CHAPTER III

GENERALIZED VARIATIONAL INEQUALITIES AND MIXED EQUILIBRIUM PROBLEMS

3.1 A generalized system of nonlinear variational inequalities in Banach space

In this section, we assume that E is a real Banach space with dual space E^*, K is a nonempty closed convex subset of E. Let $T_1, \ldots, T_N : \underbrace{K \times \ldots \times K}_{N-\text{times}} \to E^*$ be nonlinear mappings and $f : K \to E$ be a mapping. The generalized system of nonlinear variational inequality problem ($GSNVIP$) is to find $x_1^*, \ldots, x_N^* \in K$ such that for all $x \in K$,

$$\begin{cases} \langle f(x) - f(x_1^*), T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*) \rangle \geq 0 \\ \langle f(x) - f(x_2^*), T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*) \rangle \geq 0 \\ \vdots \\ \langle f(x) - f(x_N^*), T_N(x_1^*, x_2^*, \ldots, x_{N-1}^*, x_N^*) \rangle \geq 0. \end{cases} \tag{3.1.1}$$

If $N = 3$, $f = I$ and $T_1, T_2, T_3 : K \times K \times K \to E^*$ are nonlinear mappings, then the generalized system of nonlinear variational inequality problem ($GSNVIP$) reduces to the following problem: (see [13]) is to find $x_1^*, x_2^*, x_3^* \in K$ such that for all $x \in K$,

$$\begin{cases} \langle x - x_1^*, T_1(x_2^*, x_3^*, x_1^*) \rangle \geq 0 \\ \langle x - x_2^*, T_2(x_3^*, x_1^*, x_2^*) \rangle \geq 0 \\ \langle x - x_3^*, T_3(x_1^*, x_2^*, x_3^*) \rangle \geq 0. \end{cases} \tag{3.1.2}$$

If $N = 2$, $T_1, T_2 : K \times K \to E^*$ are nonlinear mappings and $f : K \to E$ is a mapping, then the generalized system of nonlinear variational inequality problem

$(GSNVIP)$ reduces to the following problem: to find $x_1^*, x_2^* \in K$ such that for all $x \in K$,

$$\begin{cases} \langle f(x) - f(x_1^*), T_1(x_2^*, x_1^*)\rangle \geq 0 \\ \langle f(x) - f(x_2^*), T_2(x_1^*, x_2^*)\rangle \geq 0. \end{cases} \tag{3.1.3}$$

If $T, S : K \times K \to E^*$ are nonlinear mappings and $g, f : K \to E$ are mappings. Define $T_1, T_2 : K \times K \to E^*$ by $T_1(x_1^*, x_2^*) = \rho_1 T(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*)$ and $T_2(x_1^*, x_2^*) = \rho_2 S(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*)$. Then the generalized system of nonlinear variational inequality problem $(GSNVIP)$ reduces to the following problem: to find $x_1^*, x_2^* \in K$ such that for all $x \in K$,

$$\begin{cases} \langle f(x) - f(x_1^*), \rho_1 T(x_2^*, x_1^*) + g(x_2^*) - g(x_1^*)\rangle \geq 0 \\ \langle f(x) - f(x_2^*), \rho_2 S(x_1^*, x_2^*) + g(x_2^*) - g(x_1^*)\rangle \geq 0, \end{cases} \tag{3.1.4}$$

where ρ_1 and ρ_2 are two positive constants.

Lemma 3.1.1. *Let E be a smooth, strictly convex and reflexive Banach space, K be a nonempty closed convex subset of E. Let $T_1, \ldots, T_N : \underbrace{K \times \ldots \times K}_{N-times} \to E^*$ be mappings, $f : K \to K$ be a bijective mapping and $\rho_1, \ldots, \rho_N$ be any positive real numbers. Then $(x_1^*, \ldots, x_N^*) \in \underbrace{K \times \ldots \times K}_{N-times}$ is a solution to problem (3.1.1) if and only if $(x_1^*, \ldots, x_N^*) \in \underbrace{K \times \ldots \times K}_{N-times}$ is a solution to the following system of operator equations:*

$$\begin{cases} x_1^* = f^{-1} \prod_K J^{-1}\Big(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\Big), \\ x_2^* = f^{-1} \prod_K J^{-1}\Big(Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\Big), \\ \qquad\qquad\vdots \\ x_{N-1}^* = f^{-1} \prod_K J^{-1}\Big(Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\Big), \\ x_N^* = f^{-1} \prod_K J^{-1}\Big(Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\Big). \end{cases} \tag{3.1.5}$$

Proof. By Lemma 2.4.19, we have $(x_1^*, \ldots, x_N^*) \in \underbrace{K \times \ldots \times K}_{N-\text{times}}$ is a solution of problem (3.1.1),

$$\Leftrightarrow \begin{cases} \langle f(x) - f(x_1^*), \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\rangle \geq 0 \\ \langle f(x) - f(x_2^*), \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\rangle \geq 0 \\ \qquad\qquad\vdots \\ \langle f(x) - f(x_{N-1}^*), \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\rangle \geq 0 \\ \langle f(x) - f(x_N^*), \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\rangle \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \langle f(x) - f(x_1^*), Jf(x_1^*) - Jf(x_1^*) + \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\rangle \geq 0 \\ \langle f(x) - f(x_2^*), Jf(x_2^*) - Jf(x_2^*) + \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\rangle \geq 0 \\ \qquad\qquad\vdots \\ \langle f(x) - f(x_{N-1}^*), Jf(x_{N-1}^*) - Jf(x_{N-1}^*) \\ \qquad\qquad + \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\rangle \geq 0 \\ \langle f(x) - f(x_N^*), Jf(x_N^*) - Jf(x_N^*) + \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\rangle \geq 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \Big\langle f(x) - f(x_1^*), Jf(x_1^*) - J\Big(J^{-1}\Big(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\Big)\Big)\Big\rangle \geq 0 \\ \Big\langle f(x) - f(x_2^*), Jf(x_2^*) - J\Big(J^{-1}\Big(Jf(x_2^*) \\ \qquad\qquad - \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\Big)\Big)\Big\rangle \geq 0 \\ \qquad\qquad\vdots \\ \Big\langle f(x) - f(x_{N-1}^*), Jf(x_{N-1}^*) - J\Big(J^{-1}\Big(Jf(x_{N-1}^*) \\ \qquad\qquad - \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\Big)\Big)\Big\rangle \geq 0 \\ \Big\langle f(x) - f(x_N^*), Jf(x_N^*) - J\Big(J^{-1}\Big(Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\Big)\Big)\Big\rangle \geq 0, \end{cases}$$

for all $x \in K$,

$$\Leftrightarrow \begin{cases} f(x_1^*) = \prod_K J^{-1}\Big(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\Big), \\ f(x_2^*) = \prod_K J^{-1}\Big(Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\Big), \\ \vdots \\ f(x_{N-1}^*) = \prod_K J^{-1}\Big(Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\Big), \\ f(x_N^*) = \prod_K J^{-1}\Big(Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\Big), \end{cases}$$

for any $\rho_1 > 0, \ldots, \rho_N > 0$,

$$\Leftrightarrow \begin{cases} x_1^* = f^{-1} \prod_K J^{-1}\Big(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\Big), \\ x_2^* = f^{-1} \prod_K J^{-1}\Big(Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\Big), \\ \vdots \\ x_{N-1}^* = f^{-1} \prod_K J^{-1}\Big(Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\Big), \\ x_N^* = f^{-1} \prod_K J^{-1}\Big(Jf(x_N^*) - \rho_N T_N(x_1^*, x_2^*, \ldots, x_N^*)\Big). \end{cases}$$

$\square$

Algorithm 3.1.2. For any given initial points $x_0^{(1)}, x_0^{(2)}, \ldots, x_0^{(N)} \in K$, compute the sequences $\{x_n^{(1)}\}$,$\{x_n^{(2)}\}$,...,$\{x_n^{(N)}\}$ by the iterative processes

$$\begin{cases} x_{n+1}^{(N)} = f^{-1}\Big[J^{-1}\Big((1-\alpha_n^{(N)})Jf(x_n^{(N)}) + \alpha_n^{(N)} J\Big(\prod_K J^{-1}(Jf(x_n^{(N)}) \\ \qquad\qquad - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}))\Big)\Big)\Big], \\ x_{n+1}^{(N-1)} = f^{-1}\Big[J^{-1}\Big((1-\alpha_n^{(N-1)})Jf(x_n^{(N-1)}) + \alpha_n^{(N-1)} J\Big(\prod_K J^{-1}(Jf(x_n^{(N-1)}) \\ \qquad\qquad - \rho_{N-1} T_{N-1}(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)}))\Big)\Big)\Big], \\ \qquad \vdots \\ x_{n+1}^{(2)} = f^{-1}\Big[J^{-1}\Big((1-\alpha_n^{(2)})Jf(x_n^{(2)}) + \alpha_n^{(2)} J\Big(\prod_K J^{-1}(Jf(x_n^{(2)}) \\ \qquad\qquad - \rho_2 T_2(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}))\Big)\Big)\Big], \\ x_{n+1}^{(1)} = f^{-1}\Big[J^{-1}\Big((1-\alpha_n^{(1)})Jf(x_n^{(1)}) + \alpha_n^{(1)} J\Big(\prod_K J^{-1}(Jf(x_n^{(1)}) \\ \qquad\qquad - \rho_1 T_1(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}))\Big)\Big)\Big], \end{cases} \tag{3.1.6}$$

where $\prod_K$ is the generalized projection and $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, ..., $\{\alpha_n^{(N)}\}$ are sequences in $[0, 1]$.

Theorem 3.1.3. *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, K be a nonempty closed and convex subset of E with $\theta \in K$. Let $f : K \to K$ be an isometry mapping. Let $T_1, \ldots, T_N : \underbrace{K \times \ldots \times K}_{N-times} \to E^*$ be continuous mappings and $\{\alpha_n^{(1)}\}$,$\{\alpha_n^{(2)}\}$,...,$\{\alpha_n^{(N)}\}$ be the sequences in (a, b) with $0 < a < b < 1$ satisfying the following conditions:*

(i) there exist a compact subset $C \subset E^*$ and constants $\rho_1 > 0, \rho_2 > 0, \ldots, \rho_N > 0$ such that

$(J(K) - \rho_N T_N(\underbrace{K \times \ldots \times K}_{N-times})) \cup (J(K) - \rho_{N-1} T_{N-1}(\underbrace{K \times \ldots \times K}_{N-times})) \cup \ldots \cup (J(K) -$

$\rho_1 T_1(\underbrace{K \times \ldots \times K}_{N-times})) \subset C$, *where* $J(x_1, x_2, \ldots, x_N) = Jx_N$, $\forall (x_1, x_2, \ldots, x_N) \in \underbrace{K \times \ldots \times K}_{N-times}$ *and*

$$\begin{cases} \langle T_1(x_1, x_2, \ldots, x_N), J^{-1}(Jx_N - \rho_1 T_1(x_1, x_2, \ldots, x_N)) \rangle \geq 0, \\ \langle T_2(x_1, x_2, \ldots, x_N), J^{-1}(Jx_N - \rho_2 T_2(x_1, x_2, \ldots, x_N)) \rangle \geq 0, \\ \vdots \\ \langle T_N(x_1, x_2, \ldots, x_N), J^{-1}(Jx_N - \rho_N T_N(x_1, x_2, \ldots, x_N)) \rangle \geq 0, \end{cases} \tag{3.1.7}$$

for all $x_1, x_2, \ldots, x_N \in K$.

(ii) $\lim_{n\to\infty} \alpha_n^{(1)} = d_1 \in (a, b)$, $\lim_{n\to\infty} \alpha_n^{(2)} = d_2 \in (a, b)$, $\ldots$, $\lim_{n\to\infty} \alpha_n^{(N)} = d_N \in (a, b)$. *Let* $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ *be the sequences defined by* (3.1.6).

Then the problem (3.1.1) *has a solution* $(x_1^*, x_2^*, \ldots, x_N^*) \in \underbrace{K \times \ldots \times K}_{N-times}$ *and the sequences* $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ *converge strongly to* $x_1^*, x_2^*, \ldots, x_N^*$, *respectively.*

Proof. **Step 1.** We first show that the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ are bounded in K. It follows from Lemma 2.4.26, J is bijective and a condition (3.4.29) that

$$\begin{aligned} & \|Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)})\|^2 \\ \leq \ & \|Jf(x_n^{(N)})\|^2 \\ & -2\rho_N \Big\langle T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}), J^{-1}\Big(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)})\Big)\Big\rangle \\ \leq \ & \|Jf(x_n^{(N)})\|^2 = \|f(x_n^{(N)})\|^2. \end{aligned} \tag{3.1.8}$$

Similarly, we note that

$$\begin{aligned} \|Jf(x_n^{(N-1)}) - \rho_{N-1} T_{N-1}(x_{n+1}^{(N)}, x_n^{(1)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)})\|^2 &\leq \|f(x_n^{(N-1)})\|^2, \\ \|Jf(x_n^{(N-2)}) - \rho_{N-2} T_{N-2}(x_{n+1}^{(N-1)}, x_{n+1}^{(N)}, x_n^{(1)}, \ldots, x_n^{(N-3)}, x_n^{(N-2)})\|^2 &\leq \|f(x_n^{(N-2)})\|^2, \\ &\vdots \end{aligned}$$

$$\|Jf(x_n^{(2)}) - \rho_2 T_2(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \dots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)})\|^2 \leq \|f(x_n^{(2)})\|^2,$$
$$\|Jf(x_n^{(1)}) - \rho_1 T_1(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \dots, x_{n+1}^{(N)}, x_n^{(1)})\|^2 \leq \|f(x_n^{(1)})\|^2. \quad (3.1.9)$$

By Lemma 2.4.21, we obtain that

$$\begin{aligned}
\|f(x_{n+1}^{(N)})\| &= \Big\| ff^{-1}\Big[J^{-1}\Big((1-\alpha_n^{(N)})Jf(x_n^{(N)}) \\
&\quad +\alpha_n^{(N)} J\Big(\prod_K J^{-1}\big(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})\big)\Big)\Big)\Big]\Big\| \\
&= \Big\| J^{-1}\Big((1-\alpha_n^{(N)})Jf(x_n^{(N)}) \\
&\quad +\alpha_n^{(N)} J\Big(\prod_K J^{-1}\big(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})\big)\Big)\Big)\Big\| \\
&= \Big\| (1-\alpha_n^{(N)})Jf(x_n^{(N)}) \\
&\quad +\alpha_n^{(N)} J\Big(\prod_K J^{-1}\big(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})\big)\Big)\Big\| \\
&\leq (1-\alpha_n^{(N)})\|Jf(x_n^{(N)})\| \\
&\quad +\alpha_n^{(N)} \Big\| J\Big(\prod_K J^{-1}\big(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})\big)\Big)\Big\| \\
&\leq (1-\alpha_n^{(N)})\|Jf(x_n^{(N)})\| \\
&\quad +\alpha_n^{(N)} \|JJ^{-1}\big(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})\big)\| \\
&= (1-\alpha_n^{(N)})\|f(x_n^{(N)})\| + \alpha_n^{(N)}\|Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(N)})\| \\
&\leq (1-\alpha_n^{(N)})\|f(x_n^{(N)})\| + \alpha_n^{(N)}\|f(x_n^{(N)})\| \\
&= \|f(x_n^{(N)})\|. \qquad (3.1.10)
\end{aligned}$$

Since f is isometry, we have $\|x_{n+1}^{(N)}\| \leq \|x_n^{(N)}\|$. By the same argument method as given above, we have $\|x_{n+1}^{(N-1)}\| \leq \|x_n^{(N-1)}\|, \dots, \|x_{n+1}^{(1)}\| \leq \|x_n^{(1)}\|$. Therefore, we note that $\lim_{n\to\infty}\|x_n^{(1)}\|, \dots, \lim_{n\to\infty}\|x_n^{(N)}\|$ exist, and hence the sequences $\{x_n^{(1)}\}$,$\{x_n^{(2)}\}$,...,$\{x_n^{(N)}\}$ are bounded in K.

