ITERATIVE APPROXIMATIONS AND EXISTENCE THEOREMS FOR FIXED POINT PROBLEMS AND GENERALIZED EQUILIBRIUM PROBLEMS FOR SOME CLASSES OF NONLINEAR MAPPINGS

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ABSTRACT

In this research, we establish the following results. Firstly, we create and prove a strong convergence theorem by using the hybrid iterative algorithm which was proposed by Yao et al. for finding the common element of fixed point set of a Lipshitz pseudo-contraction and the set of equilibrium problem in Hilbert spaces. Secondly, we construct a sequence by using some appropriated closed convex sets based on the hybrid shrinking projection methods to find a common solution of fixed point problems of a Lipschitz pseudo-contraction and generalized mixed equilibrium problems in Hilbert spaces. Thirdly, we study the new type of mappings called G-quasi-strict pseudo-contractions and to create some iterative projection techniques to find some fixed points of the mappings. Fourthly, we introduce and consider two new mixed vector equilibrium problems i.e., a new weak mixed vector equilibrium problem and a new strong mixed vector equilibrium problem which are combinations of certain vector equilibrium problems and vector variational inequality problems. We prove existence results for the problems in non-compact setting. Finally, we apply some results for solving the equilibrium problems and zeroes of Lipshitz monotone operators in Hilbert spaces.

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CHAPTER I

INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem (for short, EP) is to find $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C. \tag{1.0.1}$$

The set of solution (1.1) is denote by $EP(F)$. Given a mapping $T : C \to H$ and let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $x \in EP(F)$ if and only if $x \in C$ is a solution of the variational inequality $\langle Tx, y - x \rangle \geq 0$ for all $y \in C$. In addition, there are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an EP. In other words, the EP is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc. There are many papers have appeared in the literature on the existence of solutions of EP (see, for example [1–4]) and references therein. Motivated by the work [5–7], Takahashi and Takahashi [8] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the EP (1.0.1) and the set of fixed points of nonexpansive mappings in the setting of Hilbert space. They also studied the strong convergence of the sequences generated by their algorithm for a solution of the EP which is also a fixed point of a nonexpansive mapping defined on a closed convex subset of a Hilbert space. We use $F(T)$ to denote the set of fixed points of T.

Construction of fixed points of nonexpansive mappings via Mann's algorithm [9] has extensively been investigated in the literature, see, for example [9–15] and references therein. However we note that Mann's iterations have only weak convergence even in a Hilbert space (see e.g., [16]). If T is a nonexpansive self-mapping of C, then Mann's algorithm generates, initializing with an arbitrary $x_1 \in C$, a sequence according to the recursive manner

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \geqslant 1, \tag{1.0.2}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a real control sequence in the interval $(0,1)$. If $T : C \to C$ is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm converges weakly to a fixed point of T. Reich [11] showed that the conclusion also holds good in the setting of uniformly convex Banach spaces with a Fréchet differentiable norm. It is well know that Reich's result is on of the fundamental convergence results. Recently, Marino and Xu [17] extended Reich's result [11] to strict pseudo-contraction mapping in the setting of Hilbert spaces. From a practical point of view, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see [18]). Therefore, it is important to develop theory of iterative methods for strict pseudo-contractions. Indeed, Browder and Petryshyn [19] prove that if the sequence $\{x_n\}$ is generated by the following:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geqslant 0, \tag{1.0.3}$$

for any starting point $x_0 \in C$, α is a constant such that $\kappa < \alpha < 1$, x_n converges weakly to a fixed point of strict pseudo-contraction T. Marino and Xu [17] extended the result of Browder and Petryshyn [19] to Mann's iteration (1.0.2), they proved $\{x_n\}$ converges weakly to a fixed point of T, provided the control sequence $\{\alpha_n\}$ satisfies the conditions that $\kappa < \alpha_n < 1$, for all n and $\sum_{n=0}^{\infty}(\alpha_n - k)(1 - \alpha_n) = \infty$.

In order to obtain a strong convergence theorem for the Mann iteration method (1.0.2) to nonexpansive mapping, Nakajo and Takahashi [20] modified (1.0.2) by employing two closed convex sets that are created in order to form the sequence via metric projection so that strong convergence is guaranteed. Later, it is often referred as the hybrid algorithm or the CQ algorithm. After that, the hybrid algorithm have been studied extensively by many authors (see, for example [17, 21–26]).

From a practical point of view, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems (see [27]). Therefore, it is important to develop theory of iterative methods for strict pseudo-contractions.

A few years ago, Takahashi and Zembayashi [28, 29] proposed some hybrid methods to find the solution of fixed point problem and equilibrium problem in Banach spaces. Subsequently, many authors (see, e.g. [30–34] and references therein.) have used the hybrid methods to solve fixed point problem and equilibrium problem.

In 1994, Alber [35] introduced the generalized projections $\pi_C : E^* \to C$ and $\Pi_C : E \to C$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces and studied their properties in detail. In [36], Alber presented some applications of the generalized projections to approximately solving variational inequalities and Von Neumann intersection problem in Banach space. In addition, Li [37] extended the generalized projections from uniformly convex and uniformly smooth Banach spaces to reflexive Banach spaces, and established a Mann type iterative scheme for finding the approximate solutions for the classical variational inequality problem in compact subset of Banach spaces.

Recently, Wu and Huang [38] introduced a new generalized f-projection operator in Banach space. They extended the definition of the generalized projection operators introduced by Abler [35] and proved some properties of the generalized f-projection operator. Wu and Huang [39] continued their study and presented some properties of the generalized f-projection operator. They showed an interesting relation between the generalized f-projection operator and the resolvent operator for the subdifferential of a proper, convex and lower semicontinuous functional in reflexive and smooth Banach spaces. They also proved that the generalized f-projection operator is maximal monotone. By employing the properties of the generalized f-projection operator, Wu and Huang [40] established some new existence theorems for the generalized set-valued variational inequality and the generalized set-valued quasi-variational inequality in reflexive and smooth Banach spaces, respectively.

Very recently, Fan et al. [41] presented some basic results for the generalized f-projection operator, and discussed the existence of solutions and approximation of the solutions for generalized variational inequalities in noncompact subsets of Banach spaces by using iterative schemes.

In 2012, K. Ungchittrakool [42] provided some examples of quasi-strict pseudo-contractions related to the function ϕ in framework of smooth and strictly convex Banach space. He obtained some strong convergence results in Banach spaces.

In 2013, Saewan et al. [43] introduced and studied the modified Mann type iterative algorithm for some mappings which related to asymptotically nonexpansive mappings by using hybrid generalized f-projection method. Saewan and Kumam [44] also provided and studied the new hybrid Ishikawa iteration process by the generalized f-projection operator for finding a common element of the fixed point set for two countable families of weak relatively nonexpansive mappings and the set of solutions of the system of generalized Ky Fan inequalities in a uniformly convex and uniformly smooth Banach space. Some relevant papers, please see [43–58] for more details.

In 2014, Rahaman and Ahmad [59] considered two types of mixed vector equilibrium problems which were combinations of a vector equilibrium problem and a vector variational inequality problem. Remark that $C \subset Y$ is a pointed closed convex cone with nonempty interior i.e., $intC \neq \emptyset$. The partial ordering induced by C on Y is denoted by $\leq_C$ and is defined by $x \leq_C y$ if and only if $y - x \in C$. Let $f : K \times K \to Y$ and $T : X \to L(X, Y)$ be two mappings, where $L(X, Y)$ is the space of all linear continuous mappings from X to Y. Here $\langle T(x), y\rangle$ denotes the evaluation of the linear mapping $T(x)$ at y.

Motivated and inspired by the above work, the purposes of this thesis are to extend, to generalize, to improve existence theorems of generalized equilibrium problems and the iteration schemes of some nonlinear operators for finding a common element of the solutions of generalized vector equilibrium problems and fixed point problems.

This thesis is divided into 5 chapters. Chapter 1 is an introduction to the research problem. Chapter 2 is dealing with some preliminaries and give some useful results that will be duplicated in later Chapter.

Chapter 3 and 4 are the main results of this research. Precisely, in section 3.1, we create and prove a strong convergence theorem by using the hybrid iterative algorithm which was proposed by Yao et al. [Y.H. Yao, Y.C. Liou, G. Marino, A hybrid algorithm for pseudo-contractive mappings, Nonlinear Anal. 71 (2009) 4997-5002] for finding the common element of fixed point set of a Lipshitz pseudo-contraction and the set of equilibrium problem in Hilbert spaces. In section 3.2, we construct a sequence by using some appropriated closed convex sets based on the hybrid shrinking projection methods to find a common solution of fixed point problems of a Lipschitz pseudo-contraction and generalized mixed equilibrium problems in Hilbert spaces. In section 3.3, we study the new type of mappings called G-quasi-strict pseudo-contractions and to create some iterative projection techniques to find some fixed points of the mappings. Moreover, we also find the significant inequality related to such mappings in the framework of Banach spaces. By using the ideas of the generalized f-projection, we propose an iterative shrinking generalized f-projection method for finding a fixed point of G-quasi-strict pseudo-contractions.

Section 4.1, we introduce and consider two new mixed vector equilibrium problems i.e., a new weak mixed vector equilibrium problem and a new strong mixed vector equilibrium problem which are combinations of certain vector equilibrium problems and vector variational inequality problems. We prove existence results for the problems in non-compact setting.

The conclusion of research is in Chapter 5

CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapter.

2.1 Banach spaces and Hilbert spaces

Definition 2.1.1. [60] A *norm* on a (real or complex) vector space H is a real-valued function on H whose value at an $x \in H$ is denoted by $\|x\|$ and which has the properties

(N1) $\|x\| \geq 0$

(N2) $\|x\| = 0 \Leftrightarrow x = 0$

(N3) $\|\alpha x\| = |\alpha|\|x\|$

(N4) $\|x + y\| \leq \|x\| + \|y\|$

here x and y are arbitrary vectors in H and α is any scalar. A *normed space* H is a vector space with a norm defined on it which is denoted by $(H, \|\cdot\|)$ or simply by H.

Definition 2.1.2. [60] A *Banach space* is a complete norm space.

Definition 2.1.3. [60] A sequence $\{x_n\}$ in a normed space H is said to be a *Cauchy Sequence* if for every $\varepsilon > 0$ there is a positive integer N such that

$$\|x_m - x_n\| < \varepsilon \text{ for every } m, n > N.$$

The following definitions and lemmas are the concept of the continuous mappings.

Definition 2.1.4. [60] Let X and Y be normed spaces over the field $\mathbb{K}$ and $T : X \to Y$ be a mapping. A mapping T is said to be *continuous at* $x_0 \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|T(x) - T(x_0)\| < \varepsilon$ for all $x \in X$ satisfying $\|x - x_0\| < \delta$. Furthermore, a mapping T is *continuous*, if T is continuous at every $x \in X$.

Lemma 2.1.5. [60] A mapping T of a normed space X into a normed space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X.

Definition 2.1.6. A mapping $T : C \to C$ is said to be *closed* if for any sequence $\{x_n\} \subset C$ with $x_n \to x$, and $Tx_n \to y$, then $Tx = y$.

Lemma 2.1.7. [60] A mapping T of a normed space X into a normed space Y is continuous at a point $x_0 \in X$ if and only if $x_n \to x_0$ implies $Tx_n \to Tx_0$.

Definition 2.1.8. [60] Let X and Y be vector spaces over the field $\mathbb{K}$.

(i) A mapping $T : X \to Y$ is called *a linear operator* if $T(x + y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx$, for all $x, y \in X$, and for all $\alpha \in \mathbb{K}$.

(ii) A mapping $T : X \to \mathbb{K}$ is called *a linear functional on* X if T is a linear operator.

Definition 2.1.9. [60] Let X and Y be normed spaces over the field $\mathbb{K}$ and $T : X \to Y$ a linear operator. A mapping T is said to be *bounded* on X, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M\|x\|$, for all $x \in X$.

Let E be a real Banach space, and E^* the dual space of E. Let C be a nonempty closed convex subset of E. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction,

$\varphi : C \to \mathbb{R}$ be a real-valued function, and $A : C \to E^*$ be a nonlinear mapping. The generalized mixed equilibrium problem, is to find $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{2.1.1}$$

The solution set of (3.2.1) is denoted by $GMEP(\Theta, A, \varphi)$, i.e.,

$$GMEP(\Theta, A, \varphi) = \{x \in C : \Theta(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C\}$$

If $A = 0$, the problem (3.2.1) reduces to the mixed equilibrium problem for Θ, denoted by $MEP(\Theta, \varphi)$, which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.$$

If $\Theta = 0$, the problem (3.2.1) reduces to the mixed variational inequality of Browder type, denoted by $VI(C, A, \varphi)$, which is to find $x \in C$ such that

$$\langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.$$

If $A = 0$ and $\varphi = 0$, the problem (3.2.1) reduces to the equilibrium problem for Θ (for short, EP), denoted by $EP(\Theta)$.

Let E be a smooth Banach space and let E^* be the dual of E. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2 \tag{2.1.2}$$

for all $x, y \in E$, which was studied by Alber [36], Kamimura and Takahashi [61], and Reich [62], where J is the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\right\}, \tag{2.1.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality paring. It is well known that if E is smooth, then J is single valued and if E is strictly convex, then J is injective (one-to-one).

Lemma 2.1.10. [1] Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction of C $\times$ C into $\mathbb{R}$ satisfying $(A1)$ - $(A4)$. Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that

$$\Theta(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \text{ for all } y \in C.$$

The proof of the following lemma appears in [63, Lemma 2.8].

Lemma 2.1.11. *Let C be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E and let Θ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$ - $(A4)$. For $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows:*

$$T_r x = \{z \in C : \Theta(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \text{ for all } y \in C\}$$

for all $x \in C$. Then, the following statements hold:

(i) T_r is single-valued;

(ii) T_r is a firmly nonexpansive-type mapping, i.e., for any $x, y \in H$,
$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$;

(iii) $F(T_r) = EP(\Theta)$;

(vi) $EP(\Theta)$ is closed and convex.

Lemma 2.1.12. [64] Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, Let $A : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and Θ be a bifunction of $C \times C$ to $\mathbb{R}$ satisfying $(A1)$ - $(A4)$. For $r > 0$ and $x \in E$. Then, there exists $u \in C$ such that

$$\Theta(z,y) + \langle Au, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle, \ \forall\, y \in C.$$

Define a mapping $K_r : C \to C$ as follows:

$$\begin{aligned} K_r(x) &= \{u \in C : \Theta(u,y) + \langle Au, y-u\rangle + \varphi(y) - \varphi(u) + \tfrac{1}{r}\langle y-u, Ju - Jx\rangle \\ &\quad \leq 0, \forall y \in C\} \end{aligned}$$

for all $x \in C$. Then, the following conclusions hold:

(i) K_r is single-valued;

(ii) K_r is a firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,
$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle$;

(iii) $F(K_r) = GMEP(\Theta, A, \varphi)$;

(iv) $GMEP(\Theta, A, \varphi)$ is closed and convex;

(v) $\phi(p, K_r z) + \phi(K_r z, z) \leq \phi(p, z),\ \forall p \in F(K_r),\ z \in E.$

Remark 2.1.13. In the framework of Hilbert spaces, it is well known that $J = I$ and then K_r is firmly nonexpansive.

Definition 2.1.14. [65] A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$.

Definition 2.1.15. [66] A Banach space E is said to be *uniformly convex* if for each $0 < \varepsilon \leqslant 2$, there is $\delta > 0$ such that for all $x, y \in E$, the condition $\|x\| = \|y\| = 1$, and $\|x - y\| \geqslant \varepsilon$ imply $\|\frac{x+y}{2}\| \leqslant 1 - \delta$.

Definition 2.1.16. [66] Let E be a Banach space. Then *the modulus of convexity of* E, $\delta : [0,2] \to [0,1]$ defined as follows:

$$\delta(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| \leqslant 1, \|y\| \leqslant 1, \|x-y\| \geqslant \varepsilon\right\}.$$

Theorem 2.1.17. [66] Let E be a Banach space. Then E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$.

Definition 2.1.18. Let $S(E) := \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be *smooth* provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1.4}$$

exists for each $x, y \in S(E)$. In this case, the norm of E is said to be *Gâteaux differentiable.* The norm of E is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (2.1.4) is attained uniformly for $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (and E is said to be *uniformly smooth*) if the limit (2.1.4) is attained uniformly for $x, y \in S(E)$.

A Banach space E is said to have the *property (K)* (or *Kadec-Klee property*) if for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$. We also know the following properties are true (see [67–69] for details):

(i) if E is smooth($\Leftrightarrow$ E^* is strictly convex), then J is single-valued;

(ii) if E is strictly convex($\Leftrightarrow$ E^* is smooth), then J is one-to-one (i.e., $J(x) \cap J(y) = \emptyset$ for all $x \neq y$);

(iii) if E is reflexive($\Leftrightarrow$ E^* is reflexive), then J is surjective;

(iv) if E^* is smooth and reflexive; then $J^{-1} : E^* \to 2^E$ is single-valued and demi-continuous(i.e. if $\{x_n^*\} \subset E^*$ such that $x_n^* \to x^*$, then $J^{-1}(x_n^*) \to J^{-1}(x^*)$);

(v) If E is a reflexive, smooth and strictly convex Banach space, $J^* : E^* \to E$ is the duality mapping of E^*, then $J^{-1} = J^*, JJ^* = I_E^*, J^*J = I_E$;

(vi) E is uniformly smooth if and only if E^* is uniformly convex;

(vii) if E is uniformly convex, then

(i) it is strictly convex;

(ii) it is reflexive;

(iii) satisfy the property (K);

(viii) if E is a Hilbert space, then J is the identity operator.

