CHAPTER II

PRELIMINARIES

In this chapter, we give some notations, definitions, and some useful results that will be used in the later chapter.

2.1 Topological vector spaces

Definition 2.1.1. [16] Let X be a nonempty set and τ be a collection of subsets of X. Then τ is said to be *a topology* on X if the following conditions are satisfied:

(*i*) $\emptyset \in \tau$ and $X \in \tau$;

(*ii*) the union of every class of sets in τ is a set in τ;

(*iii*) the intersection of every finite class of sets in τ is a set in τ.

The ordered pair (X, τ) is called *a topological space* and the sets in class τ are called the *open sets* of the topological (X, τ). It is customary to denote the topological space (X, τ) by the symbol X which is used for its underlying set of points.

Definition 2.1.2. [17] Let X be a topological space, let U be a subset of X and let some $x \in X$ be a given element. The set U is called *a neighborhood of* x, if there is an open set V with $x \in V \subset U$ and x is called *an interior element of* U, if there is a neighborhood V of x contained in U. The set of all interior elements of U is called the interior of U and it is denoted by $intU$.

Definition 2.1.3. [17] A set F in a topological space X whose complement $F^C = X - F$ is open is called *a closed set.*

Definition 2.1.4. [17] Let F be a subset of a topological space X. Then the closure of F is the smallest closed set containing F. The *closure of* F is denoted by $\overline{F}$.

Theorem 2.1.5. [17] *Let F be a subset of a topological space X. Then F is closed if and only if $F = \overline{F}$.*

Definition 2.1.6. [17] *A linear space or vector space X* over the field $\mathbb{K}$ (The real field $\mathbb{R}$ or the complex field $\mathbb{C}$) is a set X together with an internal binary operation "+" called an *addition* and a *scalar multiplication* carrying (α, x) in $\mathbb{K} \times X$ to αx in X satisfying the following for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$:

(*i*) $x + y = y + x$,

(*ii*) $(x + y) + z = x + (y + z)$,

(*iii*) there exists an element $0 \in X$ called the *zero vector* of X such that $x + 0 = x$ for all $x \in X$,

(*iv*) for every element $x \in X$, there exists an element $-x \in X$ called the *additive inverse or the negative* of x such $x + (-x) = 0$,

(*v*) $\alpha(x + y) = \alpha x + \alpha y$,

(*vi*) $(\alpha + \beta)x = \alpha x + \beta x$,

(*vii*) $(\alpha\beta)x = \alpha(\beta x)$,

(*viii*) $1 \cdot x = x$.

The elements of a vector space X are called *vector* and the elements of $\mathbb{K}$ are called *scalars*. In the sequel, unless otherwise stated, X denotes a linear space over field $\mathbb{R}$.

Definition 2.1.7. [17] Let X be a linear space over a field $\mathbb{K}$ and let τ be a topology on X. Then (X, τ) is called *a topological vector space* if addition and multiplication with scalar are continuous, i.e. the maps

$$(x, y) \mapsto x + y \text{ with } x, y \in X$$

$$\text{and} \quad (\alpha, x) \mapsto \alpha x \text{ with } \alpha \in \mathbb{K} \text{ and } x \in X$$

are continuous on $X \times X$ and $\mathbb{K} \times X$, respectively. In many situations we use, for simplicity, the notation X instead of (X, τ) for a topological vector space.

Definition 2.1.8. [18] Let X be a topological vector space over the filed $\mathbb{R}$.

(*i*) A *sequence* $\{x_n\} \subset X$ *is bounded* if $\lambda_n x_n \to \theta$ whenever $\lambda_n \to 0$

in $\mathbb{R}$.

(ii) A *set* $A \subset X$ *is bounded* if every sequence in A is bounded.

Definition 2.1.9. [17] Let (X, τ) and (Y, τ') be two topological spaces. A map $f : X \to Y$ is called *continuous at some* $x \in X$, if for every neighborhood V of $f(x)$ there is a neighborhood U of x with $f(U) \subset V$ and $f : X \to Y$ is called *continuous on* X, if f is continuous at every $x \in X$.

Definition 2.1.10. [17] Let X be a topological space. Then X is said to be *Hausdorff topological space* if x and y are two distinct points in X, there exist two open sets G and H such that $x \in G$, $y \in H$, and $G \cap H = \emptyset$.

Definition 2.1.11. [16] A topological space X is said to be *compact* if every open cover has a finite subcover, i.e., if whenever $X = \bigcup_{i \in I} G_i$, where G_i is an open set, then $X = \bigcup_{i \in J} G_i$ for some finite subset J of I.

Definition 2.1.12. [16] A subset C of a topological space X is said to be *compact* if every open cover has a finite open subcover, i.e., if whenever $C \subseteq \bigcup_{i \in I} G_i$, where G_i is an open set, then $C \subseteq \bigcup_{i \in J} G_i$ for some finite subset J of I.

Remark 2.1.13. [16]

(i) Every finite set of a topological space is compact.

(ii) Every closed subset of a compact space is compact.

(iii) In a compact Hausdorff space, a set is compact if and only if it is closed.

Definition 2.1.14. [17] A subset C of a linear space X is said to be *a convex set in* X if $\lambda x + (1 - \lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0, 1]$.

