EXISTENCE AND ITERATIVE APPROXIMATION THEOREMS FOR GENERALIZED EQUILIBRIUM AND VARIATIONAL INEQUALITY PROBLEMS AND WELL-POSEDNESS FOR SOME PROBLEMS

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ABSTRACT

In this thesis, we introduce and analyze the new generalized mixed equilibrium problems (NGMEP) and the bilevel new generalized mixed equilibrium problems (BNGMEP) in Banach spaces. First, by using a minimax inequality, some new existence theorems of the solution and the behavior of solution set for the NGMEP and the BNGMEP are obtained in Banach spaces. Next, by using auxiliary principle technique, some new iterative algorithms for solving the NGMEP and the BNGMEP are suggested and analyzed. The strong convergence of the iterative sequences generated by the algorithms are also proved in Banach spaces. These results are new and generalize some recent results in this field.

Furthermore, we consider an auxiliary problem for the generalized mixed vector equilibrium problem with a relaxed monotone mapping and prove the existence and uniqueness of the solution for the auxiliary problem. We then introduce a new iterative scheme for approximating a common element of the set of solutions of a generalized mixed vector equilibrium problem with a relaxed monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings. Moreover, we introduce and study a new class of generalized nonlinear vector mixed quasi-variational-like inequality governed by a multi-valued map in Hausdorff topological vector spaces which includes generalized vector mixed general quasi-variational-like inequalities, generalized nonlinear mixed variational-like inequalities, and so on. By using the fixed point theorem, we prove some existence theorems for the purposed inequality.

On the other hand, we introduce the notion of well-posedness to the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term (HVIMN) in Banach spaces. Under very suitable conditions, we establish some metric characterizations for cheking the well-posed (HVIMN). In the setting of finite-dimensional spaces, the strongly generalized well-posedness by perturbations for (HVIMN) are established. Our results are new and improve recent existing ones in the literature.

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CHAPTER I

INTRODUCTION

One of the most important problems in nonlinear analysis is the so called equilibrium problem (abbreviated (EP)), which can be formulated as follows. Let C be a nonempty set and $f : C \times C \to \mathbb{R}$ a given function. The problem consists on finding an element $\tilde{x} \in C$ such that

$$f(\tilde{x}, y) \geq 0, \text{ for all } y \in C. \tag{EP}$$

The element $\tilde{x}$ satisfying (EP) is called equilibrium point of f on C. It is well-known that (EP) has been extensively studied in recent years (e.g. [1, 2, 3, 4, 5] and the references therein). Apart from its theoretical interest, important problems arising from economics, mechanics, electricity and other practical sciences motivate the study of (EP). Equilibrium problems include, as particular cases, optimization problems, saddle point (minimax) problems, variational inequalities, Nash equilibria problems, complementarity problems, fixed point problems, etc. As far as we know the term "equilibrium problem"was attributed in Blum and Oettli [4], but the problem itself has been investigated more than twenty years before in a paper of Ky Fan [6] in connection with the so called "intersection theorems"(i.e., results stating the nonemptiness of a certain family of sets). Ky Fan considered (EP) when C is a compact convex subset of a Hausdorff topological vector space and termed it "minimax inequality". Since that time, the existence theorems for solution of general versions of the equilibrium problem have been wildly studied by many authors, for example, mixed equilibrium problem (MEP) [7, 8], generalized equilibrium problem (GEP) [9], generalized mixed equilibrium problem (GMEP) [10], bilevel equilibrium problem (BEP) [11], vector equilibrium problem (VEP) [12] and so on.

In 2002, Moudafi [13] introduced an iterative scheme of finding the solution of nonexpansive mappings and proved a strong convergence theorem. Recently, Huang et al. [14] introduced the approximate method for solving the equilibrium problem and proved the strong convergence theorem.

It is well known that the vector equilibrium problem provides a unified model of several problems, for example, vector optimization, vector variational inequality, vector complementarity problem, and vector saddle point problem ([12, 15, 16]). In recent years, the vector equilibrium problem has been intensively studied by many authors (see, for example, [12, 15, 16, 17, 18, 19, 20] and the references therein).

Very recently, Shan and Huang [21] introduced an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the generalized mixed vector equilibrium problem and the solution set of a variational inequality problem with a monotone Lipschitz continuous mapping.

On the other hand, it is well-known that various bilevel problems, equilibrium constraint optimization problems, bilevel decision problems, mathematical program problems with equilibrium constraints represent important classes of optimization problems which have been wildly investigated, for example see [22, 23, 24, 25, 26, 27, 28, 29, 30, 31]

Recently, Moudafi [32] studied a class of bilevelmonotone equilibrium problems in Hilbert spaces and suggested an iterative algorithm to compute approximate solutions of the problem and proved the weak convergence of the iterative sequence generated by the algorithm. He pointed out that this class is very interesting because it covers mathematical programs and optimization problems over equilibrium constraints, hierarchical minimization problems, variational inequality problems, complementarity problems and so on. Ding [33] introduced a class of bilevel mixed equilibrium problems in Banach spaces. By using auxiliary principle technique, an iterative algorithm to compute the approximate solutions is suggested and analyzed. Strong convergence of the iterative sequences generated by this algorithms is also proved under quite mild assumptions.

In 1964, variational inequality problems were introduced by Stampacchia [34]. Since then these problems have witnessed explosive growth in theoretical advances, algorithmic development, and applications across all disciplines of pure and applied sciences (see [34, 35] and the references therein). In recent years, variational inequality theory has been extended and generalized in several directions, using new and powerful methods, to study a wide class of unrelated problems in a unified and general framework.

A vector variational inequality in a finite-dimensional Euclidean space was first introduced by Giannessi [36]. This is a generalization of scalar variational inequality to the vector case by virtue of multi-criterion consideration. In 1966, Browder [37] first introduced and proved the basic existence theorems of solutions to a class of nonlinear variational inequalities. The Browder's results was extended to more generalized nonlinear variational inequalities by Liu, et al. [38], Ahmad and Irfan [39], Husain and Gupta [40] and Xiao, et al. [41], Zhao, et al. [42].

On the other hand, the notion of hemivariational inequality was introduced by Panagiotopoulos [43, 44] at the beginning of the 1980s as a variational formulation for several classes of mechanical problems with nonsmooth and nonconvex energy super-potentials. In the case of convex super-potentials, hemivariational inequalities reduce to variational inequalities which were studied earlier by many authors (see e.g. Fichera [45] or Hartman and Stampacchia [46]). Wangkeeree and Preechasilp [47] also introduced and studied some existence results for the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term in reflexive Banach spaces.

Another important topic is the well-posedness which significant for both optimization theory and numerical methods of optimization problems, which guarantees that, for approximating solution sequences, there is a subsequence which converges to a solution. The study of well-posedness originates from Tikhonov [48], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Levitin-Polyak [49] introduced a new notion of well-posedness that strengthened Tykhonov's concept as it required the convergence to the optimal solution of each sequence belonging to a larger set of minimizing sequences.

The important notion of well-posedness for a minimization problem is the well-posedness by perturbations or extended well-posedness due to Zolezzi [50, 51]. The notion of well-posedness by perturbations establishes a form of continuous dependence of the solutions upon a parameter. There are many other notions of well-posedness in optimization problems. For more details, see, e.g., [50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60].

Recently Ceng, et al. [61] considered an extension of the notion of well-posedness by perturbations, introduced by Zolezzi for a minimization problem, to a class of variational-hemivariational inequalities with perturbations in Banach spaces. Under very mild conditions, they established some metric characterizations for the well-posed variational-hemivariational inequality, and proved that the well-posedness by perturbations of a variational hemivariational inequality is closely related to the well-posedness by perturbations of the corresponding inclusion problem. Furthermore, in the setting of finite-dimensional spaces they also derived some conditions under which the variational-hemivariational inequality is strongly generalized well-posed-like by perturbations.

Motivated and inspired by the above works, the purposes of this thesis are to extend, to generalize, to improve existence theorems of generalized equilibrium problems and generalized variational inequality problem and the iteration schemes of some nonlinear operators for finding a common element of the solutions of generalized vector equilibrium problems and fixed point problems and to discuss the well-posedness for generalized hemivariational inequality problem.

This thesis is divided into 6 chapters. Chapter 1 is an introduction to the research problem. Chapter 2 is dealing with some preliminaries and give some useful results that will be deplicated in later Chapter.

Chapter 3 and 4 are the main results of this research. Precisely, in section 3.1, we introduce and analyze the the new generalized mixed equilibrium problems (NGMEP) and the bilevel new generalized mixed equilibrium problems (BNGMEP) in Banach spaces. First, by using a minimax inequality, some new existence theorems of the solution and the behavior of solution set for the NGMEP and the BNGMEP are obtained in Banach spaces. Next, by using auxiliary principle technique, some new iterative algorithms for solving the NGMEP and the BNGMEP are suggested and analyzed. The strong convergence of the iterative sequences generated by the algorithms are also proved in Banach spaces. These results are new and generalize some recent results in this field. In section 3.2, first consider an auxiliary problem for the generalized mixed vector equilibrium problem with a relaxed monotone mapping and prove the existence and uniqueness of the solution for the auxiliary problem. We then introduce a new iterative scheme for approximating a common element of the set of solutions of a generalized mixed vector equilibrium problem with a relaxed monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings. The results presented in this paper can be considered as a generalization of some known results due to Wang, Marino and Wang [62]. In section 3.3, we introduce and study a new class of generalized nonlinear vector mixed quasi-variational-like inequality governed by a multi-valued map in Hausdorff topological vector spaces which includes generalized vector mixed general quasi-variational-like inequalities, generalized nonlinear mixed variational-like inequalities, and so on. By using the fixed point theorem, we prove some existence theorems for the purposed inequality.

Section 4.1, we introduce the notion of well-posedness to the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term (HVIMN) in Banach spaces. Under very suitable conditions, we establish some metric characterizations for the well-posed (HVIMN). In the setting of finite-dimensional spaces, the strongly generalized well-posedness by perturbations for (HVIMN) are established. Our results are new and improve recent existing ones in the literature.

The conclusion of research is in Chapter 5.

CHAPTER II

PRELIMINARIES

This chapter includes some notations, definitions, and some useful results.

2.1 Metric spaces and Banach spaces

In this section, we recall the basic definitions and elementary properties of metric spaces and Banach spaces.

Definition 2.1.1. [63] A metric space is a pair (X, d), where X is a set and d is a metric on X(or distance function on X), that is, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(M_1) $d(x, y) \geq 0$;

(M_2) $d(x, y) = 0$ if and only if $x = y$;

(M_3) $d(x, y) = d(y, x)$ (symmetry);

(M_4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

The element of X are called the point of the metric (X, d).

Definition 2.1.2. [63] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *convergent* if there is an $x \in X$ such that

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

x is called the limit of $\{x_n\}$ and we write

$$\lim_{n \to \infty} x_n = x \text{ or, simple, } x_n \to x. \tag{2.1.1}$$

In this case, we say that $\{x_n\}$ converges to x. If $\{x_n\}$ is not convergent, it is said to be divergent.

Definition 2.1.3. [63] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *Cauchy* if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n \geq N$.

Definition 2.1.4. [63] If every Cauchy sequence in a metric space (X, d) converges then the metric space (X, d) is said to be complete.

The concepts of open, closed and bounded subsets of normed spaces are given as follows.

Definition 2.1.5. [63] Let (X, d) be a metric space and A be a subset of E.

(i) Given a point $x_0 \in X$, the *ball centered at* x_0 *and with radius* $r > 0$ is the set $B(x_0, r) := \{x \in E : d(x_0, x) < r\}$.

(ii) A is open if for each $x_0 \in A$ there exists a $\delta > 0$ such that $B(x_0, \delta) \subseteq A$.

(iii) A is closed if the complement A^c is open.

Theorem 2.1.6. [63] For a subset A of a metric space (X, d). Then A is closed if and only if the situation $x_n \in A$, $x_n \to x$ implies that $x \in A$.

Definition 2.1.7. [63] Let A be a nonempty subset of a metric space (X, d). Then A is said to be bounded if $diam(C) := \sup_{x,y \in A} d(x, y) < +\infty$.

Definition 2.1.8. [63] A metric space (X, d) is said to be *compact* if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact considered as a subspace of X, that is, every sequence in M has a convergent subsequence whose limit is an element in M.

Definition 2.1.9. [63] A *norm* on a (real or complex) vector space E is a real-valued function on E whose valued at an $x \in E$ is denoted by $\|x\|$ and which has the properties

(N_1) $\|x\| \geq 0$;

(N_2) $\|x\| = 0 \Leftrightarrow x = 0$;

(N_3) $\|\alpha x\| = |\alpha|\|x\|$;

(N_4) $\|x + y\| \geq \|x\| + \|y\|$,

where x and y are arbitrary vectors in E and α is any scalar. A normed space E is a vector space with a norm defined on it which is denoted by $(E, \|\cdot\|)$ or simply by E.

Convergence of sequences and related concepts in normed spaces follow from the corresponding definition 2.1.2 and 2.1.3 for metric spaces and the fact that now $d(x, y) = \|x - y\|$.

Definition 2.1.10. [63] A Banach space is a complete normed space.

Definition 2.1.11. [64] Let A be a subset of normed space E. Then A is said to be *convex* if $(1 - \lambda)x + \lambda y \in A$ for all $x, y \in A$ and all scalar $\lambda \in [0, 1]$.

Next, we discuss some properties of linear operators.

Definition 2.1.12. [63] Let X and Y be linear spaces over the field $\mathbb{K}$.

(i) A mapping $T : X \to Y$ is called *a linear operator* if for all $x, y \in X$ and $\alpha \in \mathbb{K}$,

$$T(x + y) = Tx + Ty \quad \text{and} \quad T(\alpha x) = \alpha Tx,$$

(ii) A mapping $T : X \to \mathbb{K}$ is called *a linear functional on* X if T is a linear operator.

Definition 2.1.13. [63] Let X and Y be normed spaces over the field $\mathbb{K}$ and $T : X \to Y$ a linear operator. T is said to be *bounded* on X, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|, \forall x \in X$.

Definition 2.1.14. [63] Let E and Y be normed spaces over the field $\mathbb{K}$, $T : E \to Y$ an operator and $x_0 \in E$. We say that T is *continuous at* x_0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(x_0)\| < \varepsilon$ whenever $\|x - x_0\| < \delta$ and $x \in E$. If T is continuous at each $x \in E$, then T is said to be *continuous on* E.

Definition 2.1.15. [63] Let E be a normed space. Then the set of all bounded linear functionals on E is called *a dual space* of E and is denoted by E^*.

Weak convergence is defined in terms of bounded linear functionals on E as follows.

Definition 2.1.16. [63] A sequence $\{x_n\}$ is a normed space E is said to be *weakly convergent* if there exists an $x \in E$ such that for every $f \in E^*$,

$$\lim_{n\to\infty} f(x_n) = f(x).$$

This is written $x_n \rightharpoonup x$. The element x is called the weak limit of $\{x_n\}$, and we say that $\{x_n\}$ converges weakly to x.

A subset C of E is *weakly closed* if it is closed in the weak topology, that is, if it contains the weak limit of all of its weakly convergent sequences. The *weakly open* sets are now taken as those sets whose complements are weakly closed. The resulting topology on E is called the *weak topology* on E. Sets which are compact in this topology are said to be *weakly compact*.

2.2 Topological vector spaces.

Definition 2.2.1. [?] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function. Then d is called a *metric* on X if the following properties hold:

1) $d(x, y) = 0$ if and only if $x = y$,

2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The value of metric d at (x, y) is called *distance between x and y,* and the ordered pair (X, d) is called *a metric space.*

Definition 2.2.2. [?] Let X be a nonempty set and τ be a collection of subsets of X. Then τ is said to be *a topology* on X if the following conditions are satisfied:

1) $\emptyset \in \tau$ and $X \in \tau$;

2) the union of every class of sets in τ is a set in τ;

3) the intersection of every finite class of sets in τ is a set in τ.

The ordered pair (X, τ) is called *a topological space* and the sets in class τ are called the *open sets* of the topological (X, τ). It is customary to denote the topological space (X, τ) by the symbol X which is used for its underlying set of points.

Definition 2.2.3. [63] Let X be a topological space, let U be a subset of X and let some $x \in X$ be a given element. The set U is called *a neighborhood of x*, if there is an open set V with $x \in V \subset U$ and x is called *an interior element of U*, if there is a neighborhood V of x contained in U. The set of all interior elements of U is called the interior of U and it is denoted by $intU$.

Definition 2.2.4. [63] A set F in a topological space X whose complement $F^C = X - F$ is open is called *a closed set.*

Definition 2.2.5. [63] Let F be a subset of a topological space X. Then the closure of F is the smallest closed set containing F. The closure of F is denoted by $\overline{F}$.

Theorem 2.2.6. [63] *Let F be a subset of a topological space X. Then F is closed if and only if $F = \overline{F}$.*

Definition 2.2.7. [63] Let X be a topological space. Then X is said to be *Hausdorff topological space* if x and y are two distinct points in X, there exist two open sets G and H such that $x \in G$, $y \in H$, and $G \cap H = \emptyset$.

Remark 2.2.8. [63] Every metric space is a Hausdorff space.

Definition 2.2.9. A topological vector space is locally convex if it has a base of its topology consisting of convex open subsets.

Definition 2.2.10. [65] Let X be a linear topological space over the filed $\mathbb{R}$.

1) A sequence $\{x_n\} \subset X$ is bounded if $\lambda_n x_n \to \theta$ whenever $\lambda_n \to 0$ in $\mathbb{R}$.

2) A set $A \subset X$ is bounded if every sequence in A is bounded.

Definition 2.2.11. [?] A topological space X is said to be *compact* if every open cover has a finite subcover, i.e., if whenever $X = \bigcup_{i\in I} G_i$, where G_i is an open set, then $X = \bigcup_{i\in J} G_i$ for some finite subset J of I.

Definition 2.2.12. [?] A subset C of a topological space X is said to be *compact* if every open cover has a finite open subcover, i.e., if whenever $C \subseteq \bigcup_{i\in I} G_i$, where G_i is an open set, then $C \subseteq \bigcup_{i\in J} G_i$ for some finite subset J of I.

Remark 2.2.13. [?]

1) Every finite set of a topological space is compact.

2) Every closed subset of a compact space is compact.

3) In a compact Hausdorff space, a set is compact if and only if it is closed.

Definition 2.2.14. [63] *A linear space or vector space* X over the field $\mathbb{K}$ (The real field $\mathbb{R}$ or the complex field $\mathbb{C}$) is a set X together with an internal binary operation "+" called an *addition* and a *scalar multiplication* carrying (α, x) in $\mathbb{K} \times X$ to αx in X satisfying the following for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$:

1) $x + y = y + x$,

2) $(x + y) + z = x + (y + z)$,

3) there exists an element $0 \in X$ called the *zero vector* of X such that $x + 0 = x$ for all $x \in X$,

4) for every element $x \in X$, there exists an element $-x \in X$ called the *additive inverse or the negative* of x such $x + (-x) = 0$,

5) $\alpha(x+y) = \alpha x + \alpha y$,

6) $(\alpha+\beta)x = \alpha x + \beta x$,

7) $(\alpha\beta)x = \alpha(\beta x)$,

8) $1 \cdot x = x$.

The elements of a vector space X are called *vector* and the elements of $\mathbb{K}$ are called *scalars.* In the sequel, unless otherwise stated, X denotes a linear space over field $\mathbb{R}$.

Definition 2.2.15. [63] A subset C of a linear space X is said to be *a convex set in* X if $\lambda x + (1-\lambda)y \in C$ for each $x, y \in C$ and for each scalar $\lambda \in [0,1]$.

Definition 2.2.16. [63] Let X be a linear space over a field $\mathbb{K}$ and let τ be a topology on X. Then (X, τ) is called *a topological linear space or a topological vector space* if addition and multiplication with scalar are continuous, i.e. the maps

$$(x,y) \mapsto x+y \text{ with } x,y \in X$$
$$\text{and} \quad (\alpha, x) \mapsto \alpha x \text{ with } \alpha \in \mathbb{K} \text{ and } x \in X$$

are continuous on $X \times X$ and $\mathbb{K} \times X$, respectively. In many situations we use, for simplicity, the notation X instead of (X, τ) for a topological linear space.

Definition 2.2.17. [66] Let C be a nonempty subset of a real linear space X.

1) The set C is called a *cone*, if $x \in C, \lambda > 0$ then $\lambda x \in C$.

2) A cone C is called *pointed*, if $C \cap (-C) = \{0\}$.

3) A cone C is *convex*, if $C + C \subset C$.

4) A cone C is *proper* if and only if $0 \notin C$.

Definition 2.2.18. [67] Let A and B be two topological vector spaces and $T : A \to 2^B$ be a multivalued mapping, then

1. T is said to be upper semicontinuous, if for any $x_0 \in A$ and for each open set U in B containing $T(x_0)$, there is a neighborhood V of x_0 in A such that $T(x) \subset U$ for all $x \in V$.

2. T is said to have open lower sections if the set $T^{-1}(y) = \{x \in A : y \in T(x)\}$ is open in X for each $y \in B$.

3. T is said to be closed, if any net $\{x_\alpha\}$ in A such that $x_\alpha \to x$ and any $\{y_\alpha\}$ in B such that $y_\alpha \to y$ and $y_\alpha \in T(x_\alpha)$ for any α, we have $y \in T(x)$.

4. T is said to be lower semicontinuous, if for any $x_0 \in A$ and for each open set U in B containing $T(x_0)$, there is a neighborhood V of x_0 in A such that $T(x) \cap U \neq \emptyset$ for all $x \in V$.

5. T is said to be continuous if it is both lower and upper semicontinuous.

Definition 2.2.19. Let E is a topological vector space. The bifunction $\varphi : E \times E \to \mathbb{R}$ is said to be *skew-symmetric* if

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) \geq 0, \qquad \forall u, v \in E.$$

The skew-symmetric bifunctions have the properties which can be considered an analogs of monotonicity of gradient and nonnegativity of a second derivative for the convex function. For the properties and applications of the skew-symmetric bifunction, the reader may consult Antipin.

Lemma 2.2.20. [68, Aubin and Cellina] Let X and Y be topological spaces and $T : X \to 2^Y$ be a set-valued mapping.

(i) If X is compact and T is upper semicontinuous with compact values, then $T(X)$ is compact.

(ii) If Y is compact and T is closed, then T is upper semicontinuous.

(iii) If T is upper semicontinuous with closed values, then T is closed.

The following result are Theorem 7.3.11 and Theorem 7.3.14 of Klein and Thompson.

Lemma 2.2.21. *Let X, Y and Z be topological spaces. Let $F : X \to 2^Y$ and $G : Y \to 2^Z$ be set-valued mappings.*

(i) If F and G are upper semicontinuous, then so is $G \circ F : X \to 2^Z$.

(ii) If F and G are lower semicontinuous, then so is $G \circ F : X \to 2^Z$.

Lemma 2.2.22. *Let I be an index set. Let X and $Y_i, i \in I$ be all topological spaces. For each $i \in I$, let $F_i : X \to 2^{Y_i}$ be set-valued mappings. Let $F = \prod_{i\in I} F_i$ be defined by $F(x) = \prod_{i\in I} F_i(x)$. If each F_i is upper continuous with compact values, then F is also upper semicontinuous with compact values with respect to the product topology on $Y = \prod_{i\in I} Y_i$.*

Lemma 2.2.23. [66, Lin and yu] Let X and Y be two topological space. Let $F : X \times Y \to \mathbb{R}$ be a bifunction and $S : X \to 2^Y$ be a set-valued mapping with nonempty values and let $m(x) = \sup_{u\in S(x)} F(u, x)$.

(i) If F and S are both lower semicontinuous, then m is also lower semicontinuous.

(ii) If F is upper semicontinuous and S is upper semicontinuous with compact values, then m is also upper semicontinuous.

Lemma 2.2.24. [31] *Let C be a nonempty convex subset of a topological vector space and let $f : C \times C \to [-\infty, \infty]$ be such that*

(i) *$f(x, x) \geq 0$ for each $x \in C$;*

(ii) *for each $y \in C, x \mapsto f(x, y)$ is upper semicontinuous on each nonempty compact subset of C;*

(iii) *for each $x \in C, y \mapsto f(x, y)$ is convex;*

(iv) *there exist a nonempty compact subset K of C and $y \in K$ such that $f(x, y) < 0, \forall x \in C \setminus K$.*

Then there exist a point $\hat{x} \in K$ such that $f(\hat{x}, y) \geq 0$ for all $y \in C$.

Lemma 2.2.25. *[67] Let A and B be two topological spaces. Suppose $T : A \to 2^B$ and $H : A \to 2^B$ are multivalued mappings having open lower sections, then*

(i) $G : A \to 2^B$ defined by for each $x \in A$, $G(x) = co(T(x))$ has open lower sections;

(ii) $\rho : A \to 2^B$ defined by for each $x \in A$, $\rho(x) = T(x) \cap H(x)$ has open lower sections.

Lemma 2.2.26. [72] Let A and B be two topological spaces and $T : A \to 2^B$ be a upper semicontinuous mapping with compact values. Suppose $\{x_\alpha\}$ is a net in A such that $x_\alpha \to x_0$. If $y_\alpha \in T(x_\alpha)$ for each α, then there is a $y_0 \in T(x_0)$ and a subset $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.

Let I be an index set, E_i a Hausdorff topological vector space for each $i \in I$. Let K_i be a family of nonempty compact convex subsets in E_i. Let $K = \prod_{i\in I} K_i$ and $E = \prod_{i\in I} E_i$.

Lemma 2.2.27. [39] For each $i \in I$, let $T_i : K \to 2^{K_i}$ be a set-valued mapping. Assume that the following conditions hold.

(i) For each $i \in I, T_i$ is a convex set-valued mapping;

(ii) $K = \cup\{\text{int}T_i^{-1}(x_i) : x_i \in K_i\}$.

Then there exists $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x}) = \prod_{i\in I} T_i(\bar{x}_i)$, that is $\bar{x}_i \in T_i(\bar{x}_i)$ for each $i \in I$, where $\bar{x}_i$ is the projection of $\bar{x}$ onto K_i.

Definition 2.2.28. [73, 74] Let X and Y be two Hausdorff topological vector spaces, E a nonempty, convex subset of X and C a closed, convex and pointed cone of Y with int$C \neq \emptyset$. Let θ be the zero point of Y, $\mathbb{U}(\theta)$ be the neighborhood set of θ, $\mathbb{U}(x_0)$ be the neighborhood set of x_0, and $f : E \to Y$ be a mapping.

(1) If for any $V \in \mathbb{U}(\theta)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V + C, \forall x \in U \cap E,$$

then f is called *upper $C-$continuous* on x_0. If f is *upper $C-$continuous* for all $x \in E$, then f is called *upper $C-$continuous* on E.

(2) If for any $V \in \mathbb{U}(\theta)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V - C, \forall x \in U \cap E,$$

then f is called *lower $C-$continuous* on x_0. If f is *lower $C-$continuous* for all $x \in E$, then f is called *lower $C-$continuous* on E.

(3) f is called *$C-$continuous* if f is upper $C-$continuous and lower $C-$continuous.

(4) If for any $x, y \in E$ and $t \in [0,1]$, the mapping f satisfies

$$f(x) \in f(tx + (1-t)y) + C \quad \text{or} \quad f(y) \in f(tx + (1-t)y) + C,$$

then f is called *proper $C-$quasiconvex.*

(5) If for any $x_1, x_2 \in E$ and $t \in [0,1]$, the mapping f satisfies

$$tf(x_1) + (1-t)f(x_2) \in f(tx_1 - (1-t)x_2) + C,$$

then f is called *$C-$convex.*

Remark 2.2.29. How to reduce upper $C-$continuity to upper semicontinuity.

