CHAPTER IV

GENERAL ITERATIVE METHODS IN BANACH SPACES

In this chapter, we discuss the strong convergence theorems of general iterative methods for (asymptotically)nonexpansive semigroups.

4.1 The modified general iterative methods for nonexpansive semigroups in Banach spaces

Let E be a Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$ i.e. $\varphi([0,1]) \subset [0,1]$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be a nonexpansive semigroups from C into itseft. For $f \in \Pi_E$, $t \in (0,1)$, and A is a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$, the mapping $S_t : E \to E$ defined by

$$S_t(x) = t\gamma f(x) + (I - tA)T(\lambda_t)x, \forall x \in E$$

is a contraction mapping. Indeed, for any $x, y \in E$,

$$\begin{aligned} \|S_t(x) - S_t(y)\| &= \|t\gamma(f(x) - f(y)) + (I - tA)(T(\lambda_t)x - T(\lambda_t)y)\| \\ &\leq t\gamma\|f(x) - f(y)\| + \|I - tA\|\|T(\lambda_t)x - T(\lambda_t)y\| \\ &\leq t\gamma\alpha\|x - y\| + \varphi(1)(1 - t\bar{\gamma})\|x - y\| \\ &\leq (1 - t(\varphi(1)\bar{\gamma} - \gamma\alpha))\|x - y\|. \end{aligned} \tag{4.1.1}$$

Thus, by Banach contraction mapping principle, there exists a unique fixed point x_t in E that is

$$x_t = t\gamma f(x_t) + (I - tA)T(\lambda_t)x_t. \tag{4.1.2}$$

Remark 4.1.1. We note that l^p space has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. It is clear that φ is invariant on $[0, 1]$.

Lemma 4.1.2. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be a nonexpansive semigroup with $F(\mathcal{S}) \neq \emptyset$ and $f \in \Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$, and let $t \in (0,1)$ which satisfying $t \to 0$. Then the net $\{x_t\}$ defined by (4.1.2) with $\{\lambda_t\}_{0<t<1}$ is a positive real divergent sequence; converges strongly as $t \to 0$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality :*

$$\langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, z \in F(\mathcal{S}). \tag{4.1.3}$$

Proof. We first show that the uniqueness of a solution of the variational inequality (4.1.3). Suppose both $\tilde{x} \in F(\mathcal{S})$ and $x^* \in F(\mathcal{S})$ are solutions to (4.1.3), then

$$\langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - x^*)\rangle \leq 0 \tag{4.1.4}$$

and

$$\langle (A - \gamma f)x^*, J_\varphi(x^* - \tilde{x})\rangle \leq 0. \tag{4.1.5}$$

Adding (4.1.4) and (4.1.5), we obtain

$$\langle (A - \gamma f)\tilde{x} - (A - \gamma f)x^*, J_\varphi(\tilde{x} - x^*)\rangle \leq 0. \tag{4.1.6}$$

Noticing that for any $x, y \in E$,

$$\begin{aligned}
&\langle (A - \gamma f)x - (A - \gamma f)y, J_\varphi(x - y)\rangle \\
&\qquad = \langle A(x - y), J_\varphi(x - y)\rangle - \gamma\langle f(x) - f(y), J_\varphi(x - y)\rangle \\
&\qquad \geq \bar{\gamma}\|x - y\|\varphi(\|x - y\|) - \gamma\|f(x) - f(y)\|\|J_\varphi(x - y)\| \\
&\qquad \geq \bar{\gamma}\Phi(\|x - y\|) - \gamma\alpha\Phi(\|x - y\|)
\end{aligned}$$

$$= (\bar{\gamma} - \gamma\alpha)\Phi(\|x - y\|)$$
$$\geq (\bar{\gamma}\varphi(1) - \gamma\alpha)\Phi(\|x - y\|) \geq 0. \tag{4.1.7}$$

Using (4.1.6) and $0 < \bar{\gamma}\varphi(1) - \gamma\alpha$ in the last inequality, we get that $\Phi(\|\tilde{x} - x^*\|) = 0$. Therefore $\tilde{x} = x^*$ and the uniqueness is proved. Below we use $\tilde{x}$ to denote the unique solution of (4.1.3). Next, we will prove that $\{x_t\}$ is bounded. Take a $p \in F(\mathcal{S})$, then we have

$$\begin{aligned}\|x_t - p\| &= \|t\gamma f(x_t) + (I - tA)T(\lambda_t)x_t - p\| \\ &= \|(I - tA)T(\lambda_t)x_t - (I - tA)p + t(\gamma f(x_t) - A(p))\| \\ &\leq \varphi(1)(1 - t\bar{\gamma})\|x_t - p\| + t(\gamma\alpha\|x_t - p\| + \|\gamma f(p) - A(p)\|).\end{aligned}$$

It follows that

$$\|x_t - p\| \leq \frac{1}{\bar{\gamma}\varphi(1) - \gamma\alpha}\|\gamma f(p) - A(p)\|.$$

Hence $\{x_t\}$ is bounded, so are $\{f(x_t)\}$ and $\{AT(x_t)\}$. The definition of $\{x_t\}$ implies that

$$\|x_t - T(\lambda_t)x_t\| = t\|\gamma f(x_t) - A(T(\lambda_t)x_t)\| \to 0 \text{ as } t \to 0. \tag{4.1.8}$$

Next, we show that $\|x_t - T(h)x_t\| \to 0$ for all $h \geq 0$. Since $\{T(t) : t \geq 0\}$ is u.a.r. nonexpansive semigroup and $\lim_{t\to 0} \lambda_t = \infty$, then, for all $h > 0$ and for any bounded subset D of C containing $\{x_t\}$,

$$\lim_{t\to 0}\|T(h)(T(\lambda_t)x_t) - T(\lambda_t)x_t\| \leq \lim_{t\to 0}\sup_{x\in D}\|T(h)(T(\lambda_t)x_t) - T(\lambda_t)x_t\| = 0.$$

Hence, when $t \to 0$, for all $h > 0$, we have

$$\begin{aligned}\|x_t - T(h)x_t\| &\leq \|x_t - T(\lambda_t)x_t\| + \|T(\lambda_t)x_t - T(h)(T(\lambda)x_t)\| \\ &\quad + \|T(h)(T(\lambda_t)x_t) - T(h)x_t\| \\ &\leq 2\|x_t - T(\lambda_t)x_t\| + \|T(\lambda_t)x_t - T(h)(T(\lambda_t)x_t)\| \to 0.\end{aligned} \tag{4.1.9}$$

Assume that $\{t_n\}_{n=1}^{\infty} \subset (0,1)$ is such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$ and $\lambda_n := \lambda_{t_n}$. We show that $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x} \in F(\mathcal{S})$. It follows from reflexivity of E and the boundedness of sequence $\{x_n\}$ that there exists $\{x_{n_j}\}$ which is a subsequence of $\{x_n\}$ converging weakly to $w \in E$ as $n \to \infty$. Since J_φ is weakly sequentially continuous, we have by Lemma 2.1.31 that

$$\limsup_{j\to\infty} \Phi(\|x_{n_j} - x\|) = \limsup_{j\to\infty} \Phi(\|x_{n_j} - w\|) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

Let

$$H(x) = \limsup_{j\to\infty} \Phi(\|x_{n_j} - x\|), \text{ for all } x \in E.$$

It follows that

$$H(x) = H(w) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

For $h \geq 0$, from (4.1.9) we obtain

$$\begin{aligned} H(T(h)w) &= \limsup_{j\to\infty} \Phi(\|x_{n_j} - T(h)w\|) = \limsup_{j\to\infty} \Phi(\|T(h)x_{n_j} - T(h)w\|) \\ &\leq \limsup_{j\to\infty} \Phi(\|x_{n_j} - w\|) = H(w). \end{aligned} \tag{4.1.10}$$

On the other hand, however,

$$H(T(h)w) = H(w) + \Phi(\|T(h)w - w\|). \tag{4.1.11}$$

It follows from (4.1.10) and (4.1.11) that

$$\Phi(\|T(h)w - w\|) = H(T(h)w) - H(w) \leq 0.$$

This implies that $T(h)w = w$ for all $h \geq 0$, and so $w \in F(\mathcal{S})$. Next we show that $x_{n_j} \to w$ as $j \to \infty$. In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau, \forall t \geq 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function, then for $1 \geq k \geq 0$, $\varphi(kx) \leq \varphi(x)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(kx)dx \leq k\int_0^t \varphi(x)dx = k\Phi(t).$$

Following Lemma 2.1.31, we have

$$\begin{aligned}
\Phi(\|x_n - w\|) &= \Phi((I - t_nA)T(t_n)x_n - (I - t_nA)w + t_n(\gamma f(x_n) - A(w))) \\
&= \Phi(\|(I - t_nA)T(t_n)x_n - (I - t_nA)w\|) \\
&\quad + t_n\langle \gamma f(x_n) - A(w), J_\varphi(x_n - w)\rangle \\
&\leq \Phi(\varphi(1)(1 - t_n\bar{\gamma})\|x_n - w\|) + t_n\gamma\langle f(x_{t_n}) - f(w), J_\varphi(x_n - w)\rangle \\
&\quad + t_n\langle \gamma f(w) - A(w), J_\varphi(x_n - w)\rangle \\
&\leq \varphi(1)(1 - t_n\bar{\gamma})\Phi(\|x_n - w\|) + t_n\gamma\|f(x_n) - f(w)\|\|J_\varphi(x_n - w)\| \\
&\quad + t_n\langle \gamma f(w) - A(w), J_\varphi(x_n - w)\rangle \\
&\leq \varphi(1)(1 - t_n\bar{\gamma})\Phi(\|x_n - w\|) + t_n\gamma\alpha\|x_n - w\|\|J_\varphi(x_n - w)\| \\
&\quad + t_n\langle \gamma f(w) - A(w), J_\varphi(x_n - w)\rangle \\
&= \varphi(1)(1 - t_n\bar{\gamma})\Phi(\|x_n - w\|) + t_n\gamma\alpha\Phi(\|x_n - w\|) \\
&\quad + t_n\langle \gamma f(w) - A(w), J_\varphi(x_n - w)\rangle \\
&= (1 - t_n(\bar{\gamma}\varphi(1) - \gamma\alpha))\Phi(\|x_n - w\|) + t_n\langle \gamma f(w) - A(w), J_\varphi(x_n - w)\rangle.
\end{aligned} \tag{4.1.12}$$

This implies that

$$\Phi(\|x_{n_j} - w\|) \leq \frac{1}{\bar{\gamma}\varphi(1) - \gamma\alpha}\langle \gamma f(w) - A(w), J_\varphi(x_{n_j} - w)\rangle.$$

Now observing that $x_n \rightharpoonup w$ implies $J_\varphi(x_n - w) \rightharpoonup 0$, we conclude from the last inequality that

$$\Phi(\|x_{n_j} - w\|) \to 0 \text{ as } j \to \infty.$$

Hence $x_{n_j} \to w$ as $j \to \infty$. Next we prove that w solves the variational inequality (4.1.3). For any $z \in F(\mathcal{S})$, we observe that

$$\begin{aligned}
&\langle (I - T(\lambda_t))x_t - (I - T(\lambda_t))z, J_\varphi(x_t - z)\rangle \\
&\qquad = \langle x_t - z, J_\varphi(x_t - z)\rangle + \langle T(\lambda_t)x_t - T(\lambda_t)z, J_\varphi(x_t - z)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \Phi(\|x_t - z\|) - \langle T(\lambda_t)z - T(\lambda_t)x_t, J_\varphi(x_t - z)\rangle \\
&\geq \Phi(\|x_t - z\|) - \|T(\lambda_t)z - T(\lambda_t)x_t\|\|J_\varphi(x_t - z)\| \\
&\geq \Phi(\|x_t - z\|) - \|z - x_t\|\|J_\varphi(x_t - z)\| \\
&= \Phi(\|x_t - z\|) - \Phi(\|x_t - z\|) = 0. \qquad (4.1.13)
\end{aligned}
$$

Since

$$x_t = t\gamma f(x_t) + (I - tA)T(\lambda_t)x_t,$$

we can derive that

$$(A - \gamma f)(x_t) = -\frac{1}{t}(I - T(\lambda_t))x_t + (A(I - T(\lambda_t))x_t).$$

Thus

$$
\begin{aligned}
\langle (A - \gamma f)(x_t), J_\varphi(x_t - z)\rangle &= -\frac{1}{t}\langle (I - T(\lambda_t))x_t - (I - T(\lambda_t))z, J_\varphi(x_t - z)\rangle \\
&\quad + \langle A(I - T(\lambda_t))x_t, J_\varphi(x_t - z)\rangle \\
&\leq \langle A(I - T(\lambda_t))x_t, J_\varphi(x_t - z)\rangle. \qquad (4.1.14)
\end{aligned}
$$

Noticing that

$$x_{n_j} - T(\lambda_{t_{n_j}})x_{n_j} \to w - T(w) = w - w = 0.$$

Now replacing t and λ_t with n_j and t_{n_j} in (4.1.14) and letting $j \to \infty$, we have

$$\langle (A - \gamma f)w, J_\varphi(w - z)\rangle \leq 0.$$

So, $w \in F(T)$ is a solution of the variational inequality (4.1.3), and hence $w = \tilde{x}$ by the uniqueness. Applying Lemma 2.1.38, we can conclude that $x_t \to \tilde{x}$ as $t \to 0$. This completes the proof. □

Theorem 4.1.3. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0, 1]$. Let $\{T(s) : s \geq 0\}$ be a u.a.r. semigroup of nonexpansive mappings with $F(\mathcal{S}) \neq \emptyset$ and*

$f \in \Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Let the sequence $\{x_n\}$ be generated by the following :

$$\begin{cases} x_0 = x \in E, \\ y_n = \beta_n x_n + (1-\beta_n)T(t_n)x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \;\; n \geq 0 \end{cases} \tag{4.1.15}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1]$ are real sequences satisfying the following conditions :

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$

(C2) $\lim_{n\to\infty} \beta_n = 0$,

(C3) $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly to $\tilde{x}$ that is obtained in Lemma 4.1.2.