It is also very well know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. Next, we assume that E is a real smooth Banach space. Let us consider the functional defined as [70, 71] by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + y^2 \quad \text{for } x, y \in E. \tag{2.1.5}$$

Observe that, in a Hilbert space H, (2.1.5) reduces to $\phi(x,y) = \|x-y\|^2,\ x, y \in H$.

The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in X$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x), \tag{2.1.6}$$

existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, for example, [35, 36, 61, 65, 69]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(x,y) \le (\|y\| + \|x\|)^2$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle$$

for all $x, y \in E$.

Let C be a closed convex subset of E. A mapping T from C into itself is said to be a quasi-strict pseudo-contraction if there exists a constant $\kappa \in [0,1]$ and $F(T) \neq \emptyset$ such that $\phi(p, Tx) \le \phi(p, x) + \kappa\phi(x, Tx)$ for all $x \in C$ and $p \in F(T)$. In particular, T is said to be quasi-nonexpansive if $\kappa = 0$ and T is said to be quasi-pseudo-contractive if $\kappa = 1$.

Example 2.1.19. *Let E be a reflexive, strictly convex and smooth Banach space. Let $A \subset X \times X^*$ be a maximal monotone mapping such that $A^{-1}0$ is nonempty. Then, $J_r = (J + rA)^{-1} J$ is a closed and quasi-strict pseudo-contraction mapping from E onto $D(A)$ and $F(J_r) = A^{-1}0$.*

Example 2.1.20. *Let E be the generalized projection from a smooth, strictly convex and reflexive Banach space E onto a nonempty closed convex subset C of E . Then, Π_C is a closed and quasi-strict pseudo-contraction from E onto C with $F(\Pi_C) = C$.*

Next, we recall the concept of the generalized f-projection operator, together with its properties. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi \rangle + \|\varphi\|^2 + 2\rho f(\xi), \tag{2.1.7}$$

where $\xi \in C, \varphi \in E^*, \rho$ is a positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. It is obvious from the definition of function G that

$$G(x, Jy) = G(x, Jz) + G(z, Jy) + 2\langle x - z, Jz - Jy \rangle - 2\rho f(z) \tag{2.1.8}$$

for all $x, y, z \in C$.

From the definitions of G and f, it is easy to see the following properties are true:

(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed;

(ii) $G(\xi, \varphi)$ is convex and lower semicontinuous with respect to ξ when φ is fixed.

Definition 2.1.21. [38] Let E be a real Banach space with its dual E^*. Let C be a nonempty closed convex subset of E. We say that $\pi_C^f(\varphi) : E^* \to 2^C$ is a generalized f-projection operator if

$$\pi_C^f \varphi = \left\{ u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi) \right\}, \quad \forall \varphi \in E^*.$$

For the generalized f-projector operator, Wu and Huang [38] proved the following basic properties.

Lemma 2.1.22. [38] Let E be a real reflexive Banach space with its dual E^* and C is a nonempty closed convex subset of E. The following statements hold:

(i) $\pi_C^f(\varphi)$ is a nonempty closed convex subset of C for all $\varphi \in E^*$

(ii) if E is smooth, then for all $\varphi \in E^*, x \in \pi_C^f(\varphi)$ if and only if

$$\langle x - y, \varphi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C;$$

(iii) if E is strictly convex and $f : C \to \mathbb{R} \cup +\infty$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$), then π_C^f is a single valued mapping.

Recently, Fan et al. [37] shew that the condition f is positive homogeneous of (iii) in Lemma 2.1.22 can be removed.

Lemma 2.1.23. [37] Let E be a real reflexive Banach space with its dual E^* and C is a nonempty closed convex subset of E. If E is strictly convex, then π_C^f is single-valued.

Recall that the operator J is a single-valued mapping when E is a smooth Banach space. There exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$ for each $x \in E$. This substitution for (2.1.7) gives

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle \xi, Jx \rangle + \|x\|^2 + 2\rho f(\xi). \tag{2.1.9}$$

Now we consider the second generalized f-projection operator (2.1.9) in a Banach space.

Definition 2.1.24. Let E be a real smooth Banach space and C be a nonempty closed convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f x = \left\{ u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx) \right\}, \quad \forall x \in E.$$

In order to obtain our results, the following lemmas are crucial to us.

Lemma 2.1.25. [72] Let $\{a_n\}$ be a sequence of real numbers. Then, $\lim_{n\to\infty} a_n = 0$ if and only if for any subsequence $\{a_{n_i}\}$ of $\{a_n\}$, there exists a subsequence $\left\{a_{n_{i_j}}\right\}$ of $\{a_{n_i}\}$ such that $\lim_{j\to\infty} a_{n_{i_j}} = 0$.

Lemma 2.1.26. [73] Let E be a real Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex functional. Then there exist $x^* \in E^*$and $\alpha \in R$ such that

$$f(x) \geq \langle x, x^* \rangle + \alpha, \quad \forall x \in E.$$

Lemma 2.1.27. [61] Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.

Lemma 2.1.28. [74] Let E be a real reflexive and smooth Banach space and let C be a nonempty closed convex subset of E. The following statements hold:

(i) $\Pi_C^f x$ is a nonempty closed convex subset of C for all $x \in E$;

(ii) for all $x \in E, \hat{x} \in \Pi_C^f x$ if and only if

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(\hat{x}) \geq 0, \quad \forall y \in C; \tag{2.1.10}$$

(iii) if E is strictly convex , then $\Pi_C^f x$ is a single valued mapping.

Lemma 2.1.29. [74] Let E be real reflexive and smooth a Banach space, let C be a nonempty closed convex subset of E, and let $x \in E, \hat{x} \in \Pi_C^f x$. Then

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \quad \forall y \in C. \tag{2.1.11}$$

Definition 2.1.30. A mapping $T : C \to C$ is said to be G-quasi-strict pseudo-contraction if $F(T) \neq \emptyset$ and for $p \in F(T)$, then there exists $\kappa \in [0,1)$ such that

$$G(p, JTx) \leq G(p, Jx) + \kappa(G(x, JTx) - 2\rho f(p)), \quad \forall\, x \in C. \tag{2.1.12}$$

It is obvious from above definition that (2.1.12) is equivalent to

$$\phi(p, Tx) \leq \phi(p, x) + \kappa\phi(x, Tx) + 2\kappa\rho(f(x) - f(p)),\ \forall x \in C \text{ and } p \in F(T).$$

Before providing some examples of this mapping, let us consider the following remark.

Remark 2.1.31. Let α be any real number be such that $\alpha \in (-\infty, -1] \cup [1, 2)$. Then $\frac{\alpha^2-1}{(1-\alpha)^2+2} \in [0, 1)$.

Proof. Since $\alpha \leq -1$ or $\alpha \geq 1$, it is easy to obtain that $\alpha^2 - 1 \geq 0$. Notice that $(1-\alpha)^2 + 2 \geq 2$. Then $\frac{\alpha^2-1}{(1-\alpha)^2+2} \geq 0$ for any $\alpha \in (-\infty, -1] \cup [1, 2)$. It remains to show that $\frac{\alpha^2-1}{(1-\alpha)^2+2} < 1$. It can be found that if $\alpha < 2$ or $\alpha \leq -1$, then

$$0 < 2(2-\alpha) = 1 + (1-2\alpha) + 2. \tag{2.1.13}$$

Adding to both sides of (2.1.13) with α^2, we obtain

$$\alpha^2 < 1 + (1 - 2\alpha + \alpha^2) + 2 = 1 + (1-\alpha)^2 + 2.$$

By a simple calculation, we find that $\frac{\alpha^2-1}{(1-\alpha)^2+2} < 1$. This completes the proof. □

Example 2.1.32. *Let E be a smooth Banach space, $\alpha \in (-\infty, -1] \cup [1, 2)$ and $T_\alpha : E \to E$ be a mapping defined by $T_\alpha x = \alpha x$ for all $x \in E$. Then, T_α is a G-quasi-strict pseudo-contraction.*

Proof. It is easy to see that $F(T_\alpha) = \{x \in E : T_\alpha x = x\} = \{0\}$. By Remark 2.1.31, we can find $\kappa \in [0,1)$ such that $\frac{\alpha^2-1}{(1-\alpha)^2+2} \leq \kappa$. Moreover, it is found that

$$\phi(0, Tx) = \|0\|^2 - 2\langle 0, J(\alpha x)\rangle + \|\alpha x\|^2 = \alpha^2\|x\|^2 = \left(1 + \alpha^2 - 1\right)\|x\|^2$$

$$= \left(1 + \left(\frac{(1-\alpha)^2 + 2}{(1-\alpha)^2 + 2}\right)(\alpha^2 - 1)\right)\|x\|^2$$
$$= \left(1 + (1-\alpha)^2 \frac{(\alpha^2 - 1)}{(1-\alpha)^2 + 2} + 2\frac{(\alpha^2 - 1)}{(1-\alpha)^2 + 2}\right)\|x\|^2$$
$$\leq \left(1 + (1-\alpha)^2\kappa + 2\kappa\right)\|x\|^2 = \left(1 + \left(1 - 2\alpha + \alpha^2\right)\kappa + 2\kappa\right)\|x\|^2$$
$$= \|x\|^2 + \kappa\left(\|x\|^2 - 2\alpha\|x\|^2 + \alpha^2\|x\|^2\right) + 2\kappa\|x\|^2$$
$$= \phi(0,x) + \kappa\left(\|x\|^2 - 2\langle x, J(\alpha x)\rangle + \|\alpha x\|^2\right)$$
$$+ 2\kappa(1)\left(\|x\|^2 - \|0\|^2\right)$$
$$= \phi(0,x) + \kappa\left(\|x\|^2 - 2\langle x, J(T_\alpha x)\rangle + \|T_\alpha x\|^2\right)$$
$$+ 2\kappa(1)\left(\|x\|^2 - \|0\|^2\right)$$
$$= \phi(0,x) + \kappa\phi(x, T_\alpha x) + 2\kappa(1)\left(\|x\|^2 - \|0\|^2\right)$$

for all $x \in E$, where $\rho = 1$ and $f = \|\cdot\|^2$. Furthermore, if $\{x_n\} \subset E$ such that $x_n \to x$, then we have $T_\alpha x_n = \alpha x_n \to \alpha x$. Notice that $T_\alpha x = \alpha x$. This means that T_α is a closed and quasi-strict G-pseudo contraction. This completes the proof. □

Now, we recall the definitions of Hilbert spaces and also the fundamental properties of Hilbert spaces.

Definition 2.1.33. [60] An *inner product* on H is a mapping of $H \times H$ into the scalar field K of H; that is, with every pair of vectors x and y there is associated a scalar which is written by $\langle x, y\rangle$ and is called the inner product of x and y, such that for all vectors x, y, z and scalars α we have

(IP1) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$;

(IP2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$;

(IP3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;

(IP4) $\begin{cases} \langle x, x\rangle \geq 0 \\ \langle x, x\rangle = 0 \Leftrightarrow x = 0. \end{cases}$

Definition 2.1.34. [60] The space H is said to be *complete* if every Cauchy sequence in H converges (that is, has a limit which is an element of H).

Definition 2.1.35. A *Hilbert space* is an inner product space which is complete under the norm induced by its inner product.

Definition 2.1.36. A sequence of points x_n in a Hilbert space H is said to *converge weakly* to a point x in H if $\lim_{n\to\infty}\langle x_n, y\rangle = \langle x, y\rangle$ for all $y \in H$. The notation $x_n \rightharpoonup x$ is sometimes used to denote this kind of convergence.

An inner product space is a vector space H with an inner product defined on H. Let H be an inner product space. For each x in H, we define its norm $\|x\|$ by $\|x\| = \sqrt{\langle x, x\rangle}$.

Lemma 2.1.37. [72](**The Schwarz inequality**). Let H be an inner product space and let x and y be elements in H. Then, the following holds:

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

The following article is the properties of the inner product which need in this dissertation. The following equalities hold for all $x, y \in H$:

$$\begin{aligned} \|x+y\|^2 &= \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \\ \|x-y\|^2 &= \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle. \end{aligned}$$

The following lemma provides some useful properties of firmly nonexpansive mappings on Hilbert spaces.

Lemma 2.1.38. [75, Lemma 2.5] T is firmly nonexpansive if and only if $(I - T)$ is firmly nonexpansive.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a closed convex subset of H. For every point $x \in H$ there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C x\| \leqslant \|x - y\| \quad \forall y \in C,$$

where P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping. It is also known that H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

For a given sequence $\{x_n\} \subset C$, let $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denote the weak ω-limit set of $\{x_n\}$.

Now we collect some lemmas which will be used in the proof of the main result in the next section. We note that Lemma 2.1.39 and Lemma 2.1.40 are well known.

Lemma 2.1.39. *Let H be a real Hilbert space. There holds the following identities*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \quad \forall x, y \in H.$

(ii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2\ \forall x, y \in H$ *and* $\lambda \in [0,1]$

Lemma 2.1.40. *Let C be a closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relation*

$$\langle x - z, y - z\rangle \leqslant 0 \quad \forall y \in C.$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following condition:

$(A1)$ $F(x,x) = 0$ for all $x \in C$;

$(A2)$ F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

$(A3)$ for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y);$$

$(A4)$ for each $x \in C$, $y \longmapsto F(x,y)$ is convex and lower semicontinous.

The following lemma appears implicitly in [1].

Lemma 2.1.41. [1] Let C be a nonempty closed convex subset of a Hilbert space H and let F be a bifunction of C × C into $\mathbb{R}$ satisfying $(A1)$ - $(A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that

$$F(z,y) + \tfrac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

The following lemma was also given in [5].

Lemma 2.1.42. [5] Assume that $F : C \times C \to \mathbb{R}$ satisfy $(A1)$ - $(A4)$. For r > 0 and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r x = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $x \in H$. Then, the following statements hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for any x,$y \in H$,
$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;

(iii) $F(T_r) = EP(F)$;

(iv) $EP(F)$ is closed and convex.

Lemma 2.1.43. [76] Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a continuous pseudo-contractive mapping, then

(i) $F(T)$ is closed convex subset of C;

(ii) $I - T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $(I-T)x_n \to 0$, then $(I-T)z = 0$.

Lemma 2.1.44. [70] Let C be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition

$$\|x_n - u\| \leqslant \|u - q\| \quad \forall n \in \mathbb{N}.$$

Then $x_n \to q$.

2.2 Lipschitzian and convexity

Definition 2.2.1. [72] Let K be a nonempty subset of H. A single-valued mapping $T : K \to H$ is said to be *a nonexpansive mapping* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all } \ x, y \in K.$$

Definition 2.2.2. [77] A single-valued mapping $T : H \to H$ is said to be a ξ-*Lipschitzian mapping* if there exists a real number $\xi > 0$ such that

$$\|Tx - Ty\| \leq \xi\|x - y\|, \quad \text{for all } \ x, y \in H.$$

Definition 2.2.3. [72] Let K be a subset of H. A set K is called to be *convex* if the line segment joining any two points x and y in K is contained in K : algebraically, for each $x, y \in K$ and $\lambda \in [0, 1]$ such that

$$\lambda x + (1 - \lambda)y \in K. \tag{2.2.1}$$

Definition 2.2.4. [78] The smallest convex subset containing $K \subset H$ is called *a convex hull* of K.

Definition 2.2.5. [79] Let C be a nonempty subset of a real vector space H.

1. The set C is called a cone, if $x \in C,\ \lambda > 0$ then $\lambda x \in C$.
2. A cone C is called pointed, if $C \cap (-C) = \{0\}$.
3. A cone is convex, if $C + C \subset C$.
4. A cone C is proper if and only if $0 \notin C$.

Let X be a Hausdorff topological vector space, K be a subset of X, and $f : K \times K \to \mathbb{R}$ be a mapping with $f(x, x) = 0$. The classical, scalar-valued equilibrium problem deals with the existence of $\tilde{x} \in K$ such that

$$f(\tilde{x}, y) \geq 0; \quad \forall y \in K.$$

Moreover, in the case of vector valued mappings, let Y be a another Hausdorff topological vector space, $C \subset Y$ a convex cone with nonempty interior. Given a vector mapping $f : K \times K \to Y$, then the problem of finding $\tilde{x} \in K$ such that

$$f(\tilde{x}, y) \notin -intC; \quad \forall y \in K,$$

is called weak equilibrium problem and the point $\tilde{x} \in K$ is called weak equilibrium point, where $intC$ denotes the interior of the cone C in Y. Remark that $C \subset Y$ is a pointed closed convex cone with nonempty interior i.e., $intC \neq \emptyset$. The partial ordering induced by C on Y is denoted by $\leq_C$ and is defined by $x \leq_C y$ if and only if $y - x \in C$.

Definition 2.2.6. Let $g : K \to Y$ be a mapping. Then g is said to be C-convex, if for all $x, y \in K$ and $\lambda \in [0, 1]$

$$g(\lambda x + (1 - \lambda)y) \leq_C \lambda g(x) + (1 - \lambda)g(y),$$

which implies that

$$g(\lambda x + (1 - \lambda)y) \in \lambda g(x) + (1 - \lambda)g(y) - C.$$

Definition 2.2.7. A mapping $g : K \to Y$ is said to be

1. lower semicontinuous with respect to C at a point $x_0 \in K$, if for any neighborhood V of $g(x_0)$ in Y, there exists a neighborhood U of $x_0 \in X$ such that $g(U \cap K) \subseteq V + C$;

2. upper semicontinuous with respect to C at a point $x_0 \in K$, if $g(U \cap K) \subseteq V - C$;

3. continuous with respect to C at a point $x_0 \in K$, if it is lower semicontinuous and upper semicontinuous with respect to C at that point.

Remark 2.2.8. If g is lower semicontinuous, (upper semicontinuous and continuous) with respect to C at any point of K, then g is lower semicontinuous, (upper semicontinuous and continuous) with respect to C on K, respectively.