Step 2. Next, we will show that

$$\|Jf(x_{n+1}^{(N)}) - Jf(x_n^{(N)})\|$$

$$= \alpha_n^{(N)}\Big\|Jf(x_n^{(N)}) - J\prod_K J^{-1}\Big(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)})\Big)\Big\| \to 0,$$

$$\|Jf(x_{n+1}^{(N-1)}) - Jf(x_n^{(N-1)})\|$$
$$= \alpha_n^{(N-1)}\Big\|Jf(x_n^{(N-1)}) - J\prod_K J^{-1}\Big(Jf(x_n^{(N-1)})$$
$$- \rho_{N-1}T_{N-1}(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)})\Big)\Big\| \to 0,$$
$$\vdots$$
$$\|Jf(x_{n+1}^{(2)}) - Jf(x_n^{(2)})\|$$
$$= \alpha_n^{(2)}\Big\|Jf(x_n^{(2)}) - J\prod_K J^{-1}\Big(Jf(x_n^{(2)})$$
$$- \rho_2 T_2(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)})\Big)\Big\| \to 0,$$
$$\|Jf(x_{n+1}^{(1)}) - Jf(x_n^{(1)})\|$$
$$= \alpha_n^{(1)}\Big\|Jf(x_n^{(1)}) - J\prod_K J^{-1}\Big(Jf(x_n^{(1)})$$
$$- \rho_1 T_1(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)})\Big)\Big\| \to 0,$$

as $n \to \infty$.

By Lemma 2.4.25, Lemma 2.4.21, f is isometry and (3.1.8), it follows that there exists a continuous strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned}
\|f(x_{n+1}^{(N)})\|^2 \leq\ & (1 - \alpha_n^{(N)})\|Jf(x_n^{(N)})\|^2 \\
& + \alpha_n^{(N)}\Big\|J\prod_K J^{-1}(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}))\Big\|^2 \\
& - (1 - \alpha_n^{(N)})\alpha_n^{(N)} g\Big(\Big\|Jf(x_n^{(N)}) + J\prod_K J^{-1}(Jf(x_n^{(N)}) \\
& - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}))\Big\|\Big) \\
\leq\ & (1 - \alpha_n^{(N)})\|f(x_n^{(N)})\|^2 \\
& + \alpha_n^{(N)}\|Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)})\|^2 \\
& - (1 - \alpha_n^{(N)})\alpha_n^{(N)} g\Big(\Big\|Jf(x_n^{(N)}) + J\prod_K J^{-1}(Jf(x_n^{(N)}) \\
& - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)}))\Big\|\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n^{(N)})\|f(x_n^{(N)})\|^2 + \alpha_n^{(N)}\|f(x_n^{(N)})\|^2 \\
&\quad -(1-\alpha_n^{(N)})\alpha_n^{(N)} g\Big(\Big\|Jf(x_n^{(N)}) + J\prod_K J^{-1}\big(Jf(x_n^{(N)}) \\
&\quad -\rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)})\big)\Big\|\Big) \\
&= \|f(x_n^{(N)})\|^2 - (1-\alpha_n^{(N)})\alpha_n^{(N)} g\Big(\Big\|Jf(x_n^{(N)}) \\
&\quad +J\prod_K J^{-1}\big(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)})\big)\Big\|\Big).
\end{aligned} \tag{3.1.11}
$$

This implies that

$$
\begin{aligned}
(1-\alpha_n^{(N)})\alpha_n^{(N)} g\Big(\Big\|Jf(x_n^{(1)}) + J\prod_K J^{-1}\big(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)})\big)\Big\|\Big) \\
\leq \|f(x_n^{(N)})\|^2 - \|f(x_{n+1}^{(N)})\|^2.
\end{aligned} \tag{3.1.12}
$$

Since $\{\|x_n^{(k)}\|\}$ converges for all $k = 1, 2, \ldots, N$, it follows by letting $n \to \infty$ in (3.1.12), condition (ii) and the property of g that

$$
\Big\|Jf(x_n^{(N)}) - J\prod_K J^{-1}\Big(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)})\Big)\Big\| \to 0, \tag{3.1.13}
$$

as $n \to \infty$. By (3.1.6) and (3.1.13), we have

$$
\begin{aligned}
&\|Jf(x_{n+1}^{(N)}) - Jf(x_n^{(N)})\| \\
&\quad = \alpha_n^{(N)}\Big\|Jf(x_n^{(N)}) - J\prod_K J^{-1}\Big(Jf(x_n^{(N)}) - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)})\Big)\Big\| \to 0,
\end{aligned} \tag{3.1.14}
$$

as $n \to \infty$. Similarly, we can prove that

$$
\begin{aligned}
&\|Jf(x_{n+1}^{(N-1)}) - Jf(x_n^{(N-1)})\| \\
&\quad = \alpha_n^{(N-1)}\Big\|Jf(x_n^{(N-1)}) - J\prod_K J^{-1}\Big(Jf(x_n^{(N-1)}) \\
&\qquad - \rho_{N-1} T_{N-1}(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)})\Big)\Big\| \to 0,
\end{aligned}
$$

$$\vdots$$

$$\begin{aligned}
&\|Jf(x_{n+1}^{(2)}) - Jf(x_n^{(2)})\| \\
&\quad = \alpha_n^{(2)}\Big\|Jf(x_n^{(2)}) - J\prod_K J^{-1}\Big(Jf(x_n^{(2)}) \\
&\qquad\qquad - \rho_2 T_2(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)})\Big)\Big\| \to 0, \\
&\|Jf(x_{n+1}^{(1)}) - Jf(x_n^{(1)})\| \\
&\quad = \alpha_n^{(1)}\Big\|Jf(x_n^{(1)}) - J\prod_K J^{-1}\Big(Jf(x_n^{(1)}) \\
&\qquad\qquad - \rho_1 T_1(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)})\Big)\Big\| \to 0, \qquad (3.1.15)
\end{aligned}$$

as $n \to \infty$.

Step 3. We next show that $\{x_n^{(N)}\}$ converges to some $x_N^* \in E$. Let $\{x_{n_j}^{(N)}\}$ be any subsequence of $\{x_n^{(N)}\}$. Since $\{x_n^{(1)}\}$,$\{x_n^{(2)}\}$,...,$\{x_n^{(N)}\}$ are bounded, by (i) and from the compactness of C, there exists a subsequence $\{x_{n_{i(N)}}^{(N)}\}$ of $\{x_{n_j}^{(N)}\}$ such that

$$Jf(x_{n_{i(N)}}^{(N)}) - \rho_N T_N(x_{n_{i(N)}}^{(1)}, x_{n_{i(N)}}^{(2)}, \ldots, x_{n_{i(N)}}^{(N)}) \to h_1 \in E^*.$$

Since E is uniformly smooth and strictly convex, it follows by Lemma 2.4.19 (ii) and Remark 2.4.11, that $\prod_K$ and J^{-1} are continuous. Thus

$$\prod_K J^{-1}\Big(Jf(x_{n_{i(N)}}^{(N)}) - \rho_N T_N(x_{n_{i(N)}}^{(1)}, x_{n_{i(N)}}^{(2)}, \ldots, x_{n_{i(N)}}^{(N)})\Big) \to \prod_K J^{-1}(h_1) := f(x_N^*)$$

and

$$J\prod_K J^{-1}\Big(Jf(x_{n_{i(N)}}^{(N)}) - \rho_N T_N(x_{n_{i(N)}}^{(1)}, x_{n_{i(N)}}^{(2)}, \ldots, x_{n_{i(N)}}^{(N)})\Big) \to Jf(x_N^*). \quad (3.1.16)$$

From (3.1.13) and (3.1.16), we get

$$Jf(x_{n_{i(N)}}^{(N)}) \to Jf(x_N^*) \qquad (\text{as } n_{i(N)} \to \infty). \qquad (3.1.17)$$

By (3.1.14) and (3.1.17), we have

$$Jf(x_{n_{i(N)}+1}^{(N)}) \to Jf(x_N^*) \qquad (\text{as } n_{i(N)} \to \infty). \qquad (3.1.18)$$

Since E is strictly convex and reflexive, it follows by Remark 2.4.11 (iv) that J^{-1} is norm-weak-continuous. Therefore, from (3.1.17) and (3.1.18), we note that

$$f(x^{(N)}_{n_{i(N)}}) \rightharpoonup f(x^*_N), \quad f(x^{(N)}_{n_{i(N)}+1}) \rightharpoonup f(x^*_N),$$

and

$$\|f(x^{(N)}_{n_{i(N)}})\| \to \|f(x^*_N)\|, \quad \|f(x^{(N)}_{n_{i(N)}+1})\| \to \|f(x^*_N)\| \qquad (\text{as } n_{i(N)} \to \infty).$$

By the Kadec-Klee property, we have

$$f(x^{(N)}_{n_{i(N)}}) \to f(x^*_N) \text{ and } f(x^{(N)}_{n_{i(N)}+1}) \to f(x^*_N) \qquad (\text{as } n_{i(N)} \to \infty). \tag{3.1.19}$$

Since f^{-1} is a continuous mapping, we get

$$x^{(N)}_{n_{i(N)}} \to x^*_N \text{ and } x^{(N)}_{n_{i(N)}+1} \to x^*_N \qquad (\text{as } n_{i(N)} \to \infty). \tag{3.1.20}$$

This shows that $\{x^{(N)}_{n_{i(N)}}\}$ is a subsequence of $\{x^{(N)}_{n_j}\}$ such that $x^{(N)}_{n_{i(N)}} \to x^*_N \in E$. Therefore $x^{(N)}_n \to x^*_N$ as $n \to \infty$.

So, it follows from (3.1.6), (3.1.17), (3.1.19), and condition (ii) that

$$\begin{aligned} Jf(x^*_N) &= \lim_{n\to\infty} Jf(x^{(N)}_{n+1}) \\ &= \lim_{n\to\infty} \Big\{ (1-\alpha^{(N)}_n) Jf(x^{(N)}_n) + \alpha^{(N)}_n J \prod_K J^{-1}\Big(Jf(x^{(N)}_n) \\ &\qquad - \rho_N T_N(x^{(1)}_n, x^{(2)}_n, \ldots, x^{(N)}_n)\Big)\Big\} \\ &= (1-d_N) Jf(x^*_N) + d_N J \prod_K J^{-1}\Big(Jf(x^*_N) - \rho_N T_N(x^*_1, x^*_2, \ldots, x^*_N)\Big). \end{aligned}$$

Since f is a bijective mapping, we obtain that

$$x^*_N = f^{-1} \prod_K J^{-1}\Big(Jf(x^*_N) - \rho_N T_N(x^*_1, x^*_2, \ldots, x^*_N)\Big). \tag{3.1.21}$$

Similarly, we can prove that, for every subsequence $\{x_{n_j}^{(k)}\}$ of $\{x_n^{(k)}\}$ there exist a subsequence $\{x_{n_{i(k)}}^{(k)}\}$ of $\{x_{n_j}^{(k)}\}$ and $x_k^* \in E$ such that

$$f(x_{n_{i(k)}}^{(k)}) \to f(x_k^*) \quad (\text{as } n_{i(k)} \to \infty), \text{ for all } k = 1, 2, \ldots, N-1. \tag{3.1.22}$$

Since f^{-1} is a continuous mapping, we note that

$$x_{n_{i(k)}}^{(k)} \to x_k^* \quad (\text{as } n_{i(k)} \to \infty). \tag{3.1.23}$$

Hence $x_n^{(k)} \to x_k^* \in E$, for all $k = 1, 2, \ldots, N-1$. Therefore, we have

$$\begin{aligned}
x_{N-1}^* &= f^{-1} \prod_K J^{-1}\Big(Jf(x_{N-1}^*) - \rho_{N-1} T_{N-1}(x_N^*, x_1^*, x_2^*, \ldots, x_{N-2}^*, x_{N-1}^*)\Big) \\
x_{N-2}^* &= f^{-1} \prod_K J^{-1}\Big(Jf(x_{N-2}^*) - \rho_{N-2} T_{N-2}(x_{N-1}^*, x_N^*, x_1^*, \ldots, x_{N-3}^*, x_{N-2}^*)\Big) \\
&\vdots \\
x_2^* &= f^{-1} \prod_K J^{-1}\Big(Jf(x_2^*) - \rho_2 T_2(x_3^*, x_4^*, \ldots, x_N^*, x_1^*, x_2^*)\Big), \\
x_1^* &= f^{-1} \prod_K J^{-1}\Big(Jf(x_1^*) - \rho_1 T_1(x_2^*, x_3^*, \ldots, x_N^*, x_1^*)\Big).
\end{aligned} \tag{3.1.24}$$

By Lemma 3.1.1, we can conclude that $(x_1^*, x_2^*, \ldots, x_N^*)$ is a solution of (3.1.1) and $x_n^{(1)} \to x_1^*, x_n^{(2)} \to x_2^*, \ldots, x_n^{(N)} \to x_N^*$. □

Setting $N = 3$, $f = I$ in Theorem 3.1.3, we immediately obtain the following result.

Corollary 3.1.4. [13] *Let E be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, K be a nonempty closed and convex subset of E with $\theta \in K$. Let $T_1, T_2, T_3 : K \times K \times K \to E^*$ be continuous mappings and $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \{\alpha_n^{(3)}\}$ be the sequences in (a, b) with $0 < a < b < 1$ satisfying the following conditions:*

(i) there exist a compact subset $C \subset E^*$ and constants $\rho_1 > 0, \rho_2 > 0$, $\rho_3 > 0$ such that $(J(K) - \rho_3 T_3(K \times K \times K)) \cup (J(K) - \rho_2 T_2(K \times K \times K)) \cup (J(K) - \rho_1 T_1(K \times K \times K)) \subset C$, where $J(x_1, x_2, x_3) = Jx_3$, $\forall (x_1, x_2, x_3) \in K \times K \times K$ and

$$\begin{cases} \langle T_1(x_1, x_2, x_3), J^{-1}(Jx_3 - \rho_1 T_1(x_1, x_2, x_3)) \rangle \geq 0, \\ \langle T_2(x_1, x_2, x_3), J^{-1}(Jx_3 - \rho_2 T_2(x_1, x_2, x_3)) \rangle \geq 0, \\ \langle T_3(x_1, x_2, x_3), J^{-1}(Jx_3 - \rho_3 T_3(x_1, x_2, x_3)) \rangle \geq 0, \end{cases}$$

for all $x_1, x_2, x_3 \in K$.

