed* $u, v \in C$, *the mapping* $x \mapsto \langle Tv, \eta(x,u)\rangle$ *is convex and lower semicontinuous and the mapping* $x \mapsto \langle Tu, \eta(v,x)\rangle$ *is lower semicontinuous;*

(iii) $\xi : E \to \mathbb{R}$ *is weakly lower semicontinuous;*

(iv) *for any* $x, y \in C$, $\xi(x-y) + \xi(y-x) \geq 0$.

Then, the following holds:

(1) Φ_r *is single-valued;*

(2) $\langle \Phi_r x - \Phi_r y, J(\Phi_r x - x)\rangle \leq \langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle$ *for all* $x, y \in E$;

(3) $F(\Phi_r) = EP(f,T)$;

(4) $EP(f,T)$ *is nonempty closed and convex.*

Now, we ready present the existence results of solutions for $GMEP(f,T)$ under the new conditions imposed on the bifunction f.

Theorem 3.3.2. *Let C be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to E^*$ be an η-hemicontinuous and relaxed η–ξ monotone mapping. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the following conditions* (A1),(A3) *and* (A4) *as in lemma 3.3.1 and* (A2′) *as follows.*

(A2′) $f(x,y) + f(y,x) \leq \min\{\xi(x-y), \xi(y-x)\}$ *for all* $x, y \in C$;

For any $r > 0$ and $x \in E$, define a mapping $\Phi_r : E \to C$ as follows:

$$\Phi_r(x) = \Big\{ z \in C : f(z,y) + \langle Tz, \eta(y,z)\rangle + \varphi(y) + \frac{1}{r}\langle y - z, J(z-x)\rangle \geq \varphi(z), \forall y \in C \Big\} \quad (3.3.3)$$

where φ is a lower semicontinuous and convex function from C to $\mathbb{R}$. Assume that

(i) $\eta(x,y)+\eta(y,x)=0$, *for all* $x,y\in C$;

(ii) *for any fixed* $u,v\in C$, *the mapping* $x\mapsto\langle Tv,\eta(x,u)\rangle$ *is convex and lower semicontinuous and the mapping* $x\mapsto\langle Tu,\eta(v,x)\rangle$ *is lower semicontinuous;*

(iii) $\xi:E\to\mathbb{R}$ *is weakly lower semicontinuous.*

Then, the following holds:

(1) Φ_r *is single-valued;*

(2) $\langle\Phi_r x-\Phi_r y, J(\Phi_r x-x)\rangle\le\langle\Phi_r x-\Phi_r y, J(\Phi_r y-y)\rangle$ *for all* $x,y\in E$;

(3) $F(\Phi_r)=GMEP(f,T)$;

(4) $GMEP(f,T)$ *is nonempty closed and convex.*

Proof. For each $x\in E$. It follows from Lemma 3.2.2 that $\Phi_r(x)$ is nonempty.

(1) We prove that Φ_r is single-valued. Indeed, for $x\in E$ and $r>0$, let z_1, $z_2\in\Phi_r x$. Then,

$$f(z_1,z_2)+\langle Tz_2,\eta(z_2,z_1)\rangle+\varphi(z_2)+\frac{1}{r}\langle z_1-z_2,J(z_1-x)\rangle\ge\varphi(z_1)$$

and

$$f(z_2,z_1)+\langle Tz_1,\eta(z_1,z_2)\rangle+\varphi(z_1)+\frac{1}{r}\langle z_2-z_1,J(z_2-x)\rangle\ge\varphi(z_2).$$

Adding the two inequalities, from (i) we have

$$f(z_2,z_1)+f(z_1,z_2)+\langle Tz_1-Tz_2,\eta(z_2,z_1)\rangle+\frac{1}{r}\langle z_2-z_1,J(z_1-x)-J(z_2-x)\rangle\ge0. \tag{3.3.4}$$

Setting $\Delta:=\min\{\xi(z_1-z_2),\xi(z_2-z_1)\}$ and using (A2′), we have

$$\Delta+\langle Tz_1-Tz_2,\eta(z_2,z_1)\rangle+\frac{1}{r}\langle z_2-z_1,J(z_1-x)-J(z_2-x)\rangle\ge0, \tag{3.3.5}$$

That is,

$$\frac{1}{r}\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq \langle Tz_2 - Tz_1, \eta(z_2, z_1)\rangle - \Delta. \qquad (3.3.6)$$

Since T is relaxed η–ξ monotone and $r > 0$, one has

$$\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq r(\xi(z_2 - z_1) - \Delta) \geq 0 \qquad (3.3.7)$$