Proof. Since $\lim_{n\to\infty} \alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n < \varphi(1)\|A\|^{-1}$ for all n. By Lemma 2.1.33, we have $\|I - \alpha_n A\| \leq \varphi(1)(1 - \alpha_n\bar{\gamma})$. We first observe that $\{x_n\}$ is bounded. Indeed, pick any $p \in F(\mathcal{S})$ to obtain

$$\begin{aligned} \|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)T(t_n)x_n - p\| \\ &= \|\beta_n(x_n - p) + (1-\beta_n)(T(t_n)x_n - T(t_n)p)\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &= \|x_n - p\|, \end{aligned} \tag{4.1.16}$$

and so

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n - p\| \\ &= \|\alpha_n(\gamma f(x_n) - A(p)) + (I - \alpha_n A)y_n - (I - \alpha_n A)p\| \\ &\leq \alpha_n\|\gamma f(x_n) - A(p)\| + \varphi(1)(1 - \alpha_n\bar{\gamma})\|y_n - p\| \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-A(p)\|+\varphi(1)(1-\alpha_n\bar{\gamma})\|y_n-p\| \\
&\leq \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-A(p)\|+\varphi(1)(1-\alpha_n\bar{\gamma})\|x_n-p\| \\
&\leq (1-\alpha_n(\bar{\gamma}\varphi(1)-\gamma\alpha))\|x_n-p\|+\alpha_n\|\gamma f(x_n)-A(p)\| \\
&= (1-\alpha_n(\bar{\gamma}\varphi(1)-\gamma\alpha))\|x_n-p\|+\alpha_n(\bar{\gamma}\varphi(1)-\gamma\alpha)\frac{\|\gamma f(x_n)-A(p)\|}{\bar{\gamma}\varphi(1)-\gamma\alpha}.
\end{aligned}$$

It follows from induction that

$$\|x_n-p\|\leq \max\left\{\|x_0-p\|,\frac{\|\gamma f(p)-A(p)\|}{\bar{\gamma}\varphi(1)-\gamma\alpha}\right\}, n\geq 0. \tag{4.1.17}$$

The boundedness of $\{x_n\}$ implies that $\{y_n\},\{T(t_n)x_n\}$ and $\{f(x_n)\}$ are bounded. Thus by (4.1.28), (C1) and (C2), we have

$$\|y_n-T(t_n)x_n\|=\beta_n\|x_n-T(t_n)x_n\|\to 0$$

and there by,

$$\|x_{n+1}-T(t_n)x_n\|\leq\|y_n-T(t_n)x_n\|+\alpha_n\|\gamma f(x_n)-A(y_n)\|\to 0.$$

Since $\{T(t):t\geq 0\}$ is u.a.r. nonexpansive semigroup and $\lim_{n\to\infty}t_n=\infty$, then, for all $h>0$ and for any bounded subset D of C containing $\{x_n\}$,

$$\lim_{n\to\infty}\|T(h)(T(t_n)x_n)-T(t_n)x_n\|\leq\lim_{n\to\infty}\sup_{x\in D}\|T(h)(T(t_n)x_n)-T(t_n)x_n\|=0.$$

Hence, when $n\to\infty$, for all $h>0$, we have

$$\begin{aligned}
\|x_{n+1}-T(h)x_{n+1}\| &\leq \|x_{n+1}-T(t_n)x_n\|+\|T(t_n)x_n-T(h)(T(t_n)x_n)\| \\
&\quad +\|T(h)(T(t_n)x_n)-T(h)x_{n+1}\| \\
&\leq 2\|x_{n+1}-T(t_n)x_n\|+\|T(t_n)x_n-T(h)(T(t_n)x_n)\|\to 0.
\end{aligned} \tag{4.1.18}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle\gamma f(\tilde{x})-A\tilde{x},J_\varphi(x_n-\tilde{x})\rangle\leq 0, \tag{4.1.19}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n_k}-\tilde{x})\rangle = \limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n-\tilde{x})\rangle. \quad (4.1.20)$$

If follows from reflexivity of E and the boundedness of sequence $\{x_{n_k}\}$ that there exists $\{x_{n_{k_i}}\}$ which is a subsequence of $\{x_{n_k}\}$ converging weakly to $w \in E$ as $i \to \infty$. Since J_φ is weakly continuous, we have by Lemma 2.1.31 that

$$\limsup_{n\to\infty}\Phi(\|x_{n_{k_i}} - x\|) = \limsup_{n\to\infty}\Phi(\|x_{n_{k_i}} - w\|) + \Phi(\|x-w\|), \text{ for all } x \in E.$$

Let

$$H(x) = \limsup_{n\to\infty}\Phi(\|x_{n_{k_i}} - x\|), \text{ for all } x \in E.$$

It follows that

$$H(x) = H(w) + \Phi(\|x-w\|), \text{ for all } x \in E.$$

From (4.1.18), for each $h > 0$, we obtain

$$\begin{aligned} H(T(h)w) &= \limsup_{i\to\infty}\Phi(\|x_{n_{k_i}} - T(h)w\|) = \limsup_{i\to\infty}\Phi(\|T(h)x_{n_{k_i}} - T(h)w\|) \\ &\leq \limsup_{i\to\infty}\Phi(\|x_{n_{k_i}} - w\|) = H(w) \end{aligned} \quad (4.1.21)$$

On the other hand, however,

$$H(T(h)w) = H(w) + \Phi(\|T(h)w - w\|) \quad (4.1.22)$$

It follows from (4.1.21) and (4.1.22) that

$$\Phi(\|T(h)w - w\|) = H(T(h)w) - H(w) \leq 0.$$

This implies that $T(h)w = w$ for all $h > 0$, and so $w \in F(\mathcal{S})$. Since the duality map J_φ is single-valued and weakly continuous, we get that

$$\begin{aligned} \limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n-\tilde{x})\rangle &= \lim_{k\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n_k}-\tilde{x})\rangle \\ &= \lim_{i\to\infty}\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n_{k_i}}-\tilde{x})\rangle \end{aligned}$$

$$= \langle (A-\gamma f)\tilde{x}, J_\varphi(\tilde{x}-w)\rangle \le 0$$

as required.

Finally, we show that $x_n \to \tilde{x}$ as $n \to \infty$.

$$\begin{aligned}
&\Phi(\|x_{n+1}-\tilde{x}\|)\\
&\quad = \Phi(\|\alpha_n(\gamma f(x_n)) + (I-\alpha_n A)y_n - \tilde{x}\|)\\
&\quad = \Phi(\|\alpha_n(\gamma f(x_n) - A\tilde{x}) + (I-\alpha_n A)(y_n-\tilde{x})\|)\\
&\quad = \Phi(\|\alpha_n(\gamma f(x_n) - \gamma f(\tilde{x})) + \alpha_n(\gamma f(\tilde{x}) - A\tilde{x}) + (I-\alpha_n A)(y_n-\tilde{x})\|)\\
&\quad \le \Phi(\|\alpha_n(\gamma f(x_n) - \gamma f(\tilde{x})) + (I-\alpha_n A)(y_n-\tilde{x})\|)\\
&\qquad + \alpha_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n+1}-\tilde{x})\rangle\\
&\quad \le \Phi(\|\alpha_n(\gamma f(x_n) - \gamma f(\tilde{x}))\| + \|(I-\alpha_n A)(y_n-\tilde{x})\|)\\
&\qquad + + \alpha_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n+1}-\tilde{x})\rangle\\
&\quad \le \Phi(\alpha_n\gamma\alpha\|x_n-\tilde{x}\| + \varphi(1)(1-\alpha_n\bar{\gamma})\|y_n-\tilde{x}\|)\\
&\qquad + \alpha_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n+1}-\tilde{x})\rangle\\
&\quad \le \Phi(\alpha_n\gamma\alpha\|x_n-\tilde{x}\| + \varphi(1)(1-\alpha_n\bar{\gamma})\|x_n-\tilde{x}\|)\\
&\qquad + \alpha_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n+1}-\tilde{x})\rangle\\
&\quad = \Phi((\varphi(1) - \alpha_n(\varphi(1)\bar{\gamma} - \gamma\alpha))\|x_n-\tilde{x}\|) + \alpha_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n+1}-\tilde{x})\rangle\\
&\quad \le (1-\alpha_n(\varphi(1)\bar{\gamma} - \gamma\alpha))\Phi(\|x_n-\tilde{x}\|) + \alpha_n\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n+1}-\tilde{x})\rangle.
\end{aligned} \tag{4.1.23}$$

Apply Lemma 2.1.39 to (4.1.23) to conclude $\Phi(\|x_{n+1}-\tilde{x}\|) \to 0$ as $n \to \infty$; that is, $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

Corollary 4.1.4. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\{T(s) : s \ge 0\}$ be a u.a.r. semigroup of nonexpansive mappings with $F(\mathcal{S}) \neq \emptyset$ and $f \in \Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$*

and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. *Let the sequence* $\{x_n\}$ *be generated by the following :*

$$\begin{cases} u_0 = u \in E, \\ v_n = \beta_n u_n + (1-\beta_n)T(t_n)u_n, \\ u_{n+1} = \alpha_n \gamma f(T(t_n)u_n) + (I - \alpha_n A)v_n, \ \ n \geq 0 \end{cases} \tag{4.1.24}$$

where $\{\alpha_n\} \subset (0,1)$ *and* $\{\beta_n\} \subset [0,1]$ *are real sequences satisfying the following conditions :*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$

(C2) $\lim_{n\to\infty} \beta_n = 0$,

(C3) $\lim_{n\to\infty} t_n = \infty$.

Then $\{u_n\}$ *converges strongly to* $\tilde{x}$ *that is obtained in Lemma 4.1.2.*

Proof. Let $\{x_n\}$ be the sequence in given by $x_0 = u_0$ and

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)T(t_n)x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \ \ n \geq 0. \end{cases} \tag{4.1.25}$$

From Theorem 4.1.3, $x_n \to \tilde{x}$. We claim that $u_n \to \tilde{x}$. From (4.1.26) and (4.1.25), we have

$$\begin{aligned} \|y_n - v_n\| &= \|\beta_n x_n + (1-\beta_n)T(t_n)x_n - \beta_n u_n - (1-\beta_n)T(t_n)u_n\| \\ &\leq \beta_n\|x_n - u_n\| + (1-\beta_n)\|T(t_n)x_n - T(t_n)u_n\| \\ &\leq \beta_n\|x_n - u_n\| + (1-\beta_n)\|x_n - u_n\| \\ &= \|x_n - u_n\|. \end{aligned}$$

Again, it then follows that

$$\|x_{n+1} - u_{n+1}\|$$

$$
\begin{aligned}
&= \|\alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n - \alpha_n \gamma f(T(t_n) u_n) - (I - \alpha_n A) v_n\| \\
&\leq \alpha_n \gamma \|f(x_n) - f(T(t_n) u_n)\| + \|I - \alpha_n A\| \|y_n - v_n\| \\
&\leq \alpha_n \gamma \alpha \|x_n - T(t_n) u_n\| + \varphi(1)(1 - \alpha_n \bar{\gamma}) \|x_n - u_n\| \\
&\leq \alpha_n \gamma \alpha \|x_n - T(t_n) \tilde{x}\| + \alpha_n \gamma \alpha \|T(t_n) \tilde{x} - T(t_n) u_n\| \\
&\quad + \varphi(1)(1 - \alpha_n \bar{\gamma}) \|x_n - u_n\| \\
&\leq \alpha_n \gamma \alpha \|x_n - \tilde{x}\| + \alpha_n \gamma \alpha \|\tilde{x} - u_n\| + \varphi(1)(1 - \alpha_n \bar{\gamma}) \|x_n - u_n\| \\
&= \alpha_n \gamma \alpha \|x_n - \tilde{x}\| + \alpha_n \gamma \alpha \|\tilde{x} - x_n\| + \alpha_n \gamma \alpha \|x_n - u_n\| \\
&\quad + \varphi(1)(1 - \alpha_n \bar{\gamma}) \|x_n - u_n\| \\
&= (\varphi(1)(1 - \alpha_n \bar{\gamma}) + \alpha_n \gamma \alpha) \|x_n - u_n\| + (\alpha_n \gamma \alpha + \alpha_n \gamma \alpha) \|x_n - \tilde{x}\| \\
&\leq (1 - \alpha_n (\varphi(1) \bar{\gamma} - \gamma \alpha)) \|x_n - u_n\| + \alpha_n (\varphi(1) \bar{\gamma} - \gamma \alpha) \frac{2 \gamma \alpha}{(\varphi(1) \bar{\gamma} - \gamma \alpha)} \|x_n - \tilde{x}\|.
\end{aligned}
$$

It follows from $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \|x_n - \tilde{x}\| = 0$, and Lemma 2.1.39 that $\|x_n - u_n\| \to 0$. Consequently, $u_n \to \tilde{x}$ as required. □

Corollary 4.1.5. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\{T(s) : s \geq 0\}$ be a u.a.r. semigroup of nonexpansive mappings with $F(\mathcal{S}) \neq \emptyset$ and $f \in \Pi_E$, let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Let the sequence $\{x_n\}$ be generated by the following :*

$$
\begin{cases}
w_0 = w \in E, \\
v_n = \beta_n w_n + (1 - \beta_n) T(t_n) w_n, \\
w_{n+1} = T(t_n) \left(\alpha_n \gamma f(w_n) + (I - \alpha_n A) v_n \right), \ \ n \geq 0
\end{cases}
\tag{4.1.26}
$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1]$ are real sequences satisfying the following conditions :

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$

(C2) $\lim_{n\to\infty} \beta_n = 0$,

(C3) $\lim_{n\to\infty} t_n = \infty$.

Then $\{w_n\}$ converges strongly to $\tilde{x}$ that is obtained in Lemma 4.1.2.