Proof. Upper C-continuity of f if and only if for any $V \in \mathbb{U}(\theta)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V + C, \qquad \forall x \in U \cap E.$$

We consider $C = (-\infty, 0]$ and for each $\varepsilon > 0$, V is open interval $(-\varepsilon, +\varepsilon)$.

We can see that

$$V + C = (-\infty, +\varepsilon).$$

From the assumption, we can get that

$$f(x) - f(x_0) < +\varepsilon \qquad \forall x \in U.$$

This implies that f is upper semicontinuous. □

Remark 2.2.30. How to reduce lower $C-$continuity to lower semicontinuity.

Proof. Upper C-continuity of f if and only if for any $V \in \mathbb{U}(\theta)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V - C, \qquad \forall x \in U \cap E.$$

We consider $C = [0, +\infty)$ and for each $\varepsilon > 0$, V is open interval $(-\varepsilon, +\varepsilon)$.

We can see that

$$V + C = (-\varepsilon, +\infty).$$

From the assumption, we can get that

$$f(x_0) - f(x) > -\varepsilon \qquad \forall x \in U.$$

This implies that f is lower semicontinuous. □

Remark 2.2.31. How to reduce $C-$convexity to convexity.

Proof. $C-$convexity if and only if for any $x_1, x_2 \in E$ and $t \in [0, 1]$, the mapping f satisfies

$$tf(x_1) + (1-t)f(x_2) \in f(tx_1 - (1-t)x_2) + C.$$

We consider $C = [0, +\infty)$, we can get that

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 - (1-t)x_2) \in [0, +\infty).$$

That is

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 - (1-t)x_2) \geq 0.$$

This implies that

$$f(tx_1 - (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$

Hence f is convex. □

Lemma 2.2.32. [18] *Let Z and Y be two real Hausdorff topological vector spaces, X is a nonempty, compact, convex subset of Z, and C is a closed, convex and pointed cone of Y. Assume that $f : X \times X \to Y$ and $\psi : X \to Y$ are two vector valued mappings. Suppose that f and ψ satisfy*

1. *$f(x,x) \in C$, for all $x \in X$;*

2. *ψ is upper $C-$continuous on X;*

3. *$f(\cdot,y)$ is lower $C-$continuous for all $y \in X$;*

4. *$f(x,\cdot)+\psi(\cdot)$ is proper $C-$quasiconvex for all $x \in X$.*

Then there exists a point $x \in X$ satisfying

$$F(x,y) \in C \setminus \{0\}, \qquad \forall y \in X,$$

where

$$F(x,y) = f(x,y) + \psi(y) - \psi(x), \qquad \forall x,y \in X.$$

2.3 Some nonlinear operators

In this section, we first recall some definitions related to the single-valued and multi-valued operators. Throughout of this section, let E be a Banach space with the norm $\|\cdot\|$, E^* be its dual and let $\langle \cdot,\cdot \rangle$ denote the duality pairing of E^* and E.

Definition 2.3.1. [64] Let $f : E \to (-\infty,\infty]$ be a function and $\{x_n\} \subset E$. Then f is said to be

(i) *lower semicontinuous* on E if for any $x_0 \in E$, $f(x_0) \leq \liminf_{n\to\infty} f(x_n)$ whenever $x_n \to x_0$.

(ii) *upper semi (or hemi) continuous* on E if for any $x_0 \in E$, $\limsup_{n\to\infty} f(x_n) \leq f(x_0)$ whenever $x_n \to x_0$.

(iii) *weakly lower semicontinuous* on E if for any $x_0 \in E$, $f(x_0) \leq \liminf_{n\to\infty} f(x_n)$ whenever $x_n \rightharpoonup x_0$.

(iv) *weakly upper semicontinuous* on E if for any $x_0 \in E$, $\limsup_{n\to\infty} f(x_n) \leq f(x_0)$ whenever $x_n \rightharpoonup x_0$.

The following definition of continuity for multi-valued mappings can be founded in [75].

Definition 2.3.2. [64] Let $T : E \to E$ be a mapping.

(i) T is said to be *Lipschitzian* if there exists a constant $L \geq 0$ such that for all $x, y \in E$,

$$\|Tx - Ty\| \leq L\|x - y\|.$$

(ii) T is said to be *contraction* if there exists a constant $0 \leq \alpha < 1$ such that for all $x, y \in E$,

$$\|Tx - Ty\| \leq \alpha\|x - y\|.$$

(iii) T is said to be *nonexpansive* if for all $x, y \in E$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

Definition 2.3.3. [64] An element $x \in E$ is said to be

(i) a *fixed point* of a mapping $T : E \to E$ provided $Tx = x$.

(ii) a *common fixed point* of two mappings $S, T : X \to X$ provided $Sx = x = Tx$.

The set of all fixed points of T is denoted by $F(T)$.

Definition 2.3.4. Let C be a closed convex subset of a Hausdorff topological vector space X and $F : C \times C \to \mathbb{R}$ be a real valued bifunction.

(i) F is said to be monotone if

$$F(x, y) + F(y, x) \leq 0, \qquad \forall x, y \in C.$$

(ii) When $X = E$ is a Banach space, F is said to be $\alpha-$strongly monotone if there exists a $\alpha > 0$ such that

$$F(x, y) + F(y, x) \leq -\alpha\|x - y\|, \qquad \forall x, y \in C.$$

(iii) When $X = E$ is a Banach space, F is said to be $\delta-$Lipschitz if there exists a $\delta > 0$ such that

$$|F(x, y)| \leq \delta \|x - y\|, \qquad \forall x, y \in C.$$

Remark 2.3.5. Clearly, strong monotonicity of F implies monotonicity of F.

Definition 2.3.6. A mapping $G : E \to E^*$ is said to be

(i) Monotone if

$$\langle G(x) - G(y), x - y \rangle \geq 0, \qquad \forall x, y \in E;$$

(ii) $\lambda-$strongly monotone if there exists a $\lambda > 0$ such that

$$\langle G(x) - G(y), x - y \rangle \geq \lambda \|x - y\|^2, \qquad \forall x, y \in E;$$

(iii) $\beta-$Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|G(x) - G(y)\| \leq \beta \|x - y\|, \qquad \forall x, y \in E.$$

(iv) $\alpha-$strongly monotone if there exists a $\alpha > 0$ such that

$$\langle G(x) - G(y), x - y \rangle \geq \alpha \|Ax - Ay\|^2, \qquad \forall x, y \in E;$$

Definition 2.3.7. Let $G : E \to E^*$ be a bounded linear operator. G is said to be $\lambda-$strongly positive if there exists a $\lambda > 0$ such that

$$\langle G(x), x \rangle \geq \lambda \|x\|^2, \qquad \forall x \in E.$$

Remark 2.3.8. It is clear that if a bounded linear operator $G : E \to E^*$ is $\lambda-$strongly positive, then G is $\lambda-$strongly monotone and $\|G\|-$Lipschitz continuous where $\|G\|$ is the operator norm of G.

Definition 2.3.9. (see[76]) Let A, B be nonempty subsets of E. The Hausdorff metric $H(\cdot, \cdot)$ between A and B is defined by

$$H(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) := \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} \|a - b\|$.

Let $\{A_n\}$ be a sequence of nonempty subsets of E. We say that A_n converges to A in the sense of Hausdorff metric if $H(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all section $a_n \in A_n$. For more details on this topic, we refer the readers to [76].

Definition 2.3.10. Let C be a nonempty subset of a Banach space $E, T : C \to CB(E^*)$ be a set-valued mapping. T is said to be $H-$Lipschitz continuous if there exists $k > 0$ such that

$$H(T(x), T(y)) \leq k\|x - y\|, \qquad \forall x, y \in C,$$

where H is the Hausdorff metric.

Let H be a Hilbert space and X be a nonempty closed convex subset of H. A mapping $T : X \to H$ is said to be relaxed $\eta - \alpha$ monotone if there exist a mapping $\eta : X \times X \to H$ and a function $\alpha : H \to \mathbb{R}$ positively homogeneous of degree p, that is, $\alpha(tz) = t^p\alpha(z)$ for all $t > 0$ and $z \in H$ such that

$$\langle Tx - Ty, \eta(x, y)\rangle \geq \alpha(x - y), \qquad \forall x, y \in X, \tag{2.3.1}$$

where $p > 1$ is a constant; see [77]. In the case of $\eta(x, y) = x - y$ for all $x, y \in X, T$ is said to be relaxed $\alpha-$monotone. In the case of $\eta(x, y) = x - y$ for all $x, y \in X$ and $\alpha(z) = k\|z\|^p$, where $p > 1$ and $k > 0, T$ is said to be $p-$monotone; see [73, 78, 79]. In fact, in this case, if $p = 2$, then T is a $k-$strongly monotone mapping. Moreover, every monotone mapping is relaxed $\eta - \alpha$ monotone with $\eta(x, y) = x - y$ for all $x, y \in X$ and $\alpha = 0$.

Let $A : X \to H$ be a $\lambda-$inverse-strongly monotone mapping of H, that is there exists a $\lambda > 0$ such that

$$\langle A(x) - A(y), x - y\rangle \geq \lambda\|A(x) - A(y)\|^2, \qquad \forall x, y \in E.$$

For all $z, y \in X$ and $k > 0$, one has [80]

$$\|(I - kA)z - (I - kA)y\|^2 \leq \|z - y\|^2 + k(k - 2\lambda)\|Az - Ay\|^2.$$

Hence, if $k \in (0, 2\lambda)$, then $I - kA$ is a nonexpansive mapping of X into H.

For each point $z \in H$, there exists a unique nearest point of X, denoted by $P_X z$, such that

$$\|z - P_X z\| \leq \|z - y\|,$$

for all $y \in X$. Such a P_X is called the metric projection from H onto X. The well-known Browder's characterization of P_X ensures that P_X is a firmly nonexpansive mapping from H onto X, that is,

$$\|P_X z - P_X y\|^2 \leq \langle P_X z - P_X y, z - y\rangle, \qquad \forall z, y \in H.$$

Further, we know that for any $z \in H$ and $x \in X, x = P_X z$ if and only if

$$\langle z - x, x - y\rangle \geq 0, \qquad \forall y \in X.$$

Let S be a nonexpansive mapping of X into itself such that $F(S) \neq \emptyset$. Then we have

$$\hat{x} \in F(S) \Leftrightarrow \|Sx - x\|^2 \leq 2\langle x - Sx, x - \hat{x}\rangle, \qquad \forall x \in X, \tag{2.3.2}$$

which is obtained directly from the following:

$$\begin{aligned} \|x - \hat{x}\|^2 &\geq \|Sx - S\hat{x}\|^2 = \|Sx - \hat{x}\|^2 = \|Sx - x + (x - \hat{x})\|^2 \\ &= \|Sx - x\|^2 + \|x - \hat{x}\|^2 + 2\langle Sx - x, x - \hat{x}\rangle. \end{aligned}$$

This inequality is a very useful characterization of $F(S)$. Observe what is more that it immediately yields that $\mathrm{Fix}(S)$ is a convex closed set.

Let E be a Banach space with norm denoted by $\|\cdot\|$ and E^* be its dual space. Let K be a nonempty subset of E. For every $r > 0$, we define

$$B_r = \{x \in K : \|x\| < r\}.$$

We recall that a function $\varphi : E \to \mathbb{R}$ is called locally Lipschitz if for every $x \in E$ there exists a neighborhood U of x and a constant $L_x \geq 0$, so called Lipschitz constant, such that

$$|\varphi(v) - \varphi(w)| \leq L_x\|v - w\|, \quad \forall v, w \in U.$$

Recall that $g^\circ(x; v)$ denotes Clarke's generalized directional derivative of the locally Lipschitz mapping $g : E \to \mathbb{R}$ at the point $x \in E$ with respect to the direction $v \in E$, while $\partial g(x)$ is the Clarke's generalized gradient of g at $x \in E$ (see [81]), i.e.,

$$g^\circ(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{g(y + tv) - g(y)}{t}$$

and

$$\partial g(x) = \{\xi \in E^* : \langle \xi, v \rangle \leq g^\circ(x; v), \forall v \in E\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E^* and E.

The following useful results can be found in [81].

Proposition 2.3.11. *Let X be a Banach space, $x, y \in X$ and $j^\circ(x, \cdot)$ be a locally Lipschitz functional defined on X Then*

(i) The function $y \mapsto j^\circ(x, y)$ is finite, positively homogeneous, subadditive and then convex on X;

(ii) $j^\circ(x, y)$ is upper semicontinuous as a function of (x, y), as a function of y alone, is Lipschitz continuous on X;

(iii) $j^\circ(x, -y) = (-j)^\circ(x, y)$;

(iv) $\bar{\partial} j(x)$ is a nonempty, convex, bounded, weak-compact subset of X^*;*

CHAPTER III

ON THE EXISTENCE THEOREMS AND ITERATIVE ALGORITHMS OF GENERALIZED EQUILIBRIUM PROBLEMS AND GENERALIZED VARIATIONAL INEQUALITY PROBLEMS

3.1 Existence and algorithms for the bilevel new generalized mixed equilibrium problems in Banach spaces

In this work, we assume that C be a nonempty closed convex subset of Banach space B, let $K, G : C \times C \to \mathbb{R}$ be a bifunction, $A, T, Q, S : C \to C(B^*)$ be set-values mappings, $N, M : B^* \times B^* \to B^*$ and $H, F : B^* \times C \times C \to \mathbb{R}$. Let $\psi, \varphi : C \times C \to \mathbb{R}$ be a skew-symmetric bifunction and $\omega^*, \tau^* \in B^*$ We will consider the following bilevel new generalized mixed equilibrium problems :

$$\text{(BNGMEP)}\begin{cases} \text{find } \bar{x} \in S_{G,F,N}^{A,T,\varphi} \text{ and } (\bar{u}, \bar{v}) \in Q(\bar{x}) \times S(\bar{x}) \text{ such that} \\ K(\bar{x}, y) + H(M(\bar{u}, \bar{v}) - \tau^*, \bar{x}, y) + \psi(y, \bar{x}) - \psi(\bar{x}, \bar{x}) \geq 0, \end{cases} \tag{3.1.1}$$

for all $y \in S_{G,F,N}^{A,T,\varphi}$, where $S_{G,F,N}^{A,T,\varphi}$ is the solution set of the following new generalized mixed equilibrium problem involving set-valued mapping :

$$\text{(NGMEP)}\begin{cases} \text{find } x \in C \text{ and } (u, v) \in T(x) \times A(x) \text{ such that} \\ G(x, y) + F(N(u, v) - \omega^*, x, y) + \varphi(y, x) - \varphi(x, x) \geq 0, \qquad \forall y \in C. \end{cases} \tag{3.1.2}$$

Special Cases

(I) Given $\eta, \bar{\eta} : B \times B \to B$, let

$$H(M(\bar{u},\bar{v}) - \tau^*, \bar{x}, y) = \sup_{(m,n)\in Q(\bar{x})\times S(\bar{x})} \langle M(m,n) - \tau^*, \bar{\eta}(y,\bar{x})\rangle$$

and $F(N(u,v) - \omega^*, x, y) = \sup_{(q,r)\in A(x)\times T(x)} \langle N(q,r) - \omega^*, \eta(y,x)\rangle$. Then (BNGMEP) (3.1.1)−(3.1.2) reduces to the following bilevel generalized mixed equilibrium problem (BGMEP) involving generalized mixed variational-like inequality problems

find $\bar{x} \in S^{N,\eta}_{F,\varphi}$ such that

$$K(\bar{x},y) + \sup_{(m,n)\in Q(\bar{x})\times S(\bar{x})} \langle M(m,n) - \tau^*, \bar{\eta}(y,\bar{x})\rangle + \psi(y,\bar{x}) - \psi(\bar{x},\bar{x}) \geq 0,$$

for all $y \in S^{N,\eta}_{F,\varphi}$, where $S^{N,\eta}_{F,\varphi}$ is the solution set of the following generalized mixed equilibrium problem (GMEP) involving generalized mixed variational-like inequality problems:

find $x \in C$ such that

$$G(x,y) + \sup_{(q,r)\in A(x)\times T(x)} \langle N(q,r) - \omega^*, \eta(y,x)\rangle + \varphi(y,x) - \varphi(x,x) \geq 0,$$

for all $y \in C$. This problem has been considered in Ding [85].

(II) Let $K, G = 0$ and $\omega^*, \tau^* = 0$. Then (BNGMEP) (3.1.1)−(3.1.2) reduces to the following bilevel generalized mixed equilibrium problem involving set-valued mappings (BGMEP)

$$\begin{aligned} &\text{find } \bar{x} \in S^{A,T}_{F,N,\varphi} \text{ and } (\bar{u},\bar{v}) \in Q(\bar{x}) \times S(\bar{x}) \\ &\text{such that } H(M(\bar{u},\bar{v}),\bar{x},y) + \psi(y,\bar{x}) - \psi(\bar{x},\bar{x}) \geq 0, \forall y \in S^{A,T}_{F,N,\varphi}, \end{aligned} \tag{3.1.3}$$

where $S^{A,T}_{F,N,\varphi}$ is the solution set of the following generalized mixed equilibrium problem involving set-valued mapping (GMEP):

$$\begin{aligned} &\text{find } x \in C \text{ and } (u,v) \in A(x) \times T(x) \\ &\text{such that } F(N(u,v),x,y) + \varphi(y,x) - \varphi(x,x) \geq 0, \forall y \in C. \end{aligned} \tag{3.1.4}$$

The BGMEP (3.1.3) – (3.1.4) was introduced and studied by Ding [33] in Banach spaces.

(III) Given $\eta : C \times C \to B$, let $H(M(\bar{u},\bar{v}),x,y) = \langle M(\bar{u},\bar{v}),\eta(y,x)\rangle$ and

$$F(N(u,v),x,y) = \langle N(u,v),\eta(y,x)\rangle$$

for all $x,y \in C, (\bar{u},\bar{v}) \in Q(x) \times S(x)$ and $(u,v) \in A(x) \times T(x)$. Then the BGMEP (3.1.3) – (3.1.4) reduces to the following bilevel generalized mixed variational-like inequality problem (BGMVLIP):

$$\begin{aligned} &\text{find } \bar{x} \in S^{A,T}_{N,\eta,\varphi} \text{ and } (\bar{u},\bar{v}) \in Q(\bar{x}) \times S(\bar{x}) \\ &\text{such that } \langle M(\bar{u},\bar{v}),\eta(y,\bar{x})\rangle + \psi(y,\bar{x}) - \psi(\bar{x},\bar{x}) \geq 0, \forall y \in S^{A,T}_{N,\eta,\varphi}, \end{aligned} \tag{3.1.5}$$

where $S^{A,T}_{N,\eta,\varphi}$ is the solution set of the following generalized mixed variational-like inequality problem (GMVLIP):

$$\begin{aligned} &\text{find } x \in C \text{ and } (u,v) \in A(x) \times T(x) \\ &\text{such that } \langle N(u,v),\eta(y,x)\rangle + \varphi(y,x) - \varphi(x,x) \geq 0, \forall y \in C. \end{aligned} \tag{3.1.6}$$

The BGMVLIP (3.1.5) – (3.1.6) includes the most of the generalized mixed quasi variational-like inequality problems and generalized mixed quasi variational-like inequality problems studied by many authors in Hilbert and Banach spaces respectively as special cases, see [86, 87, 88, 89, 90, 91, 92].

(IV) If $N(u,v) = u$ and $M(u,v) = u$ for all $(u,v) \in B^* \times B^*$, then the BGMVLIP (3.1.5) – (3.1.6) reduces to the following bilevel generalized mixed variational-like inequality problem (BGMVLIP):

$$\begin{aligned} &\text{find } \bar{x} \in S^{A}_{N,\eta,\varphi} \text{ and } \bar{u} \in Q(\bar{x}) \\ &\text{such that } \langle \bar{u},\eta(y,\bar{x})\rangle + \psi(y,\bar{x}) - \psi(\bar{x},\bar{x}) \geq 0, \forall y \in S^{A}_{N,\eta,\varphi}, \end{aligned} \tag{3.1.7}$$

where $S^{A}_{N,\eta,\varphi}$ is the solution set of the following generalized mixed variational-like inequality problem (GMVLIP):

$$\text{find } x \in C \text{ and } u \in A(x)$$
$$\text{such that } \langle u, \eta(y,x)\rangle + \varphi(y,x) - \varphi(x,x) \geq 0, \forall y \in C. \qquad (3.1.8)$$

(V) If $H(u,x,y) = h(x,y)$ and $F(u,x,y) = f(x,y)$ for all $u \in B^*$ and $x, y \in C$, then the BGMEP (3.1.3) – (3.1.4) reduces to the following bilevel mixed equilibrium problem (BMEP):

$$\text{find } \bar{x} \in S_{f,\varphi} \text{ such that } h(\bar{x},y) + \psi(y,\bar{x}) - \psi(\bar{x},\bar{x}) \geq 0, \forall y \in S_{f,\varphi},$$

where $S_{f,\varphi} = \{u \in C : f(u,y) + \varphi(y,u) - \varphi(u,u) \geq 0, \forall y \in C\}$, i.e., $S_{f,\varphi}$ is the solution set of the following mixed equilibrium problem (MEP):

$$\text{find } u \in C \text{ such that } f(u,y) + \varphi(y,u) - \varphi(u,u) \geq 0, \forall y \in C.$$

The BMEP was introduced and studied by Ding [11] in Banach spaces. An iterative algorithm to compute the approximate solutions of the BMEP has been suggested and analyzed. Strong convergence of the iterative sequence generated by this algorithm is also proved under suitable conditions.

(VI) If $B = H$ is a Hilbert space and $\varphi = \psi = 0$, then the BMEP reduces to the following bilevel equilibrium problem (BEP):

$$\text{find } \bar{x} \in S_f \text{ such that } h(\bar{x},y) \geq 0, \forall y \in S_f, \qquad (3.1.9)$$

where $S_f = \{u \in C : f(u,y) \geq 0, \forall y \in C\}$, i.e., S_f is the solution set of the following equilibrium problem (EP):

$$\text{find } u \in C \text{ such that } f(u,y) \geq 0, \forall y \in C. \qquad (3.1.10)$$

The BEP (3.1.9) – (3.1.10) was introduced and studied by Moudafi [32] in Hilbert spaces. By using the proximal method, he also suggested an iterative algorithm to compute approximate solutions of the BEP and proved the weak convergence of the iterative sequence generated by the algorithm.

For suitable choices of $G, KF, H, N, M, Q, S, A, T, \bar{A}, \bar{T}, \psi$ and φ, it is easy to see that the BNGMEP (3.1.1) – (3.1.2) includes many bilivel generalized mixed (quasi) equilibrium problem, bilevel generalized mixed quasi-variational-like inequality problem studied by many authors in Hilbert and Banach spaces as special cases.

In this work, we introduce and analyze the NGMEP (3.1.2) and the BNGMEP (3.1.1) in Banach spaces. First, by using a minimax inequality, some new existence theorems of the solution and the behavior of solution set for the NGMEP (3.1.2) and the BNGMEP (3.1.1) are obtained in Banach spaces. Next, by using auxiliary principle technique, some new iterative algorithms for solving the NGMEP (3.1.2) and the BNGMEP (3.1.1) are suggested and analyzed. The strong convergence of the iterative sequences generated by the algorithms are also proved in Banach spaces.

Now, we will give the following new definition of ω^*-monotone and (ω^*, σ)-strongly monotone mappings.

Definition 3.1.1. Let C be a closed convex subset of a Banach space B. Let $N : B^* \times B^* \to B^*$ be a single-valued mapping and $A, T : C \to C(B^*)$ be set-valued mappings, and $F : B^* \times C \times C \to \mathbb{R}$ be real-valued function. For a given $\omega^* \in B^*$, F is said to be

(i) *ω^*-monotone with respect to N, A and T* if

$$\sup_{(u,v)\in T(x)\times A(x)} F(N(u,v) - \omega^*, x, y) + \sup_{(w,z)\in T(y)\times A(y)} F(N(w,z) - \omega^*, y, x) \leq 0,$$

for all $x, y \in C$;

(ii) *$(\omega^*, \sigma)-$strongly monotone with respect to N, A and T* if there exists a constant $\sigma > 0$ such that

$$\sup_{(u,v)\in T(x)\times A(x)} F(N(u,v)-\omega^*,x,y)+\sup_{(w,z)\in T(y)\times A(y)} F(N(w,z)-\omega^*,y,x) \leq -\sigma\|x-y\|^2,$$

for all $x, y \in C$.

Remark 3.1.2. (1) If $\omega^* \equiv 0$, then (i) and (ii) are reduced to the monotone mapping and $\sigma-$strongly monotone mapping with respect to N, A and T respectively in Definition 2.2 of Ding, Liou and Yao [33].

(ii) Clearly, $(\omega^*, \sigma)-$strong monotonicity of F with respect to N, A and T implies the ω^*-monotonicity of F with respect to N, A and T.

Example 3.1.3. For a given $w^* \in B^*$, let the mapping $F : B^* \times C \times C \to \mathbb{R}$ be real-valued function defined by

$$F(g^* - w^*, x, y) = \langle g^* - w^*, y - x\rangle, \text{ for all } x, y \in C \text{ and } g^* \in B^*.$$

Let the mappings N, A and T be defined in Definition 3.1.1. Hence

$$F(N(u,v) - \omega^*, x, y) = \langle N(u,v) - \omega^*, y - x\rangle$$

for all $x, y \in C$ and $(u, v) \in A(x) \times T(x)$. According to the definition of set-valued monotone mappings, $N(\cdot,\cdot)$ is said to be monotone with respect to A and T if

$$\langle N(u,v) - N(w,z), x-y\rangle \geq 0, \forall x, y \in C, (u,v) \in A(x)\times T(x), (w,z) \in A(y)\times T(y).$$

In this case, we have that $F : B^* \times C \times C \to \mathbb{R}$ is ω^*-monotone with respect to N, A and T if and only if $N(\cdot,\cdot)$ is monotone with respect to A and T. In fact, by definition, $N(\cdot,\cdot)$ is monotone with respect to A and T if and only if

$$\langle N(u,v), y-x\rangle \leq \langle N(w,z), y-x\rangle,$$

for all $x, y \in C, (u, v) \in A(x) \times T(x), (w, z) \in A(y) \times T(y)$, which is equivalent to

$$\langle N(u,v), y-x\rangle - \langle \omega^*, y-x\rangle \leq \langle N(w,z), y-x\rangle - \langle \omega^*, y-x\rangle,$$

for all $x, y \in C, (u,v) \in A(x) \times T(x), (w,z) \in A(y) \times T(y)$ which is equivalent to

$$\sup_{(u,v)\in A(x)\times T(x)} \langle N(u,v) - \omega^*, y - x\rangle \leq \inf_{(w,z)\in A(y)\times T(y)} \langle N(w,z) - \omega^*, y - x\rangle, \forall x, y \in C,$$

which is again equivalent to

$$\sup_{(u,v)\in A(x)\times T(x)} \langle N(u,v) - \omega^*, y - x\rangle + \sup_{(w,z)\in A(y)\times T(y)} \langle N(w,z) - \omega^*, x - y\rangle \leq 0,$$

for all $x, y \in C$. The above inequality holds if and only if

$$\sup_{(u,v)\in A(x)\times T(x)} F(N(u,v) - \omega^*, x, y) + \sup_{(w,z)\in A(y)\times T(y)} F(N(w,z) - \omega^*, y, x) \leq 0,$$

for all $x, y \in C, \omega^* \in B^*$. Hence $N(\cdot,\cdot)$ is monotone with respect to A and T if and only if $F : B^* \times C \times C \to \mathbb{R}$ is ω^*-monotone with respect to N, A and T.