Definition 2.2.9. A mapping $f : K \times K \to Y$ is said to be C-monotone, if for all $x, y \in K$

$$f(x,y) + f(y,x) \in -C.$$

Lemma 2.2.10. [80] If g is a lower semicontinuous mapping with respect to C, then the set

$$\{x \in K : g(x) \notin intC\}$$

is closed in K.

Lemma 2.2.11. [81] Let (Y, C) be an ordered topological vector space with a pointed closed convex cone C. Then for all $x, y \in Y$, we have

1. $y - x \in intC$ and $y \notin intC$ imply $x \notin intC$;
2. $y - x \in C$ and $y \notin intC$ imply $x \notin intC$;
3. $y - x \in -intC$ and $y \notin -intC$ imply $x \notin -intC$;
4. $y - x \in -C$ and $y \notin -intC$ imply $x \notin -intC$.

Definition 2.2.12. [82] Consider a subset K of a topological vector space and a topological space Y. A family $\{(C_i, Z_i)\}_{i \in I}$ of pair of sets is said to be coercing for a multivalued mapping $F : K \to 2^Y$ if and only if

(i) for each $i \in I$, C_i is contained in a compact convex subset of K and Z_i is a compact subset of Y;

(ii) for each $i, j \in I$, there exists $k \in I$ such that $C_i \cup C_j \subseteq C_k$;

(iii) for each $i \in I$, there exists $k \in I$ with $\bigcap_{x \in C_k} F(x) \subseteq Z_i$.

Definition 2.2.13. Let K be a nonempty convex subset of a topological vector space X. A multivalued mapping $F : K \to 2^X$ is said to be KKM mapping, if for every finite subset$\{x_i\}_{i \in I}$ of K,

$$Co\{x_i : i \in I\} \subseteq \bigcup_{i \in I} F(x_i),$$

where $Co\{x_i : i \in I\}$ denotes the convex hull of $\{x_i\}_{i\in I}$.

Theorem 2.2.14. [82] Let X be a Hausdorff topological vector space, Y a convex subset of X; K a nonempty subset of Y and $F : K \to 2^Y$ a KKM mapping with compactly closed values in Y (i.e., for all $x \in K$, $F(x) \cap Z$ is closed for every compact set Z of Y). If F admits a coercing family, then

$$\bigcap_{x\in K} F(x) \neq \emptyset.$$

Condition(C): We say that the cone C satisfies *Condition*(C), if there is a pointed convex closed cone $\tilde{C}$ such that $C \setminus \{0\} \subseteq int\tilde{C}$.

Lemma 2.2.15. *Let E be a Banach space and $\emptyset \neq C \subset E$ be a closed convex set, $a \in \mathbb{R}$ and*

$$K = \{v \in C : a \leq g(v)\},$$

where g is upper semicontinuous and concave functional. Then the set K is closed and convex.

Proof. Firstly, we wish to show that K is closed. Let $\{x_n\} \subset K$ be such that $x_n \to x \in C$. Thus we have $a \leq g(x_n)$ for all $n \in \mathbb{N}$ and then $a \leq \limsup_{n\to\infty} g(x_n) \leq g(x)$. Therefore, $x \in K$ and hence K is closed. For the convexity of K, we notice that for all $x, y \in K$ and $t \in [0, 1]$, we have $tx + (1-t)y \in C$, $g(x) \geq a$, $g(y) \geq a$, and then the concavity of g allows

$$g(tx + (1-t)y) \geq tg(x) + (1-t)g(y) \geq ta + (1-t)a = a.$$

This shows that K is convex. □

CHAPTER III

ITERATIVE APPROXIMATIONS VIA HYBRID ALGORITHMS FOR PSEUDO CONTRACTION TYPE MAPPING

3.1 Strong convergence by a hybrid algorithm for solving equilibrium problem and fixed point problem of a Lipschitz pseudo-contraction in Hilbert spaces

In this work, we assume that H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem (for short, EP) is to find $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C. \tag{3.1.1}$$

The set of solution (3.1.1) is denote by $EP(F)$. Given a mapping $T : C \to H$ and let $F(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $x \in EP(F)$ if and only if $x \in C$ is a solution of the variational inequality $\langle Tx, y - x \rangle \geq 0$ for all $y \in C$.

Recall, a mapping T with domain $D(T)$ and range $R(T)$ in H is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in D(T).$$

The mapping T is said to be a strict pseudo-contraction if there exists a constant $0 \leqslant \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \forall x, y \in D(T). \tag{3.1.2}$$

In this case, T may be called as κ-strict pseudo-contraction mapping. In the case that $\kappa = 1$, T is said to be a pseudo-contraction, i.e.,

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \forall x, y \in D(T). \tag{3.1.3}$$

It is easy to see that (3.1.3) is equivalent to

$$\langle x - y, (I - T)x - (I - T)y\rangle \geqslant 0, \forall x, y \in D(T).$$

By definition, it is clear that

nonexpansive $\Rightarrow$ strict pseudo-contraction $\Rightarrow$ pseudo-contraction.

Yao et al. [83] introduced the hybrid iterative algorithm which just involved one sequence of closed convex sets for pseudo-contractive mappings in Hilbert spaces as follows:

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contraction. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n &= (1 - \alpha_n)x_n + \alpha_n T z_n, \\ C_{n+1} &= \{v \in C_n : \|\alpha_n(I - T)y_n\|^2 \leqslant 2\alpha_n \langle x_n - v, (I - T)y_n\rangle\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0). \end{cases} \tag{3.1.4}$$

Theorem 3.1.1. [83] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contraction such that $F(T) \neq \varnothing$. Assume the sequence $\{\alpha_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{L+1}\right)$. Then the sequence $\{x_n\}$ generated by (3.1.4) converges strongly to $P_{F(T)}(x_0)$.

Theorem 3.1.2. [83] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a L-Lipschitz pseudo-contraction such that $F(T) \neq \varnothing$. Assume the sequence $\{\alpha_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{L+1}\right)$. Then the sequence $\{x_n\}$ generated by (3.1.4) converges strongly to $P_{F(T)}(x_0)$.

Now, by employing (3.1.4) we create a hybrid algorithm to find the common element of fixed point set of a Lipshitz pseudo-contraction and the set of equilibrium problem.

Theorem 3.1.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ be L-Lipschitz pseudo-contraction and F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$ with $\widetilde{F} := F(T) \cap EP(F) \neq \varnothing$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n &= (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n &= (1-\beta_n)x_n + \beta_n u_n, \\ u_n &\in C \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \text{ for all } y \in C, \\ C_{n+1} &= \{v \in C_n : \|\alpha_n (I-T)y_n\|^2 + (1-\beta_n)\|x_n - u_n\|^2 \\ & \quad \leqslant 2\alpha_n \langle x_n - v, (I-T)y_n\rangle + 4\langle x_n - v, (I - T_{r_n})z_n - (x_n - u_n)\rangle \\ & \quad +2\alpha_n\beta_n L\|x_n - u_n\|\|y_n - x_n + \alpha_n(I-T)y_n\| \\ & \quad +\beta_n\|x_n - u_n\|^2 + \beta_n\|(I - T_{r_n})z_n\|^2\} \\ x_{n+1} &= P_{C_{n+1}}(x_0). \end{cases} \tag{3.1.5}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are sequences such that

(1) $0 < a \leqslant \alpha_n \leqslant b < \frac{1}{L+1} < 1$ *for all* $n \in \mathbb{N}$,

(2) $0 < \beta_n \leqslant 1$ *for all* $n \in \mathbb{N}$ *with* $\lim_{n\to\infty} \beta_n = 0$,

(3) $r_n > 0$ *for all* $n \in \mathbb{N}$ *with* $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $P_{\widetilde{F}}(x_0)$.

Proof. By Lemma 2.1.43 (i) and Lemma 2.1.42 (iv), we see that $F(T)$ and $EP(F)$ are closed and convex, then $\widetilde{F}$ is also. Hence $P_{\widetilde{F}}$ is well defined. Next, we will prove by induction that $\widetilde{F} \subset C_n$ for all $n \in \mathbb{N}$. Note that $\widetilde{F} \subset C = C_1$. Assume that $\widetilde{F} \subset C_k$ holds for some $k \geqslant 1$. Let $p \in \widetilde{F}$, thus $p \in C_k$. We observe that

$$
\begin{aligned}
& \|x_k - p - \alpha_k(I-T)y_k\|^2 \\
= \;& \|x_k - p\|^2 - \|\alpha_k(I-T)y_k\|^2 - 2\alpha_k \langle (I-T)y_k, x_k - p - \alpha_k(I-T)y_k\rangle \\
= \;& \|x_k - p\|^2 - \|\alpha_k(I-T)y_k\|^2 - 2\alpha_k \langle (I-T)y_k - (I-T)p, y_k - p\rangle \\
& -2\alpha_k \langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\rangle \\
\leqslant \;& \|x_k - p\|^2 - \|\alpha_k(I-T)y_k\|^2 - 2\alpha_k \langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\rangle \\
= \;& \|x_k - p\|^2 - \|(x_k - y_k) + (y_k - x_k + \alpha_k(I-T)y_k)\|^2 \\
& -2\alpha_k \langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\rangle \\
= \;& \|x_k - p\|^2 - \|x_k - y_k\|^2 - \|y_k - x_k + \alpha_k(I-T)y_k\|^2 \\
& -2\langle x_k - y_k, y_k - x_k + \alpha_k(I-T)y_k\rangle \\
& -2\alpha_k \langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\rangle \\
\leqslant \;& \|x_k - p\|^2 - \|x_k - y_k\|^2 - \|y_k - x_k + \alpha_k(I-T)y_k\|^2 \\
& 2|\langle x_k - y_k - \alpha_k(I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k\rangle|. \qquad (3.1.6)
\end{aligned}
$$

Consider the last term of (3.1.6) we obtain

$$
\begin{aligned}
& |\langle x_k - y_k - \alpha_k(I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\rangle| \\
= \;& \alpha_k|\langle x_k - Tz_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\rangle| \\
= \;& \alpha_k|\langle x_k - Tx_k + Tx_k - Tz_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\rangle| \\
= \;& \alpha_k|\langle (I-T)x_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k\rangle \\
& + \langle Tx_k - Tz_k, y_k - x_k + \alpha_k(I-T)y_k\rangle| \\
\leqslant \;& \alpha_k(L+1)\|x_k - y_k\|\|y_k - x_k + \alpha_k(I-T)y_k\| \\
& +\alpha_k L\|x_k - z_k\|\|y_k - x_k + \alpha_k(I-T)y_k\| \\
\leqslant \;& \frac{\alpha_k(L+1)}{2}(\|x_k - y_k\|^2 + \|y_k - x_k + \alpha_k(I-T)y_k\|^2) \\
& +\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k\| + \alpha_k\|(I-T)y_k\|. \qquad (3.1.7)
\end{aligned}
$$

Substituting (3.1.7) into (3.1.6), we obtain

$$\begin{aligned}\|x_k - p - \alpha_k(I-T)y_k\|^2 &\leqslant \|x_k - p\|^2 - \|x_k - y_k\|^2 - \|y_k - x_k + \alpha_k(I-T)y_k\|^2 \\ &\quad + \alpha_k(L+1)(\|x_k - y_k\|^2 \\ &\quad + \|y_k - x_k + \alpha_k(I-T)y_k\|^2) \\ &\quad + 2\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k + \alpha_k(I-T)y_k\| \\ &\leqslant \|x_k - p\|^2 + 2\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k\| \\ &\quad + \alpha_k\|(I-T)y_k\|. \end{aligned} \tag{3.1.8}$$

Notice that

$$\|x_k - p - \alpha_k(I-T)y_k\|^2 = \|x_k - p\|^2 - 2\alpha_k\langle x_k - p, (I-T)y_k\rangle + \|\alpha_k(I-T)y_k\|^2. \tag{3.1.9}$$

Therefore, from (3.1.8) and (3.1.9), we get

$$\|\alpha_k(I-T)y_k\|^2 \leqslant 2\alpha_k\langle x_k - p, (I-T)y_k\rangle + 2\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k + \alpha_k(I-T)y_k\|. \tag{3.1.10}$$

On the other hand, we found that

$$\begin{aligned} &\|x_k - p - \beta_k(I - T_{r_k})z_k\|^2 \\ =\ &\|x_k - p\|^2 - \|\beta_k(I - T_{r_k})z_k\|^2 - 2\beta_k\langle (I - T_{r_k})z_k, x_k - p - \beta_k(I - T_{r_k})z_k\rangle \\ =\ &\|x_k - p\|^2 - \|\beta_k(I - T_{r_k})z_k\|^2 - 2\beta_k\langle (I - T_{r_k})z_k - (I - T_{r_k})p, z_k - p\rangle \\ &-2\beta_k\langle (I - T_{r_k})z_k, x_k - z_k - \beta_k(I - T_{r_k})z_k\rangle \\ \leqslant\ &\|x_k - p\|^2 - \|\beta_k(I - T_{r_k})z_k\|^2 - 2\langle \beta_k(I - T_{r_k})z_k, x_k - z_k - \beta_k(I - T_{r_k})z_k\rangle \\ =\ &\|x_k - p\|^2 - \|\beta_k(I - T_{r_k})z_k\|^2 \\ &+ \left(\|\beta_k(I - T_{r_k})z_k\|^2 - \|x_k - z_k\|^2 + \|x_k - z_k - \beta_k(I - T_{r_k})z_k\|^2\right) \\ =\ &\|(x_k - z_k) + (z_k - p)\|^2 - \|x_k - z_k\|^2 + \|\beta_k(I - T_{r_k})x_k - \beta_k(I - T_{r_k})z_k\|^2 \\ =\ &\|x_k - z_k\|^2 + 2\langle x_k - z_k, z_k - p\rangle + \|z_k - p\|^2 - \|x_k - z_k\|^2 \\ &+\|\beta_k(I - T_{r_k})x_k - \beta_k(I - T_{r_k})z_k\|^2 \end{aligned}$$

$$
\begin{aligned}
&= 2\langle x_k - z_k, (z_k - x_k) + (x_k - p)\rangle + \|(1-\beta_k)(x_k - p) + \beta_k(T_{r_k}x_k - p)\|^2 \\
&\quad + \|\beta_k(I - T_{r_k})x_k - \beta_k(I - T_{r_k})z_k\|^2 \\
&\leqslant 2\langle x_k - p, \beta_k(I - T_{r_k})x_k\rangle - 2\|x_k - z_k\|^2 + \beta_k\|T_{r_k}x_k - p\|^2 \\
&\quad - \beta_k(1-\beta_k)\|x_k - T_{r_k}x_k\|^2 + (1-\beta_k)\|x_k - p\|^2 + \beta_k^2\|x_k - z_k\|^2 \\
&\leqslant 2\langle x_k - p, \beta_k(I - T_{r_k})x_k\rangle + \beta_k\|x_k - p\|^2 + (1-\beta_k)\|x_k - p\|^2 \\
&\quad - \beta_k(1-\beta_k)\|x_k - T_{r_k}x_k\|^2 + \beta_k^2\|(I - T_{r_k})x_k\|^2 \\
&= 2\langle x_k - p, \beta_k(I - T_{r_k})x_k\rangle + \|x_k - p\|^2 - \beta_k(1-\beta_k)\|x_k - u_k\|^2 \\
&\quad + \beta_k^2\|x_k - u_k\|^2. \qquad (3.1.11)
\end{aligned}
$$

Notice that

$$
\|x_k - p - \beta_k(I - T_{r_k})z_k\|^2 = \|x_k - p\|^2 - 2\beta_k\langle x_k - p, (I - T_{r_k})z_k\rangle + \beta_k^2\|(I - T_{r_k})z_k\|^2. \qquad (3.1.12)
$$

Combining (3.1.11) and (3.1.12) and then it implies that

$$
\begin{aligned}
\beta_k(1-\beta_k)\|x_k - u_k\|^2 &\leqslant 4\beta_k\langle x_k - p, (I - T_{r_k})z_k - (x_k - u_k)\rangle + \beta_k^2\|x_k - u_k\|^2 \\
&\quad + \beta_k^2\|(I - T_{r_k})z_k\|^2.
\end{aligned}
$$

Since $\beta_n > 0$ for all n, so we get

$$
\begin{aligned}
(1-\beta_k)\|x_k - u_k\|^2 &\leqslant 4\langle x_k - p, (I - T_{r_k})z_k - (x_k - u_k)\rangle + \beta_k\|x_k - u_k\|^2 \\
&\quad + \beta_k\|(I - T_{r_k})z_k\|^2. \qquad (3.1.13)
\end{aligned}
$$

It follows form (3.1.10) and (3.1.13) we obtain

$$
\begin{aligned}
&\|\alpha_k(I - T)y_k\|^2 + (1-\beta_k)\|x_k - u_k\|^2 \\
&\leqslant 2\alpha_k\langle x_k - p, (I - T)y_k\rangle + 4\langle x_k - p, (I - T_{r_k})z_k - (x_k - u_k)\rangle \\
&\quad + 2\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k + \alpha_k(I - T)y_k\| + \beta_k\|x_k - u_k\|^2 \\
&\quad + \beta_k\|(I - T_{r_k})z_k\|^2.
\end{aligned}
$$

Therefore, $p \in C_{k+1}$. By mathematical induction, we have $\widetilde{F} \subset C_n$ for all $n \in \mathbb{N}$. It is easy to check that C_n is closed and convex and then $\{x_n\}$ is well defined. From $x_n = P_{C_n}(x_0)$, we have $\langle x_0 - x_n, x_n - y \rangle \geqslant 0$ for all $y \in C_n$. Using $\widetilde{F} \subset C_n$, we also have $\langle x_0 - x_n, x_n - u \rangle \geqslant 0$ for all $u \in \widetilde{F}$. So, for $u \in \widetilde{F}$, we have

$$\begin{aligned} 0 &\leqslant \langle x_0 - x_n, x_n - u \rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - u \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - u \rangle \\ &\leqslant -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|. \end{aligned}$$

Hence,

$$\|x_0 - x_n\| \leqslant \|x_0 - u\| \quad \text{for all} \quad u \in \widetilde{F}. \tag{3.1.14}$$

This implies that $\{x_n\}$ is bounded and then $\{y_n\}$, $\{Ty_n\}$, $\{z_n\}$, $\{T_{r_n}z_n\}$ and $\{u_n\}$ are also.

