FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIVE MAPPINGS IN GENERALIZED COMPLETE METRIC SPACES

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ABSTRACT

In this thesis, we introduce a concept of generalized Geraghty contractions in complete partial b-metric spaces. The existence of fixed point theorems for such mappings is proven omitting some condition of $\psi \in \Psi$ that is subadditive. We also prove the fixed point theorem for generalized Geraghty contractions in complete partially ordered partial b-metric spaces using our main result. Moreover, the example is presented for supporting our main result.

Furthermore, we introduce a notion of generalized contractions in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular α-orbital admissible mappings with respect to η in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result.

Moreover, the fixed point theorems and unique common fixed point theorems for generalized contractions with triangular f-α-admissible mappings on Branciari metric spaces are proven omitting some conditions of $\psi \in \Psi_1$ using Ψ_2 the set of all nondecresing and continuous functions. We prove the unique common fixed point theorem for generalized contractions in the setting of partially ordered Branciari metric spaces using our main result. Moreover, we also present the example that supports our main result.

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CHAPTER I

INTRODUCTION

The notion of metric spaces has an important role and applications potential in various branches which Frechet introduced this notion in 1906. In a large class of studying the classical concept of a metric space has been generalized in different directions by partly changing the conditions of the metric. In 2000, Branciari [1] introduced a class of generalized (rectangular) metric spaces by replacing triangular inequality by similar one which involves four or more points instead of three. In 2014, Shukla [2, 3] generalized both the concepts of b-metric spaces and partial metric spaces by introducing the partial b-metric spaces and the auther introduced a partial rectangular metric space as a generalization of the concept of a rectangular metric space and extension of the concept of a partial metric space.

On the other hand, the fixed point theorems is an important tool in analysis. It plays an important role for proving the existence and uniqueness of the solutions to various mathematical model, computer optimization theory, engineering sciences, ect. One of the most important results in fixed point theory is the Banach contraction principle introduced by Banach [4]. There were many authors have studied and proved the results for fixed point theory by generalizing the Banach contraction principle in several directions (see [5, 6, 7, 8, 9] and references contained therein).

In this research, we begin with studying the fixed point theorems in various directions as following:

One of the interesting result is Geraghty's theorem given by Geraghty [6]. In 2013, Cho et al. [5] defined the concept of α-Geraghty contractive type mappings in the setting of metric spaces. In 2014, Popescu [9] generalized α-Geraghty contractions in complete metric spaces under the new conditions concerning with triangular α-orbital admissible mappings. On the other hand, Karapinar [8] investigated the existence and uniqueness of a fixed point of a generalization of α-ψ-Geraghty contractive type mappings under the new conditions concerning with triangular α-admissible mappings. In 2015, Sastry [10] proved fixed point theorems for generalized Geraghty contractive type mappings in complete partial b-metric spaces by considering partial b-metric as in definition defined in Shukla [2].

Another interesting results were given by Jleli et al. [13] introduced a new type of contractive mappings and established a new fixed point theorem for such mappings on the setting of generalized metric spaces. Later, the authors established a new fixed point theorem in the setting of Branciari metric spaces and obtained result is an extension of the recent fixed point theorem established in Jleli et al. [14]. In 2016, Arshad et al. [20], extended the the results introduced by Jleli et al. [13, 14] by using the concept of triangular α-orbital admissible mappings obtained in [9].

This thesis is organized into 6 chapters as follows. Chapter I is an introduction. Chapter II concerns with some well-known definitions and some useful results that will be used in our main results of this thesis.

In Chapter III, we introduce a concept of generalized Geraghty contractions in complete partial b-metric spaces using altering distance functions. The existence of fixed point theorems for such mappings is proven. We also prove the fixed point theorem for generalized Geraghty contractions in complete partially ordered partial b-metric spaces using our main result. Moreover, the example is presented for supporting our main result.

In Chapter IV, the fixed point theorems and unique common fixed point theorems for generalized contractions with triangular f-α-admissible mappings on Branciari metric spaces are proven omitting some conditions of $\psi \in \Psi_1$ using Ψ_2 the set of all nondecresing and continuous functions. We prove the unique common fixed point theorem for generalized contractions in the setting of partially ordered Branciari metric spaces using our main result. Moreover, we also present the example that supports our main result.

In Chapter V, we introduce a notion of generalized contractions in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular α-orbital admissible mappings with respect to η in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result.

In Chapter VI, the conclusion of this thesis is presented.

CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapter.

Throughout this thesis, we let $\mathbb{R}$ stand for the set of all real numbers and $\mathbb{N}$ the set of all natural numbers.

2.1 Fixed point theorems for Geraghty contractions in partial b-metric spaces

For the sake of convenience, we will recall the Geraghty's Theorem, introduced by Geraghty [6].

Definition 2.1.1. Let S denote the class of the functions $\beta : [0,+\infty) \to [0,1)$ satisfying the following condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \quad \text{implies} \quad \lim_{n\to\infty} t_n = 0.$$

In 1973, Geraghty [6] generalized the Banach's contraction principle as follows.

Theorem 2.1.2. [6] *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Assume that there exists $\beta \in S$ such that for all $x, y \in X$,*

$$d(fx, fy) \leq \beta(d(x,y))d(x,y).$$

Then f has a unique fixed point $z \in X$ and for any choice of the initial point $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_n = fx_{n-1}$ for each $n \geq 1$ converges to the point z.

On the other hand, Samet et al. [15] introduced the notion of α-admissible mappings as follows.

Definition 2.1.3. [15] Let $f : X \to X$ be a mapping and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that f is α-admissible if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \text{ implies } \alpha(fx, fy) \geq 1.$$

Recently, Karapinar [8] introduced the concept of α-ψ-Geraghty contraction type mappings on complete metric spaces. Also, [8] defined the family Ψ' of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(1) ψ is nondecreasing;

(2) ψ is subadditive, that is $\psi(s + t) \leq \psi(s) + \psi(t)$;

(3) ψ is continuous;

(4) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.1.4. [8] Let (X, d) be a metric space and $\alpha : X \times X \to [0, +\infty)$ be a function. A mapping $f : X \to X$ is said to be a generalized α-ψ-Geraghty contraction if there exist $\beta \in S$ and $\psi \in \Psi'$ such that for all $x, y \in X$

$$\alpha(x, y)\psi(d(fx, fy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)),$$

where

$$M(x, y) = \max\{p(x, y), p(x, fx), p(y, fy)\}.$$

Theorem 2.1.5. [8] *Let (X, d) be a complete metric space, $\alpha : X \times X \to [0, \infty)$ be a function and $f : X \to X$ be a mapping. Suppose that there exists $\beta \in S$ and $\psi \in \Psi'$ such that for all $x, y \in X$*

$$\alpha(x, y)\psi(d(fx, fy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)),$$

where

$$M(x, y) = \max\{p(x, y), p(x, fx), p(y, fy)\}.$$

Assume that the following conditions hold:

(i) f is a triangular α-admissible mapping;

(ii) there exists $x_1 \in X$ such that $\alpha(x_1, fx_1) \geq 1$;

(iii) f is a continuous mapping.

Then f has a fixed point in X.

In 2013, Shukla [2] unified partial metrics and b-metric spaces by introducing the concept of partial b-metric spaces as follows.

Definition 2.1.6. [2] Let X be a nonempty set and $s \geq 1$ be given a real number. A function $p : X \times X \to \mathbb{R}^+$ is a partial b-metric if the following conditions are satisfied for all $x, y, z \in X$:

(p1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;

(p2) $p(x,x) \leq p(x,y)$;

(p3) $p(x,y) = p(y,x)$;

(p4) $p(x,y) \leq s[p(x,z) + p(z,y)] - p(z,z)$.

The pair (X,p) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X,p).

In the following definition, Mustafa [16] modified the definition of partial b-metric spaces defined by Shukla [2] as follows.

Definition 2.1.7. [16] Let X be a nonempty set and $s \geq 1$ be given a real number. A function $p : X \times X \to [0,+\infty)$ is a partial b-metric if the following conditions are satisfied for all $x, y, z \in X$:

(p1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;

(p2) $p(x,x) \leq p(x,y)$;

(p3) $p(x,y) = p(y,x)$;

(p4) $p(x,y) \leq s[p(x,z) + p(z,y) - p(z,z)] + \left(\frac{1-s}{2}\right)(p(x,x) + p(y,y))$.

The pair (X,p) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X,p).

In 2014, Popescu [9], introduced the concept of α-orbital admissible as follows.

Definition 2.1.8. [9] Let $f : X \to X$ be a mapping and $\alpha : X \times X \to [0, +\infty)$ be a function. Then f is said to be α-orbital admissible if $x \in X$, $\alpha(x, fx) \geq 1$ implies that $\alpha(fx, f^2x) \geq 1$.

Proposition 2.1.9. [16] *Every partial b-metric p defines a b-metric d_p, where*

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

for all $x, y \in X$.

Remark 2.1.10. The class of partial b-metric space (X, d_p) is effectively larger than the class of partial metric space, since a partial metric space is a special case of a partial b-metric space (X, d_p) when $s = 1$. Also, the class of partial b-metric space (X, d_p) is effectively larger than the class of b-metric space, since a b-metric space is a special case of a partial b-metric space (X, d_p) when the self distance $d_p(x, x) = 0$.

Definition 2.1.11. [16] Let (X, p) be a partial b-metric space. Then

(i) a sequence $\{x_n\}$ in X converges to a point $x \in X$ if

$$\lim_{n\to\infty} p(x, x_n) = p(x, x);$$

(ii) a sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite);

(iii) a partial b-metric space (X, p) is said to be a complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x, x).$$

In 2015, Sastry [11] proved the fixed point theorems for generalized Geraghty contractive type mappings in complete partial b-metric spaces by considering partial b-metric spaces defined by Shukla [2].

Definition 2.1.12. [10] Let (X, p) be a partial b-metric space with $s \geq 1$ and $f : X \to X$ be a mapping. We say that f is a generalized Geraghty contraction mapping, if there exists $\beta \in S$ such that for all $x, y \in X$,

$$sp(fx, fy) \leq \beta(M(x,y))M(x,y),$$

where

$$M(x,y) = \max\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2s}[p(x,fy) + p(fx,y)]\}.$$

Theorem 2.1.13. [10] *Let (X,p) be a complete partial b-metric space with $s \geq 1$, $\alpha : X \times X \to [0,\infty)$ be a function and $f : X \to X$ be a mapping. Suppose that there exists $\beta \in S$ such that for all $x, y \in X$,*

$$\alpha(x, fx)\alpha(y, fy)sp(fx, fy) \leq \beta(M(x,y))M(x,y),$$

where

$$M(x,y) = \max\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2s}[p(x,fy) + p(fx,y)]\}.$$

Assume that the following conditions hold:

(i) f is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, fx_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ converges to x, then $\alpha(x, fx) \geq 1$.

Then f has a fixed point in X.

Theorem 2.1.14. [10] *Let $(X, \preceq, p)$ be a complete partially ordered partial b-metric space with $s \geq 1$ and $f : X \to X$ be a nondecreasing mapping. Suppose that there exist $\beta \in S$ such that for all comparable $x, y \in X$,*

$$sp(fx, fy) \leq \beta(M(x,y))M(x,y),$$

where

$$M(x,y) = \max\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2s}[p(x,fy) + p(fx,y)]\}.$$

Assume that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;

(ii) if $\{x_n\}$ is a nondecreasing sequence that converges to x, then $x_n \leq x$ for all $n \in \mathbb{N}$.

Then f has a fixed point in X.

The following lemma shows the relationship between the concepts of Cauchy sequence and completeness in (X, p) and (X, d_p).

Lemma 2.1.15. [16]

(1) A sequence $\{x_n\}$ is a Cauchy sequence in a partial b-metric space (X, p) if and only if it is a b-Cauchy sequence in the b-metric space (X, d_p).