3.1.1 The existence of the solution set and algorithms for the BNG-MEP (3.1.1) – (3.1.2) in Banach spaces

3.1.1.1 The existence of a solution for the NGMEP

Lemma 3.1.4. *Let C be a nonempty convex subset of a Banach space B Let $\omega^* \in B^*$. Let $N : B^* \times B^* \to B^*, T : C \to C(B^*)$ and $A : C \to C(B^*)$ be upper semicontinuous and $F : B^* \times C \times C \to \mathbb{R}$ and for each $y \in C$, $(u, x, y) \mapsto F(u, x, y)$ is upper semicontinuous. Then for each $(x, y) \in C \times C$, there exists $\bar{u}, \bar{v} \in T(x) \times A(x)$ such that*

$$F(N(\bar{u},\bar{v}) - \omega^*, x, y) = \max_{(u,v)\in T(x)\times A(x)} F(N(u,v) - \omega^*, x, y).$$

Furthermore, the mapping $x \mapsto \max_{(u,v)\in T(x)\times A(x)} F(N(u,v) - \omega^*, x, y)$ is upper semicontinuous.

Proof. Since $N : B^* \times B^* \to B^*$ is upper semicontinuous, we have the mapping $N - \omega^* : B^* \times B^* \to B^*$ defined by

$$(N - \omega^*)(u,v) = N(u,v) - \omega^*, \qquad \forall (u,v) \in B^* \times B^*$$

is also upper semicontinuous. Indeed, Assume that $N : B^* \times B^* \to B^*$ is upper semicontinuous. We will show that $N - \omega^*$ which defined by

$$(N - \omega^*)(u, v) = N(u, v) - \omega^*, \qquad \forall (u, v) \in B^* \times B^*$$

is also upper semicontinuous.

Let $(u, v) \in B^* \times B^*$ and neighborhood V of $(N - \omega^*)(u, v)$. That is V is a neighborhood of $N(u, v) - \omega^*$. This implies that $V - \omega^*$ is a neighborhood of $N(u, v)$. Since $N : B^* \times B^* \to B^*$ is upper semicontinuous, there is a neighborhood U of (u, v) such that

$$\begin{aligned} N(x, y) &\subseteq V + \omega^* && \forall (x, y) \in U, \\ N(x, y) - \omega^* &\subseteq V && \forall (x, y) \in U, \\ (N - \omega^*)(x, y) &\subseteq V && \forall (x, y) \in U. \end{aligned}$$

Hence $N - \omega^*$ is upper semicontinuous.

For each $y \in C, (u, x, y) \mapsto F(u, x, y)$ is upper semicontinuous, it follows from Lemma 2.2.21 that for each $y \in C$, the mapping $((u, v), x) \mapsto F(N(u, v) - \omega^*, x, y)$ is also upper semicontinuous. Indeed, Note that $N - \omega^*$ is upper semicontinuous on $B^* \times B^*$ and for each $y \in C$, $F(\cdot, \cdot, y)$ is upper semicontinuous. For each $y \in C$, we define $g_y : (B^* \times B^*) \times C \to \mathbb{R}$ which defined by

$$g_y((u, v), x) = (F_{x,y} \circ (N - \omega^*))(u, v) = F(N(u, v) - \omega^*, x, y), \quad \forall u, v \in B^*, x \in C$$

is upper semicontinuous where

$$F_{x,y}(t) = F(t, x, y), \qquad \forall t \in B^*.$$

Since $F(\cdot, \cdot, \cdot)$ is upper semicontinuous, we have g_y is also upper semicontinuous.

By Lemma 2.2.22, the mapping $T \times A : C \to C(B^*) \times C(B^*)$ defined by $(T \times A)(x) = T(x) \times A(x)$ is upper semicontinuous with compact values. Since for each $x \in$

$C, T(x) \times A(x)$ is compact in $B^* \times B^*$, it follows that there exists $\bar{u}, \bar{v} \in T(x) \times A(x)$ such that

$$F(N(\bar{u}, \bar{v}) - \omega^*, x, y) = \max_{(u,v) \in T(x) \times A(x)} F(N(u, v) - \omega^*, x, y).$$

It follows from Lemma 2.2.23 that the mapping $x \mapsto \max_{(u,v)\in T(x)\times A(x)} F(N(u, v) - \omega^*, x, y)$ is upper semicontinuous. $\square$

Lemma 3.1.5. *Let C be closed convex subset of a Banach space B. Let $F : B^* \times C \times C \to \mathbb{R}, N : B^* \times B^* \to B^*, T, A : C \to C(B^*)$ and $\varphi : B \times B \to \mathbb{R}$. Let $G : C \times C \to \mathbb{R}$ and $\omega^* \in B^*$ satisfy the following conditions:*

(i) *$F(N(u, v) - \omega^*, x, x) \geq 0,\ \forall x \in C, (u, v) \in T(x) \times A(x)$;*

(ii) *F is ω^*-monotone with respect to N, A and T; and for each $y \in C$, $((u, v), z) \mapsto F(N(u, v) - \omega^*, z, y)$ is upper semicontinuous;*

(iii) *for each $z \in C$ and $(u, v) \in T(z) \times A(z), y \mapsto F(N(u, v) - \omega^*, z, y)$ is convex and lower semicontinuous;*

(iv) *G is monotone and $G(x, x) \geq 0$ for each $x \in C$;*

(v) *for each $y \in C, x \mapsto G(x, y)$ is upper semicontinuous and G is convex and lower semicontinuous in second argument;*

(vi) *T and A are both upper semicontinuous;*

(vii) *φ is continuous, skew symmetric and convex in first argument;*

Then there exists $x \in C$ such that

$$G(x, y) + \max_{(u,v)\in T(x)\times A(x)} F(N(u, v) - \omega^*, x, y) + \varphi(y, x) - \varphi(x, x) \geq 0, \qquad \forall y \in C.$$

if and only if

$$-G(y, x) - \max_{(w,z)\in T(y)\times A(y)} F(N(w, z) - \omega^*, y, x) + \varphi(y, y) - \varphi(x, y) \geq 0, \qquad \forall y \in C.$$

Furthermore, the solution set $S^{A,T,\varphi}_{G,F,N}$ of the NGMEP (3.1.2) is a closed convex subset of C.

Proof. For any $y \in C$, define two mappings M, P as follows:

$$M(y) = \{x \in C : G(x, y) + \max_{(u,v)\in T(x)\times A(x)} F(N(u, v) - \omega^*, x, y) + \varphi(y, x) - \varphi(x, x) \geq 0\}$$

$$P(y) = \{x \in C : -G(y,x) - \max_{(w,z)\in T(y)\times A(y)} F(N(w,z)-\omega^*, y, x) + \varphi(y,y) - \varphi(x,y) \geq 0\}$$

In order to show that the conclusion of Lemma 3.1.5 holds, we only need to show $\bigcap_{y\in C} M(y) = \bigcap_{y\in C} P(y)$, indeed, Assume that $\cap_{y\in C} M(y) = \cap_{y\in C} P(y)$ where

$$M(y) = \{x \in C : G(x,y) + \max_{(u,v)\in T(x)\times A(x)} F(N(u,v)-\omega^*, x, y) + \varphi(y,x) - \varphi(x,x) \geq 0\}$$

$$P(y) = \{x \in C : -G(y,x) - \max_{(w,z)\in T(y)\times A(y)} F(N(w,z)-\omega^*, y, x) + \varphi(y,y) - \varphi(x,y) \geq 0\}.$$

We will show that there exists $x \in C$ such that

$$G(x,y) + \max_{(u,v)\in T(x)\times A(x)} F(N(u,v)-\omega^*, x, y) + \varphi(y,x) - \varphi(x,x) \geq 0, \qquad \forall y \in C.$$

if and only if

$$-G(y,x) - \max_{(w,z)\in T(y)\times A(y)} F(N(w,z)-\omega^*, y, x) + \varphi(y,y) - \varphi(x,y) \geq 0, \qquad \forall y \in C.$$

($\Rightarrow$) Let $\bar{x} \in C$ be such that

$$G(\bar{x},y) + \max_{(u,v)\in T(\bar{x})\times A(\bar{x})} F(N(u,v)-\omega^*, \bar{x}, y) + \varphi(y,\bar{x}) - \varphi(\bar{x},\bar{x}) \geq 0, \qquad \forall y \in C.$$

That is $\bar{x} \in M(y)$ for all $y \in C$. This implies that $\bar{x} \in \cap_{y\in C} M(y) = \cap_{y\in C} P(y)$. Hence

$$-G(y,\bar{x}) - \max_{(w,z)\in T(y)\times A(y)} F(N(w,z)-\omega^*, y, \bar{x}) + \varphi(y,y) - \varphi(\bar{x},y) \geq 0, \qquad \forall y \in C.$$

In the same way, we can show another side of this conclusion.

Since F is ω^*-monotone with respect to N, A and T, and G is monotone, we have

$$\begin{aligned} & G(x,y) + \max_{(u,v)\in T(x)\times A(x)} F(N(u,v)-\omega^*, x, y) \\ \leq \; & -G(y,x) - \max_{(w,z)\in A(y)\times T(y)} F(N(w,z)-\omega^*, y, x), \end{aligned}$$

for all $x, y \in C$. Since φ is skew symmetric, we have

$$\varphi(y,x) - \varphi(x,x) \leq \varphi(y,y) - \varphi(x,y), \qquad \forall x, y \in C.$$

Hence, if $\bar{x} \in M(y)$, then $\bar{x} \in P(y)$. It follows that $\bigcap_{y\in C} M(y) \subseteq \bigcap_{y\in C} P(y)$. Conversely, if there exists $x \in \bigcap_{y\in C} P(y)$, but $x \notin \bigcap_{y\in C} M(y)$, then we have

$$\begin{aligned} &-G(y,x) - \max_{(w,z)\in T(y)\times A(y)} F(N(w,z) - \omega^*, y, x) \\ &+\varphi(y,y) - \varphi(x,y) \geq 0, \forall y \in C, \end{aligned} \tag{3.1.11}$$

and there exists $\bar{y} \in C$ such that

$$G(x,\bar{y}) + \max_{(u,v)\in T(x)\times A(x)} F(N(u,v) - \omega^*, x, \bar{y}) + \varphi(\bar{y},x) - \varphi(x,x) < 0.$$

It follows that

$$G(x,\bar{y}) + F(N(u,v) - \omega^*, x, \bar{y}) + \varphi(\bar{y},x) - \varphi(x,x) < 0, \tag{3.1.12}$$

for all $(u,v) \in T(x) \times A(x)$. Let $x_t = t\bar{y} + (1-t)x = x + t(\bar{y} - x), t \in [0,1]$. Then $x_t \in C$. It follows from (3.1.11) that

$$\begin{aligned} &-G(x_t,x) - \max_{(u_t,v_t)\in T(x_t)\times A(x_t)} F(N(u_t,v_t) - \omega^*, x_t, x) \\ &+\varphi(x_t,x_t) - \varphi(x,x_t) \geq 0, \qquad \forall t \in (0,1]. \end{aligned} \tag{3.1.13}$$

Since for each $y \in C, ((u,v),z) \mapsto F(N(u,v) - \omega^*, z, y)$ is upper semicontinuous and each $T(x_t) \times A(x_t)$ is compact, it follows that for each $t \in (0,1]$, there exists $(\bar{u}_t, \bar{v}_t) \in T(x_t) \times A(x_t)$ such that

$$F(N(\bar{u}_t,\bar{v}_t) - \omega^*, x_t, x) = \max_{(u_t,v_t)\in T(x_t)\times A(x_t)} F(N(u_t,v_t) - \omega^*, x_t, x), \qquad \forall t \in (0,1].$$

By (3.1.13), we have

$$\begin{aligned} &-G(x_t,x) - F(N(\bar{u}_t,\bar{v}_t) - \omega^*, x_t, x) \\ &+\varphi(x_t,x_t) - \varphi(x,x_t) \geq 0, \qquad \forall t \in (0,1]. \end{aligned} \tag{3.1.14}$$

Let $\Omega = \{x_t\}_{t\in[0,1]}$. Then Ω is compact. Since A and T are both upper semicontinuous with compact values, it follows from Lemma 2.2.20, that $T(\Omega)\times A(\Omega)$ is compact in $B^* \times B^*$. Noting $(\bar{u}_t, \bar{v}_t)_{t\in[0,1]} \subset T(\Omega) \times A(\Omega)$ and $x_t \to x$, without loss of generality, we can assume that there exists $(u, v) \in B^* \times B^*$ such that $(\bar{u}_t, \bar{v}_t) \to (u, v)$ and $(u, v) \in T(x) \times A(x)$. Note that for each $y \in C$, $((u, v), z) \mapsto F(N(u, v) - \omega^*, z, y)$ is upper semicontinuous and so $((u, v), z) \mapsto -F(N(u, v) - \omega^*, z, y)$ is lower semicontinuous and for any $y \in C$, the mapping $x \mapsto -G(x, y)$ is lower semicontinuous. As φ is continuous, by (3.1.12), we have that

$$\begin{aligned}&\liminf_{t\to 0}[-G(x_t, \bar{y}) - F(N(\bar{u}_t, \bar{v}_t) - \omega^*, x_t, \bar{y}) + \varphi(x_t, x_t) - \varphi(\bar{y}, x_t)]\\ &\geq -G(x, \bar{y}) - F(N(u, v) - \omega^*, x, \bar{y}) + \varphi(x, x) - \varphi(\bar{y}, x) > 0. \qquad (3.1.15)\end{aligned}$$

Hence, there exists $t^* \in (0, 1]$ such that

$$-G(x_t, \bar{y}) - F(N(\bar{u}_t, \bar{v}_t) - \omega^*, x_t, \bar{y}) + \varphi(x_t, x_t) - \varphi(\bar{y}, x_t) > 0, \qquad \forall t \in (0, t^*].$$

It follows that

$$G(x_t, \bar{y}) + F(N(\bar{u}_t, \bar{v}_t) - \omega^*, x_t, \bar{y}) + \varphi(\bar{y}, x_t) < \varphi(x_t, x_t), \qquad (3.1.16)$$

for all $t \in (0, t^*]$. By (3.1.14), we have

$$G(x_t, x) + F(N(\bar{u}_t, \bar{v}_t) - \omega^*, x_t, x) + \varphi(x, x_t) \leq \varphi(x_t, x_t), \qquad (3.1.17)$$

for all $t \in (0, t^*]$. Since for each $z \in C$ and $(u, v) \in T(z) \times A(z)$, $y \mapsto F(N(u, v) - \omega^*, z, y)$ is convex and φ is convex in first argument, G is convex in second argument and $x_t = t\bar{y} + (1-t)x$, it follows from (3.1.16) and (3.1.17) that for each $t \in (0, t^*]$,

$$\begin{aligned}& G(x_t, x_t) + F(N(\bar{u}_t, \bar{v}_t) - \omega^*, x_t, x_t) + \varphi(x_t, x_t)\\ \leq\ & t[G(x_t, \bar{y}) + F(N(\bar{u}_t, \bar{v}_t) - \omega^*, x_t, \bar{y}) + \varphi(\bar{y}, x_t)]\\ & +(1-t)[G(x_t, x) + F(N(\bar{u}_t, \bar{v}_t) - \omega^*, x_t, x) + \varphi(x, x_t)]\\ <\ & \varphi(x_t, x_t), \qquad \forall t \in (0, t^*]. \qquad (3.1.18)\end{aligned}$$

Hence for $t \in (0, t^*]$, we obtain $G(x_t, x_t) + F(N(\bar{u}_t, \bar{v}_t) - \omega^*, x_t, x_t) < 0$ which contradicts the assumption that $F(N(u,v) - \omega^*, x, x) \geq 0,\ \forall x \in C, (u,v) \in T(x) \times A(x)$ and $G(x,x) \geq 0$ for each $x \in C$. Therefore we have $\bigcap_{y \in C} M(y) = \bigcap_{y \in C} P(y)$. It is easy to see that $S^{A,T,\varphi}_{G,F,N} = \bigcap_{y \in C} M(y)$. Indeed, ($\subseteq$) Let $x \in S^{A,T,\varphi}_{G,F,N}$. There exists $(u,v) \in T(x) \times A(x)$ such that

$$G(x,y) + F(N(u,v) - \omega^*, x, y) + \varphi(y,x) - \varphi(x,x) \geq 0, \qquad \forall y \in C.$$

This implies that

$$G(x,y) + \max_{(u,v) \in T(x) \times A(x)} F(N(u,v) - \omega^*, x, y) + \varphi(y,x) - \varphi(x,x) \geq 0, \qquad \forall y \in C.$$

Hence $x \in M(y)$ for all $y \in C$. Therefore $x \in \cap_{y \in C} M(y)$.

($\supseteq$) Let $x \in \cap_{y \in C} M(y)$. That is

$$G(x,y) + \max_{(u,v) \in T(x) \times A(x)} F(N(u,v) - \omega^*, x, y) + \varphi(y,x) - \varphi(x,x) \geq 0, \qquad \forall y \in C.$$

By assumption, there exists $(u^*, v^*) \in T(x) \times A(x)$ such that

$$F(N(u^*, v^*) - \omega^*, x, y) = \max_{(u,v) \in T(x) \times A(x)} F(N(u,v) - \omega^*, x, y).$$

Hence

$$G(x,y) + F(N(u^*, v^*) - \omega^*, x, y) + \varphi(y,x) - \varphi(x,x) \geq 0, \qquad \forall y \in C.$$

Therefore $x \in S^{A,T,\varphi}_{G,F,N}$.

This implies that

$$S^{A,T,\varphi}_{G,F,N} = \bigcap_{y \in C} M(y) = \bigcap_{y \in C} P(y)$$

is the solution set of NGMEP (3.1.2). By definition of P, for each $y \in C$, we have that

$$
\begin{aligned}
P(y) &= \{x \in C : -G(y,x) - \max_{(w,z)\in T(y)\times A(y)} F(N(w,z)-\omega^*, y, x) \\
&\quad +\varphi(y,y) - \varphi(x,y) \geq 0\} \\
&= \{x \in C : -G(y,x) - F(N(w,z)-\omega^*, y, x) + \varphi(y,y) - \varphi(x,y) \geq 0, \} \\
&= \bigcap_{(w,z)\in T(y)\times A(y)} \{x \in C : G(y,x) + F(N(w,z)-\omega^*, y, x) + \varphi(x,y) \leq \varphi(y,y)\}.
\end{aligned}
$$

Since for each $z \in C$ and $(u,v) \in T(z) \times A(z), y \mapsto F(N(u,v)-\omega^*, z, y)$ is convex and lower semicontinuous, G is convex and lower semicontinuous in second argument; and φ is lower semicontinuous and convex in first argument, it follows that for each $y \in C$, the set $\{x \in C : G(y,x)+F(N(w,z)-\omega^*, y, x)+\varphi(x,y) \leq \varphi(y,y)\}$ is closed and convex. We will show the convexity, Let $a, b \in A$ and $t \in (0,1)$. That is

$$G(y,a) + F(N(w,z)-\omega^*, y, a) + \varphi(a,y) \leq \varphi(y,y)$$

and

$$G(y,b) + F(N(w,z)-\omega^*, y, b) + \varphi(b,y) \leq \varphi(y,y).$$

Consider

$$
\begin{aligned}
& G(y, ta+(1-t)b) + F(N(w,z)-\omega^*, y, ta+(1-t)b) + \varphi(ta+(1-t)b, y) \\
\leq\ & tG(y,a) + (1-t)G(y,b) + tF(N(w,z)-\omega^*, y, a) \\
& +(1-t)F(N(w,z)-\omega^*, y, b) + t\varphi(a,y) + (1-t)\varphi(b,y) \\
\leq\ & t\varphi(y,y) + (1-t)\varphi(y,y) \\
=\ & \varphi(y,y).
\end{aligned}
$$

and hence $P(y)$ and $\bigcap_{y\in C} P(y)$ are both closed convex subsets of C. Therefore the solution set $S_{G,F,N}^{A,T,\varphi}$ of NGMEP (3.1.2) is closed and convex in C. □

Next, we give the following example for illustrating the previous theorem.

Example 3.1.6. $B = \mathbb{R}, C = [0, \infty)$

$(N(u,v))(x) = 2x,$ *and* $\omega^*(x) = x$

$$F(N(u,v) - \omega^*, x, y) = \langle N(u,v) - \omega^*, y - x \rangle = 2(y-x) - (y-x) = y - x$$

$G(x,y) = x - y$ *and* $\varphi(u,v) = u + 2v.$

We have to check all conditions of Lemma 3.1.4 satisfied.

1. $F(N(u,v) - \omega^*, x, x) = x - x = 0$ *for each* $x \in C$.
2. *By the first page of this document, we can get that* F *is* ω^**-monotone with respect to* N, A *and* T*; and it easy to see that for each* $y \in C, ((u,v), z) \mapsto F(N(u,v) - \omega^*, z, y)$ *is upper semicontinuous;*
3. *for each* $z \in C$,
$$f(y) = y - z$$
is convex and lower semicontinuous, then for each $z \in C$ *and* $(u,v) \in T(z) \times A(z), y \mapsto F(N(u,v) - \omega^*, z, y)$ *is convex and lower semicontinuous;*
4. *Since* $G(x,y) = x - y$*, we have* G *is monotone and* $G(x,x) \geq 0$ *for each* $x \in C$*;*
5. *for each* $y \in C, x \mapsto G(x,y)$ *is upper semicontinuous and* G *is convex and lower semicontinuous in second argument;*
6. *Since* $\varphi(u,v) = u + 2v$*, we have* φ *is continuous, skew symmetric and convex in first argument.*

Consider

$$\begin{aligned} & G(x,y) + \max_{(u,v)\in T(x)\times A(x)} F(N(u,v)-\omega^*,x,y) + \varphi(y,x) - \varphi(x,x) \\ = \; & (x-y)+(y-x)+(y+2x)-3x \\ = \; & y-x \end{aligned}$$

and

$$\begin{aligned} & -G(y,x) - \max_{(w,z)\in T(y)\times A(y)} F(N(w,z)-\omega^*,y,x) + \varphi(y,y) - \varphi(x,y) \\ = \; & -(y-x)-(x-y)+3y-(2x+y) \\ = \; & y-x \end{aligned}$$

We must find $x \in [0,\infty)$ *such that*

$$y - x \geq 0, \qquad \forall y \in [0,\infty).$$

Hence $x = 0$. *It is easy to see that* 0 *is the unique solution of NGMEP.*

Now, we will consider an auxiliary new generalized mixed equilibrium problem (ANGMEP) for solving the NGMEP (3.1.2). Let B be a real Banach space with dual space B^*. Let C be a nonempty closed convex subset of B, let $G : C \times C \to \mathbb{R}$ be a bifunction, $A, T : C \to C(B^*)$ be set-values mappings, $N : B^* \times B^* \to B^*$ and $F : B^* \times C \times C \to \mathbb{R}$. Let $\varphi : C \times C \to \mathbb{R}$ be a skew-symmetric bifunction and $\omega^* \in B^*$. Furthermore, let $g : B \to B^*$ be a mapping. For a given $x \in C$ and $\rho > 0$, we consider the following problem :

$$\text{(ANGMEP)} \begin{cases} \text{Find } z \in C \text{ such that} \\ \rho(G(z,y) + \max_{(u,v)\in T(z)\times A(z)} F(N(u,v)-\omega^*,z,y) + \varphi(y,z) \\ -\varphi(z,z)) + \langle g(y-z), z-x\rangle \geq 0, \qquad \forall y \in C, \end{cases} \tag{3.1.19}$$

Now, we prove the existence and uniqueness of solutions of the ANGMEP (3.1.19).

Theorem 3.1.7. *Let C be a nonempty closed convex subset of a Banach space B and D be a compact subset of B with $C \cap D \neq \emptyset$. Let $G : C \times C \to \mathbb{R}$ be a bifunction, $T, A : C \to C(B^*)$ be set-values mappings, $N : B^* \times B^* \to B^*$ and $F : B^* \times C \times C \to \mathbb{R}, g : B \to B^*$ be single-valued mappings and $\varphi : B \times B \to \mathbb{R}$ be a bifunction and let $\omega^* \in B^*$. Suppose the following conditions are satisfied:*

(i) $F(N(u, v) - \omega^*, x, x) \geq 0, \ \forall x \in C, (u, v) \in T(x) \times A(x)$;

(ii) *F is ω^*-monotone with respect to N, A and T; and for each $y \in C$, $((u, v), z) \mapsto F(N(u, v) - \omega^*, z, y)$ is upper semicontinuous;*

(iii) *for each $z \in C$ and $(u, v) \in T(z) \times A(z), y \mapsto F(N(u, v) - \omega^*, z, y)$ is convex;*

(iv) *$G(x, x) \geq 0$ for each $x \in C$;*

(v) *for each $y \in C, x \mapsto G(x, y)$ is upper semicontinuous and for each $x \in C, y \mapsto G(x, y)$ is convex;*

(vi) *g is a $\lambda-$strongly positive bounded linear operator, T and A are both upper semicontinuous;*

(vii) *φ is continuous, skew symmetric and convex in first argument;*

(viii) *for each $x \in B$, there exists $y \in C \cap D$ such that*

$$\begin{aligned}&\rho(G(z, y) + \max_{(u,v) \in T(z) \times A(z)} F(N(u, v) - \omega^*, z, y) + \varphi(y, z) - \varphi(z, z)) \\ &+ \langle g(y - z), z - x \rangle < 0, \qquad \forall z \in C \setminus (C \cap D).\end{aligned}$$

Then for given $x \in B$, the ANGMEP (3.1.19) has a solution $z^* \in C \cap D$. If furthere assume that G is monotone, then the solution of ANGMEP (3.1.19) is unique.

Proof. For given $x \in B, u \in T(x), v \in A(x)$, and define the bifunction $f : C \times C \to \mathbb{R}$ by

$$\begin{aligned} f(z,y) &= \rho(G(z,y) + \max_{(u,v)\in T(z)\times A(z)} F(N(u,v)-\omega^*,z,y) + \varphi(y,z) - \varphi(z,z)) \\ &\quad + \langle g(y-z), z-x\rangle \qquad \forall z,y \in C. \end{aligned}$$

(a) By the definition of f and condition (i) and (iv), we have $f(z,z) \geq 0,\ \forall z \in C$. So condition (i) of Lemma 2.2.24 is satisfied.

(b) Since for each $y \in C$, $((u,v),z) \mapsto F(N(u,v)-\omega^*,z,y)$ is upper semicontinuous, and $T\times A : C \to C(B^*)\times C(B^*)$ is upper semicontinuous with compact values, it follows from Lemma 3.1.4 that for each $y \in C, z \mapsto \max_{(u,v)\in T(z)\times A(z)} F(N(u,v)-\omega^*,z,y)$ is upper semicontinuous. Noting that φ is continuous, g is linear and for each $y \in C, x \mapsto G(x,y)$ is upper semicontinuous, we have that for each $y \in C, z \mapsto f(z,y)$ is upper semicontinuous and so condition (ii) of Lemma 2.2.24 is satisfied.