Proof. Define the sequence $\{u_n\}$ and $\{\sigma_n\}$ by

$$u_n = \alpha_n \gamma f(w_n) + (I - \alpha_n A)w_n, \sigma_n = \alpha_{n+1} \qquad \forall n \geq 0. \tag{4.1.27}$$

Taking $p \in F(\mathcal{S})$, we have

$$\begin{aligned}
&\|w_{n+1} - p\| \\
&= \|T(t_n)u_n - T(t_n)p\| \leq \|u_n - p\| \\
&= \|\alpha_n \gamma f(w_n) + (I - \alpha_n A)w_n - (I - \alpha_n A)p - \alpha_n Ap\| \\
&\leq \alpha_n \|\gamma f(w_n) - Ap\| + \|I - \alpha_n A\| \|w_n - p\| \\
&\leq \alpha_n \|\gamma f(w_n) - Ap\| + \varphi(1)(1 - \alpha_n \bar{\gamma})\|w_n - p\| \\
&\leq \alpha_n \|\gamma f(w_n) - \gamma f(p)\| + \alpha_n \|\gamma f(p) - Ap\| + \varphi(1)(1 - \alpha_n \bar{\gamma})\|w_n - p\| \\
&\leq \alpha_n \gamma \alpha \|w_n - p\| + \alpha_n \|\gamma f(p) - Ap\| + \varphi(1)(1 - \alpha_n \bar{\gamma})\|w_n - p\| \\
&= (1 - \alpha_n(\bar{\gamma}\varphi(1) - \gamma\alpha))\|w_n - p\| + \alpha_n(\bar{\gamma}\varphi(1) - \gamma\alpha)\frac{\|\gamma f(p) - Ap\|}{(\bar{\gamma}\varphi(1) - \gamma\alpha)}.
\end{aligned}$$

It follows from induction that

$$\|w_{n+1} - p\| \leq \max\left\{\|w_0 - p\|, \frac{\|\gamma f(p) - A(p)\|}{\bar{\gamma}\varphi(1) - \gamma\alpha}\right\}, n \geq 0.$$

Thus, both $\{u_n\}$ and $\{w_n\}$ are bounded. We observe that

$$u_{n+1} = \alpha_{n+1} f(w_{n+1}) + (I - \alpha_{n+1} A)w_{n+1} = \sigma_n f(T(t_n)u_n) + (I - \sigma_n A)T(t_n)u_n.$$

Thus, Corollary 4.1.4 implies that $\{u_n\}$ converges strongly to some point $\tilde{x}$. In this case, we also have

$$\|w_n - \tilde{x}\| \leq \|w_n - u_n\| + \|u_n - \tilde{x}\| = \alpha_n \|\gamma f(w_n) - Aw_n\| + \|u_n - \tilde{x}\| \to 0.$$

Hence, the sequence $\{w_n\}$ converges strongly to some point $\tilde{x}$. This complete the proof. □

By Lemma 2.2.14, we obtain the following corollary.

Corollary 4.1.6. *Let E be a uniformly convex Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let C be a nonempty closed convex subset of E and $\mathcal{S} = \{T(s) : s \geq 0\}$ a nonexpansive semigroup from C into itself such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \prod_E$, and let A be a strongly positive linear bounded operator with a coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}\varphi(1)}{\alpha}$. Let the sequence $\{x_n\}$ be generated by the following:*

$$\begin{cases} x_0 = x \in E, \\ y_n = \beta_n x_n + (1-\beta_n)\frac{1}{t_n}\displaystyle\int_0^{t_n} T(s)x_n ds, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \ \ n \geq 0 \end{cases} \tag{4.1.28}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1]$ are real sequences satisfying the following conditions :

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$

(C2) $\lim_{n\to\infty} \beta_n = 0$,

(C3) $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly to $\tilde{x}$ that is obtained in Lemma 4.1.2.

Setting $E \equiv H$ and $\beta_n \equiv 0$ a real Hilbert space in Corollary 4.1.6, we have the following result.

Corollary 4.1.7. [90, Theorem 3.2] *Let H be a real Hilbert space. Let C be a nonempty closed convex subset of E and $\mathcal{S} = \{T(s) : s \geq 0\}$ a nonexpansive semigroup from C into itself such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \prod_E$, and let A be a strongly positive linear bounded operator with a coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let the sequence $\{x_n\}$ be generated by the following :*

$$\begin{cases} x_0 = x \in E, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\displaystyle\int_0^{t_n} T(s)x_n ds, \ \ n \geq 0 \end{cases} \tag{4.1.29}$$

where $\{\alpha_n\} \subset (0,1)$ *is a real sequences satisfying the following conditions :*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$

(C2) $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}$ *converges strongly to* $\tilde{x}$ *that is obtained in Lemma 4.1.2. Then* $\{x_n\}$ *converges strongly to* $\tilde{x}$ *which solves the variational inequality:*

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, x \in H. \tag{4.1.30}$$

4.2 The general iterative methods for asymptotically nonexpansive semigroups in Banach spaces

In this section, we prove the strong convergence theorem of general iterative methods for an asymptotically nonexpansive semigroups in Banach spaces.

Theorem 4.2.1. *Let* E *be a reflexive Banach space which admits a weakly continuous duality mapping* J_φ *with gauge* φ *such that* φ *is invariant on* $[0,1]$. *Let* $\mathcal{S} = \{T(s) : s \geq 0\}$ *be a strongly continuous semigroup of asymptotically nonexpansive mappings on* E *with a sequence* $\{L_t\} \subset [1,\infty)$ *and* $F(\mathcal{S}) \neq \emptyset$. *Let* $f \in \Pi_E$ *with coefficient* $\alpha \in (0,1)$, A *a strongly positive bounded linear operator with coefficient* $\bar{\gamma} > 0$ *and* $0 < \gamma < \frac{\varphi(1)\bar{\gamma}}{\alpha}$ *and let* $\{\alpha_n\}$ *and* $\{t_n\}$ *be sequences of real numbers such that* $0 < \alpha_n < 1, t_n > 0$. *Then the following hold:*

(i) *If* $\frac{\frac{1}{t_n}\int_0^{t_n} L_s ds - 1}{\alpha_n} < \varphi(1)\bar{\gamma} - \gamma\alpha,\ \forall n \geq 0$, *then there exists a sequence* $\{y_n\} \subset E$ *defined by*

$$y_n = \alpha_n \gamma f(y_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds, n \geq 0. \tag{4.2.1}$$

(ii) *Suppose, in addition, that* $\mathcal{S}$ *is almost uniformly asymptotically regular and the real sequences* $\{\alpha_n\}$ *and* $\{t_n\}$ *satisfy the following conditions:*

(B1) $\lim_{n\to\infty} t_n = \infty$;

(B2) $\lim_{n\to\infty} \alpha_n = 0$;

(B3) $\lim_{n\to\infty} \frac{(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1)}{\alpha_n} = 0$.

Then $\{y_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality:

$$\langle (A-\gamma f)\tilde{x}, J_\varphi(\tilde{x}-z)\rangle \leq 0, z \in F(\mathcal{S}). \tag{4.2.2}$$

Proof. We first show that the uniqueness of a solution of the variational inequality (4.2.2). Suppose both $\tilde{x} \in F(\mathcal{S})$ and $x^* \in F(\mathcal{S})$ are solutions to (4.2.2), then

$$\langle (A-\gamma f)\tilde{x}, J_\varphi(\tilde{x}-x^*)\rangle \leq 0 \tag{4.2.3}$$

and

$$\langle (A-\gamma f)x^*, J_\varphi(x^*-\tilde{x})\rangle \leq 0. \tag{4.2.4}$$

Adding (4.2.3) and (4.2.4), we obtain

$$\langle (A-\gamma f)\tilde{x} - (A-\gamma f)x^*, J_\varphi(\tilde{x}-x^*)\rangle \leq 0. \tag{4.2.5}$$

Noticing that for any $x, y \in E$,

$$\begin{aligned}
&\langle (A-\gamma f)x - (A-\gamma f)y, J_\varphi(x-y)\rangle \\
&\quad = \langle A(x-y), J_\varphi(x-y)\rangle - \gamma\langle f(x)-f(y), J_\varphi(x-y)\rangle \\
&\quad \geq \bar{\gamma}\|x-y\|\varphi(\|x-y\|) - \gamma\|f(x)-f(y)\|\|J_\varphi(x-y)\| \\
&\quad \geq \bar{\gamma}\Phi(\|x-y\|) - \gamma\alpha\Phi(\|x-y\|) \\
&\quad = (\bar{\gamma}-\gamma\alpha)\Phi(\|x-y\|) \\
&\quad \geq (\bar{\gamma}\varphi(1)-\gamma\alpha)\Phi(\|x-y\|) \geq 0.
\end{aligned}$$

Therefore $\tilde{x} = x^*$ and the uniqueness is proved. Below we use $\tilde{x}$ to denote the unique solution of (4.2.2).

Since $\lim_{n\to\infty} \alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n <$

$\varphi(1)\|A\|^{-1}$.

For each integer $n \geq 0$, define a mapping $G_n : E \to E$ by

$$G_n(y) = \alpha_n \gamma f(y) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)yds, \forall y \in E.$$

We show that G_n is a contraction mapping. For any $x, y \in E$,

$$\begin{aligned}
&\|G_n(x) - G_n(y)\| \\
&= \left\|\alpha_n\gamma f(x) + (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)xds - \alpha_n\gamma f(y) - (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)yds\right\| \\
&\leq \|\alpha_n\gamma(f(x)-f(y))\| + \left\|(I-\alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)xds - \frac{1}{t_n}\int_0^{t_n} T(s)yds\right)\right\| \\
&\leq \alpha_n\gamma\alpha\|x-y\| + \varphi(1)(1-\alpha_n\bar{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)\|x-y\| \\
&= \left(\alpha_n\gamma\alpha + \varphi(1)\frac{1}{t_n}\int_0^{t_n} L_s ds - \varphi(1)\alpha_n\bar{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s ds\right)\|x-y\| \\
&\leq \left(\frac{1}{t_n}\int_0^{t_n} L_s ds - \alpha_n\left(\varphi(1)\bar{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s ds - \gamma\alpha\right)\right)\|x-y\|.
\end{aligned}$$

Since $0 < \frac{\frac{1}{t_n}\int_0^{t_n} L_s ds - 1}{\alpha_n} < \varphi(1)\bar{\gamma} - \gamma\alpha$, we have

$$0 < \frac{\frac{1}{t_n}\int_0^{t_n} L_s ds - 1}{\alpha_n} < \varphi(1)\bar{\gamma} - \gamma\alpha \leq \varphi(1)\bar{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s ds - \gamma\alpha.$$

It then follows that $0 < \left(\frac{1}{t_n}\int_0^{t_n} L_s ds - \alpha_n\left(\varphi(1)\bar{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s ds - \gamma\alpha\right)\right) < 1$. We have G_n is a contraction map with coefficient $\left(\frac{1}{t_n}\int_0^{t_n} L_s ds - \alpha_n\left(\varphi(1)\bar{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s ds - \gamma\alpha\right)\right)$. Then, for each $n \geq 0$, there exists a unique $y_n \in E$ such that $G_n y_n = y_n$, that is,

$$y_n = \alpha_n\gamma f(y_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds, n \geq 0.$$

Hence (i) is proved.

(ii) We first show that $\{y_n\}$ is bounded. Letting $p \in F(\mathcal{S})$ and using Lemma 2.1.33, we can calculate the following

$$\begin{aligned}
&\|y_n - p\| \\
&= \left\|\alpha_n\gamma f(y_n) + (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - p\right\|
\end{aligned}$$

$$= \left\| \alpha_n \gamma f(y_n) - \alpha_n \gamma f(p) + \alpha_n \gamma f(p) + (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds \right.$$
$$\left. - (I - \alpha_n A)p - (\alpha_n A)p \right\|$$
$$\leq \alpha_n \gamma \|f(y_n) - f(p)\| + \alpha_n \|\gamma f(p) - A(p)\|$$
$$+ \varphi(1)(1 - \alpha_n \bar{\gamma}) \left\| \frac{1}{t_n} \int_0^{t_n} T(s) y_n ds - \frac{1}{t_n} \int_0^{t_n} T(s) p ds \right\|$$
$$\leq \alpha_n \gamma \alpha \|y_n - p\| + \alpha_n \|\gamma f(p) - A(p)\| + \varphi(1)(1 - \alpha_n \bar{\gamma}) \frac{1}{t_n} \int_0^{t_n} L_s ds \|y_n - p\|$$
$$\leq \alpha_n \gamma \alpha \|y_n - p\| + \alpha_n \|\gamma f(p) - A(p)\| + \varphi(1) \frac{1}{t_n} \int_0^{t_n} L_s ds \|y_n - p\|$$
$$- \varphi(1) \alpha_n \bar{\gamma} \frac{1}{t_n} \int_0^{t_n} L_s ds \|y_n - p\|$$
$$\leq \alpha_n \gamma \alpha \|y_n - p\| + \alpha_n \|\gamma f(p) - A(p)\| + \frac{1}{t_n} \int_0^{t_n} L_s ds \|y_n - p\|$$
$$- \varphi(1) \alpha_n \bar{\gamma} \frac{1}{t_n} \int_0^{t_n} L_s ds \|y_n - p\|.$$

Thus, we get that

$$\|y_n - p\| \leq \frac{\alpha_n \|\gamma f(p) - A(p)\|}{1 - \alpha_n \gamma \alpha - \frac{1}{t_n} \int_0^{t_n} L_s + \varphi(1) \alpha_n \bar{\gamma} \frac{1}{t_n} \int_0^{t_n} L_s ds}. \tag{4.2.6}$$

Calculating the righthand side of above inequality, we have

$$\frac{\alpha_n \|\gamma f(p) - A(p)\|}{1 - \alpha_n \gamma \alpha - \frac{1}{t_n} \int_0^{t_n} L_s + \varphi(1) \alpha_n \bar{\gamma} \frac{1}{t_n} \int_0^{t_n} L_s ds + \varphi(1) \alpha_n \bar{\gamma}}$$
$$= \frac{\alpha_n \|\gamma f(p) - A(p)\|}{1 - \alpha_n \gamma \alpha - \frac{1}{t_n} \int_0^{t_n} L_s + \varphi(1) \alpha_n \bar{\gamma} \frac{1}{t_n} \int_0^{t_n} L_s ds + \varphi(1) \alpha_n \bar{\gamma} - \varphi(1) \alpha_n \bar{\gamma}}$$
$$= \frac{\alpha_n \|\gamma f(p) - A(p)\|}{\varphi(1) \alpha_n \bar{\gamma} - \alpha_n \alpha \gamma + 1 - \frac{1}{t_n} \int_0^{t_n} L_s ds + \varphi(1) \alpha_n \bar{\gamma} \frac{1}{t_n} \int_0^{t_n} L_s ds - \varphi(1) \alpha_n \bar{\gamma}}$$
$$= \frac{\alpha_n \|\gamma f(p) - A(p)\|}{\alpha_n (\varphi(1) \bar{\gamma} - \alpha \gamma) - \left(\frac{1}{t_n} \int_0^{t_n} L_s ds - 1 \right) + \varphi(1) \alpha_n \bar{\gamma} \left(\frac{1}{t_n} \int_0^{t_n} L_s ds - 1 \right)}$$
$$= \frac{\alpha_n \|\gamma f(p) - A(p)\|}{\alpha_n (\varphi(1) \bar{\gamma} - \alpha \gamma) - (1 - \varphi(1) \alpha_n \bar{\gamma}) \left(\frac{1}{t_n} \int_0^{t_n} L_s ds - 1 \right)}.$$

Thus, we get that

$$\|y_n - p\| \leq \frac{\|\gamma f(p) - A(p)\|}{(\varphi(1)\bar{\gamma} - \alpha\gamma) - (1 - \alpha_n\bar{\gamma})d_n}, \tag{4.2.7}$$

where $d_n = \dfrac{\left(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1\right)}{\alpha_n}$. Thus, there exists $N > 0$ such that $\|y_n - p\| \leq \dfrac{\|\gamma f(p) - A(p)\|}{(\varphi(1)\bar{\gamma} - \gamma\alpha)}$, for all $n \geq N$. Therefore, $\{y_n\}$ is bounded and hence $\{f(y_n)\}$ and $\{\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds\}$ are also bounded.

