CHAPTER V

CONVERGENCE THEOREMS IN CAT(0) SPACES

In this chapter, we study Δ-convergence and strong convergence theorems in complete CAT(0) spaces.

5.1 Δ-convergence for generalized hybrid mappings in CAT(0) spaces

In this section, we give convergence theorems for a generalized hybrid mapping. Finally, we give an example of a (α, β)-generalized hybrid mapping in CAT(0) space which is not a nonexpansive mapping.

Let (X, d) be a CAT(0) space and C a nonempty, closed and convex subset of X. A mapping $T : C \to C$ is called (α, β)-*generalized hybrid* mapping if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha d^2(Tx, Ty) + (1-\alpha)d^2(x, Ty) \leq \beta d^2(Tx, y) + (1-\beta)d^2(x, y). \qquad (5.1.1)$$

In [91], Kocourek, Takahashi and Yao obtained the demiclosed principle of (α, β)-generalized hybrid mapping in a Hilbert space.

It is easy to see that an (α, β)-*generalized hybrid* mapping with $F(T) \neq \emptyset$ is a quasi-nonexpansive, i.e., $F(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p) \text{ for all } x \in C \text{ and } p \in F(T). \qquad (5.1.2)$$

Firstly, we present the demiclosed principle of (α, β)-generalized hybrid mapping in a CAT(0) space.

Proposition 5.1.1. *Let (X, d) be a CAT(0) space and C be a nonempty, closed and convex subset of X. Let $T : C \to C$ be an (α, β)-generalized hybrid mapping such that $\alpha \geq 1$ and $\beta \geq 0$. Let $\{x_n\}$ be a bounded sequence in C such that $lim_{n\to\infty} d(x_n, T(x_n)) = 0$ and $\{x_n\}$ Δ-converges to w. Then $T(w) = w$.*

Proof. Notice that $T : C \to C$ is an (α, β)-generalized hybrid mapping, i.e., there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha d^2(Tx, Ty) + (1-\alpha)d^2(x, Ty) \leq \beta d^2(Tx, y) + (1-\beta)d^2(x, y). \quad (5.1.3)$$

Since T is generalized hybrid mapping, we have that

$$\alpha d^2(Tx_n, Tw) + (1-\alpha)d^2(x_n, Tw) \leq \beta d^2(Tx_n, w) + (1-\beta)d^2(x_n, w). \quad (5.1.4)$$

Since $\alpha \geq 1$, $\beta \geq 0$ and (5.1.4), we get that

$$\begin{aligned} \alpha d^2(Tx_n, Tw) &\leq \beta(d(Tx_n, x_n) + d(x_n, w))^2 + (1-\beta)d^2(x_n, w) \\ &\quad +(\alpha - 1)(d(x_n, Tx_n) + d(Tx_n, Tw))^2. \end{aligned}$$

Hence

$$\begin{aligned} (\alpha - (\alpha - 1))d^2(Tx_n, Tw) &\leq (\beta + (1-\beta))d^2(x_n, w) + (\beta + \alpha - 1)d^2(x_n, Tx_n) \\ &\quad +2(\beta + \alpha - 1)(d(x_n, w) + d(Tx_n, Tw))d(Tx_n, x_n), \end{aligned}$$

and so

$$\begin{aligned} d^2(Tx_n, Tw) &\leq d^2(x_n, w) + (\beta + \alpha - 1)d^2(x_n, Tx_n) \\ &\quad +2(\beta + \alpha - 1)(d(x_n, w) + d(Tx_n, Tw))d(Tx_n, x_n). \quad (5.1.5) \end{aligned}$$

Since $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$, we get that $\{Tx_n\}$ is also bounded. Thus (5.1.5) reduces to

$$\begin{aligned} d^2(Tx_n, Tw) &\leq d^2(x_n, w) + (\beta + \alpha - 1)d^2(x_n, Tx_n) \\ &\quad +2(\beta + \alpha - 1)d(Tx_n, x_n)M, \quad (5.1.6) \end{aligned}$$

where $M \geq \sup_{n\geq 1} d(x_n, w) + d(Tx_n, Tw)$. Since $\{x_n\}$ Δ-converges to w, we then have that $A_C(\{x_n\}) = \{w\}$ and also $A(\{x_n\}) = \{w\}$. Assume that $Tw \neq w$. Then

By Opial's conditions,

$$\begin{aligned}
\limsup_{n\to\infty} d^2(x_n, w) &< \limsup_{n\to\infty} d^2(x_n, Tw) \\
&\leq \limsup_{n\to\infty} (d(x_n, Tx_n) + d(Tx_n, Tw))^2 \\
&\leq \limsup_{n\to\infty} d^2(Tx_n, Tw) \\
&\leq \limsup_{n\to\infty} \Big(d^2(x_n, w) + (\beta + \alpha - 1) d^2(x_n, Tx_n) \\
&\quad +2(\beta + \alpha - 1) d(Tx_n, x_n) M \Big) \\
&= \limsup_{n\to\infty} d^2(x_n, w).
\end{aligned}$$

This is a contradiction. So, we have $Tw = w$. □

Proposition 5.1.2. *Let (X, d) be a CAT(0) space and C be a nonempty, closed and convex subset of X. Let $T : C \to C$ be an (α, β)-generalized hybrid mapping with $F(T) \neq \emptyset$. Let γ be a real number with $0 < \gamma < 1$ and define the mapping $S : C \to C$ by:*

$$S = \gamma I \oplus (1 - \gamma) T. \tag{5.1.7}$$

Then, for each $x \in C$, $d(S^{n+1}x, S^n x)$ converges to 0.

Proof. It is easy to prove that $F(T) = F(S)$. Since $F(T) \neq \emptyset$, we have T and S are a quasi nonexpansive mapping. For any $x \in C$ and $p \in F(T)$, we get

$$\begin{aligned}
d(S^{n+1}x, p) &= d(SS^n x, p) \\
&= d(\gamma S^n x \oplus (1 - \gamma) TS^n x, p) \\
&\leq \gamma d(S^n x, p) + (1 - \gamma) d(TS^n x, p) \\
&\leq \gamma d(S^n x, p) + (1 - \gamma) d(S^n x, p) \\
&= d(S^n x, p).
\end{aligned}$$

Hence $d(S^n x, p)$ is decreasing sequence and bounded below, and so $\lim_{n\to\infty} d(S^n x, p)$ exists. Therefore $\{S^n x\}$ is bounded and $\{TS^n x\}$ is also. It follows from Lemma

2.4.9 that

$$\begin{aligned} d^2(S^{n+1},p) &= d^2(\gamma S^n x \oplus (1-\gamma)TS^n x, p) \\ &\leq \gamma d^2(S^n x,p) + (1-\gamma)d^2(TS^n x,p) - \gamma(1-\gamma)d^2(S^n x, TS^n x) \\ &\leq \gamma d^2(S^n x,p) + (1-\gamma)d^2(S^n x,p) - \gamma(1-\gamma)d^2(S^n x, TS^n x) \\ &\leq d^2(S^n x,p) - \gamma(1-\gamma)d^2(S^n x, TS^n x), \end{aligned}$$

and so

$$\gamma(1-\gamma)d^2(S^n x, TS^n x) \leq d^2(S^n x,p) - d^2(S^{n+1},p).$$

Since $\lim_{n\to\infty} d^2(S^n x,p)$ exists and $0<\gamma<1$, we have that $\lim_{n\to\infty} d(S^n x, TS^n x) = 0$. On the other hand,

$$d(S^{n+1}x, TS^n x) = d(\gamma S^n x \oplus (1-\gamma)TS^n x, TS^n x) \leq \gamma d(S^n x, TS^n x).$$

We get that

$$\begin{aligned} d(S^{n+1}x, S^n x) &\leq d(S^{n+1}x, TS^n x) + d(TS^n x, S^n x) \\ &\leq \gamma d(S^n x, TS^n x) + d(TS^n x, S^n x) \to 0 \text{ as } n\to\infty. \end{aligned}$$

This completes the proof. □

Theorem 5.1.3. *Let (X,d) be a CAT(0) space and C be a nonempty, closed and convex subset of X. Let $T: C\to C$ be an (α,β)-generalized hybrid mapping with $F(T)\neq\emptyset$. Let γ be a real number with $0<\gamma<1$ and define the mapping $S: C\to C$ by:*

$$S = \gamma I \oplus (1-\gamma)T. \tag{5.1.8}$$

Then, for each $x\in C$, $\{S^n x\}$ Δ-converges to an element in $F(T)$.