(ii) $\lim_{n\to\infty} \alpha_n^{(1)} = d_1 \in (a, b)$, $\lim_{n\to\infty} \alpha_n^{(2)} = d_2 \in (a, b)$, $\lim_{n\to\infty} \alpha_n^{(3)} = d_3 \in (a, b)$. Let $\{x_n^{(1)}\}$,$\{x_n^{(2)}\}$,$\{x_n^{(3)}\}$ be the sequences defined by

$$\begin{cases} x_{n+1}^{(3)} = J^{-1}\Big((1 - \alpha_n^{(3)})Jf(x_n^{(3)}) \\ \qquad\qquad + \alpha_n^{(3)} J\Big(\prod_K J^{-1}(Jf(x_n^{(3)}) - \rho_3 T_3(x_n^{(1)}, x_n^{(2)}, x_n^{(3)}))\Big)\Big), \\ x_{n+1}^{(2)} = J^{-1}\Big((1 - \alpha_n^{(2)})Jf(x_n^{(2)}) \\ \qquad\qquad + \alpha_n^{(2)} J\Big(\prod_K J^{-1}(Jf(x_n^{(2)}) - \rho_2 T_2(x_{n+1}^{(3)}, x_n^{(1)}, x_n^{(2)}))\Big)\Big), \\ x_{n+1}^{(1)} = J^{-1}\Big((1 - \alpha_n^{(1)})Jf(x_n^{(1)}) \\ \qquad\qquad + \alpha_n^{(1)} J\Big(\prod_K J^{-1}(Jf(x_n^{(1)}) - \rho_1 T_1(x_{n+1}^{(2)}, x_{n+1}^{(3)}, x_n^{(1)}))\Big)\Big), \quad n \geq 0. \end{cases}$$

Then the problem (3.1.2) has a solution $(x_1^*, x_2^*, x_3^*) \in K \times K \times K$ and the sequences $\{x_n^{(1)}\}$,$\{x_n^{(2)}\}$ and $\{x_n^{(3)}\}$ converge strongly to x_1^*, x_2^* and x_3^*, respectively.

Setting E is a real Hilbert space in Theorem 3.1.3, we have following result.

Corollary 3.1.5. *Let H be a real Hilbert space, K be a nonempty closed and convex subset of H. Let $f : K \to K$ be an isometry mapping. Let $T_1, \ldots, T_N : \underbrace{K \times \ldots \times K}_{N-times} \to H$ be continuous mappings and $\{\alpha_n^{(1)}\}$,$\{\alpha_n^{(2)}\}$,...,$\{\alpha_n^{(N)}\}$ be sequences in (a, b) with $0 < a < b < 1$ satisfying the following conditions:*

(i) there exist a compact subset $C \subset H$ and constants $\rho_1 > 0, \rho_2 > 0, \ldots, \rho_N > 0$ such that

$(I(K) - \rho_N T_N(\underbrace{K \times \ldots \times K}_{N-times})) \cup (I(K) - \rho_{N-1} T_{N-1}(\underbrace{K \times \ldots \times K}_{N-times})) \cup \ldots$ $\cup (I(K) - \rho_1 T_1(\underbrace{K \times \ldots \times K}_{N-times})) \subset C$, *where* $(x_1, x_2, \ldots, x_N) = x_N$, $\forall (x_1, x_2, \ldots, x_N) \in \underbrace{K \times \ldots \times K}_{N-times}$ *and*

$$\begin{cases} \langle T_1(x_1, x_2, \ldots, x_N), x_N - \rho_1 T_1(x_1, x_2, \ldots, x_N) \rangle \geq 0, \\ \langle T_2(x_1, x_2, \ldots, x_N), x_N - \rho_2 T_2(x_1, x_2, \ldots, x_N) \rangle \geq 0, \\ \vdots \\ \langle T_N(x_1, x_2, \ldots, x_N), x_N - \rho_N T_N(x_1, x_2, \ldots, x_N) \rangle \geq 0, \end{cases}$$

for all $x_1, x_2, \ldots, x_N \in K$.

(ii) $\lim_{n\to\infty} \alpha_n^{(1)} = d_1 \in (a,b)$, $\lim_{n\to\infty} \alpha_n^{(2)} = d_2 \in (a,b)$, $\ldots$, $\lim_{n\to\infty} \alpha_n^{(N)} = d_N \in (a,b)$.

Let $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ *be the sequences defined by*

$$\begin{cases} x_{n+1}^{(N)} = f^{-1}\Big((1-\alpha_n^{(N)})f(x_n^{(N)}) + \alpha_n^{(N)} P_K\big(f(x_n^{(N)}) \\ \qquad\qquad - \rho_N T_N(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N)})\big)\Big), \\ x_{n+1}^{(N-1)} = f^{-1}\Big((1-\alpha_n^{(N-1)})f(x_n^{(N-1)}) + \alpha_n^{(N-1)} P_K\big(f(x_n^{(N-1)}) \\ \qquad\qquad - \rho_{N-1} T_{N-1}(x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N-2)}, x_n^{(N-1)})\big)\Big), \\ \vdots \\ x_{n+1}^{(2)} = f^{-1}\Big((1-\alpha_n^{(2)})f(x_n^{(2)}) + \alpha_n^{(2)} P_K\big(f(x_n^{(2)}) \\ \qquad\qquad - \rho_2 T_2(x_{n+1}^{(3)}, x_{n+1}^{(4)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)}, x_n^{(2)})\big)\Big), \\ x_{n+1}^{(1)} = f^{-1}\Big((1-\alpha_n^{(1)})f(x_n^{(1)}) + \alpha_n^{(1)} P_K\big(f(x_n^{(1)}) \\ \qquad\qquad - \rho_1 T_1(x_{n+1}^{(2)}, x_{n+1}^{(3)}, \ldots, x_{n+1}^{(N)}, x_n^{(1)})\big)\Big), \quad n \geq 0, \end{cases}$$

where P_K is a metric projection on H to K.

Then the problem (3.1.1) has a solution $(x_1^*, x_2^*, \dots, x_N^*) \in \underbrace{K \times \dots \times K}_{N-times}$ *and the sequences* $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \dots, \{x_n^{(N)}\}$ *converge strongly to* $x_1^*, x_2^*, \dots, x_N^*$*, respectively.*

3.2 Existence of solutions for generalized variational inequality problems in Banach spaces

In this section, we assume that E is a reflexive and smooth Banach space, K is a closed convex subset in E. Let $T : K \to 2^{E^*}$ be a multi-valued mapping. The generalized variational inequality problem, denoted by $GVI(K,T)$, is to find a vector $x^* \in K$ such that there exists a vector $u^* \in T(x^*)$ satisfying

$$\langle u^*, y - x^* \rangle \geq 0 \text{ for all } \quad y \in K.$$

Now, we first prove the existence of solutions of generalized variational inequality for upper semicontinuous multi-valued mappings with compact contractible values over compact convex subsets in a reflexive Banach space with a Fréchet differentiable norm.

Theorem 3.2.1. *Let E be a reflexive Banach space with a Fréchet differentiable norm. Assume that*

(i) K is a nonempty compact convex in E;

(ii) $T : K \to 2^{E^*}$ is upper semicontinuous;

(iii) $T(x)$ is nonempty closed in E^* and contractible subset in E for each $x \in K$;

(iv) $T(K) = \bigcup_{x \in K} T(x)$ is compact in E^*.

Then the $GVI(K,T)$ has solution in K.

Proof. Let $C^* := \overline{co}(T(K))$. Hence, by Mazur's theorem, C^* is compact in E^*. Since $T(K)$ is compact in E^*, $\overline{T(K)}$ is also compact in E^*. By our assumption

(iv), we have $K \times C^*$ is compact in $E \times E^*$. Define $F : K \times C^* \to 2^{K \times C^*}$ by $F(x,y) = \{(u,v) : u \in \pi_K(j(x) - y), v \in T(x)\}$ for all $(x,y) \in K \times C^*$. Moreover, we note by Lemma 2.4.23 that $\pi_K(j(x) - y)$ is nonempty and hence $F(x,y)$ is nonempty for all $(x,y) \in K \times C^*$.

Step 1. Show that $F(x,y)$ is contractible for all $(x,y) \in K \times C^*$.

We note that $(u,v) \in F(x,y)$ if and only if $u \in \pi_K(j(x) - y)$ and $v \in T(x)$. By Lemma 2.4.23, we have $\pi_K(j(x) - y)$ is convex. Hence $\pi_K(j(x) - y)$ and $T(x)$ are contractible. Thus there exist $u_0 \in \pi_K(j(x) - y)$, $v_0 \in T(x)$ and continuous mappings $g_1 : \pi_K(j(x)-y) \times [0,1] \to \pi_K(j(x)-y)$ and $g_2 : T(x) \times [0,1] \to T(x)$ such that $g_1(u,0) = u$ and $g_1(u,1) = u_0$ for all $u \in \pi_K(j(x) - y)$, and $g_2(v,0) = v$ and $g_2(v,1) = v_0$ for all $v \in T(x)$. Define $h : (\pi_K(j(x)-y) \times T(x)) \times [0,1] \to \pi_K(j(x) - y) \times T(x)$ by $h((u,v),t) = (g_1(u,t), g_2(v,t))$ $\forall ((u,v),t) \in (\pi_K(j(x) - y) \times T(x)) \times [0,1]$. Thus h is a continuous mapping such that $h((u,v),0) = (g_1(u,0), g_2(v,0)) = (u,v)$ and $h((u,v),1) = (g_1(u,1), g_2(v,1)) = (u_0, v_0)$ for all $(u,v) \in \pi_K(j(x) - y) \times T(x)$. This implies that $F(x,y)$ is contractible.

Step 2. Show that $F(x,y)$ is compact subset of $K \times C^*$.

Since $K \times C^*$ is compact, we need only show that $F(x,y)$ is closed. Let $(u_n, v_n) \in F(x,y)$, $(u_n, v_n) \to (u,v)$ as $n \to \infty$. We shall show that $(u,v) \in F(x,y)$. From $(u_n, v_n) \in F(x,y)$, we have $u_n \in \pi_K(j(x) - y)$ and $v_n \in T(x)$ for all $n \in \mathbb{N}$. Since $(u_n, v_n) \to (u,v)$, we get $u_n \to u$ and $v_n \to v$. By Lemma 2.4.23, we note that $\pi_K(j(x) - y)$ is closed. Thus, we have $u \in \pi_K(j(x) - y)$ and $v \in T(x)$ by the closedness of $T(x)$. That is $(u,v) \in F(x,y)$ and hence $F(x,y)$ is closed.

Step 3. Show that F is upper semicontinuous.

Since $K \times C^*$ is compact set and $F(x,y) \subset K \times C^*$, by Lemma 2.5.4 (i), we need only show that F is a closed mapping. Let $\{(x_\alpha, y_\alpha) : \alpha \in I\} \in K \times C^*$ be given such that $(x_\alpha, y_\alpha) \to (x_0, y_0) \in K \times C^*$ and let $(u_\alpha, v_\alpha) \in F(x_\alpha, y_\alpha)$ be given such that $(u_\alpha, v_\alpha) \to (u_0, v_0)$. We shall show that $(u_0, v_0) \in F(x_0, y_0)$. Since $v_\alpha \in T(x_\alpha)$,

$x_\alpha \to x_0$, $v_\alpha \to v_0$ and T is upper semicontinuous, it follows by Lemma 2.5.4 (ii) that $v_0 \in T(x_0)$. Since $u_\alpha \in \pi_K(j(x_\alpha) - y_\alpha)$ and E is smooth, we have

$$\langle j(x_\alpha) - y_\alpha - j(u_\alpha), u_\alpha - y\rangle \geq 0, \ \forall y \in K. \tag{3.2.1}$$

From the Fréchet differentiable norm of E, we note that the duality $J : E \to E^*$ is norm to norm continuous. Thus, we have $j(x_\alpha) \to j(x_0)$ and $j(u_\alpha) \to j(u_0)$. Hence, by (3.2.1), we obtain

$$\langle j(x_0) - y_0 - j(u_0), u_0 - y\rangle \geq 0, \ \forall y \in K.$$

This implies that $u_0 \in \pi_K(j(x_0) - y_0)$ and hence $(u_0, v_0) \in F(x_0, y_0)$. Therefore F is upper semicontinuous.

Step 4. Show that the solution set of $GVI(K,T)$ is nonempty.
By Eilenberg-Montgomery Theorem, F has a fixed point. That is, there exists a point $(x^*, y^*) \in K \times C^*$ such that $(x^*, y^*) \in F(x^*, y^*)$. Hence there exists a point $x^* \in K$ and $y^* \in T(x^*)$ such that $x^* \in \pi_K(j(x^*) - y^*)$. Hence, by Lemma 2.4.24 (ii), we have

$$\langle y^*, y - x^*\rangle = \langle j(x^*) - y^* - j(x^*), x^* - y\rangle \geq 0, \ \forall y \in K.$$

□

Setting $E = \mathbb{R}^n$ in Theorem 3.2.1, we have following result.

Corollary 3.2.2. [47](Hartman-Stampacchia, Saigal). *Assume that*

(i) K is a nonempty compact convex in $\mathbb{R}^n$;

(ii) $T : K \to 2^{\mathbb{R}^n}$ is upper semicontinuous;

(iii) $T(x)$ is nonempty, compact, and contractible subset in $\mathbb{R}^n$ for each $x \in K$.

Then there is a solution (x^*, y^*) to the generalized variational inequality problem $GVI(K,T)$.

Proof. Since K is compact, $T(x)$ is nonempty compact subset in $\mathbb{R}^n$ for each $x \in K$. From $T : K \to 2^{\mathbb{R}^n}$ is upper semicontinuous, we note by [6] that $T(K) = \bigcup_{x \in K} T(x)$ is also compact. Hence, by Mazur's theorem, $\overline{co}(T(K))$ is a compact subset of $\mathbb{R}^n$. Therefore, by Theorem 3.2.1, the solution set of $GVI(K,T)$ is nonempty. □

Corollary 3.2.3. *Let E be a reflexive Banach space with a Fréchet differentiable norm. Assume that*

(i) K is a nonempty compact convex in E;

(ii) $T : K \to 2^{E^*}$ is upper semicontinuous;

(iii) $T(x)$ is nonempty closed in E^* and convex subset in E for each $x \in K$;

(iv) $T(K)$ is compact in E^*.

Then the $GVI(K,T)$ has solution in K.

Next, we prove the existence of solutions for generalized variational inequality problems for upper semicontinuous multi-valued mappings over unbounded closed convex subsets in a reflexive Banach space with a Fréchet differentiable norm.

Theorem 3.2.4. *Let E be a reflexive Banach space with a Fréchet differentiable norm and K be a closed convex set in E such that every weakly convergent sequence in K is norm convergent. Let $T : K \to 2^{E^*}$ be an upper semicontinuous multi-valued mapping such that $T(x)$ is nonempty compact and contractible in E^* for any $x \in K$. Suppose that $T(B)$ is compact in E^*, for all compact subset B of K, and*

$(C1)$ *Given $\widehat{x} \in E$ and for any $\{x_n\} \subset K$ with $\|x_n\| \to +\infty$ as $n \to +\infty$, and for any $\{u_n\}$ with $u_n \in T(x_n)$, there exist a positive integer n_0 and $y \in K$ such that $\|y - \widehat{x}\| \le \|x_{n_0} - \widehat{x}\|$ and $\langle u_{n_0}, y - x_{n_0}\rangle < 0$.*

Then the solution set of $GVI(K,T)$ is nonempty and compact.