(c) Since for each $z \in C$ and $(u,v) \in T(z)\times A(z), y \mapsto F(N(u,v)-\omega^*,z,y)$ is convex, for any $y_1, y_2 \in C$ and $t \in [0,1]$, we have

$$\begin{aligned} & \max_{(u,v)\in T(z)\times A(z)} F(N(u,v)-\omega^*,z,ty_1+(1-t)y_2) \\ \leq & \max_{(u,v)\in T(z)\times A(z)} [tF(N(u,v)-\omega^*,z,y_1) + (1-t)F(N(u,v)-\omega^*,z,y_2)] \\ \leq & \ t \max_{(u,v)\in T(z)\times A(z)} F(N(u,v)-\omega^*,z,y_1) \\ & +(1-t) \max_{(u,v)\in T(z)\times A(z)} H(N(u,v)-\omega^*,z,y_1). \end{aligned} \tag{3.1.20}$$

Hence for each $z \in C, y \mapsto \max_{(u,v)\in T(z)\times A(z)} F(N(u,v)-\omega^*,z,y)$ is convex. Note that for each $z \in C, y \mapsto \varphi(y,z)$ is convex and g is linear. We have that for each $z \in C, y \mapsto f(z,y)$ is convex. Condition (iii) of Lemma 2.2.24 is satisfied.

(d) Condition (viii) implies that for each $x \in B$, there exists $y \in C \cap D$ such that

$$f(z,y) < 0, \qquad \forall z \in C \setminus (C \cap D),$$

and hence condition (iv) of Lemma 2.2.24 is satisfied.

From (a), (b), (c) and (d), by Lemma 2.2.24, there exists a point $\hat{z} \in C \cap D$ such that

$$f(\hat{z},y) \geq 0, \qquad \forall y \in C,$$

which gives that

$$\begin{aligned}&\rho(G(\hat{z},y) + \max_{(u,v)\in T(x)\times A(x)} F(N(u,v)-\omega^*,\hat{z},y) + \varphi(y,\hat{z}) - \varphi(\hat{z},\hat{z}))\\&+\langle g(y-\hat{z}),\hat{z}-x\rangle \geq 0, \qquad \forall y \in C.\end{aligned}$$

Therefore $\hat{z} \in C$ is a solution of the ANGMEP (3.1.19). Now assume that each G is monotone and we show that for each given $x \in B$, the solution of the ANGMEP (3.1.19) is unique. Let $z_1, z_2 \in C$ be any two solutions of the ANGMEP (3.1.19). Then we have

$$\begin{aligned}&\rho(G(z_1,y) + \max_{(u,v)\in T(x)\times A(x)} F(N(u,v)-\omega^*,z_1,y)\\&+\varphi(z_1,y) - \varphi(z_1,z_1)) + \langle g(y-z_1),z_1-x\rangle \geq 0, \qquad \forall y \in C \qquad (3.1.21)\end{aligned}$$

and

$$\begin{aligned}&\rho(G(z_2,y) + \max_{(u,v)\in T(x)\times A(x)} F(N(u,v)-\omega^*,z_2,y)\\&+\varphi(z_2,y) - \varphi(z_2,z_2)) + \langle g(y-z_2),z_2-x\rangle \geq 0, \qquad \forall y \in C \qquad (3.1.22)\end{aligned}$$

Taking $y = z_2$ in (3.1.21) and $y = z_1$ in (3.1.22), we obtain

$$\begin{aligned}&\rho(G(z_1,z_2) + \max_{(u,v)\in T(x)\times A(x)} F(N(u,v)-\omega^*,z_1,z_2)\\&+\varphi(z_1,z_2) - \varphi(z_1,z_1)) + \langle g(z_2-z_1),z_1-x\rangle \geq 0, \qquad (3.1.23)\end{aligned}$$

and

$$\rho(G(z_2, z_1) + \max_{(u,v)\in T(x)\times A(x)} F(N(u,v) - \omega^*, z_2, z_1)$$
$$+\varphi(z_2, z_1) - \varphi(z_2, z_2)) + \langle g(z_1 - z_2), z_2 - x\rangle \geq 0. \quad (3.1.24)$$

Adding above two inequalities, we have

$$\rho((G(z_1, z_2) + G(z_2, z_1)) + \varphi(z_1, z_2) - \varphi(z_1, z_1) + \varphi(z_2, z_1) - \varphi(z_2, z_2)$$
$$+ \max_{(u,v)\in T(x)\times A(x)} F(N(u,v) - \omega^*, z_1, z_2) + \max_{(u,v)\in T(x)\times A(x)} F(N(u,v) - \omega^*, z_2, z_1))$$
$$\geq \langle g(z_1 - z_2), z_1 - z_2\rangle \quad (3.1.25)$$

Assume that $z_2 \neq z_1$. Noting that G is monotone, F is ω^*-monotone with respect to N, A and T; φ is skew-symmetric and g is $\lambda-$strongly positive, it follows from (3.1.25) that

$$0 \geq \langle g(z_2 - z_1), z_2 - z_1\rangle \geq \lambda\|z_2 - z_1\|^2 > 0,$$

which is a contradiction. Therefore, we must have $z_2 = z_1$. This completes the proof. □

Remark 3.1.8. We observe that if for some $\hat{x} \in C$, the $\hat{z} = \hat{x}$ is a solution of the ANGMEP (3.1.19), then $\hat{x}$ is also a solution of the NGMEP (3.1.2).

Based on this observation, we can construct the following iterative algorithm for computing approximate solution of the NGMEP (3.1.2).

3.1.1.2 Convergence theorem for NGMEP (3.1.2)

By using Theorem 3.1.7 we can construct the following iterative algorithm to compute the approximate solutions of the NGMEP (3.1.2).

Algorithm 3.1.9 (Constructive Approximation). *For given $x_0 \in B$ by Theorem* 3.1.7, *the ANGMEP* (3.1.19) *has a unique solution $x_1 \in C \cap D$, such that*

$$\rho(G(x_1, y) + \max_{(u_1,v_1)\in T(x_1)\times A(x_1)} F(N(u_1, v_1) - \omega^*, x_1, y)$$

$$+\varphi(y, x_1) - \varphi(x_1, x_1)) + \langle g(y - x_1), x_1 - x_0 \rangle \geq 0, \qquad \forall y \in C.$$

By induction, we can define the iterative sequences $\{x_n\} \subset C \cap D$ *such that*

$$\rho(G(x_{n+1}, y) + \max_{(u_{n+1}, v_{n+1}) \in T(x_{n+1}) \times A(x_{n+1})} F(N(u_{n+1}, v_{n+1}) - \omega^*, x_{n+1}, y)$$
$$+\varphi(y, x_{n+1}) - \varphi(x_{n+1}, x_{n+1})) + \langle g(y - x_{n+1}), x_{n+1} - x_n \rangle \geq 0, \qquad (3.1.26)$$

for all $y \in C$.

Next, we discuss the convergence of the iterative sequences generated by the Algorithm 3.1.9.

Theorem 3.1.10. *Let* C *be a nonempty closed convex subset of a Banach space* B *and* D *be a compact subset of* B *with* $C \cap D \neq \emptyset$. *Let* $G : C \times C \to \mathbb{R}$ *be a bifunction,* $T, A : C \to C(B^*)$ *be set-values mappings,* $N : B^* \times B^* \to B^*$ *and* $F : B^* \times C \times C \to \mathbb{R}, g : B \to B^*$ *be single-valued mappings and* $\varphi : B \times B \to \mathbb{R}$ *be a bifunction and let* $\omega^* \in B^*$. *Suppose the following conditions are satisfied:*

(i) $F(N(u, v) - \omega^*, x, x) \geq 0, \ \forall x \in C, (u, v) \in T(x) \times A(x)$;

(ii) F *is* $(\omega^*, \sigma)-$*strongly monotone with respect to* N, A *and* T; *and for each* $y \in C, ((u, v), z) \mapsto F(N(u, v), z, y)$ *is upper semicontinuous;*

(iii) *for each* $z \in C$ *and* $(u, v) \in T(z) \times A(z), y \mapsto F(N(u, v) - \omega^*, z, y)$ *is convex and lower semicontinuous;*

(iv) G *is* $\alpha-$*strongly monotone such that* $G(x, x) \geq 0$ *for each* $x \in C$;

(v) *for each* $y \in C, x \mapsto G(x, y)$ *is upper semicontinuous and for each* $x \in C, y \mapsto G(x, y)$ *is convex;*

(vi) g *is a* $\lambda-$*strongly positive bounded linear operator,* T *and* A *are both upper semicontinuous;*

(vii) φ *is continuous and skew symmetric such that for each* $y \in B, z \mapsto \varphi(z, y)$ *is convex;*

(viii) *for each $x \in B$, there exists $y \in C \cap D$ such that*

$$\rho(G(z,y) + \max_{(u,v)\in T(z)\times A(z)} F(N(u,v) - \omega^*, z, y) + \varphi(y,z) - \varphi(z,z))$$
$$+\langle g(y-z), z-x\rangle < 0, \qquad \forall z \in C \setminus (C\cap D).$$

Furthere suppose that $\frac{\|g\|}{\lambda+\rho\sigma} < 1$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1.9 converges strongly to a solution $\hat{x} \in C \cap D$ of the NGMEP (3.1.2) and the solution set $S^{A,T,\varphi}_{G,F,N}$ of the NGMEP (3.1.2) is nonempty compact convex set of $C\cap D$.

Proof. By algorithm 3.1.9

$$\rho(G(x_n,y) + \max_{(u_n,v_n)\in T(x_n)\times A(x_n)} F(N(u_n,v_n) - \omega^*, x_n, y)$$
$$+\varphi(y,x_n) - \varphi(x_n,x_n)) + \langle g(y-x_n), x_n - x_{n-1}\rangle \geq 0, \ \forall y \in C. \qquad (3.1.27)$$

and

$$\rho(G(x_{n+1},y) + \max_{(u_{n+1},v_{n+1})\in T(x_{n+1})\times A(x_{n+1})} F(N(u_{n+1},v_{n+1}) - \omega^*, x_{n+1}, y)$$
$$+\varphi(y,x_{n+1}) - \varphi(x_{n+1},x_{n+1})) + \langle g(y-x_{n+1}), x_{n+1} - x_n\rangle \geq 0, \qquad (3.1.28)$$

for all $y \in C$. Taking $y = x_{n+1}$ in (3.1.27) and $y = x_n$ in (3.1.28), respectively, we get

$$\rho(G(x_n,x_{n+1}) + \max_{(u_n,v_n)\in T(x_n)\times A(x_n)} F(N(u_n,v_n) - \omega^*, x_n, x_{n+1})$$
$$+\varphi(x_{n+1},x_n) - \varphi(x_n,x_n)) + \langle g(x_{n+1}-x_n), x_n - x_{n-1}\rangle \geq 0, \qquad (3.1.29)$$

for all $y \in C$ and

$$\rho(G(x_{n+1},x_n) + \max_{(u_{n+1},v_{n+1})\in T(x_{n+1})\times A(x_{n+1})} F(N(u_{n+1},v_{n+1}) - \omega^*, x_{n+1}, x_n)$$
$$+\varphi(x_n,x_{n+1}) - \varphi(x_{n+1},x_{n+1})) + \langle g(x_n-x_{n+1}), x_{n+1} - x_n\rangle \geq 0, \quad (3.1.30)$$

for all $y \in C$. Adding (3.1.29) and (3.1.30), we get

$$\begin{aligned}&\rho(G(x_n, x_{n+1}) + G(x_{n+1}, x_n) + \varphi(x_{n+1}, x_n) - \varphi(x_n, x_n) + \varphi(x_n, x_{n+1})\\&-\varphi(x_{n+1}, x_{n+1}) + \max_{(u_n,v_n)\in T(x_n)\times A(x_n)} F(N(u_n, v_n) - \omega^*, x_n, x_{n+1})\\&+ \max_{(u_{n+1},v_{n+1})\in T(x_{n+1})\times A(x_{n+1})} F(N(u_{n+1}, v_{n+1}) - \omega^*, x_{n+1}, x_n))\\&+\langle g(x_{n+1} - x_n), x_n - x_{n-1}\rangle + \langle g(x_n - x_{n+1}), x_{n+1} - x_n\rangle \geq 0. \qquad (3.1.31)\end{aligned}$$

Note that

$$\begin{aligned}&\rho(G(x_n, x_{n+1}) + G(x_{n+1}, x_n) + \varphi(x_{n+1}, x_n) - \varphi(x_n, x_n) + \varphi(x_n, x_{n+1})\\&-\varphi(x_{n+1}, x_{n+1}) + \max_{(u_n,v_n)\in T(x_n)\times A(x_n)} F(N(u_n, v_n) - \omega^*, x_n, x_{n+1})\\&+ \max_{(u_{n+1},v_{n+1})\in T(x_{n+1})\times A(x_{n+1})} F(N(u_{n+1}, v_{n+1}) - \omega^*, x_{n+1}, x_n))\\&+\langle g(x_{n+1} - x_n), x_n - x_{n-1}\rangle + \langle g(x_n - x_{n+1}), x_{n+1} - x_n\rangle \geq 0.\end{aligned}$$

Since G is $\alpha-$strongly monotone, F is $(\omega^*, \sigma)-$strongly monotone and φ is skew-symmetric and g is $\lambda-$strongly positive and $\|g\|-$Lipschitz continuous, we have

$$\begin{aligned}0 \ \leq\ & \rho(G(x_n, x_{n+1}) + G(x_{n+1}, x_n) + \varphi(x_{n+1}, x_n) - \varphi(x_n, x_n) + \varphi(x_n, x_{n+1})\\&-\varphi(x_{n+1}, x_{n+1}) + \max_{(u_n,v_n)\in T(x_n)\times A(x_n)} F(N(u_n, v_n) - \omega^*, x_n, x_{n+1})\\&+ \max_{(u_{n+1},v_{n+1})\in T(x_{n+1})\times A(x_{n+1})} F(N(u_{n+1}, v_{n+1}) - \omega^*, x_{n+1}, x_n))\\&+\langle g(x_{n+1} - x_n), x_n - x_{n-1}\rangle + \langle g(x_n - x_{n+1}), x_{n+1} - x_n\rangle\\ \leq\ & \rho(-\alpha\|x_n - x_{n+1}\| - \sigma\|x_n - x_{n+1}\|^2) - \lambda\|x_{n+1} - x_n\|^2\\&+\|g\|\|x_{n+1} - x_n\|\|x_n - x_{n-1}\|.\end{aligned}$$

Dividing by $\|x_{n+1} - x_n\|$, we have that

$$\begin{aligned}0 \ \leq\ & -\rho\alpha - \rho\sigma\|x_n - x_{n+1}\| - \lambda\|x_{n+1} - x_n\| + \|g\|\|x_n - x_{n-1}\|\\ <\ & -\rho\sigma\|x_n - x_{n+1}\| - \lambda\|x_{n+1} - x_n\| + \|g\|\|x_n - x_{n-1}\|.\end{aligned}$$

This implies that

$$\|x_{n+1} - x_n\| < \frac{\|g\|}{\rho\sigma + \lambda}\|x_n - x_{n-1}\|.$$

From $\frac{\|g\|}{\lambda+\rho\sigma} < 1$, we have that $\{x_n\}$ is Cauchy sequence in $C \cap D$. Indeed From

$$\|x_{n+1} - x_n\| < \frac{\|g\|}{\rho\sigma + \lambda}\|x_n - x_{n-1}\|,$$

and $q := \frac{\|g\|}{\rho\sigma+\lambda} < 1$, we consider for all n, m with $n > m$,

$$\begin{aligned}
\|x_n - x_m\| &< \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \ldots + \|x_{m+1} - x_m\| \\
&< q^{n-1}\|x_1 - x_0\| + q^{n-2}\|x_1 - x_0\| + \ldots + q^m\|x_1 - x_0\| \\
&= (q^{n-1} + q^{n-2} + \ldots + q^m)\|x_1 - x_0\| \\
&= q^m(q^{n-m-1} + q^{n-m-2} + \ldots + 1)\|x_1 - x_0\| \\
&< q^m(\sum_{i=1}^{\infty} q^i)\|x_1 - x_0\| \\
&= q^m(\frac{1}{1-q})\|x_1 - x_0\|.
\end{aligned}$$

Since $q < 1$, we have $\{x_n\}$ is a Cauchy sequence.

Let $x_n \to \hat{x} \in C \cap D$ as $n \to \infty$. By Algorithm 3.1.9, we have that for all $n \geq 0$,

$$\begin{aligned}
&\rho(G(x_{n+1}, y) + \max_{(u_{n+1}, v_{n+1}) \in T(x_{n+1}) \times A(x_{n+1})} F(N(u_{n+1}, v_{n+1}) - \omega^*, x_{n+1}, y) \\
&+\varphi(y, x_{n+1}) - \varphi(x_{n+1}, x_{n+1})) + \langle g(y - x_{n+1}), x_{n+1} - x_n\rangle \geq 0, \qquad (3.1.32)
\end{aligned}$$

for all $y \in C$. Since for each $y \in C, ((u, v), z) \mapsto F(N(u, v) - \omega^*, z, y)$ is upper semicontinuous, for each $y \in C, x \mapsto G(x, y)$ is upper semicontinuous, and $T \times A : C \to C(B^*) \times C(B^*)$ is upper semicontinuous with compact values, it follows from Lemma 3.1.4 that for each $y \in C, z \mapsto \max_{(u,v) \in T(z) \times A(z)} F(N(u, v) - \omega^*, z, y)$ is upper semicontinuous. Since φ is continuous, letting $n \to \infty$ in (3.1.32), we obtain

$$G(\hat{x}, y) + \max_{(u,v) \in T(\hat{x}) \times A(\hat{x})} F(N(u, v) - \omega^*, \hat{x}, y) + \varphi(\hat{x}, y) - \varphi(\hat{x}, \hat{x}) \geq 0 \qquad \forall y \in C$$

Therefore $\hat{x} \in C \cap D$ is a solution of NGMEP (3.1.2). and by Theorem 3.1.5 the solution set $S_{G,F,N}^{A,T,\varphi}$ of the NGMEP (3.1.2) is nonempty compact convex set of $C \cap D$. This completes the proof. $\square$

Now, we suppose the solution set $S_{G,F,N}^{A,T,\varphi}$ of the NGMEP (3.1.2) is nonempty compact convex set of $C \cap D$. Let $g : B \to B^*, x \in S_{G,F,N}^{A,T,\varphi}$ and $\rho > 0$. Related to BNGMEP (3.1.1) – (3.1.2), we consider the following auxiliary new generalized mixed equilibrium problem :

$$\text{(ABNGMEP)} \begin{cases} \text{find } z \in S_{G,F,N}^{A,T,\varphi} \text{ such that} \\ \rho(K(z,y) + \max_{(u,v)\in Q(z)\times S(z)} H(M(u,v) - \tau^*, z, y) + \psi(y,z) \\ - \psi(z,z)) + \langle g(y-z), z-x\rangle \geq 0, \qquad \forall y \in S_{G,F,N}^{A,T,\varphi}. \end{cases} \tag{3.1.33}$$

Lemma 3.1.11. *Let $S_{G,F,N}^{A,T,\varphi}$ be the solution set of the NGMEP (3.1.2) in Theorem 3.1.10. Let $K : C \times C \to \mathbb{R}$ be a bifunction, $Q, S : C \to C(B^*)$ be set-values mappings, $M : B^* \times B^* \to B^*$ and $H : B^* \times C \times C \to \mathbb{R}, g : B \to B^*$ be single-valued mappings and $\psi : B \times B \to \mathbb{R}$ be a bifunction and let $\tau^* \in B^*$. Suppose the following conditions are satisfied:*

(i) *$H(M(u,v) - \tau^*, x, x) \geq 0,\ \forall x \in C, (u,v) \in Q(x) \times S(x)$;*

(ii) *H is β^*-monotone with respect to M, S and Q; and for each $y \in C$, $((u,v), z) \mapsto H(M(u,v) - \tau^*, z, y)$ is upper semicontinuous;*

(iii) *for each $z \in C$ and $(u,v) \in Q(z) \times S(z), y \mapsto H(M(u,v) - \tau^*, z, y)$ is convex;*

(iv)*K is $\varsigma-$strongly monotone such that $K(x,x) \geq 0$ for each $x \in C$;*

(v) *for each $y \in C, x \mapsto K(x,y)$ is upper semicontinuous and for each $x \in C, y \mapsto K(x,y)$ is convex;*

(vi) *g is a $\lambda-$strongly positive bounded linear operator, T and A are both upper semicontinuous;*

(vii) *φ is continuous, skew symmetric and convex in first argument;*

(viii) *for each* $x \in B$, *there exists* $y \in C \cap S_{G,F,N}^{A,T,\varphi}$ *such that*

$$\begin{aligned}&\rho(K(z,y) + \max_{(u,v)\in Q(z)\times S(z)} H(M(u,v) - \omega^*, z, y) + \psi(y,z) - \psi(z,z))\\&+\langle g(y-z), z-x\rangle < 0, \qquad \forall z \in C \setminus (C \cap S_{G,F,N}^{A,T,\varphi}).\end{aligned}$$

Then for each $x \in B$, *there exists a unique point* $\hat{z} \in S_{G,F,N}^{A,T,\varphi}$ *such that*

$$\begin{aligned}&\rho(K(\hat{z},y) + \max_{(u,v)\in Q(\hat{z})\times S(\hat{z})} H(M(u,v) - \tau^*, \hat{z}, y) + \psi(y,\hat{z})\\&-\psi(\hat{z},\hat{z})) + \langle g(y-\hat{z}), \hat{z}-x\rangle, \qquad \forall y \in S_{G,F,N}^{A,T,\varphi},\end{aligned} \tag{3.1.34}$$

that is $\hat{z} \in S_{G,F,N}^{A,T,\varphi}$ *is a unique solution of the ANGMEP* (3.1.33).

Proof. For each fixed $x \in B$, define $f : S_{G,F,N}^{A,T,\varphi} \times S_{G,F,N}^{A,T,\varphi} \to \mathbb{R}$ by

$$\begin{aligned}f(z,y) &= \rho\left(K(z,y) + \max_{(u,v)\in Q(z)\times S(z)} H(M(u,v) - \tau^*, z, y) + \psi(y,z) - \psi(z,z)\right)\\&\quad +\langle g(y-z), z-x\rangle \geq 0, \forall z, y \in S_{G,F,N}^{A,T,\varphi}.\end{aligned}$$

By using same argument as in proof of Theorem 3.1.7 , we can show that for each $x \in B$, there exists a point $\hat{z} \in S_{G,F,N}^{A,T,\varphi}$ such that $f(\hat{z},y) \geq 0$, for all $y \in S_{G,F,N}^{A,T,\varphi}$. By definition of f, we obtain that for each $x \in B$, there exists $\hat{z} \in S_{G,F,N}^{A,T,\varphi}$ such that

$$\begin{aligned}&\rho\left(K(\hat{z},y) + \max_{(u,v)\in Q(\hat{z})\times S(\hat{z})} H(M(u,v) - \tau^*, \hat{z}, y) + \psi(y,\hat{z}) - \psi(\hat{z},\hat{z},)\right)\\&+ \ \langle g(y-\hat{z}), \hat{z}-x\rangle \geq 0, \forall y \in S_{G,F,N}^{A,T,\varphi}.\end{aligned}$$

that is $\hat{z} \in S_{G,F,N}^{A,T,\varphi}$ is a unique solution of the ANGMEP (3.1.33). □

By using Lemma 3.1.11, we can construct the following iterative algorithm for computing approximate solutions of the BNGMEP (3.1.1) – (3.1.2)

Algorithm 3.1.12. *For given $x_0 \in B$ by Lemma 3.1.11, there exists $x_1 \in S_{G,F,N}^{A,T,\varphi}$ such that*

$$\rho(K(x_1,y)+\max_{(u_1,v_1)\in Q(x_1)\times S(x_1)} H(M(u_1,v_1)-\omega^*,x_1,y)$$
$$+\psi(y,x_1)-\psi(x_1,x_1))+\langle g(y-x_1),x_1-x_0\rangle \geq 0, \qquad \forall y \in S_{G,F,N}^{A,T,\varphi}.$$

By induction, we can define the iterative sequences $\{x_n\} \subset S_{G,F,N}^{A,T,\varphi}$ such that

$$\rho(K(x_{n+1},y)+\max_{(u_{n+1},v_{n+1})\in Q(x_{n+1})\times S(x_{n+1})} H(M(u_{n+1},v_{n+1})-\omega^*,x_{n+1},y)$$
$$+\psi(y,x_{n+1})-\psi(x_{n+1},x_{n+1}))+\langle g(y-x_{n+1}),x_{n+1}-x_n\rangle \geq 0, \qquad (3.1.35)$$

for all $y \in S_{G,F,N}^{A,T,\varphi}$.

Theorem 3.1.13. *Let C be a closed convex subset of a Banach space B, D be a compact subset of B with $C \cap D \neq \emptyset$ and $\rho > 0$ be a positive number. Let $G,K : C\times C \to \mathbb{R}$ be a bifunction. Let $H,F : B^*\times C\times C \to \mathbb{R}, Q,S,T,A : C \to C(B^*), \psi,\varphi : B\times B \to \mathbb{R}$ and $g : B \to B^*$ satisfy the following conditions:*

(i) *H is $(\omega^*,\beta)-$strongly monotone with respect to M,Q and S; and F is $(\omega^*,\sigma)-$strongly monotone with respect to N,A and T; and for all $x \in C, (\bar{u},\bar{v}) \in Q(x)\times S(x)$ and $(u,v) \in A(x)\times T(x), H(M(\bar{u},\bar{v}),x,x) \geq 0$ and $F(N(u,v),x,x) \geq 0$;*

(ii) *for each $y \in C, ((u,v),z) \mapsto H(M(u,v),z,y)$ and $((u,v),x) \mapsto F(N(u,v),x,y)$ is upper semicontinuous and for each $x \in C, (\bar{u},\bar{v}) \in Q(x)\times S(x)$ and $(u,v) \in A(x)\times T(x), y \mapsto H(M(\bar{u},\bar{v}),x,y)$ and $y \mapsto F(N(u,v),x,y)$ are both convex and lower-semicontinuous;*

(iii) *G is $\alpha-$strongly monotone, K is $\varsigma-$strongly monotone such that $K(x,x) \geq 0$ and $G(x,x) \geq 0$ for each $x \in C$;*

(iv) *for each $y \in C, x \mapsto G(x,y)$ and $K(x,y)$ are upper semicontinuous and for each $x \in C, y \mapsto G(x,y)$ and $K(x,y)$ are both convex;*

(v) *Q,S,A and T are upper semicontinuous;*

(vi) *ψ and φ are both skew symmetric, continuous and convex in first argument;*

(vii) *g is a $\lambda-$strongly positive and bounded linear operator.*

(viii) *for each $x \in B$, there exists $y \in C \cap D$ such that*

$$\begin{aligned}&\rho(G(z,y) + \max_{(u,v)\in T(z)\times A(z)} F(N(u,v) - \omega^*, z, y) + \varphi(y,z) - \varphi(z,z))\\&+\langle g(y-z), z-x\rangle < 0, \qquad \forall z \in C \setminus D.\end{aligned}$$

and for each $x \in B$, there exists $y \in C \cap S^{A,T,\varphi}_{G,F,N}$ such that

$$\begin{aligned}&\rho(K(z,y) + \max_{(u,v)\in Q(z)\times S(z)} H(M(u,v) - \omega^*, z, y) + \psi(y,z) - \psi(z,z))\\&+\langle g(y-z), z-x\rangle < 0, \qquad \forall z \in C \setminus (C \cap S^{A,T,\varphi}_{G,F,N}).\end{aligned}$$

If $\max\{\frac{\|g\|}{\lambda+\rho\beta}, \frac{\|g\|}{\lambda+\rho\sigma}\} < 1$, then the iterative sequence $\{x_n\}$ defined by the Algorithm 3.1.9 converges strongly to a solutions $\hat{x} \in C \cap D$ of the NGMEP (3.1.2) and the solution set $S^{A,T,\varphi}_{G,F,N}$ of the NGMEP (3.1.2) is a nonempty compact convex in $C \cap D$. Also the iterative sequence $\{x_n\}$ defined by the Algorithm 3.1.12 converges strongly to a solution $\bar{x} \in S^{A,T,\varphi}_{G,F,N}$ of the BNGMEP (3.1.1) – (3.1.2) and the solution set of the BNGMEP (3.1.1) – (3.1.2) is also nonempty compact convex in $C \cap S^{A,T,\varphi}_{G,F,N}$.