Proof. For each $n\geq 1$, let $x_n = S^n x$. Since $F(T)\neq\emptyset$, we have that S is a quasi-nonexpansive mapping. For any $p\in F(T)$, we get that

$$d(x_{n+1},p) = d(S^{n+1}x,p)$$

$$\leq \ d(S^n x, p) = d(x_n, p).$$

This implies that $\{x_n\}$ is bounded. By Proposition 5.1.2, it follows that

$$\lim_{n\to\infty} d(Sx_n, x_n) = \lim_{n\to\infty} d(S^{n+1}x, S^n x) = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ Δ-converges to $u \in C$. By Proposition 5.1.1, $u \in F(T)$. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_j}\}$ Δ-converges to $v \in C$. Supose $u \neq v$. Again by Proposition 5.1.1, $v \in F(T)$, and so $\lim_{n\to\infty} d(x_n, u)$ and $\lim_{n\to\infty} d(x_n, v)$ exist. By Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, u) &= \limsup_{i\to\infty} d(x_{n_i}, u) \\
&< \limsup_{i\to\infty} d(x_{n_i}, v) \\
&= \lim_{n\to\infty} d(x_n, v) \\
&= \limsup_{j\to\infty} d(x_{n_j}, v) \\
&< \limsup_{j\to\infty} d(x_{n_j}, u) \\
&= \lim_{n\to\infty} d(x_n, u).
\end{aligned}$$

This is contraction. Thus $u = v$. Hence $\{x_n\}$ Δ-converges to $u \in F(T)$. □

Theorem 5.1.4. *Let (X, d) be a CAT(0) space and C be a nonempty, closed and convex subset of X. Let $T : C \to C$ be an (α, β)-generalized hybrid mapping such that $\alpha \geq 1$ and $\beta \geq 0$ with $F(T) \neq \emptyset$. Let $\{\gamma_n\}$ be a sequence of real number with $0 < a \leq \gamma_n \leq b < 1$ and defined a sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \gamma_n x_n \oplus (1-\gamma_n) T x_n, \ \forall n \in \mathbb{N} \ . \end{cases} \tag{5.1.9}$$

Then $\{x_n\}$ Δ-converges to an element $u \in F(T)$.

Proof. Since $F(T) \neq \emptyset$, we have that T is a quasi-nonexpansive mapping. For any

$p \in F(T)$, we get that

$$\begin{aligned} d(x_{n+1}, p) &= d(\gamma_n x_n \oplus (1-\gamma_n)Tx_n, p) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n) d(Tx_n, p) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n) d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

This implies that $\{x_n\}$ is bounded, and so $\{T(x_n)\}$ is also bounded since T is quasi-nonexpansive. Moreover, we have that the limit of $d(x_n, p)$ exists. By Lemma **2.4.9**,

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\gamma_n x_n \oplus (1-\gamma_n)Tx_n, p) \\ &\leq \gamma_n d^2(x_n, p) + (1-\gamma_n) d^2(Tx_n, p) - \gamma_n(1-\gamma_n) d^2(x_n, Tx_n) \\ &\leq d^2(x_n, p) - \gamma_n(1-\gamma_n) d^2(x_n, Tx_n), \end{aligned}$$

and so

$$\gamma_n(1-\gamma_n) d^2(x_n, Tx_n) \leq d^2(x_n, p) - d^2(x_{n+1}, p). \tag{5.1.10}$$

Since the limit of $d(x_n, p)$ exists and $0 < a \leq \gamma_n \leq b < 1$, it follows from (5.1.10) that,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{5.1.11}$$

It follows from the proof of Theorem 5.1.3 that $\{x_n\}$ Δ-converges to an element $z \in F(S) = F(T)$. □

Theorem 5.1.5. *Let (X, d) be a CAT(0) space and C be a nonempty, closed and convex subset of X. Let $T : C \to C$ be an (α, β)-generalized hybrid mapping such that $\alpha \geq 1$ and $\beta \geq 0$ with $F(T) \neq \emptyset$. Let $\{\gamma_n\}$ and $\{\sigma_n\}$ be sequences of real numbers with $0 < a \leq \gamma_n, \sigma_n \leq b < 1$ and define a sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \gamma_n x_n \oplus (1-\gamma_n) T(\sigma_n x_n \oplus (1-\sigma_n) Tx_n), \ \forall n \in \mathbb{N} \end{cases}. \tag{5.1.12}$$

Then $\{x_n\}$ Δ-converges to an element $u \in F(T)$.

Proof. For any n, let $y_n = \sigma_n x_n \oplus (1-\sigma_n)Tx_n$, then $x_{n+1} = \gamma_n x_n \oplus (1-\gamma_n)Ty_n$. Since $F(T) \neq \emptyset$, we have that T is a quasi-nonexpansive mapping. For any $p \in F(T)$, we get that

$$\begin{aligned} d(y_n, p) &= d(\sigma_n x_n \oplus (1-\sigma_n)Tx_n, p) \\ &\leq \sigma_n d(x_n, p) + (1-\sigma_n)d(Tx_n, p) \\ &\leq \sigma_n d(x_n, p) + (1-\sigma_n)d(x_n, p) \\ &= d(x_n, p), \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d(\gamma_n x_n \oplus (1-\gamma_n)Ty_n, p) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n)d(Ty_n, p) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n)d(y_n, p) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n)d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

This implies that $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{Tx_n\}$ and $\{Ty_n\}$. Moreover, we have that the limit of $d(x_n, p)$ exists. For any $p \in F(T)$, we have that

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\gamma_n x_n \oplus (1-\gamma_n)Ty_n, p) \\ &\leq \gamma_n d^2(x_n, p) + (1-\gamma_n)d^2(Ty_n, p) - \gamma_n(1-\gamma_n)d^2(x_n, Ty_n) \\ &\leq \gamma_n d^2(x_n, p) + (1-\gamma_n)d^2(y_n, p) - \gamma_n(1-\gamma_n)d^2(x_n, Ty_n) \\ &\leq \gamma_n d^2(x_n, p) + (1-\gamma_n)d^2(y_n, p), \end{aligned} \tag{5.1.13}$$

and

$$d^2(y_n, p) = d^2(\sigma_n x_n \oplus (1-\sigma_n)Tx_n, p)$$

$$
\begin{aligned}
&\leq \sigma_n d^2(x_n,p) + (1-\sigma_n)d^2(Tx_n,p) - \sigma_n(1-\sigma_n)d^2(x_n,Tx_n) \\
&\leq \sigma_n d^2(x_n,p) + (1-\sigma_n)d^2(x_n,p) - \sigma_n(1-\sigma_n)d^2(x_n,Tx_n) \\
&\leq d^2(x_n,p) - \sigma_n(1-\sigma_n)d^2(x_n,Tx_n). \qquad (5.1.14)
\end{aligned}
$$

Hence

$$
\begin{aligned}
d^2(x_{n+1},p) &\leq \gamma_n d^2(x_n,p) + (1-\gamma_n)(d^2(x_n,p) - \sigma_n(1-\sigma_n)d^2(x_n,Tx_n)) \\
&= \gamma_n d^2(x_n,p) + (1-\gamma_n)d^2(x_n,p) - (1-\gamma_n)\sigma_n(1-\sigma_n)d^2(x_n,Tx_n) \\
&= d^2(x_n,p) - (1-\gamma_n)\sigma_n(1-\sigma_n)d^2(x_n,Tx_n),
\end{aligned}
$$

that is,

$$
(1-\gamma_n)\sigma_n(1-\sigma_n)d^2(x_n,Tx_n) \leq d^2(x_n,p) - d^2(x_{n+1},p).
$$

Since $0 < a \leq \gamma_n, \sigma_n \leq b < 1$, we have that

$$
a(1-b)^2 d^2(x_n,Tx_n) \leq (1-\gamma_n)\sigma_n(1-\sigma_n)d^2(x_n,Tx_n) \leq d^2(x_n,p) - d^2(x_{n+1},p).
$$

It follows that

$$
\lim_{n\to\infty} d^2(x_n,Tx_n) = 0.
$$

By the same argument as in the proof of Theorem 5.1.3, $\{x_n\}$ Δ-converges to an element u in $F(T)$. □

We now present the strong convergence theorem for an (α,β)-generalized hybrid mapping. We recall a following useful lemma.

Lemma 5.1.6. [92] *The fixed point set of a quasi-nonexpansive selfmap of a metric space is always closed.*

Theorem 5.1.7. *Let (X,d) be a complete CAT(0) space and C be a nonempty, closed and convex subset of X. Let $T : C \to C$ be an (α,β)-generalized hybrid mapping such that $\alpha \geq 1$ and $\beta \geq 0$ with $F(T) \neq \emptyset$. Let $\{\gamma_n\}$ be a sequence of real* *numbers with* $0 < a \leq \gamma_n \leq b < 1$ *and a* $\{x_n\}$ *generated by* (5.1.12). *Then* $\{x_n\}$ *converges strongly to an element* $u \in F(T)$ *if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, *where* $d(x, F(T)) = \inf_{p\in F(T)} d(x,p)$.