In (3.3.6) exchanging the position of z_1 and z_2, we get

$$\frac{1}{r}\langle z_1 - z_2, J(z_2 - x) - J(z_1 - x)\rangle \geq \langle Tz_1 - Tz_2, \eta(z_1, z_2)\rangle - \Delta, \qquad (3.3.8)$$

that is,

$$\langle z_1 - z_2, J(z_2 - x) - J(z_1 - x)\rangle \geq r(\xi(z_1 - z_2) - \Delta) \geq 0 \qquad (3.3.9)$$

Now, adding the inequalities (3.3.7) and (3.3.9), we have

$$2\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle \geq 0. \qquad (3.3.10)$$

Hence,

$$0 \leq \langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\rangle = \langle (z_2 - x) - (z_1 - x), J(z_1 - x) - J(z_2 - x)\rangle.$$

Since J is monotone and E is strictly convex, we obtain that $z_1 - x = z_2 - x$ and hence $z_1 = z_2$. Therefore S_r is single-valued.

(2) For $x, y \in C$, we have

$$f(\Phi_r x, \Phi_r y) + \langle T\Phi_r x, \eta(\Phi_r y, \Phi_r x)\rangle + \varphi(\Phi_r y) - \varphi(\Phi_r x) + \frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x)\rangle \geq 0$$

and

$$f(\Phi_r y, \Phi_r x) + \langle T\Phi_r y, \eta(\Phi_r x, \Phi_r y)\rangle + \varphi(\Phi_r x) - \varphi(\Phi_r y) + \frac{1}{r}\langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle \geq 0.$$

Setting $\Lambda_{x,y} := \min\{\xi(\Phi_r x - \Phi_r y), \xi(\Phi_r y - \Phi_r x)\}$ and applying (A2), we get

$$\langle T\Phi_r x - T\Phi_r y, \eta(\Phi_r y, \Phi_r x)\rangle + \frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq -\Lambda_{x,y} \tag{3.3.11}$$

that is,

$$\begin{aligned}\frac{1}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle &\geq \langle T\Phi_r y - T\Phi_r x, \eta(\Phi_r y, \Phi_r x)\rangle - \Lambda_{x,y} \\ &\geq \xi(\Phi_r y - \Phi_r x) - \Lambda_{x,y} \geq 0. \end{aligned} \tag{3.3.12}$$

In (3.3.12) exchanging the position of $\Phi_r x$ and $\Phi_r y$, we get

$$\frac{1}{r}\langle \Phi_r x - \Phi_r y, J(\Phi_r y - y) - J(\Phi_r x - x)\rangle \geq 0. \tag{3.3.13}$$

Adding the inequalities (3.3.12) and (3.3.13), we have

$$\frac{2}{r}\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq 0. \tag{3.3.14}$$

It follows that

$$\langle \Phi_r y - \Phi_r x, J(\Phi_r x - x) - J(\Phi_r y - y)\rangle \geq 0. \tag{3.3.15}$$

Hence

$$\langle \Phi_r x - \Phi_r y, J(\Phi_r x - x)\rangle \leq \langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle. \tag{3.3.16}$$

The conclusion (3), (4) follows from Lemma 3.3.1. □

Example 3.3.3. *Define* $\xi : \mathbb{R} \to \mathbb{R}$ *and* $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *by*

$$f(x,y) = \frac{(x-y)^2}{2} \text{ and } \xi(x) = x^2 \text{ for all } x, y \in \mathbb{R}.$$

It is easy to see that f *satisfies* (A1), (A3), (A4), *and* (A2$'$) : $f(x,y) + f(y,x) \leq \min\{\xi(x-y), \xi(x-y)\}, \ \forall (x,y) \in \mathbb{R} \times \mathbb{R}$.

Remark 3.3.4. Theorem 3.1 generalizes and improves [9, Lemma 3.3] in the following manners.

(i) The condition (A2), *i.e.*, $f(x,y)+f(y,x)\leq 0$ has been weakened by (A2′), *i.e.*, $f(x,y)+f(y,x)\leq \min\{\xi(x-y),\xi(y-x)\}$ for all $x,y\in C$.

(ii) The control condition $\xi(x-y)+\xi(y-x)\geq 0$ imposed on the mapping ξ in [9, Lemma 3.3] can be removed.

If T is monotone i.e. T is relaxed η–ξ monotone with $\eta(x,y)=x-y$ for all $x,y\in C$ and $\xi=0$, we have the following results.