Definition 2.1.15. [16] Let C be an arbitrary subset (not necessarily convex) of a linear space X. Then the *convex hull* of C in X is the intersection of all convex subsets of X containing C. It is denoted by $co(C)$. Hence

$$co(C) = \bigcap \{D \subseteq X : C \subseteq D,\ D \text{ is convex}\}$$

Thus, $co(C)$ is the unique smallest convex set containing C. Clearly,

$$co(C) = \left\{ \alpha_1 x_1 + \alpha_1 x_1 + \ldots + \alpha_1 x_1 : x_i \in C,\ \alpha_i \geq 0 \text{ and } \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Definition 2.1.16. Let X be a Hausdorff topological vector space with dual space X^*, K a nonempty compact convex subset of X. A mapping $T : K \to X^*$ is said to be *relaxed η-α monotone* if there exist a mapping $\eta : K \times K \to K$ and a function $\alpha : X \to \mathbb{R}$ positively homogeneous of degree p, i.e., $\alpha(tz) = t^p \alpha(z)$ for all $t > 0$ and $z \in X$ such that

$$\langle Tx - Ty, \eta(x, y) \rangle \geq \alpha(x - y), \quad \forall x, y \in K,$$

where $p > 1$ is a constant; see [19]. In the case of $\eta(x, y) = x - y$ for all $x, y \in K$, T is said to be *relaxed α-monotone*. Moreover, every monotone mapping is relaxed η-α monotone with $\eta(x, y) = x - y$ for all $x, y \in K$ and $\alpha \equiv 0$.

Definition 2.1.17. [20, 21] Let X be a topological space. A subset D of X is call *contractible*, if there exist a point $v \in D$ and a continuous mapping $g : D \times [0, 1] \to D$ such that $g(u, 0) = u$ for all $u \in D$ and $g(u, 1) = v$, for all $u \in D$.

We note that if D is convex, it is contractible since for any $v \in D$, the mapping $g(u, t) = tu + (1 - t)v$ would satisfy the above property. In addition a set star shaped at v also contractible to v.

2.2 Normed spaces and Banach spaces

Definition 2.2.1. [22] Let X be a linear space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). A function $\|\cdot\| : X \to \mathbb{K}$ is said to be *a norm on X* if it satisfies the following conditions:

(*i*) $\|x\| \geq 0, \forall x \in X$;

(*ii*) $\|x\| = 0 \Leftrightarrow x = 0$;

(*iii*) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$;

(*iv*) $\|\alpha x\| = |\alpha| \|x\|, \forall x \in X$ and $\forall \alpha \in \mathbb{K}$.

Definition 2.2.2. [22] Let $(X, \|\cdot\|)$ be a normed space.

(*i*) A sequence $\{x_n\} \subset X$ is said to *converge strongly* in X if there exists $x \in X$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. That is, if for any $\varepsilon > 0$ there exists a positive integer N such that $\|x_n - x\| < \varepsilon, \forall n \geq N$. We often write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ to mean that x is the limit of the sequence $\{x_n\}$.

(*ii*) A sequence $\{x_n\} \subset X$ is said to be *a Cauchy sequence* if for any $\varepsilon > 0$ there exists a positive integer N such that $\|x_m - x_n\| < \varepsilon, \forall\, m, n \geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in X if and only if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$.

(*iii*) A sequence $\{x_n\} \subset X$ is said to be *a bounded sequence* if there exists $M > 0$ such that $\|x_n\| \leqslant M, \forall n \in \mathbb{N}$.

Definition 2.2.3. [22] A normed space X is called to be *complete* if every Cauchy sequence in X converges to an element in X.

Definition 2.2.4. [22] A complete normed linear space over field $\mathbb{K}$ is called *a Banach space over* $\mathbb{K}$

Definition 2.2.5. [22] Let X and Y be linear spaces over the field $\mathbb{K}$.

(*i*) A mapping $T : X \to Y$ is called *a linear operator* if $T(x + y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx$, $\forall x, y \in X$, and $\forall \alpha \in \mathbb{K}$.

(*ii*) A mapping $T : X \to \mathbb{K}$ is called *a linear functional on* X if T is a linear operator.

Definition 2.2.6. [22] Let X and Y be normed spaces over the field $\mathbb{K}$ and $T : X \to Y$ a linear operator. T is said to be *bounded* on X, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

Definition 2.2.7. [22] Let X and Y be normed spaces over the field $\mathbb{K}$, $T : X \to Y$ an operator and $x_0 \in X$. We say that T is *continuous at* x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(x_0)\| < \varepsilon$ whenever $\|x - x_0\| < \delta$ and $x \in X$. If T is continuous at each $x \in X$, then T is said to be *continuous on* X.

Definition 2.2.8. [22] Let X be normed spaces. A mapping $T : X \to X$ is said to be *Lipschitzian* if there exists a constant $k \geq 0$ such that for all $x, y \in X$

$$\|Tx - Ty\| \leq k\|x - y\|. \tag{2.2.1}$$

The smallest number k for which (2.2.1) holds is called the *Lipschitz constant* of T and T is called *a contraction mapping* if $k \in (0, 1)$.

Definition 2.2.9. [22] Let X be normed spaces. A mapping T of X into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in X$.

Definition 2.2.10. [22] An element $x \in X$ is said to be

(*i*) a *fixed point* of a mapping $T : X \to X$ provided $Tx = x$.

(*ii*) a *common fixed point* of two mappings $S, T : X \to X$ provided $Sx = x = Tx$. The set of all fixed points of T is denoted by $F(T)$.

Definition 2.2.11. [22] Let X be a normed space. Then the set of all bounded linear functionals on X is called *a dual space* of X and is denoted by X^*.

Definition 2.2.12. [22] A normed space X is said to be *reflexive* if the *canonical mapping* $G : X \to X^{**}$ (i.e. $G(x) = g_x$ for all $x \in X$ where $g_x(f) = f(x)$ for all $f \in X^*$) is surjective.