Proof. We observe that the necessity is obvious. Next, we prove the sufficiency. As proved in Theorem 5.1.5, we have $d(x_{n+1}, p) \leq d(x_n, p)$, for all $p \in F(T)$. This implies that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)),$$

and so the limit of $d(x_n, F(T))$ exists.
It follows by our assumption that $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

We claim that $\{x_n\}$ is a Cauchy sequence in C. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for any $\varepsilon > 0$ there exists $N \geq 1$ such that

$$d(x_n, F(T)) < \frac{\varepsilon}{4}, \ \forall n \geq N.$$

Therefore $d(x_N, F(T)) < \frac{\varepsilon}{4}$. From the definition of $d(x_N, F(T))$, there exists $q \in F(T)$ such that $d(x_N, q) < \frac{\varepsilon}{2}$. For any $n, m \geq N \geq 1$, we have

$$d(x_n, x_m) \leq d(x_n, q) + d(x_m, q) \leq 2d(x_N, q) < 2\frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset of a complete CAT(0) space. Hence $\{x_n\}$ converges to some $p^* \in C$. Note that (α, β)-*generalized hybrid* mapping with nonempty fixed point set is a quasi-nonexpansive. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $F(T)$ is closed, then $p^* \in F(T)$. This completes the proof. □

Remark 5.1.8. Consider $\mathbb{R}^2$ with the usual Euclidean meter $d(\cdot,\cdot)$ and $\|\cdot\|$ are defined by

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We define the radial metric d_r by

$$d_r(x, y) = \begin{cases} d(x, y), & \text{if } y = tx \text{ for some } t \in \mathbb{R}; \\ d(x, 0) + d(y, 0), & \text{otherwise.} \end{cases}$$

Then $X := (\mathbb{R}^2, d_r)$ is an $\mathbb{R}$-tree with the radial meter d_r (see [93] and [94, page 65]). We show that X is not inner product space. We prove this by showing that the norm does not satisfy the parallelogram equality. Indeed, if we take $x = (0, 1)$, $y_1 = (-1, 0), y_2 = (1, 0)$ and $y_0 = (0, 0)$, a midpoint of y_1 and y_2, then

$$1 = d_r^2(x, y_0) < \frac{1}{2}d_r^2(x, y_1) + \frac{1}{2}d_r^2(x, y_2) - \frac{1}{4}d_r^2(y_1, y_2) = \frac{1}{2}4 + \frac{1}{2}4 - \frac{1}{4}4 = 3.$$

Therefore X does not satisfy Parallelogram law, and so X is not an inner product space.

The following is an example of a (α, β)-generalized hybrid mapping in CAT(0) spaces which is not a nonexpansive mapping.

Example 5.1.9. *Let X be an $\mathbb{R}$-tree with the radial meter d_r. We put*

$$C = \left\{(t, 0) : t \in [0, 2] \cup \left[4, 5\frac{1}{2}\right]\right\} \cup \left\{(0, t) : t \in [0, 2] \cup \left[4, 5\frac{1}{2}\right]\right\} \subset \mathbb{R}^2$$

and define $T : C \to C$ by

$$T(t, 0) = \begin{cases} (0, 0), & \text{if } t \in [0, 2]; \\ \left(0, \frac{(t-4)^2}{2}\right), & \text{if } t \in \left[4, 5\frac{1}{2}\right], \end{cases}$$

and

$$T(0, t) = \begin{cases} (0, 0), & \text{if } t \in [0, 2]; \\ \left(\frac{(t-4)^2}{2}, 0\right), & \text{if } t \in [4, 5\frac{1}{2}]. \end{cases}$$

Clearly, $F(T) = \{(0, 0)\}$. Claim that T is (2,1)-generalized hybrid mapping, i.e.,

$$2d_r^2(Tx, Ty) \leq d_r^2(Tx, y) + d_r^2(x, Ty).$$

Case 1. $x = (s, 0), y = (t, 0)$.

If $s, t \in [0, 2]$, *then we have done.*

If $s, t \in \left[4, 5\frac{1}{2}\right]$, *then* $Tx = \left(0, \frac{(s-4)^2}{2}\right)$ *and* $Ty = \left(0, \frac{(t-4)^2}{2}\right)$, *and so*

$$\begin{aligned} 2d_r^2(Tx, Ty) &= 2\left(\frac{(s-4)^2}{2} - \frac{(t-4)^2}{2}\right)^2 \leq \left(\frac{9}{8}\right)^2 + \left(\frac{9}{8}\right)^2 \\ &\leq 4^2 + 4^2 \\ &\leq \left(\frac{(s-4)^2}{2} + t\right)^2 + \left(s + \frac{(t-4)^2}{2}\right)^2 \\ &= d_r^2(Tx, y) + d_r^2(x, Ty). \end{aligned}$$

If $s \in [0, 2], t \in [4, 5\frac{1}{2}]$, *then* $Tx = (0, 0)$ *and* $Ty = (0, \frac{(t-4)^2}{2})$, *and so*

$$\begin{aligned} 2d_r^2(Tx, Ty) = 2\left(\frac{(t-4)^2}{2}\right)^2 &\leq \left(\frac{(t-4)^2}{2}\right)^2 + \left(\frac{(t-4)^2}{2}\right)^2 \\ &\leq (t)^2 + \left(s + \frac{(t-4)^2}{2}\right)^2 \\ &= d_r^2(Tx, y) + d_r^2(x, Ty) \end{aligned}$$

If $s \in [4, 5\frac{1}{2}], t \in [0, 2]$, *then* $Tx = (0, \frac{(s-4)^2}{2})$ *and* $Ty = (0, 0)$, *and so*

$$\begin{aligned} 2d_r^2(Tx, Ty) = 2\left(\frac{(s-4)^2}{2}\right)^2 &= \left(\frac{(s-4)^2}{2}\right)^2 + \left(\frac{(s-4)^2}{2}\right)^2 \\ &\leq \left(t + (\frac{(s-4)^2}{2})\right)^2 + (s)^2 \\ &= d_r^2(Tx, y) + d_r^2(x, Ty). \end{aligned}$$

Case 2. $x = (s, 0), y = (0, t)$.

If $s, t \in [0, 2]$, *then we have done.*

If $s, t \in [4, 5\frac{1}{2}]$, *then* $Tx = \left(0, \frac{(s-4)^2}{2}\right)$ *and* $Ty = \left(\frac{(t-4)^2}{2}, 0\right)$, *and so*

$$\begin{aligned} 2d_r^2(Tx, Ty) &= 2\left(\frac{(s-4)^2}{2} + \frac{(t-4)^2}{2}\right)^2 \leq \left(\frac{9}{4}\right)^2 + \left(\frac{9}{4}\right)^2 \\ &\leq \left(4 - \frac{9}{8}\right)^2 + \left(4 - \frac{9}{8}\right)^2. \end{aligned}$$

$$
\begin{aligned}
&\leq \left(4-\frac{(s-4)^2}{2}\right)^2+\left(4-\frac{(t-4)^2}{2}\right)^2\\
&\leq \left(t-\frac{(s-4)^2}{2}\right)^2+\left(s-\frac{(t-4)^2}{2}\right)^2\\
&= d_r^2(Tx,y)+d_r^2(x,Ty).
\end{aligned}
$$

If $s\in[0,2], t\in[4,5\frac{1}{2}]$, *then* $Tx=(0,0)$ *and* $Ty=\left(\frac{(t-4)^2}{2},0\right)$, *and so*

$$
\begin{aligned}
2d_r^2(Tx,Ty) &= 2\left(\frac{(t-4)^2}{2}\right)^2=\left(\frac{(t-4)^2}{2}\right)^2+\left(\frac{(t-4)^2}{2}\right)^2\\
&\leq (t)^2+\left(\frac{9}{8}\right)^2\\
&\leq (t)^2+\left(4-\frac{9}{8}\right)^2\\
&\leq (t)^2+\left(4-\frac{(t-4)^2}{2}\right)^2\\
&\leq (t)^2+\left(s-\frac{(t-4)^2}{2}\right)^2=d_r^2(Tx,y)+d_r^2(x,Ty).
\end{aligned}
$$

If $s\in\left[4,5\frac{1}{2}\right], t\in[0,2]$, *then* $Tx=\left(0,\frac{(s-4)^2}{2}\right)$ *and* $Ty=(0,0)$, *and so*

$$
\begin{aligned}
2d_r^2(Tx,Ty) &= 2\left(\frac{(s-4)^2}{2}\right)^2=\left(\frac{(s-4)^2}{2}\right)^2+\left(\frac{(s-4)^2}{2}\right)^2\\
&\leq \left(\frac{9}{8}\right)^2+(s)^2\\
&\leq \left(4-\frac{9}{8}\right)^2+(s)^2\\
&\leq \left(4-\frac{(s-4)^2}{2}\right)^2+(s)^2\\
&\leq \left(t-\frac{(s-4)^2}{2}\right)^2+(s)^2=d_r^2(Tx,y)+d_r^2(x,Ty).
\end{aligned}
$$

Case 3. $x=(0,s), y=(0,t)$ *the proof is similarly to case 1.*

Case 4. $x=(0,s), y=(t,0)$ *the proof is similarly to case 2.*

Hence we have the claim. By the similar argument, we can conclude that T *is also* $(1,1)$*-generalized hybrid.*

But T is not nonexpansive. Indeed, if $x = \left(0, 5\frac{1}{4}\right), y = \left(0, 5\frac{1}{2}\right)$, then $x = \left(\frac{25}{32}, 0\right), y = \left(\frac{9}{8}, 0\right)$. Thus, we have

$$d_r(Tx, Ty) = \left|\frac{25}{32} - \frac{9}{8}\right| = \frac{11}{32} > \frac{1}{4} = \left|5\frac{1}{4} - 5\frac{1}{2}\right| = d_r(x, y).$$

□

5.2 Viscosity approximation methods for nonexpansive mappings in CAT(0) spaces

In this section, we present strong convergence theorems of Moudafi's viscosity methods in CAT(0) spaces. Our first result is the continuous version of Theorem 2.2 of Shi and Chen [55]. By using the concept of quasilinearization, we note that the proof given below is different from that of Shi and Chen.

The following two key lemmas can be obtained from elementary computation. For convenience of the readers, we include the details.