Let $\delta_{t_n}(y_n) = \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds$. Then, from (4.2.1), we get

$$\|y_n - \delta_{t_n}(y_n)\| = \alpha_n\|\gamma f(y_n) - A\delta_{t_n}(y_n)\| \to 0 \text{ as } n \to \infty. \tag{4.2.8}$$

Moreover, the fact that $\mathcal{S}$ is almost uniformly asymptotically regular and (4.2.8) implies that,

$$\begin{aligned} \|y_n - T(h)y_n\| &\leq \|y_n - \delta_{t_n}(y_n)\| + \|\delta_{t_n}(y_n) - T(h)\delta_{t_n}(y_n)\| \\ &\quad + \|T(h)\delta_{t_n}(y_n) - T(h)y_n\| \\ &\leq \|y_n - \delta_{t_n}(y_n)\| + \|\delta_{t_n}(y_n) - T(h)\delta_{t_n}(y_n)\| \\ &\quad + L_s\|\delta_{t_n}(y_n) - y_n\| \to 0. \end{aligned} \tag{4.2.9}$$

It follows from reflexivity of E and the boundedness of sequence $\{y_n\}$ that there exists $\{y_{n_j}\}$ which is a subsequence of $\{y_n\}$ converging weakly to $w \in E$ as $j \to \infty$. Since J_φ is weakly sequentially continuous, we have by Lemma 2.1.31 that

$$\limsup_{j\to\infty} \Phi(\|y_{n_j} - y\|) = \limsup_{j\to\infty} \Phi(\|y_{n_j} - w\|) + \Phi(\|y - w\|), \text{ for all } x \in E.$$

Let

$$H(x) = \limsup_{j\to\infty} \Phi(\|y_{n_j} - y\|), \text{ for all } y \in E.$$

It follows that

$$H(y) = H(w) + \Phi(\|y - w\|), \text{ for all } y \in E.$$

For $h \geq 0$, from (4.2.9) we obtain

$$\begin{aligned} H(T(h)w) &= \limsup_{j\to\infty} \Phi(\|y_{n_j} - T(h)w\|) = \limsup_{j\to\infty} \Phi(\|T(h)y_{n_j} - T(h)w\|) \\ &\leq \limsup_{j\to\infty} \Phi(\|y_{n_j} - w\|) = H(w). \end{aligned} \tag{4.2.10}$$

On the other hand, however,

$$H(T(h)w) = H(w) + \Phi(\|T(h)w - w\|). \tag{4.2.11}$$

It follows from (4.2.10) and (4.2.11) that

$$\Phi(\|T(h)w - w\|) = H(T(h)w) - H(w) \leq 0.$$

This implies that $T(h)w = w$ for all $h \geq 0$, and so $w \in F(\mathcal{S})$. Next we show that $y_{n_j} \to w$ as $j \to \infty$. In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau, \forall t \geq 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function, then for $1 \geq k \geq 0$, $\varphi(kx) \leq \varphi(x)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(kx)dx \leq k\int_0^t \varphi(x)dx = k\Phi(t).$$

Following Lemma 2.1.31, we have

$$\begin{aligned} &\Phi(\|y_n - w\|) \\ &= \Phi(\|(I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - (I-\alpha_n A)w \\ &\quad + \alpha_n(\gamma f(y_n) - \gamma f(w) + \gamma f(w) - A(w))\|) \\ &\leq \Phi\left(\left\|(I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - (I-\alpha_n A)w + \alpha_n\gamma(f(y_n) - f(w))\right\|\right) \\ &\quad + \alpha_n\langle\gamma f(w) - A(w), J_\varphi(y_n - w)\rangle \\ &\leq \Phi\left(\varphi(1)(1-\alpha_n\bar{\gamma})\left\|\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)w ds\right\| + \alpha_n\gamma\alpha\|y_n - w\|\right) \\ &\quad + \alpha_n\gamma\langle f(w) - f(w), J_\varphi(y_n - w)\rangle \\ &\leq \Phi\left(\varphi(1)(1-\alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds)\|y_n - w\| + \alpha_n\gamma\alpha\|y_n - w\|\right) \\ &\quad + \alpha_n\langle\gamma f(w) - A(w), J_\varphi(y_n - w)\rangle \\ &\leq \Phi\left(\left[\varphi(1)(1-\alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds) + \alpha_n\gamma\alpha\right]\|y_n - w\|\right) \\ &\quad + \alpha_n\langle\gamma f(w) - A(w), J_\varphi(y_n - w)\rangle \end{aligned}$$

$$\leq \left[\varphi(1)(1-\alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds) + \alpha_n\gamma\alpha\right]\Phi(\|y_n - w\|)$$
$$+\alpha_n\langle \gamma f(w) - A(w), J_\varphi(y_n - w)\rangle.$$

This implies that

$$\Phi(\|y_n - w\|) \leq \frac{1}{1-\varphi(1)(1-\alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds) + \alpha_n\gamma\alpha}\alpha_n\langle \gamma f(w) - A(w), J_\varphi(y_n - w)\rangle,$$

also

$$\Phi(\|y_n - w\|) \leq \frac{1}{(\varphi(1)\bar{\gamma} - \alpha\gamma) - (1-\alpha_n\bar{\gamma})d_n}\langle \gamma f(w) - A(w), J_\varphi(y_n - w)\rangle,$$

where $d_n = \dfrac{\left(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1\right)}{\alpha_n}$. Now observing that $y_{n_j} \rightharpoonup w$ implies $J_\varphi(y_{n_j} - w) \rightharpoonup^* 0$, we conclude from the above inequality that

$$\Phi(\|y_{n_j} - w\|) \to 0 \text{ as } j \to \infty.$$

Hence $y_{n_j} \to w$ as $j \to \infty$. Next we prove that w solves the variational inequality (4.2.2). For any $z \in F(\mathcal{S})$, we observe that

$$\begin{aligned}
&\langle (y_n - \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds) - (z - \frac{1}{t_n}\int_0^{t_n} T(s)z ds), J_\varphi(y_n - z)\rangle \\
&= \langle y_n - z, J_\varphi(y_n - z)\rangle \\
&\quad - \langle \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)z ds, J_\varphi(y_n - z)\rangle \\
&\geq \Phi(\|y_n - z\|) - \left\|\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)z ds\right\| \|J_\varphi(y_n - z)\| \\
&\geq \Phi(\|y_n - z\|) - \frac{1}{t_n}\int_0^{t_n} L_s ds \, \|y_n - z\| \, \|J_\varphi(y_n - z)\| \\
&= \Phi(\|y_n - z\|) - \frac{1}{t_n}\int_0^{t_n} L_s ds \Phi(\|y_n - z\|) \\
&= \left(1 - \frac{1}{t_n}\int_0^{t_n} L_s ds\right)\Phi(\|y_n - z\|).
\end{aligned} \tag{4.2.12}$$

Since

$$y_n = \alpha_n\gamma f(y_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds,$$

we can derive that

$$(A-\gamma f)(y_n) = -\frac{1}{\alpha_n}\left(y_n - \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds\right) + \left(A(y_n) - A(\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds)\right).$$

Since Φ is strictly increasing and $\|y_n - p\| \leq M$ for some $M > 0$, we have $\Phi(\|y_n - p\|) \leq \Phi(M)$. Thus

$$\begin{aligned}
&\langle (A-\gamma f)(y_n), J_\varphi(y_n - z)\rangle \\
&\quad = -\frac{1}{\alpha_n}\left\langle \left(y_n - \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds\right) - \left(z - \frac{1}{t_n}\int_0^{t_n} T(s)z ds\right), J_\varphi(y_n - z)\right\rangle \\
&\qquad + \left\langle A(y_n) - A(\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds), J_\varphi(y_n - z)\right\rangle \\
&\quad \leq \frac{\left(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1\right)}{\alpha_n}\Phi(\|y_n - z\|) \\
&\qquad + \left\langle A(y_n) - A(\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds), J_\varphi(y_n - z)\right\rangle \\
&\quad \leq \frac{\left(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1\right)}{\alpha_n}\Phi(M) \\
&\qquad + \left\langle A\left(y_n - \frac{1}{t_n}\int_0^{t_n} T(s)y_n ds\right), J_\varphi(y_n - z)\right\rangle \qquad (4.2.13)
\end{aligned}$$

Noticing that

$$y_{n_j} - \frac{1}{t_{n_j}}\int_0^{t_{n_j}} T(s)y_{n_j} ds \to w - \frac{1}{t_{n_j}}\int_0^{t_{n_j}} T(s)w ds = w - w = 0.$$

Now using (B3) and replacing n with n_j in (4.2.13) and letting $j \to \infty$, we have

$$\langle (A-\gamma f)w, J_\varphi(w - z)\rangle \leq 0.$$

So, $w \in F(\mathcal{S})$ is a solution of the variational inequality (4.2.2), and hence $w = \tilde{x}$ by the uniqueness. Applying Lemma 2.1.38, we can conclude that $y_n \to \tilde{x}$ as $n \to \infty$. This completes the proof.

□

If $A = I$, the identity mapping on E, and $\gamma \equiv 1$, then Theorem 4.2.1 reduces to the following corollary.

Corollary 4.2.2. *Let E be a reflexive Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings on E with a sequence $\{L_t\} \subset [1,\infty)$ and $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$ and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1$ and $t_n > 0$. Then the following hold:*

(i) *If $\frac{1}{t_n}\int_0^{t_n} L_s ds - 1 < \alpha_n(1-\alpha), \ \forall n \in \mathbb{N}$, then there exists a sequence $\{y_n\}$ defined by*

$$y_n = \alpha_n f(y_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)y_n ds, n \geq 0.$$

(ii) *Suppose, in addition, that $\mathcal{S}$ is almost uniformly asymptotically regular and the real sequences $\{\alpha_n\}$ and $\{t_n\}$ satisfy the following:*

(B1) $\lim_{n\to\infty} t_n = \infty$;

(B2) $\lim_{n\to\infty} \alpha_n = 0$;

(B3) $\lim_{n\to\infty} \frac{(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1)}{\alpha_n} = 0$.

Then $\{y_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality :

$$\langle (I-f)\tilde{x}, J_\varphi(\tilde{x}-z)\rangle \leq 0, z \in F(\mathcal{S}).$$

If $E := H$ is a Hilbert space and $\mathcal{S} = \{T(s) : s \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on H, then we have $L_t \equiv 1$ and Theorem 4.2.1 reduces to the following corollary.

Corollary 4.2.3. [90, Theorem 3.1] *Let H be a real Hilbert space. Suppose that $f : H \to H$ is a contraction with coefficient $\alpha \in (0,1)$ and $\mathcal{S} = \{T(s) : s \geq 0\}$ a strongly continuous semigroup of nonexpansive mappings on H such that $F(\mathcal{S}) \neq \emptyset$. Let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1, t_n > 0$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$, then for any $0 < \gamma < \bar{\gamma}/\alpha$, there is a unique $\{y_n\}$ in H such that*

$$y_n = \alpha_n \gamma f(y_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s) y_n ds, n \geq 0$$

and the iterative sequence $\{y_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality :

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z\rangle \leq 0, \quad \text{for } z \in F(\mathcal{S}).$$

Theorem 4.2.4. *Let E be a reflexive strictly convex Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings on E with a sequence $\{L_t\} \subset [1,\infty)$ and $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$, A a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\varphi(1)\bar{\gamma}}{\alpha}$. For any $x_0 \in C$, let the sequence $\{x_n\}$ be defined by*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s) x_n ds, n \geq 0. \tag{4.2.14}$$

Suppose, in addition, that $\mathcal{S}$ is almost uniformly asymptotically regular. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1, t_n > 0$,

(C1) $\lim_{n\to\infty} t_n = \infty$;

(C2) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty} \frac{(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1)}{\alpha_n} = 0$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality (4.2.2).