From $x_n = P_{C_n}(x_0)$ and $x_{n+1} = P_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geqslant 0. \tag{3.1.15}$$

Hence,

$$\begin{aligned} 0 &\leqslant \langle x_0 - x_n, x_n - x_{n+1} \rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leqslant -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|, \end{aligned}$$

and therefore

$$\|x_0 - x_n\| \leqslant \|x_0 - x_{n+1}\|,$$

which implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. From Lemma 2.1.39 and (3.1.15), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leqslant \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\begin{aligned}
& \|\alpha_n(I-T)y_n\|^2 + (1-\beta_n)\|x_n - u_n\|^2 \\
\leqslant\ & 2\alpha_n \langle x_n - x_{n+1}, (I-T)y_n\rangle + 4\langle x_n - x_{n+1}, (I-T_{r_n})z_n - (x_n - u_n)\rangle \\
& +2\alpha_n\beta_n L\|x_n - u_n\|\|y_n - x_n + \alpha_n(I-T)y_n\| + \beta_n\|x_n - u_n\|^2 \\
& +\beta_n\|(I-T_{r_n})z_n\|^2 \\
& \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Therefore, we obtian

$$\|y_n - Ty_n\| \to 0 \quad \text{and} \quad \|x_n - u_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.1.16}$$

We note that

$$\begin{aligned}
\|x_n - Tx_n\| &\leqslant \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \\
&\leqslant (L+1)\|x_n - y_n\| + \|y_n - Ty_n\| \\
&\leqslant \alpha_n(L+1)\|x_n - Tz_n\| + \|y_n - Ty_n\| \\
&\leqslant \alpha_n(L+1)\|x_n - Tx_n\| + \alpha_n(L+1)\|Tx_n - Tz_n\| + \|y_n - Ty_n\| \\
&\leqslant \alpha_n(L+1)\|x_n - Tx_n\| + \alpha_n\beta_n L(L+1)\|x_n - u_n\| + \|y_n - Ty_n\|,
\end{aligned} \tag{3.1.17}$$

that is,

$$\|x_n - Tx_n\| \leqslant \frac{\alpha_n\beta_n L(L+1)}{1-\alpha_n(L+1)}\|x_n - u_n\| + \frac{1}{1-\alpha_n(L+1)}\|y_n - Ty_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Next, we will show that

$$\omega_w(x_n) \subset \widetilde{F}. \tag{3.1.18}$$

Since $\{x_n\}$ is bounded, the reflexivity of H guarantees that $\omega_w(x_n) \neq \varnothing$. Let $p \in \omega_w(x_n)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$ and by Lemma 2.1.43 (ii) we have $p \in F(T)$. On the other hand, since $\|x_n - u_n\| \to 0$ and $x_{n_i} \rightharpoonup p$, so we have $u_{n_i} \rightharpoonup p$. It follows from $u_n = T_{r_n}x_n$ and $(A2)$ that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n) \quad \text{for all} \quad y \in C.$$

Replacing n by n_i, we have

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq F(y, u_{n_i}).$$

By using (A4) and the assumption (3) on $\{r_n\}$, we obtain $0 \geq F(y,p)$ for all $y \in C$. So, from $(A1)$ and $(A4)$ we have

$$0 = F(y_t, y_t) = F(y_t, ty + (1-t)p) \leq tF(y_t, y) + (1-t)F(y_t, p) \leq tF(y_t, y).$$

Dividing by t, we have

$$F(y_t, y) \geq 0 \quad \text{for all} \quad y \in C.$$

From $(A3)$ we have $0 \leq \lim_{t\to 0} F(y_t, y) = \lim_{t\to 0} F(ty + (1-t)p, y) \leq F(p, y)$ for all $y \in C$, and hence $p \in EP(F)$. So, $p \in F(T) \cap EP(F) = \widetilde{F}$ and then we have (3.1.18). Therefore, by inequality (3.1.14) and Lamma 2.1.44, we obtain $\{x_n\}$ converges strongly to $P_{\widetilde{F}}(x_0)$. This completes the proof. $\square$

Corollary 3.1.4. [83] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T: C \to C$ be an L-Lipschitz pseudo-contraction such that $F(T) \neq \varnothing$. Assume that $\{\alpha_n\}$ is a sequence such that $0 < a \leqslant \alpha_n \leqslant b < \frac{1}{L+1} < 1$ for all n. Then the sequence $\{x_n\}$ generated by (3.1.4) converges strongly to $P_{F(T)}(x_0)$.

Proof. Put $F(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \geqslant 1$ in Theorem 3.1.3. Then, $T_{r_n} = P_C$ for all $n \geqslant 1$. So, $u_n = P_C x_n$ for all $n \geqslant 1$(Note that $x_1 = P_C x_0$). Since $x_n = P_{C_n} x_0 \in C_n \subset C$ for all $n \geqslant 1$, so we have $u_n = x_n$ and then $z_n = x_n$ for all $n \geqslant 1$. Thus $(I - T_{r_n})z_n = x_n - P_C x_n = 0$ for all $n \geqslant 1$. For this reason, (3.1.4) is a special case of (3.1.5). Applying Theorem 3.1.3, we have the desired result. $\square$

Recall that a mapping A is said to be *monotone*, if $\langle x - y, Ax - Ay \rangle \geqslant 0$ for all $x, y \in H$ and *inverse strongly monotone* if there exists a real number $\gamma > 0$ such that $\langle x - y, Ax - Ay \rangle \geqslant \gamma \|Ax - Ay\|^2$ for all $x, y \in H$. For the second case

A is said to be γ-inverse strongly monotone. It follows immediately that if A is γ-inverse strongly monotone, then A is monotone and *Lipschitz continuous*, that is, $\|Ax - Ay\| \leqslant \frac{1}{\gamma}\|x - y\|$. The pseudo-contractive mapping and strictly pseudo-contractive mapping are strongly related to the monotone mapping and inverse strongly monotone mapping, respectively. It is well known that

1. A is monotone $\Longleftrightarrow T := (I - A)$ is pseudo-contractive.

2. A is inverse strongly monotone $\Longleftrightarrow T := (I - A)$ is strictly pseudo-contractive.

Indeed, for (ii), we notice that the following equality always holds in a real Hilbert space

$$\|(I-A)x-(I-A)y\|^2 = \|x-y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle \quad \forall x, y \in H, \tag{3.1.19}$$

with out loss of generality we can assume that $\gamma \in (0, \frac{1}{2}]$ and then it yields

$$\langle x - y, Ax - Ay\rangle \geqslant \gamma\|Ax - Ay\|^2 \Longleftrightarrow -2\langle x - y, Ax - Ay\rangle \leqslant -2\gamma\|Ax - Ay\|^2$$

$$\Longleftrightarrow \quad \|(I - A)x - (I - A)y\|^2 \leqslant \|x - y\|^2 + (1 - 2\gamma)\|Ax - Ay\|^2 \quad \text{(via (3.1.19))}$$

$$\Longleftrightarrow \quad \|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \tag{3.1.20}$$

$$\text{(where } T := (I - A) \text{ and } \kappa := 1 - 2\gamma).$$

Corollary 3.1.5. *Let $A : H \to H$ be an L-Lipschitz monotone mapping and F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$ which $A^{-1}(0) \cap EP(F) \neq \varnothing$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n &= x_n - \alpha_n(x_n - z_n) - \alpha_n A z_n, \\ z_n &= (1-\beta_n)x_n + \beta_n u_n, \\ u_n &\in C \text{ such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \\ C_{n+1} &= \{v \in C_n : \|\alpha_n A y_n\|^2 + (1-\beta_n)\|x_n - u_n\|^2 \\ & \quad \leqslant 2\alpha_n \langle x_n - v, Ay_n\rangle + 4\langle x_n - v, (I - T_{r_n})z_n - (x_n - u_n)\rangle \\ & \quad +2\alpha_n\beta_n L\|x_n - u_n\|\|y_n - x_n + \alpha_n A y_n\| \\ & \quad +\beta_n\|x_n - u_n\|^2 + \beta_n\|(I - T_{r_n})z_n\|^2\} \\ x_{n+1} &= P_{C_{n+1}}(x_0). \end{cases} \tag{3.1.21}$$

Assume $0 < a \leqslant \alpha_n \leqslant b < \frac{1}{L+2} < 1$ *for all* $n \in \mathbb{N}$, $\{\beta_n\}$ *and* $\{r_n\}$ *are as in Theorem 3.1.3. Then* $\{x_n\}$ *converges strongly to* $P_{A^{-1}(0)\cap EP(F)}(x_0)$.

Proof. Let $T := (I - A)$. Then T is pseudo-contractive and $(L+2)$-Lipschitz. Hence, it follows from Theorem 3.1.5, we have the desired result. □

3.2 Strong convergence by a hybrid algorithm for solving generalized mixed equilibrium problems and fixed point problems of a Lipschitz pseudo-contraction in Hilbert spaces

Let E be a real Banach space, and E^* the dual space of E. Let C be a nonempty closed convex subset of E. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be a real-valued function, and $A : C \to E^*$ be a nonlinear mapping. The generalized mixed equilibrium problem, is to find $x \in C$ such that

$$\Theta(x,y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{3.2.1}$$

The solution set of (3.2.1) is denoted by $GMEP(\Theta, A, \varphi)$, i.e.,
$GMEP(\Theta, A, \varphi) = \{x \in C : \Theta(x,y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \forall y \in C\}$.

If $A = 0$, the problem (3.2.1) reduces to the mixed equilibrium problem for Θ, denoted by $MEP(\Theta, \varphi)$, which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.$$

If $\Theta = 0$, the problem (3.2.1) reduces to the mixed variational inequality of Browder type, denoted by $VI(C, A, \varphi)$, which is to find $x \in C$ such that

$$\langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C.$$

If $A = 0$ and $\varphi = 0$, the problem (3.2.1) reduces to the equilibrium problem for Θ (for short, EP), denoted by $EP(\Theta)$, which is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C. \tag{3.2.2}$$

Recall, a mapping T with domain $D(T)$ and range $R(T)$ in H is called firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \forall x, y \in D(T),$$

nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in D(T).$$

Throughout this paper, I stands for an identity mapping. The mapping T is said to be a strict pseudo-contraction if there exists a constant $0 \leqslant \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \forall x, y \in D(T).$$

In this case, T may be called as a κ-strict pseudo-contraction mapping. In the even that $\kappa = 1$, T is said to be a pseudo-contraction, i.e.,

$$\|Tx - Ty\|^2 \leqslant \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \forall x, y \in D(T). \tag{3.2.3}$$

It is easy to see that (3.2.3) is equivalent to

$$\langle x - y, (I - T)x - (I - T)y \rangle \geqslant 0, \forall x, y \in D(T).$$

By definition, it is clear that

firmly nonexpansive $\Rightarrow$ nonexpansive $\Rightarrow$ strict pseudo-contraction $\Rightarrow$ pseudo-contraction.

However, the following examples show that the converse is not true.

Example 3.2.1 (Chidume and Mutangadura [84]). *Take* $H = \mathbb{R}^2$, $B = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, $B_1 = \{x \in B : \|x\| \leq \frac{1}{2}\}$, $B_2 = \{x \in B : \frac{1}{2} \leq \|x\| \leq 1\}$. *If* $x = (a, b) \in X$ *we define* $x^{\perp}$ *to be* $(b, -a) \in X$. *Define* $T : B \to B$ *by*

$$Tx = \begin{cases} x + x^{\perp}, & x \in B_1, \\ \frac{x}{\|x\|} - x + x^{\perp}, & x \in B_2. \end{cases}$$

Then, T *is a Lipschitz and pseudo-contraction but not a strict pseudo-contraction.*

Example 3.2.2. *Take* $H = \mathbb{R}^1$ *and define* $T : X \to X$ *by* $Tx = -3x$. *Then,* T *is a strict pseudo-contraction but not a nonexpansive mapping.*

Indeed, it is clear that T *is not nonexpansive. On the other hand, let us consider*

$$\begin{aligned} \|Tx - Ty\|^2 &= \|(-3x) - (-3y)\|^2 = 9\|x - y\|^2 = \|x - y\|^2 + 8\|x - y\|^2 \\ &= \|x - y\|^2 + \frac{16}{2}\|x - y\|^2 = \|x - y\|^2 + \frac{1}{2}\|4x - 4y\|^2 \\ &= \|x - y\|^2 + \frac{1}{2}\|(1 - (-3))x - (1 - (-3))y\|^2 \\ &= \|x - y\|^2 + \frac{1}{2}\|(I - T)x - (I - T)y\|^2 \\ &\leqslant \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \end{aligned}$$

for all $\kappa \in [\frac{1}{2}, 1)$. Thus T is a strict pseudo-contraction.

Example 3.2.3. *Take* $H \neq \{0\}$ *and let* $T = -I$, *it is not hard to verify that* T *is nonexpansive but not firmly nonexpansive.*

Yao et al. [83] introduced the hybrid iterative algorithm which just involved one sequence of closed convex sets for pseudo-contractive mappings in Hilbert spaces as follows:

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a pseudo-contraction. Let $\{\alpha_n\}$ be a sequence in $(0,1)$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n &= (1-\alpha_n)x_n + \alpha_n T z_n, \\ C_{n+1} &= \{v \in C_n : \|\alpha_n(I-T)y_n\|^2 \leqslant 2\alpha_n \langle x_n - v, (I-T)y_n\rangle\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0). \end{cases} \tag{3.2.4}$$

Theorem 3.2.4. [83] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be an L-Lipschitz pseudo-contraction such that $F(T) \neq \varnothing$. Assume the sequence $\{\alpha_n\} \subset [a,b]$ for some $a, b \in \left(0, \frac{1}{L+1}\right)$. Then the sequence $\{x_n\}$ generated by (3.2.4) converges strongly to $P_{F(T)}(x_0)$.

Very recently, Tang et al. [85] generalized the hybrid algorithm (3.2.4) in the case of the Ishikawa iterative process as follows:

$$\begin{cases} y_n &= (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n &= (1-\beta_n)x_n + \beta_n T x_n, \\ C_{n+1} &= \{v \in C_n : \|\alpha_n(I-T)y_n\|^2 \leqslant 2\alpha_n \langle x_n - v, (I-T)y_n\rangle \\ & \quad +2\alpha_n\beta_n L\|x_n - Tx_n\|\|y_n - x_n + \alpha_n(I-T)y_n\|\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0). \end{cases} \tag{3.2.5}$$

Under some appropriate conditions of $\{\alpha_n\}$ and $\{\beta_n\}$, they proved that (3.2.5) converges strongly to $P_{F(T)}(x_0)$.

Now, by employing (3.2.4) and (3.2.5) we construct a sequence by using some appropriated closed convex sets based on the hybrid shrinking projection methods to find a common solution of fixed point problems of a Lipschitz pseudo-contraction and generalized mixed equilibrium problems in Hilbert spaces.

Theorem 3.2.5. *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ be an L-Lipschitz pseudo-contraction. Let Θ be a bifunction from*

$C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$, $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function, $A : C \to E^*$ be a continuous and monotone mapping such that $\Omega := F(T) \cap GMEP(\Theta, A, \varphi) \neq \emptyset$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as follows:

$$\begin{cases} y_n &= (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n &= (1-\beta_n)x_n + \beta_n u_n, \\ u_n &\in C \text{ such that } \Theta(u_n, y) + \langle Au_n, y-u_n\rangle + \varphi(y) - \varphi(u_n) \\ & \qquad + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \geq 0, \\ C_{n+1} &= \{v \in C_n : \|\alpha_n(I-T)y_n\|^2 + \|x_n - u_n\| \leq 2\alpha_n \langle x_n - v, (I-T)y_n\rangle \\ & \qquad + \sqrt{\langle x_n - v, x_n - u_n\rangle}\,(2\alpha_n\beta_n L\,\|y_n - x_n + \alpha_n(I-T)y_n\| + 1)\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0). \end{cases} \tag{3.2.6}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are sequences such that

(1) *$0 < a \leqslant \alpha_n \leqslant b < \frac{1}{L+1} < 1$ for all $n \in \mathbb{N}$,*

(2) *$0 \leqslant \beta_n \leqslant 1$ for all $n \in \mathbb{N}$,*

(3) *$r_n > 0$ for all $n \in \mathbb{N}$ with $\liminf_{n\to\infty} r_n > 0$.*

Then $\{x_n\}$ converges strongly to $P_\Omega(x_0)$.