Lemma 5.2.1. *Let X be a CAT(0) space. Then for all $u, x, y \in X$, the following inequality holds*

$$d^2(x, u) \leq d^2(y, u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle.$$

Proof.

$$\begin{aligned} d^2(y,u) - d^2(x,u) - 2\langle \overrightarrow{yx}, \overrightarrow{xu} \rangle &= d^2(y,u) - d^2(x,u) - 2\langle \overrightarrow{yu}, \overrightarrow{xu} \rangle - 2\langle \overrightarrow{ux}, \overrightarrow{xu} \rangle \\ &= d^2(y,u) - d^2(x,u) - 2\langle \overrightarrow{yu}, \overrightarrow{xu} \rangle + 2d^2(x,u) \\ &= d^2(y,u) + d^2(x,u) - 2\langle \overrightarrow{yu}, \overrightarrow{xu} \rangle \\ &\geq d^2(y,u) + d^2(x,u) - 2d(y,u)d(x,u) \\ &= (d(y,u) - d(x,u))^2 \geq 0. \end{aligned}$$

The proof is completes. □

Lemma 5.2.2. *Let X be a CAT(0) space. For any $u, v \in X$ and $t \in [0,1]$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,*

(i) $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{uy} \rangle$ *and*
$\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$.

Proof. (i) It follows from (CN)-inequality that,

$$\begin{aligned}
2\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle &= d^2(u_t, y) + d^2(x, u_t) - d^2(u_t, u_t) - d^2(x, y) \\
&\leq td^2(u, y) + (1-t)d^2(v, y) - t(1-t)d^2(u, v) \\
&\quad + d^2(x, u_t) - d^2(u_t, u_t) - d^2(x, y) \\
&= td^2(u, y) + td^2(x, u_t) - td^2(u, u_t) - td^2(x, y) \\
&\quad + (1-t)d^2(v, y) + (1-t)d^2(x, u_t) - (1-t)d^2(v, u_t) \\
&\quad - (1-t)d^2(x, y) + td^2(u, u_t) + (1-t)d^2(v, u_t) - t(1-t)d^2(u, v) \\
&= t\left[d^2(u, y) + d^2(x, u_t) - d^2(u, u_t) - d^2(x, y)\right] \\
&\quad + (1-t)\left[d^2(v, y) + d^2(x, u_t) - d^2(v, u_t) - d^2(x, y)\right] \\
&\quad + t(1-t)^2 d^2(u, v) + (1-t)t^2 d^2(u, v) - t(1-t)d^2(u, v) \\
&= t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle.
\end{aligned}$$

(ii) The proof is similar to (i). □

Now, we ready to present the convergence theorems of viscosity approximation methods for nonexpansive mappings.

For any $t \in (0,1]$, and a contraction f with coefficient $\alpha \in (0,1)$. Define the mapping $S_t : C \to C$ by

$$G_t = tf(x) \oplus (1-t)Tx, \quad \forall x \in C. \tag{5.2.1}$$

It is not hard to see that G_t is a contraction on C. Indeed, for $x, y \in C$, we have

$$\begin{aligned} d(G_t(x), G_t(y)) &= d(tf(x) \oplus (1-t)Tx, tf(y) \oplus (1-t)Ty) \\ &\leq d(tf(x) \oplus (1-t)Tx, tf(y) \oplus (1-t)Tx) \\ &\quad + d(tf(y) \oplus (1-t)Tx, tf(y) \oplus (1-t)Ty) \\ &\leq td(f(x), f(y)) + (1-t)d(Tx, Ty) \\ &\leq t\alpha d(x, y) + (1-t)d(x, y) \\ &= (1 - t(1-\alpha))d(x, y). \end{aligned}$$

This implies that G_t is a contraction mapping. Then there exists a unique $u \in C$ such that

$$u = G_t(u) = tf(u) \oplus (1-t)Tu.$$

Let $x_t \in C$ be the unique fixed point of G_t. Thus

$$x_t = tf(x_t) \oplus (1-t)Tx_t. \tag{5.2.2}$$

Theorem 5.2.3. *Let C be a closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let f be a contraction on C with coefficient $0 < \alpha < 1$. For each $t \in (0, 1]$, let $\{x_t\}$ be given by*

$$x_t = tf(x_t) \oplus (1-t)Tx_t. \tag{5.2.3}$$

Then $\{x_t\}$ converges strongly as $t \to 0$ to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T). \tag{5.2.4}$$

Proof. We first show that $\{x_t\}$ is bounded. For any $p \in F(T)$, we have that

$$d(x_t, p) = d(tf(x_t) \oplus (1-t)Tx_t, p)$$

$$
\begin{aligned}
&\le\ td(f(x_t),p)+(1-t)d(Tx_t,p)\\
&\le\ td(f(x_t),p)+(1-t)d(x_t,p).
\end{aligned}
$$

Then

$$
\begin{aligned}
d(x_t,p)\ &\le\ d(f(x_t),p)\le d(f(x_t),f(p))+d(f(p),p)\\
&\le\ \alpha d(x_t,p)+d(f(p),p).
\end{aligned}
$$

This implies that

$$
d(x_t,p)\le\frac{1}{1-\alpha}d(f(p),p).
$$

Hence $\{x_t\}$ is bounded, so are $\{Tx_t\}$ and $\{f(x_t)\}$. We get that

$$
\begin{aligned}
\lim_{t\to 0}d(x_t,Tx_t)\ &=\ \lim_{t\to 0}d(tf(x_t)\oplus(1-t)Tx_t,Tx_t)\\
&\le\ \lim_{t\to 0}[td(f(x_t),Tx_t)+(1-t)d(Tx_t,Tx_t)]\\
&\le\ \lim_{t\to 0}td(f(x_t),Tx_t)=0.
\end{aligned}
$$

Assume that $\{t_n\}\subset(0,1)$ is such that $t_n\to 0$ as $n\to\infty$. Put $x_n:=x_{t_n}$. We will show that $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x}$ such that $\tilde{x}=P_{F(T)}f(\tilde{x})$ which is equivalent to the following variational inequality:

$$
\langle\overrightarrow{\tilde{x}f(\tilde{x})},\overrightarrow{x\tilde{x}}\rangle\ge 0,\quad x\in F(T).
$$

Since $\{x_n\}$ is bounded, by Lemma 2.4.11, 2.4.13, we may assume that $\{x_n\}$ Δ-converges to a point $\tilde{x}$, and $\tilde{x}\in F(T)$. It follows from Lemma 5.2.2 (i) that

$$
\begin{aligned}
d^2(x_n,\tilde{x})\ &=\ \langle\overrightarrow{x_n\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle\\
&\le\ \alpha_n\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle+(1-\alpha_n)\langle\overrightarrow{Tx_n\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle\\
&\le\ \alpha_n\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle+(1-\alpha_n)d(Tx_n,\tilde{x})d(x_n,\tilde{x})\\
&\le\ \alpha_n\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle+(1-\alpha_n)d^2(x_n,\tilde{x}).
\end{aligned}
$$

It follows that,

$$
\begin{aligned}
d^2(x_n,\tilde{x}) &\leq \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \\
&= \langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_n\tilde{x}} \rangle + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \\
&\leq d(f(x_n), f(\tilde{x}))d(x_n,\tilde{x}) + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \\
&\leq \alpha d^2(x_n,\tilde{x}) + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle,
\end{aligned}
$$

and thus

$$
d^2(x_n,\tilde{x}) \leq \frac{1}{1-\alpha}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle. \tag{5.2.5}
$$

Since $\{x_n\}$ Δ-converges to $\tilde{x}$, by Lemma 2.4.16, we have

$$
\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \leq 0. \tag{5.2.6}
$$

It follows from (5.2.5) that $\{x_n\}$ converge strongly to $\tilde{x}$.

Next, we show that $\tilde{x}$ solves the variational inequality (5.2.4). Applying Lemma 2.4.9, for any $q \in F(T)$,

$$
\begin{aligned}
d^2(x_t,q) &= d^2(tf(x_t)\oplus(1-t)Tx_t, q) \\
&\leq td^2(f(x_t),q) + (1-t)d^2(Tx_t,q) - t(1-t)d^2(f(x_t),Tx_t) \\
&\leq td^2(f(x_t),q) + (1-t)d^2(x_t,q) - t(1-t)d^2(f(x_t),Tx_t).
\end{aligned}
$$

It implies that

$$
d^2(x_t,q) \leq d^2(f(x_t),q) - (1-t)d^2(f(x_t),Tx_t)
$$

Taking the limit through $t = t_n \to 0$, we can get that

$$
d^2(\tilde{x},q) \leq d^2(f(\tilde{x}),q) - d^2(f(\tilde{x}),\tilde{x}).
$$

Hence

$$
0 \leq \frac{1}{2}\left[d^2(\tilde{x},\tilde{x}) + d^2(f(\tilde{x}),q) - d^2(\tilde{x},q) - d^2(f(\tilde{x}),\tilde{x})\right] = \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{q\tilde{x}} \rangle, \ \ \forall q \in F(T).
$$

That is, $\tilde{x}$ solves the inequality (5.2.4).