(2) A partial b-metric space (X, p) is complete if and only if the b-metric space (X, d_p) is b-complete. Moreover, $\lim_{n\to\infty} d_p(x, x_n) = 0$ if and only if

$$\lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = p(x, x).$$

The following important lemma is useful in proving our main results.

Lemma 2.1.16. [16] *Let (X, p) be a partial b-metric space with the coefficient $s > 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y, respectively. Then we have*

$$\begin{aligned}\frac{1}{s^2}p(x,y) - \frac{1}{s}p(x,x) - p(y,y) \leq \liminf_{n\to\infty} p(x_n, y_n) &\leq \limsup_{n\to\infty} p(x_n, y_n)\\ &\leq sp(x,x) + s^2p(y,y) + s^2p(x,y).\end{aligned}$$

In particular, if $p(x, y) = 0$, then we have $\lim_{n\to\infty} p(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\begin{aligned}\frac{1}{s}p(x,z) - p(x,x) \leq \liminf_{n\to\infty} p(x_n, z) &\leq \limsup_{n\to\infty} p(x_n, z)\\ &\leq sp(x,z) + sp(x,x).\end{aligned}$$

In particular, if $p(x, x) = 0$, then we have

$$\frac{1}{s}p(x,z) \leq \liminf_{n\to\infty} p(x_n, z) \leq \limsup_{n\to\infty} p(x_n, z) \leq sp(z,z).$$

Lemma 2.1.17. [11] *Let (X, p) be a partial b-metric space. Then the following hold.*

(1) If $p(x, y) = 0$ then $x = y$.

(2) If $x \neq y$ then $p(x, y) = p(y, x) > 0$.

In 2016, Chuadchawna [17] introduced the concept of α-orbital admissible mappings with respect to η as follows.

Definition 2.1.18. [17] Let $f : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Then f is said to be α-orbital admissible with respect to η if

$$\alpha(x, fx) \geq \eta(x, fx) \text{ implies } \alpha(fx, f^2x) \geq \eta(fx, f^2x).$$

Remark 2.1.19. If we suppose that $\eta(x, y) = 1$ for all $x, y \in X$, then Definition 2.1.18 reduces to Definition 2.1.8.

Let Ψ denote the family of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following:

(1) ψ is nondecreasing;

(2) ψ is continuous;

(3) $\psi(t) = 0$ if and only if $t = 0$.

In this paper, we introduce a concept of generalized Geraghty contractions in complete partial b-metric spaces. The existence of fixed point theorems for such mappings is proven omitting some condition of $\psi \in \Psi'$ that is subadditive. We also prove the fixed point theorem for generalized Geraghty contractions in complete partially ordered partial b-metric spaces using our main result. Moreover, the example is presented for supporting our main result.

2.2 Generalized contractions with triangular α-orbital admissible mappings on Branciari metric spaces

In 2000, Branciari [1] introduced the concept of Branciari metric spaces as follows.

Definition 2.2.1. [1] Let X be a nonempty set. We say that a mapping $d : X \times X \rightarrow \mathbb{R}$ is a Branciari metric on X if d satisfies the following:

(d1) $0 \leq d(x, y)$, for all $x, y \in X$;

(d2) $d(x, y) = 0$ if and only if $x = y$;

(d3) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d4) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$, for all $x, y \in X$ and for all distinct points $w, z \in X \setminus \{x, y\}$.

If d is a Branciari metric, then (X, d) is called a Branciari metric space or a regtangular metric space (or for short BMS). By the definition, we see that a Branciari metric space is a generalization of a metric space.

Definition 2.2.2. Let (X, d) be a BMS, $\{x_n\}$ be a sequence in X, and $x \in X$. We say that

(i) $\{x_n\}$ is convergent to x if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ and denoted by $x_n \rightarrow x$ as $n \rightarrow \infty$.

(ii) $\{x_n\}$ is a Cauchy sequence if $d(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$.

(iii) (X, d) is a complete BMS if every Cauchy sequence in X converges to some element in X.

Lemma 2.2.3. [18] *Let (X, d) be a BMS and $\{x_n\}$ be a Cauchy sequence in (X, d) such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ for some $x \in X$. Then $d(x_n, y) \rightarrow d(x, y)$ as $n \rightarrow \infty$ for all $y \in X$. In particular, $\{x_n\}$ does not converge to y if $x \neq y$.*

Lemma 2.2.4. [12] *Let (X, d) be a BMS, $\{x_n\}$ be a Cauchy sequence in (X, d) and $x, y \in X$. Suppose that there exists a positive integer N such that*

(i) $x_n = x_m$ for all $n, m \geq N$;

(ii) x_n *and* x *are distinct points in* X *for all* $n \geq N$;

(iii) x_n *and* y *are distinct points in* X *for all* $n \geq N$;

(iv) $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x_n, y)$.

Then we have $x = y$.

Shukla [2] introduced a concept of the partial rectangular metric spaces as the following:

Definition 2.2.5. [2] Let X be a nonempty set. We say that a mapping $p : X \times X \to \mathbb{R}$ is a partial rectangular metric on X if p satisfies the following:

(p1) $p(x, y) \geq 0$, for all $x, y \in X$;

(p2) $x = y$ if and only if $p(x, y) = p(x, x) = p(y, y)$, for all $x, y \in X$;

(p3) $p(x, x) \leq p(x, y)$, for all $x, y \in X$;

(p4) $p(x, y) = p(y, x)$, for all $x, y \in X$;

(p5) $p(x, y) \leq p(x, w) + p(w, z) + p(z, y) - p(w, w) - p(z, z)$, for all $x, y \in X$ and for all distinct points $w, z \in X \backslash \{x, y\}$.

If p is a partial rectangular metric on X, then a pair (X, p) is called a partial rectangular metric space.

Remark 2.2.6. [2] In a partial rectangular metric space (X, p), if $x, y \in X$ and $p(x, y) = 0$, then $x = y$ but the converse may not be true.

Remark 2.2.7. [2] It is clear that every rectangular metric space is a partial rectangular metric space with zero self-distance. However, the converse of this fact need not hold.

Proposition 2.2.8. [2] *For each partial rectangular metric space* (X, p), *the pair* (X, d_p) *is rectangular metric space where*

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

for all $x, y \in X$.

Definition 2.2.9. [2] Let (X,p) be a partial rectangular metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then,

(i) the sequence $\{x_n\}$ is said to converges to $x \in X$ if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$;

(ii) the sequence $\{x_n\}$ is said to be a Cauchy sequence in (X,p) if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite;

(iii) (X,p) is said to be a complete partial rectangular metric space if for every Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x) = p(x,x).$$

Lemma 2.2.10. [2] *Let (X,p) be a partial rectangular metric space and let $\{x_n\}$ be a sequence in X. Then $\lim_{n\to\infty} d_p(x_n,x) = 0$ if and only if $\lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x_n,x_n) = p(x,x)$.*

Lemma 2.2.11. [2] *Let (X,p) be a partial rectangular metric space and let $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in (X,d_p).*

Lemma 2.2.12. [2] *A partial rectangular metric space (X,p) is complete if and only if a rectangular metric space (X,d_p) is complete.*

In 2012, Samet et al. [15] introduced the notion of α-admissible mappings as follows.

Definition 2.2.13. [15] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0,\infty)$ be a function. We say that T is α-admissible if for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } \alpha(Tx,Ty) \geq 1.$$

Karapinar et al. [8] defined the concept of triangular α-admissible mappings.

Definition 2.2.14. [8] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0,\infty)$ be a function. We say that T is a triangular α-admissible mapping if:

(T1) T is α-admissible;

(T2) for all $x, y, z \in X$, $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ imply that $\alpha(x,y) \geq 1$.

Later, Popescu et al. [9] introduced the notions of α-orbital admissible, triangular α-orbital admissible and α-orbital attractive mappings as follows.

Definition 2.2.15. [9] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0,\infty)$ be a function. Then T is said to be α-orbital admissible if for all $x \in X$,

$$\alpha(x, Tx) \geq 1 \text{ implies } \alpha(Tx, T^2x) \geq 1.$$

Definition 2.2.16. [9] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0,\infty)$ be a function. Then T is said to be triangular α-orbital admissible if:

(T3) T is α-orbital admissible;

(T4) for all $x, y \in X$, $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$ imply that $\alpha(x,Ty) \geq 1$.

Definition 2.2.17. [9] Let $T : X \to X$ be a mapping and $\alpha : X \times X \to [0,\infty)$ be a function. Then T is said to be α-orbital attractive if for all $x \in X$,

$$\alpha(x, Tx) \geq 1 \text{ implies } \alpha(x,y) \geq 1 \text{ or } \alpha(y,Tx) \geq 1, \text{ for all } \ y \in X.$$

Lemma 2.2.18. [9] *Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Suppose that T is a triangular α-orbital admissible mapping and assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Denote by Ψ_1 the set of all functions $\psi : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(1) ψ is nondecreasing;

(2) for each sequence $\{t_n\} \subset (0,\infty)$,

$$\lim_{n\to\infty} \psi(t_n) = 1 \text{ if and only if } \lim_{n\to\infty} t_n = 0;$$

(3) there exist $r \in (0,1)$ and $l \in (0,\infty]$ such that $\lim_{t\to 0} \frac{\psi(t)-1}{t^r} = l$.

Li and Jiang [19] introduced Ψ_2 the set of all functions $\psi : (0,\infty) \to (1,\infty)$ which is nondecreasing and continuous. They also gave some examples illustrating the relationship between Ψ_1 and Ψ_2 as follows:

Example 2.2.19. [19] Let $f(t) = e^{te^t}$ for $t \geq 0$. Then $f \in \Psi_2$ but $f \notin \Psi_1$ since $\lim_{t\to 0} \frac{e^{te^t}-1}{t^r} = 0$ for each $r \in (0,1)$.

Example 2.2.20. [19] Let $g(t) = e^{t^a}$ for $t \geq 0$, where $a > 0$. If $a \in (0,1)$, then $g \in \Psi_1 \cap \Psi_2$. If $a = 1$, then $g \in \Psi_2$ but $g \notin \Psi_1$ since $\lim_{t\to 0} \frac{e^t-1}{t^r} = 0$ for each $r \in (0,1)$. If $a > 1$, then $g \in \Psi_2$ but $g \notin \Psi_1$ since $\lim_{t\to 0} \frac{e^{t^a}-1}{t^r} = 0$ for each $r \in (0,1)$.

Remark 2.2.21. From Example 2.2.19 and Example 2.2.20, we can conclude that $\Psi_2 \not\subset \Psi_1$ and $\Psi_1 \cap \Psi_2 \neq \varnothing$. Moreover, it is clear that if $\psi \in \Psi_1$ and ψ is continuous, then $\psi \in \Psi_2$.

Jleli et al. [14] established the following theorem by adding the continuity to a function $\psi \in \Psi_1$ on Branciari metric spaces.

Theorem 2.2.22. [14] *Let (X,d) be a complete BMS and $T : X \to X$ be a mapping. Suppose that there exist $\psi \in \Psi_1$ that is continuous and $\lambda \in (0,1)$ such that for all $x, y \in X$,*

$$d(Tx,Ty) \neq 0 \text{ implies } \psi(d(Tx,Ty)) \leq [\psi(R(x,y))]^\lambda,$$

where

$$R(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Then T has a fixed point z in X and $\{T^n x_1\}$ converges to z.

Arshad et al. [20] extended the results proved by Jleli et al. [13] and [14] by using the concept of triangular α-orbital admissible mappings obtained in [9] by adding the continuity to a function $\psi \in \Psi_1$.

Theorem 2.2.23. [20] *Let (X,d) be a complete BMS, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0,\infty)$ be a function. Suppose that the following conditions hold:*
(1) there exist $\psi \in \Psi_1$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \neq 0 \text{ implies } \alpha(x,y)\psi(d(Tx,Ty)) \leq [\psi(R(x,y))]^\lambda,$$

where

$$R(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\};$$

(2) there exists $x_1\in X$ such that $\alpha(x_1,Tx_1)\geq 1$ and $\alpha(x_1,T^2x_1)\geq 1$;

(3) T is a triangular α-orbital admissible mapping;

(4) T is continuous.