Corollary 3.3.5. *Let C be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Let $T: C\to E^*$ be a monotone mapping and f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying the following conditions* (i)-(iv):

(i) $f(x,x)=0$ *for all* $x\in C$;

(ii) $f(x,y)+f(y,x)\leq 0$ *for all* $x,y\in C$;

(iii) *for all* $y\in C$, $f(\cdot,y)$ *is weakly upper semicontinuous;*

(iv) *for all* $x\in C$, $f(x,\cdot)$ *is convex.*

For any $r>0$ and $x\in E$, define a mapping $\Phi_r: E\to C$ as follows:

$$\Phi_r(x)=\left\{z\in C: f(z,y)+\langle Tz,y-z\rangle+\varphi(y)+\frac{1}{r}\langle y-z,J(z-x)\rangle\geq\varphi(z),\forall y\in C\right\} \tag{3.3.17}$$

where φ is a lower semicontinuous and convex function from C to $\mathbb{R}$. Then, the following holds:

(1) Φ_r *is single-valued;*

(2) $\langle \Phi_r x - \Phi_r y, J(\Phi_r x - x)\rangle \leq \langle \Phi_r x - \Phi_r y, J(\Phi_r y - y)\rangle$ *for all* $x, y \in E$*;*

(3) $F(\Phi_r) = GEP(f)$*;*

(4) $GEP(f)$ *is nonempty closed and convex.*

3.3.2 Strong convergence theorems

In this section, we prove the strong convergence theorem of the sequence $\{x_n\}$ defined by (3.3.2) for solving a common element in the solution set of a generalized mixed equilibrium problem and the common fixed point set of a finite family of asymptotically nonexpansive mappings.

Theorem 3.3.6. *Let E be a uniformly convex and smooth Banach space and let C be a nonempty, bounded, closed and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1),(A2′),(A3) and (A4). Let $T : C \to E^*$ be an η-hemicontinuous and relaxed η–ξ monotone mapping and φ a lower semicontinuous and convex function from C to $\mathbb{R}$. Let, for each $1 \leq i \leq N, S_i : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_{n,i}\}_{n=1}^{\infty}$ respectively, such that $k_{n,i} \to 1$ as $n \to \infty$. Assume that $\Omega := \cap_{i=1}^{N} F(S_i) \cap GMEP(f, T)$ is nonempty. Let $\{x_n\}$ be a sequence generated by (3.3.2), where $\{t_n\}$ and $\{r_n\}$ are real sequences in $(0, 1)$ satisfying $\lim_{n\to\infty} t_n = 0$ and $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly, as $n \to \infty$, to $P_\Omega x_0$, where P_Ω is the metric projection of E onto Ω.*

Proof. First, define the sequence $\{k_n\}$ by $k_n := \max\{k_{n,i} : 1 \leq i \leq N\}$ and so $k_n \to 1$ as $n \to \infty$ and

$$\|S_{i(n)}^{h(n)} x - S_{i(n)}^{h(n)} y\| \leq k_n \|x - y\| \text{ for all } x, y \in C,$$

where $h(n) = j + 1$ if $jN < n \leq (j + 1)N, j = 1, 2, \ldots$ and $n = jN + i(n); i(n) \in$

$\{1, 2, \ldots, N\}$. Next, we rewrite the algorithm (3.3.2) as the following relation :

$$\begin{cases} x_0 \in C, D_0 = C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - S_{i(n)}^{h(n)} z\| \leq t_n \|x_n - S_{i(n)}^{h(n)} x_n\|\},\ \ n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle \Phi_{r_n} x_n - z, J(x_n - \Phi_{r_n} x_n)\rangle \geq 0\},\ \ n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0,\ \ n \geq 0, \end{cases} \tag{3.3.18}$$

where Φ_r is the mapping defined by (3.3.17). We show that the sequence $\{x_n\}$ is well defined. It is easy to verify that $C_n \cap D_n$ is closed and convex and $\Omega \subset C_n$ for all $n \geq 0$. Next, we prove that $\Omega \subset C_n \cap D_n$. Indeed, since $D_0 = C$, we also have $\Omega \subset C_0 \cap D_0$. Assume that $\Omega \subset C_{k-1} \cap D_{k-1}$ for $k \geq 2$. Utilizing Theorem 3.3.2 (2), we obtain

$$\langle \Phi_{r_k} x_k - \Phi_{r_k} u, J(\Phi_{r_k} u - u) - J(\Phi_{r_k} x_k - x_k)\rangle \geq 0, \text{ for all } u \in \Omega,$$

which gives that

$$\langle \Phi_{r_k} x_k - u, J(x_k - \Phi_{r_k} x_k)\rangle \geq 0, \text{ for all } u \in \Omega,$$

Hence $\Omega \subset D_k$. By the mathematical induction. we get that $\Omega \subset C_n \cap D_n$ for each $n \geq 0$ and hence $\{x_n\}$ is well defined. Now, we show that

$$\lim_{n \to \infty} \|x_n - x_{n+j}\| = 0,\ \ \forall j = 1, 2, \ldots, N.$$

Put $w = P_\Omega x_0$. Since $\Omega \subset C_n \cap D_n$ and $x_{n+1} = P_{C_n \cap D_n}$, we have

$$\|x_{n+1} - x_0\| \leq \|w - x_0\|, \forall\ n \geq 0. \tag{3.3.19}$$

Since $x_{n+2} \in D_{n+1} \subset D_n$ and $x_{n+1} = P_{C_n \cap D_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|x_{n+2} - x_0\|.$$