Definition 2.2.13. [16] A sequence $\{x_n\}$ in a normed space X is said to be *converge weakly* to $x \in X$ if $f(x_n) \to f(x)$ for all $f \in X^*$. In this case, we write $x_n \rightharpoonup x$ or weak-$\lim_{n\to\infty} x_n = x$.

Definition 2.2.14. [23] Let X be a normed space, $\{x_n\} \subset X$ and $f : X \to (-\infty, \infty]$. Then f is said to be

(*i*) *lower semicontinuous* on X if for any $x_0 \in X$, $f(x_0) \leqslant \liminf_{n\to\infty} f(x_n)$ whenever $x_n \to x_0$.

(*ii*) *upper semicontinuous* on X if for any $x_0 \in X$, $\limsup_{n\to\infty} f(x_n) \leqslant f(x_0)$ whenever $x_n \to x_0$.

Remark 2.2.15. [23] Let X be a normed space and $f : X \to (-\infty, \infty]$. Then f is continuous if and only if f is lower semicontinuous and upper semicontinuous.

Definition 2.2.16. [23] Let X be a normed space and let C be a convex subset of X. A function $f : C \to (-\infty, \infty]$ is *convex* on X if for any $x_1, x_2 \in X$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leqslant tf(x_1) + (1-t)f(x_2)$$

and f is *concave* if $-f$ is convex.

Definition 2.2.17. [16] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function. Then d is called a *metric* on X if the following properties hold:

(*i*) $d(x, y) = 0$ if and only if $x = y$,

(*ii*) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(*iii*) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The value of metric d at (x, y) is called *distance between x and y*, and the ordered pair (X, d) is called *a metric space.*

Remark 2.2.18. [17] Every metric space is a Hausdorff space.

Remark 2.2.19. [22] Every normed space is a metric space with respect to the metric $d(x, y) = \|x - y\|, x, y \in X$.

Definition 2.2.20. [24] Let (X, d) be a metric space. Let 2^X be the collection of all nonempty subset of X and $CB(X)$ be the collection of all nonempty closed bounded subset of X. Define *Hausdorff metric* on $CB(X)$ by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}, \quad \text{for all } A, B \in CB(X),$$

where $d(x, A) = \inf_{y\in A} d(x, y)$.

For any a Banach space E, define a *Hausdorff pseudo-metric* $D : 2^E \times 2^E \to [0, +\infty]$ by

$$D(U, V) = \max\left\{\sup_{u\in U}\inf_{v\in V} \|u - v\|, \sup_{u\in V}\inf_{v\in U} \|u - v\|\right\}$$

for any given $U, V \in 2^E$. Note that if the domain of D is restricted to closed bounded subsets, then D is the Hausdorff metric.

Definition 2.2.21. [19] Let E be a Banach space with the dual space E^* and K be a nonempty subset of E. Let $T : K \to E^*$ and $\eta : K \times K \to E$ be two mappings. The mapping $T : K \to E^*$ is said to be *η-hemicontinuous* if, for any fixed $x, y \in K$, the function $f : [0,1] \to (-\infty, \infty)$ defined by $f(t) = \langle T((1-t)x + ty), \eta(x,y)\rangle$ is continuous at 0^+.

Definition 2.2.22. [23] Let E be a Banach space with the dual space E^* and K be a nonempty subset of E. The mapping $A : K \to E^*$ is said to be *hemicontinuous* if for any $x, y \in K$, the mapping $f : [0,1] \to E^*$ defined by $f(t) = \langle A((1-t)x+ty), z\rangle$ is continuous, for all $z \in E$.

Lemma 2.2.23. [25] *Let K be a nonempty closed convex subset of a strictly convex Banach space E and $S : K \to K$ a nonexpansive mapping with $F(S) \neq \emptyset$, where $F(S)$ is the set of all fixed points of S. Then $F(S)$ is closed convex.*

Theorem 2.2.24. (Mazur's Theorem, [16]). *The closed convex hull $\overline{co}(C)$ of a compact set C of a Banach space is compact.*

2.3 Inner product spaces and Hilbert spaces

Definition 2.3.1. [22] The real-valued function of two variables $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is called *inner product* on a real vector space X if it satisfies the following conditions:

(*i*) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$ for all $x, y, z \in X$ and all real number α and β;

(*ii*) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in X$; and

(*iii*) $\langle x, x\rangle \geq 0$ for each $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

A *real inner product space* is a real vector space equipped with an inner product.

Remark 2.3.2. [22] Every inner product space is a normed space with respect to the norm $\|x\| = |\langle x, x\rangle|^{\frac{1}{2}}, x, y \in X$.

Definition 2.3.3. [22] A *Hilbert space* is an inner product space which is complete under the norm induced by its inner product.

Definition 2.3.4. [22] A sequence $\{x_n\}$ in a Hilbert space H is said to *converge weakly* to a point x in H if $\lim_{n\to\infty}\langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in H$. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.