Proof. **Step 1.** Show that the solution set of $GVI(K,T)$ is nonempty.
Let $\widehat{x} \in E$. For any $n = 1,2,...$, let $K_n = \{x \in K : \|x - \widehat{x}\| \leq n\}$. Thus, we note that K_n is nonempty closed convex bounded subset of E. Since E is reflexive, we have K_n is a weakly compact subset of E. We shall show that K_n is compact. Let $\{x_n^k\}_{k=1}^{\infty}$ be any sequence in K_n. Thus there exists a subsequence $\{x_n^{k_j}\}_{j=1}^{\infty}$ of $\{x_n^k\}_{k=1}^{\infty}$ such that $x_n^{k_j} \rightharpoonup u_n \in K_n$ as $j \to \infty$. Since $\{x_n^{k_j}\} \subset K$, it follows by our assumption that the sequence $\{x_n^{k_j}\}$ converges strongly to u_n. Hence K_n is compact and therefore $T(K_n)$ is compact in E^*. By Theorem 3.2.1, the solution set of $GVI(K_n,T)$ is nonempty, that is, there exists $x_n \in K_n$ and $v_n \in T(x_n)$ such that

$$\langle v_n, y - x_n \rangle \geq 0 \quad \forall y \in K_n.$$

If the sequence $\{x_n\}$ is unbounded, by without loss of generality, we assume that $\|x_n\| \to \infty$ as $n \to +\infty$. Then, by condition $(C1)$, there exist a positive integer n_0 and $y \in K$ such that $\|y - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$ and $\langle v_{n_0}, y - x_{n_0} \rangle < 0$. This implies that $y \in K$ and $\langle v_{n_0}, y - x_{n_0} \rangle < 0$, which is a contradiction. Hence $\{x_n\}$ is bounded. i.e., there exists a positive integer N such that $\{x_n\}_{n=1}^{\infty} \subset K_N \subset K$. From the compactness of K_N, there exists a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ of $\{x_n\}$ such that $x_{n_i} \to x^* \in K_N$. We note that $v_{n_i} \in T(x_{n_i})$ for all $i = 1,2,3,\ldots$. Since T is upper semicontinuous, it follows by Lemma 2.5.5, that a subsequence $\{v_{n_j}\}$ of $\{v_{n_i}\}$ such that $v_{n_j} \to v^* \in T(x^*)$ as $j \to \infty$. Let $y \in K$. Since $K_1 \subset K_2 \subset \ldots$, there exists a positive integer n_1 such that $y \in K_n$ for all $n \geq n_1$. From $\langle v_{n_j}, y - x_{n_j} \rangle \geq 0$ for all $y \in K$ and for all $n_j \geq n_1$, we get $\langle v^*, y - x^* \rangle \geq 0$ for all $y \in K$. Therefore the solution set of $GVI(K,T)$ is nonempty.

Step 2. Show that the solution set of $GVI(K,T)$ is compact.

Let $M = \{x \in K : \exists u \in T(x) \text{ such that } \langle u, y - x \rangle \geq 0 \ \ \forall y \in K\}$.

We first show that M is closed. Let $\{x_n\}$ be any sequence in M and $x_n \to x$. Since $x_n \in M$ there exists a $u_n \in T(x) \subset K$ such that $\langle u_n, y - x_n \rangle \geq 0$ for any $y \in K$. Since $T(x)$ is compact and T is upper semicontinuous at x, it follows by Lemma

2.5.5, that $\{u_n\}$ has a cluster point $u \in T(x)$. Then, without loss of generality, we assume that $u_n \to u \in T(x)$. It follows from $\langle u_n, y - x_n \rangle \geq 0 \ \ \forall y \in K$ that $\langle u, y - x \rangle \geq 0 \ \ \forall y \in K$ as $n \to \infty$. Hence $x \in M$ and therefore M is closed.

Next, we show that M is bounded. Suppose that M is unbounded. Thus there exists a sequence $\{x_n\} \subset M$ such that $\|x_n\| \to +\infty$. This implies that there is a sequence $\{u_n\} \subset K$ such that $u_n \in T(x_n)$ and $\langle u_n, y - x_n \rangle \geq 0 \ \ \forall y \in K$. By condition $(C1)$, there exists a positive integer n_0 and $y \in K$ such that $\|y - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$ and $\langle u_{n_0}, y - x_{n_0} \rangle < 0$. This is a contradiction. Therefore M is a bounded subset of K.

Finally, we show that M is compact. Let $\{v_n\} \subset M \subset K$. Since E is reflexive and M is bounded, there exists a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that $v_{n_j} \rightharpoonup v \in E$. By our assumption, we have $v_n \to v$ and so $v \in M$. Hence M is a compact subset of K. □

Corollary 3.2.5. *Let E be a reflexive Banach space with a Fréchet differentiable norm, K be a closed convex set in E such that every weakly convergent sequence in K is norm convergent. Let $T : K \to 2^{E^*}$ be an upper semicontinuous multi-valued mapping such that $T(x)$ is nonempty compact and convex in E^* for any $x \in K$. Suppose that $T(B)$ is compact in E^*, for all compact subset B of K, and*

$(C1)$ *Given $\widehat{x} \in E$ and for any $\{x_n\} \subset K$ with $\|x_n\| \to +\infty$ as $n \to +\infty$, and for any $\{u_n\}$ with $u_n \in T(x_n)$, there exist a positive integer n_0 and $y \in K$ such that $\|y - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$ and $\langle u_{n_0}, y - x_{n_0} \rangle < 0$.*

Then the solution set of $GVI(K,T)$ is nonempty and compact.

Theorem 3.2.6. *Let E be a reflexive Banach space with a Fréchet differentiable norm, K be a closed convex set in E such that every weakly convergent sequence in K is norm convergent. Let $T : K \to 2^{E^*}$ be an upper semicontinuous multi-valued mapping such that $T(x)$ is nonempty compact and contractible for any $x \in K$.*

Suppose that $T(B)$ is compact in E^*, for all compact subset B of K, and one of the following conditions hold:

$(C2)$ *Given $\widehat{x} \in E$ and for any $\{x_n\} \subset K$ with $\|x_n\| \to +\infty$ as $n \to +\infty$, and for any sequence $\{u_n\}$ with $u_n \in T(x_n)$, there exist a positive integer n_0 and $y \in K$ such that $\|y - \widehat{x}\| < \|x_{n_0} - \widehat{x}\|$ and $\langle u_{n_0}, y - x_{n_0}\rangle \leq 0$.*

$(C3)$ *Given $\widehat{x} \in E$, there exists a constant $\rho > 0$ such that, for any $x \in K$ with $\|x - \widehat{x}\| > \rho$, there exist $y \in K$ and $u \in T(x)$ satisfying $\|y - \widehat{x}\| \leq \|x - \widehat{x}\|$ and $\langle u, y - x\rangle < 0$.*

$(C4)$ *Given $\widehat{x} \in E$, there exists a constant $\rho > 0$ such that, for any $x \in K$ with $\|x - \widehat{x}\| > \rho$, there exists $y \in K$ and $u \in T(x)$ satisfying $\|y - \widehat{x}\| < \|x - \widehat{x}\|$ and $\langle u, y - x\rangle \leq 0$.*

Then there exists a solution to $GVI(K,T)$ and the solution set is compact.

Proof. We note by Yu and Yang [8] that $(C2)$ implies $(C1)$ and $(C3)$ implies $(C1)$. We will show that $(C4)$ implies $(C2)$. In fact, for any $\{x_n\} \subset K$ with $\|x_n\| \to +\infty$ as $n \to +\infty$, and for any sequence $\{u_n\}$ with $u_n \in T(x_n)$. For given $\widehat{x} \in E$, we note that $\|x_n - \widehat{x}\| \to +\infty$ as $n \to +\infty$. Since ρ is a constant, there exists a positive integer n_0 such that $\|x_{n_0} - \widehat{x}\| > \rho$. By $(C4)$, there exists $y \in K$ and $u_{n_0} \in T(x_{n_0})$ satisfying $\|y - \widehat{x}\| < \|x_{n_0} - \widehat{x}\|$ and $\langle u_{n_0}, y - x_{n_0}\rangle \leq 0$. Hence the condition $(C2)$ holds. □

Setting $E = \mathbb{R}^n$ in Theorem 3.2.4 and Theorem 3.2.6, we have following result.

Corollary 3.2.7. [8] *Let $K \subset \mathbb{R}^n$ be a nonempty a closed convex subset, $T : K \to 2^{\mathbb{R}^n}$ be an upper semicontinuous multi-valued mapping, where $T(x)$ is nonempty* *compact contractible in* $\mathbb{R}^n$ *for any* $x \in K$. *Suppose that one of the following conditions hold:*

$(C1)'$ *Given* $\widehat{x} \in \mathbb{R}^n$ *and for any* $\{x_n\} \subset K$ *with* $\|x_n\| \to +\infty$ *as* $n \to +\infty$, *and for any* $\{u_n\}$ *with* $u_n \in T(x_n)$, *there exist a positive integer* n_0 *and* $y \in K$ *such that* $\|y - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$ *and* $\langle u_{n_0}, y - x_{n_0} \rangle < 0$.

$(C2)'$ *Given* $\widehat{x} \in \mathbb{R}^n$ *and for any* $\{x_n\} \subset K$ *with* $\|x_n\| \to +\infty$ *as* $n \to +\infty$, *and for any sequence* $\{u_n\}$ *with* $u_n \in T(x_n)$, *there exist a positive integer* n_0 *and* $y \in K$ *such that* $\|y - \widehat{x}\| < \|x_{n_0} - \widehat{x}\|$ *and* $\langle u_{n_0}, y - x_{n_0} \rangle \leq 0$.

$(C3)'$ *Given* $\widehat{x} \in \mathbb{R}^n$, *there exists a constant* $\rho > 0$ *such that, for any* $x \in K$ *with* $\|x - \widehat{x}\| > \rho$, *there exist* $y \in K$ *and* $u \in T(x)$ *satisfying* $\|y - \widehat{x}\| \leq \|x - \widehat{x}\|$ *and* $\langle u, y - x \rangle < 0$.

$(C4)'$ *Given* $\widehat{x} \in \mathbb{R}^n$, *there exists a constant* $\rho > 0$ *such that, for any* $x \in K$ *with* $\|x - \widehat{x}\| > \rho$, *there exists* $y \in K$ *and* $u \in T(x)$ *satisfying* $\|y - \widehat{x}\| < \|x - \widehat{x}\|$ *and* $\langle u, y - x \rangle \leq 0$.

Then the solution set of $GVI(K, T)$ *is nonempty and compact.*

Proof. It is easy to see that every weakly convergent sequences in $\mathbb{R}^n$ is norm convergent. Moreover, we note as in the proof of Corollary 3.2.2 that $T(B)$ is compact in E^*, for all compact subset B of K. □

As special cases, we obtain the following two existence theorems of solutions for variational inequality problems.

Theorem 3.2.8. *Let* E *be a reflexive Banach space with a Fréchet differentiable norm,* K *be a closed convex set in* E *such that every weakly convergent sequence in* K *is norm convergent. Let* $f : K \to E^*$ *be a continuous mapping. Suppose that one of the following conditions hold:*

(*C*5) *Given* $\widehat{x} \in E$, *for any* $\{x_n\} \in K$ *where* $\|x_n\| \to +\infty$ *there exists a positive integer* n_0 *and* $y \in K$ *with* $\|y - \widehat{x}\| < \|x_{n_0} - \widehat{x}\|$ *such that* $\langle f(x_{n_0}), y - x\rangle \leq 0$.

(*C*6) *Given* $\widehat{x} \in E$, *for any* $\{x_n\} \in K$ *where* $\|x_n\| \to +\infty$ *there exists a positive integer* n_0 *and* $y \in K$ *with* $\|y - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$ *such that* $\langle f(x_{n_0}), y - x\rangle < 0$.

(*C*7) *Given* $\widehat{x} \in E$, *there exists a constant* $\rho > 0$ *such that, for any* $x \in K$ *with* $\|x - \widehat{x}\| > \rho$, *there exists* $y \in K$ *satisfying* $\|y - \widehat{x}\| \leq \|x - \widehat{x}\|$ *and* $\langle f(x), y - x\rangle < 0$.

(*C*8) *Given* $\widehat{x} \in E$, *there exists a constant* $\rho > 0$ *such that, for any* $x \in K$ *with* $\|x - \widehat{x}\| > \rho$, *there exists* $y \in K$ *satisfying* $\|y - \widehat{x}\| < \|x - \widehat{x}\|$ *and* $\langle f(x), y - x\rangle \leq 0$.

Then the solution set of variational inequality $VI(K, f)$ *is nonempty, closed and bounded.*

3.3 System of nonlinear set-valued variational inclusions involving a finite family of $H(\cdot,\cdot)$-accretive operators in Banach spaces

In this section, we assume that E is q-uniformly smooth real Banach space and $C(E)$ is a nonempty closed convex set. Let $S_i, H_i : E \times E \to E$, $A_i, B_i : E \to E$ be single-valued operators, for all $i = 1, 2, \ldots, N$. For any fix $i \in \{1, 2, \ldots, N\}$, we let $M_i : E \to 2^E$, $H_i(A_i, B_i)$-accretive set-valued operator and $U_i : E \to 2^E$ be a set-valued mapping which nonempty values. The system of nonlinear set-valued variational inclusions is to find $a_1, \ldots, a_N \in E$, $u_1 \in U_1(a_N), \ldots, u_N \in U_N(a_1)$ such that

$$0 \in S_i(a_i, u_i) + M_i(a_i), \quad \text{for all } i = 1, 2, \ldots, N. \tag{3.3.1}$$

If $N = 2$, then system of nonlinear set-valued variational inclusions (3.3.1) becomes to the following system of variational inclusions: finding $a_1, a_2 \in E$, $u_1 \in U_1(a_2)$ and $u_2 \in U_2(a_1)$ such that

$$\begin{cases} 0 \in S_1(a_1, u_1) + M_1(a_1) \\ 0 \in S_2(a_2, u_2) + M_2(a_2). \end{cases} \tag{3.3.2}$$

If $N = 1$, then system of nonlinear set-valued variational inclusions (3.3.1) becomes to the following the class of nonlinear set-valued variational inclusions [48]: finding $a \in E$, $u \in U(a)$ such that

$$0 \in S(a, u) + M(a). \tag{3.3.3}$$

For solving the system of nonlinear set-valued variational inclusions involving a finite family of $H(\cdot,\cdot)$-accretive operators in Banach spaces, let us give the following assumptions.

For any $i = 1, 2, \ldots, N$, we suppose that

($A1$) $H(A_i, B_i)$ is α_i-strongly accretive with respect to A_i, β_i-relaxed accretive with respect to B_i and $\alpha_i > \beta_i$

($A2$) $M_i : E \to 2^E$ is an $H_i(\cdot,\cdot)$-accretive single-valued mapping,

($A3$) $U_i : E \to C(E)$ is a contraction set-valued mapping with $0 \leq L_i < 1$ and nonempty values,

($A4$) $H_i(A_i, B_i)$ is r_i-Lipschitz continuous with respect to A_i and t_i-Lipschitz continuous with respect to B_i,

($A5$) $S_i : E \times E \to E$ is l_i-Lipschitz continuous with respect to its first argument and m_i-Lipschitz continuous with respect to its second argument,

($A6$) $S_i(\cdot, u)$ is s_i-strongly accretive with respect to $H_i(A_i, B_i)$.

Lemma 3.3.1. *For given $a_1, \ldots, a_N \in E$, $u_1 \in U_1(a_N), \ldots, u_N \in U_N(a_1)$, it is a solution of problem* (3.3.1) *if and only if*

$$a_i = R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}[H_i(A_i(a_i), B_i(a_i)) - \lambda_i S_i(a_i, u_i)] \tag{3.3.4}$$

where $\lambda_i > 0$ are constants.