Proof. By Lemma 2.1.43 (i) and Lemma 2.1.12 (iv), we see that $F(T)$ and $GMEP(\Theta, A, \varphi)$ are closed and convex respectively, then Ω is also. Hence P_Ω is well defined. Next, we will prove by induction that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. Note that $\Omega \subset C = C_1$. Assume that $\Omega \subset C_k$ holds for some $k \geqslant 1$. Let $p \in \Omega$, thus $p \in C_k$. We observe that

$$\begin{aligned} & \|x_k - p - \alpha_k(I-T)y_k\|^2 \\ = \;& \|x_k - p\|^2 - \|\alpha_k(I-T)y_k\|^2 - 2\alpha_k \langle (I-T)y_k, x_k - p - \alpha_k(I-T)y_k\rangle \\ = \;& \|x_k - p\|^2 - \|\alpha_k(I-T)y_k\|^2 - 2\alpha_k \langle (I-T)y_k - (I-T)p, y_k - p\rangle \end{aligned}$$

$$
\begin{aligned}
& -2\alpha_k \langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k \rangle \\
\leqslant \;& \|x_k - p\|^2 - \|\alpha_k(I-T)y_k\|^2 - 2\alpha_k \langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k \rangle \\
= \;& \|x_k - p\|^2 - \|(x_k - y_k) + (y_k - x_k + \alpha_k(I-T)y_k)\|^2 \\
& -2\alpha_k \langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k \rangle \\
= \;& \|x_k - p\|^2 - \|x_k - y_k\|^2 - \|y_k - x_k + \alpha_k(I-T)y_k\|^2 \\
& -2 \langle x_k - y_k, y_k - x_k + \alpha_k(I-T)y_k \rangle \\
& -2\alpha_k \langle (I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k \rangle \\
\leqslant \;& \|x_k - p\|^2 - \|x_k - y_k\|^2 - \|y_k - x_k + \alpha_k(I-T)y_k\|^2 \\
& +2| \langle x_k - y_k - \alpha_k(I-T)y_k, x_k - y_k - \alpha_k(I-T)y_k \rangle |. \qquad (3.2.7)
\end{aligned}
$$

Consider the last term of (3.2.7) we obtain

$$
\begin{aligned}
& | \langle x_k - y_k - \alpha_k(I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k \rangle | \\
= \;& \alpha_k | \langle x_k - Tz_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k \rangle | \\
= \;& \alpha_k | \langle x_k - Tx_k + Tx_k - Tz_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k \rangle | \\
= \;& \alpha_k | \langle (I-T)x_k - (I-T)y_k, y_k - x_k + \alpha_k(I-T)y_k \rangle \\
& + \langle Tx_k - Tz_k, y_k - x_k + \alpha_k(I-T)y_k \rangle | \\
\leqslant \;& \alpha_k(L+1)\|x_k - y_k\|\|y_k - x_k + \alpha_k(I-T)y_k\| \\
& +\alpha_k L\|x_k - z_k\|\|y_k - x_k + \alpha_k(I-T)y_k\| \\
\leqslant \;& \frac{\alpha_k(L+1)}{2}(\|x_k - y_k\|^2 + \|y_k - x_k + \alpha_k(I-T)y_k\|^2) \\
& +\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k + \alpha_k(I-T)y_k\|. \qquad (3.2.8)
\end{aligned}
$$

By connecting (3.2.7) and (3.2.8), we obtain

$$
\begin{aligned}
\|x_k - p - \alpha_k(I-T)y_k\|^2 \leqslant \;& \|x_k - p\|^2 - \|x_k - y_k\|^2 - \|y_k - x_k + \alpha_k(I-T)y_k\|^2 \\
& +\alpha_k(L+1)(\|x_k - y_k\|^2 + \|y_k - x_k + \alpha_k(I-T)y_k\|^2) \\
& +2\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k + \alpha_k(I-T)y_k\| \\
& \leqslant \|x_k - p\|^2
\end{aligned}
$$

$$+2\alpha_k\beta_k L\|x_k - u_k\|\|y_k - x_k + \alpha_k(I-T)y_k\|. \quad (3.2.9)$$

Notice that $u_k = K_{r_k}x_k$ and by Lemma 2.1.38, we observe that

$$\begin{aligned} \|x_k - u_k\|^2 &= \|(I-K_{r_k})x_k - (I-K_{r_k})p\|^2 \\ &\leq \langle (I-K_{r_k})x_k - (I-K_{r_k})p, x_k - p\rangle \\ &= \langle (I-K_{r_k})x_k, x_k - p\rangle. \end{aligned}$$

So, we have

$$\|x_k - u_k\| \leq \sqrt{\langle x_k - p, x_k - u_k\rangle}. \quad (3.2.10)$$

Joining (3.2.9) and (3.2.10), we obtain

$$\begin{aligned} \|x_k - p - \alpha_k(I-T)y_k\|^2 &\leqslant \|x_k - p\|^2 + 2\alpha_k\beta_k L\sqrt{\langle x_k - p, x_k - u_k\rangle}\|y_k - x_k \\ &\quad +\alpha_k(I-T)y_k\|. \quad (3.2.11) \end{aligned}$$

Notice that

$$\|x_k - p - \alpha_k(I-T)y_k\|^2 = \|x_k - p\|^2 - 2\alpha_k\langle x_k - p, (I-T)y_k\rangle + \|\alpha_k(I-T)y_k\|^2. \quad (3.2.12)$$

By (3.2.11) and (3.2.12), we have

$$\begin{aligned} \|\alpha_k(I-T)y_k\|^2 &\leqslant 2\alpha_k\langle x_k - p, (I-T)y_k\rangle + 2\alpha_k\beta_k L\sqrt{\langle x_k - p, x_k - u_k\rangle}\|y_k - x_k \\ &\quad +\alpha_k(I-T)y_k\|. \quad (3.2.13) \end{aligned}$$

Combining (3.2.13) and (3.2.10) we obtain

$$\begin{aligned} &\|\alpha_k(I-T)y_k\|^2 + \|x_k - u_k\| \\ &\leq 2\alpha_k\langle x_k - p, (I-T)y_k\rangle \\ &+ \sqrt{\langle x_k - p, x_k - u_k\rangle}\left(2\alpha_k\beta_k L\|y_k - x_k + \alpha_k(I-T)y_k\| + 1\right). \end{aligned}$$

Therefore, $p \in C_{k+1}$. By mathematical induction, we have $\Omega \subset C_n$ for all $n \in \mathbb{N}$.

Let $f_n(\cdot) := 2\alpha_n \langle x_n - (\cdot), (I-T)y_n \rangle$ $+\sqrt{\langle x_n - (\cdot), x_n - u_n \rangle}\,(2\alpha_n\beta_n L \|y_n - x_n + \alpha_n(I-T)y_n\| + 1)$, it is not hard to see that the linearity of $\langle x_n - (\cdot), (I-T)y_n \rangle$ and $\langle x_n - (\cdot), x_n - u_n \rangle$ together with the continuity and concavity of $\sqrt{(\cdot)}$ allow f_n to be continuous and concave. By Lemma 2.2.15, C_n is closed and convex for all $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is well defined. From $x_n = P_{C_n}(x_0)$, we have $\langle x_0 - x_n, x_n - y \rangle \geqslant 0$ for all $y \in C_n$. Using $\Omega \subset C_n$, we also have $\langle x_0 - x_n, x_n - u \rangle \geqslant 0$ for all $u \in \Omega$. So, for $u \in \Omega$, we have

$$\begin{aligned} 0 &\leqslant \langle x_0 - x_n, x_n - u \rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - u \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - u \rangle \\ &\leqslant -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|. \end{aligned}$$

Hence,

$$\|x_0 - x_n\| \leqslant \|x_0 - u\| \quad \text{for all} \quad u \in \Omega. \tag{3.2.14}$$

This implies that $\{x_n\}$ is bounded and then $\{y_n\}$, $\{Ty_n\}$ and $\{u_n\}$ are also.

From $x_n = P_{C_n}(x_0)$ and $x_{n+1} = P_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geqslant 0. \tag{3.2.15}$$

Hence,

$$\begin{aligned} 0 &\leqslant \langle x_0 - x_n, x_n - x_{n+1} \rangle = \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \\ &= -\|x_0 - x_n\|^2 + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leqslant -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|, \end{aligned}$$

and therefore

$$\|x_0 - x_n\| \leqslant \|x_0 - x_{n+1}\|,$$

which implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. From Lemma 2.1.39 and (3.2.15), we obtain

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leqslant \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$\begin{aligned}
&\|\alpha_n(I-T)y_n\|^2 + \|x_n - u_n\| \\
&\leq 2\alpha_n \langle x_n - x_{n+1}, (I-T)y_n \rangle \\
&+ \sqrt{\langle x_n - x_{n+1}, x_n - u_n \rangle}\,(2\alpha_n\beta_n L\,\|y_n - x_n + \alpha_n(I-T)y_n\| + 1) \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Therefore, we obtain

$$\|y_n - Ty_n\| \to 0 \quad \text{and} \quad \|x_n - u_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

We note that

$$\begin{aligned}
\|x_n - Tx_n\| &\leqslant \|x_n - y_n\| + \|y_n - Ty_n\| + \|Ty_n - Tx_n\| \\
&\leqslant (L+1)\|x_n - y_n\| + \|y_n - Ty_n\| \\
&\leqslant \alpha_n(L+1)\|x_n - Tz_n\| + \|y_n - Ty_n\| \\
&\leqslant \alpha_n(L+1)\|x_n - Tx_n\| + \alpha_n(L+1)\|Tx_n - Tz_n\| + \|y_n - Ty_n\| \\
&\leqslant \alpha_n(L+1)\|x_n - Tx_n\| + \alpha_n\beta_n L(L+1)\|x_n - u_n\| + \|y_n - Ty_n\|,
\end{aligned}$$

that is,

$$\|x_n - Tx_n\| \leqslant \frac{\alpha_n\beta_n L(L+1)}{1-\alpha_n(L+1)}\|x_n - u_n\| + \frac{1}{1-\alpha_n(L+1)}\|y_n - Ty_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Next, we will show that

$$\omega_w(x_n) \subset \Omega. \tag{3.2.16}$$

Since $\{x_n\}$ is bounded, the reflexivity of H guarantees that $\omega_w(x_n) \neq \varnothing$. Let $p \in \omega_w(x_n)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$ and by Lemma 2.1.43 (ii) we have $p \in F(T)$. On the other hand, since $\|x_n - u_n\| \to 0$ and $x_{n_i} \rightharpoonup p$, so we have $u_{n_i} \rightharpoonup p$. Define $G : C \times C \to \mathbb{R}$ by $G(x,y) = \Theta(x,y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x)$ for all $x, y \in C$. It is not hard to verify that G satisfies conditions $(A1) - (A4)$. It follows from $u_n = K_{r_n}x_n$ and $(A2)$ that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq G(y, u_n) \quad \text{for all} \quad y \in C.$$

Replacing n by n_i, we have

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq G(y, u_{n_i}).$$

By using (A4), we obtain $0 \geq G(y,p)$ for all $y \in C$. For $t \in (0,1]$ and $y \in C$, let $y_t = ty + (1-t)p$. So, from $(A1)$ and $(A4)$ we have

$$0 = G(y_t, y_t) = G(y_t, ty + (1-t)p) \leq tG(y_t, y) + (1-t)G(y_t, p) \leq tG(y_t, y).$$

Dividing by t, we have

$$G(y_t, y) \geq 0 \text{ for all } y \in C.$$

From $(A3)$ we have $0 \leq \lim_{t\to 0} G(y_t, y) = \lim_{t\to 0} G(ty + (1-t)p, y) \leq G(p, y)$ for all $y \in C$, and hence $p \in GMEP\,(\Theta, A, \varphi)$. So, $p \in F(T) \cap GMEP\,(\Theta, A, \varphi) = \Omega$ and then we have (3.2.16). Therefore, by inequality (3.2.14) and Lamma 2.1.44, we obtain $\{x_n\}$ converges strongly to $P_\Omega(x_0)$. This completes the proof. □

Remark 3.2.6. It is interesting that the assumption on a sequence of scalars $\{\beta_n\}$ is very mild condition. This is a direct result of the firmly nonexpansiveness of $I - K_{r_n}$ together with the structure and the definition of the set C_n. If $\beta_n = 0$ for all n, then $z_n = x_n$ and the sequence $\{y_n\}$ and $\{u_n\}$ are independent. However, the properties of C_n still force to produce the sequence $\{x_n\}$ to cause a convergence to the common solution $P_\Omega(x_0)$.

If $A = 0$ and $\varphi = 0$, then we have the following corollary.

Corollary 3.2.7. *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ be an L-Lipschitz pseudo-contraction. Let Θ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$, such that $\Omega := F(T) \cap EP(\Theta) \neq \varnothing$. Let*

$x_0 \in H$. *For* $C_1 = C$ *and* $x_1 = P_{C_1}(x_0)$, *define a sequence* $\{x_n\}$ *of* C *as follows:*

$$\begin{cases} y_n &= (1-\alpha_n)x_n + \alpha_n T z_n, \\ z_n &= (1-\beta_n)x_n + \beta_n u_n, \\ u_n &\in C \text{ such that } \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ C_{n+1} &= \{v \in C_n : \|\alpha_n(I-T)y_n\|^2 + \|x_n - u_n\| \leq 2\alpha_n \langle x_n - v, (I-T)y_n\rangle \\ & \qquad + \sqrt{\langle x_n - v, x_n - u_n\rangle}\,(2\alpha_n\beta_n L\,\|y_n - x_n + \alpha_n(I-T)y_n\| + 1)\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0). \end{cases} \tag{3.2.17}$$

Assume that $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{r_n\}$ *are as in Theorem 3.2.5. Then* $\{x_n\}$ *converges strongly to* $P_\Omega(x_0)$.

Corollary 3.2.8. [83] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be an L-Lipschitz pseudo-contraction such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence such that $0 < a \leqslant \alpha_n \leqslant b < \frac{1}{L+1} < 1$ for all n. Then the sequence $\{x_n\}$ generated by (3.2.4) converges strongly to $P_{F(T)}(x_0)$.

Proof. Put $\Theta = 0$, $A = 0$, $\varphi = 0$ and $r_n = 1$ for all $n \geq 1$ in Theorem 3.2.5. Then, $K_{r_n} = P_C$ for all $n \geqslant 1$. So, $u_n = P_C x_n$ for all $n \geqslant 1$(Note that $x_1 = P_C x_0$). Since $x_n = P_{C_n} x_0 \in C_n \subset C$ for all $n \geqslant 1$, so we have $u_n = x_n$ and then $z_n = x_n$ for all $n \geqslant 1$. Thus $x_n - u_n = 0$ for all $n \geq 1$. For this reason, (3.2.4) is a special case of (3.2.6). Applying Theorem 3.2.5, we have the desired result. □

Recall that a mapping B is said to be *monotone*, if $\langle x - y, Bx - By\rangle \geqslant 0$ for all $x, y \in H$ and *inverse strongly monotone* if there exists a real number $\gamma > 0$ such that $\langle x - y, Bx - By\rangle \geqslant \gamma\|Bx - By\|^2$ for all $x, y \in H$. For the second case B is said to be γ-inverse strongly monotone. It follows immediately that if B is γ-inverse strongly monotone, then B is monotone and *Lipschitz continuous*, that is, $\|Bx - By\| \leqslant \frac{1}{\gamma}\|x - y\|$. The pseudo-contractive mapping and strictly pseudo-contractive mapping are strongly related to the monotone mapping and inverse strongly monotone mapping, respectively. It is well known that

(i) B is monotone $\Longleftrightarrow T := (I - B)$ is pseudo-contractive.

(ii) B is inverse strongly monotone $\Longleftrightarrow T := (I - B)$ is strictly pseudo-contractive.

Indeed, for (ii), we notice that the following equality always holds in a real Hilbert space

$$\|(I-B)x-(I-B)y\|^2 = \|x-y\|^2+\|Bx-By\|^2-2\langle x-y, Bx-By\rangle \quad \forall x, y \in H. \tag{3.2.18}$$

With out loss of generality we can assume that $\gamma \in (0, \frac{1}{2}]$ and then it yields

$$\langle x-y, Bx-By\rangle \geqslant \gamma\|Bx-By\|^2$$

$$\Longleftrightarrow -2\langle x-y, Bx-By\rangle \leqslant -2\gamma\|Bx-By\|^2$$

$$\Longleftrightarrow \|(I-B)x-(I-B)y\|^2 \leqslant \|x-y\|^2+(1-2\gamma)\|Bx-By\|^2$$

(via (3.2.18))

$$\Longleftrightarrow \|Tx-Ty\|^2 \leqslant \|x-y\|^2+\kappa\|(I-T)x-(I-T)y\|^2$$

(where $T := (I-B)$ and $\kappa := 1-2\gamma$).

Corollary 3.2.9. *Let C, H, Θ, A and φ be as in Theorem 3.2.5 and let $B : H \to H$ be an L-Lipschitz monotone mapping such that $\Omega = B^{-1}(0) \cap GMEP(\Theta, A, \varphi) \neq \varnothing$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C as* *follows:*

$$\begin{cases} y_n &= x_n - \alpha_n(x_n - z_n) - \alpha_n B z_n, \\ z_n &= (1-\beta_n)x_n + \beta_n u_n, \\ u_n &\in C \text{ such that } \Theta(u_n, y) + \langle Au_n, y - u_n\rangle + \varphi(y) - \varphi(u_n) \\ & \quad + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \\ C_{n+1} &= \{v \in C_n : \|\alpha_n B y_n\|^2 + \|x_n - u_n\| \leq 2\alpha_n \langle x_n - v, By_n\rangle \\ & \quad + \sqrt{\langle x_n - v, x_n - u_n\rangle}\,(2\alpha_n\beta_n L\|y_n - x_n + \alpha_n By_n\| + 1)\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0). \end{cases} \tag{3.2.19}$$

Assume $0 < a \leqslant \alpha_n \leqslant b < \frac{1}{L+2} < 1$ *for all* $n \in \mathbb{N}$, $\{\beta_n\}$ *and* $\{r_n\}$ *are as in Theorem 3.2.5. Then* $\{x_n\}$ *converges strongly to* $P_\Omega(x_0)$.