Definition 2.3.5. [22] The *metric (nearest point) projection* P_K from a Hilbert space H to a closed convex subset K of H is defined as follows: given $x \in H$, $P_K x$ is the only point in K with the property

$$\|x - P_K x\| = \inf\{\|x - y\| : y \in K\}.$$

Lemma 2.3.6. [23] *Let H be a real Hilbert space, K a closed convex subset of H. Given $x \in H$ and $y \in K$. Then $y = P_K x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \forall z \in K.$$

Remark 2.3.7. It is well known that P_K is a nonexpansive mapping of H onto K and satisfies

$$\langle x - y, P_K x - P_K y\rangle \geq \|P_K x - P_K y\|^2 \tag{2.3.1}$$

for every $x, y \in H$. Moreover, $P_K x$ is characterized by the following properties: $P_K x \in K$ and

$$\langle x - P_K x, y - P_K x\rangle \leq 0, \tag{2.3.2}$$

$$\|x - y\|^2 \geq \|x - P_K x\|^2 + \|y - P_K x\|^2 \tag{2.3.3}$$

for all $x \in H, y \in K$.

Definition 2.3.8. [14] A mapping T of a Hilbert space H into itself is said to be *firmly nonexpansive* if for each $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle.$$

Remark 2.3.9. Every firmly nonexpansive mapping is a nonexpansive mapping.

Definition 2.3.10. [26] Let K be a subset of a Hilbert space H and A is a mapping of K into H.

(*i*) A is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0.$$

(*ii*) A is called *α-inverse-strongly monotone* if there exists a positive number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$$

for all $u, v \in K$.

Remark 2.3.11.

monotone $\Leftarrow$ α-inverse-strongly monotone $\Rightarrow$ Lipschitz continuous

Figure 1: Relation between monotone and Lipschitz continuous mappings

Definition 2.3.12. Let H be a real Hilbert space, K be a nonempty bounded closed convex subset of H. Let $g, h : K \times K \to \mathbb{R}$, $A : K \to H$ be a monotone mapping, and let $T : K \to H$ be a relaxed η-α monotone mapping. For any $r > 0$, *the resolvent operator* $T_r : H \to 2^K$ defined by

$$T_r(z) = \Big\{x \in K : g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y - x\rangle + \frac{1}{r}\langle x - z, y - x\rangle \geq 0, \text{for all} \quad y \in K\Big\}. \tag{2.3.4}$$

Lemma 2.3.13. [27] *Let $\{c_n\}$ and $\{k_n\}$ be two real sequences of nonnegative numbers that satisfy the following conditions:*

(*i*) $0 < k_n < 1$ *for* $n = 0, 1, 2, \ldots,$ *and* $\limsup_n k_n < 1$;

(*ii*) $c_{n+1} \leq k_n c_n$ *for* $n = 0, 1, 2, \ldots$.

Then, c_n converges to 0 as $n \to \infty$.

2.4 Convexity, smoothness and duality mappings

Definition 2.4.1. [23] A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$

Definition 2.4.2. [28] A Banach space E is said to be *uniformly convex* if for each $0 < \varepsilon \leqslant 2$, there is $\delta > 0$ such that $\forall x, y \in E$, the condition $\|x\| = \|y\| = 1$, and $\|x - y\| \geqslant \varepsilon$ imply $\|\frac{x+y}{2}\| \leqslant 1 - \delta$.

Definition 2.4.3. [28] Let E be a Banach space. Then *the modulus of convexity of* E is $\delta : [0, 2] \to [0, 1]$ defined as follows:

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leqslant 1, \|y\| \leqslant 1, \|x - y\| \geqslant \varepsilon\right\}.$$

Theorem 2.4.4. [28] *Let E be a Banach space. Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$.*

Definition 2.4.5. [29] Let E be a Banach space and $S = \{x \in E : \|x\| = 1\}$. Then E is said to be *locally uniformly convex* if for each $\varepsilon > 0$ and $x \in S$, there exists $\delta(\varepsilon, x) > 0$ for $y \in S$,

$$\|x - y\| \geq \varepsilon \ \text{ implies } \ \|\frac{x+y}{2}\| < 1 - \delta(\varepsilon, x), \tag{2.4.1}$$

where $\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\right\}$.

Remark 2.4.6.

$$\boxed{\text{uniformly convex}} \Rightarrow \boxed{\text{locally uniformly convex}} \Rightarrow \boxed{\text{strictly convex}}$$

Figure 2: Relation on convex spaces

Definition 2.4.7. [23] Let E be a Banach space and $S = \{x \in E : \|x\| = 1\}$. Then E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.4.2}$$

exists for all $x, y \in S$. The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (2.4.2) is attained uniformly for $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (E is said to be *uniformly smooth*) if the limit (2.4.2) is attained uniformly for $x, y \in S(E)$.

Theorem 2.4.8. [23] *Let E be a Banach space with a Fréchet differentiable norm. Then, the duality mapping $J : E \to E^*$ is norm to norm continuous.*

Definition 2.4.9. [23] A Banach space E is said to have *Kadec-Klee property* if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

Remark 2.4.10. It is known that if E is uniformly convex, then E has the Kadec-Klee property.

Remark 2.4.11. The following properties are well-known (see [29] for details):

(*i*) If E is a uniformly smooth Banach space, then the normalized duality mapping J is uniformly continuous on each bounded subset of E.

(*ii*) If E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is a single valued bijective mapping.

(*iii*) If E is a smooth, strictly convex and reflexive Banach space and $J^* : E^* \to E$ is the duality mapping in E^*, then $J^{-1} = J^*, JJ^* = I_{E^*}$ and $J^*J = I_E$.

(*iv*) If E is a strictly convex and reflexive Banach space, then J^{-1} is hemi-continuous, i.e., J^{-1} is norm-weak-continuous.

(*v*) E is uniformly smooth if and only if E^* is uniformly convex.

(*vi*) If E is a uniformly smooth and strictly convex Banach space with the Kadec-Klee property, then both the normalized duality mappings $J : E \to E^*$ and $J^* = J^{-1} : E^* \to E$ are continuous.