Proof. First we show that $\{x_n\}$ is bounded. By condition (C3) and given $0 < \varepsilon < \varphi(1)\bar{\gamma} - \alpha\gamma$ there exists $N > 0$ such that $\frac{\frac{1}{t_n}\int_0^{t_n} L_s ds - 1}{\alpha_n} < \varepsilon$ for all $n \geq N$. Thus

$$(1 - \varphi(1)\bar{\gamma}\alpha_n)(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1) \leq \frac{1}{t_n}\int_0^{t_n} L_s ds - 1 < \varepsilon\alpha_n,$$

for all $n \geq N$. Since $\lim_{n\to\infty}\alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n < \varphi(1)\|A\|^{-1}$. We claim that $\|x_n - p\| \leq M, n \geq 0$ where

$$M := \max\left\{\|x_0 - p\|, \ldots, \|x_N - p\|, \frac{\|f(p) - p\|}{(\varphi(1)\bar{\gamma} - \alpha\gamma - \varepsilon)}\right\}.$$

Let $p \in F(\mathcal{S})$. Then from (4.2.23) we get that

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \left\|\alpha_n\gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - p\right\| \\
&\leq \left\|\alpha_n\gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - \alpha_n A(p) - (I - \alpha_n A)p\right\| \\
&\leq \|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(p) + \alpha_n\gamma f(p) - \alpha_n A(p)\| \\
&\quad + \left\|(I - \alpha_n A)\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)p ds\right)\right\| \\
&\leq \|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(p) + \alpha_n\gamma f(p) - \alpha_n A(p)\| \\
&\quad + \|I - \alpha_n A\|\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)p ds\right\| \\
&\leq \alpha_n\|\gamma f(p) - Ap\| + \alpha_n\alpha\gamma\|x_n - p\| \\
&\quad + \varphi(1)(1 - \alpha_n\bar{\gamma})\left(\frac{1}{t_n}\int_0^{t_n} L_s ds\right)\|x_n - p\| \\
&= \alpha_n\|\gamma f(p) - Ap\| + \left(\alpha_n\alpha\gamma + \varphi(1)(1 - \alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds)\right)\|x_n - p\| \\
&\leq \alpha_n\|\gamma f(p) - Ap\| \\
&\quad + \left(\alpha_n\alpha\gamma + \frac{1}{t_n}\int_0^{t_n} L_s ds - \varphi(1)\alpha_n\bar{\gamma}\frac{1}{t_n}\int_0^{t_n} L_s ds\right)\|x_n - p\| \\
&\leq \alpha_n\|\gamma f(p) - Ap\| \\
&\quad + \left(1 + (1 - \varphi(1)\alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1) - \alpha_n(\varphi(1)\bar{\gamma} - \alpha\gamma)\right)\|x_n - p\|
\end{aligned}$$

$$\leq \alpha_n \|f(p) - p\| + (1 - \alpha_n(\varphi(1)\bar{\gamma} - \alpha\gamma) + \varepsilon\alpha_n) \|x_n - p\|$$
$$= \alpha_n \|f(p) - p\| + (1 - \alpha_n(\varphi(1)\bar{\gamma} - \alpha\gamma - \varepsilon)) \|x_n - p\|$$
$$\leq \max\left\{\frac{\|f(p) - p\|}{(\varphi(1)\bar{\gamma} - \alpha\gamma - \varepsilon)}, \|x_n - p\|\right\}.$$

By induction,

$$\|x_n - p\| \leq \max\left\{\frac{\|f(p) - p\|}{(\varphi(1)\bar{\gamma} - \alpha\gamma - \varepsilon)}, \|x_N - p\|\right\} \text{ for all } n \geq N,$$

and hence $\{x_n\}$ is bounded, so are $\{f(x_n)\}$ and $\left\{\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds\right\}$.
Let $\delta_{t_n}(x_n) := \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds$. Then, since $\alpha_n \to 0$ as $n \to \infty$, we obtain that

$$\|x_{n+1} - \delta_{t_n}(x_n)\| = \alpha_n \|\gamma f(x_n) - A\delta_{t_n}(x_n)\| \to 0 \text{ as } n \to \infty. \tag{4.2.15}$$

For any $h > 0$, we have

$$\begin{aligned}
&\|T(h)x_{n+1} - x_{n+1}\| \\
&\quad \leq \|T(h)x_{n+1} - T(h)\delta_{t_n}(x_n)\| + \|T(h)\delta_{t_n}(x_n) - \delta_{t_n}(x_n)\| \\
&\qquad + \|\delta_{t_n}(x_n) - x_{n+1}\| \\
&\quad \leq L_h \|x_{n+1} - \delta_{t_n}(x_n)\| + \|T(h)\delta_{t_n}(x_n) - \delta_{t_n}(x_n)\| + \|\delta_{t_n}(x_n) - x_{n+1}\|,
\end{aligned}$$

it follows from (4.2.15) and $\mathcal{S}$ is almost uniformly asymptotically regular that

$$\|T(h)x_{n+1} - x_{n+1}\| \to 0 \text{ as } n \to \infty. \tag{4.2.16}$$

Next, we prove that

$$\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \leq 0, \tag{4.2.17}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n_k} - \tilde{x})\rangle = \limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle. \tag{4.2.18}$$

If follows from reflexivity of E and the boundedness of sequence $\{x_{n_k}\}$ that there exists $\{x_{n_{k_i}}\}$ which is a subsequence of $\{x_{n_k}\}$ converging weakly to $w \in E$ as

$i \to \infty$. Since J_φ is weakly continuous, we have by Lemma 2.1.31 that

$$\limsup_{i\to\infty} \Phi(\|x_{n_{k_i}} - x\|) = \limsup_{i\to\infty} \Phi(\|x_{n_{k_i}} - w\|) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

Let

$$H(x) = \limsup_{i\to\infty} \Phi(\|x_{n_{k_i}} - x\|), \text{ for all } x \in E.$$

It follows that

$$H(x) = H(w) + \Phi(\|x - w\|), \text{ for all } x \in E.$$

From (4.2.16), for each $h > 0$, we obtain

$$\begin{aligned} H(T(h)w) &= \limsup_{i\to\infty} \Phi(\|x_{n_{k_i}} - T(h)w\|) = \limsup_{i\to\infty} \Phi(\|T(h)x_{n_{k_i}} - T(h)w\|) \\ &\le \limsup_{i\to\infty} \Phi(\|x_{n_{k_i}} - w\|) = H(w). \end{aligned} \tag{4.2.19}$$

On the other hand, however,

$$H(T(h)w) = H(w) + \Phi(\|T(h)w - w\|). \tag{4.2.20}$$

It follows from (4.2.19) and (4.2.20) that

$$\Phi(\|T(h)w - w\|) = H(T(h)w) - H(w) \le 0.$$

This implies that $T(h)w = w$ for all $h > 0$, and so $w \in F(\mathcal{S})$. Since the duality map J_φ is single-valued and weakly continuous, we get that

$$\begin{aligned} \limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle &= \lim_{k\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n_k} - \tilde{x})\rangle \\ &= \lim_{i\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_{n_{k_i}} - \tilde{x})\rangle \\ &= \langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - w)\rangle \le 0 \end{aligned}$$

as required. Finally, we show that $x_n \to \tilde{x}$ as $n \to \infty$.

$$\Phi(\|x_{n+1} - \tilde{x}\|)$$

$$
\begin{aligned}
&= \Phi(\|(I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - (I-\alpha_n A)\tilde{x} \\
&\quad + \alpha_n(\gamma f(x_n) - \gamma f(\tilde{x}) + \gamma f(\tilde{x}) - A(\tilde{x}))\|) \\
&\leq \Phi\left(\left\|(I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - (I-\alpha_n A)\tilde{x} + \alpha_n\gamma(f(x_n) - f(\tilde{x}))\right\|\right) \\
&\quad + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(y_n - \tilde{x})\rangle \\
&\leq \Phi\left(\varphi(1)(1-\alpha_n\bar{\gamma})\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)\tilde{x} ds\right\| + \alpha_n\gamma\alpha\|x_n - \tilde{x}\|\right) \\
&\quad + \alpha_n\gamma\langle f(\tilde{x}) - f(\tilde{x}), J_\varphi(y_n - \tilde{x})\rangle \\
&\leq \Phi\left(\varphi(1)(1-\alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds)\|x_n - \tilde{x}\| + \alpha_n\gamma\alpha\|x_n - \tilde{x}\|\right) \\
&\quad + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\leq \Phi\left(\left[\varphi(1)(1-\alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds) + \alpha_n\gamma\alpha\right]\|x_n - \tilde{x}\|\right) \\
&\quad + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\leq \left[\varphi(1)(1-\alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds) + \alpha_n\gamma\alpha\right]\Phi\left(\|x_n - \tilde{x}\|\right) \\
&\quad + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\leq \left[(1-\varphi(1)\alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1) + 1 - \alpha_n(\varphi(1)\bar{\gamma} - \gamma\alpha)\right]\Phi\left(\|x_n - \tilde{x}\|\right) \\
&\quad + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\leq (1-\alpha_n(\varphi(1)\bar{\gamma} - \gamma\alpha))\Phi\left(\|x_n X - \tilde{x}\|\right) + (1-\varphi(1)\alpha_n\bar{\gamma})(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1)M \\
&\quad + \alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \qquad (4.2.21)
\end{aligned}
$$

where $M > 0$ such that $\Phi(\|x_n - \tilde{x}\|) \leq M$. Putting

$$s_n = \alpha_n(\varphi(1)\bar{\gamma} - \gamma\alpha)$$

and

$$
\begin{aligned}
t_n &= \left(\frac{1-\varphi(1)\alpha_n\bar{\gamma}}{\varphi(1)\bar{\gamma} - \gamma\alpha}\right)\left(\frac{\frac{1}{t_n}\int_0^{t_n} L_s ds - 1}{\alpha_n}\right)M \\
&\quad + \frac{1}{(\varphi(1)\bar{\gamma} - \gamma\alpha)}\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle.
\end{aligned}
$$

Then (4.2.21) is reduced to

$$\Phi(\|x_{n+1} - \tilde{x}\|) \leq (1-s_n)\Phi(\|x_n - \tilde{x}\|) + s_n t_n. \qquad (4.2.22)$$

Applying Lemma 2.1.39 to (4.2.22), we conclude that $\Phi(\|x_{n+1}-\tilde{x}\|) \to 0$ as $n \to \infty$, that is, $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

If $A = I$, the identity mapping on E, and $\gamma \equiv 1$, then Theorem 4.2.4 reduces to the following corollary.

Corollary 4.2.5. *Let E be a reflexive strictly convex Banach space which admits a weakly continuous duality mapping J_φ with gauge φ such that φ is invariant on $[0,1]$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings from C into C with a sequence $\{L_t\} \subset [1,\infty)$, $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_E$ with coefficient $\alpha \in (0,1)$ and the sequence $\{x_n\}$ be defined by $x_0 \in C$,*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, n \geq 0. \tag{4.2.23}$$

Suppose, in addition, that $\mathcal{S}$ is almost uniformly asymptotically regular. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1, t_n > 0$,

(C1) $\lim_{n\to\infty} t_n = \infty$;

(C2) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;

(C3) $\lim_{n\to\infty} \frac{(\frac{1}{t_n}\int_0^{t_n} L_s ds - 1)}{\alpha_n} = 0$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality

$$\langle (I-f)\tilde{x}, J_\varphi(\tilde{x}-z)\rangle \leq 0, z \in F(\mathcal{S}).$$

If $E := H$ is a Hilbert space and $\mathcal{S} = \{T(s) : s \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on H, then we have $L_t \equiv 1$ and Theorem 4.2.4 reduces to the following corollary.

Corollary 4.2.6. [90, Theorem 3.2] *Let H be a real Hilbert space. Suppose that $f : H \to H$ is a contraction with coefficient $\alpha \in (0,1)$ and $\mathcal{S} = \{T(s) : s \geq 0\}$ a* *strongly continuous semigroup of nonexpansive mappings on H such that $F(\mathcal{S}) \neq \emptyset$. Let A be a strongly positive bounded linear operator with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \bar{\gamma}/\alpha$ and the sequence $\{x_n\}$ be defined by $x_0 \in C$,*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, n \geq 0.$$

Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1, t_n > 0$,

(C1) $\lim_{n\to\infty} t_n = \infty$;

(C2) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty$;

Then $\{x_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z\rangle \leq 0, \quad \text{for } z \in F(\mathcal{S}).$$

4.3 The general iterative methods for asymptotically nonexpansive semigroups in Banach spaces

In this section, we obtain the strongly convergence theorems of general iterative schemes for asymptotically nonexpansive semigroups.

Theorem 4.3.1. *Let E be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, C be a nonempty, bounded, closed and convex subset of E such that $C \pm C \subset C$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on C with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < \sqrt{N(E)}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_C$ with coefficient $\alpha \in (0,1)$, A a strongly positive bounded linear operator on C with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\varphi(1)\bar{\gamma}}{\alpha}$ and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1, t_n > 0$. Then the following hold:*

(i) *If* $\dfrac{k_n - 1}{\alpha_n} < \varphi(1)\bar{\gamma} - \gamma\alpha,\ \forall n \geq 1$, *then there exists a sequence* $\{y_n\} \subset E$ *defined by*

$$y_n = \alpha_n \gamma f(y_n) + (I - \alpha_n A)T^n(t_n)y_n, \quad n \geq 1. \tag{4.3.1}$$

(ii) *Suppose, in addition,* $\lim_{n\to\infty} \|y_n - T(t)y_n\| = 0$ *uniformly in* $t \in [0, \infty)$ *and the real sequence* $\{\alpha_n\}$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0$.