Proof. We note from the Definition 2.5.13 that $a_1, \ldots, a_N \in E$, $u_1 \in U_1(a_N), \ldots, u_N \in U_N(a_1)$ is a solution of (3.3.1) if and only if, for each $i = 1, 2, \ldots, N$, we have

$$\begin{aligned} & a_i = R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}[H_i(A_i(a_i), B_i(a_i)) - \lambda_i S_i(a_i, u_i)] \\ \Leftrightarrow\ & a_i = [H_i(A_i, B_i) + \lambda_i M_i]^{-1}[H_i(A_i(a_i), B_i(a_i)) - \lambda_i S_i(a_i, u_i)] \\ \Leftrightarrow\ & [H_i(A_i(a_i), B_i(a_i)) - \lambda_i S_i(a_i, u_i)] \in [H_i(A_i, B_i) + \lambda_i M_i](a_i) \\ \Leftrightarrow\ & -\lambda_i S_i(a_i, u_i) \in \lambda_i M_i(a_i) \\ \Leftrightarrow\ & 0 \in S_i(a_i, u_i) + M_i(a_i). \end{aligned}$$

□

Algorithm 3.3.2. For given $a_0^1, \ldots, a_0^N \in E$, $u_0^1 \in U_1(a_0^N), \ldots, u_0^N \in U_N(a_0^1)$, we let

$$a_1^i = \sigma_0 a_0^i + (1 - \sigma_0) R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}[H_i(A_i(a_0^i), B_i(a_0^i)) - \lambda_i S_i(a_0^i, u_0^i)]$$

for all $i = 1, 2, \ldots, N$, where $0 < \sigma_0 \leq 1$ is a constant. By Nadler theorem [49], there exists $u_1^1 \in U_1(a_1^N), \ldots, u_1^N \in U_N(a_1^1)$ such that

$$\|u_1^i - u_0^i\| \leq (1+1) D(U_i(a_1^{N-(i-1)}), U_i(a_0^{N-(i-1)})), \quad \text{for all } \ i = 1, 2, \ldots, N,$$

where $D(\cdot, \cdot)$ is the Hausdorff pseudo metric on 2^E. Continuing the above process inductively, we can obtain the sequences $\{a_n^i\}$ and $\{u_n^i\}$ such that

$$a_{n+1}^i = \sigma_n a_n^i + (1 - \sigma_n) R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}[H_i(A_i(a_n^i), B_i(a_n^i)) - \lambda_i S_i(a_n^i, u_n^i)] \tag{3.3.5}$$

for all $n = 1, 2, 3, \ldots,$ $i = 1, 2, \ldots, N$ where $0 < \sigma_n \leq 1$ are constant with $\limsup_{n\to\infty} \sigma_n < 1$. Therefore, by Nadler theorem [49], there exists $u^1_{n+1} \in U_1(a^N_{n+1}), \ldots, u^N_{n+1} \in U_N(a^1_{n+1})$ such that

$$\|u^i_{n+1} - u^i_n\| \leq (1 + (1+n)^{-1}) D(U_i(a^{N-(i-1)}_{n+1}), U_i(a^{N-(i-1)}_n)), \quad (3.3.6)$$

for all $n = 1, 2, 3, \ldots,$ $i = 1, 2, \ldots, N$.

The idea of the proof of the next theorem is contained in the paper of Verma [48] and Zou and Huang [12].

Theorem 3.3.3. *Let E be q-uniformly smooth real Banach space. Let $A_i, B_i : E \to E$ be single-valued operators, $H_i : E \times E \to E$ be a single-valued operator satisfying (A1) and M_i, U_i, $H_i(A_i, B_i)$, S_i, $S_i(\cdot, u)$ satisfy conditions (A2)-(A6), respectively. If there exists a constant $c_{q,i}$ such that*

$$\frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i} + \frac{\lambda_i m_i}{\alpha_i - \beta_i} < 1 \quad (3.3.7)$$

for all $i = 1, 2, \ldots, N$, then problem (3.3.1) has a solution $a_1, \ldots, a_N$, $u_1 \in U_1(a_N)$, $\ldots$,$u_N \in U_N(a_1)$.

Proof. For any $i \in \{1, 2, \ldots, N\}$ and $\lambda_i > 0$, we define $F_i : E \times E \to E$ by

$$F_i(u, v) = R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}[H_i(A_i(u), B_i(u)) - \lambda_i S_i(u, v)], \quad (3.3.8)$$

for all $u, v \in E$. Let $J_i(x, y) = H_i(A_i(x), B_i(y))$. For any $(u_1, v_1), (u_2, v_2) \in E \times E$, we note by (3.3.8) and Lemma 2.5.14, that

$$\begin{aligned}\|F_i(u_1, v_1) - F_i(u_2, v_2)\| &= \|R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}[H_i(A_i(u_1), B_i(u_1)) - \lambda_i S_i(u_1, v_1)] \\ &\quad - R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}[H_i(A_i(u_2), B_i(u_2)) - \lambda_i S_i(u_2, v_2)]\|\end{aligned}$$

$$
\begin{aligned}
&= \|R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}[J_i(u_1,u_1) - \lambda_i S_i(u_1,v_1)] \\
&\quad -R^{H_i(\cdot,\cdot)}_{M_i,\lambda_i}[J_i(u_2,u_2) - \lambda_i S_i(u_2,v_2)]\| \\
&\le \frac{1}{\alpha_i - \beta_i}\|[J_i(u_1,u_1) - \lambda_i S_i(u_1,v_1)] \\
&\quad -[J_i(u_2,u_2) - \lambda_i S_i(u_2,v_2)]\| \\
&= \frac{1}{\alpha_i - \beta_i}\|[J_i(u_1,u_1) - J_i(u_2,u_2)] \\
&\quad -\lambda_i[S_i(u_1,v_1) - S_i(u_2,v_2)]\| \\
&\le \frac{1}{\alpha_i - \beta_i}\|[J_i(u_1,u_1) - J_i(u_2,u_2)] \\
&\quad -\lambda_i[S_i(u_1,v_1) - S_i(u_2,v_1)]\| \\
&\quad +\frac{\lambda_i}{\alpha_i - \beta_i}\|[S_i(u_2,v_1) - S_i(u_2,v_2)]\|. \qquad (3.3.9)
\end{aligned}
$$

By Lemma 2.4.17, we have

$$
\begin{aligned}
&\|J_i(u_1,u_1) - J_i(u_2,u_2) - \lambda_i[S_i(u_1,v_1) - S_i(u_2,v_1)]\|^q \\
&\quad \le \|J_i(u_1,u_1) - J_i(u_2,u_2)\|^q \\
&\quad\quad - q\lambda_i\langle S_i(u_1,v_1) - S_i(u_2,v_1), J_q(J_i(u_1,u_1) - J_i(u_2,u_2))\rangle \\
&\quad\quad + c_{q,i}\lambda_i^q\|S_i(u_1,v_1) - S_i(u_2,v_1)\|^q. \qquad (3.3.10)
\end{aligned}
$$

Moreover, by $(A4)$, we obtain

$$
\begin{aligned}
\|J_i(u_1,u_1) - J_i(u_2,u_2)\| &\le \|J_i(u_1,u_1) - J_i(u_2,u_1)\| + \|J_i(u_2,u_1) - J_i(u_2,u_2)\| \\
&\le r_i\|u_1 - u_2\| + t_i\|u_1 - u_2\| \\
&\le (r_i + t_i)\|u_1 - u_2\|. \qquad (3.3.11)
\end{aligned}
$$

From $(A6)$, we have

$$
\begin{aligned}
&-q\lambda_i\langle S_i(u_1,v_1) - S_i(u_2,v_1), J_q(J_i(u_1,u_1) - J_i(u_2,u_2))\rangle \\
&\quad \le -q\lambda_i s_i\|u_1 - u_2\|^q. \qquad (3.3.12)
\end{aligned}
$$

Moreover, from $(A5)$, we obtain

$$\|S_i(u_1, v_1) - S_i(u_2, v_1)\| \leq l_i\|u_1 - u_2\| \tag{3.3.13}$$

and

$$\|S_i(u_2, v_1) - S_i(u_2, v_2)\| \leq m_i\|v_1 - v_2\|. \tag{3.3.14}$$

From (3.3.10)-(3.3.13), we have

$$\begin{aligned}\|J_i(u_1, u_1) - J_i(u_2, u_2) - \lambda_i[S_i(u_1, v_1) - S_i(u_2, v_1)]\|^q \\ \leq \sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}\|u_1 - u_2\|. \end{aligned} \tag{3.3.15}$$

It follows from (3.3.9), (3.3.14) and (3.3.15) that

$$\begin{aligned}\|F_i(u_1, v_1) - F_i(u_2, v_2)\| \leq & \frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i}\|u_1 - u_2\| \\ & + \frac{\lambda_i m_i}{\alpha_i - \beta_i}\|v_1 - v_2\|. \end{aligned} \tag{3.3.16}$$

Put

$$\theta_1^i = \frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i}, \quad \text{and} \quad \theta_2^i = \frac{\lambda_i m_i}{\alpha_i - \beta_i}.$$

Define $\|\cdot\|$ on $\underbrace{E \times \ldots \times E}_{N-\text{times}}$ by $\|(x_1, \ldots, x_N)\| = \|x_1\| + \ldots + \|x_N\|$ for all $(x_1, \ldots, x_N)$ $\in \underbrace{E \times \ldots \times E}_{N-\text{times}}$. It is easy to see that $(\underbrace{E \times \ldots \times E}_{N-\text{times}}, \|\cdot\|)$ is a Banach space. For any given $x_1, \ldots, x_N \in E$, we choose a finite sequence $w_1 \in U_1(x_N), \ldots, w_N \in U_N(x_1)$. Define $Q : \underbrace{E \times \ldots \times E}_{N-\text{times}} \to \underbrace{E \times \ldots \times E}_{N-\text{times}}$ by $Q(x_1, \ldots, x_N) = (F_1(x_1, w_1), \ldots,$ $F_N(x_N, w_N))$. Set $k = \max\{(\theta_1^1 + \theta_2^N L_N), \ldots, (\theta_2^1 L_1 + \theta_1^N)\}$ where $L_1, \ldots, L_N$ are contraction constants of $U_1, \ldots, U_N$, respectively. We note that $\theta_1^i + \theta_2^i L_i < \theta_1^i + \theta_2^i < 1$, for all $i = 1, 2, \ldots, N$ and so $k < 1$. Let $x_1, \ldots, x_N \in E$, $w_1 \in U_1(x_N), \ldots, w_N \in U_N(x_1)$ and $y_1, \ldots, y_N \in E$, $z_1 \in U_1(y_N), \ldots, z_N \in U_N(y_1)$.

By $(A3)$, we get

$$\begin{aligned}
&\|Q(x_1,\ldots,x_N)-Q(y_1,\ldots,y_N)\| \\
&\quad = \|(F_1(x_1,w_1),\ldots,F_N(x_N,w_N))-(F_1(y_1,z_1),\ldots,F_N(y_N,z_N))\| \\
&\quad = \|F_1(x_1,w_1)-F_1(y_1,z_1)\|+\ldots+\|F_N(x_N,w_N)-F_N(y_N,z_N)\| \\
&\quad \leq (\theta_1^1\|x_1-y_1\|+\theta_2^1\|w_1-z_1\|)+\ldots+(\theta_1^N\|x_N-y_N\|+\theta_2^N\|w_N-z_N\|) \\
&\quad \leq (\theta_1^1\|x_1-y_1\|+\theta_2^1L_1\|x_N-y_N\|)+\ldots+(\theta_1^N\|x_N-y_N\|+\theta_2^NL_N\|x_1-y_1\|) \\
&\quad = (\theta_1^1+\theta_2^NL_N)\|x_1-y_1\|+\ldots+(\theta_1^N+\theta_2^1L_1)\|x_N-y_N\| \\
&\quad \leq k\|x_1-y_1\|+\ldots+k\|x_N-y_N\| \\
&\quad = k(\|x_1-y_1\|+\ldots+\|x_N-y_N\|) \\
&\quad = k\|(x_1,\ldots,x_N)-(y_1,\ldots,y_N)\|, \qquad (3.3.17)
\end{aligned}$$

and so Q is a contraction on $\underbrace{E\times\ldots\times E}_{N-\text{times}}$. Hence there exists $a_1,\ldots,a_N\in E$, $u_1\in U_1(a_N),\ldots,u_N\in U_N(a_1)$ such that $a_1=F_1(a_1,u_1),\ldots,a_N=F_N(a_N,u_N)$. From Lemma 3.3.1, $a_1,\ldots,a_N\in E$, $u_1\in U_1(a_N),\ldots,u_N\in U_N(a_1)$ is the solution of the problem (3.3.1). □

Theorem 3.3.4. *Let E be q-uniformly smooth real Banach space. For $i=1,2,\ldots,N$. Let $A_i,B_i:E\to E$ be two single-valued operators, $H_i:E\times E\to E$ be a single-valued operator satisfying (A1) and suppose that M_i, U_i, $H_i(A_i,B_i)$, S_i, $S_i(\cdot,u)$ satisfy conditions (A2)-(A6), respectively. Then, for any $i\in\{1,2,\ldots,N\}$, the sequence $\{a_n^1\}_{n=1}^\infty$ and $\{u_n^i\}_{n=1}^\infty$, generated by Algorithm 3.3.2, converge strongly to a_i, $u_i\in U_i(a_{N-(i-1)})$, respectively.*

Proof. By Theorem 3.3.3, the problem (3.3.1) has a solution $a_1,\ldots,a_N\in E$, $u_1\in U_1(a_N),\ldots,u_N\in U_N(a_1)$. From Lemma 3.3.1, we note that

$$a_i=\sigma_na_i+(1-\sigma_n)R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}[H_i(A_i(a_i),B_i(a_i)-\lambda_iS_i(a_i,u_i))], \qquad (3.3.18)$$

for all $i = 1, 2, \ldots, N$. Hence by (3.3.5) and (3.3.18), we have

$$\begin{aligned}
\|a_{n+1}^i - a_n^i\| &= \|\sigma_n a_n^i + (1-\sigma_n) R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}[H_i(A_i(a_n^i), B_i(a_n^i)) - \lambda_i S_i(a_n^i, u_n^i)] \\
&\quad -[\sigma_n a_{n-1}^i + (1-\sigma_n) R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}[H_i(A_i(a_{n-1}^i), B_i(a_{n-1}^i)) \\
&\quad -\lambda_i S_i(a_{n-1}^i, u_{n-1}^i)]]\| \\
&\leq \sigma_n \|a_n^i - a_{n-1}^i\| \\
&\quad +(1-\sigma_n)\|R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}[H_i(A_i(a_n^i), B_i(a_n^i)) - \lambda_i S_i(a_n^i, u_n^i)] \\
&\quad -R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}[H_i(A_i(a_{n-1}^i), B_i(a_{n-1}^i)) - \lambda_i S_i(a_{n-1}^i, u_{n-1}^i)]\| \\
&= \sigma_n \|a_n^i - a_{n-1}^i\| + (1-\sigma_n)\|R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}[J_i(a_n^i, a_n^i) - \lambda_i S_i(a_n^i, u_n^i)] \\
&\quad -R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}[J_i(a_{n-1}^i, a_{n-1}^i) - \lambda_i S_i(a_{n-1}^i, u_{n-1}^i)]\| \\
&\leq \sigma_n \|a_n^i - a_{n-1}^i\| + (1-\sigma_n)\frac{1}{\alpha_i - \beta_i}\|[J_i(a_n^i, a_n^i) - \lambda_i S_i(a_n^i, u_n^i)] \\
&\quad -[J_i(a_{n-1}^i, a_{n-1}^i) - \lambda_i S_i(a_{n-1}^i, u_{n-1}^i)]\| \\
&= \sigma_n \|a_n^i - a_{n-1}^i\| + (1-\sigma_n)\frac{1}{\alpha_i - \beta_i}\|[J_i(a_n^i, a_n^i) - J_i(a_{n-1}^i, a_{n-1}^i)] \\
&\quad -\lambda_i[S_i(a_n^i, u_n^i) - S_i(a_{n-1}^i, u_{n-1}^i)]\| \\
&\leq \sigma_n \|a_n^i - a_{n-1}^i\| + (1-\sigma_n)\frac{1}{\alpha_i - \beta_i}\|[J_i(a_n^i, a_n^i) - J_i(a_{n-1}^i, a_{n-1}^i)] \\
&\quad -\lambda_i[S_i(a_n^i, u_n^i) - S_i(a_{n-1}^i, u_n^i)]\| \\
&\quad +(1-\sigma_n)\frac{1}{\alpha_i - \beta_i}\|S_i(a_{n-1}^i, u_n^i) - S_i(a_{n-1}^i, u_{n-1}^i)\|. \qquad (3.3.19)
\end{aligned}$$