Then T has a fixed point z in X and $\{T^nx_1\}$ converges to z.

Theorem 2.2.24. [20] *Let (X,d) be a complete BMS, $T:X\to X$ be a mapping and $\alpha:X\times X\to[0,\infty)$ be a function. Suppose that the following conditions hold:*

(1) there exist $\psi\in\Psi_1$ and $\lambda\in(0,1)$ such that for all $x,y\in X$,

$$d(Tx,Ty)\neq 0 \text{ implies } \alpha(x,y)\psi(d(Tx,Ty))\leq[\psi(R(x,y))]^{\lambda},$$

where

$$R(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\};$$

(2) there exists $x_1\in X$ such that $\alpha(x_1,Tx_1)\geq 1$ and $\alpha(x_1,T^2x_1)\geq 1$;

(3) T is a triangular α-orbital admissible mapping;

(4) if $\{T^nx_1\}$ is a sequence in X such that $\alpha(T^nx_1,T^{n+1}x_1)\geq 1$ for all $n\in\mathbb{N}$ and $x_n\to x\in X$ as $n\to\infty$, then there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^nx_1\}$ such that $\alpha(T^{n(k)}x_1,x)\geq 1$ for all $k\in\mathbb{N}$;

(5) ψ is continuous.

Then T has a fixed point z in X and $\{T^nx_1\}$ converges to z.

Theorem 2.2.25. [20] *Let (X,d) be a complete BMS, $T:X\to X$ be a mapping and $\alpha:X\times X\to[0,\infty)$ be a function. Suppose that the following conditions hold:*

(1) there exist $\psi\in\Psi_1$ and $\lambda\in(0,1)$ such that for all $x,y\in X$,

$$d(Tx,Ty)\neq 0 \text{ implies } \alpha(x,y)\psi(d(Tx,Ty))\leq[\psi(R(x,y))]^{\lambda},$$

where

$$R(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$ and $\alpha(x_1, T^2x_1) \geq 1$;

(3) T is an α-orbital admissible mapping;

(4) T is an α-orbital attractive mapping;

(5) ψ is continuous.

Then T has unique fixed point z in X and $\{T^n x_1\}$ converges to z.

In 2016, Chuadchawna [17] introduced the notion of triangular α-orbital admissible mappings with respect to η and proved the lemma which will be used for proving our main results.

Definition 2.2.26. [17] Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Then T is said to be α-orbital admissible with respect to η if for all $x \in X$,

$$\alpha(x, Tx) \geq \eta(x, Tx) \text{ implies } \alpha(Tx, T^2x) \geq \eta(Tx, T^2x).$$

Definition 2.2.27. [17] Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Then T is said to be triangular α-orbital admissible with respect to η if

(T1) T is α-orbital admissible with respect to η;

(T2) for all $x, y \in X, \alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$ imply $\alpha(x, Ty) \geq \eta(x, Ty)$.

Remark 2.2.28. If we suppose that $\eta(x, y) = 1$ for all $x, y \in X$, then Definition 2.2.26 and Definition 2.2.27 reduces to Definition 2.2.15 and Definition 2.2.16, respectively.

Lemma 2.2.29. [17] *Let $T : X \to X$ be a triangular α-orbital admissible mapping with respect to η. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Definition 2.2.30. Let $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Then T is said to be α-orbital attractive with respect to η if for all

$x \in X$,

$$\alpha(x, Tx) \geq \eta(x, Tx) \text{ implies } \alpha(x, y) \geq \eta(x, y) \text{ or } \alpha(y, Tx) \geq \eta(y, Tx)$$

for all $y \in X$.

Definition 2.2.31. Let $T, f : X \to X$. If $\omega = Tx = fx$ for some $x \in X$, then x is called a coincidence point of T and f, and ω is called a point of coincidence of T and f.

Definition 2.2.32. Let $T, f : X \to X$. The pair $\{T, f\}$ is said to be weakly compatible if $Tfx = fTx$ whenever $Tx = fx$ for some $x \in X$.

Proposition 2.2.33. [2] *Let $T, f : X \to X$ and $\{T, f\}$ is weakly compatible. If T and f have a unique point of coincidence $\omega = Tx = fx$, then ω is the unique common fixed point of T and f.*

In this thesis, we introduce a notion of generalized contractions in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular α-orbital admissible mappings with respect to η in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result. And the fixed point theorems and unique common fixed point theorems for generalized contractions with triangular f-α-admissible mappings on Branciari metric spaces are proven omitting some conditions of $\psi \in \Psi_1$ using Ψ_2 the set of all nondecresing and continuous functions. We prove the unique common fixed point theorem for generalized contractions in the setting of partially ordered Branciari metric spaces using our main result. Moreover, we also present the example that supports our main result.

CHAPTER III

FIXED POINT THEOREMS FOR GENERALIZED GERAGHTY CONTRACTIONS IN COMPLETE PARTIAL b-METRIC SPACES

In this chapter, the existence of fixed point theorems for generalized Geraghty contractions is proven. We also prove the fixed point theorem for generalized Geraghty contractions in complete partially ordered partial b-metric spaces using our main result. Moreover, the example is presented for supporting our main result.

3.1 The fixed point theorems in complete partial b-metric spaces

We now prove the existence of fixed point theorems for generalized Geraghty contractions using altering distance functions in complete partial b-metric spaces.

Theorem 3.1.1. *Let (X, p) be a complete partial b-metric space with $s \geq 1$ and let $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Let $f : X \to X$ be a mapping. Suppose that there exist $\beta \in S$ and $\psi \in \Psi$ such that for all $x, y \in X$,*
$\alpha(x, fx) \geq \eta(x, fx)$ and $\alpha(y, fy) \geq \eta(y, fy)$ imply that

$$\psi(sp(fx, fy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)), \tag{3.1.1}$$

where

$$M(x, y) = \max\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2s}[p(x, fy) + p(fx, y)]\}.$$

Assume that

(i) f is α-orbital admissible with respect to η;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, fx_n) \geq \eta(x_n, fx_n)$ for all $n \in \mathbb{N} \cup \{0\}$

and $\{x_n\}$ *converges to* x, *then* $\alpha(x, fx) \geq \eta(x, fx)$.

Then f *has a fixed point in* X.

Proof. By condition (ii), there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$. Define the sequence $\{x_n\}$ in X such that $x_n = fx_{n-1}$, for all $n \in \mathbb{N}$. Since f is an α-orbital admissible mapping with respect to η, we obtain that

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.1.2}$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then we have $x_n = x_{n+1} = fx_n$ and so x_n is a fixed point of f. Now, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. By Lemma 2.1.17, we have $p(x_n, x_{n+1}) > 0$. By (3.1.1) and (3.1.2), for each $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} \psi(sp(x_{n+1}, x_{n+2})) &= \psi(sp(fx_n, fx_{n+1})) \\ &\leq \beta(\psi(M(x_n, x_{n+1})))\psi(M(x_n, x_{n+1})) \\ &< \psi(M(x_n, x_{n+1})), \end{aligned} \tag{3.1.3}$$

where

$$\begin{aligned} &M(x_n, x_{n+1}) \\ &= \max\left\{p(x_n, x_{n+1}), p(x_n, fx_n), p(x_{n+1}, fx_{n+1}), \frac{p(x_n, fx_{n+1}) + p(fx_n, x_{n+1})}{2s}\right\} \\ &= \max\left\{p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{p(x_n, x_{n+2}) + p(x_{n+1}, x_{n+1})}{2s}\right\} \\ &\leq \max\Big\{p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})}{2} \\ &\quad + \frac{(1-s)}{4s}(p(x_n, x_n) + 2p(x_{n+1}, x_{n+1}) + p(x_{n+2}, x_{n+2}))\Big\} \\ &\leq \max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\}. \end{aligned}$$

If $\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_{n+1}, x_{n+2})$, then by (3.1.3) we have

$$\psi(sp(x_{n+1}, x_{n+2})) < \psi(M(x_n, x_{n+1})) \leq \psi(p(x_{n+1}, x_{n+2})).$$

Since ψ is nondecreasing, we have $sp(x_{n+1}, x_{n+2}) < p(x_{n+1}, x_{n+2})$, which is a contradiction. Hence $\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_n, x_{n+1})$. By (3.1.3), we obtain that

$$\psi(sp(x_{n+1}, x_{n+2})) < \psi(M(x_n, x_{n+1})) \leq \psi(p(x_n, x_{n+1})).$$

Since ψ is nondecreasing, we have $p(x_{n+1}, x_{n+2}) < sp(x_{n+1}, x_{n+2}) < p(x_n, x_{n+1})$. Hence the sequence $\{p(x_n, x_{n+1})\}$ is strictly decreasing and bounded below. Therefore it converges to some $r \geq 0$, that is

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r.$$

Suppose that $r > 0$. By (3.1.3), we have

$$\begin{aligned}\psi(p(x_{n+1}, x_{n+2})) &\leq \beta(\psi(M(x_n, x_{n+1})))\psi(M(x_n, x_{n+1}))\\ &\leq \beta(\psi(M(x_n, x_{n+1})))\psi(p(x_n, x_{n+1})).\end{aligned}$$

Therefore

$$\frac{\psi(p(x_{n+1}, x_{n+2}))}{\psi(p(x_n, x_{n+1}))} \leq \beta(\psi(M(x_n, x_{n+1}))) < 1.$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} \beta(\psi(M(x_n, x_{n+1}))) = 1.$$

Since $\beta \in S$, we have $\lim_{n\to\infty} \psi(M(x_n, x_{n+1})) = 0$ and so

$$\psi(r) = \lim_{n\to\infty} \psi(p(x_n, x_{n+1})) = 0. \tag{3.1.4}$$

Hence $r = 0$. Next, we will prove that $\{x_n\}$ is a Cauchy sequence in (X, p) which is equivalent to show that $\{x_n\}$ is a b-Cauchy sequence in (X, d_p). Suppose that $\{x_n\}$ is not a b-Cauchy sequence in (X, d_p). Then there exists $\varepsilon > 0$ such that, for $k > 0$, there exist $n_k > m_k > k$ for which we can find two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which,

$$d_p(x_{m_k}, x_{n_k}) \geq \varepsilon,$$

and

$$d_p(x_{m_k}, x_{n_k-1}) < \varepsilon.$$

Then we have

$$\begin{aligned}\varepsilon \le d_p(x_{m_k}, x_{n_k}) &\le s d_p(x_{m_k}, x_{n_k-1}) + s d_p(x_{n_k-1}, x_{n_k}) \\ &< s\varepsilon + s d_p(x_{n_k-1}, x_{n_k}).\end{aligned} \tag{3.1.5}$$

Letting the lower limit for (3.1.5) as $k \to \infty$, we have

$$\varepsilon \le s \liminf_{k\to\infty} d_p(x_{m_k}, x_{n_k-1}) \le s \limsup_{k\to\infty} d_p(x_{m_k}, x_{n_k-1}) \le s\varepsilon.$$

This implies that

$$\frac{\varepsilon}{s} \le \liminf_{k\to\infty} d_p(x_{m_k}, x_{n_k-1}) \le \limsup_{k\to\infty} d_p(x_{m_k}, x_{n_k-1}) \le \varepsilon. \tag{3.1.6}$$

Similarly, from (3.1.5) and (3.1.6), we have

$$\varepsilon \le \limsup_{k\to\infty} d_p(x_{m_k}, x_{n_k}) \le s\varepsilon.$$

By using the triangular inequality, we deduce that

$$d_p(x_{m_k+1}, x_{n_k}) \le s d_p(x_{m_k+1}, x_{m_k}) + s d_p(x_{m_k}, x_{n_k}). \tag{3.1.7}$$

