# CHAPTER IV

# CONCLUSION

The following results are all main theorems of this dissertation:

1. Let $E$ be a real uniformly smooth and strictly convex Banach space with Kadec-Klee property, $K$ be a nonempty closed and convex subset of $E$ with $\theta \in K$. Let $f : K \to K$ be an isometry mapping. Let $T_1, \ldots, T_N : \underbrace{K \times \ldots \times K}_{N-\text{times}} \to E^*$ be continuous mappings and $\{\alpha_n^{(1)}\}, \{\alpha_n^{(2)}\}, \ldots, \{\alpha_n^{(N)}\}$ be the sequences in $(a, b)$ with $0 < a < b < 1$ satisfying the following conditions:

(*i*) there exist a compact subset $C \subset E^*$ and constants $\rho_1 > 0, \rho_2 > 0, \ldots, \rho_N > 0$ such that
$(J(K) - \rho_N T_N(\underbrace{K \times \ldots \times K}_{N-\text{times}})) \cup (J(K) - \rho_{N-1} T_{N-1}(\underbrace{K \times \ldots \times K}_{N-\text{times}})) \cup \ldots \cup (J(K) - \rho_1 T_1(\underbrace{K \times \ldots \times K}_{N-\text{times}})) \subset C$, where $J(x_1, x_2, \ldots, x_N) = Jx_N$, $\forall (x_1, x_2, \ldots, x_N) \in \underbrace{K \times \ldots \times K}_{N-\text{times}}$ and

$$\begin{cases} \langle T_1(x_1, x_2, \ldots, x_N), J^{-1}(Jx_N - \rho_1 T_1(x_1, x_2, \ldots, x_N)) \rangle \geq 0, \\ \langle T_2(x_1, x_2, \ldots, x_N), J^{-1}(Jx_N - \rho_2 T_2(x_1, x_2, \ldots, x_N)) \rangle \geq 0, \\ \qquad \vdots \\ \langle T_N(x_1, x_2, \ldots, x_N), J^{-1}(Jx_N - \rho_N T_N(x_1, x_2, \ldots, x_N)) \rangle \geq 0, \end{cases} \tag{3.4.29}$$

for all $x_1, x_2, \ldots, x_N \in K$.

(*ii*) $\lim_{n\to\infty} \alpha_n^{(1)} = d_1 \in (a, b)$, $\lim_{n\to\infty} \alpha_n^{(2)} = d_2 \in (a, b)$, $\ldots$, $\lim_{n\to\infty} \alpha_n^{(N)} = d_N \in (a, b)$. Let $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ be the sequences defined by (3.1.6).

Then the problem (3.1.1) has a solution $(x_1^*, x_2^*, \ldots, x_N^*) \in \underbrace{K \times \ldots \times K}_{N-\text{times}}$ and the sequences $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(N)}\}$ converge strongly to $x_1^*, x_2^*, \ldots, x_N^*$, respectively.

2. Let $E$ be a reflexive Banach space with a Fréchet differentiable norm. Assume that

$(i)$ $K$ is a nonempty compact convex in $E$;

$(ii)$ $T : K \to 2^{E^*}$ is upper semicontinuous;

$(iii)$ $T(x)$ is nonempty closed in $E^*$ and contractible subset in $E$ for each $x \in K$;

$(iv)$ $T(K) = \bigcup_{x \in K} T(x)$ is compact in $E^*$.

Then the $GVI(K,T)$ has solution in $K$.

3. Let $E$ be a reflexive Banach space with a Fréchet differentiable norm and $K$ be a closed convex set in $E$ such that every weakly convergent sequence in $K$ is norm convergent. Let $T : K \to 2^{E^*}$ be an upper semicontinuous multi-valued mapping such that $T(x)$ is nonempty compact and contractible in $E^*$ for any $x \in K$. Suppose that $T(B)$ is compact in $E^*$, for all compact subset $B$ of $K$, and

$(C1)$ Given $\widehat{x} \in E$ and for any $\{x_n\} \subset K$ with $\|x_n\| \to +\infty$ as $n \to +\infty$, and for any $\{u_n\}$ with $u_n \in T(x_n)$, there exist a positive integer $n_0$ and $y \in K$ such that $\|y - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$ and $\langle u_{n_0}, y - x_{n_0} \rangle < 0$.

Then the solution set of $GVI(K,T)$ is nonempty and compact.