Proof. It is easy to check that G, F, N, T, A and φ satisfy all conditions of Theorem 3.1.10. By Theorem 3.1.10, the iterative sequence $\{x_n\}$ defined by the Algorithm 3.1.9 converges strongly to a solutions $\hat{x} \in C \cap D$ of the NGMEP (3.1.2) and the solution set $S^{A,T,\varphi}_{G,F,N}$ of the NGMEP (3.1.2) is a nonempty compact convex in $C \cap D$. By using similar argument as in the proof of Theorem 3.1.10, it is easy to show that the iterative sequence $\{x_n\}$ defined by the Algorithm 3.1.12 converges strongly to a solution $\bar{x} \in S^{A,T,\varphi}_{G,F,N}$ of the BNGMEP (3.1.1) – (3.1.2) and the solution set of the BNGMEP (3.1.1) – (3.1.2) is also nonempty compact convex in $C \cap S^{A,T,\varphi}_{G,F,N}$. □

3.2 Existence and Iterative Approximation Methods for Generalized Mixed Vector Equilibrium Problems with Relaxed Monotone Mappings

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let X be a nonempty closed convex subset of H. Let $\varphi : X \times X \to \mathbb{R} = (-\infty, +\infty)$ be a bifunction. The equilibrium problem $EP(\varphi)$ is to find $x \in X$ such that

$$\varphi(x, y) \geq 0, \qquad \forall y \in X. \tag{3.2.1}$$

In 2002, Moudafi [13] introduced an iterative scheme of finding the solution of nonexpansive mappings and proved a strong convergence theorem. Recently, Huang, et al. [14] introduced the approximate method for solving the equilibrium problem and proved the strong convergence theorem.

Let $\varphi : X \times X \to \mathbb{R}$ be a bifunction and $T, A : X \to H$ nonlinear mappings. In 2010, Wang, Marino and Wang [62] introduced the following generalized mixed equilibrium problem with a relaxed monotone mapping:

$$\text{Find } z \in C \text{ such that } \varphi(z, y) + \langle Tz, \eta(y, z) \rangle + \langle Az, y - z \rangle \geq 0, \qquad \forall y \in C. \tag{3.2.2}$$

Problem (3.2.2) is very general setting and it includes as special cases of Nash equilibrium problems, complementarity problems, fixed point problems, optimization problems and variational inequalities (see, for example [14, 17] and the references therein). Moreover, Wang, Marino and Wang [62] studied the existence of solutions for the proposed problem and introduced a new iterative scheme for finding a common element of the set of solutions of a generalized equilibrium problem with a relaxed monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert space.

Let Y be a Hausdorff topological vector space and C be a closed, convex and pointed cone of Y with $\text{int}C \neq \emptyset$. Let $\varphi : X \times X \to Y$ be a vector-valued bifunction. The strong vector equilibrium problem (for short, SVEP(φ)) is to find $z \in X$ such that

$$\varphi(z,y) \in C, \qquad \forall y \in X \tag{3.2.3}$$

and the weak vector equilibrium problem (for short, WVEP(φ)) is to find $z \in X$ such that

$$\varphi(z,y) \notin \text{-int}C, \qquad \forall y \in X. \tag{3.2.4}$$

In this work, we consider the following generalized mixed vector equilibrium problem with a relaxed monotone mapping (for short, GVEPR(φ, T)): find $z \in X$ such that

$$\varphi(z,y) + e\langle Tz, \eta(y,z)\rangle + e\langle Az, y-z\rangle \in C, \qquad \forall y \in X, \tag{3.2.5}$$

where $e \in \text{int}C$, $\varphi : X \times X \to Y$, and $T, A : X \to H$ are the mappings. The set of all solutions of the generalized mixed vector equilibrium problem with a relaxed monotone mapping is denoted by SGVEPR(φ, T), that is

$$\text{SGVEPR}(\varphi,T) = \{z \in X : \varphi(z,y) + e\langle Tz, \eta(y,z)\rangle + e\langle Az, y-z\rangle \in C,\ \forall y \in X\}.$$

If $A = 0$, we denote the set ASGVEPR(φ, T) by

$$\text{ASGVEPR}(\varphi,T) = \{z \in X : \varphi(z,y) + e\langle Tz, \eta(y,z)\rangle \in C,\ \forall y \in X\}.$$

Some special cases of the problem (3.2.5) are as follows:

(1) If $Y = \mathbb{R}, C = \mathbb{R}^+$ and $e = 1$, then GVEPR(φ, T) (3.2.5) reduces to the generalized mixed equilibrium problem with a relaxed monotone mapping which introduced by Wang, Marino and Wang [62].

(2) If $T = 0$ and $A = 0$, then GVEPR(φ, T) (3.2.5) reduces to the classic vector equilibrium problem (3.2.3).

We consider the auxiliary problem of GVEPR(φ, T) and prove the existence and uniqueness of the solutions of auxiliary problem of GVEPR(φ, T) under some proper conditions. By using the result for the auxiliary problem, we introduce a new iterative scheme for finding a common element of the set of solutions of a generalized mixed vector equilibrium problem with a relaxed monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings and then obtain a strong convergence theorem. The results presented in this paper improve and generalize some known results of Wang, Marino and Wang [62].

3.2.1 The existence of solutions for the generalized mixed vector equilibrium problem with a relaxed monotone mapping

For solving the generalized mixed vector equilibrium problem with a relaxed monotone mapping, we give the following assumptions. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, repspectively. Assume that $X \subseteq H$ is nonempty, compact, convex subset and Y is real Hausdorff topological vector space, $C \subseteq Y$ is a closed, convex and pointed cone. Let $\varphi : X \times X \to Y, T : X \to H$ be two mappings. For any $x \in H$, define a mapping $\Phi_x : X \times X \to Y$ as follows:

$$\Phi_x(z, y) = \varphi(z, y) + e\langle Tz, \eta(y, z)\rangle + \frac{e}{r}\langle y - z, z - x\rangle,$$

where r is a positive number in $\mathbb{R}$ and $e \in C \setminus \{0\}$. Let Φ_x, φ and T satisfy the following conditions:

(A_1) for all $z \in X, \varphi(z, z) = \theta$;

(A_2) φ is monotone, that is, $\varphi(z, y) + \varphi(y, z) \in -C$ for all $z, y \in X$;

(A_3) $\varphi(\cdot, y)$ is $C-$continuous for all $y \in X$;

(A_4) $\varphi(z, \cdot)$ is $C-$convex, that is,

$$t\varphi(z, y_1) + (1-t)\varphi(z, y_2) \in \varphi(z, ty_1 + (1-t)y_2) + C, \ \ \forall z, y_1, y_2 \in X, \forall t \in [0, 1];$$

(A_5) $\langle T(\cdot), \eta(\cdot,\cdot)\rangle$ is continuous and

$$y \mapsto \langle Tu, \eta(y,v)\rangle \text{ is convex;}$$

(A_6) $\Phi_x(z,\cdot)$ is proper $C-$quasiconvex for all $z \in X$ and $x \in H$.

Remark 3.2.1. Let $Y = \mathbb{R}, C = \mathbb{R}^+$ and $e = 1$. For any $y \in X$, if $\varphi(\cdot, y)$ is upper semicontinuous and $z \mapsto \langle Tz, \eta(y,z)\rangle$ is continuous, then $\Phi_x(\cdot, y)$ is lower $C-$continuous. In fact, since $\varphi(\cdot, y)$ is upper semicontinuous and $z \mapsto \langle Tz, \eta(y,z)\rangle$ is continuous, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $z \in \{z \in X, \|z - z_0\| < \delta\}$, we have

$$\Phi_x(z,y) < \Phi_x(z_0,y) + \epsilon,$$

where z_0 is a point in X. This means $\Phi_x(\cdot, y)$ is lower $C-$continuous.

Remark 3.2.2. Let $Y = \mathbb{R}, C = \mathbb{R}^+$ and $e = 1$. Assume that $\varphi(z,\cdot)$ is a convex mapping for all $z \in X$. Then for any $y_1, y_2 \in X$ and $t \in [0,1]$, we have

$$\begin{aligned}
& \Phi_x(z, ty_1 + (1-t)y_2) \\
= \;& \varphi(z, ty_1 + (1-t)y_2) + \langle Tz, \eta(ty_1 + (1-t)y_2, z)\rangle + \frac{1}{r}\langle ty_1 + (1-t)y_2 - z, z - x\rangle \\
\leq \;& t\varphi(z,y_1) + (1-t)\varphi(z,y_2) + t\langle Tz, \eta(y_1,z)\rangle + (1-t)\langle Tz, \eta(y_2,z)\rangle \\
& + \frac{t}{r}\langle y_1 - z, z - x\rangle + \frac{1-t}{r}\langle y_2 - z, z - x\rangle \\
= \;& t(\varphi(z,y_1) + \langle Tz, \eta(y_1,z)\rangle + \frac{1}{r}\langle y_1 - z, z - x\rangle) \\
& + (1-t)(\varphi(z,y_2) + \langle Tz, \eta(y_2,z)\rangle + \frac{1}{r}\langle y_2 - z, z - x\rangle) \\
= \;& t\Phi_x(z,y_1) + (1-t)\Phi_x(z,y_2) \\
\leq \;& \max\{\Phi_x(z,y_1), \Phi_x(z,y_2)\},
\end{aligned}$$

which implies that $\Phi_x(z,\cdot)$ is proper $C-$quasiconvex.

Now we are the position to state and prove the existence of solutions for the generalized mixed vector equilibrium problem with a relaxed monotone mapping.

Theorem 3.2.3. *Let X be a nonempty, compact, convex subset of a real Hilbert space H. Let C be a closed, convex and pointed cone of a Hausdorff topological vector space Y. Let $T : X \to H$ be a relaxed $\eta - \alpha-$monotone mapping. Let $\varphi : X \times X \to Y$ be a vector-valued bifunction. Suppose that all the conditions (A_1) - (A_6) are satisfied. Let $r > 0$ and define a mapping $Z_r : H \to X$ as follows:*

$$Z_r(x) = \{z \in X : \varphi(z,y) + e\langle Tz, \eta(y,z)\rangle + \frac{e}{r}\langle y-z, z-x\rangle \in C, \quad \forall y \in X\}. \quad (3.2.6)$$

for all $x \in H$. Assume that

(i) $\eta(x,y) + \eta(y,x) = 0$, for all $x, y \in X$;

(ii) for any $x, y \in X, \alpha(x-y) + \alpha(y-x) \geq 0$.

Then, the following holds:

(1) $Z_r(z) \neq \emptyset$ for all $z \in X$;

(2) Z_r is single-value;

(3) Z_r is a firmly nonexpansive mapping, that is, for all $x, y \in X$,

$$\|Z_r x - Z_r y\|^2 \leq \langle Z_r x - Z_r y, x - y\rangle;$$

(4) $F(Z_r) = ASGVEPR(\varphi, T)$;

(5) $ASGVEPR(\varphi, T)$ is closed and convex.

Proof. (1) In Lemma 2.2.32, let $f(z,y) = \Phi_x(z,y)$ and $\psi(z) = \theta$ for all $z, y \in X$ and $x \in H$. Then it is easy to check that f and ψ satisfy all the conditions of Lemma 2.2.32. Indeed,

(i) Show that $f(x,x) \in C$, for all $x \in X$;

Since $\eta(x,y)+\eta(y,x)=0$, for all $x,y \in X$, we have $\eta(x,x)=0$ for all $x \in X$. From A_1, that is for all $z \in X, \varphi(z,z)=\theta$, we can get that

$$f(x,x)=0 \in C, \ \ \forall x \in X.$$

(ii) Show that ψ is upper $C-$ *continuous* on X.

Since $\psi(z)=\theta$, we have ψ is upper $C-$ *continuous* on X.

(iii) Show that $f(\cdot,y)$ is lower $C-$ *continuous* for all $y \in X$.

It is sufficient to show that if g is continuous, then εg is $C-$continuous, where $e \in C \setminus \{0\}$.

Let V be a neighborhood of 0 in Y. We can choose $\varepsilon > 0$ such that

$$e\varepsilon \in V.$$

Since f is continuous, there is a neighborhood U_δ of x_0 such that

$$|g(x)-g(x_0)| < \varepsilon, \forall x \in U_\delta.$$

For all $x \in U_\delta$,

$$g(x)-g(x_0) < \varepsilon.$$

This implies that

$$\varepsilon-(g(x)-g(x_0)) > 0.$$

Therefore

$$e\varepsilon-(eg(x)-eg(x_0)) \in C.$$

Hence

$$-(eg(x)-eg(x_0)) \in -e\varepsilon + C,$$

which gives us that

$$eg(x)-eg(x_0) \in V-C.$$

Hence eg is $C-$continuous.

This can show that $f(\cdot, y)$ is lower $C-$continuous for all $y \in X$.

(iv) Show that $f(x, \cdot) + \psi(\cdot)$ is proper $C-$quasiconvex for all $x \in X$.

It is sufficient to show that if g is convex, then eg is proper $C-$quasiconvex, where $e \in C \setminus \{0\}$.

Let $x, y \in X$ and $t \in [0, 1]$. Since g is convex, we have

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y) \leq \max g(x), g(y).$$

This implies that

$$g(tx + (1-t)y) \leq g(x) \text{ or } g(tx + (1-t)g(y)) \leq g(y).$$

Hence

$$eg(x) - e(g(tx + (1-t)y)) \in C \text{ or } eg(y) - e(g(tx + (1-t)y)) \in C.$$

Therefore eg is proper $C-$quasiconvex..

We can complete the proof of conclusion.

Thus, there exists a point $z \in X$ such that

$$f(z, y) + \psi(z) - \psi(y) \in C, \qquad \forall y \in X, x \in H$$

which gives that, for any $x \in H$,

$$\varphi(z, y) + e\langle Tz, \eta(y, z)\rangle + \frac{e}{r}\langle y - z, z - x\rangle \in C, \qquad \forall y \in X.$$

Therefor we conclude that $Z_r(x) \neq \emptyset$ for all $x \in H$.

(2) For $x \in H$ and $r > 0$, let $z_1, z_2 \in Z_r(x)$. Then

$$\varphi(z_1, y) + e\langle Tz_1, \eta(y, z_1)\rangle + \frac{e}{r}\langle y - z_1, z_1 - x\rangle \in C, \qquad \forall y \in X \qquad (3.2.7)$$

and

$$\varphi(z_2, y) + e\langle Tz_2, \eta(y, z_2)\rangle + \frac{e}{r}\langle y - z_2, z_2 - x\rangle \in C, \qquad \forall y \in X. \tag{3.2.8}$$

Letting $y = z_2$ in (3.2.7) and $y = z_1$ in (3.2.8), adding (3.2.7) and (3.2.8), we have

$$\varphi(z_2, z_1) + \varphi(z_1, z_2) + e\langle Tz_1 - Tz_2, \eta(z_2, z_1)\rangle + \frac{e}{r}\langle z_1 - z_2, z_2 - z_1\rangle \in C.$$

By the monotonicity of φ, we have

$$e\langle Tz_1 - Tz_2, \eta(z_2, z_1)\rangle + \frac{e}{r}\langle z_1 - z_2, z_2 - z_1\rangle \in C.$$

Thus

$$\frac{e}{r}\langle z_1 - z_2, z_2 - z_1\rangle - e\langle Tz_2 - Tz_1, \eta(z_2, z_1)\rangle \in C. \tag{3.2.9}$$

Since T is relaxed $\eta - \alpha-$monotone, we have

$$\langle Tz_2 - Tz_1, \eta(z_2, z_1)\rangle - \alpha(z_2 - z_1) \geq 0,$$

this implies that

$$r\langle Tz_2 - Tz_1, \eta(z_2, z_1)\rangle - r\alpha(z_2 - z_1) \geq 0.$$

Then

$$er\langle Tz_2 - Tz_1, \eta(z_2, z_1)\rangle - er\alpha(z_2 - z_1) \in C.$$

Then we have that

$$e\langle z_1 - z_2, z_2 - z_1\rangle - er\alpha(z_2 - z_1) \in C. \tag{3.2.10}$$

In (3.2.9) exchanging the position of z_1 and z_2, we get

$$\frac{e}{r}\langle z_2 - z_1, z_1 - z_2\rangle - e\alpha(z_1 - z_2) \in C,$$

that is,

$$e\langle z_2 - z_1, z_1 - z_2\rangle - er\alpha(z_1 - z_2) \in C. \tag{3.2.11}$$

Now, adding the inequalities (3.2.10) and (3.2.11),

$$e\langle z_1 - z_2, z_2 - z_1\rangle - er\alpha(z_2 - z_1) + e\langle z_2 - z_1, z_1 - z_2\rangle - er\alpha(z_1 - z_2) \in C.$$

By using (ii), we have

$$2e\langle z_2 - z_1, z_1 - z_2\rangle \in C. \tag{3.2.12}$$

If $\langle z_2 - z_1, z_1 - z_2\rangle < 0$, then

$$-2\langle z_2 - z_1, z_1 - z_2\rangle > 0.$$

This implies that

$$-2e\langle z_2 - z_1, z_1 - z_2\rangle \in C. \tag{3.2.13}$$

From (3.2.12) and (3.2.13), we have $z_1 = z_2$ which is a contradiction. Thus

$$\langle z_2 - z_1, z_1 - z_2\rangle \geq 0,$$

so

$$-\|z_1 - z_2\|^2 = \langle z_1 - z_2, z_2 - z_1\rangle \geq 0.$$

Hence $z_1 = z_2$. Therefore Z_r is single-value.

(3) For any $x_1, x_2 \in H$, let $z_1 = Z_r(x_1)$ and $z_2 = Z_r(x_2)$. Then

$$\varphi(z_1, y) + e\langle Tz_1, \eta(y, z_1)\rangle + \frac{e}{r}\langle y - z_1, z_1 - x_1\rangle \in C, \qquad \forall y \in X \tag{3.2.14}$$

and

$$\varphi(z_2, y) + e\langle Tz_2, \eta(y, z_2)\rangle + \frac{e}{r}\langle y - z_2, z_2 - x_2\rangle \in C, \qquad \forall y \in X. \tag{3.2.15}$$

Letting $y = z_2$ in (3.2.14) and $y = z_1$ in (3.2.15), adding (3.2.14) and (3.2.15), we have

$$\begin{aligned}&\varphi(z_1, z_2) + \varphi(z_2, z_1) + e\langle Tz_1, \eta(z_2, z_1)\rangle + e\langle Tz_2, \eta(z_1, z_2)\rangle \\ &+\frac{e}{r}\langle z_2 - z_1, z_1 - z_2 - (x_1 - x_2)\rangle \in C.\end{aligned} \tag{3.2.16}$$

Since φ is monotone and C is closed convex cone, we get

$$\langle Tz_1 - Tz_2, \eta(z_2, z_1)\rangle + \frac{1}{r}\langle z_2 - z_1, z_1 - z_2 - x_1 + x_2\rangle \geq 0,$$

that is,

$$\frac{1}{r}\langle z_2 - z_1, z_1 - z_2 - x_1 + x_2\rangle \geq \langle Tz_2 - Tz_1, \eta(z_2, z_1)\rangle \geq \alpha(z_2 - z_1). \quad (3.2.17)$$

In (3.2.17) exchanging the position of z_1 and z_2, we get

$$\frac{1}{r}\langle z_1 - z_2, z_2 - z_1 - x_2 + x_1\rangle \geq \alpha(z_1 - z_2). \quad (3.2.18)$$

Adding the inequalities (3.2.17) and (3.2.18), we have

$$2\langle z_1 - z_2, z_2 - z_1 - x_2 + x_1\rangle \geq r(\alpha(z_1 - z_2) + \alpha(z_2 - z_1)). \quad (3.2.19)$$

It follows from (ii) that

$$\langle z_1 - z_2, z_2 - z_1 - x_2 + x_1\rangle \geq 0.$$

This implies that

$$\|Z_r x_1 - Z_r x_2\|^2 \leq \langle Z_r x_1 - Z_r x_2, x_1 - x_2\rangle.$$

This shows that Z_r is firmly nonexpansive.

(4) We claim that $F(Z_r) = \text{ASGVEPR}(\varphi, T)$. Indeed, we have the following:

$$\begin{aligned} x \in F(Z_r) &\Leftrightarrow x = Z_r x \\ &\Leftrightarrow \varphi(x, y) + e\langle Tx, \eta(y, x)\rangle + \frac{e}{r}\langle y - x, x - x\rangle \in C, \qquad \forall y \in X \\ &\Leftrightarrow \varphi(x, y) + e\langle Tx, \eta(y, x)\rangle \in C, \qquad \forall y \in X \\ &\Leftrightarrow x \in \text{ASGVEPR}(\varphi, T). \end{aligned}$$

(5) Since every firmly nonexpansive mapping is nonexpansive, we see that Z_r is nonexpansive. Since the set of fixed point of every nonexpansive mapping is closed and convex, we have that $\text{ASGVEPR}(\varphi, T)$ is closed and convex. This completes the proof.

3.2.2 Convergence Analysis

In this section, we prove a strong convergence theorem which is one of our main results.

Theorem 3.2.4. *Let X be a nonempty, compact, convex subset of a real Hilbert space H. Let C be a closed, convex cone of a real Hausdorff topological vector space Y and $e \in C \setminus \{0\}$. Let $\varphi : X \times X \to Y$ and $T : X \to H$ be a relaxed $\eta-\alpha-$monotone mapping which satisfy (A_1) - (A_6). Let $A : X \to H$ be a $\lambda-$inverse-strongly monotone mapping, and let $\{S_n\}_{n=1}^{\infty}$ be a countable family of nonexpansive mappings from X onto itself such that*

$$F := \cap_{n=1}^{\infty} Fix\ (S_n) \cap SGVEPR(\varphi, T) \neq \emptyset.$$

Assume that the conditions (i)-(ii) of Theorem 3.2.3 are satisfied. Put $\alpha_0 = 1$ and assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ is a strictly decreasing sequence. Assume that $\{\beta_n\}_{n=1}^{\infty} \subset (c,d)$ with some $c, d \in (0,1)$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [a,b]$ with some $a, b \in (0, 2\lambda)$. Then, for any $x_1 \in X$, the sequence $\{x_n\}$ generated by

$$\begin{aligned}
&\varphi(u_n, y) + e\langle Tu_n, \eta(y, u_n)\rangle + e\langle Ax_n, y - u_n\rangle + \frac{e}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \in C, \forall y \in X,\\
&y_n = \alpha_n x_n + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n S_i x_n + (1-\alpha_n)(1-\beta_n)u_n,\\
&C_n = \{z \in X : \|y_n - z\| \leq \|x_n - z\|\}, \qquad (3.2.20)\\
&D_n = \cap_{j=1}^{n} C_j,\\
&x_{n+1} = P_{D_n} x_1, \qquad n \geq 1,
\end{aligned}$$

converges strongly to $x^* \in P_F x_1$. In particular, if X contains the origin 0 and taking $x_1 = 0$, then the sequence $\{x_n\}$ generated by (3.2.20) converges strongly to the minimum norm element in F, that is $x^* = P_F 0$.

Proof. We devide the proof into several steps.

Step 1. F is closed and convex, the sequence $\{x_n\}$ generated by (3.2.20) is well defined, and $F \subset D_n$, for all $n \geq 1$.

First, we prove that F is closed and convex. It suffices to prove that SGVEPR(φ, T) is closed and convex. Indeed, it is easy to prove the conclusion by the following fact:

$$\begin{aligned}
\forall p \in \text{SGVEPR}(\varphi, T) \quad &\Leftrightarrow \quad \varphi(p, y) + e\langle Tp, \eta(y, p)\rangle + \frac{e}{\lambda_n}\langle y - p, \lambda_n Ap\rangle \in C, \ \forall y \in X \\
&\Leftrightarrow \quad \varphi(p, y) + e\langle Tp, \eta(y, p)\rangle + \frac{e}{\lambda_n}\langle y - p, p - (p - \lambda_n Ap)\rangle \in C, \\
&\Leftrightarrow \quad p = Z_{\lambda_n}(I - \lambda_n A)p.
\end{aligned}$$

This implies that

$$\text{SGVEPR}(\varphi, T) = \text{Fix}[Z_{\lambda_n}(I - \lambda_n A)].$$

Since $Z_{\lambda_n}(I - \lambda_n A)$ is a nonexpansive mapping for $\lambda_n < 2\lambda$ and the set of fixed points of a nonexpansive mapping is closed and convex, we have SGVEPR(φ, T) is closed and convex.

Next, we prove that the sequence $\{x_n\}$ generated by (3.2.20) is well defined and $F \subset D_n$ for all $n \geq 1$. By Definition of C_n, for all $z \in X$, the inequality

$$\|y_n - z\| \leq \|x_n - z\|$$

is equivalent to

$$\langle y_n - x_n, y_n + x_n\rangle - 2\langle y_n - x_n, z\rangle \leq 0.$$

Indeed, We consider

$$\begin{aligned}
&\|y_n - z\| \leq \|x_n - z\| \\
\Leftrightarrow \quad &\|y_n - z\|^2 \leq \|x_n - z\|^2 \\
\Leftrightarrow \quad &\|y_n - z\|^2 - \|x_n - z\|^2 \leq 0 \\
\Leftrightarrow \quad &\langle y_n - z, y_n - z\rangle - \langle x_n - z, x_n - z\rangle \leq 0
\end{aligned}$$

$$\Leftrightarrow \quad \langle y_n - z, y_n - z\rangle + \langle z - x_n, y_n - z\rangle + \langle y_n - z, x_n - z\rangle - \langle x_n - z, x_n - z\rangle \leq 0$$

$$\Leftrightarrow \quad \langle y_n - z, y_n - z\rangle + \langle z - x_n, y_n - z\rangle + \langle y_n - z, x_n - z\rangle + \langle z - x_n, x_n - z\rangle \leq 0$$

$$\Leftrightarrow \quad \langle y_n - z + z - x_n, y_n - z\rangle + \langle y_n - z + z - x_n, x_n - z\rangle \leq 0$$

$$\Leftrightarrow \quad \langle y_n - x_n, y_n - z\rangle + \langle y_n - x_n, x_n - z\rangle \leq 0$$

$$\Leftrightarrow \quad \langle y_n - x_n, y_n + x_n - z - z\rangle \leq 0$$

$$\Leftrightarrow \quad \langle y_n - x_n, y_n + x_n\rangle - \langle y_n - x_n, 2z\rangle \leq 0$$

$$\Leftrightarrow \quad \langle y_n - x_n, y_n + x_n\rangle - 2\langle y_n - x_n, z\rangle \leq 0$$

It is easy to see that C_n is closed and convex for all $n \in \mathbb{N}$. That is For each $n \in \mathbb{N}$, we let $z_1, z_2 \in C_n$ and $t \in (0,1)$. Then we have

$$\|y_n - z_1\| \leq \|x_n - z_1\|$$

and

$$\|y_n - z_2\| \leq \|x_n - z_2\|.$$

That is

$$\langle y_n - x_n, y_n + x_n\rangle - 2\langle y_n - x_n, z_1\rangle \leq 0$$

and

$$\langle y_n - x_n, y_n + x_n\rangle - 2\langle y_n - x_n, z_2\rangle \leq 0.$$

Since $0 < t < 1$, we have

$$t\langle y_n - x_n, y_n + x_n\rangle - 2t\langle y_n - x_n, z_1\rangle \leq 0$$

and

$$(1-t)t\langle y_n - x_n, y_n + x_n\rangle - 2(1-t)\langle y_n - x_n, z_2\rangle \leq 0.$$

Consider

$$\begin{aligned} & \langle y_n - x_n, y_n + x_n\rangle - 2\langle y_n - x_n, tz_1 + (1-t)z_2\rangle \\ = \; & \langle y_n - x_n, y_n + x_n\rangle - 2\langle y_n - x_n, tz_1\rangle - 2\langle y_n - x_n, (1-t)z_2\rangle \end{aligned}$$

$$
\begin{aligned}
&= \langle y_n - x_n, y_n + x_n\rangle - 2t\langle y_n - x_n, z_1\rangle - 2(1-t)\langle y_n - x_n, z_2\rangle \\
&= (t + (1-t))(\langle y_n - x_n, y_n + x_n\rangle) - 2t\langle y_n - x_n, z_1\rangle - 2(1-t)\langle y_n - x_n, z_2\rangle \\
&\leq 0.
\end{aligned}
$$

It is equivalent to

$$\|y_n - tz_1 - (1-t)z_2\| \leq \|x_n - tz_1 - (1-t)z_2\|.$$

Hence $tz_1 + (1-t)z_2 \in C_n$. Therefore C_n is convex for all $n \in \mathbb{N}$.