Finally, We show that the entire net $\{x_t\}$ converges to $\tilde{x}$, assume $x_{s_n} \to \hat{x}$, where $s_n \to 0$. By the same argument, we get that $\hat{x} \in F(T)$ and solves the variational inequality (5.2.4), i.e.,

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \leq 0, \tag{5.2.7}$$

and

$$\langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle \leq 0. \tag{5.2.8}$$

Adding up (5.2.7) and (5.2.8), we get that

$$\begin{aligned}
0 &\geq \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \\
&= \langle \overrightarrow{\tilde{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle + \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\hat{x}\tilde{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\tilde{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \\
&= \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle \\
&\geq \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - d(f(\hat{x}), f(\tilde{x}))d(\hat{x}, \tilde{x}) \\
&\geq d^2(\tilde{x}, \hat{x}) - \alpha d(\hat{x}, \tilde{x})d(\hat{x}, \tilde{x}) \\
&= d^2(\tilde{x}, \hat{x}) - \alpha d^2(\hat{x}, \tilde{x}) \\
&\geq (1 - \alpha)d^2(\tilde{x}, \hat{x}).
\end{aligned}$$

Since $0 < \alpha < 1$, we have that $d(\tilde{x}, \hat{x}) = 0$, and so $\tilde{x} = \hat{x}$. Hence the net x_t converge strongly to $\tilde{x}$ which is the unique solution to the variational inequality (5.2.4). This completes the proof. □

Remark 5.2.4. We give the different proof of [55, Theorem 2.2]. In fact, the property $\mathcal{P}$ imposed on a CAT(0) space X is removed.

If $f \equiv u$, then the following result can be obtained directly from Theorem 5.2.3.

Corollary 5.2.5. [95, Lemma 2.2] *Let C be a closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. For each $t \in (0, 1]$, let u be fixed and $\{x_t\}$ be given by*

$$x_t = tu \oplus (1-t)Tx_t. \tag{5.2.9}$$

Then $\{x_t\}$ converges strongly as $t \to 0$ to $\tilde{x} \in F(T)$ which is nearest to u which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}u}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T). \tag{5.2.10}$$

Theorem 5.2.6. *Let C be a closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let f be a contraction on C with coefficient $0 < \alpha < 1$. For the arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \ \ \forall n \geq 0, \tag{5.2.11}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ which is equivalent to the variational inequality (5.2.4).

Proof. We first show that the sequence $\{x_n\}$ is bounded. For any $p \in F(T)$, we have that

$$d(x_{n+1}, p) \ = \ d(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, p)$$

$$
\begin{aligned}
&\leq \alpha_n d(f(x_n),p) + (1-\alpha_n)d(Tx_n,p) \\
&\leq \alpha_n \left(d(f(x_n),f(p)) + d(f(p),p)\right) + (1-\alpha_n)d(Tx_n,p) \\
&\leq \max\left\{d(x_n,p), \frac{1}{1-\alpha}d(f(p),p)\right\}.
\end{aligned}
$$

By induction, we have

$$
d(x_n,p) \leq \max\left\{d(x_0,p), \frac{1}{1-\alpha}d(f(p),p)\right\},
$$

for all $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded, so are $\{Tx_n\}$ and $\{f(x_n)\}$. Next, we claim that $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$. To this end, we observe that

$$
\begin{aligned}
d(x_{n+1},x_n) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \alpha_{n-1}f(x_{n-1}) \oplus (1-\alpha_{n-1})Tx_{n-1}) \\
&\leq d(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_{n-1}) \\
&\quad + d(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_{n-1}, \alpha_n f(x_{n-1}) \oplus (1-\alpha_n)Tx_{n-1}) \\
&\quad + d(\alpha_n f(x_{n-1}) \oplus (1-\alpha_n)Tx_{n-1}, \alpha_{n-1}f(x_{n-1}) \oplus (1-\alpha_{n-1})Tx_{n-1}) \\
&\leq (1-\alpha_n)d(Tx_n,Tx_{n-1}) + \alpha_n d(f(x_n),f(x_{n-1})) + |\alpha_n \\
&\quad - \alpha_{n-1}|d(f(x_{n-1}),Tx_{n-1}) \\
&\leq (1-\alpha_n)d(x_n,x_{n-1}) + \alpha_n d(f(x_n),f(x_{n-1})) + |\alpha_n \\
&\quad - \alpha_{n-1}|d(f(x_{n-1}),Tx_{n-1}) \\
&\leq (1-\alpha_n)d(x_n,x_{n-1}) + \alpha_n \alpha d(x_n,x_{n-1}) + |\alpha_n \\
&\quad - \alpha_{n-1}|d(f(x_{n-1}),Tx_{n-1}) \\
&= (1-\alpha_n(1-\alpha))d(x_n,x_{n-1}) + |\alpha_n - \alpha_{n-1}|d(f(x_{n-1}),Tx_{n-1}).
\end{aligned}
$$

By the conditions (ii) and (iii) and Lemma 2.1.39, we have

$$
\lim_{n\to\infty} d(x_{n+1},x_n) = 0. \tag{5.2.12}
$$

It follows from (5.2.12) and condition (i) that

$$
\begin{aligned}
d(x_n,Tx_n) &\leq d(x_n,x_{n+1}) + d(x_{n+1},Tx_n) \\
&= d(x_n,x_{n+1}) + d(\alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, Tx_n)
\end{aligned}
$$

$$\leq \quad d(x_n, x_{n+1}) + \alpha_n d(f(x_n), Tx_n) \to 0 \text{ as } n \to \infty. \tag{5.2.13}$$

Let $\{x_t\}$ be a net in C such that

$$x_t = tf(x_t) \oplus (1-t)Tx_t.$$

By Theorem 5.2.3, we have that $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\tilde{x} \in F(T)$ which solves the variational inequality (5.2.4). Now, we claim that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}} \rangle \leq 0.$$

It follows from Lemma 5.2.2 (i) that

$$\begin{aligned}
d^2(x_t, x_n) &= \langle \overrightarrow{x_t x_n}, \overrightarrow{x_t x_n} \rangle \\
&\leq t\langle \overrightarrow{f(x_t)x_n}, \overrightarrow{x_t x_n} \rangle + (1-t)\langle \overrightarrow{Tx_t x_n}, \overrightarrow{x_t x_n} \rangle \\
&= t\langle \overrightarrow{f(x_t)f(\tilde{x})}, \overrightarrow{x_t x_n} \rangle + t\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_t x_n} \rangle + t\langle \overrightarrow{\tilde{x}x_t}, \overrightarrow{x_t x_n} \rangle + t\langle \overrightarrow{x_t x_n}, \overrightarrow{x_t x_n} \rangle \\
&\quad + (1-t)\langle \overrightarrow{Tx_t Tx_n}, \overrightarrow{x_t x_n} \rangle + (1-t)\langle \overrightarrow{Tx_n x_n}, \overrightarrow{x_t x_n} \rangle \\
&\leq t\alpha d(x_t, \tilde{x})d(x_t, x_n) + t\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_t x_n} \rangle + td(\tilde{x}, x_t)d(x_t, x_n) + td^2(x_t, x_n) \\
&\quad + (1-t)d^2(x_t, x_n) + (1-t)d(Tx_n, x_n)d(x_t, x_n) \\
&\leq t\alpha d(x_t, \tilde{x})M + t\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_t x_n} \rangle + td(\tilde{x}, x_t)M + td^2(x_t, x_n) \\
&\quad + (1-t)d^2(x_t, x_n) + (1-t)d(Tx_n, x_n)M \\
&\leq d^2(x_t, x_n) + t\alpha d(x_t, \tilde{x})M + td(\tilde{x}, x_t)M + d(Tx_n, x_n)M \\
&\quad + t\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_t x_n} \rangle,
\end{aligned}$$

where $M \geq \sup_{m,n\geq 1}\{d(x_t, x_n)\}$. This implies that

$$\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t} \rangle \leq (1+\alpha)d(x_t, \tilde{x})M + \frac{d(Tx_n, x_n)}{t}M. \tag{5.2.14}$$

Taking the limit as $n \to \infty$ first, and then $t \to 0$ inequality (5.2.14) yields that

$$\limsup_{t\to 0}\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t} \rangle \leq 0.$$

Since $x_t \to \tilde{x}$ as $t \to 0$ and continuity of metric distance d, we have for any fixed $n \geq 0$,

$$\begin{aligned}\lim_{t\to 0}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t}\rangle &= \lim_{t\to 0}\frac{1}{2}[d^2(f(\tilde{x}), x_t) + d^2(\tilde{x}, x_n) - d^2(f(\tilde{x}), x_n) - d^2(\tilde{x}, x_t)] \\ &= \frac{1}{2}[d^2(f(\tilde{x}), \tilde{x}) + d^2(\tilde{x}, x_n) - d^2(f(\tilde{x}), x_n) - d^2(\tilde{x}, \tilde{x})] \\ &= \langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle.\end{aligned}$$

It implies that, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle < \langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n x_t}\rangle + \varepsilon, \ \ \forall t \in (0, \delta). \tag{5.2.15}$$

Thus, by the upper limit as $n \to \infty$ first, and then $t \to 0$ inequality in (5.2.15), we get that

$$\limsup_{n\to\infty}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle \leq \varepsilon.$$

Since ε is arbitrary, it follows that

$$\limsup_{n\to\infty}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_n\tilde{x}}\rangle \leq 0.$$