(*vii*) Each uniformly convex Banach space E has the Kadec-Klee property.

(*viii*) If E is a uniformly smooth, then E is reflexive and smooth.

Lemma 2.4.12. [23] *Let E be a strictly convex, smooth, and reflexive Banach space, and let J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from E^* into E.*

Lemma 2.4.13. [30] *Let E be a reflexive Banach space and E^* be strictly convex. Then the following statements are hold.*

(i) The duality mapping $J : E \to E^*$ is single-valued, surjective and bounded.

(ii) If E and E^* are locally uniformly convex, then J is a homeomorphism, that is, J and J^{-1} are continuous single-valued mappings.

Definition 2.4.14. [23] Let E^* be dual space of a Banach space E. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad \text{for all} \quad x \in E.$$

Definition 2.4.15. [23] For each $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \text{for all} \quad x \in E. \qquad (2.4.3)$$

It is known that, in general, $J_q(x) = \|x\|^{q-1} J_2(x)$ for all $x \neq 0$ and J_q is single-valued if E^* is strictly convex. In the sequel, we always assume that E is a real Banach space such that J_q is single-valued.

Definition 2.4.16. Let E be a norm linear space with $\dim E \geq 2$. The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = t\right\}.$$

The space E is said to be *smooth* if $\rho_E(t) > 0$, $\forall t > 0$. E is called *uniformly smooth* if and only if $\lim_{t \to 0^+} \frac{\rho_E(t)}{t} = 0$. Let $q > 1$. E is said to be *q-uniformly smooth* (or to have a modulus of smoothness of power type q) if there exists a constant $c > 0$ such that $\rho_E(t) \leq ct^q$, $t > 0$. Note that J_q is single-valued if E is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [31] proved the following result.

Lemma 2.4.17. *Let E be a real uniformly smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in E$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Let E be a Banach space. Define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.4.4}$$

for all $x, y \in E$. We see that

(*i*) $(\|y\| - \|x\|)^2 \leqslant \phi(y,x) \leqslant (\|y\| + \|x\|)^2$, for all $x, y \in E$.

(*ii*) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle$, for all $x, y, z \in E$.

(*iii*) $\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$, for all $x, y \in E$.

(*iv*) In a Hilbert space H (the next section), we have $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$.

Definition 2.4.18. [32, 33] Let K be a nonempty closed convex subset of a Banach space E. For any $x \in E$, the mapping $\Pi_K : E \to K$ defined by $\Pi_K x = x_0$, where $x_0 \in K$ such that $\phi(x_0, x) = \min_{y\in K} \phi(y,x)$, is called the *generalized projection*.

Lemma 2.4.19. [33] *Let E be a smooth, strictly convex and reflexive Banach space and K be a nonempty closed convex subset of E. Then the following conclusions hold:*

(*i*) *If $x \in E$ and $z \in K$, then*

$$z = \prod_K x \Leftrightarrow \langle y - z, Jz - Jx\rangle \geq 0, \quad \text{for all} \quad y \in K; \tag{2.4.5}$$

(*ii*) *$\prod_K$ is a continuous mapping from E onto K.*

Remark 2.4.20. If E is a real Hilbert space, then $J = I$ (identity mapping) and $\prod_K$ is the metric projection P_K from E onto K.

Lemma 2.4.21. [13] *Let E be a real Banach space, K be a nonempty closed convex subset of E with $0 \in K$ and $\prod_K : E \to K$ be the generalized projection. Then for each $x \in E$, we have $\|\prod_K x\| \leq \|x\|$.*

For any a real Banach space E with dual space E^*, consider the functional $V : E^* \times E \to \mathbb{R}$ defined by

$$V(\varphi, x) = \|\varphi\|^2 - 2\langle \varphi, x\rangle + \|x\|^2, \text{ for all } \varphi \in E^*, \text{and } x \in E.$$

It is clear that $V(\varphi, x)$ is continuous and the map $x \mapsto V(\varphi, x)$ and $\varphi \mapsto V(\varphi, x)$ are convex and $(\|\varphi\| - \|x\|)^2 \leq V(\varphi, x) \leq (\|\varphi\| + \|x\|)^2$. We remark that the main Lyapunov functional V was first introduced by Alber [33] and its properties were studied there. By using this functional, Alber defined a generalized projection operator on uniformly convex and uniformly smooth Banach spaces which is further extended by Li [34] on reflexive Banach spaces.

Definition 2.4.22. [34] Let E be reflexive Banach space with its dual E^* and K be a nonempty, closed and convex subset of E. The operator $\pi_K : E^* \to K$ defined by

$$\pi_K(\varphi) = \{x \in K : V(\varphi, x) = \inf_{y \in K} V(\varphi, y)\}, \text{ for all } \varphi \in E^*, \tag{2.4.6}$$

is said to be a *generalized projection operator.* For each $\varphi \in E^*$, the set $\pi_K(\varphi)$ is called the *generalized projection of φ on K.*

Lemma 2.4.23. [34] *If E is a reflexive Banach space with dual space E^* and K is a nonempty closed convex subset of E, then $\pi_K(\varphi)$ is a nonempty, closed, convex and bounded subset of K, for any point $\varphi \in E^*$.*

Lemma 2.4.24. [34] *Let E be a reflexive Banach space with its dual E^* and K be a nonempty closed convex subset of E, then the following properties hold:*

(i) The operator $\pi_K : E^* \to 2^K$ is single-valued if and only if E is strictly convex.