Letting the upper limit for (3.1.7) as $k \to \infty$, we get

$$\limsup_{k\to\infty} d_p(x_{m_k+1}, x_{n_k}) \le s^2\varepsilon.$$

Using triangular inequality again, this yields

$$\begin{aligned}d_p(x_{m_k+1}, x_{n_k-1}) &\le s d_p(x_{m_k+1}, x_{m_k}) + s d_p(x_{m_k}, x_{n_k-1}) \\ &\le s d_p(x_{m_k+1}, x_{m_k}) + s\varepsilon.\end{aligned} \tag{3.1.8}$$

Letting the upper limit for (3.1.8) as $k \to \infty$, we have

$$\limsup_{k\to\infty} d_p(x_{m_k+1}, x_{n_k-1}) \le s\varepsilon.$$

By Proposition 2.1.9, we get that

$$2 \limsup_{k\to\infty} p(x_{m_k}, x_{n_k-1}) = \limsup_{k\to\infty} d_p(x_{m_k}, x_{n_k-1}).$$

Hence,

$$\frac{\varepsilon}{2s} \leq \liminf_{k\to\infty} p(x_{m_k}, x_{n_k-1}) \leq \limsup_{k\to\infty} p(x_{m_k}, x_{n_k-1}) \leq \frac{\varepsilon}{2}. \tag{3.1.9}$$

Similarly, we can derive

$$\limsup_{k\to\infty} p(x_{m_k}, x_{n_k}) \leq \frac{s\varepsilon}{2}, \tag{3.1.10}$$

$$\frac{\varepsilon}{2s} \leq \limsup_{k\to\infty} p(x_{m_k+1}, x_{n_k}) \leq \frac{s^2\varepsilon}{2}, \tag{3.1.11}$$

$$\limsup_{k\to\infty} p(x_{m_k+1}, x_{n_k-1}) \leq \frac{s\varepsilon}{2}. \tag{3.1.12}$$

Since f is α-orbital admissible with respect to η and using (3.1.2). By using (3.1.1), we have

$$\begin{aligned} \psi(sp(x_{m_k+1}, x_{n_k})) &= \psi(sp(fx_{m_k}, fx_{n_k-1})) \\ &\leq \beta(\psi(M(x_{m_k}, x_{n_k-1})))\psi(M(x_{m_k}, x_{n_k-1})) \\ &< \psi(M(x_{m_k}, x_{n_k-1})), \end{aligned} \tag{3.1.13}$$

where

$$\begin{aligned} M(x_{m_k}, x_{n_k-1}) &= \max\Big\{p(x_{m_k}, x_{n_k-1}), p(x_{m_k}, fx_{m_k}), p(x_{n_k-1}, fx_{n_k-1}), \\ &\qquad \frac{p(x_{m_k}, fx_{n_k-1}) + p(fx_{m_k}, x_{n_k-1})}{2s}\Big\} \\ &= \max\Big\{p(x_{m_k}, x_{n_k-1}), p(x_{m_k}, x_{m_k+1}), p(x_{n_k-1}, x_{n_k}), \\ &\qquad \frac{p(x_{m_k}, x_{n_k}) + p(x_{m_k+1}, x_{n_k-1})}{2s}\Big\}. \end{aligned}$$

Letting the upper limit as $k \to \infty$ in the above inequality using (3.1.4), (3.1.10) and (3.1.12), we get that

$$\begin{aligned} \limsup_{k\to\infty} M(x_{m_k}, x_{n_k-1}) &= \max\Big\{\limsup_{k\to\infty} p(x_{m_k}, x_{n_k-1}), \limsup_{k\to\infty} p(x_{m_k}, x_{m_k+1}), \\ &\qquad \limsup_{k\to\infty} p(x_{n_k-1}, x_{n_k}), \frac{\limsup_{k\to\infty} p(x_{m_k}, x_{n_k})}{2s} \\ &\qquad + \frac{\limsup_{k\to\infty} p(x_{m_k+1}, x_{n_k-1})}{2s}\Big\} \\ &\leq \max\Big\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big\} \end{aligned}$$

$$= \frac{\varepsilon}{2}.$$

By letting the upper limit in (3.1.13) as $k \to \infty$ we have

$$\begin{aligned}
\psi\Big(s\frac{\varepsilon}{2s}\Big) &\leq \beta(\psi(\limsup_{k\to\infty} M(x_{m_k}, x_{n_k-1})))\psi(\limsup_{k\to\infty} M(x_{m_k}, x_{n_k-1})) \\
&\leq \beta(\psi(\limsup_{k\to\infty} M(x_{m_k}, x_{n_k-1})))\psi(\frac{\varepsilon}{2}) \\
\frac{\psi\Big(\frac{\varepsilon}{2}\Big)}{\psi\Big(\frac{\varepsilon}{2}\Big)} &\leq \beta(\psi(\limsup_{k\to\infty} M(x_{m_k}, x_{n_k-1}))) \leq 1.
\end{aligned}$$

This implies that

$$\frac{\psi\Big(\frac{\varepsilon}{2}\Big)}{\psi\Big(\frac{\varepsilon}{2}\Big)} \leq \beta(\psi(\limsup_{k\to\infty} M(x_{m_k}, x_{n_k-1}))) \leq 1.$$

Thus we have

$$\beta(\psi(\limsup_{k\to\infty} M(x_{m_k}, x_{n_k-1}))) = 1.$$

Since $\beta \in S$, we get that

$$\psi(\limsup_{k\to\infty} M(x_{m_k}, x_{n_k-1})) = 0.$$

Therefore

$$\limsup_{k\to\infty} M(x_{m_k}, x_{n_k-1}) = 0.$$

which is a contradiction with (3.1.9). So the sequence $\{x_n\}$ is a Cauchy sequence in the b-metric space (X, d_p). Since (X, p) is complete, then (X, d_p) is a complete b-metric space. It follows from the completeness that there exists $z \in X$ such that

$$\lim_{n\to\infty} d_p(x_n, z) = 0.$$

Since

$$\begin{aligned}
2p(x_n, z) &= d_p(x_n, z) + p(x_n, x_n) + p(z, z) \\
&\leq d_p(x_n, z) + p(x_n, x_{n+1}) + p(x_n, z),
\end{aligned}$$

we obtain that

$$p(x_n, z) \leq d_p(x_n, z) + p(x_n, x_{n+1}).$$

Letting the limit as $n \to \infty$, we get that

$$\lim_{n\to\infty} p(x_n, z) = 0.$$

By Lemma 2.1.15, we have

$$0 = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(z, z).$$

By condition (iii), we have $\alpha(z, fz) \geq \eta(z, fz)$. This implies that

$$\psi(sp(fx_n, fz)) \leq \beta(\psi(M(x_n, z)))\psi(M(x_n, z)), \tag{3.1.14}$$

where

$$\begin{aligned} M(x_n, z) &= \max\left\{p(x_n, z), p(x_n, fx_n), p(z, fz), \frac{p(x_n, fz) + p(fx_n, z)}{2s}\right\} \\ &= \max\left\{p(x_n, z), p(x_n, x_{n+1}), p(z, fz), \frac{p(x_n, fz) + p(x_{n+1}, z)}{2s}\right\} \\ &\leq \max\Big\{p(x_n, z), p(x_n, x_{n+1}), p(z, fz), \\ &\qquad \frac{sp(x_n, z) + sp(z, fz) + p(x_{n+1}, z)}{2s}\Big\}. \end{aligned} \tag{3.1.15}$$

Letting the upper limit as $n \to \infty$ in above inequality, we get that

$$\limsup_{n\to\infty} M(x_n, z) \leq p(z, fz). \tag{3.1.16}$$

From (3.1.14) and using Lemma 2.1.16, then letting the upper limit as $n \to \infty$, we obtain that

$$\begin{aligned} \psi(p(z, fz)) &= \psi(s\frac{1}{s}p(z, fz)) \\ &\leq \psi(s \liminf_{n\to\infty} p(x_{n+1}, fz)) \\ &\leq \psi(s \limsup_{n\to\infty} p(x_{n+1}, fz)) \end{aligned}$$

$$\leq \beta(\psi(\limsup_{n\to\infty} M(x_n, z)))\psi(\limsup_{n\to\infty} M(x_n, z))$$
$$\leq \beta(\psi(\limsup_{n\to\infty} M(x_n, z)))\psi(p(z, fz)).$$

This implies that,

$$\lim_{n\to\infty} \beta(\psi(\limsup_{n\to\infty} M(x_n, z))) = 1.$$

Therefore

$$\psi(\limsup_{n\to\infty} M(x_n, z)) = 0.$$

Thus we have

$$\limsup_{n\to\infty} M(x_n, z) = 0. \tag{3.1.17}$$

Using Lemma 2.1.16 and (3.1.17), we get that

$$\begin{aligned}
\frac{\frac{p(z,fz)}{2s}}{s} &\leq \liminf_{n\to\infty} \frac{p(x_n, fz)}{2s} \\
&\leq \liminf_{n\to\infty} \frac{p(x_n, fz) + p(x_n, z)}{2s} \\
&\leq \liminf_{n\to\infty} M(x_n, z) \\
&\leq \limsup_{n\to\infty} M(x_n, z) \\
&\leq p(z, fz).
\end{aligned}$$

This implies that $p(z, fz) = 0$. Since $p(fz, fz) \leq sp(fz, z) + sp(z, fz)$, we have $p(fz, fz) = 0$. Therefore $p(z, z) = p(z, fz) = p(fz, fz)$ which implies that $z = fz$. Hence z is a fixed point of f. □

Next, we now present a numerical example to support Theorem 3.1.1.

Example 3.1.2. Let $X = [0, \infty)$ and $p : X \times X \to [0, \infty)$ defined by $p(x, y) = 2|x - y|^2 + 3$ for all $x, y \in X$. Clearly, (X, p) is a partial b-metric space with $s = 2$. Define the mapping $f : X \to X$ by

$$fx = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x \leq 1; \\ x & \text{if } x > 1. \end{cases}$$

Then f is nondecreasing. Define $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 3 & \text{if } x, y \in [0,1]; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\eta(x,y) = \begin{cases} 2 & \text{if } x, y \in [0,1]; \\ 1 & \text{otherwise.} \end{cases}$$

Define functions $\psi : [0, \infty) \to [0, \infty)$ and $\beta : [0, \infty) \to [0, 1)$ by $\psi(t) = \frac{t}{2}$ and

$$\beta(t) = \begin{cases} \frac{(t-1)^2}{2} & \text{if } t \in [0,1]; \\ t & \text{if } t > 1. \end{cases}$$

We check that all conditions in Theorem 3.1.1 hold. Let $\alpha(x, fx) \geq \eta(x, fx)$ and $\alpha(y, fy) \geq \eta(y, fy)$. Therefore $x, y, fx, fy \in [0, 1]$. This implies that

$$\psi(sp(fx, fy)) = \psi(2p(\tfrac{x}{2}, \tfrac{y}{2})) = \frac{2p(\frac{x}{2}, \frac{y}{2})}{2} = 2|\tfrac{x}{2} - \tfrac{y}{2}|^2 + 3 = \tfrac{1}{2}|x - y|^2 + 3,$$

and

$$\begin{aligned} M(x,y) &= \max\{p(x,y), p(x,fx), p(y,fy), \frac{p(x,fy) + p(y,fx)}{2}\} \\ &= \max\{2|x-y|^2 + 3, 2|x - \frac{x}{2}|^2 + 3, 2|y - \frac{y}{2}|^2 + 3, \\ &\quad \frac{|x - \frac{y}{2}|^2 + |y - \frac{x}{2}|^2 + 3}{2}\} \\ &= 2|x-y|^2 + 3. \end{aligned}$$

Without loss of generality, we may assume that $y \leq x$. This implies that

$$\begin{aligned} \psi(sp(fx, fy)) &= \psi(2p(\frac{x}{2}, \frac{y}{2})) \\ &= \frac{1}{2}|x-y|^2 + 3 \\ &\leq \frac{25}{4} = \frac{5}{2} \cdot \frac{5}{2} \\ &= \beta(\psi(5))\psi(5) \\ &\leq \beta(\psi(M(x,y)))\psi(M(x,y)). \end{aligned}$$

We will prove the following:

(i) f is α-orbital admissible with respect to η ;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ converges to x, then $\alpha(x, fx) \geq \eta(x, fx)$.