Hence D_n is closed and convex for all $n \in \mathbb{N}$. Next, we will show that

$$\|y_n - p\| = \|\alpha_n(x_n - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n(S_i x_n - p) + (1-\alpha_n)(1-\beta_n)(u_n - p)\|$$

where $y_n = \alpha_n x_n + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n S_i x_n + (1-\alpha_n)(1-\beta_n)u_n$. First, we can easy to see that

$$y_n - p = \alpha_n x_n + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n S_i x_n + (1-\alpha_n)(1-\beta_n)u_n - p.$$

In another considering, we can get that

$$
\begin{aligned}
&\alpha_n(x_n - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n(S_i x_n - p) + (1-\alpha_n)(1-\beta_n)(u_n - p) \\
&= \alpha_n x_n - \alpha_n p + (1-\alpha_n)(1-\beta_n)u_n - (1-\alpha_n)(1-\beta_n)p \\
&\quad +(1-\alpha_1)\beta_n(S_1 x_n - p) + (\alpha_1 - \alpha_2)\beta_n(S_2 x_n - p) + \ldots \\
&\quad +(\alpha_{n-1} - \alpha_n)\beta_n(S_n x_n - p) \\
&= \alpha_n x_n - \alpha_n p + (1-\alpha_n)(1-\beta_n)u_n - p + \alpha_n p + \beta_n p - \alpha_n\beta_n p \\
&\quad +(1-\alpha_1)\beta_n S_1 x_n + (\alpha_1 - \alpha_2)\beta_n S_2 x_n + \ldots + (\alpha_{n-1} - \alpha_n)\beta_n S_n x_n \\
&\quad -(1-\alpha_1)\beta_n(p) - (\alpha_1 - \alpha_2)\beta_n(p) - \ldots - (\alpha_{n-1} - \alpha_n)\beta_n(p) \\
&= \alpha_n x_n + (1-\alpha_n)(1-\beta_n)u_n - p + \beta_n p - \alpha_n\beta_n p + \sum_{i=1}^{n}(\alpha_{i=1} - \alpha_i)\beta_n S_i x_n \\
&\quad -\beta_n p + \alpha_1\beta_n(p) - \alpha_1\beta_n(p) + \alpha_2\beta_n(p) - \alpha_2\beta_n(p) + \ldots + \alpha_{n-1}\beta_n(p) \\
&\quad -\alpha_{n-1}\beta_n(p) + \alpha_n\beta_n p
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_n x_n + (1-\alpha_n)(1-\beta_n)u_n + \sum_{i=1}^{n}(\alpha_{i=1}-\alpha_i)\beta_n S_i x_n - p \\
&= y_n - p.
\end{aligned}
$$

This completes the proof.

For any $p \in F$, since $u_n = Z_{\lambda_n}(x_n - \lambda_n A x_n)$ and $I - \lambda_n A$ is nonexpansive, we have

$$
\begin{aligned}
& \|y_n - p\| \\
&= \|\alpha_n(x_n - p) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n(S_i x_n - p) + (1-\alpha_n)(1-\beta_n)(u_n - p)\| \\
&\leq \alpha_n\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n\|S_i x_n - p\| + (1-\alpha_n)(1-\beta_n)\|u_n - p\| \\
&\leq \alpha_n\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n\|x_n - p\| \\
&\quad +(1-\alpha_n)(1-\beta_n)\|Z_{\lambda_n}(x_n - \lambda_n A x_n) - Z_{\lambda_n}(p - \lambda_n A p)\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad +(1-\alpha_n)(1-\beta_n)\|(x_n - \lambda_n A x_n) - (p - \lambda_n A p)\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\beta_n\|x_n - p\| + (1-\alpha_n)(1-\beta_n)\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}
$$

This implies that $F \subset C_n$ for all $n \in \mathbb{N}$. Hence $F \subset \cap_{j=1}^{n} C_j$. That is

$$F \subset D_n, \qquad \text{for all } n \in \mathbb{N}.$$

Since D_n is nonempty closed convex, we get that the sequence $\{x_n\}$ is well defined. This completes the proof of Step 1.

Step 2. $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and there is $x^* \in X$ such that $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

It easy to see that $D_{n+1} \subset D_n$ for all $n \in \mathbb{N}$ from the construction of D_n. Hence

$$x_{n+2} = P_{D_{n+1}} x_1 \in D_{n+1} \subset D_n.$$

Since $x_{n+1} = P_{D_n}x_1$, we have

$$\|x_{n+1} - x_1\| \leq \|x_{n+2} - x_1\|,$$

for all $n \geq 1$. This implies that $\{\|x_n - x_1\|\}$ is increasing. Note that X is bounded, we get that $\{\|x_n - x_1\|\}$ is bounded. This shows that $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

Since $x_{n+1} = P_{D_n}x_1$ and $x_{m+1} = P_{D_m}x_1 \in D_m \subset D_n$ for all $m \geq n$, we have

$$\langle x_{n+1} - x_1, x_{m+1} - x_{n+1}\rangle \geq 0. \tag{3.2.21}$$

It follows from (3.2.21) that

$$\begin{aligned}
\|x_{m+1} - x_{n+1}\|^2 &= \|x_{m+1} - x_1 - (x_{n+1} - x_1)\|^2 \\
&= \|x_{m+1} - x_1\|^2 + \|x_{n+1} - x_1\|^2 - 2\langle x_{m+1} - x_1, x_{n+1} - x_1\rangle \\
&= \|x_{m+1} - x_1\|^2 + \|x_{n+1} - x_1\|^2 \\
&\quad -2\langle x_{n+1} - x_1, x_{m+1} - x_{n+1} + x_{n+1} - x_1\rangle \\
&= \|x_{m+1} - x_1\|^2 - \|x_{n+1} - x_1\|^2 - 2\langle x_{n+1} - x_1, x_{m+1} - x_{n+1}\rangle \\
&\leq \|x_{m+1} - x_1\|^2 - \|x_{n+1} - x_1\|^2. \qquad (3.2.22)
\end{aligned}$$

By taking $m = n + 1$ in (3.2.22), we have

$$\|x_{n+2} - x_{n+1}\|^2 \leq \|x_{n+2} - x_1\|^2 - \|x_{n+1} - x_1\|^2. \tag{3.2.23}$$

Since the limits of $\|x_n - x_1\|$ exists, we get that

$$\|x_{n+2} - x_{n+1}\| \to 0, \qquad \text{as } n \to \infty. \tag{3.2.24}$$

This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Moreover, from (3.2.22) we also have

$$\lim_{m,n\to\infty} \|x_{m+1} - x_{n+1}\| = 0. \tag{3.2.25}$$

This shows that the sequence $\{x_n\}$ is a Cauchy sequence. Hence there is $x^* \in X$ such that

$$x_n \to x^* \in X, \qquad \text{as } n \to \infty.$$

Step 3. Show that $\|x_n - u_n\| \to 0$ as $n \to \infty$.

Since $x_{n+1} \in C_n$ and $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \to 0 \qquad \text{as } n \to \infty,$$

and hence

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \to 0 \qquad \text{as } \to \infty.$$

Note that u_n can be rewritten as $u_n = Z_{\lambda_n}(x_n - \lambda_n Ax_n)$ for all $n \geq 1$. Taking $p \in F$. Since $p = Z_{\lambda_n}(p - \lambda_n Ap)$, A is $\lambda-$inverse-strongly monotone, and $0 < \lambda_n < 2\lambda$, we know that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|u_n - p\|^2 &= \|Z_{\lambda_n}(x_n - \lambda_n Ax_n) - Z_{\lambda_n}(p - \lambda_n Ap)\|^2 \\
&\leq \|x_n - \lambda_n Ax_n - p + \lambda_n Ap\|^2 \\
&= \|(x_n - p) - \lambda_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2\lambda_n \langle x_n - p, Ax_n - Ap\rangle + \lambda_n^2 \|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - 2\lambda_n \lambda \|Ax_n - Ap\|^2 + \lambda_n^2 \|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\lambda)\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2. \qquad (3.2.26)
\end{aligned}$$

Using (3.2.20) and (3.2.26), we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
= \;& \|\alpha_n(x_n - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n(S_i x_n - p) + (1 - \alpha_n)(1 - \beta_n)(u_n - p)\|^2 \\
\leq \;& \alpha_n \|x_n - p\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n \|S_i x_n - p\|^2 + (1 - \alpha_n)(1 - \beta_n)\|u_n - p\|^2
\end{aligned}$$

$$\leq \alpha_n\|x_n - p\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n\|x_n - p\|^2$$
$$+(1-\alpha_n)(1-\beta_n)(\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\lambda)\|Ax_n - Ap\|^2)$$
$$= \|x_n - p\|^2 + (1-\alpha_n)(1-\beta_n)\lambda_n(\lambda_n - 2\lambda)\|Ax_n - Ap\|^2,$$

and we will show that

$$\|x_n - p\|^2 - \|y_n - p\|^2 \leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|).$$

For each $x, y \in X$, we know that

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\|.$$

Then

$$\|x\| - \|y\| \leq \|x - y\|, \qquad \forall x, y \in X.$$

This implies that

$$(\|x\| - \|y\|)(\|x\| + \|y\|) \leq \|x - y\|(\|x\| + \|y\|), \qquad \forall x, y \in X.$$

Hence

$$\|x\|^2 - \|y\|^2 \leq \|x - y\|(\|x\| + \|y\|), \qquad \forall x, y \in X.$$

Consider $x = x_n - p$ and $y = y_n - p$ in the proposed inequality, we can completes the proof. Hence

$$(1-\alpha_n)(1-d)a(2\lambda - b)\|Ax_n - Ap\|^2$$
$$\leq (1-\alpha_n)(1-\beta_n)\lambda_n(2\lambda - \lambda_n)\|Ax_n - Ap\|^2$$
$$\leq \|x_n - p\|^2 - \|y_n - p\|^2$$
$$\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|).$$

Note that $\{x_n\}$ and $\{y_n\}$ are bounded, $\alpha_n \to 0$ and $x_n - y_n$ converges to 0, we get that

$$\lim_{n\to\infty} \|Ax_n - Ap\| \to 0. \tag{3.2.27}$$

Using Theorem 3.2.3, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|Z_{\lambda_n}(x_n - \lambda_n Ax_n) - Z_{\lambda_n}(p - \lambda_n Ap)\|^2 \\ &\leq \langle x_n - \lambda_n Ax_n - (p - \lambda_n Ap), u_n - p\rangle \\ &= \frac{1}{2}(\|x_n - \lambda_n Ax_n - (p - \lambda_n Ap)\|^2 + \|u_n - p\|^2 \\ &\quad -\|x_n - \lambda_n Ax_n - (p - \lambda_n Ap) - (u_n - p)\|^2) \\ &\leq \frac{1}{2}(\|x_n - p\|^2 + \lambda_n^2\|Ax_n - Ap\|^2 + \|u_n - p\|^2 \\ &\quad -\|x_n - u_n - \lambda_n(Ax_n - Ap)\|^2) \\ &= \frac{1}{2}(\|x_n - p\|^2 + \lambda_n^2\|Ax_n - Ap\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\ &\quad +2\lambda_n\langle x_n - u_n, Ax_n - Ap\rangle - \lambda_n^2\|Ax_n - Ap\|^2).\end{aligned}$$

This implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda_n\langle x_n - u_n, Ax_n - Ap\rangle.$$

From above inequality, we have

$$\begin{aligned}&\|y_n - p\|^2 \\ = \ &\|\alpha_n(x_n - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n(S_i x_n - p) + (1 - \alpha_n)(1 - \beta_n)(u_n - p)\|^2 \\ \leq \ &\alpha_n\|x_n - p\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n\|S_i x_n - p\|^2 + (1 - \alpha_n)(1 - \beta_n)\|u_n - p\|^2 \\ \leq \ &\alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\beta_n\|x_n - p\|^2 + (1 - \alpha_n)(1 - \beta_n) \\ &\times(\|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda_n\langle x_n - u_n, Ax_n - Ap\rangle) \\ \leq \ &\|x_n - p\|^2 - (1 - \alpha_n)(1 - \beta_n)\|x_n - u_n\|^2 \\ &+2(1 - \alpha_n)(1 - \beta_n)\lambda_n\langle x_n - u_n, Ax_n - Ap\rangle,\end{aligned}$$

and hence

$$\begin{aligned}&(1 - d)(1 - \alpha_n)\|x_n - u_n\|^2 \\ \leq \ &(1 - \beta_n)(1 - \alpha_n)\|x_n - u_n\|^2\end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\quad +2(1-\alpha_n)(1-\beta_n)\lambda_n\langle x_n - u_n, Ax_n - Ap\rangle \\
&\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) \\
&\quad +2(1-\alpha_n)(1-\beta_n)\lambda_n\|x_n - u_n\|\|Ax_n - A_p\|.
\end{aligned}$$

From (3.2.27) and $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0.$$

Step 4. Show that $\lim_{n\to\infty}\|x_n - S_i x_n\| = 0$, for all $i = 0, 1, \ldots$.

It follows from definition of scheme (3.2.20) that

$$y_n + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n(x_n - S_i x_n) - (1-\alpha_n)\beta_n x_n = \alpha_n x_n + (1-\alpha_n)(1-\beta_n)u_n,$$

that is,

$$\begin{aligned}
&\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n(x_n - S_i x_n) \\
&= x_n - y_n - x_n + \alpha_n x_n + (1-\alpha_n)\beta_n x_n \\
&\quad +(1-\alpha_n)(1-\beta_n)u_n \\
&= x_n - y_n + (1-\alpha_n)(\beta_n - 1)x_n + (1-\alpha_n)(1-\beta_n)u_n \\
&= x_n - y_n + (1-\alpha_n)(1-\beta_n)(u_n - x_n).
\end{aligned}$$

Hence, for any $p \in F$, one has

$$\begin{aligned}
\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n\langle x_n - S_i x_n, x_n - p\rangle &= (1-\alpha_n)(1-\beta_n)\langle u_n - x_n, x_n - p\rangle \\
&\quad +\langle x_n - y_n, x_n - p\rangle. \qquad (3.2.28)
\end{aligned}$$

Since each S_i is nonexpansive and by (2.3.2) we get that

$$\|S_i x_n - x_n\|^2 \leq 2\langle x_n - S_i x_n, x_n - p\rangle. \qquad (3.2.29)$$

Hence, combining this inequality with (3.2.28), we have

$$\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n\|S_ix_n-x_n\|^2 \leq (1-\alpha_n)(1-\beta_n)\langle u_n-x_n, x_n-p\rangle+\langle x_n-y_n, x_n-p\rangle,$$

that is

$$\begin{aligned}
&\|S_ix_n - x_n\|^2 \\
\leq\ & \frac{2(1-\alpha_n)(1-\beta_n)}{(\alpha_{i-1}-\alpha_i)\beta_n}\langle u_n - x_n, x_n - p\rangle + \frac{2}{(\alpha_{i-1}-\alpha_i)\beta_n}\langle x_n - y_n, x_n - p\rangle \\
\leq\ & \frac{2(1-\alpha_n)(1-\beta_n)}{(\alpha_{i-1}-\alpha_i)\beta_n}\|u_n - x_n\|\|x_n - p\| + \frac{2}{(\alpha_{i-1}-\alpha_i)\beta_n}\|x_n - y_n\|\|x_n - p\|.
\end{aligned}$$

Since $\|u_n - x_n\| \to 0$ and $\|x_n - y_n\| \to 0$, we have

$$\lim_{n\to\infty}\|S_ix_n - x_n\| = 0, \qquad \forall i = 1, 2, \ldots.$$

Step 5. We show that $x_n \to x^* = P_F x_1$.

First, we show that $x^* \in \cap_{i=1}^{\infty}\mathrm{Fix}(S_i)$. Since

$$\lim_{n\to\infty} x_n = x^* \text{ and } \lim_{n\to\infty}\|S_ix_n - x_n\| = 0,$$

we have

$$x^* \in \mathrm{Fix}(S_i) \qquad \text{for each } i = 1, 2, \ldots.$$

Hence $x^* \in \cap_{i=1}^{\infty}\mathrm{Fix}(S_i)$. Next, we show that $x^* \in \mathrm{SGVEPR}(\varphi, T)$. Note that $u_n = Z_{\lambda_n}(x_n - \lambda_n Ax_n)$, one obtains

$$\varphi(u_n, y)+e\langle Tu_n, \eta(y, u_n)\rangle+e\langle Ax_n, y-u_n\rangle+\frac{e}{\lambda_n}\langle y-u_n, u_n-x_n\rangle \in C, \qquad \forall y \in X,$$

which implies that

$$0 \in \varphi(y, u_n)-\{e\langle Tu_n, \eta(y, u_n)\rangle+e\langle Ax_n, y-u_n\rangle+\frac{e}{\lambda_n}\langle y-u_n, u_n-x_n\}+C, \forall y \in X. \tag{3.2.30}$$

Put $v_t = ty + (1-t)x^*$, for all $t \in (0, 1)$ and $y \in X$. Then, we have $v_t \in X$. So, from (3.2.30), we have

$$e\langle v_t - u_n, Av_t\rangle \ \in\ e\langle v_t - u_n, Av_t\rangle - e\langle v_t - u_n, Ax_n\rangle - e\langle v_t - u_n, \frac{u_n - x_n}{\lambda_n}\rangle$$

$$
\begin{aligned}
& +\varphi(v_t, u_n) + e\langle Tu_n, \eta(u_n, v_t)\rangle + C. \\
= \quad & e\langle v_t - u_n, Av_t - Au_n\rangle + e\langle v_t - u_n, Au_n - Ax_n\rangle \\
& -e\langle v_t - u_n, \frac{u_n - x_n}{\lambda_n}\rangle + \varphi(v_t, u_n) \\
& +e\langle Tu_n, \eta(u_n, v_t)\rangle + C. \qquad (3.2.31)
\end{aligned}
$$

Since $\|x_n - u_n\| \to 0$ and the properties of A , we have

$$\|Au_n - Ax_n\| \to 0, \frac{u_n - x_n}{\lambda_n} \to 0, \langle v_t - u_n, Au_n - Ax_n\rangle \to 0. \qquad (3.2.32)$$

From the monotonicity of A, we have

$$\langle v_t - u_n, Av_t - Au_n\rangle \geq 0.$$

Thus

$$e\langle v_t - u_n, Av_t - Au_n\rangle \in C. \qquad (3.2.33)$$

So, from (3.2.31)-(3.2.33) and condition (A5), we have

$$e\langle v_t - x^*, Av_t\rangle \in \varphi(v_t, x^*) + e\langle Tx^*, \eta(x^*, v_t)\rangle + C. \qquad (3.2.34)$$

Since φ is $C-$convex, we have

$$t\varphi(v_t, y) + (1-t)\varphi(v_t, x^*) \in \varphi(v_t, v_t) + C. \qquad (3.2.35)$$

Since for any $u, v \in X$, the mapping $x \mapsto \langle Tv, \eta(x, u)\rangle$ is convex, we have

$$\langle Tx^*, \eta(v_t, v_t)\rangle \leq t\langle Tx^*, \eta(y, v_t)\rangle + (1-t)\langle Tx^*, \eta(x^*, v_t)\rangle.$$

This implies that

$$et\langle Tx^*, \eta(y, v_t)\rangle + e(1-t)\langle Tx^*, \eta(x^*, v_t)\rangle \in e\langle Tx^*, \eta(v_t, v_t)\rangle + C. \qquad (3.2.36)$$

From (3.2.35) and (3.2.36), we get that

$$t\varphi(v_t, y) + (1-t)\varphi(v_t, x^*) + et\langle Tx^*, \eta(y, v_t)\rangle + e(1-t)\langle Tx^*, \eta(x^*, v_t)\rangle$$

$$\in \quad e\langle Tx^*, \eta(v_t, v_t)\rangle + \varphi(v_t, v_t) + C = C,$$

which implies that

$$-t(\varphi(v_t, y)+e\langle Tx^*, \eta(y, v_t)\rangle)-(1-t)(\varphi(v_t, x^*)+e\langle Tx^*, \eta(x^*, v_t)\rangle) \in -C. \quad (3.2.37)$$

From (3.2.34) and (3.2.37), we have

$$\begin{aligned} -t(\varphi(v_t, y) + e\langle Tx^*, \eta(y, v_t)\rangle) &\in (1-t)(\varphi(v_t, x^*) + e\langle Tx^*, \eta(x^*, v_t)\rangle) - C \\ &\in (1-t)e\langle v_t - x^*, Av_t\rangle - C. \end{aligned}$$

This implies that

$$-t(\varphi(v_t, y) + e\langle Tx^*, \eta(y, v_t)\rangle) - e(1-t)t\langle y - x^*, Av_t\rangle \in -C.$$

It follows that

$$(\varphi(v_t, y) + e\langle Tx^*, \eta(y, v_t)\rangle) + e(1-t)\langle y - x^*, Av_t\rangle \in C.$$

As $t \to 0$, we obtain that for each $y \in X$,

$$(\varphi(x^*, y) + e\langle Tx^*, \eta(y, x^*)\rangle) + e(1-t)\langle y - x^*, Ax^*\rangle \in C.$$

Hence $x^* \in \text{SGVEPR}(\varphi, T)$. Finally, we prove that $x^* = P_F x$. From $x_{n+1} = P_{D_n} x$ and $F \subset D_n$, we have

$$\langle x - x_{n+1}, x_{n+1} - v\rangle \geq 0, \qquad \forall v \in F. \quad (3.2.38)$$

Note that $\lim_{n\to\infty} x_n = x^*$, we take the limit in (3.2.38), then we have

$$\langle x - x^*, x^* - v\rangle \geq 0, \qquad \forall v \in F.$$

We see that $x^* = P_F x$ by (3.2.8). This completes the proof.

Remark 3.2.5. If $Y = \mathbb{R}, C = \mathbb{R}^+$ and $e = 1$, then Theorem 3.2.4 extends and improves Theorem 3.1 of Wang, Marino and Wang [62].

3.3 Existence Results for Generalized Nonlinear Vector Mixed Quasi-Variational-Like Inequality Governed by a Multi-Valued Map

In this paper, we consider a generalized nonlinear vector mixed quasi-variational-like inequality governed by a multi-valued map and establish some existence results in locally convex topological vector spaces by using the fixed point theorem.

Let Y be a locally convex Hausdorff topological vector space (l.c.s., in short) and K be a nonempty convex subset of a Hausdorff topological vector space (t.v.s., in short) E. We denote by $L(E,Y)$ the space of all continuous linear operators from E into Y and $\langle l, x\rangle$ the evaluation of $l \in L(E,Y)$ at $x \in E$. Let $X \subseteq L(E,Y)$. From corollary of the Schaefer [93], $L(E,Y)$ becomes a l.c.s.. By Ding and Tarafdar [94], we have the bilinear map $\langle \cdot,\cdot\rangle : L(K,Y) \times K \to Y$ is continuous. Let int$A$ and co(A) represent the interior and convex hull of a set A, respectively. Let $C : K \to 2^Y$ be a set-valued mapping such that $\text{int}C(x) \neq \emptyset$ for each $x \in K$, $\eta : K \times K \to E$ be a vector-valued mapping.

Let $N : L(E,Y) \times L(E,Y) \times L(E,Y) \to 2^{L(E,Y)}$ be a set-valued mapping, $H : K \times K \to 2^Y$, $D : K \to 2^K$ and $T, A, M : K \to 2^X$ be set-valued mappings and $g : K \to K$ a single-valued mapping. For each $\omega^* \in L(E,Y)$, we consider the following class of generalized nonlinear vector mixed quasi-variational-like inequality governed by a multi-valued map :

$$(\mathcal{P}) \begin{cases} \text{find } u \in K \text{ such that } u \in D(u) \text{ and for each } v \in D(u), \\ \text{there exist } x \in T(u), y \in A(u) \text{ and } z \in M(u) \text{ satisfying} \\ \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \nsubseteq -\text{int}C(u). \end{cases} \quad (3.3.1)$$

The problem $(\mathcal{P})$ encompass many models of variational inequality problems. The following problems are the special cases of $(\mathcal{P})$.

(a) If $N : L(E,Y) \times L(E,Y) \times L(E,Y) \to L(E,Y)$ and $H : K \times K \to Y$ are two single-valued mappings, $N(x,y,z) = A(x)$ where $A : L(E,Y) \to L(E,Y)$ and $\omega^* = 0$, then the problem $(\mathcal{P})$ reduces to the following generalized vector mixed general quasi-variational-like inequality problem for finding $u \in K$ such that $u \in D(u)$ and for each $v \in D(u)$, there exists $x \in T(u)$ satisfying

$$\langle A(x), \eta(v, g(u))\rangle + H(g(u), v) \notin -\text{int}C(u). \tag{3.3.2}$$

The problem (3.3.2) was studied by Ding and Salahuddin [95]. Some existence results of solutions are established under suitable assumptions without monotonicity and compactness.

(b) If g is an identity mapping and $\omega^* = 0$, then the problem $(\mathcal{P})$ reduces to the following generalized nonlinear vector quasi-variational-like inequality problem for finding $(u,x,y,z) \in K \times U \times V \times W$ such that $u \in D(u)$ and for each $v \in D(u)$, there exist $x \in T(u), y \in A(u)$ and $z \in M(u)$ satisfying

$$\langle N(x,y,z), \eta(v,u)\rangle + H(u,v) \nsubseteq -\text{int}C(u). \tag{3.3.3}$$

The problem (3.3.3) was studied by Husain and Gupta [96].

(c) If $D(u) = K$, then the problem (3.3.3) reduces to the problem of finding $u \in K$ such that there exist $x \in T(u), y \in A(u)$ and $z \in M(u)$ satisfying

$$\langle N(x,y,z), \eta(v,u)\rangle + H(u,v) \nsubseteq -\text{int}C(u), \forall v \in K, \tag{3.3.4}$$

which is introduced and studied by Xiao, Fan and Qi [36]. When $N : L(E,Y) \times L(E,Y) \times L(E,Y) \to L(E,Y)$ and $H : K \times K \to Y$ are two single-valued mappings The problem (3.3.4) includes some generalized variational inequality problems investigated in [39, 42, 97, 98, 99, 100] as special cases.