Proof. Let $T := (I - B)$. Then T is pseudo-contractive and $(L+2)$-Lipschitz. Hence, it follows from Theorem 3.2.6, we have the desired result. □

3.3 An iterative shrinking generalized f-projection method for G-quasi-strict pseudo-contractions in Banach spaces

Let E be a real Banach space with its dual E^*, and let C be a nonempty closed convex subset of E. In 1994, Alber [35] introduced the generalized projections $\pi_C : E^* \to C$ and $\Pi_C : E \to C$ from Hilbert spaces to uniformly convex and uniformly smooth Banach spaces and studied their properties in detail.

Let E be a smooth Banach space and let E^* be the dual of E. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 \tag{3.3.1}$$

for all $x, y \in E$, which was studied by Alber [36], Kamimura and Takahashi [61], and Reich [62], where J is the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \tag{3.3.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality paring. It is well known that if E is smooth, then J is single-valued and if E is strictly convex, then J is injective (one-to-one).

In 2005, Matsushita and Takahashi [71] applied (3.3.1) to define the mapping $T : C \to C$ called the relatively nonexpansive mapping where C is a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and they proposed the following projection algorithm based on the ideas of Nakajo and Takahashi [86] to find a fixed point of T:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}\left(\alpha_n J x_n + (1-\alpha_n) JT x_n\right), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ which satisfies some appropriate conditions and $\Pi_{C_n \cap Q_n}$ is the generalized projection from E onto $C_n \cap Q_n$.

In 2007, Takahashi et al. [87] studied a strong convergence theorem for a family of nonexpansive mappings in Hilbert spaces as follows: $x_0 \in H$, $C_1 = C$ and $x_1 = P_{C_1} x_0$, and let

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \; n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$ and $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. They proved that if $\{T_n\}$ satisfies some appropriate conditions, then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} F(T_n)} x_0$.

In 2010, Zhou and Gao [88] introduced the definition of a quasi-strict pseudo contraction related to the function ϕ and proposed a projection algorithm for finding a fixed point of a closed and quasi-strict pseudo contraction in more general framework than uniformly smooth and uniformly convex Banach spaces as follows:

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily,} \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ C_{n+1} = \left\{ z \in C_n \,\middle|\, \begin{array}{l} \phi(x_n, Tx_n) \\ \quad \leqslant \dfrac{2}{1-k} \langle x_n - z, Jx_n - JTx_n \rangle \end{array} \right\} \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{3.3.3}$$

where $k \in [0, 1)$ and $\Pi_{C_{n+1}}$ is the generalized projection from E onto C_{n+1}.

In 2012, K. Ungchittrakool [42] provided some examples of quasi-strict pseudo-contractions related to the function ϕ in framework of smooth and strictly convex Banach space. He obtained some strong convergence results in Banach spaces.

In 2013, Saewan et al. [43] introduced and studied the modified Mann type iterative algorithm for some mappings which related to asymptotically nonexpansive mappings by using hybrid generalized f-projection method. Saewan and Kumam [44] also provided and studied the new hybrid Ishikawa iteration process by the generalized f-projection operator for finding a common element of the fixed point set for two countable families of weak relatively nonexpansive mappings and the set of solutions of the system of generalized Ky Fan inequalities in a uniformly convex and uniformly smooth Banach space. Some relevant papers, please see [43–58] for more details.

Recently, Li et al. [74] studied the following hybrid iterative scheme for a relatively nonexpansive mapping by using the generalized f-projection operator in

Banach spaces as follows:

$$\begin{cases} x_0 \in C, \ C_0 = C, \\ y_n = J^{-1} \left(\alpha_n J x_n + (1 - \alpha_n) JT x_n \right), \\ C_{n+1} = \{ w \in C_n : G(w, Jy_n) \leq G(w, Jx_n) \}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \ n \geq 1. \end{cases}$$

Under some appropriate assumptions, they obtained strong convergence theorems in Banach spaces.

We introduce a mapping called G-quasi-strict pseudo-contractions (2.1.12) in the framework of smooth Banach spaces and also provide an inequality related to such a mappings. The inequality was taken to create an iterative shrinking projection method for finding fixed point problems of closed and G-quasi-strict pseudo-contractions. Its results hold in reflexive, strictly convex and smooth Banach spaces with the property (K).

Lemma 3.3.1. *Let C be a nonempty closed convex subset of a smooth Banach space E and $T : C \to C$ be a G-quasi-strict pseudo-contraction. Then the fixed point set $F(T)$ of T is closed and convex.*

Proof. Firstly, we wish to show that $F(T)$ is closed. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n \to p \in C$ as $n \to \infty$. From the definition of T, we have

$$G(p_n, JTp) \leq G(p_n, Jp) + \kappa(G(p, JTp) - 2\rho f(p_n)).$$

By using (2.1.8), we obtain

$$\begin{aligned} G(p_n, Jp) + G(p, JTp) + 2\langle p_n - p, Jp - JTp\rangle - 2\rho f(p) \\ \leq G(p_n, Jp) + \kappa(G(p, JTp) - 2\rho f(p_n)). \end{aligned}$$

By simple calculation, we have

$$(1 - \kappa)G(p, JTp) \leq 2\langle p - p_n, Jp - JTp\rangle + 2\rho f(p) - 2\kappa\rho f(p_n).$$

Next, it becomes

$$(1-\kappa)\phi(p,Tp)+(1-\kappa)2\rho f(p)\leq 2\langle p-p_n, Jp-JTp\rangle+2\rho f(p)-2\kappa\rho f(p_n).$$

And hence

$$\phi(p,Tp)\leq \frac{2}{1-\kappa}\langle p-p_n, Jp-JTp\rangle+\frac{2\kappa\rho}{1-\kappa}(f(p)-f(p_n)). \tag{3.3.4}$$

Take $\limsup_{n\to\infty}$ on the both sides of (3.3.4), so we have

$$\begin{aligned}\phi(p,Tp) &= \limsup_{n\to\infty}\phi(p,Tp)\\ &= \limsup_{n\to\infty}\left(\frac{2}{1-\kappa}\langle p-p_n, Jp-JTp\rangle+\frac{2\kappa\rho}{1-\kappa}(f(p)-f(p_n))\right)\\ &\leq \frac{2}{1-\kappa}\limsup_{n\to\infty}\langle p-p_n, Jp-JTp\rangle+\frac{2\kappa\rho}{1-\kappa}\limsup_{n\to\infty}(f(p)-f(p_n))\\ &\leq \frac{2\kappa\rho}{1-\kappa}\left(\limsup_{n\to\infty}f(p)+\limsup_{n\to\infty}(-f(p_n))\right)\\ &= \frac{2\kappa\rho}{1-\kappa}\left(f(p)-\liminf_{n\to\infty}f(p_n)\right)\leq 0.\end{aligned}$$

This means that $p=Tp$.

We next show that $F(T)$ is convex. For arbitrary $p_1,p_2\in F(T)$ and $t\in(0,1)$, we let $p_t=tp_1+(1-t)p_2$. By the definition of T, we have

$$G(p_1,JTp_t)\leq G(p_1,Jp_t)+\kappa(G(p_t,JTp_t)-2\rho f(p_1)) \tag{3.3.5}$$

and

$$G(p_2,JTp_t)\leq G(p_2,Jp_t)+\kappa(G(p_t,JTp_t)-2\rho f(p_2)). \tag{3.3.6}$$

By (2.1.8) it is easy to see that (3.3.5) and (3.3.6) are equivalent to

$$\phi(p_t,Tp_t)\leq \frac{2}{1-\kappa}\langle p_t-p_1, Jp_t-JTp_t\rangle+\frac{2\kappa\rho}{1-\kappa}(f(p_t)-f(p_1)) \tag{3.3.7}$$

and

$$\phi(p_t,Tp_t)\leq \frac{2}{1-\kappa}\langle p_t-p_2, Jp_t-JTp_t\rangle+\frac{2\kappa\rho}{1-\kappa}(f(p_t)-f(p_2)), \tag{3.3.8}$$

respectively. Multiply into both sides of (3.3.7) and (3.3.8) with t and $(1-t)$, respectively. And then adding two equations together with the property of convexity of f, we have

$$\phi(p_t, Tp_t) \leq \frac{2}{1-\kappa}\langle p_t - p_t, Jp_t - JTp_t\rangle + \frac{2\kappa\rho}{1-\kappa}(f(p_t) - tf(p_1) - (1-t)f(p_2)) \leq 0.$$

Hence $Tp_t = p_t$. This completes the proof. □

Theorem 3.3.2. *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E, $T : C \to C$ is closed and G-quasi-strict pseudo-contraction and $f : E \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous mapping. Define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} x_0 \in C, \\ C_1 = C, \\ x_1 = \Pi^f_{C_1}(x_0), \\ C_{n+1} = \left\{ z \in C_n \;\middle|\; \phi(x_n, Tx_n) \leq \frac{2}{1-\kappa}\langle x_n - z, Jx_n - JTx_n\rangle + \frac{2\kappa\rho}{1-\kappa}(f(x_n) - f(z)) \right\}, \\ x_{n+1} = \Pi^f_{C_{n+1}}(x_0), \qquad n \geq 0, \end{cases} \tag{3.3.9}$$

where $\kappa \in [0, 1)$. Then $\{x_n\}$ converges strongly to $\Pi^f_{F(T)}(x_0)$.

Proof. We split the proof into seven steps.

Step 1. Show that $F(T)$ is closed and convex.

Since T is a G-quasi-strict pseudo-contraction, $F(T) \neq \emptyset$. It follows from Lemma 3.3.1 that $F(T)$ is closed and convex. Therefore, $\Pi^f_{F(T)}(x_0)$ is well defined for every $x_0 \in E$.

Step 2. Show that C_n is closed and convex for all $n \geq 1$.

For $k = 1, C_1 = C$ is closed and convex. Assume that C_k is closed and convex for some $k \in \mathbb{N}$. For $z \in C_{k+1}$, we have that

$$\phi(x_k, Tx_k) \leq \frac{2}{1-\kappa}\langle x_k - z, Jx_k - JTx_k\rangle + \frac{2\kappa\rho}{1-\kappa}(f(x_k) - f(z)).$$

Define $g_k(\cdot) := \frac{1}{1-\kappa}2\langle x_k - (\cdot), Jx_k - JTx_k\rangle + \frac{2\kappa\rho}{1-\kappa}(f(x_k) - f(\cdot))$. It is not hard to see that the linearity of $\langle x_k - (\cdot), Jx_k - JTx_k\rangle$ together with the upper semicontinuity and concavity of $-f(\cdot)$ allow g_k to be upper semicontinuous and concave. By applying Lemma 2.2.15, C_{k+1} is closed and convex. By mathematical induction, we obtain that C_n is convex for all $n \in \mathbb{N}$.

Step 3. Show that $F(T) \subset C_n$ for all $n \geq 1$.

It is obvious that $F(T) \subset C = C_1$. Suppose that $F(T) \subset C_k$ for some $k \in \mathbb{N}$. For any $p' \in F(T)$, one has $p' \in C_k$. By using the definition of T, we have

$$G(p', JTx_k) \leq G(p', Jx_k) + \kappa(G(x_k, JTx_k) - 2\rho f(p')).$$

Using (2.1.8) and by a simple calculation, we obtain

$$\phi(x_k, Tx_k) \leq \frac{2}{1-\kappa}\langle x_k - p', Jx_k - JTx_k\rangle + \frac{2\kappa\rho}{1-\kappa}(f(x_k) - f(p')),$$

which implies that $p' \in C_{k+1}$. This implies that $F(T) \subset C_n$ for all $n \geq 1$. Therefore, $F(T) \subset \bigcap_{n=1}^{\infty} C_n \neq \emptyset := D$.

Step 4. Show that $\{x_n\}$ is bounded and the limit of $G(x_n, Jx_0)$ exists.

By the properties of f together with Lemma 2.1.26, we see that there exists $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, x^*\rangle + \alpha, \quad \forall y \in E.$$

It follows that

$$\begin{aligned}G(x_n, Jx_0) &= \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_n)\\ &\geq \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2\\ &\quad + 2\rho\langle x_n, x^*\rangle + 2\rho\alpha\\ &= \|x_n\|^2 - 2\langle x_n, Jx_0 - \rho x^*\rangle + \|x_0\|^2 + 2\rho\alpha\\ &\geq \|x_n\|^2 - 2\|Jx_0 - \rho x^*\|\|x_n\| + \|x_0\|^2 + 2\rho\alpha\\ &= (\|x_n\| - \|Jx_0 - \rho x^*\|)^2\\ &\quad + \|x_0\|^2 - \|Jx_0 - \rho x^*\|^2 + 2\rho\alpha. \end{aligned} \tag{3.3.10}$$

Since $x_n = \Pi^f_{C_n}(x_0)$, it follows from (3.3.10) that

$$\begin{aligned}G(u, Jx_0) &\geq G(x_n, Jx_0)\\ &\geq (\|x_n\| - \|Jx_0 - \rho x^*\|)^2\\ &\quad + \|x_0\|^2 - \|Jx_0 - \rho x^*\|^2 + 2\rho\alpha\end{aligned}$$

for each $u \in F(T)$. This implies that $\{x_n\}$ is bounded and so is $\{G(x_n, Jx_0)\}$. By the fact that $x_{n+1} \in C_{n+1} \subset C_n$ and (2.1.11) of Lemma 2.1.29, we obtain

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0).$$

Since $\phi(x_{n+1}, x_n) \geq 0$, $\{G(x_n, Jx_0)\}$ is nondecreasing. Therefore, the limit of $\{G(x_n, Jx_0)\}$ exists.

Step 5. Show that $x_n \to p$ as $n \to \infty$, where $p = \Pi^f_D x_0$.

Let $\{x_{n_k}\} \subset \{x_n\}$. From the boundedness of $\{x_{n_k}\}$ there exists $\{x_{n_{k_j}}\} \subset \{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup p$. Write $\tilde{x}_j := x_{n_{k_j}}$, it is easy to see that $p \in \tilde{C}_j$ where $\tilde{C}_j := C_{n_{k_j}}$. Note that

$$G(\tilde{x}_j, Jx_0) = \inf_{\xi \in \tilde{C}_j} G(\xi, Jx_0) \leq G(p, Jx_0). \tag{3.3.11}$$

On the other hand, since $\tilde{x}_j \rightharpoonup p$, the weakly lower semicontinuity of $\|\cdot\|^2$ and f yields

$$\phi(p, x_0) \leq \liminf_{j\to\infty} \phi(\tilde{x}_j, x_0), \tag{3.3.12}$$

and

$$f(p) \leq \liminf_{j\to\infty} f(\tilde{x}_j). \tag{3.3.13}$$

By (3.3.12) and (3.3.13), we obtain

$$\begin{aligned} G(p, Jx_0) &= \phi(p, x_0) + 2\rho f(p) \\ &\leq \liminf_{j\to\infty} \phi(\tilde{x}_j, x_0) + 2\rho \liminf_{j\to\infty} f(\tilde{x}_j) \\ &\leq \liminf_{j\to\infty} (\phi(\tilde{x}_j, x_0) + 2\rho f(\tilde{x}_j)) \\ &= \liminf_{j\to\infty} G(\tilde{x}_j, Jx_0). \end{aligned} \tag{3.3.14}$$

By connecting (3.3.11) and (3.3.14), we have

$$\begin{aligned} G(p, Jx_0) &\leq \liminf_{j\to\infty} G(\tilde{x}_j, Jx_0) \leq \limsup_{j\to\infty} G(\tilde{x}_j, Jx_0) \\ &\leq G(p, Jx_0), \end{aligned}$$

and then

$$\lim_{j\to\infty} G(\tilde{x}_j, Jx_0) = G(p, Jx_0).$$

Next, we consider

$$\begin{aligned} \limsup_{j\to\infty} \phi(\tilde{x}_j, x_0) &= \limsup_{j\to\infty} (G(\tilde{x}_j, Jx_0) - 2\rho f(\tilde{x}_j)) \\ &\leq G(p, Jx_0) - 2\rho \liminf_{j\to\infty} f(\tilde{x}_j) \\ &\leq G(p, Jx_0) - 2\rho f(p) = \phi(p, x_0). \end{aligned} \tag{3.3.15}$$

Combine (3.3.12) and (3.3.15), we obtain

$$\phi(p, x_0) \leq \liminf_{j\to\infty} \phi(\tilde{x}_j, x_0) \leq \limsup_{j\to\infty} \phi(\tilde{x}_j, x_0) \leq \phi(p, x_0),$$

and then

$$\lim_{j\to\infty} \phi(\tilde{x_j}, x_0) = \phi(p, x_0).$$

Note that $f(\tilde{x_j}) = \frac{1}{2\rho}(G(\tilde{x_j}, Jx_0) - \phi(\tilde{x_j}, x_0))$. Then, we have

$$\begin{aligned}\lim_{j\to\infty} f(\widetilde{x}_j) &= \frac{1}{2\rho}\lim_{j\to\infty} (G(\widetilde{x}_j, Jx_0) - \phi(\widetilde{x}_j, x_0)) \\ &= \frac{1}{2\rho}(G(p, Jx_0) - \phi(p, x_0)) \\ &= \frac{1}{2\rho}(2\rho f(p)) = f(p).\end{aligned}$$

The virtue of Lemma 2.1.25 implies that

$$\lim_{n\to\infty} f(x_n) = f(p).$$

Notice that $\widetilde{x}_j = \Pi^f_{\widetilde{C}_j} x_0$, by using Lemma 2.1.29 we obtain

$$\phi(p, \widetilde{x}_j) \leq G(p, Jx_0) - G(\widetilde{x}_j, Jx_0). \tag{3.3.16}$$

Taking $j \to \infty$ in (3.3.16), we obtain

$$\lim_{j\to\infty} \phi(p, \tilde{x_j}) = 0.$$

By virtue of Lemma 2.1.27, it follows that $\tilde{x_j} \to p$ as $j \to \infty$. This implies by Lemma 2.1.25 that $x_n \to p$ as $n \to \infty$. It follows from $x_n = \Pi^f_{C_n} x_0$ and (2.1.10) of Lemma 2.1.28 that

$$\langle x_n - y, Jx_0 - Jx_n\rangle + \rho f(y) - \rho f(x_n) \geq 0, \quad \forall y \in C_n.$$

In particular, because we know that $D = \bigcap_{n=1}^{\infty} C_n \subset C_n$ for all $n \geq 0$ so we have

$$\langle x_n - y, Jx_0 - Jx_n\rangle + \rho f(y) - \rho f(x_n) \geq 0, \quad \forall y \in D. \tag{3.3.17}$$

Taking $n \to \infty$ on (3.3.17) to get

$$\langle p - y, Jx_0 - Jp\rangle + \rho f(y) - \rho f(p) \geq 0, \quad \forall y \in D. \tag{3.3.18}$$

By applying (2.1.10) of Lemma 2.1.28 to (3.3.18) we obtain $p = \Pi^f_D x_0$.