(ii) If E is smooth, then for any given $\varphi \in E^*, x \in \pi_K\varphi$ if and only if

$$\langle \varphi - J(x), x - y \rangle \geq 0, \quad \forall y \in K.$$

(iii) If E is strictly convex, then the generalized projection operator $\pi_K : E^* \to K$ is continuous.

Lemma 2.4.25. [31, 35] *Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0) := \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then for any given finite subset $\{x_1, x_2, \ldots, x_N\} \subset B_r(0)$ and for any given positive numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ with $\sum_{n=1}^{N} \lambda_n = 1$, there exists a continuous, strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any $i, j \in \{1, 2, \ldots, N\}$ with $i < j$, the following holds:*

$$\left\| \sum_{n=1}^{N} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{N} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.4.7}$$

Lemma 2.4.26. [36] *Let E be a real reflexive, smooth and strictly convex Banach space. Then the following inequality holds:*

$$\|f + g\|^2 \leq \|f\|^2 + 2\langle g, J^{-1}(f + g)\rangle, \quad \textit{for all} \quad f, g \in E^*. \tag{2.4.8}$$

Definition 2.4.27. Let E be a Banach space. Let $H, \eta : E \times E \to E$ be two single-valued mappings and $A, B : E \to E$ be two single-valued mappings.

(*i*) A is said to be *accretive* if

$$\langle Ax - Ay, J_q(x - y)\rangle \geq 0, \quad \text{for all } x, y \in E;$$

(ii) A is said to be *strictly accretive* if A is accretive and

$$\langle Ax - Ay, J_q(x-y)\rangle = 0, \quad \text{for all } x, y \in E;$$

if and only if $x = y$;

(iii) $H(A,\cdot)$ is said to be *α-strongly accretive with respect to A* if there exists a constant $\alpha > 0$ such that

$$\langle H(Ax,u) - H(Ay,u), J_q(x-y)\rangle \geq \alpha\|x-y\|^q, \quad \text{for all } x, y, u \in E;$$

(iv) $H(\cdot, B)$ is said to be *β-relaxed accretive with respect to B* if there exists a constant $\beta > 0$ such that

$$\langle H(u,Bx) - H(u,By), J_q(x-y)\rangle \geq -\beta\|x-y\|^q, \quad \text{for all } x, y, u \in E;$$

(v) $H(\cdot,\cdot)$ is said to be *γ-Lipschitz continuous with respect to A* if there exists a constant $\gamma > 0$ such that

$$\|H(Ax,u) - H(Ay,u)\| \leq \gamma\|x-y\|, \quad \text{for all } x, y, u \in E;$$

(vi) $\eta(\cdot,\cdot)$ is said to be *strongly accretive with respect to $H(A,B)$* if there exists a constant $\rho > 0$ such that

$$\langle \eta(x,u) - \eta(y,u), J_q(H(Ax,Bx) - H(Ay,By))\rangle \geq \rho\|x-y\|^q, \quad \text{for all } x, y, u \in E.$$

2.5 Multi-valued mappings

In this section, we let X and Y be topological vector spaces.

Definition 2.5.1. [37] Let $T : X \to 2^Y$. The *graph of T*, denoted by $\mathcal{G}(T)$, is

$$\mathcal{G}(T) = \{(x,y) \in X \times Y \mid x \in X,\ y \in T(x)\}.$$

Definition 2.5.2. [37] A multi-valued mapping $T : X \to 2^Y$ is called:

(*i*) *closed* if the graph of T is a closed subset of $X \times Y$.

(*ii*) *upper semicontinuous at* $x \in X$ if for every open set V containing $T(x)$, there is an open set U containing x such that $T(u) \subseteq V$ for all $u \in U$. T is upper semicontinuous if T is upper semicontinuous at x for all $x \in X$.

(*iii*) *lower semicontinuous at* $x \in X$ if for every open set V with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that $T(u) \cap V \neq \emptyset$ for all $u \in U$. T is lower semicontinuous if T is lower semicontinuous at x for all $x \in X$.

(*iv*) *continuous* if it is both upper semicontinuous and lower semicontinuous.

Definition 2.5.3. [16] A multi-valued mapping $T : X \to 2^Y$ is called *convex* [*concave*] if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, $T(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda T(x_1) + (1-\lambda)T(x_2)$ [resp. $\lambda T(x_1) + (1-\lambda)T(x_2) \subseteq T(\lambda x_1 + (1-\lambda)x_2)$].

Lemma 2.5.4. [38] *Let X and Y be two Hausdorff topological vector spaces and $T : X \to 2^Y$ be a multi-valued mapping. Then the following properties hold:*

(*i*) *If T is closed and $\overline{T(X)}$ is compact, then T is upper semicontinuous, where $T(X) = \bigcup_{x \in X} T(x)$ and $\overline{E}$ denotes the closure of the set E.*

(*ii*) *If T is upper semicontinuous and for any $x \in X, T(x)$ is closed, then T is closed.*

(*iii*) *T is lower semicontinuous at $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}$, $x_\alpha \to x$, there exists a subnet $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \to y$.*

Lemma 2.5.5. [8] *Let M and N be two metric space and $T : M \to 2^N$ be a multi-valued mapping. Given any $x \in M$, if $T(x)$ is compact and T is upper semicontinuous at x, then $\forall x_n \to x$, $\forall u_n \in T(x_n), \{u_n\}$ must have a cluster point $u^* \in T(x)$.*

Definition 2.5.6. (*KKM* mapping, [39]). Let K be a nonempty subset of a topological vector space X. A multi-valued mapping $G : K \to 2^X$ is said to be a *KKM* *mapping* if for any finite subset $\{y_1, y_2, ..., y_n\}$ of K, we have

$$co\{y_1, y_2, ..., y_n\} \subset \bigcup_{i=1}^{n} G(y_i)$$

where $co\{y_1, y_2, ..., y_n\}$ denotes the convex hull of $\{y_1, y_2, ..., y_n\}$.