Finally, we prove that $x_n \to \tilde{x}$ as $n \to \infty$. For any $n \in \mathbb{N}$, we set $y_n = \alpha_n\tilde{x} \oplus (1 - \alpha_n)Tx_n$. It follows from Lemma 5.2.1 and Lemma 5.2.2 (i), (ii) that

$$\begin{aligned}d^2(x_{n+1}, \tilde{x}) &\leq d^2(y_n, \tilde{x}) + 2\langle\overrightarrow{x_{n+1}y_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\ &\leq (\alpha_n d(\tilde{x}, \tilde{x}) + (1-\alpha_n)d(Tx_n, \tilde{x}))^2 + 2\Big[\alpha_n\langle\overrightarrow{f(x_n)y_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\ &\quad + (1-\alpha_n)\langle\overrightarrow{Tx_n y_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\Big] \\ &\leq (1-\alpha_n)^2 d^2(x_n, \tilde{x}) + 2\Big[\alpha_n\alpha_n\langle\overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\ &\quad + \alpha_n(1-\alpha_n)\langle\overrightarrow{f(x_n)Tx_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + (1-\alpha_n)\alpha_n\langle\overrightarrow{Tx_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\ &\quad + (1-\alpha_n)(1-\alpha_n)\langle\overrightarrow{Tx_nTx_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\Big] \\ &\leq (1-\alpha_n)^2 d^2(x_n, \tilde{x}) + 2\Big[\alpha_n\alpha_n\langle\overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\ &\quad + \alpha_n(1-\alpha_n)\langle\overrightarrow{f(x_n)Tx_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + (1-\alpha_n)\alpha_n\langle\overrightarrow{Tx_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\ &\quad + (1-\alpha_n)^2 d(Tx_n, Tx_n)d(x_{n+1}\tilde{x})\Big]\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\Big[\alpha_n^2 \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad + \alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle\Big] \\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n \langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&= (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n \langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leq (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + 2\alpha_n \alpha d(x_n,\tilde{x}) d(x_{n+1},\tilde{x}) + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leq (1-\alpha_n)^2 d^2(x_n,\tilde{x}) + \alpha_n \alpha \left(d^2(x_n,\tilde{x}) + d^2(x_{n+1},\tilde{x})\right) \\
&\quad + 2\alpha_n \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
d^2(x_{n+1},\tilde{x}) &\leq \frac{1-(2-\alpha)\alpha_n + \alpha_n^2}{1-\alpha\alpha_n} d^2(x_n,\tilde{x}) + \frac{2\alpha_n}{1-\alpha\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\leq \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n} d^2(x_n,\tilde{x}) + \frac{2\alpha_n}{1-\alpha\alpha_n} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + \alpha_n^2 M,
\end{aligned}
$$

where $M \geq \sup_{n\geq 0}\{d^2(x_n,\tilde{x})\}$. It then follows that

$$
d^2(x_{n+1},\tilde{x}) \leq (1-\alpha_n')d^2(x_n,\tilde{x}) + \alpha_n'\beta_n',
$$

where

$$
\alpha_n' = \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n} \quad \text{and} \quad \beta_n' = \frac{(1-\alpha\alpha_n)\alpha_n}{2(1-\alpha)} M + \frac{1}{(1-\alpha)} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle
$$

Applying Lemma 2.1.39, we can conclude that $x_n \to \tilde{x}$. This complete the proof.

□

Remark 5.2.7. We give the different proof of [55, Theorem 2.3]. In fact, the property $\mathcal{P}$ imposed on a CAT(0) space X is removed.

If $f \equiv u$, then the following corollary can be obtained directly from Theorem 5.2.6.

Corollary 5.2.8. [95, Theorem 2.3] *Let C be a closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$.*

Let $u, x_0 \in C$ *are arbitrary chosen and* $\{x_n\}$ *be generated by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T x_n, \ \forall n \geq 0, \tag{5.2.16}$$

where $\{\alpha_n\} \subset (0, 1)$ *satisfies the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Then $\{x_n\}$ *converges strongly as* $n \to \infty$ *to* $\tilde{x} \in F(T)$ *which is nearest to* u *which is equivalent to the following variational inequality* (5.2.10).

5.3 Viscosity approximation methods for nonexpansive semigroups in CAT(0) spaces

A family $\mathcal{S} := \{T(t) : t \in \mathbb{R}^+\}$ of self-mappings of C is called a one-parameter continuous semigroup of nonexpansive mappings if the following conditions hold:

(i) for each $t \in \mathbb{R}^+$, $T(t)$ is a nonexpansive mapping on C, i.e.,

$$d(T(t)x, T(t)y) \leq d(x, y), \ \forall x, y \in C;$$

(ii) $T(s + t) = T(t) \circ T(s)$ for all $t, s \in \mathbb{R}^+$;

(iii) for each $x \in X$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into C is continuous.

A family $\mathcal{S} := \{T(t) : t \in \mathbb{R}^+\}$ of mappings is called a one-parameter strongly continuous semigroup of nonexpansive mappings if conditions (i),(ii), and (iii) and the following condition are satisfied:

(iv) $T(0)x = x$ for all $x \in C$.

We shall denote by $\mathcal{F}$ the common fixed point set of $\mathcal{S}$, that is,

$$\mathcal{F} := F(\mathcal{S}) = \{x \in C : T(t)x = x, t \in \mathbb{R}^+\} = \bigcap_{t \in \mathbb{R}^+} F(T(t)).$$

Next, we present the strong convergence theorems of the Moudafi's viscosity approximation methods for a one-parameter continuous semigroup of nonexpansive mappings $\mathcal{S} := \{T(t) : t \in \mathbb{R}^+\}$ in CAT(0) spaces.

For any $\alpha_n \in (0,1)$, $t_n \in [0,\infty)$, and a contraction f with coefficient $\alpha \in (0,1)$. Define the mapping $G_n : C \to C$ by

$$G_n(x) = \alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x, \quad \forall x \in C. \tag{5.3.1}$$

It is not hard to see that G_n is a contraction on C. Indeed, for $x, y \in C$, we have

$$\begin{aligned} d(G_n(x), G_n(y)) &= d(\alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1-\alpha_n)T(t_n)y) \\ &\leq d(\alpha_n f(x) \oplus (1-\alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1-\alpha_n)T(t_n)x) \\ &\quad + d(\alpha_n f(y) \oplus (1-\alpha_n)T(t_n)x, \alpha_n f(y) \oplus (1-\alpha_n)T(t_n)y) \\ &\leq \alpha_n d(f(x), f(y)) + (1-\alpha_n)d(T(t_n)x, T(t_n)y) \\ &\leq \alpha_n \alpha d(x,y) + (1-\alpha_n)d(x,y) \\ &= (1-\alpha_n(1-\alpha))d(x,y). \end{aligned}$$

Therefore we have that G_n is a contraction mapping. Let $x_n \in C$ be the unique fixed point of G_n; that is

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, \quad \text{for all } n \geq 0. \tag{5.3.2}$$

Now we are a position to state and prove our main results.

Theorem 5.3.1. *Let C be a closed convex subset of a complete CAT(0) space X, and let $\{T(t)\}$ be one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular (in short, u.a.r.) on C, that is, for all $h \geq 0$ and any bounded subset B of C,*

$$\lim_{t\to\infty} \sup_{x \in B} d(T(h)(T(t)x), T(t)x) = 0.$$

Let f be a contraction on C with coefficient $0 < \alpha < 1$. Suppose $t_n \in [0, \infty)$, $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and let $\{x_n\}$ be given by (5.3.2). Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$ which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x} f \tilde{x}}, \overrightarrow{x \tilde{x}} \rangle \geq 0, \ \forall x \in \mathcal{F}. \tag{5.3.3}$$

Proof. We first show that $\{x_n\}$ is bounded. For any $p \in \mathcal{F}$, we have that

$$\begin{aligned} d(x_n, p) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n) T(t_n) x_n, p) \\ &\leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d(T(t_n)x_n, p) \\ &\leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d(x_n, p). \end{aligned}$$

Then

$$\begin{aligned} d(x_n, p) &\leq d(f(x_n), p) \leq d(f(x_n), f(p)) + d(f(p), p) \\ &\leq \alpha d(x_n, p) + d(f(p), p). \end{aligned}$$

This implies that

$$d(x_n, p) \leq \frac{1}{1-\alpha} d(f(p), p).$$

Hence $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. We get that

$$\begin{aligned} d(x_n, T(t_n)x_n) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n) T(t_n)x_n, T(t_n)x_n) \\ &\leq \alpha_n d(f(x_n), T(t_n)x_n) + (1-\alpha_n) d(T(t_n)x_n, T(t_n)x_n) \\ &\leq \alpha_n d(f(x_n), T(t_n)x_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $\{T(t)\}$ is u.a.r. and $\lim_{n\to\infty} t_n = \infty$, then for all $h > 0$,

$$\lim_{n\to\infty} d\left(T(h)(T(t_n)x_n), T(t_n)x_n\right) \leq \lim_{n\to\infty} \sup_{x \in B} d\left(T(h)(T(t_n)x), T(t_n)x\right) = 0,$$

where B is any bounded subset of C containing $\{x_n\}$. Hence

$$\begin{aligned} d(x_n, T(h)x_n) &\leq d(x_n, T(t_n)x_n) + d(T(t_n)x_n, T(h)(T(t_n)x_n)) + d(T(h)(T(t_n)x_n), T(h)x_n) \\ &\leq 2d(x_n, T(t_n)x_n) + d(T(t_n)x_n, T(h)(T(t_n)x_n)) \to 0 \text{ as } n \to \infty. \end{aligned} \quad (5.3.4)$$