By Lemma 2.4.17, we obtain

$$\begin{aligned}
&\|J_i(a_n^i, a_n^i) - J_i(a_{n-1}^i, a_{n-1}^i) - \lambda_i[S_i(a_n^i, u_n^i) - S_i(a_{n-1}^i, u_n^i)]\|^q \\
&\leq \|J_i(a_n^i, a_n^i) - J_i(a_{n-1}^i, a_{n-1}^i)\|^q \\
&\quad - q\lambda_i \langle S_i(a_n^i, u_n^i) - S_i(a_{n-1}^i, u_n^i), J_{q,i}(J_i(a_n^i, a_n^i) - J_i(a_{n-1}^i, a_{n-1}^i)\rangle \\
&\quad + c_{q,i}\lambda_i^q \|S_i(a_n^i, u_n^i) - S_i(a_{n-1}^i, u_n^i)\|^q. \qquad (3.3.20)
\end{aligned}$$

From $(A4)$, we note that

$$\|J_i(a_n^i, a_n^i) - J_i(a_{n-1}^i, a_{n-1}^i)\|$$

$$
\begin{aligned}
&= \|H_i(A_i(a_n^i), B_i(a_n^i)) - H_i(A_i(a_{n-1}^i), B_i(a_{n-1}^i))\| \\
&\leq \|H_i(A_i(a_n^i), B_i(a_n^i)) - H_i(A_i(a_{n-1}^i), B_i(a_n^i))\| \\
&\quad + \|H_i(A_i(a_{n-1}^i), B_i(a_n^i)) - H_i(A_i(a_{n-1}^i), B_i(a_{n-1}^i))\| \\
&\leq (r_i + t_i)\|a_n^i - a_{n-1}^i\|. \qquad (3.3.21)
\end{aligned}
$$

From (3.3.20) and $(A6)$, it follows that

$$
\begin{aligned}
&-q\lambda_i\langle S_i(a_n^i, u_n^i) - S_i(a_{n-1}^i, u_n^i), J_{q,i}(J_i(a_n^i, a_n^i) - J_i(a_{n-1}^i, a_{n-1}^i))\rangle \\
&\qquad \leq -q\lambda_i s_i\|a_n^i - a_{n-1}^i\|^q. \qquad (3.3.22)
\end{aligned}
$$

By (3.3.19), (3.3.20) and $(A5)$, we have

$$
\begin{aligned}
\|S_i(a_{n-1}^i, u_n^i) - S_i(a_{n-1}^i, u_{n-1}^i)\| &\leq m_i\|u_n^i - u_{n-1}^i\| \\
&\leq m_i d_i(1 + n^{-1})\|a_n^i - a_{n-1}^i\| \qquad (3.3.23)
\end{aligned}
$$

and

$$
\|S_i(a_n^i, u_n^i) - S_i(a_{n-1}^i, u_n^i)\| \leq l_i\|a_n^i - a_{n-1}^i\|. \qquad (3.3.24)
$$

From (3.3.19)-(3.3.24), we obtain

$$
\begin{aligned}
&\|J_i(a_n^i, a_n^i) - J_i(a_{n-1}^i, a_{n-1}^i)) - \lambda_i[S_i(a_n^i, u_n^i) - S_i(a_{n-1}^i, u_n^i)]\|^q \\
&\qquad \leq \frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i}\|a_n^i - a_{n-1}^i\| \\
&\qquad\quad + \frac{\lambda_i m_i}{\alpha_i - \beta_i} d_i(1 + n^{-1})\|a_n^i - a_{n-1}^i\|. \qquad (3.3.25)
\end{aligned}
$$

Hence by (3.3.19), (3.3.24) and (3.3.25), we have

$$
\begin{aligned}
\|a_{n+1}^i - a_n^i\| &\leq \sigma_n\|a_n^i - a_{n-1}^i\| + (1 - \sigma_n)\frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i}\|a_n^i - a_{n-1}^i\| \\
&\quad + (1 - \sigma_n)\frac{\lambda_i m_i}{\alpha_i - \beta_i} d_i(1 + n^{-1})\|a_n^i - a_{n-1}^i\|. \qquad (3.3.26)
\end{aligned}
$$

Put $k = \max\{\pi_1 \ldots, \pi_N\}$,

where

$$\pi_i = \frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i} + \frac{\lambda_i m_i d_i(1 + n^{-1})}{\alpha_i - \beta_i}.$$

It follows from (3.3.26)that

$$\begin{aligned} \|a_{n+1}^1 - a_n^1\| + \ldots + \|a_{n+1}^N - a_n^N\| \;\leq\; & \sigma_n\|a_n^1 - a_{n-1}^1\| + (1-\sigma_n)k\|a_n^1 - a_{n-1}^1\| \\ & + \ldots + \sigma_n\|a_n^N - a_{n-1}^N\| \\ & +(1-\sigma_n)k\|a_n^N - a_{n-1}^N\|. \qquad (3.3.27) \end{aligned}$$

Set $c_n = \|a_n^1 - a_{n-1}^1\| + \ldots + \|a_n^N - a_{n-1}^N\|$ and $k_n = k + (1-k)\sigma_n$. From (3.3.27), we obtain

$$c_{n+1} \;\leq\; k_n c_n, \quad \forall\, n = 0, 1, 2, \ldots.$$

Since $\limsup_{n\to\infty} \sigma_n < 1$, we have $\limsup_{n\to\infty} k_n < 1$. Thus, it follows from Lemma 2.3.13 that $c_{n+1} \to 0$ and hence $\lim_{n\to\infty} \|a_{n+1}^i - a_n^i\| = 0$. Therefore $\{a_n^i\}$ is a Cauchy sequence and hence there exists $a_i \in E$ such that $a_n^i \to a_i$ as $n \to \infty$ for all $i = 1, 2, \ldots, N$. Next, we will show that $u_n^1 \to u_1 \in U_1(a_N)$ as $n \to \infty$. Hence, it follows from (3.3.6) that $\{u_n^1\}$ is also a Cauchy sequence. Thus there exists $u_1 \in E$ such that $u_n^1 \to u_1$ as $n \to \infty$. Consider,

$$\begin{aligned} d(u_1, U_1(a_N)) &= \inf\{\|u_1 - q\| : q \in U_1(a_N)\} \\ &\leq \|u_1 - u_n^1\| + d(u_n^1, U_1(a_N)) \\ &\leq \|u_1 - u_n^1\| + D(U_1(a_n^N), U_1(a_N)) \\ &\leq \|u_1 - u_n^1\| + d_1\|a_n^N - a_N\| \to 0. \end{aligned}$$

as $n \to \infty$. Since $U_1(a_N)$ is closed and $d(u_1, U_1(a_N)) = 0$, we have $u_1 \in U_1(a_N)$. By continuing the above process, there exist $u_2 \in U_2(a_{N-1}), \ldots, u_N \in U_N(a_1)$ such that $u_n^2 \to u_2, \ldots, u_n^N \to u_N$ as $n \to \infty$. Hence, by (3.3.5), we obtain

$$a_i = R_{M_i,\lambda_i}^{H_i(\cdot,\cdot)}[H_i(A_i(a_i), B_i(a_i)) - \lambda_i S_i(a_i, u_i)].$$

Therefore, it follows from Lemma 3.3.1 that $a_1, \ldots, a_N$ is a solution of problem (3.3.1). $\square$

Setting $N = 2$ in Theorem 3.3.3, we have the following result.

Corollary 3.3.5. *Let E be q-uniformly smooth real Banach spaces. Let $A_i, B_i : E \to E$ be two singled valued operators, $H_i : E \times E \to E$ a single-valued operator such that $H(A_i, B_i)$ is α_i-strongly accretive with respect to A_i, β_i-relaxed accretive with respect to B_i and $\alpha_i > \beta_i$ and suppose that $M_i : E \to 2^E$ is an $H_i(\cdot,\cdot)$-accretive set-valued mapping and $U_i : E \to C(E)$ is a contraction set-valued mapping with $0 \leq L_i < 1$ and nonempty values for all $i = 1, 2$. Assume that $H_i(A_i, B_i)$ is r_i-Lipschitz continuous with respect to A_i and t_i-Lipschitz continuous with respect to B_i, $S_i : E \times E \to E$ is l_i-Lipschitz continuous with respect to its first argument and m_i-Lipschitz continuous with respect to its second argument, $S_1(\cdot, y)$ is s_1-strongly accretive with respect to $H_1(A_1, B_1)$ and $S_2(x, \cdot)$ is s_2-strongly accretive with respect to $H_2(A_2, B_2)$, for all $i = 1, 2$. If*

$$\frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i} + \frac{\lambda_i m_i}{\alpha_i - \beta_i} < 1,$$

for all $i \in \{1, 2\}$, then problem (3.3.2) has a solution $a_1, a_2 \in E$, $u_1 \in U_1(a_2)$, $u_2 \in U_2(a_1)$.

Setting $N = 1$ in Theorem 3.3.3, we have the following result.

Corollary 3.3.6. *Let E be q-uniformly smooth real Banach spaces. Let $A, B : E \to E$ be four singled valued operators, $H : E \times E \to E$ be a single-valued operator such that $H(A, B)$ is α-strongly accretive with respect to A, β-relaxed accretive with respect to B, and $\alpha > \beta$ and suppose that $M : E \to 2^E$ is an $H(\cdot,\cdot)$-accretive set-valued mapping, $U : E \to C(E)$ is a contraction set-valued mapping with $0 \leq L < 1$ and nonempty values. Assume that $H(A, B)$ is r-Lipschitz continuous with respect to A and t-Lipschitz continuous with respect to B, $S : E \times E \to E$ is l-Lipschitz continuous with respect to its first argument and m-Lipschitz continuous* *with respect to its second argument, $S(\cdot, y)$ is s-strongly accretive with respect to $H(A, B)$. If*

$$\frac{\sqrt[q]{(r+t)^q - q\lambda s + c_q\lambda^q l^q}}{\alpha - \beta} + \frac{\lambda m}{\alpha - \beta} < 1$$

then problem (3.3.3) *has a solution $a \in E$, $u \in U(a)$.*

3.4 Existence and algorithm for generalized mixed equilibrium problem with a relaxed monotone mapping

In this section, let X be a Hausdorff topological vector space, K be a nonempty closed convex subset of X. Let $g, h : K \times K \to \mathbb{R}$, $A : K \to X^*$ be a monotone mapping, and $T : K \to X^*$ a relaxed η-α monotone mapping. We consider the following generalized mixed equilibrium problem with a relaxed monotone mapping: finding $x \in K$ such that

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y - x\rangle \geq 0 \text{ for all } y \in K. \quad (3.4.1)$$

The set of solution of (3.4.1) is denoted by $GMEPRM(g, h, T, A)$.

If $h \equiv 0$, then generalized mixed equilibrium problem with a relaxed monotone mapping (3.4.1) becomes to the following the generalized equilibrium problem with a relaxed monotone mapping [50]: find $x \in K$ such that

$$g(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y - x\rangle \geq 0 \text{ for all } y \in K, \quad (3.4.2)$$

where K is a nonempty closed convex subset of a real Hilbert space H, $A : K \to H$ is a λ-inverse-strongly mapping, and $g : K \times K \to \mathbb{R}$ is a bifunction mapping.

For proving our main result, let us give the following assumptions:

($\widehat{A}1$) $g(x,x) = 0$ for all $x \in K$;

($\widehat{A}2$) g is monotone, i.e. $g(x,y) + g(y,x) \leq 0$ for all $x, y \in K$;

$(\widehat{A}3)$ for each $x \in K$, $y \mapsto g(x,y)$ is convex and lower semicontinuous;

$(\widehat{A}4)$ for each $x, y, z \in K$, $\limsup_{t\to 0} g(tz + (1-t)x, y) \le g(x,y)$;

$(\widehat{B}1)$ $h(x,x) = 0$ for all $x \in K$;

$(\widehat{B}2)$ for each $x \in K$, $y \mapsto h(x,y)$ is lower semicontinuous;

$(\widehat{B}3)$ for each $x \in K$, $y \mapsto h(x,y)$ is convex;

$(\widehat{B}4)$ for each $x, y, z \in K$, $\limsup_{t\to 0} h(tz + (1-t)x, y) \le h(x,y)$;

$(\widehat{C}1)$ $\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in K$;

$(\widehat{C}2)$ for each $u, v \in K$, $z \mapsto \langle Tv, \eta(z,u)\rangle$ is convex and lower semicontinuous and $z \mapsto \langle Tu, \eta(v,z)\rangle$ is lower semicontinuous;

$(\widehat{C}3)$ for each $x, y \in K$, $\alpha(x-y) + \alpha(y-x) \ge 0$;

$(\widehat{C}4)$ for each $u, v, x, z \in K$, $\limsup_{t\to 0}\langle Tu, \eta(v, tx + (1-t)z)\rangle \le \langle Tu, \eta(v,z)\rangle$;

$(\widehat{D}1)$ for each $u, v \in K$, $z \mapsto \langle Av, z-u\rangle$ is convex and lower semicontinuous and $z \mapsto \langle Au, v-z\rangle$ is lower semicontinuous;

$(\widehat{D}2)$ for each $u, v, x, z \in K$, $\limsup_{t\to 0}\langle Au, v - (tx + (1-t)z)\rangle \le \langle Au, v-z\rangle$;

$(\widehat{D}3)$ $\langle Tx, \eta(y,x)\rangle + \langle Ty, \eta(x,y)\rangle + \langle Ax, y-x\rangle + \langle Ay, x-y\rangle \le 0$ for all $x, y \in K$.

The idea of the proof of the next theorem is contained in the paper of Peng and Yao [51], Wang, et al. [50], and Combettes and Hirstoaga [52].