(d) If $T(u) = A(u) = \emptyset$ for all $u \in K$, and N is an identity mapping, the problem (3.3.3) reduces to the problem of finding $u \in K$ such that $u \in D(u)$ and for all $v \in D(u)$,

$$\langle T(u), \eta(v,u)\rangle + H(u,v) \nsubseteq -\mathrm{int}C(u),$$

which is introduced and studied by Peng and Yang [101].

For suitable and appropriate conditions imposed on the mappings $C, N, H, D, T, A, M, \eta$ and g and by means of the fixed point theorem, we establish some existence results of solutions for the problem $(\mathcal{P})$. It is clear that the problem $(\mathcal{P})$ is the most general and unifying one, which is also one of the main motivations of this paper.

3.3.1 Existence Results

In this section, we shall derive the solvability for the problem $(\mathcal{P})$ under certain conditions.

First, we give the concept of 0−diagonally convex which is useful for establishing existence theorem for the problem $(\mathcal{P})$.

Definition 3.3.1. Let K a convex subset of a t.v.s E and Y be a t.v.s.. Let $C : K \to 2^Y$ be a set-valued mapping and $g : K \to K$ a single-valued mapping. Then the multi-valued mapping $H : K \times K \to 2^Y$ is said to be 0−diagonally convex with respect to g in second variable, if for any finite subset $\{x_1, \ldots, x_n\}$ of K and any $x = \sum_{i=1}^n \alpha_i x_i$ with $\alpha_i \geq 0$ for $i = 1, \ldots, n$, and $\sum_{i=1}^n \alpha_i = 1$,

$$\sum_{i=1}^n \alpha_i H(g(x), x_i) \nsubseteq -\mathrm{int}C(x).$$

Remark 3.3.2. (i) If g is an identity mapping, then the concept in Definition 3.3.1 reduces to the corresponding concept of 0−diagonally convexity in [102].

(ii) If $H : K \times K \to Y$ is a single-valued mapping, then the concept in Definition 3.3.1 reduces to the corresponding concept of 0−diagonally convex with respect to g in the second variable in [95].

Example 3.3.3. *Let* $Y = \mathbb{R}, K = \mathbb{R}^+, C(x) = \mathbb{R}^+$ *for all* $x \in K$,

$$g(x) = 2x, \ \ \forall x \in K$$

and

$$H(x, y) = x + y, \ \ \forall x, y \in K.$$

Let $\{x_1, \ldots, x_n\} \subseteq K$ *and* $x = \sum_{i=1}^{n} \alpha_i x_i$ *with* $\alpha_i \geq 0$ *for* $i = 1, \ldots, n$ *and* $\sum_{i=1}^{n} \alpha_i = 1$.

We consider

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i H(g(x), x_i) &= \sum_{i=1}^{n} \alpha_i H(2x, x_i)\\ &= \alpha_1 H(2x, x_1) + \alpha_2 H(2x, x_2) + \ldots + \alpha_n H(2x, x_n)\\ &= \alpha_1(2x + x_1) + \alpha_2(2x + x_2) + \ldots + \alpha_n(2x + x_n)\\ &\geq 0.\end{aligned}$$

This implies that

$$\sum_{i=1}^{n} \alpha_i H(g(x), x_i) \notin (-\infty, 0) = -int\, \mathbb{R}^+ = -int\, C(x).$$

Theorem 3.3.4. *Let* Y *be a l.c.s.,* K *be a nonempty compact convex subset of a Hausdorff t.v.s.* E, X *a nonempty compact convex subset of* $L(E, Y)$, *which is equipped with a* $\sigma-$*topology. Let* $g : K \to K$, $\omega^* \in L(E, Y)$ *and* $T, A, M : K \to 2^X$ *be upper semicontinuous set-valued mappings with nonempty compact values. Assume that the following conditions are satisfied:*

(i) $D : K \to 2^K$ *is nonempty convex set-valued mappings and have open lower sections;*

(ii) *for each* $v \in K$, *the mapping*

$$\langle N(\cdot, \cdot, \cdot) - \omega^*, \eta(v, \cdot)\rangle + H(\cdot, v) : L(E, Y) \times L(E, Y) \times L(E, Y) \times K \times K \to 2^Y$$

is a upper semicontinuous set-valued mapping with compact values;

(iii) $C : K \to 2^Y$ *is a convex set-valued mapping with* $\text{int}C(u) \neq \emptyset$ *for all* $u \in K$;

(iv) $\eta : K \times K \to E$ *is affine in the first argument and for all* $u \in K$, $\eta(u, g(u)) = 0$;

(v) $H : K \times K \to 2^Y$ *is generalized vector* $0-$*diagonally convex with respect to* g;

(vi) $g : K \to K$ *is continuous;*

(vii) *for each* $u \in K$, *the set* $\{u \in K : co\Lambda(u) \cap D(u) \neq \emptyset\}$ *is closed in* K, *where* $\Lambda(u)$ *defined as*

$$\Lambda(u) = \left\{ \begin{array}{c} v \in K : \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \subseteq -\text{int}C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u) \end{array} \right\}.$$

(viii) $Y \setminus \{-\text{int}C(u)\}$ *is an upper semicontinuous set-valued mapping.*

Then the problem $(\mathcal{P})$ *admits at least one solution.*

Proof. Let $\omega^* \in L(E, Y)$. Define a set-valued mapping $Q : K \to 2^K$ by

$$Q(u) = \left\{ \begin{array}{c} v \in K : \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \subseteq -\text{int}C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u) \end{array} \right\},$$

for all $u \in K$. We first prove that $u \notin coQ(u)$ for all $u \in K$. To see this, suppose, by the method of contradiction, that there exists some point $\bar{u} \in K$ such that $\bar{u} \in coQ(\bar{u})$. Then there exists a finite subset $\{v_1, v_2, \ldots, v_n\} \subset Q(\bar{u})$, for $\bar{u} \in co\{v_1, v_2, \ldots, v_n\}$, such that

$$\langle N(\bar{x}, \bar{y}, \bar{z}) - \omega^*, \eta(v_i, g(\bar{u}))\rangle + H(g(\bar{u}), v_i) \subseteq -\text{int}C(\bar{u}), \qquad i = 1, 2, \ldots, n.$$

Since $\text{int}C(\bar{u})$ is convex set and η is affine in the first argument, for $i = 1, 2, \ldots, n$, $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1, \bar{u} = \sum_{i=1}^n \alpha_i v_i$, we have

$$\langle N(\bar{x}, \bar{y}, \bar{z}) - \omega^*, \eta(\sum_{i=1}^n \alpha_i v_i, g(\bar{u}))\rangle + \sum_{i=1}^n \alpha_i H(g(\bar{u}), v_i) \subseteq -\text{int}C(\bar{u}).$$

Since $\eta(u, g(u)) = 0$ for all $u \in K$, we have

$$\sum_{i=1}^{n} \alpha_i H(g(\bar{u}), v_i) \subseteq -\text{int}C(\bar{u}),$$

which contradictis the condition (v), so that $u \notin coQ(u)$ for all $u \in K$.

We now prove that

$$Q^{-1}(v) \;=\; \left\{ \begin{array}{c} u \in K : \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \subseteq -\text{int}C(u) \\ \forall x \in T(u), y \in A(u), z \in M(u) \end{array} \right\}$$

is open for all $v \in K$. That is Q has open lower sections.

Consider a set-valued mapping $J : K \to 2^K$ is defined by

$$J(v) \;=\; \left\{ \begin{array}{l} u \in K : \exists x \in T(u), y \in A(u), z \in M(u) \;\text{ such that} \\ \qquad\qquad \langle N(x,y,z,) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \nsubseteq -\text{int}C(u) \end{array} \right\}.$$

We only need to prove that $J(v)$ is closed for all $v \in K$. Let $\{u_\alpha\}$ be a net in $J(v)$ such that

$$u_\alpha \to u^*.$$

Since g is continuous, we have

$$g(u_\alpha) \to g(u^*).$$

Then there exists $x_\alpha \in T(u_\alpha), y_\alpha \in A(u_\alpha)$ and $z_\alpha \in M(u_\alpha)$ such that

$$\langle N(x_\alpha, y_\alpha, z_\alpha,) - \omega^*, \eta(v, g(u_\alpha))\rangle + H(g(u_\alpha), v_\alpha) \nsubseteq -\text{int}C(u_\alpha).$$

Since T, A, M are upper semicontinuous set-valued mappings with compact values, by Lemma 2.2.26, $\{x_\alpha\}$, $\{y_\alpha\}, \{z_\alpha\}$ have convergent subnets with limits, say x^*, y^*, z^* and $x^* \in T(u^*), y^* \in A(u^*)$ and $z^* \in M(u^*)$. Without loss of generality we may assume that $x_\alpha \to x^*, y_\alpha \to y^*$ and $z_\alpha \to z^*$. Suppose that

$$m_\alpha \in \{\langle N(x_\alpha, y_\alpha, z_\alpha,) - \omega^*, \eta(v, g(u_\alpha))\rangle + H(g(u_\alpha), v)\} \nsubseteq -\text{int}C(u_\alpha).$$

Since $\langle N(\cdot,\cdot,\cdot)-\omega^*,\eta(v,\cdot)\rangle+H(\cdot,v)$ is upper semicontinuous with compact values, by Lemma 2.2.26, there exists $m^*\in\langle N(x^*,y^*,z^*)-\omega^*,\eta(v^*,g(u^*))\rangle+H(g(u^*),v^*)$ and a subnet $\{m_\beta\}$ of $\{m_\alpha\}$ such that $m_\beta\to m^*$. From (viii), we can get that

$$m^*\in Y\setminus \mathrm{int}C(u^*).$$

That is

$$\langle N(x^*,y^*,z^*)-\omega^*,\eta(v^*,g(u^*))\rangle+H(g(u^*),v^*)\nsubseteq \mathrm{int}C(u^*).$$

Hence $J(v)$ is closed in K. So that $Q^{-1}(v)$ is open for each $v\in K$. Therefore Q has open lower sections.

Consider a set-valued mapping $G:K\to 2^K$ defined by

$$G(u)=coQ(u)\cap D(u),\qquad \forall u\in K.$$

Since D has open lower sections by hypothesis (i), we may apply Lemma 2.2.25 to assert that the set-valued mapping G has also open lower sections. Let

$$Z=\{u\in K:G(u)\neq\emptyset\}.$$

There are two cases to consider. In the case $Z=\emptyset$, we have

$$coQ(u)\cap D(u)=\emptyset,\qquad \text{for each } u\in K.$$

This implies that, for each $u\in K$,

$$Q(u)\cap D(u)=\emptyset.$$

On the other hand, by condition (i), and the fact K is a compact subset of E, we can apply Lemma 2.2.27, in this case that $I=\{1\}$, to assert the existence of a fixed point $u^*\in D(u^*)$, we have

$$Q(u^*)\cap D(u^*)=\emptyset.$$

This implies $\forall v\in D(u^*), v\notin Q(u^*)$. Hence, in this particular case, the assertion of the theorem holds.

We now consider the case $Z \neq \emptyset$. Define a set-valued mapping $S : K \to 2^K$ by

$$S(u) = \begin{cases} G(u), & u \in Z; \\ D(u), & u \in K \setminus Z. \end{cases}$$

Then, for each $u \in K$, $S(u)$ is convex set and for each $t \in K$,

$$S^{-1}(t) = G^{-1}(t) \cup ((K \setminus Z) \cap (D^{-1}(t))).$$

Since $D^{-1}(t), coQ^{-1}(t)$ are open in K and $K \setminus Z$ is open in K by condition (vii), we have $S^{-1}(t)$ is open in K. This implies that S has open lower sections. Therefore, there exists $u^* \in K$ such that $u^* \in S(u^*)$. Suppose that $u^* \in Z$, then

$$u^* \in coQ(u^*) \cap D(u^*),$$

so that $u^* \in coQ(u^*)$. This is a contradiction. Hence, $u^* \notin Z$. Therefore,

$$u^* \in D(u^*), \text{ and } G(u^*) = \emptyset.$$

Thus

$$u^* \in D(u^*), \text{ and } coQ(u^*) \cap D(u^*) = \emptyset.$$

This implies

$$Q(u^*) \cap D(u^*) = \emptyset.$$

Consequently, the assertion of the theorem holds in this case. The the problem $(\mathcal{P})$ admits at least one solution. □

Example 3.3.5. $E, Y = \mathbb{R}, K = [0, \infty)$.

$C(x) = [0, \infty)$ *for each* $x \in K$.

$D(x) = [0, 2x)$ *for each* $x \in K$.

$N(x, y, z) = I$, *for all* $x, y, z \in X$ *and* $\omega^* = I$

$g(u) = 2u$ *and* $H(x, y) = x + y$

$\eta(u,v) = 2u - v.$

First, we show that D has open lower section. Consider

$$D^{-1}(y) = \{x \in [0,\infty) : y \in D(x)\}, \ \forall y \in K.$$

$$D^{-1}(0) = \{x \in [0,\infty) : 0 \in [0,2x)\} = (0,\infty).$$

$$D^{-1}(1) = \{x \in [0,\infty) : 0 \in [0,2x)\} = (\frac{1}{2},\infty).$$

$$D^{-1}(y) = (\frac{y}{2},\infty)$$

It is easy to see that $D^{-1}(y)$ is open for all $y \in K$. That is D is nonempty convex set valued mapping and has open lower section.

Next, we show that C satisfies condition (iii) and (viii). It is easy to see that condition (iii) satisfied. We just show that $Y \setminus -\mathrm{int}C : K \to 2^Y$ which defied by

$$(Y \setminus -\mathrm{int}C)(x) = Y \setminus \{-\mathrm{int}C(x)\}$$

for each $x \in K$ is an upper semicontinuous set-valued mapping. If we define

$$F(x) = (Y \setminus -\mathrm{int}C)(x).$$

Then

$$\begin{aligned} F(x) &= (Y \setminus -\mathrm{int}C)(x) \\ &= (\mathbb{R} \setminus -\mathrm{int}C)(x) \\ &= \mathbb{R} \setminus -\mathrm{int}C(x) \\ &= [0,\infty). \end{aligned}$$

It is easy to show that F is upper semicontinuous.

Next, we show that condition (iv) satisfied. Since $\eta(u,v) = 2u - v$, we can see that η is affine in the first argument. Since $g(u) = 2u$, we have g is continuous *and*

$$\eta(u, g(u)) = 2u - 2u = 0.$$

The previous example can get that H is $0-$diagonally convex with respect to g.

We consider

$$\begin{aligned}\Lambda(u) &= \left\{\begin{array}{c} v \in K : \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \subseteq -\mathrm{int}C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u) \end{array}\right\}. \\ &= \{v \in [0,\infty) : 2u + v \in (-\infty, 0)\} \\ &= \{v \in [0,\infty) : v \in (-\infty, -2u)\} = \emptyset.\end{aligned}$$

We can see that condition (vii) satisfied.

We would like to find $u \in [0,\infty)$ such that $u \in [0, 2u)$ and for each $v \in [0, 2u)$

$$2u + v \notin (-\infty, 0)$$

That is

$$2u + v \in [0, \infty).$$

Then it is very easy to see that $(\mathcal{P})$ has at least one solution.

Corollary 3.3.6. *Let Y be a l.c.s., K be a nonempty compact convex subset of a Hausdorff t.v.s. E, X a nonempty compact convex subset of $L(E,Y)$, which is equipped with a $\sigma-$topology. Assume that N and H are single-valued mappings and $T, A, M : K \to 2^X$ are upper semicontinuous set-valued mappings with nonempty compact values. Let $\omega^* \in L(E,Y)$ and $g : K \to K$. Assume that the following conditions are satisfied:*

(i) *$D : K \to 2^K$ is a nonempty convex set-valued mappings and have open lower sections;*

(ii) *for each $v \in K$, the mapping*

$$\langle N(\cdot,\cdot,\cdot)-\omega^*, \eta(v,\cdot)\rangle + H(\cdot,v) : L(E,Y)\times L(E,Y)\times L(E,Y)\times K\times K \to 2^Y$$

is continuous;

(iii) $C : K \to 2^Y$ *is a convex set-valued mapping with* $\mathrm{int}C(u) \neq \emptyset$ *for all* $u \in K$;

(iv) $\eta : K\times K \to E$ *is affine in the first argument and for all* $u \in K$, $\eta(u, g(u)) = 0$;

(v) $H : K \times K \to 2^Y$ *is vector* $0-$*diagonally convex with respect to* g;

(vi) $g : K \to K$ *is continuous;*

(vii) *for each* $u \in K$, *the set* $\{u \in K : co\Lambda(u) \cap D(u) \neq \emptyset\}$ *is closed in* K, *where* $\Lambda(u)$ *defined as*

$$\Lambda(u) = \left\{ \begin{array}{c} v \in K : \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \subseteq -\mathrm{int}C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u) \end{array} \right\};$$

(viii) $Y \setminus \{-\mathrm{int}C(u)\}$ *is an upper semicontinuous set-valued mapping.*

Then there exists a point $\bar{u} \in K$ *such that* $\bar{u} \in D(\bar{u})$ *and for each* $v \in D(\bar{u})$, *there exists* $\bar{x} \in T(\bar{u}), \bar{y} \in A(\bar{u})$ *and* $\bar{z} \in M(\bar{u})$ *such that*

$$\langle N(\bar{x},\bar{y},\bar{z}) - \omega^*, \eta(v, g(\bar{u}))\rangle + H(g(\bar{u}), v) \not\subseteq -\mathrm{int}C(\bar{u}).$$

Proof. Define a set-valued mapping $Q : K \to 2^K$ by

$$Q(u) = \left\{ \begin{array}{c} v \in K : \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \in -\mathrm{int}C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u) \end{array} \right\},$$

for all $u \in K$. We now prove that $Q^{-}(v)$ is open for each $v \in K$, that is,

$$(Q^{-1}(v))^c = \left\{ \begin{array}{l} u \in K : \exists x \in T(u), y \in A(u), z \in M(u) \text{ such that} \\ \langle N(x,y,z) - \omega^*, \eta(v, g(u))\rangle + H(g(u), v) \cap Y \setminus \{-\mathrm{int}C(u)\} \neq \emptyset \end{array} \right\},$$

is closed in K. Let $\{u_t\}$ be a net in $(Q^{-1}(v))^c$ such that

$$g(u_t) \to g(u^*) \in K.$$

Then there exist $x_t \in T(u_t)$, $y_t \in A(u_t)$ and $z_t \in M(u_t)$ such that

$$\langle N(x_t, y_t, z_t) - \omega^*, \eta(v, g(u_t))\rangle + H(g(u_t), v) \in Y \setminus \{-\text{int}C(u_t)\}.$$

The upper semicontinuity, compact values of T, A, M and Lemma 2.2.26 imply that there exist convergent subnets $\{x_{t_j}\}$, $\{y_{t_j}\}$ and $\{z_{t_j}\}$ such that

$$x_{t_j} \to x^*, y_{t_j} \to y^* \text{ and } z_{t_j} \to z^*$$

for some $x^* \in T(u), y^* \in A(u)$ and $z^* \in M(u)$. Since $\langle N(\cdot,\cdot,\cdot) - \omega^*, \eta(v,\cdot)\rangle + H(\cdot, v)$ is continuous, we have

$$\begin{aligned}&\langle N(x_{t_j}, y_{t_j}, z_{t_j}) - \omega^*, \eta(v, g(u_{t_j}))\rangle + H(g(u_{t_j}), v)\\ &\to \langle N(x^*, y^*, z^*) - \omega^*, \eta(v, g(u^*))\rangle + H(g(u^*), v).\end{aligned}$$

From Lemma 2.2.20 and upper semicontinuity of $Y \setminus (-\text{int}C(u))$, we have

$$\langle N(x^*, y^*, z^*) - \omega^*, \eta(v, g(u^*))\rangle + H(g(u^*), v) \in Y \setminus (-\text{int}C(u^*)),$$

and hence $u^* \in (Q^{-1}(v))^c$, which gives that $(Q^{-1}(v))^c$ is closed. Therefore Q has open lower sections. For the remainder of the proof, we can just follow that of Theorem 3.3.4. This completes the proof. □

Theorem 3.3.7. *Let Y be a l.c.s., K be a nonempty convex subset of a Hausdorff t.v.s. E, X a nonempty compact convex subset of $L(E, Y)$, which is equipped with a $\sigma-$topology. Let $\omega^* \in L(E, Y)$, $g : K \to K$ and $T, A, M : K \to 2^X$ be upper semicontinuous set-valued mappings. Assume that the following conditions are satisfied.*

(i) *$D : K \to 2^K$ is a nonempty convex set-valued mapping and has open lower sections;*

(ii) *for each $y \in K$, the mapping*

$$\langle N(\cdot,\cdot,\cdot)-\omega^*,\eta(v,\cdot)\rangle+H(\cdot,v):L(E,Y)\times L(E,Y)\times L(E,Y)\times K\times K\to 2^Y$$

is upper semicontinuous;

(iii) $C: K \to 2^Y$ *is a convex set-valued mapping with* $\text{int}C(u) \neq \emptyset$ *for all* $u \in K$;

(iv) $\eta: K\times K \to E$ *is affine in the first argument and for all* $x \in K$, $\eta(u, g(u)) = 0$;

(v) $H: K \times K \to 2^Y$ *is generalized vector* $0-$*diagonally convex with respect to* g;

(vi) $g: K \to K$ *is continuous;*

(vii) *For each* $u \in K$, *the set* $\{u \in K : co\Lambda(u) \cap D(u) \neq \emptyset\}$ *is closed in* K, *where* $\Lambda(u)$ *defined as*

$$\Lambda(u) \;=\; \left\{ \begin{array}{c} v \in K : \langle N(x,y,z)-\omega^*, \eta(v,g(u))\rangle + H(g(u),v) \subseteq -\text{int}C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u) \end{array} \right\};$$

(viii) *for a given* $u \in K$, *and a neighborhood* O *of* u, *for all* $t \in O$, $\text{int}C(u) = \text{int}C(t)$.

Then the problem $(\mathcal{P})$ *admits at least one solution.*

Proof. Define a set-valued mapping $Q: K \to 2^K$ by

$$Q(u) \;=\; \left\{ \begin{array}{c} v \in K : \langle N(x,y,z)-\omega^*, \eta(v,g(u))\rangle + H(g(u),v) \subseteq -\text{int}C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u) \end{array} \right\},$$

for all $u \in K$. We now prove that for each $v \in K$,,

$$Q^{-1}(v) = \left\{ \begin{array}{c} u \in K : \langle N(x,y,z)-\omega^*, \eta(v,g(u))\rangle + H(g(u),v) \subseteq -\text{int}C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u) \end{array} \right\},$$

is open. That is, Q has open lower sections in K. Indeed, let $\bar{u} \in Q^-(v)$, that is,

$$\langle N(x,y,z)-\omega^*, \eta(v,g(\bar{u}))\rangle + H(g(\bar{u}),v) \subseteq -\text{int}C(\bar{u}).$$

Since $\langle N(\cdot,\cdot,\cdot)-\omega^*, \eta(y,g(\cdot))\rangle + H(g(\cdot),y)$ is upper semicontinuous, there exists a neighborhood O of $\bar{u}$ such that

$$\langle N(x,y,z)-\omega^*, \eta(v,g(u))\rangle + H(g(u),v) \subseteq -\text{int}C(u), \qquad \forall u \in O.$$

By (vii),

$$\langle N(x,y,z)-\omega^*, \eta(v,g(u))\rangle + H(g(u),v) \subseteq -\text{int}C(\bar{u}), \qquad \forall u \in O.$$

Hence, $O \subset Q^-(v)$. This implies $Q^-(v)$ is open for each $v \in K$, and so Q has open lower sections. For the remainder of the proof, we can just follow that of Theorem 3.3.4. This completes the proof. □

Corollary 3.3.8. *Let Y be a l.c.s., K be a nonempty convex subset of a Hausdorff t.v.s. E, X a nonempty compact convex subset of $L(E,Y)$, which is equipped with a $\sigma-$topology. Let $\omega^* \in L(E,Y), g: K \to K$ and $T, A, M : K \to 2^X$ be upper semicontinuous set-valued mappings Assume that the following conditions are satisfied.*

(i) *$D: K \to 2^K$ is a nonempty convex set-valued mapping and have open lower sections;*

(ii) *for each $y \in K$, the mapping*

$$\langle N(\cdot,\cdot,\cdot)-\omega^*, \eta(v,g(\cdot))\rangle + H(g(\cdot),v) : L(E,Y)\times L(E,Y)\times L(E,Y)\times K\times K \to 2^Y$$

is upper semicontinuous;

(iii) *$C: K \to 2^Y$ is a convex set-valued mapping such that for each $u \in K, C(u) = C$ is a convex cone with $\text{int}C(u) \neq \emptyset$ for all $u \in K$;*

(iv) *$\eta: K\times K \to E$ is affine in the first argument and for all $u \in K$, $\eta(u,g(u)) = 0$;*

(v) *$H: K \times K \to 2^Y$ is generalized vector $0-$diagonally convex with respect to g;*

(vi) $g : K \to K$ *is continuous;*

(vii) *for each* $u \in K$, *the set* $\{u \in K : co\Lambda(u) \cap D(u) \neq \emptyset\}$ *is closed in* K, *where* $\Lambda(u)$ *defined as*

$$\Lambda(u) = \left\{ \begin{array}{c} v \in K : \langle N(x,y,z) - \omega^*, \eta(v, g(u)) \rangle + H(g(u), v) \subseteq -\text{int}C(u), \\ \forall x \in T(u), y \in A(u), z \in M(u) \end{array} \right\}.$$

Then the problem $(\mathcal{P})$ *admits at least one solution.*

Proof. By hypothesis (iii), the condition (vii) in Theorem 3.3.7 is satisfied. Hence, all the conditions in Theorem 3.3.7 are satisfied. □

CHAPTER IV

WELL-POSEDNESS FOR GENERALIZED VARIATIONAL INEQUALITY PROBLEM

4.1 Well-posedness by perturbations for the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term

The well-posedness which significant for both optimization theory and numerical methods of optimization problems, which guarantees that, for approximating solution sequences, there is a subsequence which converges to a solution. This means that if there is a iterative sequence which satisfies property of approximating solution sequence, then it will converges to a solution or there is a subsequence which converges to a solution.

Let K be a nonempty subset of a real Banach space E with its dual E^*, $F : K \to 2^{E^*}$ a multivalued mapping. Let $T : E \to L^p(\Omega; \mathbb{R}^k)$ be a linear compact operator, where $1 < p < \infty$ and $k \geq 1$, and Ω a bounded open subset of $\mathbb{R}^N$. Let $j : \Omega \times \mathbb{R}^k \to \mathbb{R}$ be a function such that the mapping

$$j(\cdot, y) : \Omega \to \mathbb{R} \text{ is measurable}, \quad \forall y \in \mathbb{R}^k. \tag{j1}$$

We shall denote $\hat{u} := Tu, j^\circ(x, y; h)$ denotes the Clarke's generalized directional derivative of a locally Lipschitz mapping $j(x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to direction $h \in \mathbb{R}^k$, where $x \in \Omega$.