Proof. (i) Let $\alpha(x, fx) \geq \eta(x, fx)$. Then we have $x, fx \in [0,1]$ and so, $f^2x = f(fx) \in [0,1]$. Thus $\alpha(fx, f^2x) \geq \eta(fx, f^2x)$. Therefore f is α-orbital admissible with respect to η.

(ii) Letting $x_0 = 1 \in X$, we have $fx_0 = f1 = \frac{1}{2}$. So $\alpha(x_0, fx_0) = \alpha(1, f1) = \alpha(1, \frac{1}{2}) = 3 \geq 2 = \eta(1, \frac{1}{2}) = \eta(1, f1) = \eta(x_0, fx_0)$.

(iii) Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Then $\{x_n\} \subseteq [0,1]$ for all $n \in \mathbb{N} \cup \{0\}$ and so $x \in [0,1]$. By the definition of f we have $fx \in [0,1]$, which we have $\alpha(x, fx) \geq \eta(x, fx)$.

Hence all assumptions in Theorem 3.1.1 are satisfied and thus f has a fixed point.

□

3.2 The fixed point theorems in partially ordered partial b-metric spaces

We next prove the fixed point theorem in complete partially ordered partial b-metric spaces using Theorem 3.1.1.

Theorem 3.2.1. *Let $(X, \preceq, p)$ be a complete partially ordered partial b-metric space with $s \geq 1$. Let $f : X \to X$ be a nondecreasing mapping. Suppose that there exist $\beta \in S$ and $\psi \in \Psi$ such that for all comparable $x, y \in X$,*

$$\psi(sp(fx, fy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)), \tag{3.2.1}$$

where

$$M(x,y)=\max\{p(x,y),p(x,fx),p(y,fy),\frac{1}{2s}[p(x,fy)+p(fx,y)]\}. \qquad (3.2.2)$$

Assume that

(i) there exists $x_0 \in X$ *such that* $x_0 \preceq fx_0$*;*

(ii) if $\{x_n\}$ *is a nondecreasing sequence that converges to* x *such that* $x_n \preceq fx_n$ *for all* $n \in \mathbb{N}\cup\{0\}$*, then* $x \preceq fx$*.*

Then f *has a fixed point* z *in* X*.*

Proof. Define mappings $\alpha, \eta : X \times X \to [0,\infty)$ by

$$\alpha(x,y)=\begin{cases}1 & \text{if } x \preceq y;\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\eta(x,y)=\begin{cases}1 & \text{if } x \preceq y;\\ 2 & \text{otherwise.}\end{cases}$$

For each $x \in X$ such that $\alpha(x,fx) \geq \eta(x,fx)$, by definitions of α and η, we have $x \preceq fx$. Since f is nondecreasing, we have $fx \preceq f(fx) = f^2x$. Thus $\alpha(fx,f^2x) \geq \eta(fx,f^2x)$. Therefore f is α-orbital admissible with respect to η. By (ii), we have $\alpha(x_0,fx_0) \geq \eta(x_0,fx_0)$ for some $x_0 \in X$. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$, for all $n \in \mathbb{N}\cup\{0\}$ and $x_n \to x$ as $n \to \infty$. By definitions of α and η, we have $x_n \preceq x_{n+1}$, for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is a nondecreasing. By (ii), we have $x \preceq fx$. By the definitions of α and η, we obtain that $\alpha(x,fx) \geq \eta(x,fx)$. Thus all assumptions in Theorem 3.1.1 are satisfied. Hence f has a fixed point in X. □

In Theorem 3.1.1 and Theorem 3.2.1, if we put $\eta(x,y)=1$ and $\psi(t)=t$, then we obtain the following result proved by Sastry [10].

Corollary 3.2.2. [10] *Let (X,p) be a complete partial b-metric space with $s \geq 1$ and let $\alpha : X \times X \to [0,\infty)$ be a function. Let $f : X \to X$ be a mapping. Suppose that there exists $\beta \in S$ such that for all $x, y \in X$,*

$$\alpha(x,fx)\alpha(y,fy)sp(fx,fy) \leq \beta(M(x,y))M(x,y),$$

where

$$M(x,y) = \max\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2s}[p(x,fy) + p(fx,y)]\}.$$

Assume that

(i) f is α-orbital admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, fx_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ converges to x, then $\alpha(x,fx) \geq 1$.

Then f has a fixed point in X.

Corollary 3.2.3. [10] *Let $(X, \preceq, p)$ be a complete partially ordered partial b-metric space with $s \geq 1$. Let $f : X \to X$ be a nondecreasing mapping. Suppose that there exists $\beta \in S$ such that for all comparable $x, y \in X$,*

$$sp(fx,fy) \leq \beta(M(x,y))M(x,y),$$

where

$$M(x,y) = \max\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2s}[p(x,fy) + p(fx,y)]\}.$$

Assume that

(i) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$;

(ii) if $\{x_n\}$ is a nondecreasing sequence that converges to x such that $x_n \preceq fx_n$ for all $n \in \mathbb{N} \cup \{0\}$, then $x \preceq fx$.

Then f has a fixed point z in X.

CHAPTER IV

FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS WITH TRIANGULAR α-ORBITAL ADMISSIBLE MAPPING ON BRANCIARI METRIC SPACES

In this chapter, the fixed point theorems and unique common fixed point theorems for generalized contractions with triangular f-α-admissible mappings on Branciari metric spaces are proven omitting some conditions of $\psi \in \Psi_1$ using Ψ_2 the set of all nondecresing and continuous functions. We prove the unique common fixed point theorem for generalized contractions in the setting of partially ordered Branciari metric spaces using our main result. Moreover, we also present the example that supports our main result.

4.1 The fixed point theorems

We now prove the existence of fixed point theorems for triangular α-orbital admissible mappings omitting some conditions of $\psi \in \Psi_1$ using Ψ_2 the set of all nondecreasing and continuous functions on $(0, \infty)$ to $(1, \infty)$.

Theorem 4.1.1. *Let (X, d) be a complete BMS, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Suppose that the following conditions hold:*

(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(x, y)\psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}, \tag{4.1.1}$$

where

$$R(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\};$$

(ii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;

(iii) T is a triangular α-orbital admissible mapping;

(iv) T is continuous;

Then T has a fixed point.

Proof. Let $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define the iterative sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n, \quad \text{for all } n \in \mathbb{N}.$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point T. We now suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By condition (ii), we have $\alpha(x_1, Tx_1) \geq 1$. Using Lemma 2.2.18, we obtain that

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{4.1.2}$$

From (4.1.1) and (4.1.2), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &= \psi(d(Tx_{n-1}, Tx_n)) \\ &\leq \alpha(x_{n-1}, x_n)\psi(d(Tx_{n-1}, Tx_n)) \\ &\leq [\psi(R(x_{n-1}, x_n))]^{\lambda}, \end{aligned} \tag{4.1.3}$$

where

$$\begin{aligned} R(x_{n-1}, x_n) &= \max\Big\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\Big\} \\ &= \max\Big\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\Big\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

If $R(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then by (4.1.3) we obtain that

$$\psi(d(x_n, x_{n+1})) \leq [\psi(d(x_n, x_{n+1}))]^{\lambda} < \psi(d(x_n, x_{n+1})),$$

which is a contradiction. Hence $R(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Using (4.1.3), we have

$$\psi(d(x_n, x_{n+1})) \leq [\psi(d(x_{n-1}, x_n))]^{\lambda} < \psi(d(x_{n-1}, x_n)).$$

Since ψ is nondecreasing, we have $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$. Hence the sequence $\{d(x_n, x_{n+1})\}$ is decreasing. Hence $\{d(x_n, x_{n+1})\}$ converges to a nonnegative real number. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$ and

$$d(x_n, x_{n+1}) \geq r. \tag{4.1.4}$$

We will prove that $r = 0$. Suppose that $r > 0$. Since ψ is nondecreasing and by using (4.1.3) and (4.1.4), we obtain that

$$1 < \psi(r) \leq \psi(d(x_n, x_{n+1})) \leq [\psi(d(x_{n-1}, x_n))]^{\lambda} \leq \cdots \leq [\psi(d(x_0, x_1))]^{\lambda^n}, \tag{4.1.5}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in this inequality, we get that $\psi(r) = 1$, which contradicts the assumption that $\psi(t) > 1$ for each $t > 0$. Consequently, we have $r = 0$ and therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4.1.6}$$

Suppose that there exist $n, p \in \mathbb{N}$ such that $x_n = x_{n+p}$. We prove that $p = 1$. Assume that $p > 1$. Using (4.1.1) and (4.1.2), we obtain that

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &= \psi(d(x_{n+p}, x_{n+p+1})) \\ &= \psi(d(Tx_{n+p-1}, Tx_{n+p})) \\ &\leq \alpha(x_{n+p-1}, x_{n+p})\psi(d(Tx_{n+p-1}, Tx_{n+p})) \\ &\leq [\psi(R(x_{n+p-1}, x_{n+p}))]^{\lambda}, \end{aligned} \tag{4.1.7}$$

where

$$\begin{aligned} R(x_{n+p-1}, x_{n+p}) &= \max\left\{d(x_{n+p-1}, x_{n+p}), d(x_{n+p-1}, Tx_{n+p-1}), d(x_{n+p}, Tx_{n+p})\right\} \\ &= \max\left\{d(x_{n+p-1}, x_{n+p}), d(x_{n+p-1}, x_{n+p}), d(x_{n+p}, x_{n+p+1})\right\} \\ &= \max\left\{d(x_{n+p-1}, x_{n+p}), d(x_{n+p}, x_{n+p+1})\right\}. \end{aligned}$$

If $R(x_{n+p}, x_{n+p+1}) = d(x_{n+p}, x_{n+p+1})$, then from (4.1.7) we obtain that

$$\psi(d(x_n, x_{n+1})) = \psi(d(x_{n+p}, x_{n+p+1})) \leq [\psi(d(x_{n+p}, x_{n+p+1}))]^{\lambda} < \psi(d(x_{n+p}, x_{n+p+1})),$$

which is a contradiction. Hence $R(x_{n+p}, x_{n+p+1}) = d(x_{n+p-1}, x_{n+p})$. By (4.1.7), we obtain that

$$\psi(d(x_n, x_{n+1})) = \psi(d(x_{n+p}, x_{n+p+1})) \leq [\psi(d(x_{n+p-1}, x_{n+p}))]^\lambda < \psi(d(x_{n+p-1}, x_{n+p})).$$

Since ψ is nondecreasing, we have $d(x_n, x_{n+1}) < d(x_{n+p-1}, x_{n+p})$. By using (4.1.1), we get that

$$\begin{aligned}\psi(d(x_{n+p-1}, x_{n+p})) &\leq \alpha(x_{n+p-2}, x_{n+p-1})\psi(d(Tx_{n+p-2}, Tx_{n+p-1})) \\ &\leq [\psi(R(x_{n+p-2}, x_{n+p-1}))]^\lambda, \end{aligned} \tag{4.1.8}$$

where

$$\begin{aligned} &R(x_{n+p-2}, x_{n+p-1}) \\ &= \max\left\{d(x_{n+p-2}, x_{n+p-1}), d(x_{n+p-2}, Tx_{n+p-2}), d(x_{n+p-1}, Tx_{n+p-1}\right\} \\ &= \max\left\{d(x_{n+p-2}, x_{n+p-1}), d(x_{n+p-2}, x_{n+p-1}), d(x_{n+p-1}, x_{n+p})\right\} \\ &= \max\left\{d(x_{n+p-2}, x_{n+p-1}), d(x_{n+p-1}, x_{n+p})\right\}. \end{aligned}$$