Hence the sequence $\{\|x_n - x_0\|\}$ is bounded and monotone increasing and hence there exists a constant d such that

$$\lim_{n \to \infty} \|x_n - x_0\| = d.$$

Moreover, by the convexity of D_n, we also have $\frac{1}{2}(x_{n+1}+x_{n+2}) \in D_n$ and hence

$$\|x_0 - x_{n+1}\| \le \left\| x_0 - \frac{x_{n+1}+x_{n+2}}{2} \right\| \le \frac{1}{2}\Big(\|x_0 - x_{n+1}\| + \|x_0 - x_{n+2}\|\Big).$$

This implies that

$$\lim_{n\to\infty} \left\| \frac{1}{2}(x_0 - x_{n+1}) + \frac{1}{2}(x_0 - x_{n+2}) \right\| = \lim_{n\to\infty} \left\| x_0 - \frac{x_{n+1}+x_{n+2}}{2} \right\| = d.$$

By Lemma 2.1.25, we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.3.20}$$

Furthermore, we can easily see that

$$\lim_{n\to\infty} \|x_n - x_{n+j}\| = 0, \ \forall j = 1, 2, \ldots, N. \tag{3.3.21}$$

Next, we show that

$$\lim_{n\to\infty} \|x_n - S_{i(n-\kappa)}^{h(n-\kappa)} x_n\| = 0, \text{ for any } \kappa \in \{1, 2, \ldots, N\}. \tag{3.3.22}$$

Fix $\kappa \in \{1, 2, \ldots, N\}$ and put $m = n - \kappa$. Since $x_n = P_{C_{n-1}\cap D_{n-1}} x$, we have $x_n \in C_{n-1} \subseteq \cdots \subseteq C_m$. Since $t_m > 0$, there exists $y_1, \ldots, y_P \in C$ and a nonnegative number $\lambda_1, \ldots, \lambda_P$ with $\lambda_1 + \cdots + \lambda_P = 1$ such that

$$\left\| x_n - \sum_{i=1}^{P} \lambda_i y_i \right\| < t_m, \tag{3.3.23}$$

and

$$\|y_i - S_{i(m)}^{h(m)} y_i\| \le t_m \|x_m - S_{i(m)}^{h(m)} x_m\|, \text{ for all } i \in \{1, \ldots, P\}.$$

By the boundedness of C and $\{k_n\}$, we can put the following

$$M = \sup_{x\in C} \|x\|, u = P_{\cap_{i=1}^N F(S_i)} x_0 \text{ and } r_0 = \sup_{n\ge1} (1 + k_n)\|x_n - u\|.$$

This together with (3.3.23) implies that

$$\left\| x_n - \frac{1}{k_m} \sum_{i=1}^{P} \lambda_i y_i \right\| \le \left(1 - \frac{1}{k_m}\right) \|x\| + \frac{1}{k_m} \left\| x_n - \sum_{i=1}^{P} \lambda_i y_i \right\| \le \left(1 - \frac{1}{k_m}\right) M + t_m$$

$$(3.3.24)$$

and

$$\begin{aligned} \|y_i - S_{i(m)}^{h(m)} y_i\| &\leq t_m \|x_m - S_{i(m)}^{h(m)} x_m\| \\ &\leq t_m \|x_m - S_{i(m)}^{h(m)} u\| + t_m \|S_{i(m)}^{h(m)} u - S_{i(m)}^{h(m)} x_m\| \\ &\leq t_m \|x_m - u\| + t_m k_m \|u - x_m\| \\ &\leq t_m (1 + k_m) \|x_m - u\| \\ &\leq t_m r_0, \end{aligned} \tag{3.3.25}$$

for all $i \in \{1, \ldots, N\}$. Therefore, for each $i \in \{1, \ldots, P\}$, we get

$$\begin{aligned} \left\| y_i - \frac{1}{k_m} S_{i(m)}^{h(m)} y_i \right\| &\leq \left\| y_i - S_{i(m)}^{h(m)} y_i \right\| + \left\| S_{i(m)}^{h(m)} y_i - \frac{1}{k_m} S_{i(m)}^{h(m)} y_i \right\| \\ &\leq r_0 t_m + \left(1 - \frac{1}{k_m}\right) M. \end{aligned} \tag{3.3.26}$$