4. Let $E$ be a reflexive Banach space with a Fréchet differentiable norm, $K$ be a closed convex set in $E$ such that every weakly convergent sequence in $K$ is norm convergent. Let $T : K \to 2^{E^*}$ be an upper semicontinuous multi-valued mapping such that $T(x)$ is nonempty compact and contractible for any $x \in K$. Suppose that $T(B)$ is compact in $E^*$, for all compact subset $B$ of $K$, and one of the following conditions hold:

$(C2)$ Given $\widehat{x} \in E$ and for any $\{x_n\} \subset K$ with $\|x_n\| \to +\infty$ as $n \to +\infty$, and for any sequence $\{u_n\}$ with $u_n \in T(x_n)$, there exist a positive integer $n_0$ and $y \in K$ such that $\|y - \widehat{x}\| < \|x_{n_0} - \widehat{x}\|$ and $\langle u_{n_0}, y - x_{n_0}\rangle \leq 0$.

$(C3)$ Given $\widehat{x} \in E$, there exists a constant $\rho > 0$ such that, for any $x \in K$ with $\|x - \widehat{x}\| > \rho$, there exist $y \in K$ and $u \in T(x)$ satisfying $\|y - \widehat{x}\| \leq \|x - \widehat{x}\|$ and $\langle u, y - x\rangle < 0$.

$(C4)$ Given $\widehat{x} \in E$, there exists a constant $\rho > 0$ such that, for any $x \in K$ with $\|x - \widehat{x}\| > \rho$, there exists $y \in K$ and $u \in T(x)$ satisfying $\|y - \widehat{x}\| < \|x - \widehat{x}\|$ and $\langle u, y - x\rangle \leq 0$.

Then there exists a solution to $GVI(K, T)$ and the solution set is compact.

5. Let $E$ be a reflexive Banach space with a Fréchet differentiable norm, $K$ be a closed convex set in $E$ such that every weakly convergent sequence in $K$ is norm convergent. Let $f : K \to E^*$ be a continuous mapping. Suppose that one of the following conditions hold:

$(C5)$ Given $\widehat{x} \in E$, for any $\{x_n\} \in K$ where $\|x_n\| \to +\infty$ there exists a positive integer $n_0$ and $y \in K$ with $\|y - \widehat{x}\| < \|x_{n_0} - \widehat{x}\|$ such that $\langle f(x_{n_0}), y - x\rangle \leq 0$.

$(C6)$ Given $\widehat{x} \in E$, for any $\{x_n\} \in K$ where $\|x_n\| \to +\infty$ there exists a positive integer $n_0$ and $y \in K$ with $\|y - \widehat{x}\| \leq \|x_{n_0} - \widehat{x}\|$ such that $\langle f(x_{n_0}), y - x\rangle < 0$.

$(C7)$ Given $\widehat{x} \in E$, there exists a constant $\rho > 0$ such that, for any $x \in K$ with $\|x - \widehat{x}\| > \rho$, there exists $y \in K$ satisfying $\|y - \widehat{x}\| \leq \|x - \widehat{x}\|$ and $\langle f(x), y - x\rangle < 0$.

$(C8)$ Given $\widehat{x} \in E$, there exists a constant $\rho > 0$ such that, for any $x \in K$ with $\|x - \widehat{x}\| > \rho$, there exists $y \in K$ satisfying $\|y - \widehat{x}\| < \|x - \widehat{x}\|$ and $\langle f(x), y - x\rangle \leq 0$.

Then the solution set of variational inequality $VI(K,f)$ is nonempty, closed and bounded.

6. Let $E$ be $q$-uniformly smooth real Banach space. Let $A_i, B_i : E \to E$ be single-valued operators, $H_i : E \times E \to E$ be a single-valued operator satisfying $(A1)$ and $M_i$, $U_i$, $H_i(A_i, B_i)$, $S_i$, $S_i(\cdot, u)$ satisfy conditions $(A2)$-$(A6)$, respectively. If there exists a constant $c_{q,i}$ such that

$$\frac{\sqrt[q]{(r_i + t_i)^q - q\lambda_i s_i + c_{q,i}\lambda_i^q l_i^q}}{\alpha_i - \beta_i} + \frac{\lambda_i m_i}{\alpha_i - \beta_i} < 1 \tag{3.4.30}$$

for all $i = 1, 2, \ldots, N$, then problem (3.3.1) has a solution $a_1, \ldots, a_N$, $u_1 \in U_1(a_N)$, $\ldots, u_N \in U_N(a_1)$.