If $R(x_{n+p-2}, x_{n+p-1}) = d(x_{n+p-1}, x_{n+p})$, then by (4.1.8) we obtain that

$$\psi(d(x_{n+p-1}, x_{n+p})) \leq [\psi(d(x_{n+p-1}, x_{n+p}))]^\lambda < \psi(d(x_{n+p-1}, x_{n+p})),$$

which is a contradiction. Hence $R(x_{n+p-2}, x_{n+p-1}) = d(x_{n+p-2}, x_{n+p-1})$. By (4.1.8), we have

$$\psi(d(x_{n+p-1}, x_{n+p})) \leq [\psi(d(x_{n+p-2}, x_{n+p-1}))]^\lambda < \psi(d(x_{n+p-2}, x_{n+p-1})).$$

Since ψ is nondecreasing, we have $d(x_{n+p-1}, x_{n+p}) < d(x_{n+p-2}, x_{n+p-1})$. By continuing this process, we obtain the following inequality

$$d(x_n, x_{n+1}) < d(x_{n+p-1}, x_{n+p}) < d(x_{n+p-2}, x_{n+p-1}) < \ldots < d(x_n, x_{n+1}),$$

which is a contradiction and hence $p = 1$. We deduce that T has a fixed point. We can assume that $x_n \neq x_m$ for $n \neq m$. We now prove that $\{d(x_n, x_{n+2})\}$ is bounded. Since $\{d(x_n, x_{n+1})\}$ is bounded, there exists $M > 0$ such that

$$d(x_n, x_{n+1}) \leq M \text{ for all } n \in \mathbb{N}.$$

If $d(x_n, x_{n+2}) > M$ for all $n \in \mathbb{N}$, then from

$$\begin{aligned} R(x_{n-1}, x_{n+1}) &= \max\Big\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, Tx_{n-1}), d(x_{n+1}, Tx_{n+1})\Big\} \\ &= \max\Big\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2})\Big\} \\ &= d(x_{n-1}, x_{n+1}), \end{aligned}$$

and Lemma 2.2.18, we obtain that

$$\begin{aligned} \psi(d(x_n, x_{n+2})) &= \psi(d(Tx_{n-1}, Tx_{n+1})) \\ &\le \alpha(x_{n-1}, x_{n+1})\psi(d(Tx_{n-1}, Tx_{n+1})) \\ &\le [\psi(R(x_{n-1}, x_{n+1}))]^{\lambda} \\ &= [\psi(d(x_{n-1}, x_{n+1}))]^{\lambda} \\ &< \psi(d(x_{n-1}, x_{n+1}). \end{aligned}$$

This implies that $\{d(x_n, x_{n+2})\}$ is decreasing. Therefore $\{d(x_n, x_{n+2})\}$ is bounded. If $d(x_n, x_{n+2}) \le M$ for some $n \in \mathbb{N}$, then from

$$\begin{aligned} R(x_n, x_{n+2}) &= \max\Big\{d(x_n, x_{n+2}), d(x_n, Tx_n), d(x_{n+2}, Tx_{n+2})\Big\} \\ &= \max\Big\{d(x_n, x_{n+2}), d(x_n, x_{n+1}), d(x_{n+2}, x_{n+3})\Big\}, \end{aligned}$$

and Lemma 2.2.18, we obtain that

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+3})) &= \psi(d(Tx_n, Tx_{n+2})) \\ &\le \alpha(x_n, x_{n+2})\psi(d(Tx_n, Tx_{n+2})) \\ &\le [\psi(R(x_n, x_{n+2}))]^{\lambda} \\ &\le [\psi(M)]^{\lambda} \\ &< \psi(M), \end{aligned}$$

we obtain that $d(x_{n+1}, x_{n+3}) < M$. This implies that $\{d(x_n, x_{n+2})\}$ is bounded. We next prove that $\lim_{n\to\infty} d(x_n, x_{n+2}) = 0$. Suppose that $\lim_{n\to\infty} d(x_n, x_{n+2}) \neq 0$. So there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} d(x_{n_k}, x_{n_k+2}) = a \text{ for some } a > 0.$$

Using (4.1.1) and Lemma 2.2.18, we have

$$\begin{aligned}\psi(d(x_{n_k}, x_{n_k+2})) &= \psi(d(Tx_{n_k-1}, Tx_{n_k+1})) \\ &\leq \alpha(x_{n_k-1}, x_{n_k+1})\psi(d(Tx_{n_k-1}, Tx_{n_k+1})) \\ &\leq [\psi(R(x_{n_k-1}, x_{n_k+1}))]^{\lambda},\end{aligned}$$

where

$$\begin{aligned}R(x_{n_k-1}, x_{n_k+1}) &= \max\left\{d(x_{n_k-1}, x_{n_k+1}), d(x_{n_k-1}, Tx_{n_k-1}), d(x_{n_k+1}, Tx_{\mathbf{n_k+1}}\right\} \\ &= \max\left\{d(x_{n_k-1}, x_{n_k+1}), d(x_{n_k-1}, x_{n_k}), d(x_{n_k+1}, x_{n_k+2})\right\}.\end{aligned}$$

Letting $k \to \infty$ in the above inequality, we obtain that

$$\psi(a) = \lim_{k\to\infty} \psi(d(x_{n_k}, x_{n_k+2})) \leq \lim_{k\to\infty} [\psi(R(x_{n_k-1}, x_{n_k+1}))]^{\lambda} = [\psi(a)]^{\lambda} < \psi(a),$$

which is a contradiction. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{4.1.9}$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that n_k is the smallest index with $n_k > m_k > k$ for which

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon, \tag{4.1.10}$$

and

$$d(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{4.1.11}$$

By applying the rectangular inequality and using (4.1.10) and (4.1.11), we obtain that

$$\begin{aligned}\varepsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k-2}) + d(x_{n_k-2}, x_{n_k}) \\ &< \varepsilon + d(x_{n_k-1}, x_{n_k-2}) + d(x_{n_k-2}, x_{n_k}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality, using (4.1.6) and (4.1.9), we get that

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{4.1.12}$$

For each $k \in \mathbb{N}$, we have

$$\begin{aligned} R(x_{n_k}, x_{m_k}) &= \max\Big\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, Tx_{n_k}), (x_{m_k}, Tx_{m_k})\Big\} \\ &= \max\Big\{d(x_{n_k}, x_{m_k}), d(x_{n_k}, x_{n_k+1}), (x_{m_k}, x_{m_k+1})\Big\}. \end{aligned}$$

By using (4.1.6) and (4.1.12), we obtain that

$$\lim_{k\to\infty} R(x_{n_k}, x_{m_k}) = \varepsilon. \tag{4.1.13}$$

By (4.1.12) and (4.1.13), there exists a positive integer k_0 such that

$$d(x_{n_k+1}, x_{m_k+1}) > 0 \quad \text{and} \quad R(x_{n_k}, x_{m_k}) > 0, \quad \text{for all} \quad k \geq k_0.$$

By Lemma 2.2.18 and using (4.1.1), we get that

$$\begin{aligned} \psi(d(x_{n_k+1}, x_{m_k+1})) &= \psi(d(Tx_{n_k}, Tx_{m_k})) \\ &= \psi(d(Tx_{m_k}, Tx_{n_k})) \\ &\leq \alpha(x_{m_k}, x_{n_k})\psi(d(Tx_{m_k}, Tx_{n_k})) \\ &\leq [\psi(R(x_{m_k}, x_{n_k}))]^{\lambda} \\ &= [\psi(R(x_{n_k}, x_{m_k}))]^{\lambda}, \end{aligned}$$

for all $n_k > m_k > k \geq k_0$. Letting $k \to \infty$ in this inequality, by (4.1.12), (4.1.13) and the continuity of ψ, we obtain that

$$\psi(\varepsilon) = \lim_{k\to\infty} \psi(d(x_{n_k+1}, x_{m_k+1})) \leq \lim_{k\to\infty} [\psi(R(x_{n_k}, x_{m_k}))]^{\lambda} = [\psi(\varepsilon)]^{\lambda} < \psi(\varepsilon),$$

which is a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is a complete BMS, it follows that $\{x_n\}$ converges to $x \in X$. Since T is continuous, we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx.$$

Therefore x is a fixed point of T. □

We now replace the continuity of T in Theorem 4.1.1 by some appropriate conditions to obtain the following theorem.

Theorem 4.1.2. *Let (X,d) be a complete BMS, $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$. Suppose that the following conditions hold :*

(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(Tx,Ty) \neq 0 \text{ implies } \alpha(x,y)\psi(d(Tx,Ty)) \leq [\psi(R(x,y))]^{\lambda}, \tag{4.1.14}$$

where

$$R(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\};$$

(ii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;

(iii) T is a triangular α-orbital admissible mapping;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point.

Proof. As in the proof of Theorem 4.1.1, we can construct the sequence $\{x_n\}$ in X such that

$$x_{n+1} = Tx_n, \quad \text{for all } n \in \mathbb{N},$$

$\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$. By (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$. We can suppose that $x_{n_k} \neq Tx$. Applying inequality (4.1.14), we obtain that

$$\begin{aligned}\psi(d(Tx_{n_k}, Tx)) &\leq \alpha(x_{n_k}, x)\psi(d(Tx_{n_k}, Tx)) \\ &\leq [\psi(R(x_{n_k}, x))]^{\lambda},\end{aligned}$$

where

$$R(x_{n_k}, x) = \max\left\{d(x_{n_k}, x), d(x_{n_k}, Tx_{n_k}), d(x, Tx)\right\}$$

$$= \max\Big\{d(x_{n_k}, x), d(x_{n_k}, x_{n_k+1}), d(x, Tx)\Big\}.$$

Taking the limit as $k \to \infty$ and since ψ is continuous, we obtain that

$$\lim_{k\to\infty} R(x_{n_k}, x) = d(x, Tx).$$

We will prove that $x = Tx$. Suppose that $x \neq Tx$. Therefore

$$d(x, Tx) \leq d(x, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) + d(x_{n_k}, Tx).$$

It follows that

$$d(x, Tx) \leq \lim_{k\to\infty} d(x_{n_k}, Tx).$$

Since ψ is continuous and nondecreasing, we obtain that

$$\psi(d(x, Tx)) \leq \lim_{k\to\infty} \psi(d(x_{n_k}, Tx)) \leq [\psi(d(x, Tx))]^{\lambda} < \psi(d(x, Tx)),$$

which is a contradiction. Thus $x = Tx$ and hence x is a fixed point of T. □

We now present some examples that supporting our main result.

Example 4.1.3. Let $X = \{0, 1, 2, 3\}$. Define $d : X \times X \to [0, \infty)$ as follows:

$d(x, x) = 0$ for all $x \in X$,

$d(0, 2) = d(2, 0) = d(0, 3) = d(3, 0) = d(2, 3) = d(3, 2) = 2$,

$d(0, 1) = d(1, 0) = d(1, 2) = d(2, 1) = 4$,

$d(1, 3) = d(3, 1) = 1$, and

$d(x, y) = |x - y|$, otherwise.