Moreover, since each S_i, $i \in \{1, 2, \ldots, N\}$, is asymptotically nonexpansive, we can obtain that

$$\begin{aligned} &\left\| \frac{1}{k_m} S_{i(m)}^{h(m)} \left(\sum_{i=1}^{P} \lambda_i y_i \right) - S_{i(m)}^{h(m)} x_n \right\| \\ &\leq \left\| \frac{1}{k_m} S_{i(m)}^{h(m)} \left(\sum_{i=1}^{P} \lambda_i y_i \right) - \frac{1}{k_m} S_{i(m)}^{h(m)} x_n \right\| + \left\| \frac{1}{k_m} S_{i(m)}^{h(m)} x_n - S_{i(m)}^{h(m)} x_n \right\| \\ &\leq \left\| \sum_{i=1}^{P} \lambda_i y_i - x_n \right\| + \left(1 - \frac{1}{k_m}\right) M \\ &= t_m + \left(1 - \frac{1}{k_m}\right) M. \end{aligned} \tag{3.3.27}$$

It follows from Theorem 2.2.10 and the inequalities (3.3.24) - (3.3.27) that

$$\begin{aligned} &\|x_n - S_{i(m)}^{h(m)} x_n\| \\ &\leq \left\| x_n - \frac{1}{k_m} \sum_{i=1}^{P} \lambda_i y_i \right\| + \frac{1}{k_m} \left\| \sum_{i=1}^{P} \lambda_i (y_i - S_{i(m)}^{h(m)} y_i) \right\| \end{aligned}$$

$$+\frac{1}{k_m}\left\|\sum_{i=1}^{P}\lambda_i S_{i(m)}^{h(m)}y_i - S_{i(m)}^{h(m)}\left(\sum_{i=1}^{P}\lambda_i y_i\right)\right\| + \left\|\frac{1}{k_m}S_{i(m)}^{h(m)}\left(\sum_{i=1}^{P}\lambda_i y_i\right) - S_{i(m)}^{h(m)}x_n\right\|$$

$$\leq 2\left[\left(1-\frac{1}{k_m}\right)M + t_m\right] + \frac{r_0 t_m}{k_m}$$

$$+\gamma^{-1}\left(\max_{1\leq i\leq j\leq N}\left(\|y_i - y_j\| - \frac{1}{k_m}\|S_{i(m)}^{h(m)}y_i - S_{i(m)}^{h(m)}y_j\|\right)\right)$$

$$= 2\left(1-\frac{1}{k_m}\right)M + 2t_m + \frac{r_0 t_m}{k_m}$$

$$+\gamma^{-1}\left(\max_{1\leq i\leq j\leq N}\left(\|y_i - y_j\| - \frac{1}{k_m}\|S_{i(m)}^{h(m)}y_i - S_{i(m)}^{h(m)}y_j\|\right)\right)$$

$$\leq 2\left(1-\frac{1}{k_m}\right)M + 2t_m + \frac{r_0 t_m}{k_m}$$

$$+\gamma^{-1}\left(\max_{1\leq i\leq j\leq N}\left(\|y_i - \frac{1}{k_m}S_{i(m)}^{h(m)}y_i\| + \|y_j - \frac{1}{k_m}S_{i(m)}^{h(m)}y_j\|\right)\right)$$

$$\leq 2\left(1-\frac{1}{k_m}\right)M + 2t_m + \frac{r_0 t_m}{k_m} + \gamma^{-1}\left(2\left(1-\frac{1}{k_m}\right)M + 2r_0 t_m\right).$$

Since $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} t_n = 0$, it follows from the above inequality that

$$\lim_{n\to\infty}\|x_n - S_{i(m)}^{h(m)}x_n\| = 0.$$

Hence (3.3.22) is proved. Next, we show that

$$\lim_{n\to\infty}\|x_n - S_l x_n\| = 0;\ \ \forall l = 1, 2, \ldots, N. \tag{3.3.28}$$

From the construction of C_n, one can easily see that

$$\|x_{n+1} - S_{i(n)}^{h(n)}x_{n+1}\| \leq t_n\|x_n - S_{i(n)}^{h(n)}x_n\|.$$

The boundedness of C and $\lim_{n\to\infty} t_n = 0$ implies that

$$\lim_{n\to\infty}\|x_{n+1} - S_{i(n)}^{h(n)}x_{n+1}\| = 0. \tag{3.3.29}$$

On the other hand, since for any positive integer $n > N$, $n = (n-N) \pmod N$ and $n = (h(n)-1)N + i(n)$, we have