For the given bifunction $f : K \times K \to [-\infty, +\infty]$ imposed the condition that the set $\mathcal{D}_1(f) = \{u \in K : \ f(u, v) \neq -\infty, \ \forall v \in K\}$ is nonempty, Wangkeeree and Preechasilp [47] introduced and studied the existence of a solution for the following hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term

$$\text{(HVIMN)} \begin{cases} \text{Find } u \in \mathcal{D}_1(f) \text{ and } u^* \in F(u) \text{ such that} \\ \langle u^*, v-u\rangle + f(u,v) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \\ \forall v \in K. \end{cases} \quad (4.1.1)$$

Now, let us consider some special cases of the problem (4.1.1). If $f(u,v) = \phi(v) - \phi(u)$, where $\phi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function such that $K_\phi = K \cap \mathrm{dom}\phi \neq \emptyset$, then $\mathcal{D}_1(f) = K_\phi$ and (4.1.1) is reduced to the following *variational-hemivariational inequality problem*: Find $u \in K_\phi$ such that

$$\langle u^*, v-u\rangle + \phi(v) - \phi(u) + \int_\Omega j(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq 0, \quad \forall v \in K. \quad (4.1.2)$$

The problem (4.1.2) was studied by Costea and Lupu [103] by assuming that F is monotone and lower hemicontinuous and several existence results were obtained. Furthermore, if $F \equiv 0$ and $f(u,v) = \Lambda(u,v) - \langle g^*, v-u\rangle$, where $\Lambda : K \times K \to \mathbb{R}$ and $g^* \in X^*$, then (4.1.1) reduces to the problem: Find $u \in K$ such that

$$\Lambda(u,v) + \int_\Omega j(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq \langle g^*, v-u\rangle, \quad \forall v \in K. \quad (4.1.3)$$

The problem (4.1.3) was studied by Costea and Radulescu [104] and it was called nonlinear hemivariational inequality (see also Andrei and Costea [105] for some applications of nonlinear hemivariational inequalities to Nonsmooth Mechanics).

Now, suppose that L is a normed space, $P \subset L$ is a closed ball with positive radius $p^* \in P$ is a fixed point. Let $\tilde{F} : P \times K \to 2^{E^*}$ be multivalued mapping. Let $\tilde{T} : P \times E \to L^p(\Omega; \mathbb{R}^k)$ be a linear continuous mapping, where $1 < p < \infty, k \geq 1$ and $\tilde{j} : P \times \Omega \times \mathbb{R}^k \to \mathbb{R}$ a function. We denote $\tilde{j}_p^\circ(x, y; h)$ denotes the Clarke's generalized directional derivative of a locally Lipschitz mapping $\tilde{j}(p, x, \cdot)$ at the point $y \in \mathbb{R}^k$ with respect to direction $h \in \mathbb{R}^k$. For the given bifunction $\tilde{f} : P \times K \times K \to [-\infty, +\infty]$, we assume the condition

$$\tilde{\mathcal{D}}_1(\tilde{f}) = \{u \in K | \tilde{f}(p,u,v) \neq -\infty, \forall v \in K\} \neq \emptyset.$$

The perturbed problem of the HVIMN (4.1.1) is given by

$$(\text{HVIMN}_p)\begin{cases}\text{Find } u \in \tilde{\mathcal{D}}_1(\tilde{f}) \text{ and } u^* \in \tilde{F}(p,u) \text{ such that} \\ \langle u^*, v-u\rangle + \tilde{f}(p,u,v) + \int_\Omega \tilde{j}_p^\circ(x,\hat{u}(x);\hat{v}(x)-\hat{u}(x))dx \geq 0, \\ \forall v \in K.\end{cases}$$

where $\tilde{f}(p^*,u,v) = f(u,v), \tilde{F}(p^*,u) = F(u), \tilde{j}(p,x,\cdot) = j(x,\cdot)$.

4.1.1 Well-posedness by perturbations and metric characterizations

In this section, we generalize the concepts of well-posedness by perturbations to the variationalhemivariational inequality and establish their metric characterizations. In the sequel we always denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. Let $\alpha \geq 0$ be a fixed number.

Definition 4.1.1. Let $\{p_n\} \subset P$ be such that $p_n \to p^*$. A sequence $\{u_n\} \subset E$ is called an α-approximating sequence corresponding to $\{p_n\}$ for HVIMN (4.1.1) if there exist a sequence $\{\varepsilon_n\}$ of nonnegative numbers with $\varepsilon_n \to 0$, $u_n^* \in \tilde{F}(p_n,u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$, and

$$\begin{aligned}&\langle u_n^*, v-u_n\rangle + \tilde{f}(p_n,u_n,v) + \int_\Omega \tilde{j}_{p_n}^\circ(x,\hat{u}_n(x);\hat{v}(x)-\hat{u}_n(x))dx \\ &\geq -\frac{\alpha}{2}\|v-u_n\|^2 - \varepsilon_n, \qquad \forall v \in K.\end{aligned}$$

for each $n \geq 1$. Whenever $\alpha = 0$, we say that $\{u_n\}$ is an approximating sequence corresponding to $\{p_n\}$ for HVIMN (4.1.1). Clearly, every α_2-approximating sequence corresponding to $\{p_n\}$ is α_1-approximating sequence corresponding to $\{p_n\}$ whenever $\alpha_1 > \alpha_2 \geq 0$.

Definition 4.1.2. We say that HVIMN (4.1.1) is strongly (resp., weakly) $\alpha-$well-posed by perturbations if

(i) HVIMN (4.1.1) has a unique solution

(ii) for any $\{p_n\} \subset P$ with $p_n \to p^*$, every $\alpha-$approximating sequence corresponding to $\{p_n\}$ converges strongly (resp., weakly) to the unique solution.

In the sequel, strong (resp., weak) $0-$well-posedness by perturbations is always called as strong (resp., weak) well-posedness by perturbations. If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) α_1-well-posedness by perturbations implies strong (resp., weak) α_2-well-posedness by perturbations.

Definition 4.1.3. We say that HVIMN (4.1.1) is strongly (resp., weakly) generalized $\alpha-$well-posed by perturbations if

(i) HVIMN (4.1.1) has a nonempty solution set S

(ii) for any $\{p_n\} \subset P$ with $p_n \to p^*$, every $\alpha-$approximating sequence corresponding to $\{p_n\}$ has some subsequence which converges strongly (resp., weakly) to some point of S

In the sequel, strong (resp., weak) generalized $0-$well-posedness by perturbations is always called as strong (resp., weak) generalized well-posedness by perturbations.

If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp., weak) generalized α_1-well-posedness by perturbations implies strong (resp., weak) generalized α_2-well-posedness by perturbations.

To derive the metric characterizations of α-well-posedness by perturbations, we consider the following approximating solution set of HVIMN (4.1.1):

$$\begin{aligned}\Omega_\alpha(\varepsilon) \;=\; & \bigcup_{p\in B(p^*,\varepsilon)} \{u \in \tilde{D}_1(\tilde{f}), u^* \in \tilde{F}(p,u) : \langle u^*, v-u\rangle + \tilde{f}(p,u,v) \\ & + \int_\Omega \tilde{j}_p^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq -\frac{\alpha}{2}\|v-u\|^2 - \varepsilon, \forall v \in K.\}\end{aligned}$$

when $B(p^*, \varepsilon)$ denotes the closed ball centered at p^* with radius ε. In this section, we assume that $\bar{u}$ is a fixed solution of HVIMN (4.1.1). Define

$$\theta(\varepsilon) = \sup\{\|u - \bar{u}\| : u \in \Omega_\alpha(\varepsilon)\}, \ \forall \varepsilon \geq 0.$$

It is easy to see that $\theta(\varepsilon)$ is the radius of the smallest closed ball centered at $\bar{u}$ containing $\Omega_\alpha(\varepsilon)$. Now, we give a metric characterization of strong α-well-posedness by perturbations by considering the behavior of $\theta(\varepsilon)$ when $\varepsilon \to 0$.

Theorem 4.1.4. *HVIMN (4.1.1) is strongly $\alpha-$well-posed by perturbations if and only if $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Proof. Assume that HVIMN (4.1.1) is strongly $\alpha-$well-posed by perturbations. Then $\bar{u} \in E$ is the unique solution of HVIMN (4.1.1). Suppose to the contrary that $\theta(\varepsilon) \not\to 0$ as $\varepsilon \to 0$. There exist $\delta > 0$ and $0 < \varepsilon_n \to 0$ such that

$$\theta(\varepsilon_n) > \delta > 0.$$

By the definition of θ, there exists $u_n \in \Omega_\alpha(\varepsilon_n)$ such that

$$\|u_n - \bar{u}\| \geq \delta. \tag{4.1.4}$$

Since $u_n \in \Omega_\alpha(\varepsilon_n)$, there exist $p_n \in B(p^*, \varepsilon_n), u_n^* \in \tilde{F}(p_n, u_n)$ such that

$$\langle u_n^*, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon,$$

for all $v \in K$ and $n \geq 1$. Since $p_n \in B(p^*, \varepsilon_n)$, we have $p_n \to p^*$. Then $\{u_n\}$ is an α approximating sequence corresponding to $\{p_n\}$ for HVIMN (4.1.1). Since HVIMN (4.1.1) is strongly $\alpha-$well-posed by perturbations, we can get that $\|u_n - \bar{u}\| \to 0$, which leads to a contradiction with (4.1.4).

Conversely, suppose that $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then $\bar{u} \in E$ is the unique solution of HVIMN (4.1.1). Indeed, if $\hat{u}$ is another solution of HVIMN (4.1.1) with $\hat{u} \neq \bar{u}$, then by definition,

$$\theta(\varepsilon) \geq \|\bar{u} - \hat{u}\| > 0, \ \forall \varepsilon \geq 0,$$

a contradiction. Let $p_n \in P$ be such that $p_n \to p^*$ and let $\{u_n\}$ be an $\alpha-$approximating sequence corresponding to $\{p_n\}$ for HVIMN (4.1.1). Then there exist $0 < \varepsilon_n \to 0, u_n^* \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\begin{aligned}&\langle u_n^*, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}_{p_n}^\circ(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \\ &\geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon_n,\end{aligned}$$

for all $v \in K$ and $n \geq 1$. Take $\delta_n = \|p_n - p^*\|$ and $\varepsilon_n' = \max\{\delta_n, \varepsilon_n\}$. It is easy to verify that $u_n \in \Omega_\alpha(\varepsilon_n')$ with $\varepsilon_n' \to 0$. Put

$$t_n = \|u_n - \bar{u}\|,$$

by definition of θ, we can get that

$$\theta(\varepsilon_n') \geq t_n = \|u_n - \bar{u}\|.$$

Since $\theta(\varepsilon_n') \to 0$, we have $\|u_n - \bar{u}\| \to 0$ as $n \to \infty$. So, HVIMN (4.1.1) is strongly $\alpha-$well-posed by perturbations. □

Now, we give an example to illustrate Theorem 4.1.4.

Example 4.1.5. *Let $E = \mathbb{R}, P = [-1, 1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p, u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p, u, v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. Clearly $\bar{u} = 0$ is a solution of HVIMN (4.1.1). For any $\varepsilon > 0$, it follows that*

$$\begin{aligned}\Omega_\alpha^p(\varepsilon) &= \{u \in \tilde{D}_1(\tilde{f}), u^* \in \tilde{F}(p, u) : \langle u^*, v - u\rangle + u^2 - \frac{(p^2+1)^2}{4}u^2 \\ &\quad \geq -(v-u)^2 - \varepsilon, \;\; \forall v \in K\} \\ &= \{u \in \mathbb{R} : 2u(v - u) + u^2 - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon, \;\; \forall v \in \mathbb{R}\} \\ &= \{u \in \mathbb{R} : -u^2 + 2uv - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon, \;\; \forall v \in \mathbb{R}\} \\ &= \{u \in \mathbb{R} : v^2 - (v-u)^2 - \frac{(p^2+1)^2}{4}u^2 \geq -(v-u)^2 - \varepsilon, \;\; \forall v \in \mathbb{R}\} \\ &= \{u \in \mathbb{R} : -v^2 + \frac{(p^2+1)^2}{4}u^2 \leq +\varepsilon, \;\; \forall v \in \mathbb{R}\}\end{aligned}$$

$$= \Big[-\frac{2\sqrt{\varepsilon}}{p^2+1}, \frac{2\sqrt{\varepsilon}}{p^2+1}\Big].$$

Therefore,

$$\Omega_\alpha(\varepsilon) = \bigcup_{p\in B(0,\varepsilon)} \Omega_\alpha^p(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}\Big],$$

for $\varepsilon > 0$. *By trivial computation, we have*

$$\theta(\varepsilon) = \sup\{u - \bar{u} : u \in \Omega_\alpha(\varepsilon)\} = 2\sqrt{\varepsilon} \to 0 \ \text{as}\ \varepsilon \to 0.$$

By Theorem 4.1.4, HVIMN (4.1.1) is 2-well-posed by perturbations

To derive a characterization of strong generalized α−well-posedness by perturbations, we need another function q which is defined by

$$q(\varepsilon) = e(\Omega_\alpha(\varepsilon), S), \ \ \forall\varepsilon \geq 0,$$

where S is the solution set of HVIMN (4.1.1) and e is defined as in definition 2.3.9.

Theorem 4.1.6. *HVIMN (4.1.1) is strongly generalized* $\alpha-$*well-posed by perturbations if and only if* S *is nonempty compact and* $q(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. Assume that HVIMN (4.1.1) is strongly generalized α−well-posed by perturbations. Clearly, S is nonempty. Let $\{u_n\}$ be any sequence in S and $\{p_n\} \subset P$ be such that $p_n = p^*$. Then $\{u_n\}$ is an α-approximating sequence corresponding to $\{p_n\}$ for HVIMN (4.1.1). Since HVIMN (4.1.1) is strongly generalized α−well-posed by perturbations, we have $\{u_n\}$ has a subsequence which converges strongly to some point of S. Thus S is compact. Next, we suppose that $q(\varepsilon) \not\to 0$ as $\varepsilon \to 0$, then there exist $l > 0, 0 < \varepsilon_n \to 0$ and $u_n \in \Omega_\alpha(\varepsilon_n)$ such that

$$u_n \notin S + B(0,l), \ \ \forall n \geq 1. \tag{4.1.5}$$

Since $u_n \in \Omega_\alpha(\varepsilon_n)$, there exist $p_n \in B(p^*,\varepsilon), u_n^* \in \tilde{F}(p_n,u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\langle u_n^*, v - u_n\rangle + \tilde{f}(p_n,u_n,v) + \int_\Omega \tilde{j}^\circ_{p_n}(x,\hat{u}_n(x);\hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v-u_n\|^2 - \varepsilon,$$

for all $v \in K$ and $n \geq 1$. Since $p_n \in B(p^*, \varepsilon_n)$, we have $p_n \to p^*$. Then $\{u_n\}$ is an α approximating sequence corresponding to $\{p_n\}$ for HVIMN (4.1.1). Since HVIMN (4.1.1) is strongly generalized $\alpha-$well-posed by perturbations, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converging strongly to some point of S, which leads to a contradiction with (4.1.5) and so $q(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Conversely, we assume that S is nonempty compact and $q(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let $\{p_n\} \subset P$ be such that $p_n \to p^*$ and let $\{u_n\}$ be an $\alpha-$approximating sequence corresponding to $\{p_n\}$. Take $\varepsilon'_n = \max\{\varepsilon_n, \|p_n - p^*\|\}$. Thus $\varepsilon'_n \to 0$ and $u_n \in \Omega_\alpha(\varepsilon'_n)$. It follows that

$$d(u_n, S) \geq e(\Omega_\alpha(\varepsilon'_n), S) = q(\varepsilon'_n) \to 0.$$

Since S is compact, there exists $\bar{u}_n \in S$ such that

$$\|u_n - \bar{u}_n\| = d(x_n, S) \to 0.$$

Again from the compactness of S, $\{\bar{u}_n\}$ has a subsequence $\{\bar{u}_{n_k}\}$ which converges to $\bar{u}$. Thus HVIMN (4.1.1) is strongly generalized $\alpha-$well-posed by perturbations. $\square$

The following example is shown for illustrating the metric characterizations in Theorem 4.1.6.

Example 4.1.7. *Let $E = \mathbb{R}, P = [-1, 1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p, u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p, u, v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. It is easy to see that $u = 0$ is a solution of HVIMN (4.1.1). Repeating the same argument as in Example 4.1.5, we obtain that*

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega_\alpha^p(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}\Big],$$

for $\epsilon > 0$. By trivial computation, we have

$$q(\epsilon) = e(\Omega_\alpha(\epsilon), S) = \sup_{u(\epsilon) \in \Omega_\alpha(\epsilon)} d(u(\epsilon), S) \to 0 \text{ as } \epsilon \to 0.$$

By Theorem 4.1.6, HVIMN (4.1.1) is strongly generalized $\alpha-$well-posed by perturbations.

The strong generalized α-well-posedness by perturbations can be also characterized by the behavior of the noncompactness measure $\mu(\Omega_\alpha(\epsilon))$.

Theorem 4.1.8. *Let L be finite-dimensional, $\tilde{j}_p^\circ(x,y)$ be upper semicontinuous as a functional of $(p,x,y) \in P \times E \times E$ and f is convex. Let $\tilde{F}$ is closed on $P \times K$ and $\tilde{f}$ be continuous on $P \times K \times K$. Then HVIMN (4.1.1) is strongly generalized $\alpha-$well-posed by perturbations if and only if $\Omega_\alpha(\varepsilon) \neq \emptyset, \forall \varepsilon > 0$ and $\mu(\Omega_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$.*

Proof. First, we will prove that $\Omega_\alpha(\varepsilon)$ is closed for all $\varepsilon \geq 0$. Let $\{u_n\} \subset \Omega_\alpha(\varepsilon)$ with $u_n \to \bar{u}$. Then there exist $p_n \in B(p^*, \varepsilon), u_n^* \in \tilde{F}(p_n, u_n)$ such that $u_n \in \tilde{D}_1(\tilde{f})$ and

$$\langle u_n^*, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}_{p_n}^\circ(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \geq -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon,$$

for all $v \in K$ and $n \geq 1$. Without loss of generality, we may assume that $p_n \to \bar{p} \in B(p^*, \varepsilon)$ because L is finite dimensional. Since $\tilde{j}_p(x,y)$ is upper semicontinuous as a functional of $(p,x,y) \in P \times E \times E$. Hence by the continuity of $\tilde{f}$ that

$$\begin{aligned}
&\langle u^*, v - \bar{u}\rangle + \tilde{f}(\bar{p}, \bar{u}, v) + \int_\Omega \tilde{j}_{\bar{p}}^\circ(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \\
&\geq \limsup_{n\to\infty} \langle u_n^*, v - u_n\rangle + \tilde{f}(p_n, u_n, v) + \int_\Omega \tilde{j}_{p_n}^\circ(x, \hat{u}_n(x); \hat{v}(x) - \hat{u}_n(x))dx \\
&\geq \limsup_{n\to\infty} -\frac{\alpha}{2}\|v - u_n\|^2 - \varepsilon, \\
&= -\frac{\alpha}{2}\|v - \bar{u}\|^2 - \varepsilon \ \forall v \in K.
\end{aligned}$$

Thus $\bar{u} \in \Omega_\alpha(\varepsilon)$. Hence $\Omega_\alpha(\varepsilon)$ is closed.

Next, we show that

$$S = \bigcap_{\varepsilon>0} \Omega_\alpha(\varepsilon). \tag{4.1.6}$$

It is easy to see that $S \subseteq \cap_{\varepsilon>0}\Omega_\alpha(\varepsilon)$. Thus, we show that $\cap_{\varepsilon>0}\Omega_\alpha(\varepsilon) \subseteq S$. Let $\bar{u} \in \cap_{\varepsilon>0}\Omega_\alpha(\varepsilon)$. Let $\{\varepsilon_n\}$ be a sequence of positive real numbers such that $\varepsilon_n \to 0$.

Thus

$$\bar{u} \in \Omega_\alpha(\varepsilon_n)$$

and so there exist $p_n \in B(p^*, \varepsilon_n)$ and $u^* \in \tilde{F}(p_n, \bar{u})$ such that $\bar{u} \in \tilde{D}_1(\tilde{f})$ and

$$\langle u^*, v - \bar{u}\rangle + \tilde{f}(p_n, \bar{u}, v) + \int_\Omega \tilde{j}^\circ_{p_n}(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \geq -\frac{\alpha}{2}\|v - \bar{u}\|^2 - \varepsilon_n,$$

for all $v \in K$ and $n \geq 1$. It is easy to verify that $p_n \to p^*$. Taking limit as $n \to \infty$, we can get that

$$\begin{aligned} &\langle u^*, v - \bar{u}\rangle + f(\bar{u}, v) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \\ &= \langle u^*, v - \bar{u}\rangle + \tilde{f}(p^*, \bar{u}, v) + \int_\Omega \tilde{j}^\circ_{p^*}(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \\ &\geq -\frac{\alpha}{2}\|v - \bar{u}\|^2, \quad \forall v \in K \end{aligned} \tag{4.1.7}$$

Since $\tilde{F}$ is closed on $P \times K$, we have $u^* \in F(\bar{u})$ and for any $z \in K$ and $t \in (0, 1)$, letting $v = \bar{u} + t(z - \bar{u})$ in (4.1.7), we can get from T is linear, f is convex and definition of j° that

$$\begin{aligned} &t\langle u^*, z - \bar{u}\rangle + tf(\bar{u}, z) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \\ &\geq t\langle u^*, z - \bar{u}\rangle + f(\bar{u}, \bar{u} + t(z - \bar{u})) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{z}(x) - \hat{\bar{u}}(x))dx \\ &\geq -\frac{\alpha t^2}{2}\|z - \bar{u}\|^2. \end{aligned}$$

This implies that

$$\langle u^*, z - \bar{u}\rangle + tf(\bar{u}, z) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \geq -\frac{\alpha t}{2}\|z - \bar{u}\|^2 \ \ \forall z \in K.$$

As $t \to 0$ in the last inequality, we get

$$\langle u^*, z - \bar{u}\rangle + tf(\bar{u}, z) + \int_\Omega j^\circ(x, \hat{\bar{u}}(x); \hat{v}(x) - \hat{\bar{u}}(x))dx \geq 0 \ \ \forall z \in K.$$

Hence $\bar{u} \in S$ and thus it completes the proof. Next, we suppose that HVIMN (4.1.1) is strongly generalized $\alpha-$well-posed by perturbations. By Theorem 4.1.6, we can get that S is nonempty compact and $q(\varepsilon) \to 0$. Since $S \subset \Omega_\alpha(\varepsilon)$ for all $\varepsilon > 0$, we have

$$\Omega_\alpha(\varepsilon) \neq \emptyset, \quad \forall \varepsilon > 0.$$

We observe that for each $\varepsilon > 0$,

$$H(\Omega_\alpha(\varepsilon), S) = \max\{e(\Omega_\alpha(\varepsilon), S), e(S, \Omega_\alpha(\varepsilon))\} = e(\Omega_\alpha(\varepsilon), S).$$

By the compactness of S, we have

$$\mu(\Omega_\alpha(\varepsilon)) \leq 2H(\Omega_\alpha(\varepsilon), S) = 2q(\varepsilon) \to 0.$$

Conversely, we suppose that $\Omega_\alpha(\varepsilon) \neq \emptyset, \quad \forall \varepsilon > 0$ and $\mu(\Omega_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Since $\Omega_\alpha(\cdot)$, by the Kuratowski theorem, we can get from (4.1.6) that

$$q(\varepsilon) = H(\Omega_\alpha(\varepsilon), S) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and S is nonempty compact. Hence HVIMN (4.1.1) is strongly generalized $\alpha-$well-posed by perturbations by Theorem 4.1.6. □

The following example is given for illustrating the measure in Theorem 4.1.8.

Example 4.1.9. *Let $E = \mathbb{R}, P = [-1,1], K = \mathbb{R}, p^* = 0, \alpha = 2, \tilde{F}(p,u) = \{2u\}, \tilde{j} = 0, \tilde{f}(p,u,v) = (1 - \frac{(p^2+1)^2}{4})u^2$ for all $p \in P, u, v \in K$. It is easy to see that $u = 0$ is a solution of HVIMN (4.1.1). Repeating the same argument as in Example 4.1.5, we obtain that*

$$\Omega_\alpha(\varepsilon) = \bigcup_{p \in B(0,\varepsilon)} \Omega_\alpha^p(\varepsilon) = \Big[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon} \Big].$$

We will show that $\mu(\Omega_\alpha(\epsilon)) = 0$ for each $\epsilon > 0$. Let $\epsilon > 0$. Consider

$$\mu(\Omega_\alpha(\epsilon)) = \inf\{\lambda > 0 : [-2\sqrt{\epsilon}, 2\sqrt{\epsilon}] \subseteq \bigcup_{k=1}^{n}[a_k, b_k], \text{ with } \operatorname{diam}[a_k, b_k] < \lambda, \forall i = 1, \ldots, n, \exists n \in \mathbb{N}\}.$$

For every $\lambda > 0$, we can find $n \in \mathbb{N}$ with $a_1 = -2\sqrt{\epsilon}, b_n = 2\sqrt{\epsilon}$ such that

$$[-2\sqrt{\epsilon}, 2\sqrt{\epsilon}] \subseteq \bigcup_{k=1}^{n}[a_k, b_k] \text{ and } \operatorname{diam}[a_k, b_k] < \lambda.$$

This implies that $\mu(\Omega_\alpha(\epsilon)) = 0$ for each $\epsilon > 0$. Then HVIMN (4.1.1) is strongly generalized $\alpha-$well-posed by perturbations.

Remark 4.1.10. Any solution of HVIMN (4.1.1) is a solution of the α problem: find $u \in D_1(f)$ and $u^* \in F(u)$ such that

$$\langle u^*, v - u\rangle + f(u, v) + \int_\Omega j^\circ(x, \hat{u}(x); \hat{v}(x) - \hat{u}(x))dx \geq -\frac{\alpha}{2}\|y - x\|^2, \ \forall v \in K,$$

but the converse is not true in general. To show this, let $K = \mathbb{R}$,

$$F(u) = \{u\}, f(u, v) = 2u^2 - v \text{ and } j = 0,$$

for all $u, v \in K$. It is easy to see that the solution set of HVIMN (4.1.1) is empty and $u^* = u = 0$ is the unique solution of the corresponding α problem with $\alpha = 2$.

CHAPTER V

CONCLUSION

In this section, we summarize and give some concluding remarks. Finally we delineate some important questions that are related to the work in this dissertation.

We introduce and analyze the new generalized mixed equilibrium problems (NGMEP) and the bilevel new generalized mixed equilibrium problems (BNGMEP) in Banach spaces. First, by using a minimax inequality, some new existence theorems of the solution and the behavior of solution set for the NGMEP and the BNGMEP are obtained in Banach spaces. Next, by using auxiliary principle technique, some new iterative algorithms for solving the NGMEP and the BNGMEP are suggested and analyzed. The strong convergence of the iterative sequences generated by the algorithms are also proved in Banach spaces. These results are new and generalize some recent results in this field.

Furthermore, we consider an auxiliary problem for the generalized mixed vector equilibrium problem with a relaxed monotone mapping and prove the existence and uniqueness of the solution for the auxiliary problem. We then introduce a new iterative scheme for approximating a common element of the set of solutions of a generalized mixed vector equilibrium problem with a relaxed monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings. Moreover, we introduce and study a new class of generalized nonlinear vector mixed quasi-variational-like inequality governed by a multi-valued map in Hausdorff topological vector spaces which includes generalized vector mixed general quasi-variational-like inequalities, generalized nonlinear mixed variational-like inequalities, and so on. By using the fixed point theorem, we prove some existence theorems for the purposed inequality.

On the other hand, we introduce the notion of well-posedness to the hemivariational inequality governed by a multi-valued map perturbed with a nonlinear term (HVIMN) in Banach spaces. Under very suitable conditions, we establish some metric characterizations for cheking the well-posed (HVIMN). In the setting of finite-dimensional spaces, the strongly generalized well-posedness by perturbations for (HVIMN) are established. Our results are new and improve recent existing ones in the literature.

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