Lemma 3.4.1. *Let X be a Hausdorff topological vector space, K be a nonempty closed convex subset of X. Let $g : K \times K \to \mathbb{R}$ be a mapping satisfying $(\widehat{A}1)$ and $(\widehat{A}3)$, and $h : K \times K \to \mathbb{R}$ be a mapping satisfying $(\widehat{B}1)$ and $(\widehat{B}3)$. Let $T : K \to X^*$ be an η-hemicontinuous and relaxed η-α monotone mapping satisfying $(\widehat{C}2)$. Let $A : K \to X^*$ be a monotone and hemicontinuous mapping satisfying $(\widehat{D}1)$ and assume that $\eta(x,x) = 0$ for all $x \in K$. Then for all $r > 0$ and $z \in K$ the following problems are equivalent;*

(*i*) *find* $x \in K$ *such that*

$$g(x,y)+h(x,y)+\langle Tx,\eta(y,x)\rangle+\langle Ax,y-x\rangle+\frac{1}{r}\langle y-x,x-z\rangle \geq 0 \quad \textit{for all} \quad y \in K;$$

(*ii*) *find* $x \in K$ *such that*

$$g(x,y)+h(x,y)+\langle Ty,\eta(y,x)\rangle+\langle Ay,y-x\rangle+\frac{1}{r}\langle y-x,x-z\rangle \geq \alpha(y-x) \quad \textit{for all} \quad y \in K.$$

Proof. Let $x \in K$ be a solution of the problem (i). Since T is relaxed η-α monotone and A is monotone, we get

$$\begin{aligned} g(x,y) &+ h(x,y) + \langle Ty,\eta(y,x)\rangle + \langle Ay,y-x\rangle + \frac{1}{r}\langle y-x,x-z\rangle \\ &\geq g(x,y) + h(x,y) + \langle Tx,\eta(y,x)\rangle + \alpha(y-x) + \langle Ax,y-x\rangle \\ &\quad + \frac{1}{r}\langle y-x,x-z\rangle \\ &\geq \alpha(y-x), \quad \text{for all} \quad y \in K. \end{aligned}$$

Hence x is a solution of the problem (ii).
Conversely, let $x \in K$ be a solution of the problem (ii). Setting $y_t = (1-t)x + ty$ for all $t \in (0,1)$, then $y_t \in K$. Thus, it follows that

$$\begin{aligned} g(x,y_t) + h(x,y_t) + \langle Ty_t,\eta(y_t,x)\rangle + \langle Ay_t,y_t-x\rangle + \frac{1}{r}\langle y_t-x,x-z\rangle \\ \geq \alpha(y_t-x) \\ = t^p\alpha(y-x). \quad (3.4.3) \end{aligned}$$

From the conditions $(\widehat{A}1)$, $(\widehat{A}3)$, $(\widehat{B}1)$, $(\widehat{B}3)$, $(\widehat{C}2)$ and $(\widehat{D}1)$, we obtain

$$g(x,y_t) \leq (1-t)g(x,x) + tg(x,y) = tg(x,y), \quad (3.4.4)$$

$$h(x,y_t) \leq (1-t)h(x,x) + th(x,y) = th(x,y), \quad (3.4.5)$$

$$\langle Ty_t,\eta(y_t,x)\rangle \leq (1-t)\langle Ty_t,\eta(x,x)\rangle + t\langle Ty_t,\eta(y,x)\rangle$$

$$= \quad t\langle T(x+t(y-x)), \eta(y,x)\rangle, \tag{3.4.6}$$

and

$$\langle Ay_t, y_t - x\rangle = \langle Ay_t, x+t(y-x)-x\rangle = t\langle A(x+t(y-x)), y-x\rangle. \tag{3.4.7}$$

Since

$$\langle y_t - x, x - z\rangle = \langle x+t(y-x)-x, x-z\rangle = t\langle y-x, x-z\rangle, \tag{3.4.8}$$

it follows from (3.4.3)-(3.4.8) that

$$\begin{aligned} g(x,y) + h(x,y) + \langle T(x+t(y-x)), \eta(y,x)\rangle + \langle A(x+t(y-x)), y-x\rangle \\ + \frac{1}{r}\langle y-x, x-z\rangle \ \geq \ t^{p-1}\alpha(y-x), \end{aligned} \tag{3.4.9}$$

for all $y \in K$. Letting $t \to 0$ in (3.4.9), we get

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0, \tag{3.4.10}$$

for all $y \in K$. Hence x is a solution of the problem (i). This completes the proof. □

Theorem 3.4.2. *Let X be a Hausdorff topological vector space, K be a nonempty compact convex subset of X. Let $g : K \times K \to \mathbb{R}$ be a mapping satisfying $(\widehat{A}1)$ and $(\widehat{A}3)$ and let $h : K \times K \to \mathbb{R}$ be a mapping satisfying $(\widehat{B}1)$ and $(\widehat{B}3)$. Let $T : K \to X^*$ be an η-hemicontinuous and relaxed η-α monotone mapping satisfying $(\widehat{C}1)$-$(\widehat{C}3)$. Let $A : K \to X^*$ be a monotone and hemicontinuous mapping satisfying $(\widehat{D}1)$. Then, for all $r > 0$ and $z \in K$ there exists $x \in K$ such that*

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0, \quad \textit{for all} \quad y \in K.$$

Proof. Let z be any given point in K and let $r > 0$. We will show that $T_r(z) \neq \emptyset$. Define $M_z, N_z : K \to 2^K$ by

$$M_z(y) = \Big\{x \in K : g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y - x\rangle + \frac{1}{r}\langle y - x, x - z\rangle \geq 0\Big\}, \quad \text{for all} \quad y \in K$$

and

$$N_z(y) = \Big\{x \in K : g(x,y) + h(x,y) + \langle Ty, \eta(y,x)\rangle + \langle Ay, y - x\rangle + \frac{1}{r}\langle y - x, x - z\rangle \geq \alpha(y - x)\Big\}, \quad \text{for all} \quad y \in K.$$

Note that, for each $y \in K$, $M_z(y)$ is nonempty because $y \in M_z(y)$. We claim that M_z is a KKM mapping. Assume that M_z is not a KKM mapping. Then there exists $\{y_1, y_2, ..., y_n\} \subset K$ and $t_i > 0$, $i = 1, 2, ..., n$ with $\sum_{i=1}^{n} t_i = 1$ such that $\widehat{z} = \sum_{i=1}^{n} t_i y_i \notin \bigcup_{i=1}^{n} M_z(y_i)$ for each $i = 1, 2, ..., n$. This implies that

$$g(\widehat{z}, y_i) + h(\widehat{z}, y_i) + \langle T\widehat{z}, \eta(y_i, \widehat{z})\rangle + \langle A\widehat{z}, y_i - \widehat{z}\rangle + \frac{1}{r}\langle y_i - \widehat{z}, \widehat{z} - z\rangle < 0,$$

for each $i = 1, 2, ..., n$. By $(\widehat{A}1)$, $(\widehat{A}3)$, $(\widehat{B}1)$, $(\widehat{B}3)$, $(\widehat{C}2)$ and $(\widehat{D}1)$, we have

$$\begin{aligned}
0 &= g(\widehat{z}, \widehat{z}) + h(\widehat{z}, \widehat{z}) \\
&= g\left(\widehat{z}, \sum_{i=1}^{n} t_i y_i\right) + h\left(\widehat{z}, \sum_{i=1}^{n} t_i y_i\right) + \left\langle T\widehat{z}, \eta\left(\sum_{i=1}^{n} t_i y_i, \widehat{z}\right)\right\rangle \\
&\quad + \left\langle A\widehat{z}, \sum_{i=1}^{n} t_i y_i - \widehat{z}\right\rangle \\
&\leq \sum_{i=1}^{n} t_i g(\widehat{z}, y_i) + \sum_{i=1}^{n} t_i h(\widehat{z}, y_i) + \sum_{i=1}^{n} t_i \langle T\widehat{z}, \eta(y_i, \widehat{z})\rangle + \sum_{i=1}^{n} t_i \langle A\widehat{z}, y_i - \widehat{z}\rangle \\
&< \sum_{i=1}^{n} t_i \frac{1}{r}\langle \widehat{z} - y_i, \widehat{z} - z\rangle \\
&= 0,
\end{aligned}$$

which is a contradiction. Hence M_z is a KKM mapping. We now show that

$M_z(y) \subset N_z(y)$ for all $y \in K$. For any $y \in K$, we let $x \in M_z(y)$. Thus, we have

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0.$$

Since T is relaxed η-α monotone and A is monotone, we get

$$\begin{aligned} g(x,y) + h(x,y) + \langle Ty, \eta(y,x)\rangle + \langle Ay, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \\ \geq\ & g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \alpha(y-x) + \langle Ax, y-x\rangle \\ & + \frac{1}{r}\langle y-x, x-z\rangle \\ \geq\ & \alpha(y-x). \end{aligned}$$

This implies that $x \in N_z(y)$ and hence $M_z(y) \subset N_z(y)$ for all $y \in K$. Since $z \mapsto \langle Ty, \eta(y,z)\rangle$ and $z \mapsto \langle Ay, y-z\rangle$ are the lower semicontinuous function, we have $z \mapsto \langle Ty, \eta(y,z)\rangle$ and $z \mapsto \langle Ay, y-z\rangle$ are weakly lower semicontinuous. Thus $M_z(y)$ is weakly closed for all $y \in K$ implies that $M_z(y)$ is closed for all $y \in K$. Since K is compact, we have $M_z(y)$ is compact in K for all $y \in K$. By Lemma 3.4.1 and Lemma 2.5.7, we get

$$\bigcap_{y\in K} M_z(y) = \bigcap_{y\in K} N_z(y) \neq \emptyset.$$

Therefore, there exists $x \in K$ such that

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0.$$

□

Theorem 3.4.3. *Let H be a real Hilbert space, K be a nonempty bounded closed convex subset of H. Let $g : K \times K \to \mathbb{R}$ be a mapping satisfying $(\widehat{A}1)$-$(\widehat{A}3)$ and let $h : K \times K \to \mathbb{R}$ be a monotone mapping satisfying $(\widehat{B}1)$-$(\widehat{B}3)$. Let $T : K \to H$ be an η-hemicontinuous and relaxed η-α monotone mapping satisfying $(\widehat{C}1)$-$(\widehat{C}3)$. Let $A : K \to H$ be a λ-inverse-strongly monotone and hemicontinuous mapping satisfying $(\widehat{D}1)$. For $r > 0$ and $z \in K$, define $T_r : K \to 2^K$ by*

$$T_r(z) = \Big\{ x \in K : g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0, \quad \text{for all} \quad y \in K \Big\}.$$

Then, the following results hold:

(i) $\mathrm{dom}T_r = H$*;*

(ii) T_r *is single-valued;*

(iii) T_r *is firmly nonexpansive i.e., for any* $x, y \in K$*,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x-y\rangle;$$

(iv) $F(T_r) = GMEPRM(g,h,T,A)$*;*

(v) $GMEPRM(g,h,T,A)$ *is closed and convex.*

Proof. **Step 1.** We first show that $\mathrm{dom}T_r = H$. Since K is bounded closed and convex, we note that K is weakly compact. Hence, for every $r > 0$ and $z \in K$ there exists $x \in K$ such that

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0, \quad \text{for all} \quad y \in K.$$

Step 2. We will show that T_r is single-valued. For each $z \in K$ and $r > 0$, let $x_1, x_2 \in T_r(z)$. Thus, we have

$$g(x_1,x_2) + h(x_1,x_2) + \langle Tx_1, \eta(x_2,x_1)\rangle + \langle Ax_1, x_2-x_1\rangle + \frac{1}{r}\langle x_2-x_1, x_1-z\rangle \geq 0$$

and $g(x_2,x_1) + h(x_2,x_1) + \langle Tx_2, \eta(x_1,x_2)\rangle + \langle Ax_2, x_1-x_2\rangle + \frac{1}{r}\langle x_1-x_2, x_2-z\rangle \geq 0.$

Adding the two inequalities, we obtain

$$g(x_1,x_2) + h(x_1,x_2) + g(x_2,x_1) + h(x_2,x_1) + \langle Tx_1 - Tx_2, \eta(x_2,x_1)\rangle + \langle Ax_1 - Ax_2, x_2 - x_1\rangle + \frac{1}{r}\langle x_2-x_1, x_1-x_2\rangle \geq 0.$$

From the monotonicity of H and $(\widehat{A}2)$, we have

$$\langle Tx_1 - Tx_2, \eta(x_2, x_1)\rangle + \langle Ax_1 - Ax_2, x_2 - x_1\rangle + \frac{1}{r}\langle x_2 - x_1, x_1 - x_2\rangle \geq 0.$$

This implies that

$$\frac{1}{r}\langle x_2 - x_1, x_1 - x_2\rangle \geq \langle Tx_2 - Tx_1, \eta(x_2, x_1)\rangle + \langle x_2 - x_1, Ax_2 - Ax_1\rangle. \quad (3.4.11)$$

Since T is relaxed η-α monotone, A is λ-inverse-strongly monotone and $r > 0$, it follows that

$$\langle x_2 - x_1, x_1 - x_2\rangle \geq r\Big[\alpha(x_2 - x_1) + \lambda\|Ax_2 - Ax_1\|^2\Big] \geq r\alpha(x_2 - x_1). \quad (3.4.12)$$

By exchanging the position of x_1 and x_2 in (3.4.11), we get

$$\frac{1}{r}\langle x_1 - x_2, x_2 - x_1\rangle \geq \langle Tx_1 - Tx_2, \eta(x_1, x_2)\rangle + \langle x_1 - x_2, Ax_1 - Ax_2\rangle \geq \alpha(x_1 - x_2).$$

Hence $\langle x_1 - x_2, x_2 - x_1\rangle \geq r\alpha(x_1 - x_2)$ and therefore

$$\langle x_2 - x_1, x_1 - x_2\rangle = \langle x_1 - x_2, x_2 - x_1\rangle \geq r\alpha(x_1 - x_2). \quad (3.4.13)$$

Adding the inequalities (3.4.12) and (3.4.13) and using $(\widehat{C}3)$, we have

$$-2\|x_1 - x_2\|^2 = 2\langle x_2 - x_1, x_1 - x_2\rangle \geq 0.$$

Hence $x_1 = x_2$ and therefore T_r is a single-valued mapping.

Step 3. We will show that T_r is a firmly nonexpansive mapping. For $x, y \in H$, we note that

$$g(T_r(x), T_r(y)) + h(T_r(x), T_r(y)) + \langle TT_r(x), \eta(T_r(y), T_r(x))\rangle + \langle AT_r(x), T_r(y) - T_r(x)\rangle + \frac{1}{r}\langle T_r(y) - T_r(x), T_r(x) - x\rangle \geq 0$$

and

$$g(T_r(y), T_r(x)) + h(T_r(y), T_r(x)) + \langle TT_r(y), \eta(T_r(x), T_r(y))\rangle$$
$$+ \langle AT_r(y), T_r(x) - T_r(y)\rangle + \frac{1}{r}\langle T_r(x) - T_r(y), T_r(y) - y\rangle \geq 0.$$

By $(\widehat{A}2)$, $(\widehat{C}1)$, $r > 0$, and h is monotone we obtain

$$\langle TT_r(x) - TT_r(y), \eta(T_r(y), T_r(x))\rangle + \langle AT_r(x) - AT_r(y), T_r(y) - T_r(x)\rangle$$
$$+ \frac{1}{r}\langle T_r(y) - T_r(x), T_r(x) - x - T_r(y) + y\rangle \geq 0.$$

Thus, we have

$$\begin{aligned}
\frac{1}{r}\langle T_r(y) - T_r(x), T_r(x) - T_r(y) + y - x\rangle \\
&\geq \langle TT_r(y) - TT_r(x), \eta(T_r(y), T_r(x))\rangle \\
&\quad + \langle T_r(y) - T_r(x), AT_r(y) - AT_r(x)\rangle \\
&\geq \alpha(T_r(y) - T_r(x)) + \lambda\|AT_r(y) - AT_r(x)\|^2 \\
&\geq \alpha(T_r(y) - T_r(x)). \qquad (3.4.14)
\end{aligned}$$

By exchanging the position of x and y in (3.4.14), we note that

$$\frac{1}{r}\langle T_r(x) - T_r(y), T_r(y) - T_r(x) + x - y\rangle \geq \alpha(T_r(x) - T_r(y)). \qquad (3.4.15)$$

From (3.4.14) and (3.4.15), we get

$$2\langle T_r(x) - T_r(y), T_r(y) - T_r(x) + x - y\rangle \geq r\Big[\alpha(T_r(y) - T_r(x)) + \alpha(T_r(x) - T_r(y))\Big].$$

By $(\widehat{C}3)$, we obtain

$$\begin{aligned}
\langle T_r(x) - T_r(y), T_r(y) - T_r(x) + x - y\rangle &= \langle T_r(x) - T_r(y), T_r(y) - T_r(x)\rangle \\
&\quad + \langle T_r(x) - T_r(y), x - y\rangle \\
&\geq 0.
\end{aligned}$$

Thus, we have $\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle$. Hence T_r is a firmly nonexpansive mapping.