Therefore (X, d) is complete BMS but (X, d) is not a metric space because it lacks the triangular property as the following:

$$d(1, 2) = 4 > 1 + 2 = d(1, 3) + d(3, 2).$$

Let $T : X \to X$ be the mapping defined by

$$Tx = \begin{cases} 1 & \text{if } x \neq 2; \\ 3 & \text{if } x = 2. \end{cases}$$

Let $\alpha : X \times X \to [0, \infty)$ be given by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in X \backslash \{2\}; \\ \frac{3}{5} & \text{otherwise.} \end{cases}$$

Define a function $\psi : (0, \infty) \to (1, \infty)$ by $\psi(t) = e^t$. By Example 2.2.20, we obtain that $\psi \in \Psi_2$ but $\psi \notin \Psi_1$. We next illustrate that all conditions in Theorem 4.1.2 hold. Taking $x_1 = 1$, we have $\alpha(1, T1) = \alpha(1, 1) = 1 \geq 1$. We next prove that T is α-orbital admissible. Let $x \in X$ such that $\alpha(x, Tx) \geq 1$. Therefore $x, Tx \in X \backslash \{2\}$ and then $x \in \{0, 1, 3\}$. By the definition of α, we obtain that

$$\alpha(T0, T^2 0) = \alpha(1, 1) \geq 1,$$

$$\alpha(T1, T^2 1) = \alpha(1, 1) \geq 1,$$

$$\alpha(T3, T^2 3) = \alpha(1, 1) \geq 1.$$

It follows that T is α-orbital admissible. Let $x, y \in X$ such that $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$. By the definition of α, we have $x, y, Ty \in X \backslash \{2\}$. This yields

$$\alpha(0, 1) \geq 1 \text{ and } \alpha(1, T1) \geq 1 \text{ imply } \alpha(0, T1) \geq 1,$$

$$\alpha(0, 3) \geq 1 \text{ and } \alpha(3, T3) \geq 1 \text{ imply } \alpha(0, T3) \geq 1,$$

$$\alpha(1, 3) \geq 1 \text{ and } \alpha(3, T3) \geq 1 \text{ imply } \alpha(1, T3) \geq 1,$$

$$\alpha(1, 0) \geq 1 \text{ and } \alpha(0, T0) \geq 1 \text{ imply } \alpha(1, T0) \geq 1,$$

$$\alpha(3, 0) \geq 1 \text{ and } \alpha(0, T0) \geq 1 \text{ imply } \alpha(3, T0) \geq 1,$$

$$\alpha(3, 1) \geq 1 \text{ and } \alpha(1, T1) \geq 1 \text{ imply } \alpha(3, T1) \geq 1.$$

This implies that T is triangular α-orbital admissible. Let $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. By the definition of α, for each $n \in \mathbb{N}$, we get that $x_n \in X\backslash\{2\} = \{0,1,3\}$. We obtain that $x \in \{0,1,3\}$. Thus we have $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$. We next prove that (4.1.14) holds. Let $x, y \in X$ be such that $d(Tx, Ty) \neq 0$. So we consider the following cases:

- $x = 2$ and $y \in \{0,1,3\}$ or
- $y = 2$ and $x \in \{0,1,3\}$.

We divide the proof into three cases as follows:

(1) If $(x, y) \in \{(0,2),(2,0)\}$, then

$$\begin{aligned} R(0,2) &= \max\Big\{d(0,2), d(0,1), d(2,3)\Big\} \\ &= \max\Big\{2,4,2\Big\} \\ &= 4. \end{aligned}$$

This implies that

$$\psi(d(T0,T2)) = \psi(d(1,3)) = \psi(1) = e^1 \leq [e^4]^{0.3} = [\psi(4)]^{0.3} \leq [\psi(R(0,2))]^{0.3}.$$

Therefore

$$\alpha(0,2)\psi(d(T0,T2)) = \frac{3}{5}\psi(d(T0,T2)) \leq \psi(d(T0,T2)) \leq [\psi(R(0,2))]^{0.3}.$$

Since $d(x,y) = d(y,x)$ for all $x, y \in X$, we also obtain that

$$\alpha(2,0)\psi(d(T2,T0)) \leq [\psi(R(2,0))]^{0.3}.$$

(2) If $(x, y) \in \{(2,1),(1,2)\}$, then

$$\begin{aligned} R(2,1) &= \max\Big\{d(2,1), d(2,3), d(1,1)\Big\} \\ &= \max\Big\{1,2,0\Big\} \\ &= 2. \end{aligned}$$

This implies that

$$\psi(d(T2,T1)) = \psi(d(3,1)) = \psi(1) = e^1 \leq [e^2]^{0.7} = [\psi(2)]^{0.7} \leq [\psi(R(2,1))]^{0.7}.$$

Therefore

$$\alpha(2,1)\psi(d(T2,T1)) = \frac{3}{5}\psi(d(T2,T1)) \leq \psi(d(T2,T1)) \leq [\psi(R(2,1))]^{0.7}.$$

Since $d(x,y) = d(y,x)$ for all $x, y \in X$, we also obtain that

$$\alpha(1,2)\psi(d(T1,T2)) \leq [\psi(R(1,2))]^{0.7}.$$

(3) If $(x,y) \in \{(2,3),(3,2)\}$, then

$$\begin{aligned} R(2,3) &= \max\Big\{d(2,3), d(2,3), d(3,1)\Big\} \\ &= \max\Big\{2,2,1\Big\} \\ &= 2. \end{aligned}$$

This implies that

$$\psi(d(T2,T3)) = \psi(d(3,1)) = \psi(1) = e^1 \leq [e^2]^{0.7} = [\psi(2)]^{0.7} \leq [\psi(R(2,3))]^{0.7}.$$

Therefore

$$\alpha(2,3)\psi(d(T2,T3)) = \frac{3}{5}\psi(d(T2,T3)) \leq \psi(d(T2,T3)) \leq [\psi(R(2,3))]^{0.7}.$$

Since $d(x,y) = d(y,x)$ for all $x, y \in X$, we also obtain that

$$\alpha(3,2)\psi(d(T3,T2)) \leq [\psi(R(3,2))]^{0.7}.$$

It follows that if $x, y \in X$ and $d(Tx,Ty) \neq 0$, then

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq [\psi(R(x,y))]^{\lambda}.$$

Hence all assumptions in Theorem 4.1.2 are satisfied and thus T has a fixed point which is $x = 1$.

4.2 The unique common fixed point theorems

We now introduce the notion of triangular f-α-admissible mappings and prove a key lemma that will be used for proving our results.

Definition 4.2.1. Let $T, f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is said to be a triangular f-α-admissible mapping if

(i) T is an f-α-admissible mapping;

(ii) for all $x, y \in X, \alpha(fx, fy) \geq 1$ and $\alpha(fy, Ty) \geq 1$ imply $\alpha(fx, Ty) \geq 1$.

Lemma 4.2.2. *Let $T, f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Suppose that $T : X \to X$ is a triangular f-α-admissible mapping and assume that there exists $x_1 \in X$ such that $\alpha(fx_1, Tx_1) \geq 1$. Define a sequence $\{fx_n\}$ by $fx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $\alpha(fx_n, fx_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Proof. Since T is a triangular f-α-admissible mapping and $\alpha(fx_1, Tx_1) \geq 1$, we have $\alpha(fx_2, fx_3) = \alpha(Tx_1, Tx_2) \geq 1$. By continuing this process, we obtain that

$$\alpha(fx_n, fx_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}.$$

Suppose that $\alpha(fx_n, fx_m) \geq 1$. We will prove that $\alpha(fx_n, fx_{m+1}) \geq 1$ where $n < m$. Since T is triangular f-α-admissible and $\alpha(fx_m, fx_{m+1}) \geq 1$, we obtain that $\alpha(fx_n, fx_{m+1}) \geq 1$. Hence $\alpha(fx_n, fx_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$. □

Theorem 4.2.3. *Let (X, d) be a BMS and $T, f : X \to X$ be such that $TX \subseteq fX$ where one of these two subsets of X being complete. Assume that $\alpha : X \times X \to [0, \infty)$ and suppose that the following conditions hold:*

(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(fx, fy)\psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}, \tag{4.2.1}$$

where

$$R(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\};$$

(ii) there exists $x_1 \in X$ such that $\alpha(fx_1, Tx_1) \geq 1$;

(iii) T is a triangular f-α-admissible mapping;

(iv) T is continuous with respect to f;

(v) either $\alpha(fu, fv) \geq 1$ or $\alpha(fv, fu) \geq 1$ whenever $fu = Tu$ and $fv = Tv$.

Then T and f have a unique point of coincidence. Moreover, if the pair $\{T, f\}$ is weakly compatible, then T and f have a unique common fixed point.

Proof. Let $x_1 \in X$ such that $\alpha(fx_1, Tx_1) \geq 1$. Define the iterative sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_n = fx_{n+1} = Tx_n, \quad \text{for all } n \in \mathbb{N}.$$

Moreover, we assume that if $Tx_n = y_n = y_m = Tx_m$ for some $n \neq m$, then we choose $x_{n+1} = x_{m+1}$, this can be done since $TX \subseteq fX$. It follows that $y_{n+1} = y_{m+1}$. If $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then y_{n_0+1} is a point of coincidence of T and f. Consequently, we can suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. By condition (ii), we have $\alpha(fx_1, Tx_1) \geq 1$. Using Lemma 4.2.2, we obtain that

$$\alpha(fx_n, fx_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N}. \tag{4.2.2}$$

From (4.2.1) and (4.2.2), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \psi(d(y_n, y_{n+1})) &= \psi(d(Tx_n, Tx_{n+1})) \\ &\leq \alpha(fx_n, fx_{n+1})\psi(d(Tx_n, Tx_{n+1})) \\ &\leq [\psi(R(x_n, x_{n+1}))]^{\lambda}, \end{aligned} \tag{4.2.3}$$

where

$$\begin{aligned} R(x_n, x_{n+1}) &= \max\left\{d(fx_n, fx_{n+1}), d(fx_n, Tx_n), d(fx_{n+1}, Tx_{n+1})\right\} \\ &= \max\left\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1})\right\} \\ &= \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \end{aligned}$$

If $R(x_n, x_{n+1}) = d(y_n, y_{n+1})$, then by (4.2.3) we obtain that

$$\psi(d(y_n, y_{n+1})) \leq [\psi(d(y_n, y_{n+1}))]^{\lambda} < \psi(d(y_n, y_{n+1})),$$

which is a contradiction. Hence $R(x_n, x_{n+1}) = d(y_{n-1}, y_n)$. Using (4.2.3), we have

$$\psi(d(y_n, y_{n+1})) \leq [\psi(d(y_{n-1}, y_n))]^{\lambda} < \psi(d(y_{n-1}, y_n)).$$

Since ψ is nondecreasing, we have $d(y_n, y_{n+1}) < d(y_{n-1}, y_n)$. Hence the sequence $\{d(y_n, y_{n+1})\}$ is decreasing. Hence $\{d(y_n, y_{n+1})\}$ converges to a nonnegative real number. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(y_n, y_{n+1}) = r$ and

$$d(y_n, y_{n+1}) \geq r \quad \text{for all } n \in \mathbb{N}. \tag{4.2.4}$$

We will prove that $r = 0$. Suppose that $r > 0$. Since ψ is nondecreasing and by using (4.2.3) and (4.2.4), we obtain that

$$1 < \psi(r) \leq \psi(d(y_n, y_{n+1})) \leq [\psi(d(y_{n-1}, y_n))]^{\lambda} \leq \cdots \leq [\psi(d(y_0, y_1))]^{\lambda^n}, \tag{4.2.5}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in this inequality, we get that $\psi(r) = 1$ which contradicts the assumption that $\psi(t) > 1$ for each $t > 0$. Consequently, we have $r = 0$ and therefore

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{4.2.6}$$

Suppose that there exist $n, p \in \mathbb{N}$ such that $y_n = y_{n+p}$. We prove that $p = 1$. Assume that $p > 1$. By using (4.2.1), we obtain that

$$\begin{aligned}
\psi(d(y_n, y_{n+1})) &= \psi(d(y_{n+p}, y_{n+p+1})) \\
&= \psi(d(Tx_{n+p}, Tx_{n+p+1})) \\
&\leq \alpha(fx_{n+p}, fx_{n+p+1})\psi(d(Tx_{n+p}, Tx_{n+p+1})) \\
&\leq [\psi(R(x_{n+p}, x_{n+p+1}))]^{\lambda},
\end{aligned} \tag{4.2.7}$$

where

$$\begin{aligned}
&R(x_{n+p}, x_{n+p+1}) \\
&= \max\Big\{d(fx_{n+p}, fx_{n+p+1}), d(fx_{n+p}, Tx_{n+p}), d(fx_{n+p+1}, Tx_{n+p+1})\Big\} \\
&= \max\Big\{d(y_{n+p-1}, y_{n+p}), d(y_{n+p-1}, y_{n+p}), d(y_{n+p}, y_{n+p+1})\Big\} \\
&= \max\Big\{d(y_{n+p-1}, y_{n+p}), d(y_{n+p}, y_{n+p+1})\Big\}.
\end{aligned}$$