7. Let $E$ be $q$-uniformly smooth real Banach space. For $i = 1, 2, \ldots, N$. Let $A_i, B_i : E \to E$ be two single-valued operators, $H_i : E \times E \to E$ be a single-valued operator satisfying $(A1)$ and suppose that $M_i$, $U_i$, $H_i(A_i, B_i)$, $S_i$, $S_i(\cdot, u)$ satisfy conditions $(A2)$-$(A6)$, respectively. Then, for any $i \in \{1, 2, \ldots, N\}$, the sequence $\{a_n^1\}_{n=1}^{\infty}$ and $\{u_n^i\}_{n=1}^{\infty}$, generated by Algorithm 3.3.2, converge strongly to $a_i$, $u_i \in U_i(a_{N-(i-1)})$, respectively.

8. Let $X$ be a Hausdorff topological vector space, $K$ be a nonempty compact convex subset of $X$. Let $g : K \times K \to \mathbb{R}$ be a mapping satisfying $(\widehat{A}1)$ and $(\widehat{A}3)$ and let $h : K \times K \to \mathbb{R}$ be a mapping satisfying $(\widehat{B}1)$ and $(\widehat{B}3)$. Let $T : K \to X^*$ be an $\eta$-hemicontinuous and relaxed $\eta$-$\alpha$ monotone mapping satisfying $(\widehat{C}1)$-$(\widehat{C}3)$. Let $A : K \to X^*$ be a monotone and hemicontinuous mapping satisfying $(\widehat{D}1)$. Then, for all $r > 0$ and $z \in K$ there exists $x \in K$ such that

$$g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y - x\rangle + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \quad \text{for all} \quad y \in K.$$

9. Let $H$ be a real Hilbert space, $K$ be a nonempty bounded closed convex subset of $H$. Let $g : K \times K \to \mathbb{R}$ be a mapping satisfying $(\widehat{A}1)$-$(\widehat{A}3)$ and let $h : K \times K \to \mathbb{R}$ be a monotone mapping satisfying $(\widehat{B}1)$-$(\widehat{B}3)$. Let $T : K \to H$

be an $\eta$-hemicontinuous and relaxed $\eta$-$\alpha$ monotone mapping satisfying $(\widehat{C}1)$-$(\widehat{C}3)$. Let $A : K \to H$ be a $\lambda$-inverse-strongly monotone and hemicontinuous mapping satisfying $(\widehat{D}1)$. For $r > 0$ and $z \in K$, define $T_r : K \to 2^K$ by

$$T_r(z) = \Big\{x \in K : g(x,y) + h(x,y) + \langle Tx, \eta(y,x)\rangle + \langle Ax, y-x\rangle + \frac{1}{r}\langle y-x, x-z\rangle \geq 0, \quad \text{for all} \quad y \in K\Big\}.$$

Then, the following results holds:

$(i)$ $\mathrm{dom}T_r = H$;

$(ii)$ $T_r$ is single-valued;

$(iii)$ $T_r$ is firmly nonexpansive i.e., for any $x, y \in K$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x-y\rangle;$$

$(iv)$ $F(T_r) = GMEPRM(g,h,T,A)$;

$(v)$ $GMEPRM(g,h,T,A)$ is closed and convex.

10. Let $H$ be a real Hilbert space, $K$ be a nonempty bounded closed convex subset of $H$. Assume that $g : K \times K \to \mathbb{R}$ satisfies $(\widehat{A}1)$-$(\widehat{A}4)$, and $h : K \times K \to \mathbb{R}$ is a monotone mapping satisfying $(\widehat{B}1)$-$(\widehat{B}4)$. Suppose that $T : K \to H$ satisfies $(\widehat{C}2)$ and $(\widehat{C}4)$, $A : K \to H$ satisfies $(\widehat{D}1)$-$(\widehat{D}3)$ and that the set $GMEPRM(g,h,T,A)$ of solutions (3.4.1) is nonempty. Let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence generated by the form

$$x_0 \in K \quad \text{and} \quad x_{n+1} = T_{r_n}x_n, \quad \text{where} \quad r_n \in (0,+\infty), \quad \text{for all} \quad n \in \mathbb{N}, \tag{3.4.31}$$

where $\sum_{n\in\mathbb{N}} r_n^2 = +\infty$. Then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a point in $GMEPRM(g,h,T,A)$.