Step 6. Show that $p \in F(T)$.

Firstly, we wish to prove that $\{Tx_n\}$ is bounded. Indeed, take $q \in F(T) \subset C_{n+1}$, we have

$$\phi(x_n, Tx_n) \leq \frac{2}{1-\kappa}\langle x_n - q, Jx_n - JTx_n\rangle + \frac{2\kappa\rho}{1-\kappa}(f(x_n) - f(q)).$$

i.e.,

$$\|x_n\|^2 - 2\langle x_n, JTx_n\rangle + \|Tx_n\|^2 \leq \frac{2}{1-\kappa}\|x_n - q\|(\|x_n\| + \|Tx_n\|) + \frac{2\kappa\rho}{1-\kappa}(f(x_n) - f(q)).$$

It follows that

$$\begin{aligned} \|Tx_n\|^2 \quad \leq \quad & \frac{2}{1-\kappa}\|x_n - q\|\|x_n\| - \|x_n\|^2 + \left(\frac{2}{1-\kappa}\|x_n - q\| + 2\|x_n\|\right)\|Tx_n\| \\ & + \frac{2\kappa\rho}{1-\kappa}(f(x_n) - f(q)). \end{aligned}$$

Since $\{\|x_n\|\}$ and $\{f(x_n)\}$ are bounded, we obtain that $\{\|Tx_n\|\}$ is bounded. From $x_{n+1} \in C_{n+1}$, one has

$$\phi(x_n, Tx_n) \leq \frac{1}{1-\kappa}2\langle x_n - x_{n+1}, Jx_n - JTx_n\rangle + \frac{\kappa}{1-\kappa}2\rho(f(x_n) - f(x_{n+1})). \quad (3.3.19)$$

By step 5, we obtain that $x_{n+1} - x_n \to 0$ and $\lim_{n\to\infty} f(x_n) = f(p)$. Taking limit on the both sides of (3.3.19), we obtain that $\phi(x_n, Tx_n) \to 0$ as $n \to \infty$. Noting that $0 \leq (\|x_n\| - \|Tx_n\|)^2 \leq \phi(x_n, Tx_n)$. Hence $\|Tx_n\| \to \|p\|$ and consequently $\|J(Tx_n)\| \to \|Jp\|$. This implies that $\{\|J(Tx_n)\|\}$ is bounded. Since E is reflexive, E^* is also reflexive. So we can assume that

$$J(Tx_n) \rightharpoonup f_0 \in E^*.$$

On the other hand, in view of the reflexivity of E, one has $J(E) = E^*$, which means that for $f_0 \in E^*$, there exists $x \in E$, such that $Jx = f_0$. It follows that

$$\begin{aligned} \phi(x_n, Tx_n) &= \|x_n\|^2 - 2\langle x_n, JTx_n\rangle + \|Tx_n\|^2 \\ &= \|x_n\|^2 - 2\langle x_n, JTx_n\rangle + \|J(Tx_n)\|^2, \end{aligned}$$

taking $\liminf_{n\to\infty}$ on the both sides of equality above, we have

$$\begin{aligned} 0 &\geq \|p\|^2 - 2\langle p, f_0\rangle + \|f_0\|^2 \\ &= \|p\|^2 - 2\langle p, Jx\rangle + \|Jx\|^2 \\ &= \phi(p, x). \end{aligned}$$

We have $\phi(p, x) = 0$ and consequently $p = x$, which implies that $f_0 = Jp$. Hence

$$J(Tx_n) \rightharpoonup Jp \in E^*.$$

Since $\|J(Tx_n)\| \to \|Jp\|$ and E^* has the property (K), we have

$$\|J(Tx_n) - Jp\| \to 0.$$

Noting that $J^{-1} : E^* \to E$ is demi-continuous, we have

$$Tx_n \rightharpoonup p \in E.$$

Since $\|Tx_n\| \to \|p\|$ and E has the property (K), we obtain that $Tx_n \to p$ as $n \to \infty$. From $x_n \to p$ and the closeness property of T, we have $p \in F(T)$.

Step 7. Show that $p = \Pi^f_{F(T)}x_0$.

It follows from steps 5 and 6 that

$$\begin{aligned} G(p, x_0) = G\left(\Pi^f_D x_0, x_0\right) &= \inf_{\xi\in D} G(\xi, x_0) \\ &\leq G\left(\Pi^f_{F(T)}x_0, x_0\right) \\ &\leq G(p, x_0), \end{aligned}$$

which implies that $G\left(\Pi^f_{F(T)}x_0, x_0\right) = G(p, x_0)$. It follows from the uniqueness, we can conclude that $p = \Pi^f_{F(T)}x_0$. This completes the proof. □

If $f(x) = \|x\|^2$ for all $x \in E$, then $G(\xi, Jx) = \phi(\xi, x) + 2\rho\|\xi\|^2$ and $\Pi^f_C x = \Pi^{\|\cdot\|^2}_C x$. By Theorem 3.3.2, we obtain the following corollary.

Corollary 3.3.3. *Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E, $T : C \to C$ is closed and G-quasi-strict pseudo-contraction(where $f(\cdot) = \|\cdot\|^2$). Define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} x_0 \in C, \ C_1 = C, \\ x_1 = \Pi_{C_1}^f(x_0), \\ C_{n+1} = \left\{ z \in C_n \middle| \begin{array}{l} \phi(x_n, Tx_n) \\ \quad \leq \dfrac{2}{1-\kappa}\langle x_n - z, Jx_n - JTx_n\rangle \\ \quad\quad + \dfrac{2\kappa\rho}{1-\kappa}(\|x_n\|^2 - \|z\|^2) \end{array} \right\}, \\ x_{n+1} = \Pi_{C_{n+1}}^{\|\cdot\|^2}(x_0), \qquad n \geq 0, \end{cases} \tag{3.3.20}$$

where $\kappa \in [0,1)$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)}^{\|\cdot\|^2}(x_0)$.

If $f(x) = 0$ for all $x \in E$, then $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi_C^f x = \Pi_C x$. By Theorem 3.3.2, we obtain the following corollary.

Corollary 3.3.4. [88] Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E. Let $T : C \to C$ be a closed and quasi-strict pseudo-contraction. Define a sequence $\{x_n\}$ as in (3.3.3). Then $\{x_n\}$ converges strongly to $p_0 = \Pi_{F(T)}x_0$.

CHAPTER IV

EXISTENCE RESULT FOR VECTOR EQUILIBRIUM PROBLEMS

4.1 Existence results for new weak and strong mixed vector equilibrium problems on non-compact domain

Let X be a Hausdorff topological vector space, K be a subset of X, and $f : K \times K \to \mathbb{R}$ be a mapping with $f(x,x) = 0$. The classical, scalar-valued equilibrium problem deals with the existence of $\tilde{x} \in K$ such that

$$f(\tilde{x}, y) \geq 0; \quad \forall y \in K.$$

Moreover, in the case of vector valued mappings, let Y be a another Hausdorff topological vector space, $C \subset Y$ a convex cone with nonempty interior. Given a vector mapping $f : K \times K \to Y$, then the problem of finding $\tilde{x} \in K$ such that

$$f(\tilde{x}, y) \notin -intC; \quad \forall y \in K,$$

is called weak equilibrium problem and the point $\tilde{x} \in K$ is called weak equilibrium point, where $intC$ denotes the interior of the cone C in Y. In 2014, Rahaman and Ahmad [59] considered two types of mixed vector equilibrium problems which were combinations of a vector equilibrium problem and a vector variational inequality problem. Remark that $C \subset Y$ is a pointed closed convex cone with nonempty interior i.e., $intC \neq \emptyset$. The partial ordering induced by C on Y is denoted by $\leq_C$ and is defined by $x \leq_C y$ if and only if $y - x \in C$. Let $f : K \times K \to Y$ and $T : X \to L(X,Y)$ be two mappings, where $L(X,Y)$ is the space of all linear continuous mappings from X to Y. Here $\langle T(x), y\rangle$ denotes the evaluation of the linear mapping $T(x)$ at y. They considered the following two problems:

Find $\tilde{x} \in K$ such that

$$f(\tilde{x}, y) + \langle T(\tilde{x}), y - \tilde{x}\rangle \notin -intC; \quad \forall y \in K, \tag{4.1.1}$$

and

$$f(\tilde{x}, y) + \langle T(\tilde{x}), y - \tilde{x}\rangle \notin -C \setminus \{0\}; \quad \forall y \in K. \tag{4.1.2}$$

It is clear that the solution set of (4.1.2) is a subset of the solution set of (4.1.1). Also if we consider $Y = \mathbb{R}^2$ and $C = \{(x, x) : x \in \mathbb{R}\}$ then $intC = \emptyset$ and the solution set of (4.1.1) is always the whole set K. They called problem (4.1.1) as weak mixed vector equilibrium problem and problem (4.1.2) as strong mixed vector equilibrium problem. Problems (4.1.1) and (4.1.2) are unified models of several known problems used in applied sciences, for instance, vector variational inequality problem, vector complementarity problem, vector optimization problem and vector saddle point problem, see e.g. [80, 89–93] and references therein.

With the inspiration from the notice of some characteristics of the mappings of the original problem, we are interested and motivated in the development of the existing problems to the new weak mixed vector equilibrium problem and the new strong mixed vector equilibrium as follows:

Find $\tilde{x} \in K$ such that

$$f(\tilde{x}, y) + \langle T(\tilde{x}), y - \tilde{x}\rangle + \tau(\tilde{x}, y) + \langle y - \tilde{x}, D\tilde{x} - Dz\rangle \notin -intC; \quad \forall y \in K, \tag{4.1.3}$$

and

$$f(\tilde{x}, y) + \langle T(\tilde{x}), y - \tilde{x}\rangle + \tau(\tilde{x}, y) + \langle y - \tilde{x}, D\tilde{x} - Dz\rangle \notin -C \setminus \{0\}; \quad \forall y \in K, \tag{4.1.4}$$

where $\tau : K \times K \to Y$ is a bifunction, $D : X \to L(X, Y)$, and $z \in X$.

We prove the following existence results for new weak and strong mixed vector equilibrium problems (4.1.3) and (4.1.4) for non-compact domains.

Theorem 4.1.1. *Let K be a nonempty closed convex subset of a Hausdorff topological vector space X, Y a Hausdorff topological vector space and C a closed convex pointed cone in Y with $intC \neq \emptyset$. Let $f : K \times K \to Y$, $\tau : K \times K \to Y$, $T : K \to L(X,Y)$ and $D : X \to L(X,Y)$ be four mappings satisfy the following conditions:*

(i) f *and* τ *are* C*-monotone;*

(ii) $f(x,x) = 0$, *and* $\tau(x,x) = 0$ *for all* $x \in K$*;*

(iii) *for any fixed* $x, y \in K$*;* $t \in [0,1] \mapsto f(ty + (1-t)x, y) \in Y$ *and* $t \in [0,1] \mapsto \tau(ty + (1-t)x, y) \in Y$ *are upper semicontinuous with respect to* C *at* $t = 0$*;*

(iv) *for any fixed* $x \in K$, $f(x,\cdot), \tau(x,\cdot) : K \to Y$ *are* C*-convex, lower semicontinuous with respect to* C *on* K*;*

(v) D *and* T *are upper semicontinuous with respect to* C *with nonempty closed values;*

(vi) *there exists a family* $\{C_i, Z_i\}_{i \in I}$ *satisfying conditions* (i) *and* (ii) *of Definition 2.2.12 and the following condition: For each* $i \in I$*, there exists* $k \in I$ *such that*

$$\{x \in K : f(y,x) - \langle Tx, y-x \rangle + \tau(y,x) - \langle y-x, Dx - Dz \rangle \notin intC,\ \forall y \in C_k\} \subset Z_i.$$

Then, there exists a point $\tilde{x} \in K$ *such that*

$$f(\tilde{x}, y) + \langle T(\tilde{x}), y - \tilde{x} \rangle + \tau(\tilde{x}, y) + \langle y - x, Dx - Dz \rangle \notin -intC; \quad \forall y \in K.$$

For the proof of the Theorem 4.1.1, we need the following proposition, for which the assumptions remain same as in Theorem 4.1.1.

Proposition 4.1.2. *The following two problems are equivalent:*

(i) *Find* $\tilde{x} \in K$ *such that* $f(y,\tilde{x}) - \langle T(\tilde{x}), y-\tilde{x}\rangle + \tau(y,\tilde{x}) - \langle y-\tilde{x}, D\tilde{x}-Dz\rangle \notin intC$; $\forall y \in K$;

(ii) *Find* $\tilde{x} \in K$ *such that* $f(\tilde{x},y) + \langle T(\tilde{x}), y-\tilde{x}\rangle + \tau(\tilde{x},y) + \langle y-\tilde{x}, D\tilde{x}-Dz\rangle \notin -intC$; $\forall y \in K$.

Proof. Suppose (i) holds. Then for fixed $y \in K$, set $x_t = ty + (1-t)\tilde{x}$, for $t \in [0,1]$. It is clear that $x_t \in K$, for all $t \in [0,1]$ and hence

$$f(x_t,\tilde{x}) - \langle T(\tilde{x}), x_t - \tilde{x}\rangle + \tau(x_t,\tilde{x}) - \langle x_t - \tilde{x}, D\tilde{x} - Dz\rangle \notin intC. \tag{4.1.5}$$

Since $f(x,x) = 0$ and $f(x,\cdot)$ is $C-$convex, we have

$$\begin{aligned} &0 = f(x_t,x_t) \leq_C tf(x_t,y) + (1-t)f(x_t,\tilde{x}) \\ &\Rightarrow tf(x_t,y) + (1-t)f(x_t,\tilde{x}) \in C. \end{aligned} \tag{4.1.6}$$

On the other hand, the convexity of τ in the second variable implies that

$$\begin{aligned} &0 = \tau(x_t,x_t) \leq_C t\tau(x_t,y) + (1-t)\tau(x_t,\tilde{x}) \\ &\Rightarrow t\tau(x_t,y) + (1-t)\tau(x_t,\tilde{x}) \in C. \end{aligned} \tag{4.1.7}$$

Also,

$$\begin{aligned} &\langle T(\tilde{x}), x_t - \tilde{x}\rangle = t\langle T(\tilde{x}), y - \tilde{x}\rangle \\ &\Rightarrow (1-t)t\langle T(\tilde{x}), y-\tilde{x}\rangle - (1-t)\langle T(\tilde{x}), x_t - \tilde{x}\rangle = 0. \end{aligned} \tag{4.1.8}$$

And

$$\begin{aligned} &\langle x_t - \tilde{x}, D\tilde{x} - Dz\rangle = t\langle y - \tilde{x}, D\tilde{x} - Dz\rangle \\ &\Rightarrow (1-t)t\langle y-\tilde{x}, D\tilde{x} - Dz\rangle - (1-t)\langle x_t - \tilde{x}, D\tilde{x} - Dz\rangle = 0 \end{aligned} \tag{4.1.9}$$

Combining (4.1.6), (4.1.7), (4.1.8) and (4.1.9), we obtain

$$\begin{aligned} &t(f(x_t,y) + \tau(x_t,y)) + (1-t)\{f(x_t,\tilde{x}) + \tau(x_t,\tilde{x}) - \langle T(\tilde{x}), x_t - \tilde{x}\rangle \\ &- \langle x_t - \tilde{x}, D\tilde{x} - Dz\rangle\} + (1-t)t\{\langle T(\tilde{x}), y-\tilde{x}\rangle + \langle y - \tilde{x}, D\tilde{x} - Dz\rangle\} \in C, \end{aligned} \tag{4.1.10}$$

for all $t \in [0,1]$. It is not hard to see that (4.1.10) equivalent to

$$\begin{aligned}(1-t)\{f(x_t,\tilde{x}) + \tau(x_t,\tilde{x}) - \langle T(\tilde{x}), x_t - \tilde{x}\rangle - \langle x_t - \tilde{x}, D\tilde{x} - Dz\rangle\} \qquad (4.1.11)\\ - (-t(f(x_t,y) + \tau(x_t,y)) - (1-t)t\{\langle T(\tilde{x}), y - \tilde{x}\rangle\\ + \langle y - \tilde{x}, D\tilde{x} - Dz\rangle\}) \in C.\end{aligned}$$

By using (4.1.5) and (4.1.11) and (ii) of Lemma 2.2.11, we have

$$\begin{aligned}&t(f(x_t,y) + \tau(x_t,y)) + (1-t)t\{\langle T(\tilde{x}), y-\tilde{x}\rangle + \langle y-\tilde{x}, D\tilde{x}-Dz\rangle\} \notin -intC\\ &\Rightarrow f(x_t,y) + \tau(x_t,y) + (1-t)\langle T(\tilde{x}), y-\tilde{x}\rangle + (1-t)\langle y-\tilde{x}, D\tilde{x}-Dz\rangle\\ &\notin -intC,\ \forall t \in (0,1]. \qquad (4.1.12)\end{aligned}$$

By condition (iii) of Theorem 4.1.1 as $t \mapsto f(ty+(1-t)x, y)$ and $t \mapsto \tau(ty+(1-t)x, y)$ are upper semicontinuous with respect to C at $t=0$, therefore from (4.1.12) we have

$$f(\tilde{x},y) + \langle T(\tilde{x}), y-\tilde{x}\rangle + \tau(\tilde{x},y) + \langle y-\tilde{x}, D\tilde{x}-Dz\rangle \notin -intC,$$

and hence (ii) holds.