Step 4. We will show that $F(T_r) = GMEPRM(g, h, T, A)$. Indeed, we have the following

$$\begin{aligned} u \in F(T_r) &\Leftrightarrow u = T_r(u) \\ &\Leftrightarrow g(u, y) + h(u, y) + \langle Tu, \eta(y, u)\rangle + \langle Au, y - u\rangle \geq 0, \\ &\quad \text{for all} \quad y \in K \\ &\Leftrightarrow u \in GMEPRM(g, h, T, A). \end{aligned}$$

Step 5. We will show that $GMEPRM(g, h, T, A)$ is closed and convex. Since T_r is firmly nonexpansive, it follows by Lemma 2.2.23 that $GMEPRM(g, h, T, A)$ is closed and convex. This completes the proof. □

Corollary 3.4.4. [50] *Let H be a real Hilbert space, K be a nonempty bounded closed convex subset of H. Let $T : K \to H$ be an η-hemicontinuous and relaxed η-α monotone satisfying $(\widehat{C}1)$-$(\widehat{C}3)$ and let $g : K \times K \to \mathbb{R}$ be a mapping satisfying $(\widehat{A}1)$-$(\widehat{A}3)$. For $r > 0$ and $z \in K$, define $\widetilde{T}_r : K \to 2^K$ by*

$$\widetilde{T}_r(z) = \left\{x \in K : g(x, y) + \langle Tx, \eta(y, x)\rangle + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \quad \text{for all} \quad y \in K\right\}.$$

Then, the following results hold:

(i) $\widetilde{T}_r$ is single-valued;

(ii) $\widetilde{T}_r$ is firmly nonexpansive i.e., for any $x, y \in K$,

$$\|\widetilde{T}_r(x) - \widetilde{T}_r(y)\|^2 \leq \langle \widetilde{T}_r(x) - \widetilde{T}_r(y), x - y\rangle;$$

(iii) $F(\widetilde{T}_r) = GEP(g, T)$;

(iv) $GEP(g, T)$ is closed and convex.

Proof. It is easy to see by setting $h \equiv 0$ and $A \equiv 0$ in Theorem 3.4.3. □

3.4.1 Weak convergence theorems

In the section, we introduce an iterative sequence and prove weak convergence theorem for solving a generalized mixed equilibrium problem with a relaxed monotone mapping.

We note that dom$T_r = H$ under certain condition in Theorem 3.4.3.

Lemma 3.4.5. *Let H be a real Hilbert space, K be a nonempty bounded closed convex subset of H. Let $g : K \times K \to \mathbb{R}$ be a mapping satisfying $(\widehat{A}1)$-$(\widehat{A}4)$, and $h : K \times K \to \mathbb{R}$ be a monotone mapping satisfying $(\widehat{B}1)$-$(\widehat{B}4)$. Let $T : K \to H$ be an η-hemicontinuous and relaxed η-α monotone mapping satisfying $(\widehat{C}2)$ and $(\widehat{C}4)$. Let $A : K \to H$ be a monotone mapping satisfying $(\widehat{D}1)$ and $(\widehat{D}2)$ and assume that $\eta(x, x) = 0$ for all $x \in K$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in H for all $n \geq 1$ and $\{T_r\}$ a sequence of mapping defined in (2.3.4) which* dom$T_r = H$. *Define*

$$z_n = T_r x_n \quad and \quad u_n = x_n - z_n, \quad \forall n \in \mathbb{N}, \tag{3.4.16}$$

and suppose that

$$z_n \rightharpoonup x \quad and \quad u_n \to u. \tag{3.4.17}$$

If $r > 0$, then

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y - x\rangle + \frac{1}{r}\langle u, x - y\rangle \geq 0, \quad for\ all \quad y \in K.$$

Proof. Since dom$T_r = H$, we note that the sequence $\{z_n\}_{n\in\mathbb{N}}$ is well defined in K. By g, h, A are monotone and T is relaxed η-α monotone, we get

$$\begin{aligned} g(x,y) + g(y,x) + h(x,y) + h(y,x) + \langle Tx, \eta(y,x)\rangle + \langle Ty, \eta(x,y)\rangle \\ + \langle Ax, y - x\rangle + \langle Ay, x - y\rangle \leq 0, \quad \forall\, x, y \in K. \end{aligned}$$

This implies that

$$\begin{aligned} &g(y,x)+h(y,x)+\langle Ty,\eta(x,y)\rangle+\langle Ay,x-y\rangle \\ &\qquad \le\ -g(x,y)-h(x,y)-\langle Tx,\eta(y,x)\rangle-\langle Ax,y-x\rangle,\quad \forall\, x,y\in K. \quad (3.4.18)\end{aligned}$$

It follows from $(\widehat{A}3)$, $(\widehat{B}2)$, $(\widehat{C}2)$, and $(\widehat{D}1)$ that $y\mapsto g(x,y)$, $y\mapsto h(x,y)$, $z\mapsto \langle Tv,\eta(z,u)\rangle$, and $z\mapsto\langle Av,z-u\rangle$ are weak lower semicontinuous for every $y\in K$. Therefore, we derive from (2.3.4), (3.4.16), (3.4.17) and (3.4.18) that

$$\begin{aligned} g(y,x)&+h(y,x)+\langle Ty,\eta(x,y)\rangle+\langle Ay,x-y\rangle \\ &\le \liminf_{n\to\infty} g(y,z_n)+\liminf_{n\to\infty} h(y,z_n)+\liminf_{n\to\infty}\langle Ty,\eta(z_n,y)\rangle \\ &\quad +\liminf_{n\to\infty}\langle Ay,z_n-y\rangle \\ &\le \liminf_{n\to\infty}\Big[g(y,z_n)+h(y,z_n)+\langle Ty,\eta(z_n,y)\rangle+\langle Ay,z_n-y\rangle\Big] \\ &\le \liminf_{n\to\infty}\Big[-g(z_n,y)-h(z_n,y)-\langle Tz_n,\eta(y,z_n)\rangle-\langle Az_n,y-z_n\rangle\Big] \\ &\le \frac{1}{r}\liminf_{n\to\infty}\langle u_n,z_n-y\rangle \\ &= \frac{1}{r}\langle u,x-y\rangle. \qquad (3.4.19)\end{aligned}$$

Fix $y\in K$ and define $x_t=(1-t)x+ty$ for all $t\in(0,1)$, then $x_t\in K$. Thus, by $(\widehat{A}1)$, $(\widehat{B}1)$, $(\widehat{A}3)$, $(\widehat{B}3)$, $(\widehat{C}2)$, $(\widehat{D}1)$ and (3.4.19), we have that

$$\begin{aligned} 0 &= g(x_t,x_t)+h(x_t,x_t)+\langle Tx_t,\eta(x_t,x_t)\rangle+\langle Ax_t,x_t-x_t\rangle \\ &\le (1-t)g(x_t,x)+tg(x_t,y)+(1-t)h(x_t,x)+th(x_t,y)+(1-t)\langle Tx_t,\eta(x,x_t)\rangle \\ &\quad +t\langle Tx_t,\eta(y,x_t)\rangle+(1-t)\langle Ax_t,x-x_t\rangle+t\langle Ax_t,y-x_t\rangle \\ &= (1-t)\Big[g(x_t,x)+h(x_t,x)+\langle Tx_t,\eta(x,x_t)\rangle+\langle Ax_t,x-x_t\rangle\Big] \\ &\quad +t\Big[g(x_t,y)+h(x_t,y)+\langle Tx_t,\eta(y,x_t)\rangle+\langle Ax_t,y-x_t\rangle\Big] \\ &\le (1-t)\frac{1}{r}\langle u,x-x_t\rangle+t\Big[g(x_t,y)+h(x_t,y)+\langle Tx_t,\eta(y,x_t)\rangle+\langle Ax_t,y-x_t\rangle\Big] \\ &= t(1-t)\frac{1}{r}\langle u,x-y\rangle+t\Big[g(x_t,y)+h(x_t,y)+\langle Tx_t,\eta(y,x_t)\rangle+\langle Ax_t,y-x_t\rangle\Big]. \end{aligned}$$

(3.4.20)

Hence,

$$g(x_t, y) + h(x_t, y) + \langle Tx_t, \eta(y, x_t)\rangle + \langle Ax_t, y - x_t\rangle \geq (1-t)\frac{1}{r}\langle u, y - x\rangle.$$

By $(\widehat{A}4)$, $(\widehat{B}4)$, $(\widehat{C}4)$, and $(\widehat{D}2)$, we obtain that

$$\begin{aligned} & g(x, y) + h(x, y) + \langle Tx, \eta(y, x)\rangle + \langle Ax, y - x\rangle \\ & \quad \geq \limsup_{t\to\infty} g(x_t, y) + \limsup_{t\to\infty} h(x_t, y) + \limsup_{t\to\infty}\langle Tx_t, \eta(y, x_t)\rangle \\ & \quad\quad + \limsup_{t\to\infty}\langle Ax_t, y - x_t\rangle \\ & \quad \geq \frac{1}{r}\langle u, y - x\rangle. \end{aligned}$$

□

Theorem 3.4.6. *Let H be a real Hilbert space, K be a nonempty bounded closed convex subset of H. Assume that $g : K \times K \to \mathbb{R}$ satisfies $(\widehat{A}1)$-$(\widehat{A}4)$, and $h : K \times K \to \mathbb{R}$ is a monotone mapping satisfying $(\widehat{B}1)$-$(\widehat{B}4)$. Suppose that $T : K \to H$ satisfies $(\widehat{C}2)$ and $(\widehat{C}4)$, $A : K \to H$ satisfies $(\widehat{D}1)$-$(\widehat{D}3)$ and that the set $GMEPRM(g, h, T, A)$ of solutions (3.4.1) is nonempty. Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence generated by the form*

$$x_0 \in K \quad \text{and} \quad x_{n+1} = T_{r_n}x_n, \quad \text{where} \quad r_n \in (0, +\infty), \quad \text{for all} \quad n \in \mathbb{N}, \qquad (3.4.21)$$

where $\sum_{n\in\mathbb{N}} r_n^2 = +\infty$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a point in $GMEPRM(g, h, T, A)$.

Proof. Since $GMEPRM(g, h, T, A) \neq \emptyset$, it follows that $\mathrm{dom}T_{r_n} = H$ for all $n \geq 1$. For any $n \in \mathbb{N}$, we note from (3.4.21) and (2.3.4) that

$$\begin{cases} 0 \leq g(x_{n+1}, x_{n+2}) + h(x_{n+1}, x_{n+2}) + \langle Tx_{n+1}, \eta(x_{n+2}, x_{n+1})\rangle \\ \qquad + \langle Ax_{n+1}, x_{n+2} - x_{n+1}\rangle + \frac{1}{r_n}\langle x_{n+1} - x_n, x_{n+2} - x_{n+1}\rangle \\ 0 \leq g(x_{n+2}, x_{n+1}) + h(x_{n+2}, x_{n+1}) + \langle Tx_{n+2}, \eta(x_{n+1}, x_{n+2})\rangle \\ \qquad + \langle Ax_{n+2}, x_{n+1} - x_{n+2}\rangle + \frac{1}{r_{n+1}}\langle x_{n+2} - x_{n+1}, x_{n+1} - x_{n+2}\rangle. \end{cases} \qquad (3.4.22)$$

Setting $z_n = T_{r_n}x_n$ and $u_n = (x_n - z_n)/r_n$. Then (3.4.22) yields

$$\begin{cases} \langle u_n, x_{n+2} - x_{n+1}\rangle \leq g(x_{n+1}, x_{n+2}) + h(x_{n+1}, x_{n+2}) + \langle Tx_{n+1}, \eta(x_{n+2}, x_{n+1})\rangle \\ \qquad + \langle Ax_{n+1}, x_{n+2} - x_{n+1}\rangle \\ \langle u_{n+1}, x_{n+1} - x_{n+2}\rangle \leq g(x_{n+2}, x_{n+1}) + h(x_{n+2}, x_{n+1}) + \langle Tx_{n+2}, \eta(x_{n+1}, x_{n+2})\rangle \\ \qquad + \langle Ax_{n+2}, x_{n+1} - x_{n+2}\rangle \end{cases} \tag{3.4.23}$$

and by $(\widehat{A}2)$, $(\widehat{D}3)$, and the monotonicity of h that

$$\begin{aligned} &\langle u_n - u_{n+1}, x_{n+2} - x_{n+1}\rangle \\ &\quad \leq g(x_{n+1}, x_{n+2}) + g(x_{n+2}, x_{n+1}) + h(x_{n+1}, x_{n+2}) + h(x_{n+2}, x_{n+1}) \\ &\quad + \langle Tx_{n+1}, \eta(x_{n+2}, x_{n+1})\rangle + Tx_{n+2}, \eta(x_{n+1}, x_{n+2})\rangle \\ &\quad + \langle Ax_{n+1}, x_{n+2} - x_{n+1}\rangle + \langle Ax_{n+2}, x_{n+1} - x_{n+2}\rangle \leq 0. \end{aligned} \tag{3.4.24}$$

Thus $\langle u_{n+1} - u_n, u_{n+1}\rangle \leq 0$ and, by Cauchy-Schwarz, $\|u_{n+1}\| \leq \|u_n\|$. Therefore

$$\{\|u_n\|\}_{n\in\mathbb{N}} \text{ is a convergent sequence.} \tag{3.4.25}$$

Since T_{r_n} is firmly nonexpansive, it follows by Theorem 2.6 in [52] that $\sum_{n\in\mathbb{N}} r_n^2\|u_n\|^2$ $\sum_{n\in\mathbb{N}} r_n^2\|z_n - x_n\|^2 < +\infty$. Since $\sum_{n\in\mathbb{N}} r_n^2 = +\infty$, we have $\liminf_{n\to\infty}\|u_n\| = 0$ and, consequently, (3.4.25) yields $u_n \to 0$. Since $\{x_n\}$ is bounded, we may assume that there exists a sequence $\{x_{k_n}\}$ of $\{x_n\}$ such that $x_{k_n} \rightharpoonup x$ and

$$u_{k_n} \to 0. \tag{3.4.26}$$

On the other hand, since $z_n - x_n \to 0$, we have

$$z_{k_n} \rightharpoonup x. \tag{3.4.27}$$

Combining (3.4.26), (3.4.27), and Lemma 3.4.5, we conclude that x is a solution of (3.4.1). □

In the case of $h \equiv 0$, $T \equiv 0$, and $A \equiv 0$ in (3.4.1), $GMEPRM(g, h, T, A)$ deduced to equilibrium problem (for short, $EP(g)$)

Corollary 3.4.7. [52] *Let H be a real Hilbert space, K be a nonempty bounded closed convex subset of H. Assume that $g : K \times K \to \mathbb{R}$ satisfies $(\widehat{A}1)$-$(\widehat{A}4)$ and that the set $EP(g)$ of solutions to (2.6.1) is nonempty. Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence generated by the form*

$$x_0 \in K \quad and \quad x_{n+1} = J_{r_n}x_n, \quad where \quad r_n \in (0, +\infty), \quad for\ all \quad n \in \mathbb{N}, \tag{3.4.28}$$

where $\sum_{n\in\mathbb{N}} r_n^2 = +\infty$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a point in $EP(g)$.

Proof. It follows from Theorem 3.4.6 by setting $h \equiv 0$, $T \equiv 0$, and $A \equiv 0$. □