Then $\{y_n\}$ *converges strongly as* $n \to \infty$ *to a common fixed point* $\tilde{x}$ *in* $F(\mathcal{S})$ *which solves the variational inequality:*

$$\langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0,\ \forall z \in F(\mathcal{S}). \tag{4.3.2}$$

Proof. We first show that the uniqueness of a solution of the variational inequality (4.3.2). Suppose both $\tilde{x} \in F(\mathcal{S})$ and $x^* \in F(\mathcal{S})$ are solutions to (4.3.2), then

$$\langle (A - \gamma f)\tilde{x}, J_\varphi(\tilde{x} - x^*)\rangle \leq 0 \tag{4.3.3}$$

and

$$\langle (A - \gamma f)x^*, J_\varphi(x^* - \tilde{x})\rangle \leq 0. \tag{4.3.4}$$

Adding (4.3.3) and (4.3.4), we obtain

$$\langle (A - \gamma f)\tilde{x} - (A - \gamma f)x^*, J_\varphi(\tilde{x} - x^*)\rangle \leq 0. \tag{4.3.5}$$

Noticing that for any $x, y \in C$,

$$\begin{aligned}
&\langle (A - \gamma f)x - (A - \gamma f)y, J_\varphi(x - y)\rangle \\
&\quad = \langle A(x - y), J_\varphi(x - y)\rangle - \gamma\langle f(x) - f(y), J_\varphi(x - y)\rangle \\
&\quad \geq \bar{\gamma}\|x - y\|\varphi(\|x - y\|) - \gamma\|f(x) - f(y)\|\|J_\varphi(x - y)\| \\
&\quad \geq \bar{\gamma}\Phi(\|x - y\|) - \gamma\alpha\Phi(\|x - y\|) \\
&\quad = (\bar{\gamma} - \gamma\alpha)\Phi(\|x - y\|)
\end{aligned}$$

$$\geq (\bar{\gamma}\varphi(1) - \gamma\alpha)\Phi(\|x-y\|) \geq 0. \tag{4.3.6}$$

Therefore $\tilde{x} = x^*$ and the uniqueness is proved. Below, we use $\tilde{x}$ to denote the unique solution of (4.3.2). Since $\lim_{n\to\infty} \alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n < \varphi(1)\|A\|^{-1}$. For each integer $n \geq 1$, define a mapping $G_n : C \to C$ by

$$G_n(y) = \alpha_n \gamma f(y) + (I - \alpha_n A)T^n(t_n)y, \quad \forall y \in C.$$

We shall show that G_n is a contraction mapping. For any $x, y \in C$,

$$\begin{aligned}
\|G_n(x) - G_n(y)\| &= \|\alpha_n\gamma f(x) + (I-\alpha_n A)T^n(t_n)x - \alpha_n\gamma f(y) - (I-\alpha_n A)T^n(t_n)yds\| \\
&\leq \|\alpha_n\gamma(f(x) - f(y))\| + \|(I-\alpha_n A)\left(T^n(t_n)x - T^n(t_n)y\right)\| \\
&\leq \alpha_n\gamma\alpha\|x-y\| + \varphi(1)(1-\alpha_n\bar{\gamma})k_n\|x-y\| \\
&= (\alpha_n\gamma\alpha + \varphi(1)(1-\alpha_n\bar{\gamma})k_n)\,\|x-y\| \\
&\leq (k_n - \alpha_n\gamma\alpha + \varphi(1)\alpha_n\bar{\gamma}k_n)\,\|x-y\| \\
&\leq (k_n - \alpha_n(\varphi(1)\bar{\gamma}k_n - \gamma\alpha))\,\|x-y\|.
\end{aligned}$$

Since $0 < \frac{k_n - 1}{\alpha_n} < \varphi(1)\bar{\gamma} - \gamma\alpha$, we have

$$0 < \frac{k_n - 1}{\alpha_n} < \varphi(1)\bar{\gamma} - \gamma\alpha \leq \varphi(1)\bar{\gamma}k_n - \gamma\alpha.$$

It then follows that $0 < (k_n - \alpha_n\left(\varphi(1)\bar{\gamma}k_n - \gamma\alpha\right)) < 1$. We have G_n is a contraction map with coefficient $(k_n - \alpha_n\left(\varphi(1)\bar{\gamma}k_n - \gamma\alpha\right))$. Then, for each $n \geq 1$, there exists a unique $y_n \in K$ such that $G_n(y_n) = y_n$, that is,

$$y_n = \alpha_n\gamma f(y_n) + (I - \alpha_n A)T^n(t_n)y_n, n \geq 1.$$

Hence (i) is proved.

(ii) Define $\mu : K \to \mathbb{R}$ by

$$\mu(y) = \mathbf{LIM}_n\Phi(\|y_n - y\|), \;\; y \in C,$$

where $\mathbf{LIM}_n$ is a Banach limit on l^∞. Since μ is continuous and convex and $g(z) \to \infty$ as $\|z\| \to \infty$, and E is reflexive, by Lemma 2.1.37, g attains its infimum over E. Let $u \in C$ be such that

$$\mathbf{LIM}_n\Phi(\|y_n - u\|) = \inf_{y\in E} \mathbf{LIM}_n\Phi(\|y_n - y\|). \tag{4.3.7}$$

Let

$$C^* := \left\{ z \in E : \mu(z) = \inf_{y\in K} \mu(y) \right\}.$$

We have that C^* is a nonempty, bounded, closed and convex subset of C and also has the property (P), indeed, if $x \in C^*$ and $w \in \omega_w(x)$, i.e. $w =$ weak $-\lim_{j\to\infty} T^{m_j}x$ as $j \to \infty$. Notice that, $\lim_{n\to\infty} \|y_n - T(t)y_n\| = 0$ uniformly in $t \in [0, \infty)$, by induction we can prove that for all $m \geq 1$

$$\lim_{n\to\infty} \|y_n - T^m(t)y_n\| = 0 \text{ uniformly in } t \in [0, \infty). \tag{4.3.8}$$

From (4.3.8) and weakly lower semicontinuous of μ. For each $h \geq 0$, we have that

$$\begin{aligned}
\mu(w) &\leq \liminf_{j\to\infty} \mu(T^{m_j}(h)x) \leq \limsup_{m\to\infty} \mu(T^m(h)x) \\
&= \limsup_{m\to\infty} \mathbf{LIM}_n\Phi(\|y_n - T^m(h)x\|) \\
&\leq \limsup_{m\to\infty} [\mathbf{LIM}_n\Phi(\|y_n - T^m(h)y_n\| + \|T^m(h)y_n - T^m(h)x\|)] \\
&= \limsup_{m\to\infty} \mathbf{LIM}_n\Phi(\|T^m(h)y_n - T^m(h)x\|) \\
&\leq \limsup_{m\to\infty} \mathbf{LIM}_n\Phi(k_m\|y_n - x\|) \\
&= \mathbf{LIM}_n\Phi(\|y_n - x\|) \\
&= \mu(x) = \inf_{y\in K} \mu(y),
\end{aligned}$$

which implies that C^* satisfies the property (P). By Theorem 2.2.9, there exists a element $z \in C$ such that $z \in F(\mathcal{S}) \cap C^*$.

Since $C \pm C \subset C$, we have $z + \gamma f(z) - Az \in C$. By Proposition 2.1.36,

$$\mathbf{LIM}_n\langle z + \gamma f(z) - Az - z, J_\varphi(y_n - z)\rangle \leq 0,$$

it implies that

$$\mathrm{LIM}_n\langle \gamma f(z) - Az, J_\varphi(y_n - z)\rangle \le 0. \tag{4.3.9}$$

In fact, since $\Phi(t) = \int_0^t \varphi(\tau)d\tau, \forall t \ge 0$, and $\varphi : [0,\infty) \to [0,\infty)$ is a gauge function, then for $1 \ge k \ge 0$, $\varphi(kx) \le \varphi(x)$ and

$$\Phi(kt) = \int_0^{kt} \varphi(\tau)d\tau = k\int_0^t \varphi(kx)dx \le k\int_0^t \varphi(x)dx = k\Phi(t).$$

It follows from Lemma 2.1.31 that

$$\begin{aligned}
\Phi(\|y_n - z\|) &= \Phi(\|(I-\alpha_n A)T^n(t_n)y_n - (I-\alpha_n A)z \\
&\quad + \alpha_n(\gamma f(y_n) - \gamma f(z) + \gamma f(z) - Az)\|) \\
&\le \Phi\left(\|(I-\alpha_n A)T^n(t_n)y_n - (I-\alpha_n A)z + \alpha_n\gamma(f(y_n) - f(z))\|\right) \\
&\quad + \alpha_n\langle \gamma f(z) - Az, J_\varphi(y_n - z)\rangle \\
&\le \Phi\left(\varphi(1)(1-\alpha_n\bar{\gamma})\,\|T^n(t_n)y_n - T^n(t_n)z\| + \alpha_n\gamma\alpha\|y_n - z\|\right) \\
&\quad + \alpha_n\gamma\langle f(z) - f(z), J_\varphi(y_n - z)\rangle \\
&\le \Phi\left(\varphi(1)(1-\alpha_n\bar{\gamma})(k_n)\|y_n - z\| + \alpha_n\gamma\alpha\|y_n - z\|\right) \\
&\quad + \alpha_n\langle \gamma f(z) - Az, J_\varphi(y_n - z)\rangle \\
&\le \Phi\left([\varphi(1)(1-\alpha_n\bar{\gamma})(k_n) + \alpha_n\gamma\alpha]\,\|y_n - z\|\right) \\
&\quad + \alpha_n\langle \gamma f(z) - Az, J_\varphi(y_n - z)\rangle \\
&\le [\varphi(1)(1-\alpha_n\bar{\gamma})k_n + \alpha_n\gamma\alpha]\,\Phi\left(\|y_n - z\|\right) \\
&\quad + \alpha_n\langle \gamma f(z) - Az, J_\varphi(y_n - z)\rangle.
\end{aligned}$$

This implies that

$$\Phi(\|y_n - z\|) \le \frac{1}{1 - \varphi(1)(1-\alpha_n\bar{\gamma})k_n + \alpha_n\gamma\alpha}\alpha_n\langle \gamma f(z) - Az, J_\varphi(y_n - z)\rangle,$$

also

$$\Phi(\|y_n - z\|) \le \frac{1}{(\varphi(1)\bar{\gamma} - \alpha\gamma) - (1-\alpha_n\bar{\gamma})d_n}\langle \gamma f(z) - Az, J_\varphi(y_n - z)\rangle,$$

where $d_n = \dfrac{k_n - 1}{\alpha_n}$. Thus

$$\begin{aligned}\mathbf{LIM}_n\Phi(\|y_n - z\|) &\leq \mathbf{LIM}_n\left(\frac{1}{(\varphi(1)\bar{\gamma} - \alpha\gamma) - (1-\alpha_n\bar{\gamma})d_n}\langle \gamma f(z) - Az, J_\varphi(y_n - z)\rangle\right)\\ &\leq 0,\end{aligned}$$

and hence

$$(\varphi(1)\bar{\gamma} - \gamma\alpha)\mathbf{LIM}_n\Phi(\|y_n - z\|) \leq 0.$$

Since $\varphi(1)\bar{\gamma} > \gamma\alpha$, $\mathbf{LIM}_n\Phi(\|y_n - z\|) = 0$, and then there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $y_{n_j} \to z$ as $j \to \infty$, we shall denoted by $\{y_j\}$.

Next, we prove that z solves the variational inequality (4.3.2). From (4.3.1), we have

$$(A - \gamma f)y_n = -\frac{1}{\alpha_n}(I - \alpha_n A)(I - T^n(t_n))y_n.$$

On the other hand, note for all $x, y \in C$,

$$\begin{aligned}&\langle (I - T^n(t_n))x - (I - T^n(t_n))y, J_\varphi(x-y)\rangle\\ &\quad = \langle x - y, J_\varphi(x-y)\rangle - \langle T^n(t_n)x - T^n(t_n)y, J_\varphi(x-y)\rangle\\ &\quad = \|x-y\|\varphi(\|x-y\|) - \langle T^n(t_n)x - T^n(t_n)y, J_\varphi(x-y)\rangle\\ &\quad \geq \Phi(\|x-y\|) - k_n\|x-y\|\varphi(\|x-y\|)\\ &\quad \geq \Phi(\|x-y\|) - k_n\Phi(\|x-y\|)\\ &\quad = (1-k_n)\Phi(\|x-y\|).\end{aligned}$$

For $p \in F(\mathcal{S})$, we have

$$\begin{aligned}\langle (A-\gamma f)y_n, J_\varphi(y_n - p)\rangle &= -\frac{1}{\alpha_n}\langle (I - \alpha_n A)(I - T^n(t_n)y_n, J_\varphi(y_n - p)\rangle\\ &= -\frac{1}{\alpha_n}\langle (I - T^n(t_n))y_n - (I - T^n(t_n))p, J_\varphi(y_n - p)\rangle\\ &\quad +\langle A(I - T^n(t_n))y_n, J_\varphi(y_n - p)\rangle\\ &\leq \frac{k_n - 1}{\alpha_n}\Phi(\|x-y\|) + \langle A(I - T^n(t_n))y_n, J_\varphi(y_n - p)\rangle\end{aligned}$$

$$\leq \frac{k_n - 1}{\alpha_n}\Phi(\|x - y\|) + \|A\|\|y_n - T^n(t_n)y_n\|M,$$

where $M \geq \sup_{n\geq 1} \varphi(\|y_n - p\|)$. Replacing y_n with y_{n_j} and letting $j \to \infty$, note that $\|y_n - T^n(t_n)y_n\| \to 0$ and $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0$, we have that

$$\langle (A - \gamma f)z, J_\varphi(z - p)\rangle \leq 0, \ \ \forall p \in F(\mathcal{S}).$$

That is, $z \in F(\mathcal{S})$ is a solution of (4.3.2). Then $z = \tilde{x}$. Applying Lemma 2.1.38, we can conclude that $\{y_n\}$ converges strongly to $\tilde{x}$ as $n \to \infty$. This completes the proof. □

If $A \equiv I$, the identity mapping on C, and $\gamma = 1$, then Theorem 4.3.1 reduces to the following corollary.

Corollary 4.3.2. *Let E be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, C be a nonempty, bounded, closed and convex subset of E. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on C with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < \sqrt{N(E)}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_C$ with coefficient $\alpha \in (0, 1)$ and let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1, t_n > 0$. Then the following hold:*

(i) *If $\frac{k_n-1}{\alpha_n} < 1 - \alpha$, $\forall n \geq 1$, then there exists a sequence $\{y_n\} \subset K$ defined by*

$$y_n = \alpha_n f(y_n) + (1 - \alpha_n)T^n(t_n)y_n, \ \ n \geq 1. \tag{4.3.10}$$

(ii) *Suppose, in addition, $\lim_{n\to\infty} \|y_n - T(t)y_n\| = 0$ uniformly in $t \in [0, \infty)$ and the real sequences $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0$.*

Then $\{y_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality:

$$\langle (I - f)\tilde{x}, J_\varphi(\tilde{x} - z)\rangle \leq 0, \ \ z \in F(\mathcal{S}). \tag{4.3.11}$$

If $f \equiv u$, the constant mapping on C, then Corollary 4.3.2 reduces to the following corollary.