We will show that $\{x_n\}$ contains a subsequence converging strongly to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \mathcal{F}. \quad (5.3.5)$$

Since $\{x_n\}$ is bounded, by Lemma 2.4.11, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which Δ-converges to a point $\tilde{x}$, denoted the $\{x_{n_j}\}$ by $\{x_j\}$. We claim that $\tilde{x} \in \mathcal{F}$. Since every CAT(0) space has Opial's property, for any $h \geq 0$, if $T(h)\tilde{x} \neq \tilde{x}$ we have

$$\begin{aligned} \limsup_{j\to\infty} d(x_j, T(h)\tilde{x}) &\leq \limsup_{j\to\infty} \{d(x_j, T(h)x_j) + d(T(h)x_j, T(h)\tilde{x})\} \\ &\leq \limsup_{j\to\infty} \{d(x_j, T(h)x_j) + d(x_j, \tilde{x})\} \\ &= \limsup_{j\to\infty} d(x_j, \tilde{x}) \\ &< \limsup_{j\to\infty} d(x_j, T(h)\tilde{x}). \end{aligned}$$

This is a contradiction, and hence $\tilde{x} \in \mathcal{F}$. So we have the claim. It follows from Lemma 5.2.2 (i) that

$$\begin{aligned} d^2(x_j, \tilde{x}) &= \langle \overrightarrow{x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &\leq \alpha_j \langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle + (1-\alpha_j) \langle \overrightarrow{T(t_j)x_j\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &\leq \alpha_j \langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle + (1-\alpha_j) d(T(t_j)x_j, \tilde{x}) d(x_j, \tilde{x}) \\ &\leq \alpha_j \langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle + (1-\alpha_j) d^2(x_j, \tilde{x}). \end{aligned}$$

It follows that,

$$\begin{aligned} d^2(x_j, \tilde{x}) &\leq \langle \overrightarrow{f(x_j)\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &= \langle \overrightarrow{f(x_j)f(\tilde{x})}, \overrightarrow{x_j\tilde{x}} \rangle + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &\leq d(f(x_j), f(\tilde{x}))d(x_j, \tilde{x}) + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \\ &\leq \alpha d^2(x_j, \tilde{x}) + \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle, \end{aligned}$$

and thus

$$d^2(x_j, \tilde{x}) \leq \frac{1}{1-\alpha} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle. \tag{5.3.6}$$

Since $\{x_j\}$ Δ-converges to $\tilde{x}$, by Lemma 2.4.16, we have

$$\limsup_{n\to\infty} \langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_j\tilde{x}} \rangle \leq 0.$$

It follows from (5.3.6) that $\{x_j\}$ converges strongly to $\tilde{x}$. Next, we show that $\tilde{x}$ solves the variational inequality (5.2.4). Applying Lemma 2.4.9, for any $q \in \mathcal{F}$,

$$\begin{aligned} d^2(x_j, q) &= d^2(\alpha_j f(x_j) \oplus (1-\alpha_j)T(t_j)x_j, q) \\ &\leq \alpha_j d^2(f(x_j), q) + (1-\alpha_j)d^2(T(t_j)x_j, q) - \alpha_j(1-\alpha_j)d^2(f(x_j), T(t_j)x_j) \\ &\leq \alpha_j d^2(f(x_j), q) + (1-\alpha_j)d^2(x_j, q) - \alpha_j(1-\alpha_j)d^2(f(x_j), T(t_j)x_j). \end{aligned}$$

It implies that

$$d^2(x_j, q) \leq d^2(f(x_j), q) - (1-\alpha_j)d^2(f(x_j), T(t_j)x_j).$$

Taking the limit through $j \to \infty$, we can get that

$$d^2(\tilde{x}, q) \leq d^2(f(\tilde{x}), q) - d^2(f(\tilde{x}), \tilde{x}).$$

Hence

$$0 \leq \frac{1}{2}\left[d^2(\tilde{x}, \tilde{x}) + d^2(f(\tilde{x}), q) - d^2(\tilde{x}, q) - d^2(f(\tilde{x}), \tilde{x})\right] = \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{q\tilde{x}} \rangle, \ \ \forall q \in \mathcal{F}.$$

That is, $\tilde{x}$ solves the inequality (5.3.3). Finally, We show that the sequence $\{x_n\}$ converges to $\tilde{x}$, assume $x_{n_i} \to \hat{x}$, where $i \to \infty$. By the same argument, we get that $\hat{x} \in \mathcal{F}$ and solves the variational inequality (5.3.3), i.e.,

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle \leq 0, \tag{5.3.7}$$

and

$$\langle \overrightarrow{\hat{x}f\hat{x}}, \overrightarrow{\hat{x}\tilde{x}} \rangle \leq 0. \tag{5.3.8}$$

Adding up (5.3.7) and (5.3.8), we get that

$$\begin{aligned}
0 &\geq \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \\
&= \langle \overrightarrow{\tilde{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle + \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{\tilde{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \\
&= \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \\
&\geq \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - d(f(\hat{x}), f(\tilde{x}))d(\hat{x}, \tilde{x}) \\
&\geq d^2(\tilde{x}, \hat{x}) - \alpha d(\hat{x}, \tilde{x})d(\hat{x}, \tilde{x}) \\
&\geq d^2(\tilde{x}, \hat{x}) - \alpha d^2(\hat{x}, \tilde{x}) \\
&\geq (1 - \alpha)d^2(\tilde{x}, \hat{x}).
\end{aligned}$$

Since $0 < \alpha < 1$, we have that $d(\tilde{x}, \hat{x}) = 0$, and so $\tilde{x} = \hat{x}$. It follows from Lemma 2.1.38 that the sequence x_n converge strongly to $\tilde{x}$ which is the unique solution to the variational inequality (5.3.3). This completes the proof. □

If $f \equiv u$, then the following result can be obtained directly from Theorem 5.3.1.

Corollary 5.3.2. *Let C be a closed convex subset of a complete CAT(0) space X, and let $\{T(t)\}$ be one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular (in short, u.a.r.) on*

C, that is, for all $h \geq 0$ and any bounded subset B of C,

$$\lim_{t\to\infty} \sup_{x\in B} d(T(h)(T(t)x), T(t)x) = 0.$$

Let u be any element in C. Suppose $t_n \in [0, \infty)$, $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$ and let $\{x_n\}$ be given by

$$x_n = \alpha_n u \oplus (1 - \alpha_n)T(t_n)x_n.$$

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}}\tilde{x}$ which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}u}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in \mathcal{F}. \tag{5.3.9}$$

Theorem 5.3.3. *Let C be a closed convex subset of a complete CAT(0) space X, and let $\{T(t)\}$ be one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular (in short, u.a.r.) on C, that is, for all $h \geq 0$ and any bounded subset B of C,*

$$\lim_{t\to\infty} \sup_{x\in B} d(T(h)(T(t)x), T(t)x) = 0.$$

Let f be a contraction on C with coefficient $0 < \alpha < 1$. Suppose $t_n \in [0, \infty)$, $\alpha_n \in (0, 1)$, $x_0 \in C$, and $\{x_n\}$ be given by

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T(t_n)x_n, \ \forall n \geq 0, \tag{5.3.10}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and*

(iii) $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{\mathcal{F}} f(\tilde{x})$ which is equivalent to the variational inequality (5.3.3).

Proof. We first show that the sequence $\{x_n\}$ is bounded. For any $p \in \mathcal{F}$, we have that

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n f(x_n) \oplus (1-\alpha_n)T(t_n)x_n, p) \\ &\leq \alpha_n d(f(x_n), p) + (1-\alpha_n) d(T(t_n)x_n, p) \\ &\leq \alpha_n \left(d(f(x_n), f(p)) + d(f(p), p) \right) + (1-\alpha_n) d(T(t_n)x_n, p) \\ &\leq \max\left\{ d(x_n, p), \frac{1}{1-\alpha} d(f(p), p) \right\}. \end{aligned}$$

By induction, we have

$$d(x_n, p) \leq \max\left\{ d(x_0, p), \frac{1}{1-\alpha} d(f(p), p) \right\},$$

for all $n \in \mathbb{N}$. Hence $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. Using the assumption that $\lim_{n\to\infty} \alpha_n = 0$, we get that

$$d(x_{n+1}, T(t_n)x_n) \leq \alpha_n d(f(x_n), T(t_n)x_n) \to 0 \text{ as } n \to \infty.$$

Since $\{T(t)\}$ is u.a.r. and $\lim_{n\to\infty} t_n = \infty$, then for all $h \geq 0$,

$$\lim_{n\to\infty} d\left(T(h)(T(t_n)x_n), T(t_n)x_n\right) \leq \lim_{n\to\infty} \sup_{x \in B} d\left(T(h)(T(t_n)x), T(t_n)x\right) = 0,$$

where B is any bounded subset of C containing $\{x_n\}$. Hence

$$\begin{aligned} & d(x_{n+1}, T(h)x_{n+1}) \\ & \quad \leq d(x_{n+1}, T(t_n)x_n) + d(T(t_n)x_n, T(h)(T(t_n)x_n)) \\ & \qquad + d(T(h)(T(t_n)x_n), T(h)x_{n+1}) \\ & \quad \leq 2d(x_{n+1}, T(t_n)x_n) + d(T(t_n)x_n, T(h)(T(t_n)x_n)) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{5.3.11}$$