Lemma 2.5.7. (*Fan-KKM* Theorem, [39]). *Let K be a nonempty convex subset of a Hausdorff topological vector space X and let $G : K \to 2^X$ be a KKM mapping with closed values. If there exists a point $y_0 \in K$ such that $G(y_0)$ is a compact subset of K, then $\bigcap_{y \in K} G(y) \neq \emptyset$.*

Lemma 2.5.8. [40] (Eilenberg-Montgomery) *Let K be a compact contractible subset of a complete metric space M and let $T : K \to 2^K$ be a upper semicontinuous such that for every $x \in K$ the set $T(x)$ is nonempty, compact and contractible subset of M, then T has a fixed point.*

Definition 2.5.9. Let E be a Banach space. Let $\eta : E \times E \to E$ be a single-valued mapping. Let $M : E \to 2^E$ be a set-valued mapping.

(*i*) η is said to be *τ-Lipschitz continuous* if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \quad \text{for all } x, y \in E;$$

(*ii*) M is said to be *accretive* if

$$\langle u - v, J_q(x - y) \rangle \geq 0, \quad \text{for all } x, y \in E,\ u \in M(x),\ v \in M(y);$$

(*iii*) M is said to be *η-accretive* if

$$\langle u - v, J_q(\eta(x, y)) \rangle \geq 0, \quad \text{for all } x, y \in E,\ u \in M(x),\ v \in M(y);$$

(*iv*) M is said to be *strictly η-accretive* if M is η-accretive and equality holds if and only if $x = y$;

(*v*) M is said to be *γ-strongly η-accretive* if there exists a positive constant $\gamma > 0$ such that

$$\langle u - v, J_q(\eta(x, y)) \rangle \geq \gamma \|x - y\|^q, \quad \text{for all } x, y \in E,\ u \in M(x),\ v \in M(y);$$

(*vi*) M is said to be *α-relaxed η-accretive* if there exists a positive constant $\alpha > 0$ such that

$$\langle u - v, J_q(\eta(x, y))\rangle \geq -\alpha\|x - y\|^q, \quad \text{for all } x, y \in E,\ u \in M(x),\ v \in M(y);$$

Remark 2.5.10.

accretive $\Rightarrow$ η-accretive $\Leftarrow$ strictly η-accretive

$\Uparrow$

γ-strongly η-accretive $\Rightarrow$ α-relaxed η-accretive

Figure 3: Relation of accretive mappings

Definition 2.5.11. [41] Let E be a Banach space. Let $A, B : E \to E$, $H : E \times E \to E$ be three single-valued mappings. Let $M : E \to 2^E$ be a set-valued mapping. M is said to be *$H(\cdot,\cdot)$-accretive with respect to A and B* (or simply $H(\cdot,\cdot)$-accretive in the sequel), if M is accretive and $(H(A, B) + \lambda M)(E) = E$ for every $\lambda > 0$.

Lemma 2.5.12. [41] *Let $H(A, B)$ be α-strongly accretive with respect to A, β-relaxed accretive with respect to B, and $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-accretive operator with respect to A and B. Then, the operator $H((A, B) + \lambda M)^{-1}$ is single-valued.*

Definition 2.5.13. [41] Let E be a Banach space. Let H, A, B, M be defined as in Definition 2.5.11. Let $H(A, B)$ be α-strongly accretive with respect to A, β-relaxed accretive with respect to B, and $\alpha > \beta$. Let M be an $H(\cdot,\cdot)$-accretive operator with respect to A and B. The *resolvent operator* $R^{H(\cdot,\cdot)}_{M,\lambda} : E \to E$ is defined by

$$R^{H(\cdot,\cdot)}_{M,\lambda}(z) = (H(A, B) + \lambda M)^{-1}(z), \quad \text{for all } z \in E, \tag{2.5.1}$$

where $\lambda > 0$ is a constant.

Lemma 2.5.14. [41] *Let E be a Banach space. Let H, A, B, M be defined as in* Definition 2.5.11. *Let $H(A,B)$ be α-strongly accretive with respect to A, β-relaxed accretive with respect to B, and $\alpha > \beta$. Suppose that $M : E \to 2^E$ is an $H(\cdot,\cdot)$-accretive operator. Then resolvent operator $R_{M,\lambda}^{H(\cdot,\cdot)}$ defined by* (2.5.1) *is $\frac{1}{\alpha-\beta}$ Lipschitz continuous. That is,*

$$\|R_{M,\lambda}^{H(\cdot,\cdot)}(x) - R_{M,\lambda}^{H(\cdot,\cdot)}(y)\| \leq \frac{1}{\alpha-\beta}\|x-y\|, \ \ \textit{for all } x, y \in E.$$

2.6 Equilibrium problems and variational inequality problems

Definition 2.6.1. [42] Let X be a Hausdorff topological vector space, K a nonempty compact convex subset of X. Let g be a bifunction of $K \times K$ into $\mathbb{R}$. The *equilibrium problem* is to find $x \in K$ such that

$$g(x,y) \geq 0 \text{ for all } y \in K. \tag{2.6.1}$$

The set of solutions of (2.6.1) is denoted by $EP(g)$.