Corollary 4.3.3. *Let E be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, C be a nonempty bounded closed convex subset of E. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on C with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < \sqrt{N(E)}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \alpha_n < 1, t_n > 0$. Then the following hold:*

(i) *If $\frac{k_n - 1}{\alpha_n} < 1$, $\forall n \geq 1$, then there exists a sequence $\{y_n\} \subset C$ defined by*

$$y_n = \alpha_n u + (1 - \alpha_n) T^n(t_n) y_n, \quad n \geq 1. \tag{4.3.12}$$

(ii) *Suppose, in addition, $\lim_{n\to\infty} \|y_n - T(t)y_n\| = 0$ uniformly in $t \in [0, \infty)$ and the real sequences $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$.*

Then $\{y_n\}$ converges strongly as $n \to \infty$ to a common fixed point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality:

$$\langle \tilde{x} - u, J_\varphi(\tilde{x} - z) \rangle \leq 0, \quad z \in F(\mathcal{S}). \tag{4.3.13}$$

Next, we present the convergence theorem for the explicit scheme.

Theorem 4.3.4. *Let E be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, C be a nonempty, bounded, closed and convex subset of E such that $C \pm C \subset C$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on C with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < \sqrt{N(E)}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_C$ with coefficient $\alpha \in (0, 1)$, A a strongly positive bounded linear operator on C with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\varphi(1)\bar{\gamma}}{\alpha}$. Let $\{\beta_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \beta_n < 1, t_n \geq 0$,* (C1) $\lim_{n\to\infty} \beta_n = 0$;

(C2) $\lim_{n\to\infty} \frac{k_n-1}{\beta_n} = 0$;

(C3) $\sum_{n=0}^{\infty} \beta_n = \infty$.

For any $x_0 \in K$, let the sequences $\{x_n\}$ be defined by

$$x_{n+1} = \beta_n \gamma f(x_n) + (I - \beta_n A)T^n(t_n)x_n ds, \ n \geq 0. \tag{4.3.14}$$

Suppose, in addition, $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$ uniformly in $t \in [0,\infty)$. Then $\{x_n\}$ converge strongly as $n \to \infty$ to the same point $\tilde{x}$ in $F(S)$ which solves the variational inequality (4.3.2).

Proof. By Theorem 4.3.1, there exists a unique solution $\tilde{x}$ in $F(S)$ which solves the variational inequality (4.3.2) and $y_m \to \tilde{x}$ as $m \to \infty$. Next, we prove that

$$\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x})\rangle \leq 0. \tag{4.3.15}$$

For all $m \geq 1, n \geq 1$, we have

$$\begin{aligned} y_m - x_n &= \alpha_m \gamma f(y_m) + (I - \alpha_m A)T^m(t_m)y_m - x_n \\ &= \alpha_m(\gamma f(y_m) - Ay_m) + (T^m(t_m)y_m - T^m(t_m)x_n) \\ &\quad +(T^m(t_m)x_n - x_n) + \alpha_m(Ay_m - AT^m(t_m)y_m). \end{aligned}$$

It follows from Lemma 2.1.31 that

$$\begin{aligned} \Phi(\|y_m - x_n\|) &= \Phi(\|(I - \alpha_m A)T^m(t_m)y_m - (I - \alpha_m A)x_n \\ &\quad +\alpha_m(\gamma f(y_m) - \gamma f(\tilde{x}) + \gamma f(\tilde{x}) - A\tilde{x} + A\tilde{x} - Ax_n)\|) \\ &\leq \Phi\big(\big\|(I - \alpha_m A)T^m(t_m)x_n - (I - \alpha_m A)x_n \\ &\quad +\alpha_m \gamma(f(x_m) - f(\tilde{x})) + A\tilde{x} - Ax_n\big\|\big) \\ &\quad +\alpha_n \langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(y_n - \tilde{x})\rangle \end{aligned}$$

$$
\begin{aligned}
&\leq \Phi\left(\varphi(1)(1-\alpha_n\bar{\gamma})\left\|T^n(t_n)x_n - T^n(t_n)\tilde{x}\right\| + \alpha_n\gamma\alpha\|x_n - \tilde{x}\|\right) \\
&\quad +\alpha_n\gamma\langle f(\tilde{x}) - f(\tilde{x}), J_\varphi(y_n - \tilde{x})\rangle \\
&\leq \Phi\left(\varphi(1)(1-\alpha_n\bar{\gamma})k_n\|x_n - \tilde{x}\| + \alpha_n\gamma\alpha\|x_n - \tilde{x}\|\right) \\
&\quad +\alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\leq \Phi\left(\left[\varphi(1)(1-\alpha_n\bar{\gamma})k_n + \alpha_n\gamma\alpha\right]\|x_n - \tilde{x}\|\right) \\
&\quad +\alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\leq \left[\varphi(1)(1-\alpha_n\bar{\gamma})k_n + \alpha_n\gamma\alpha\right]\Phi\left(\|x_n - \tilde{x}\|\right) \\
&\quad +\alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\leq \left[(1-\varphi(1)\alpha_n\bar{\gamma})(k_n - 1) + 1 - \alpha_n(\varphi(1)\bar{\gamma} - \gamma\alpha)\right]\Phi\left(\|x_n - \tilde{x}\|\right) \\
&\quad +\alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle \\
&\leq \left(1-\alpha_n(\varphi(1)\bar{\gamma} - \gamma\alpha)\right)\Phi\left(\|x_n - \tilde{x}\|\right) + (1-\varphi(1)\alpha_n\bar{\gamma})(k_n - 1)M \\
&\quad +\alpha_n\langle \gamma f(\tilde{x}) - A(\tilde{x}), J_\varphi(x_n - \tilde{x})\rangle. \qquad (4.3.16)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|y_m - x_n\|\varphi(\|y_m - x_n\|) \\
&\quad = \langle \alpha_m(\gamma f(y_m) - Ay_m) + (T^m(t_m)y_m - T^m(t_m)x_n) \\
&\qquad +(T^m(t_m)x_n - x_n) + \alpha_m(Ay_m - AT^m(t_m)y_m), J_\varphi(y_m - x_n)\rangle \\
&\quad = \alpha_m\langle \gamma f(y_m) - Ay_m, J_\varphi(y_m - x_n)\rangle + \langle T^m(t_m)y_m - T^m(t_m)x_n, J_\varphi(y_m - x_n)\rangle \\
&\qquad +\langle T^m(t_m)x_n - x_n, J_\varphi(y_m - x_n)\rangle + \alpha_m\langle Ay_m - AT^m(t_m)y_m, J_\varphi(y_m - x_n)\rangle \\
&\quad \leq \alpha_m\langle \gamma f(y_m) - Ay_m, J_\varphi(y_m - x_n)\rangle + \|T^m(t_m)y_m - T^m(t_m)x_n\|\varphi(\|y_m - x_n\|) \\
&\qquad +\|T^m(t_m)x_n - x_n\|\varphi(\|y_m - x_n\|) + \alpha_m\|Ay_m - AT^m(t_m)y_m\|\varphi(\|y_m - x_n\|) \\
&\quad \leq \alpha_m\langle \gamma f(y_m) - Ay_m, J_\varphi(y_m - x_n)\rangle + k_m\|y_m - x_n\|\varphi(\|y_m - x_n\|) \\
&\qquad +\|T^m(t_m)x_n - x_n\|\varphi(\|y_m - x_n\|) + \alpha_m\|A(y_m - T^m(t_m)y_m)\|\varphi(\|y_m - x_n\|).
\end{aligned}
$$

Since C is bounded, so that $\{x_n\}$ and $\{y_m\}$ are all bounded, and hence

$$\langle \gamma f(y_m) - Ay_m, J_\varphi(x_n - y_m)\rangle$$

$$\leq \frac{k_m - 1}{\alpha_m} M^2 + \frac{\|T^m(t_m)x_n - x_n\|}{\alpha_m} M + \|A(y_m - T^m(t_m)y_m)\| M, \tag{4.3.17}$$

where M is a constant satisfying $M \geq \sup_{n,m\in\mathbb{N}} \varphi(\|x_n - y_m\|)$. By our hypothesis, $\lim_{n\to\infty} \|x_n - T(t)x_n\| = 0$, uniformly in $t \in [0, \infty)$. By induction, we can prove that for all $m \geq 1$

$$\lim_{n\to\infty} \|x_n - T^m(t)x_n\| = 0, \text{ uniformly in } t \in [0, \infty).$$

Hence for all $m \geq 1$, we have

$$\lim_{n\to\infty} \|x_n - T^m(t_m)x_n\| = 0, \text{ as } n \to \infty. \tag{4.3.18}$$

Therefore, taking upper limit as $n \to \infty$ in (4.3.17), we have

$$\begin{aligned} &\limsup_{n\to\infty} \langle \gamma f(y_m) - Ay_m, J_\varphi(x_n - y_m) \rangle \\ &\qquad \leq \limsup_{n\to\infty} \frac{k_m - 1}{\alpha_m} M^2 + \limsup_{n\to\infty} \|A(y_m - T^m(t_m)y_m)\| M. \end{aligned} \tag{4.3.19}$$

Since C is bounded, it follows from (C1) that

$$\|y_m - T^m(t_m)y_n\| = \alpha_n \|\gamma f(y_m) + AT^m(t_m)y_m\| \to 0 \text{ as } m \to \infty. \tag{4.3.20}$$

And then, taking upper limit as $m \to \infty$ in (4.3.19), by (C3) and (4.3.20), we get

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \langle \gamma f(y_m) - Ay_m, J_\varphi(x_n - y_m) \rangle \leq 0. \tag{4.3.21}$$

On the other hand, since $\lim_{m\to\infty} y_m = \tilde{x}$ due to the fact the duality mapping J_φ is norm-to-weak* uniformly continuous on bounded subset of E, it implies that

$$\begin{aligned} &|\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x}) \rangle - \langle \gamma f(y_m) - Ay_m, J_\varphi(x_n - y_m) \rangle| \\ &\quad = |\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x}) - J_\varphi(x_n - y_m) \rangle \\ &\qquad + \langle \gamma f(\tilde{x}) - \gamma f(y_m) + Ay_m - A\tilde{x}, J_\varphi(x_n - y_m) \rangle| \\ &\quad \leq |\langle \gamma f(\tilde{x}) - A\tilde{x}, J_\varphi(x_n - \tilde{x}) - J_\varphi(x_n - y_m) \rangle| \end{aligned}$$

$$+\left(\|\gamma f(\tilde{x})-\gamma f(y_m)\|+\|A(y_m-\tilde{x})\|\right)\varphi(\|x_n-y_m\|)\to 0 \text{ as } m\to\infty.$$

Therefore, for any given $\varepsilon>0$, there exists a positive number N such that for all $m\geq N$

$$\langle \gamma f(\tilde{x})-A\tilde{x}, J_\varphi(x_n-\tilde{x})\rangle \leq \langle \gamma f(y_m)-Ay_m, J_\varphi(x_n-y_m)\rangle+\varepsilon.$$

It follows from (4.3.21) that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle \gamma f(\tilde{x})-A\tilde{x}, J_\varphi(x_n-\tilde{x})\rangle\\
&\quad=\limsup_{m\to\infty}\limsup_{n\to\infty}\langle \gamma f(\tilde{x})-A\tilde{x}, J_\varphi(x_n-\tilde{x})\rangle\\
&\quad\leq\limsup_{m\to\infty}\limsup_{n\to\infty}\langle \gamma f(y_m)-Ay_m, J_\varphi(x_n-y_m)\rangle+\varepsilon\\
&\quad\leq\varepsilon.
\end{aligned}$$

Since ε is arbitrary, we have

$$\limsup_{n\to\infty}\langle \gamma f(\tilde{x})-A\tilde{x}, J_\varphi(x_n-\tilde{x})\rangle \leq 0. \tag{4.3.22}$$

Finally, we show that $x_n\to\tilde{x}$ as $n\to\infty$.