Conversely, we assume that (ii) holds. In order to prove (i), on contrary suppose that there exists a point $\tilde{y} \in K$ such that

$$\begin{aligned}&f(\tilde{y},\tilde{x}) - \langle T(\tilde{x}), \tilde{y}-\tilde{x}\rangle + \tau(\tilde{y},\tilde{x}) - \langle \tilde{y}-\tilde{x}, D\tilde{x}-Dz\rangle \in intC\\ &\Rightarrow f(\tilde{y},\tilde{x}) + \tau(\tilde{y},\tilde{x}) - \langle \tilde{y}-\tilde{x}, D\tilde{x}-Dz\rangle = \langle T(\tilde{x}), \tilde{y}-\tilde{x}\rangle + w; \qquad (4.1.13)\end{aligned}$$

for some $w \in intC$.

On the other hand, since f and τ are C-monotone, we have

$$f(\tilde{x},\tilde{y}) + f(\tilde{y},\tilde{x}) \in -C \Rightarrow f(\tilde{y},\tilde{x}) = -f(\tilde{x},\tilde{y}) - v; \qquad (4.1.14)$$

for some $v \in C$ and

$$\tau(\tilde{x},\tilde{y}) + \tau(\tilde{y},\tilde{x}) \in -C \Rightarrow \tau(\tilde{y},\tilde{x}) = -\tau(\tilde{x},\tilde{y}) - u; \qquad (4.1.15)$$

for some $u \in C$. Combining (4.1.13), (4.1.14) and (4.1.15), we have

$$f(\tilde{x},\tilde{y}) + \langle T(\tilde{x}), \tilde{y} - \tilde{x}\rangle + \tau(\tilde{x},\tilde{y}) + \langle \tilde{y} - \tilde{x}, D\tilde{x} - Dz\rangle = -w - v - u \in -intC;$$

which contradicts assumption (ii). Therefore (i) holds. □

Now, we are able to prove Theorem 4.1.1 which has the following details:

Proof. For each $y \in K$, consider the set

$$F(y) = \{x \in K : f(y,x) - \langle T(x), y - x\rangle + \tau(y,x) \\ -\langle y - x, Dx - Dz\rangle \notin intC\}.$$

By Lemma 2.2.10, $F(y)$ is closed in K and hence F has compactly closed values in K. Now, we show that F is a KKM map. For this, let $\{y_i : i \in I\}$ be a finite subset of K and $u \in Co\{y_i : i \in I\}$. We claim that

$$Co\{y_i : i \in I\} \subseteq \bigcup_{i\in I} F(y_i).$$

In contrary, suppose that $u \notin \bigcup_{i\in I} F(y_i)$. As $u \in Co\{y_i : i \in I\}$, we have $u = \sum_{i\in I}\lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i\in I}\lambda_i = 1$. This follows that

$$f(y_i,u) - \langle T(u), y_i - u\rangle + \tau(y_i,u) - \langle y_i - x, Dx - Dz\rangle \in intC.$$

Since $intC$ is convex, therefore

$$\sum_{i\in I}\lambda_i\{f(y_i,u) - \langle T(u), y_i - u\rangle + \tau(y_i,u) - \langle y_i - x, Dx - Dz\rangle\} \in intC$$

Since $f(x,\cdot)$ is C-convex and C-monotone, we have

$$\sum_{i\in I}\lambda_i f(y_i,u) \leq_C \sum_{i,j\in I}\lambda_i\lambda_j f(y_i,y_j) \\ = \frac{1}{2}\sum_{i,j\in I}\lambda_i\lambda_j\{f(y_i,y_j) + f(y_j,y_i)\} \leq_C 0. \tag{4.1.16}$$

On the other hand, the convexity of τ in the second variable and C-monotonicity, implies that

$$\sum_{i\in I}\lambda_i\tau(y_i,u) \leq_C \sum_{i,j\in I}\lambda_i\lambda_j\tau(y_i,y_j)$$

$$= \frac{1}{2}\sum_{i,j\in I}\lambda_i\lambda_j\{\tau(y_i,y_j)+\tau(y_j,y_i)\} \leq_C 0. \quad (4.1.17)$$

Furthermore,

$$\begin{aligned} 0 &= \langle T(u), u-u\rangle = \langle T(u), \sum_{i\in I}\lambda_i y_i - \sum_{i\in I}\lambda_i u\rangle \\ &= \langle T(u), \sum_{i\in I}\lambda_i (y_i-u)\rangle = \sum_{i\in I}\lambda_i\langle T(u),(y_i-u)\rangle. \end{aligned} \quad (4.1.18)$$

And

$$\begin{aligned} 0 &= \langle u-u, Dx-Dz\rangle = \langle \sum_{i\in I}\lambda_i y_i - \sum_{i\in I}\lambda_i u, Dx-Dz\rangle \\ &= \langle \sum_{i\in I}\lambda_i (y_i-u), Dx-Dz\rangle = \sum_{i\in I}\lambda_i\langle (y_i-u), Dx-Dz\rangle. \end{aligned} \quad (4.1.19)$$

Combining (4.1.16), (4.1.17), (4.1.18) and (4.1.19), we have

$$\begin{aligned} &\sum_{i\in I}\lambda_i\langle (y_i-u), Dx-Dz\rangle + \sum_{i\in I}\lambda_i\langle T(u),(y_i-u)\rangle - \sum_{i\in I}\lambda_i f(y_i,u) \\ &- \sum_{i\in I}\lambda_i\tau(y_i,u) \in C \\ &\Rightarrow \sum_{i\in I}\lambda_i\{f(y_i,u)+\tau(y_i,u) - \langle T(u),(y_i-u)\rangle - \langle (y_i-u), Dx-Dz\rangle\} \in -C. \end{aligned} \quad (4.1.20)$$

From (4.1.16) and (4.1.20), we conclude that

$$\begin{aligned} &\sum_{i\in I}\lambda_i\{f(y_i,u) \\ &+ \tau(y_i,u) - \langle T(u),(y_i-u)\rangle - \langle (y_i-u), Dx-Dz\rangle\} \in intC \cap (-C) = \emptyset, \end{aligned}$$

which is a contradiction. This follows that $u \in \bigcup_{i\in I} F(y_i)$ and hence $Co\{y_i : i \in I\} \subseteq \bigcup_{i\in I} F(y_i)$. Thus, F is a KKM mapping. From the assumption (vi), we can see that the family $\{(C_i, Z_i)\}_{i\in I}$ satisfies the condition which is for all $i \in I$, there exists $k \in I$ such that

$$\bigcap_{y\in C_k} F(y) \subset Z_i;$$

and therefore it is a coercing family for F. We deduce that F satisfies all the hypothesis of Theorem 2.2.14. Therefore, we have

$$\bigcap_{y\in C_k} F(y) \neq \emptyset.$$

Hence, there exists $\tilde{x} \in K$ such that for any $y \in K$

$$f(y,\tilde{x}) - \langle T(\tilde{x}), y-\tilde{x}\rangle + \tau(y,\tilde{x}) - \langle y-\tilde{x}, D\tilde{x} - Dz\rangle \notin intC.$$

Now applying Proposition 4.1.2, we obtain that there exists $\tilde{x} \in K$ such that for all $y \in K$

$$f(\tilde{x},y) + \langle T(\tilde{x}), y-\tilde{x}\rangle + \tau(\tilde{x},y) + \langle y-\tilde{x}, D\tilde{x} - Dz\rangle \notin -intC.$$

Hence problem (4.1.3) admits a solution. This completes the proof. □

Corollary 4.1.3. *Let $K, C, \{(C_i, Z_i)\}_{i\in I}$, f, τ, T and D satisfy all the assumptions of Theorem 4.1.1. In addition, if C satisfies Condition(C), then the problem* (4.1.4) *is solvable i.e., there exists $\tilde{x} \in K$ such that for any $y \in K$*

$$f(\tilde{x},y) + \langle T(\tilde{x}), y-\tilde{x}\rangle + \tau(\tilde{x},y) + \langle y-\tilde{x}, D\tilde{x} - Dz\rangle \notin -(C\setminus\{0\}).$$

Proof. Suppose that C satisfies $Condition(C)$. Then there is a pointed convex and closed cone $\tilde{C}$ in Y such that $C\setminus\{0\} \subseteq int\tilde{C}$. Therefore, it is not hard to see that $K, C, \{(C_i, Z_i)\}_{i\in I}$, f, τ, T and D satisfy all the assumptions of Theorem 4.1.1. It follows from Theorem 4.1.1 that

$$f(\tilde{x},y) + \langle T(\tilde{x}), y-\tilde{x}\rangle + \tau(\tilde{x},y) + \langle y-\tilde{x}, D\tilde{x} - Dz\rangle \notin -int\tilde{C}; \quad \forall y \in K. \tag{4.1.21}$$

Since $-(C\setminus\{0\}) \subseteq -int\tilde{C}$, (4.1.21), yields that there exists $\tilde{x} \in K$ such that

$$f(\tilde{x},y) + \langle T(\tilde{x}), y-\tilde{x}\rangle + \tau(\tilde{x},y) + \langle y-\tilde{x}, D\tilde{x} - Dz\rangle \notin -(C\setminus\{0\}); \quad \forall y \in K.$$

Therefore, problem (4.1.4) admits a solution. This completes the proof. □

In the case of $\tau \equiv 0$ and $D \equiv 0$, we obtain the following corollaries.

Corollary 4.1.4. [59] Let K be a nonempty closed convex subset of a Hausdorff topological vector space X, Y a Hausdorff topological vector space and C a closed convex pointed cone in Y with $intC \neq \emptyset$. Let $f : K \times K \to Y$ and $T : K \to L(X,Y)$ be two mappings satisfying the following conditions:

(i) f is C-monotone;

(ii) $f(x,x) = 0$, $x \in K$;

(iii) for any fixed $x, y \in K$; $t \in [0,1] \mapsto f(ty + (1-t)x, y) \in Y$ is upper semicontinuous with respect to C at $t = 0$;

(iv) for any fixed $x \in K$, $f(x, \cdot) : K \to Y$ are C-convex, lower semicontinuous with respect to C on K;

(v) T is upper semicontinuous with respect to C with nonempty closed values;

(vi) there exists a family $\{C_i, Z_i\}_{i \in I}$ satisfying conditions (i) and (ii) of Definition 2.2.12 and the following condition: For each $i \in I$, there exists $k \in I$ such that

$$\{x \in K : f(y,x) - \langle Tx, y - x\rangle \notin intC,\ \forall y \in C_k\} \subset Z_i.$$

Then, there exists a point $\tilde{x} \in K$ such that

$$f(\tilde{x}, y) + \langle T(\tilde{x}), y - \tilde{x}\rangle \notin -intC; \quad \forall y \in K.$$

Corollary 4.1.5. [59] Let $K, C, \{(C_i, Z_i)\}_{i \in I}, f$ and T satisfy all the assumptions of Corollary (4.1.4). In addition, if C satisfies *Condition*(C), then the problem (4.1.2) is solvable i.e., there exists $\tilde{x} \in K$ such that for any $y \in K$

$$f(\tilde{x}, y) + \langle T(\tilde{x}), y - \tilde{x}\rangle \notin -(C \setminus \{0\}).$$

CHAPTER V

CONCLUSION

The following results are all main theorems of this thesis:

1. Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ be L-Lipschitz pseudo-contraction and F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$ with $\widetilde{F} := F(T) \cap EP(F) \neq \varnothing$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C by (3.1.5). Assume the sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ be such that

(1) $0 < a \leqslant \alpha_n \leqslant b < \frac{1}{L+1} < 1$ for all $n \in \mathbb{N}$,

(2) $0 < \beta_n \leqslant 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} \beta_n = 0$,

(3) $r_n > 0$ for all $n \in \mathbb{N}$ with $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $P_{\widetilde{F}}(x_0)$.

2. Let $A : H \to H$ be L-Lipschitz monotone mapping and F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$ which $A^{-1}(0) \cap EP(F) \neq \varnothing$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C by (3.1.21). Assume $0 < a \leqslant \alpha_n \leqslant b < \frac{1}{L+2} < 1$ for all $n \in \mathbb{N}$, $\{\beta_n\}$ and $\{r_n\}$ be as in Theorem 3.1.3. Then $\{x_n\}$ converges strongly to $P_{A^{-1}(0)\cap EP(F)}(x_0)$.

3. Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ be an L-Lipschitz pseudo-contraction. Let Θ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$, $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function, $A : C \to E^*$ be a continuous and monotone mapping such that $\Omega := F(T) \cap GMEP\,(\Theta, A, \varphi) \neq \emptyset$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C by (3.2.6). Assume the sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ be such that

(1) $0 < a \leqslant \alpha_n \leqslant b < \frac{1}{L+1} < 1$ for all $n \in \mathbb{N}$,

(2) $0 \leqslant \beta_n \leqslant 1$ for all $n \in \mathbb{N}$,

(3) $r_n > 0$ for all $n \in \mathbb{N}$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_\Omega(x_0)$.

4. Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to C$ be an L-Lipschitz pseudo-contraction. Let Θ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$, such that $\Omega := F(T) \cap EP(\Theta) \neq \varnothing$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C by (3.2.17). Assume the sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are as in Theorem 3.2.5. Then $\{x_n\}$ converges strongly to $P_\Omega(x_0)$.

5. Let C, H, Θ, A and φ be as in Theorem 3.2.5 and let $B : H \to H$ be an L-Lipschitz monotone mapping such that $\Omega = B^{-1}(0) \cap GMEP(\Theta, A, \varphi) \neq \varnothing$. Let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define a sequence $\{x_n\}$ of C by (3.1.21). Assume $0 < a \leqslant \alpha_n \leqslant b < \frac{1}{L+2} < 1$ for all $n \in \mathbb{N}$, $\{\beta_n\}$ and $\{r_n\}$ be as in Theorem 3.2.5. Then $\{x_n\}$ converges strongly to $P_\Omega(x_0)$.

6. Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E, $T : C \to C$ is closed and G-quasi-strict pseudo-contraction and $f : E \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semicontinuous mapping. Define a sequence $\{x_n\}$ of C by (3.3.9), where $\kappa \in [0, 1)$. Then $\{x_n\}$ converges strongly to $\Pi^f_{F(T)}(x_0)$.

7. Let E be a reflexive, strictly convex and smooth Banach space such that E and E^* have the property (K). Assume that C is a nonempty closed convex subset of E, $T : C \to C$ is closed and G-quasi-strict pseudo-contraction(where $f(\cdot) = \|\cdot\|^2$). Define a sequence $\{x_n\}$ of C by (3.3.20), where $\kappa \in [0, 1)$. Then $\{x_n\}$ converges strongly to $\Pi^{\|\cdot\|^2}_{F(T)}(x_0)$.

8. Let K be a nonempty closed convex subset of a Hausdorff topological vector space X, Y a Hausdorff topological vector space and C a closed convex pointed cone in Y with $intC \neq \emptyset$. Let $f : K \times K \to Y$, $\tau : K \times K \to Y$,

$T : K \to L(X,Y)$ and $D : X \to L(X,Y)$ be four mappings satisfy the following conditions:

1. f and τ are C-monotone;
2. $f(x,x) = 0$, and $\tau(x,x) = 0$ for all $x \in K$;
3. for any fixed $x, y \in K$; $t \in [0,1] \mapsto f(ty + (1-t)x, y) \in Y$ and $t \in [0,1] \mapsto \tau(ty + (1-t)x, y) \in Y$ are upper semicontinuous with respect to C at $t = 0$;
4. for any fixed $x \in K$, $f(x,\cdot), \tau(x,\cdot) : K \to Y$ are C-convex, lower semicontinuous with respect to C on K;
5. D and T are upper semicontinuous with respect to C with nonempty closed values;
6. there exists a family $\{C_i, Z_i\}_{i\in I}$ satisfying conditions (i) and (ii) of Definition 2.2.12 and the following condition: For each $i \in I$, there exists $k \in I$ such that

   $\{x \in K : f(y,x) - \langle Tx, y-x\rangle + \tau(y,x) - \langle y-x, Dx - Dz\rangle \notin intC,\ \forall y \in C_k\}$
   $\subset Z_i$.

Then, there exists a point $\tilde{x} \in K$ such that

$$f(\tilde{x}, y) + \langle T(\tilde{x}), y - \tilde{x}\rangle + \tau(\tilde{x}, y) + \langle y - x, Dx - Dz\rangle \notin -intC;\quad \forall y \in K.$$

9. Let $K, C, \{(C_i, Z_i)\}_{i\in I}$, f, τ, T and D satisfy all the assumptions of Theorem 4.1.1. In addition, if C satisfies *Condition*(C), then the problem (4.1.4) is solvable i.e., there exists $\tilde{x} \in K$ such that for any $y \in K$

$$f(\tilde{x}, y) + \langle T(\tilde{x}), y - \tilde{x}\rangle + \tau(\tilde{x}, y) + \langle y - \tilde{x}, D\tilde{x} - Dz\rangle \notin -(C \setminus \{0\}).$$

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