$$n - N = (h(n)-1)N + i(n) = (h(n-N)-1)N + i(n-N).$$

That is

$$h(n-N) = h(n) - 1,\ \ i(n-N) = i(n).$$

Thus

$$\begin{aligned}
\|x_n - S_n x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - S_{i(n)}^{h(n)} x_{n+1}\| + \|S_{i(n)}^{h(n)} x_{n+1} - S_n x_n\| \\
&\le \|x_n - x_{n+1}\| + \|x_{n+1} - S_{i(n)}^{h(n)} x_{n+1}\| + \|S_{i(n)}^{h(n)} x_{n+1} - S_n x_{n+1}\| \\
&\quad + \|S_n x_{n+1} - S_n x_n\| \\
&\le (1+k_1)\|x_n - x_{n+1}\| + \|x_{n+1} - S_{i(n)}^{h(n)} x_{n+1}\| + k_1\|S_{i(n)}^{h(n)-1} x_{n+1} - x_{n+1}\| \\
&\le (1+k_1)\|x_n - x_{n+1}\| + \|x_{n+1} - S_{i(n)}^{h(n)} x_{n+1}\| \\
&\quad + k_1[\|S_{i(n)}^{h(n)-1} x_{n+1} - S_{i(n)}^{h(n)-1} x_n\| + \|S_{i(n)}^{h(n)-1} x_n - x_n\| + \|x_n - x_{n+1}\|] \\
&\le (1+2k_1)\|x_n - x_{n+1}\| + \|x_{n+1} - S_{i(n)}^{h(n)} x_{n+1}\| \\
&\quad + k_1\|S_{i(n-N)}^{h(n-N)} x_{n+1} - S_{i(n-N)}^{h(n-N)} x_n\| + k_1\|S_{i(n-N)}^{h(n-N)} x_n - x_n\| \\
&\le (1+2k_1)\|x_n - x_{n+1}\| + \|x_{n+1} - S_{i(n)}^{h(n)} x_{n+1}\| \\
&\quad + k_1 k_{n-N}\|x_{n+1} - x_n\| + k_1\|S_{i(n-N)}^{h(n-N)} x_n - x_n\| \\
&\le (1+2k_1+k_1 k_{n-N})\|x_n - x_{n+1}\| + \|x_{n+1} - S_{i(n)}^{h(n)} x_{n+1}\| \\
&\quad + k_1\|S_{i(n-N)}^{h(n-N)} x_n - x_n\|.
\end{aligned}$$

Applying the facts (3.3.20), (3.3.22), and (3.3.29) to the above inequality, we obtain

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0.$$

Therefore, for any $j = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\|x_n - S_{n+j} x_n\| &\le \|x_n - x_{n+j}\| + \|x_{n+j} - S_{n+j} x_{n+j}\| + \|S_{n+j} x_{n+j} - S_{n+j} x_n\| \\
&\le \|x_n - x_{n+j}\| + \|x_{n+j} - S_{n+j} x_{n+j}\| + k_1\|x_{n+j} - x_n\| \\
&= (1+k_1)\|x_n - x_{n+j}\| + \|x_{n+j} - S_{n+j} x_{n+j}\| \to 0 \text{ as } n \to \infty,
\end{aligned}$$

which gives that

$$\lim_{n\to\infty} \|x_n - S_l x_n\| = 0;\ \ \forall\, l = 1, 2, \ldots, N$$

as required. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x} \in C$. It follows from Lemma 2.2.11 that $\tilde{x} \in F(S_l)\ \forall\, l = 1, 2, \ldots, N$.

That is $x \in \cap_{i=1}^{N} F(S_i)$.

Next, we show that $\tilde{x} \in GMEP(f,T)$. By the construction of D_n, we see from Theorem 2.1.35 that $\Phi_{r_n} x_n = P_{D_n} x_n$. Since $x_{n+1 \in D_n}$, we get

$$\|x_n - \Phi_{r_n} x_n\| \leq \|x_n - x_{n+1}\| \to 0. \tag{3.3.30}$$

Furthermore, since $\liminf_{n\to\infty} r_n > 0$, we have

$$\frac{1}{r_n}\Big\|J(x_n - \Phi_{r_n} x_n)\Big\| = \frac{1}{r_n}\|x_n - \Phi_{r_n} x_n\| \to 0, \tag{3.3.31}$$

as $n \to \infty$. By (3.3.31), we also have $\Phi_{r_{n_i}} x_{n_i} \rightharpoonup \tilde{x}$. By the definition of $\Phi_{r_{n_i}}$, for each $y \in C$, we obtain

$$\begin{aligned} f\left(\Phi_{r_{n_i}} x_{n_i}, y\right) + \left\langle T\Phi_{r_{n_i}} x_{n_i}, \eta\left(y, \Phi_{r_{n_i}} x_{n_i}\right)\right\rangle + \varphi(y) \\ + \frac{1}{r_{n_i}}\left\langle y - \Phi_{r_{n_i}} x_{n_i}, J\left(\Phi_{r_{n_i}} x_{n_i} - x_{n_i}\right)\right\rangle \geq \varphi\left(\Phi_{r_{n_i}} x_{n_i}\right). \end{aligned}$$