Let $\{z_m\}$ be a sequence in C such that

$$z_m = \alpha_m f(z_m) \oplus (1-\alpha_m) T(t_m) z_m.$$

It follows from 5.3.1 that $\{z_m\}$ converges strongly as $m \to \infty$ to a fixed point $\tilde{x} \in \mathcal{F}$ which solves the variational inequality (5.3.3). Now, we claim that

$$\limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \leq 0.$$

It follows from Lemma 5.2.2 (i) that

$$\begin{aligned}
&d^2(z_m, x_{n+1})\\
&\quad= \langle \overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}}\rangle\\
&\quad\leq \alpha_m\langle \overrightarrow{f(z_m)x_{n+1}}, \overrightarrow{z_m x_{n+1}}\rangle + (1-\alpha_m)\langle \overrightarrow{T(t_m)z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}}\rangle\\
&\quad= \alpha_m\langle \overrightarrow{f(z_m)f(\tilde{x})}, \overrightarrow{z_m x_{n+1}}\rangle + \alpha_m\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}}\rangle + \alpha_m\langle \overrightarrow{\tilde{x}z_m}, \overrightarrow{z_m x_{n+1}}\rangle\\
&\qquad + \alpha_m\langle \overrightarrow{z_m x_{n+1}}, \overrightarrow{z_m x_{n+1}}\rangle + (1-\alpha_m)\langle \overrightarrow{T(t_m)z_m T(t_m)x_{n+1}}, \overrightarrow{z_m x_{n+1}}\rangle\\
&\qquad + (1-\alpha_m)\langle \overrightarrow{T(t_m)x_{n+1}x_{n+1}}, \overrightarrow{z_m x_{n+1}}\rangle\\
&\quad\leq \alpha_m\alpha d(z_m,\tilde{x})d(z_m,x_{n+1}) + \alpha_m\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}}\rangle + \alpha_m d(\tilde{x},z_m)d(z_m,x_{n+1})\\
&\qquad + \alpha_m d^2(z_m,x_{n+1}) + (1-\alpha_m)d^2(z_m,x_{n+1})\\
&\qquad + (1-\alpha_m)d(T(t_m)x_{n+1},x_{n+1})d(z_m,x_{n+1})\\
&\quad\leq \alpha_m\alpha d(z_m,\tilde{x})M + \alpha_m\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}}\rangle + \alpha_m d(\tilde{x},z_m)M\\
&\qquad + \alpha_m d^2(z_m,x_{n+1}) + (1-\alpha_m)d^2(z_m,x_{n+1})\\
&\qquad + (1-\alpha_m)d(T(t_m)x_{n+1},x_{n+1})M\\
&\quad\leq d^2(z_m,x_{n+1}) + \alpha_m\alpha d(z_m,\tilde{x})M + \alpha_m d(\tilde{x},z_m)M\\
&\qquad + d(T(t_m)x_{n+1},x_{n+1})M + \alpha_m\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{z_m x_{n+1}}\rangle,
\end{aligned}$$

where $M \geq \sup_{m,n\geq 1}\{d(z_m,x_n)\}$. This implies that

$$\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m}\rangle \leq (1+\alpha)d(z_m,\tilde{x})M + \frac{d(T(t_m)x_{n+1},x_{n+1})}{\alpha_m}M. \tag{5.3.12}$$

Taking the upper limit as $n \to \infty$ first, and then $m \to \infty$ inequality (5.3.12) yields that

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}z_m}\rangle \leq 0. \tag{5.3.13}$$

Since

$$\begin{aligned}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle &= \langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}z_m}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{z_m\tilde{x}}\rangle\\ &\leq \langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}z_m}\rangle + d(f(\tilde{x}),\tilde{x})d(z_m,\tilde{x}).\end{aligned}$$

Thus, by taking the upper limit as $n\to\infty$ first, and then $m\to\infty$ the last inequality, it follows from $z_m\to\tilde{x}$ and (5.3.13) that

$$\limsup_{n\to\infty}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle \leq 0.$$

Finally, we prove that $x_n\to\tilde{x}$ as $n\to\infty$. For any $n\in\mathbb{N}$, we set $y_n=\alpha_n\tilde{x}\oplus(1-\alpha_n)T(t_n)x_n$. It follows from Lemma 5.2.1 and Lemma 5.2.2 (i), (ii) that

$$\begin{aligned}
&d^2(x_{n+1},\tilde{x})\\
&\quad= d^2(y_n,\tilde{x}) + 2\langle\overrightarrow{x_{n+1}y_n},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\quad\leq (\alpha_n d(\tilde{x},\tilde{x}) + (1-\alpha_n)d(T(t_n)x_n,\tilde{x}))^2\\
&\qquad + 2\Big[\alpha_n\langle\overrightarrow{f(x_n)y_n},\overrightarrow{x_{n+1}\tilde{x}}\rangle + (1-\alpha_n)\langle\overrightarrow{T(t_n)x_ny_n},\overrightarrow{x_{n+1}\tilde{x}}\rangle\Big]\\
&\quad\leq (1-\alpha_n)^2d^2(x_n,\tilde{x}) + 2\Big[\alpha_n\alpha_n\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\qquad + \alpha_n(1-\alpha_n)\langle\overrightarrow{f(x_n)T(t_n)x_n},\overrightarrow{x_{n+1}\tilde{x}}\rangle + (1-\alpha_n)\alpha_n\langle\overrightarrow{T(t_n)x_n\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\qquad + (1-\alpha_n)(1-\alpha_n)\langle\overrightarrow{T(t_n)x_nT(t_n)x_n},\overrightarrow{x_{n+1}\tilde{x}}\rangle\Big]\\
&\quad\leq (1-\alpha_n)^2d^2(x_n,\tilde{x}) + 2\Big[\alpha_n\alpha_n\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\qquad + \alpha_n(1-\alpha_n)\langle\overrightarrow{f(x_n)T(t_n)x_n},\overrightarrow{x_{n+1}\tilde{x}}\rangle + (1-\alpha_n)\alpha_n\langle\overrightarrow{T(t_n)x_n\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\qquad + (1-\alpha_n)^2d(T(t_n)x_n,T(t_n)x_n)d(x_{n+1}\tilde{x})\Big]\\
&\quad= (1-\alpha_n)^2d^2(x_n,\tilde{x}) + 2\Big[\alpha_n^2\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle + \alpha_n(1-\alpha_n)\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\Big]\\
&\quad= (1-\alpha_n)^2d^2(x_n,\tilde{x}) + 2\alpha_n\langle\overrightarrow{f(x_n)\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\quad= (1-\alpha_n)^2d^2(x_n,\tilde{x}) + 2\alpha_n\langle\overrightarrow{f(x_n)f(\tilde{x})},\overrightarrow{x_{n+1}\tilde{x}}\rangle + 2\alpha_n\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\quad\leq (1-\alpha_n)^2d^2(x_n,\tilde{x}) + 2\alpha_n\alpha d(x_n,\tilde{x})d(x_{n+1},\tilde{x}) + 2\alpha_n\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle\\
&\quad\leq (1-\alpha_n)^2d^2(x_n,\tilde{x}) + \alpha_n\alpha\left(d^2(x_n,\tilde{x}) + d^2(x_{n+1},\tilde{x})\right) + 2\alpha_n\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned} d^2(x_{n+1},\tilde{x}) &\leq \frac{1-(2-\alpha)\alpha_n+\alpha_n^2}{1-\alpha\alpha_n}d^2(x_n,\tilde{x})+\frac{2\alpha_n}{1-\alpha\alpha_n}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle \\ &\leq \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}d^2(x_n,\tilde{x})+\frac{2\alpha_n}{1-\alpha\alpha_n}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle+\alpha_n^2M, \end{aligned}$$

where $M \geq \sup_{n\geq 0}\{d^2(x_n,\tilde{x})\}$. It then follows that

$$d^2(x_{n+1},\tilde{x}) \leq (1-\alpha_n')d^2(x_n,\tilde{x})+\alpha_n'\beta_n',$$

where

$$\alpha_n' = \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n} \quad \text{and} \quad \beta_n' = \frac{(1-\alpha\alpha_n)\alpha_n}{2(1-\alpha)}M+\frac{1}{(1-\alpha)}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle.$$

Applying Lemma 2.1.39, we can conclude that $x_n \to \tilde{x}$. This complete the proof.

□

If $f \equiv u$, then the following corollary can be obtained directly from Theorem 5.3.3.

Corollary 5.3.4. *Let C be a closed convex subset of a complete* CAT(0) *space X, and let $\{T(t)\}$ be one-parameter continuous semigroup of nonexpansive mappings on C satisfying $\mathcal{F} \neq \emptyset$ and uniformly asymptotically regular (in short, u.a.r.) on C, that is, for all $h \geq 0$ and any bounded subset B of C,*

$$\lim_{t\to\infty}\sup_{x\in B} d(T(h)(T(t)x),T(t)x)=0.$$

Suppose $t_n \in [0,\infty)$, $\alpha_n \in (0,1)$, $x_0 \in C$ and $\{x_n\}$ be given by

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)T(t_n)x_n, \ \ \forall n \geq 0, \tag{5.3.14}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and*

(iii) $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}$ *converges strongly as* $n \to \infty$ *to* $\tilde{x}$ *such that* $\tilde{x} = P_{\mathcal{F}}\tilde{x}$ *which is equivalent to the variational inequality* (5.3.9).