$$\begin{aligned}
\Phi(\|x_{n+1}-\tilde{x}\|) &= \Phi(\|(I-\beta_nA)T^n(t_n)x_n-(I-\beta_nA)\tilde{x}\\
&\quad+\beta_n(\gamma f(x_n)-\gamma f(\tilde{x})+\gamma f(\tilde{x})-A(\tilde{x}))\|)\\
&\leq \Phi\left(\|(I-\beta_nA)T^n(t_n)x_n-(I-\beta_nA)\tilde{x}+\beta_n\gamma(f(x_n)-f(\tilde{x}))\|\right)\\
&\quad+\beta_n\langle \gamma f(\tilde{x})-A(\tilde{x}), J_\varphi(y_n-\tilde{x})\rangle\\
&\leq \Phi\left(\varphi(1)(1-\beta_n\bar{\gamma})\|T^n(t_n)x_n-T^n(t_n)\tilde{x}\|+\beta_n\gamma\alpha\|x_n-\tilde{x}\|\right)\\
&\quad+\beta_n\gamma\langle f(\tilde{x})-f(\tilde{x}), J_\varphi(y_n-\tilde{x})\rangle\\
&\leq \Phi\left(\varphi(1)(1-\beta_n\bar{\gamma})k_n\|x_n-\tilde{x}\|+\beta_n\gamma\alpha\|x_n-\tilde{x}\|\right)\\
&\quad+\beta_n\langle \gamma f(\tilde{x})-A(\tilde{x}), J_\varphi(x_n-\tilde{x})\rangle\\
&\leq \Phi\left([\varphi(1)(1-\beta_n\bar{\gamma})k_n+\beta_n\gamma\alpha]\,\|x_n-\tilde{x}\|\right)\\
&\quad+\beta_n\langle \gamma f(\tilde{x})-A(\tilde{x}), J_\varphi(x_n-\tilde{x})\rangle\\
&\leq [\varphi(1)(1-\beta_n\bar{\gamma})k_n+\beta_n\gamma\alpha]\,\Phi\left(\|x_n-\tilde{x}\|\right)
\end{aligned}$$

$$
\begin{aligned}
&+\beta_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_n-\tilde{x})\rangle\\
\leq\ & [(1-\varphi(1)\beta_n\bar{\gamma})(k_n-1)+1-\beta_n(\varphi(1)\bar{\gamma}-\gamma\alpha)]\,\Phi\left(\|x_n-\tilde{x}\|\right)\\
&+\beta_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_n-\tilde{x})\rangle\\
\leq\ & (1-\beta_n(\varphi(1)\bar{\gamma}-\gamma\alpha))\,\Phi\left(\|x_n-\tilde{x}\|\right)+(1-\varphi(1)\beta_n\bar{\gamma})(k_n-1)M''\\
&+\beta_n\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_n-\tilde{x})\rangle \qquad (4.3.23)
\end{aligned}
$$

where $M''>0$ such that $\sup_{n\geq 1}\Phi(\|x_n-\tilde{x}\|)\leq M''$. Put

$$s_n=\beta_n(\varphi(1)\bar{\gamma}-\gamma\alpha)$$

and

$$\sigma_n=\left(\frac{1-\varphi(1)\beta_n\bar{\gamma}}{\varphi(1)\bar{\gamma}-\gamma\alpha}\right)\left(\frac{k_n-1}{\beta_n}\right)M''+\frac{1}{(\varphi(1)\bar{\gamma}-\gamma\alpha)}\langle\gamma f(\tilde{x})-A(\tilde{x}),J_\varphi(x_n-\tilde{x})\rangle.$$

Then (4.3.23) is reduced to

$$\Phi(\|x_{n+1}-\tilde{x}\|)\leq(1-s_n)\Phi(\|x_n-\tilde{x}\|)+s_n\sigma_n. \qquad (4.3.24)$$

Applying Lemma 2.1.39 to (4.3.24), we conclude that $\Phi(\|x_{n+1}-\tilde{x}\|)\to 0$ as $n\to\infty$; that is, $x_n\to\tilde{x}$ as $n\to\infty$. This completes the proof. □

Using Theorem 4.3.4, we obtain the following two strong convergence theorems of new iterative approximation methods for an asymptotically nonexpansive semigroup $\mathcal{S}$.

Theorem 4.3.5. *Let E be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, C be a nonempty, bounded, closed and convex subset of E such that $C\pm C\subset C$. Let $\mathcal{S}=\{T(s):s\geq 0\}$ be an asymptotically nonexpansive semigroup on C with a sequence $\{k_n\}\subset[1,+\infty)$, $\lim_{n\to\infty}k_n=1$ and $\sup_{n\geq 1}k_n<\sqrt{N(E)}$ such that $F(\mathcal{S})\neq\emptyset$. Let $f\in\Pi_K$ with coefficient $\alpha\in(0,1)$, A a strongly positive bounded linear operator on C with coefficient $\bar{\gamma}>0$ and $0<\gamma<\frac{\varphi(1)\bar{\gamma}}{\alpha}$. Let $\{\beta_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0<\beta_n<1, t_n\geq 0$,* (C1) $\lim_{n\to\infty} \beta_n = 0$;

(C2) $\lim_{n\to\infty} \frac{k_n-1}{\beta_n} = 0$;

(C3) $\sum_{n=0}^{\infty} \beta_n = \infty$.

For any $w_0 \in C$, let the sequence $\{w_n\}$ be defined by

$$w_{n+1} = \beta_n \gamma f(T^n(t_n)w_n) + (I - \beta_n A)T^n(t_n)w_n, \ \ n \geq 0, \tag{4.3.25}$$

Then $\{w_n\}$ converges strongly as $n \to \infty$ to a point $\tilde{x}$ in $F(\mathcal{S})$ which solves the variational inequality (4.3.2).

Proof. Let $\{x_n\}$ be the sequence given by $x_0 = w_0$ and

$$x_{n+1} = \beta_n \gamma f(x_n) + (I - \beta_n A)T^n(t_n)x_n, \ \forall n \geq 0.$$

By Theorem 4.3.4, $x_n \to \tilde{x}$. We claim that $w_n \to \tilde{x}$. We calculate the following

$$\begin{aligned}
\|x_{n+1} - w_{n+1}\| &= \beta_n \gamma \|f(x_n) - f(T^n(t_n)w_n)\| + \|I - \beta_n A\| \|T^n(t_n)x_n - T^n(t_n)w_n\| \\
&\leq \beta_n \gamma \alpha \|x_n - T^n(t_n)w_n\| + \varphi(1)(1 - \beta_n \bar{\gamma})(k_n)\|x_n - w_n\| \\
&\leq \beta_n \gamma \alpha \|x_n - T^n(t_n)\tilde{x}\| + \beta_n \gamma \alpha \|T^n(t_n)\tilde{x} - T^n(t_n)w_n\| \\
&\quad + \varphi(1)(1 - \beta_n \bar{\gamma})(k_n)\|x_n - w_n\| \\
&\leq \beta_n \gamma \alpha \|x_n - \tilde{x}\| + \beta_n \gamma \alpha k_n \|\tilde{x} - w_n\| \\
&\quad + \varphi(1)(1 - \beta_n \bar{\gamma})(k_n)\|x_n - w_n\| \\
&\leq \beta_n \gamma \alpha \|x_n - \tilde{x}\| + \beta_n \gamma \alpha \|w_n - x_n\| + \beta_n \gamma \alpha k_n \|\tilde{x} - x_n\| \\
&\quad + \varphi(1)(1 - \beta_n \bar{\gamma})(k_n)\|x_n - w_n\| \\
&= \varphi(1)(1 - \beta_n \bar{\gamma})(k_n)\|x_n - w_n\| + \beta_n \gamma \alpha \|w_n - x_n\| \\
&\quad + \beta_n(\gamma \alpha + k_n)\|x_n - \tilde{x}\| \\
&= (\varphi(1)(1 - \beta_n \bar{\gamma})(k_n) + \beta_n \gamma \alpha)\, \|x_n - w_n\| \\
&\quad + \beta_n(\gamma \alpha + k_n)\|x_n - \tilde{x}\| \\
&\leq [(1 - \varphi(1)\beta_n \bar{\gamma})(k_n - 1) + 1 - \beta_n(\varphi(1)\bar{\gamma} - \gamma \alpha)]\, \|x_n - w_n\|
\end{aligned}$$

$$+\beta_n(\gamma\alpha + k_n)\|x_n - \tilde{x}\|$$
$$\leq \ (1-\beta_n(\varphi(1)\bar{\gamma} - \gamma\alpha))\,\|x_n - w_n\| + (1-\varphi(1)\beta_n\bar{\gamma})(k_n - 1)M$$
$$+\beta_n(\gamma\alpha + k_n)\|x_n - \tilde{x}\|,$$

where $M > 0$ such that $\sup_{n\geq 1}\|x_n - w_n\| \leq M$. Put

$$s_n = \beta_n(\varphi(1)\bar{\gamma} - \gamma\alpha)$$

and

$$\sigma_n = \left(\frac{1-\varphi(1)\beta_n\bar{\gamma}}{\varphi(1)\bar{\gamma} - \gamma\alpha}\right)\left(\frac{k_n - 1}{\beta_n}\right)M + \frac{(\gamma\alpha + k_n)}{(\varphi(1)\bar{\gamma} - \gamma\alpha)}\|x_n - \tilde{x}\|.$$

Then we have that

$$\|x_{n+1} - w_{n+1}\| \leq (1 - s_n)\|x_n - w_n\| + s_n\sigma_n. \tag{4.3.26}$$

It follows from (C3), $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$ and Lemma 2.1.39 that $\|x_n - w_n\| \to 0$ as $n \to \infty$; that is, $w_n \to \tilde{x}$ as $n \to \infty$. □

Theorem 4.3.6. *Let E be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping J_φ, C be a nonempty, bounded, closed and convex subset of E such that $C \pm C \subset C$. Let $\mathcal{S} = \{T(s) : s \geq 0\}$ be an asymptotically nonexpansive semigroup on C with a sequence $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ and $\sup_{n\geq 1} k_n < \sqrt{N(E)}$ such that $F(\mathcal{S}) \neq \emptyset$. Let $f \in \Pi_C$ with coefficient $\alpha \in (0,1)$, A a strongly positive bounded linear operator on C with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\varphi(1)\bar{\gamma}}{\alpha}$. Let $\{\beta_n\}$ and $\{t_n\}$ be sequences of real numbers such that $0 < \beta_n < 1, t_n \geq 0$,*

(C1) $\lim_{n\to\infty}\beta_n = 0$;

(C2) $\lim_{n\to\infty}\frac{k_n - 1}{\beta_n} = 0$;

(C3) $\sum_{n=0}^{\infty}\beta_n = \infty$.

For any $z_0 \in C$, *let the sequence* $\{z_n\}$ *be defined by*

$$z_{n+1} = T^n(t_n)\left(\beta_n \gamma f(z_n) + (I - \beta_n A)z_n\right), \;\; n \geq 0, \tag{4.3.27}$$

Then $\{z_n\}$ *converges strongly as* $n \to \infty$ *to a point* $\tilde{x}$ *in* $F(\mathcal{S})$ *which solves the variational inequality* (4.3.2).

Proof. Define the sequences $\{w_n\}$ and $\{\sigma_n\}$ by

$$w_n = \beta_n \gamma f(z_n) + (I - \beta_n A)z_n \text{ and } \sigma_n = \beta_{n+1}, \;\; n \geq 0.$$

We have that

$$w_{n+1} = \beta_{n+1}\gamma f(z_{n+1}) + (I - \beta_{n+1}A)z_{n+1} = \sigma_n \gamma f(T^n(t_n)w_n) + (I - \sigma_n A)T^n(t_n)w_n.$$

It follows from Corollary 4.3.5 that $\{w_n\}$ converges strongly to $\tilde{x}$. Thus we have

$$\begin{aligned} \|z_n - \tilde{x}\| &\leq \|z_n - w_n\| + \|w_n - \tilde{x}\| = \beta_n \|\gamma f(z_n) - Az_n\| + \|w_n - \tilde{x}\| \\ &\leq \beta_n M + \|w_n - \tilde{x}\| \to 0 \text{ as } n \to \infty, \end{aligned}$$

where $M > 0$ such that $M \geq \sup_{n\geq 1} \|\gamma f(z_n) - Az_n\|$. Hence $\{z_n\}$ converges strongly to $\tilde{x}$. □

If $A \equiv I$, the identity mapping on E, and $\gamma = 1$, then Theorem 4.3.4 reduces to the following corollary.

Corollary 4.3.7. *Let* E *be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping* J_φ, C *be a nonempty, bounded, closed and convex subset of* E. *Let* $\mathcal{S} = \{T(s) : s \geq 0\}$ *be an asymptotically nonexpansive semigroup on* C *with a sequence* $\{k_n\} \subset [1, +\infty)$, $\lim_{n\to\infty} k_n = 1$ *and* $\sup_{n\geq 1} k_n < \sqrt{N(E)}$ *such that* $F(\mathcal{S}) \neq \emptyset$. *Let* $f \in \Pi_C$ *with coefficient* $\alpha \in (0,1)$. *Let* $\{\beta_n\}$ *and* $\{t_n\}$ *be sequences of real numbers such that* $0 < \beta_n < 1, t_n \geq 0$, (C1) $\lim_{n\to\infty}\beta_n = 0$;

(C2) $\lim_{n\to\infty}\frac{k_n-1}{\beta_n} = 0$;

(C3) $\sum_{n=0}^{\infty}\beta_n = \infty$.

For any $x_0 \in K$, *let the sequence* $\{x_n\}$ *be defined by*

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)T^n(t_n)x_n ds,\ n \geq 0. \tag{4.3.28}$$

Suppose, in addition, $\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0$ *uniformly in* $t \in [0,\infty)$. *Then* $\{x_n\}$ *converges strongly as* $n \to \infty$ *to a point* $\tilde{x}$ *in* $F(\mathcal{S})$ *which solves the variational inequality* (4.3.11).

If $f \equiv u$, then Corollary 4.3.7 reduces to the following corollary.

Corollary 4.3.8. *Let* E *be a real Banach space with uniform normal structure which has a uniformly Gateaux differentiable norm and admits the duality mapping* J_φ, C *be a nonempty, bounded, closed and convex subset of* E *such that* $C \pm C \subset C$. *Let* $\mathcal{S} = \{T(s) : s \geq 0\}$ *be an asymptotically nonexpansive semigroup on* C *with a sequence* $\{k_n\} \subset [1,+\infty)$, $\lim_{n\to\infty} k_n = 1$ *and* $\sup_{n\geq 1} k_n < \sqrt{N(E)}$ *such that* $F(\mathcal{S}) \neq \emptyset$. *Let* $f \in \Pi_C$ *with coefficient* $\alpha \in (0,1)$. *Let* $\{\beta_n\}$ *and* $\{t_n\}$ *be sequences of real numbers such that* $0 < \beta_n < 1, t_n \geq 0$,

(C1) $\lim_{n\to\infty}\beta_n = 0$;

(C2) $\lim_{n\to\infty}\frac{k_n-1}{\beta_n} = 0$;

(C3) $\sum_{n=0}^{\infty}\beta_n = \infty$.

For any $x_0 \in C$, *let the sequence* $\{x_n\}$ *be defined by*

$$x_{n+1} = \beta_n u + (1-\beta_n)T^n(t_n)x_n ds,\ n \geq 0. \tag{4.3.29}$$

Suppose, in addition, $\lim_{n\to\infty}\|x_n - T(t)x_n\| = 0$ *uniformly in* $t \in [0,\infty)$. *Then* $\{x_n\}$ *converges strongly as* $n \to \infty$ *to a point* $\tilde{x}$ *in* $F(\mathcal{S})$ *which solves the variational inequality* (4.3.13).