Hence

$$\begin{aligned} f\left(\Phi_{r_{n_i}} x_{n_i}, y\right) + \varphi(y) + \frac{1}{r_{n_i}}\left\langle y - \Phi_{r_{n_i}} x_{n_i}, J\left(\Phi_{r_{n_i}} x_{n_i} - x_{n_i}\right)\right\rangle \\ \geq \left\langle T\Phi_{r_{n_i}} x_{n_i}, \eta\left(\Phi_{r_{n_i}} x_{n_i}, y\right)\right\rangle + \varphi\left(\Phi_{r_{n_i}} x_{n_i}\right). \end{aligned}$$

By (A3), (3.3.31), (ii), the weakly lower semicontinuity of φ and ξ and relaxed η-ξ-monotonicity of T we have

$$\begin{aligned} &\varphi(\tilde{x}) + \langle Ty, \eta(\tilde{x}, y)\rangle + \xi(y - \tilde{x}) \\ &\quad \leq \liminf_{i\to\infty} \varphi\left(\Phi_{r_{n_i}} x_{n_i}\right) + \liminf_{i\to\infty} \langle Ty, \eta(\Phi_{r_{n_i}} x_{n_i}, y)\rangle + \liminf_{i\to\infty} \xi(y - \Phi_{r_{n_i}} x_{n_i}) \\ &\quad \leq \liminf_{i\to\infty} \left(\varphi\left(\Phi_{r_{n_i}} x_{n_i}\right) + \langle Ty, \eta(\Phi_{r_{n_i}} x_{n_i}, y)\rangle + \xi(y - \Phi_{r_{n_i}} x_{n_i})\right) \\ &\quad \leq \liminf_{i\to\infty} \left(\varphi\left(\Phi_{r_{n_i}} x_{n_i}\right) + \langle T\Phi_{r_{n_i}} x_{n_i}, \eta(\Phi_{r_{n_i}} x_{n_i}, y)\rangle\right) \\ &\quad \leq \limsup_{i\to\infty} \left(\varphi\left(\Phi_{r_{n_i}} x_{n_i}\right) + \langle T\Phi_{r_{n_i}} x_{n_i}, \eta(\Phi_{r_{n_i}} x_{n_i}, y)\rangle\right) \end{aligned}$$

$$\leq \limsup_{i\to\infty}\left(f\left(\Phi_{r_{n_i}}x_{n_i},y\right)+\varphi(y)+\frac{1}{r_{n_i}}\left\langle y-\Phi_{r_{n_i}}x_{n_i},J\left(\Phi_{r_{n_i}}x_{n_i}-x_{n_i}\right)\right\rangle\right)$$
$$=\limsup_{i\to\infty} f\left(\Phi_{r_{n_i}}x_{n_i},y\right)+\varphi(y)$$
$$\leq f(\tilde{x},y)+\varphi(y).$$

Hence,

$$f(\tilde{x},y)+\varphi(y)+\langle Ty,\eta(y,\tilde{x})\rangle \geq \varphi(\tilde{x})+\xi(y-\tilde{x}),\ \forall y\in C.$$

By Lemma 3.2.1, we have that

$$f(\tilde{x},y)+\varphi(y)+\langle T\tilde{x},\eta(y,\tilde{x})\rangle \geq \varphi(\tilde{x}),\ \forall y\in C.$$

This shows that $\tilde{x}\in EP(f,T)$ and hence $\tilde{x}\in\Omega:=\bigcap_{i=1}^{N}F(S_i)\cap GMEP(f,T)$.

Finally, we show that $x_n\to w$ as $n\to\infty$, where $w:=P_\Omega x_0$. By the weakly lower semicontinuity of the norm, it follows from (3.3.19) that

$$\|x_0-w\|\leq\|x_0-\tilde{x}\|\leq\liminf_{i\to\infty}\|x_0-x_{n_i}\|\leq\limsup_{i\to\infty}\|x_0-x_{n_i}\|\leq\|x_0-w\|.$$

This shows that

$$\lim_{i\to\infty}\|x_0-x_{n_i}\|=\|x_0-w\|=\|x_0-\tilde{x}\|$$

and $\tilde{x}=w$. Since E is uniformly convex, we obtain that $x_0-x_{n_i}\to x_0-w$. It follows that $x_{n_i}\to w$. By Lemma 2.1.38, we have $x_n\to w$ as $n\to\infty$. This completes the proof. □

Setting $S_i\equiv S$, an asymptotically nonexpansive mapping, in Theorem 3.3.6 then we have the following result.