Theorem 2.6.2. [42] *Let K be a compact convex subset of a topological vector space X and let g be a real valued function on $K \times K$ satisfying the following conditions:*

(i) for each $y \in K$, the function $x \mapsto g(x,y)$ is upper semicontinuous;

(ii) for each $x \in K$, the function $y \mapsto g(x,y)$ is convex;

(iii) $g(x,x) \geq 0$, for all $x \in K$.

Then, there exists an element $x_0 \in K$ such that

$$g(x_0, y) \geq 0, \ \ \textit{for all} \ \ y \in K.$$

Definition 2.6.3. [14] Let X be a Hausdorff topological vector space with dual space X^*, K a nonempty compact convex subset of X. Let g, h be bifunctions of $K \times K$ into $\mathbb{R}$. The *equilibrium problem* is to find $x \in K$ such that

$$g(x,y) + h(x,y) \geq 0 \text{ for all } y \in K. \tag{2.6.2}$$

The set of solutions of (2.6.2) is denoted by $EP(g,h)$.

Theorem 2.6.4. [14] *Let the following assumptions (i)-(iv) hold:*

(i) X is a real topological vector space:

K is a nonempty closed convex subset of X;

(ii) $g : K \times K \to \mathbb{R}$ has the following properties:

$g(x,x) = 0$ for all $x \in K$;

$g(x,y) + g(y,x) \leq 0$ for all $x, y \in K$ (monotonicity);

for all $x, y \in K$ the function $t \in [0,1] \mapsto g(ty + (1-t)x, y)$ is upper semicontinuous at $t = 0$ (hemicontinuity);

g is convex and lower semicontinuous in the second argument;

(iii) $h : K \times K \to \mathbb{R}$ has the following properties:

$h(x,x) = 0$ for all $x \in K$;

h is upper semicontinuous in the first argument;

h is convex in the second argument;

(iv) There exists $C \subset K$ nonempty compact convex such that for every $x \in C \setminus \mathrm{core}_K C$ there exists $a \in \mathrm{core}_K C$ such that

$$g(x,a) + h(x,a) \leq 0 \quad \textit{(coercivity)}.$$

Then there exists $\overline{x} \in C$ such that

$$0 \leq g(\overline{x}, y) + h(\overline{x}, y), \ \textit{for all} \ \ y \in K.$$

Definition 2.6.5. [43] Let K be a subset of a real Banach space E with dual space E^*. Let $A : K \to E^*$ be a mapping. The *classical variational inequality*, denoted by $VI(A,K)$, is to find $x^* \in K$ such that

$$\langle Ax^*, v - x^* \rangle \geq 0 \text{ for all } v \in K. \tag{2.6.3}$$

Definition 2.6.6. [35] Let E be a real Banach space, E^* be the dual space of E, K be a nonempty closed convex subset of E, and $\langle \cdot, \cdot \rangle$ be the pairing between E and E^*. Let $F : K \times K \to \mathbb{R}$ be a bifunction, $\psi : K \to \mathbb{R}$ be a real-valued function, and $A : K \to E^*$ be a nonlinear mapping. The *generalized mixed equilibrium problem* is to find $u \in K$ such that

$$F(u,y) + \langle Au, y-u\rangle + \psi(y) - \psi(u) \geq 0,\ \forall y \in K. \tag{2.6.4}$$

The set of solutions of (2.6.4) is denoted by $GMEP(F, A, \psi)$.

Special examples

(*i*) If $A = 0$, then the problem (2.6.4) is equivalent to find $u \in K$ such that

$$F(u,y) + \psi(y) - \psi(u) \geq 0,\ \forall y \in K, \tag{2.6.5}$$

which is called the *mixed equilibrium problem* [44]. The set of solutions to (2.6.5) is denoted by $MEP(F, \psi)$.

(*ii*) If $F = 0$, then the problem (2.6.4) is equivalent to find $u \in K$ such that

$$\langle Au, y-u\rangle + \psi(y) - \psi(u) \geq 0,\ \forall y \in K, \tag{2.6.6}$$

which is called the *mixed variational inequality of Browder type* [45]. The set of solutions to (2.6.6) is denoted by $VI(K, A, \psi)$.

(*iii*) If $\psi = 0$, then the problem (2.6.4) is equivalent to find $u \in K$ such that

$$F(u,y) + \langle Au, y-u\rangle \geq 0,\ \forall y \in K, \tag{2.6.7}$$

which is called the *generalized equilibrium problem* [46]. The set of solutions to (2.6.7) is denoted by $GEP(F, A)$.

(*iv*) If $A = 0$ and $\psi = 0$, then the problem (2.6.4) is equivalent to the equilibrium problem (2.6.1).

(*v*) If $F = 0$ and $\psi = 0$, then the problem (2.6.4) is equivalent to the classical variational inequality (2.6.3).

Lemma 2.6.7. [35] *Let K be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and E^* be a dual space of E. Let F be a bifunction of $K \times K$ into $\mathbb{R}$ satisfying conditions (A1)-(A4), $A : K \to E^*$ be a continuous monotone mapping and $\psi : K \to \mathbb{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in E$, define a mapping $T_r : E \to K$ as follows:*

$$T_r(x) = \left\{ z \in K : F(z,y) + \langle Az, y-z\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle y-z, Jz-Jx\rangle \geq 0, \forall y \in K \right\}.$$

Then, the following hold:

(i) $T_r(x) \neq \emptyset$, $\forall x \in E$;

(ii) T_r is single- valued;

(iii) T_r is firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(iv) $F(T_r) = GMEP(F, A, \psi)$;

(v) $GMEP(F, A, \psi)$ is closed and convex.