Theorem 3.3.7. *Let E be a uniformly convex and smooth Banach space and let C be a nonempty, bounded, closed and convex subset of E. Let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying* (A1),(A2′),(A3) *and* (A4). *Let $T:C\to E^*$ be an η-hemicontinuous and relaxed η–ξ monotone mapping and φ a lower semicontinuous and convex function from C to $\mathbb{R}$. Let S be an asymptotically nonexpansive mapping* *with a sequence* $\{k_n\}$, *such that* $k_n \to 1$ *as* $n \to \infty$. *Assume that* $\Omega := F(S) \cap GMEP(f,T)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0 \in C, D_0 = C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - S^n z\| \le t_n \|x_n - S^n x_n\|\}, \ \ n \ge 1, \\ u_n \in C \text{ such that} \\ f(u_n, y) + \varphi(y) + \langle Tu_n, \eta(y, u_n)\rangle + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle, \forall y \in C, n \ge 1, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \ge 0\}, \ \ n \ge 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \ \ n \ge 0. \end{cases} \tag{3.3.32}$$

where $\{t_n\}$ *and* $\{r_n\}$ *are real sequences in* $(0,1)$ *satisfying* $\lim_{n\to\infty} t_n = 0$ *and* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *converges strongly, as* $n \to \infty$, *to* $P_\Omega x_0$, *where* P_Ω *is the metric projection of* E *onto* Ω.

It's well known that each nonexpansive mapping is an asymptotically nonexpansive mapping, then Theorem 3.3.6 works for nonexpansive mapping.

Theorem 3.3.8. *Let* E *be a uniformly convex and smooth Banach space and let* C *be a nonempty, bounded, closed and convex subset of* E. *Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1),(A2′),(A3) *and* (A4). *Let* $T : C \to E^*$ *be an* η-*hemicontinuous and relaxed* η-ξ *monotone mapping and* φ *a lower semicontinuous and convex function from* C *to* $\mathbb{R}$. *Let* S *be a nonexpansive mapping of* C *into itself such that* $\Omega := F(S) \cap GMEP(f,T) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence in* C *generated* *by*

$$\begin{cases} x_0 \in C, D_0 = C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - Sz\| \le t_n \|x_n - Sx_n\|\}, \quad n \ge 1, \\ u_n \in C \quad \text{such that } f(u_n, y) + \varphi(y) + \langle Tu_n, \eta(y, u_n)\rangle \\ \qquad\qquad + \frac{1}{r}\langle y - u_n, J(u_n - x_n)\rangle \ge \varphi(u_n), \forall y \in C, n \ge 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \ge 0\}, \quad n \ge 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \ge 0, \end{cases}$$

where $\{t_n\}$ *and* $\{r_n\}$ *are real sequences in* $(0,1)$ *satisfying* $\lim_{n\to\infty} t_n = 0$ *and* $\liminf_{n\to\infty} r_n > 0$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

If we take $T \equiv 0$ and $\varphi \equiv 0$ in Theorem 3.3.6, then we obtain the following result concerning an equilibrium problem in a Banach space setting.

Theorem 3.3.9. *Let* E *be a uniformly convex and smooth Banach space and let* C *be a nonempty, bounded, closed and convex subset of* E. *Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* (A1),(A2′),(A3) *and* (A4). *and let* S *be an asympotically nonexpansive mapping of* C *into itself such that* $\Omega := \bigcap_{n=0}^{\infty} F(S_n) \cap EP(f) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence in* C *generated by*

$$\begin{cases} x_0 \in C, D_0 = C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - S^n z\| \le t_n \|x_n - S^n x_n\|\}, \quad n \ge 1, \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J(u_n - x_n)\rangle \ge 0, \forall y \in C, n \ge 0, \\ D_n = \{z \in D_{n-1} : \langle u_n - z, J(x_n - u_n)\rangle \ge 0\}, \quad n \ge 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \ge 0, \end{cases} \tag{3.3.33}$$

where $\{t_n\}$ *and* $\{r_n\}$ *are real sequences in* $(0,1)$ *satisfying* $\lim_{n\to\infty} t_n = 0$ *and* $\liminf_{n\to\infty} r_n > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

If we take $f \equiv 0$ and $T \equiv 0$ and $\varphi \equiv 0$ in Theorem 3.3.6, then we obtain the following result.

Theorem 3.3.10. *Let E be a uniformly convex and smooth Banach space, C a nonempty, bounded, closed and convex subset of E and S an asympotically nonexpansive mapping of C into itself such that $\Omega := \bigcap_{n=0}^{\infty} F(S_n) \neq \emptyset$. Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, C_0 = C, \\ C_n = \overline{co}\{z \in C_{n-1} : \|z - S^n z\| \leq t_n \|x_n - S^n x_n\|\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n} x_0, \quad n \geq 0. \end{cases} \tag{3.3.34}$$

If $\{t_n\} \subset (0,1)$ and $\lim_{n\to\infty} t_n = 0$, then $\{x_n\}$ converges strongly to $P_\Omega x_0$.