If $R(x_{n+p}, x_{n+p+1}) = d(y_{n+p}, y_{n+p+1})$, then from (4.2.7) we obtain that

$$\psi(d(y_n, y_{n+1})) = \psi(d(y_{n+p}, y_{n+p+1})) \leq [\psi(d(y_{n+p}, y_{n+p+1}))]^{\lambda} < \psi(d(y_{n+p}, y_{n+p+1})),$$

which is a contradiction. Hence $R(x_{n+p}, x_{n+p+1}) = d(y_{n+p-1}, y_{n+p})$. By (4.2.7), we obtain that

$$\psi(d(y_n, y_{n+1})) = \psi(d(y_{n+p}, y_{n+p+1})) \leq [\psi(d(y_{n+p-1}, y_{n+p}))]^\lambda < \psi(d(y_{n+p-1}, y_{n+p})).$$

Since ψ is nondecreasing, we have $d(y_n, y_{n+1}) < d(y_{n+p-1}, y_{n+p})$. Next, by using (4.2.1), we get that

$$\begin{aligned}\psi(d(y_{n+p-1}, y_{n+p})) &\leq \alpha(fx_{n+p-1}, fx_{n+p})\psi(d(Tx_{n+p-1}, Tx_{n+p}))\\ &\leq [\psi(R(x_{n+p-1}, x_{n+p}))]^\lambda, \end{aligned} \tag{4.2.8}$$

where

$$\begin{aligned}&R(x_{n+p-1}, x_{n+p})\\ &= \max\left\{d(fx_{n+p-1}, fx_{n+p}), d(fx_{n+p-1}, Tx_{n+p-1}), d(fx_{n+p}, Tx_{n+p})\right\}\\ &= \max\left\{d(y_{n+p-2}, y_{n+p-1}), d(y_{n+p-2}, y_{n+p-1}), d(y_{n+p-1}, y_{n+p})\right\}\\ &= \max\left\{d(y_{n+p-2}, y_{n+p-1}), d(y_{n+p-1}, y_{n+p})\right\}.\end{aligned}$$

If $R(x_{n+p-1}, x_{n+p}) = d(y_{n+p-1}, y_{n+p})$, then by (4.2.8) we obtain that

$$\psi(d(y_{n+p-1}, y_{n+p})) \leq [\psi(d(y_{n+p-1}, y_{n+p}))]^\lambda < \psi(d(y_{n+p-1}, y_{n+p})),$$

which is a contradiction. Hence $R(x_{n+p-1}, x_{n+p}) = d(y_{n+p-2}, y_{n+p-1})$. By (4.2.8), we have

$$\psi(d(y_{n+p-1}, y_{n+p})) \leq [\psi(d(y_{n+p-2}, y_{n+p-1}))]^\lambda < \psi(d(y_{n+p-2}, y_{n+p-1})).$$

Since ψ is nondecreasing, we have $d(y_{n+p-1}, y_{n+p}) < d(y_{n+p-2}, y_{n+p-1})$. By continuing this process, we obtain the following inequality

$$d(y_n, y_{n+1}) < d(y_{n+p-1}, y_{n+p}) < d(y_{n+p-2}, y_{n+p-1}) < \ldots < d(y_n, y_{n+1}),$$

which is a contradiction and hence $p = 1$. We deduce that T and f have a point of coincidence. We can assume that $y_n \neq y_m$ for $n \neq m$. We now prove that

$\{d(y_n, y_{n+2})\}$ is bounded. Since $\{d(y_n, y_{n+1})\}$ is bounded, there exists $M > 0$ such that

$$d(y_n, y_{n+1}) \leq M \text{ for all } n \in \mathbb{N}.$$

If $d(y_n, y_{n+2}) > M$ for all $n \in \mathbb{N}$, then from

$$\begin{aligned} R(x_n, x_{n+2}) &= \max\Big\{d(fx_n, fx_{n+2}), d(fx_n, Tx_n), d(fx_{n+2}, Tx_{n+2}\Big\} \\ &= \max\Big\{d(y_{n-1}, y_{n+1}), d(y_{n-1}, y_n), d(y_{n+1}, y_{n+2})\Big\} \\ &= d(y_{n-1}, y_{n+1}), \end{aligned}$$

and Lemma 4.2.2, we obtain that

$$\begin{aligned} \psi(d(y_n, y_{n+2})) &= \psi(d(Tx_n, Tx_{n+2})) \\ &\leq \alpha(fx_n, fx_{n+2})\psi(d(Tx_n, Tx_{n+2})) \\ &\leq [\psi(R(x_n, x_{n+2}))]^\lambda \\ &= [\psi(d(y_{n-1}, y_{n+1}))]^\lambda \\ &< \psi(d(y_{n-1}, y_{n+1})). \end{aligned}$$

This implies that $\{d(y_n, y_{n+2})\}$ is decreasing. Therefore $\{d(y_n, y_{n+2})\}$ is bounded. If $d(y_n, y_{n+2}) \leq M$ for some $n \in \mathbb{N}$, then from

$$\begin{aligned} R(x_{n+1}, x_{n+3}) &= \max\Big\{d(fx_{n+1}, fx_{n+3}), d(fx_{n+1}, Tx_{n+1}), d(fx_{n+3}, Tx_{n+3}\Big\} \\ &= \max\Big\{d(y_n, y_{n+2}), d(y_n, y_{n+1}), d(y_{n+2}, y_{n+3})\Big\}, \end{aligned}$$

and Lemma 4.2.2, we obtain that

$$\begin{aligned} \psi(d(y_{n+1}, y_{n+3})) &= \psi(d(Tx_{n+1}, Tx_{n+3})) \\ &\leq \alpha(fx_{n+1}, fx_{n+3})\psi(d(Tx_{n+1}, Tx_{n+3})) \\ &\leq [\psi(R(x_{n+1}, x_{n+3}))]^\lambda \\ &\leq [\psi(M)]^\lambda \\ &< \psi(M), \end{aligned}$$

It follows that $d(y_{n+1}, y_{n+3}) < M$. This implies that $\{d(y_n, y_{n+2})\}$ is bounded. We next prove that $\lim_{n\to\infty} d(y_n, y_{n+2}) = 0$. Suppose that $\lim_{n\to\infty} d(y_n, y_{n+2}) \neq 0$. So there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$\lim_{k\to\infty} d(y_{n_k}, y_{n_k+2}) = a \text{ for some } a > 0.$$

Using (4.2.1), we have

$$\begin{aligned}\psi(d(y_{n_k}, y_{n_k+2})) &= \psi(d(Tx_{n_k}, Tx_{n_k+2}))\\ &\leq \alpha(fx_{n_k}, fx_{n_k+2})\psi(d(Tx_{n_k}, Tx_{n_k+2}))\\ &\leq [\psi(R(x_{n_k}, x_{n_k+2}))]^{\lambda},\end{aligned}$$

where

$$\begin{aligned}R(x_{n_k}, x_{n_k+2}) &= \max\Big\{d(fx_{n_k}, fx_{n_k+2}), d(fx_{n_k}, Tx_{n_k}), d(fx_{n_k+2}, Tx_{n_k+2})\Big\}\\ &= \max\Big\{d(y_{n_k-1}, y_{n_k+1}), d(y_{n_k-1}, y_{n_k}), d(y_{n_k+1}, y_{n_k+2})\Big\}.\end{aligned}$$

Letting $k \to \infty$ in the above inequality, we obtain that

$$\psi(a) = \lim_{k\to\infty} \psi(d(y_{n_k}, y_{n_k+2})) \leq \lim_{k\to\infty} [\psi(R(x_{n_k}, x_{n_k+2}))]^{\lambda} = [\psi(a)]^{\lambda} < \psi(a),$$

which is a contradiction. Therefore

$$\lim_{n\to\infty} d(y_n, y_{n+2}) = 0. \tag{4.2.9}$$

We now prove that $\{y_n\}$ is a Cauchy sequence. Suppose that $\{y_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and two subsequences $\{y_{n_k}\}$ and $\{y_{m_k}\}$ of $\{y_n\}$ such that n_k is the smallest index with $n_k > m_k > k$ for which

$$d(y_{m_k}, y_{n_k}) \geq \varepsilon, \tag{4.2.10}$$

and

$$d(y_{m_k}, y_{n_k-1}) < \varepsilon. \tag{4.2.11}$$

By applying the rectangular inequality and using (4.2.10) and (4.2.11), we obtain that

$$
\begin{aligned}
\varepsilon &\leq d(y_{m_k}, y_{n_k}) \\
&\leq d(y_{m_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k}) \\
&< \varepsilon + d(y_{n_k-1}, y_{n_k-2}) + d(y_{n_k-2}, y_{n_k}).
\end{aligned}
$$

Letting $k \to \infty$ in the above inequality and using (4.2.6), we get that

$$\lim_{k\to\infty} d(y_{m_k}, y_{n_k}) = \varepsilon. \tag{4.2.12}$$

For each $k \in \mathbb{N}$, we have

$$
\begin{aligned}
R(x_{n_k}, x_{m_k}) &= \max\left\{d(fx_{n_k}, fx_{m_k}), d(fx_{n_k}, Tx_{n_k}), (fx_{m_k}, Tx_{m_k})\right\} \\
&= \max\left\{d(y_{n_k-1}, y_{m_k-1}), d(y_{n_k-1}, y_{n_k}), (y_{m_k-1}, y_{m_k})\right\}.
\end{aligned}
$$

By using (4.2.6) and (4.2.12), we obtain that

$$\lim_{k\to\infty} R(x_{n_k}, x_{m_k}) = \varepsilon. \tag{4.2.13}$$

By (4.2.12) and (4.2.13), there exists a positive integer k_0 such that

$$d(y_{n_k}, y_{m_k}) > 0 \quad \text{and} \quad R(x_{n_k}, x_{m_k}) > 0, \quad \text{for all} \quad k \geq k_0.$$

By Lemma 4.2.2 and using (4.2.1), we get

$$
\begin{aligned}
\psi(d(y_{n_k}, y_{m_k})) &= \psi(d(Tx_{n_k}, Tx_{m_k})) \\
&= \psi(d(Tx_{m_k}, Tx_{n_k})) \\
&\leq \alpha(fx_{m_k}, fx_{n_k})\psi(d(Tx_{m_k}, Tx_{n_k})) \\
&\leq [\psi(R(x_{m_k}, x_{n_k}))]^{\lambda} \\
&= [\psi(R(x_{n_k}, x_{m_k}))]^{\lambda},
\end{aligned}
$$

for all $n_k > m_k > k \geq k_0$. Letting $k \to \infty$ in this inequality, by (4.2.12) and (4.2.13) and the continuity of ψ, we obtain that

$$\psi(\varepsilon) = \lim_{k\to\infty} \psi(d(y_{n_k}, y_{m_k})) \leq \lim_{k\to\infty} [\psi(R(x_{n_k}, x_{m_k}))]^{\lambda} = [\psi(\varepsilon)]^{\lambda} < \psi(\varepsilon),$$

which is a contradiction. Therefore $\{y_n\}$ is a Cauchy sequence in X. Assume that fX is a complete BMS. It follows that $\{y_n\}$ converges to $z \in fX$. Thus there exists $x \in X$ such that $fx \in fX$ and $\lim_{n\to\infty} y_n = fx$. Therefore $\lim_{n\to\infty} fx_{n+1} = fx$. Since T is continuous with respect to f, we have

$$fx = \lim_{n\to\infty} fx_{n+2} = \lim_{n\to\infty} Tx_{n+1} = Tx.$$

Therefore x is a coincidence point of T and f. In the case of completeness of TX, we obtain that $\{y_n\}$ converges to $z \in TX \subseteq fX$.

We now prove that the point of coincidence of T and f is unique. Suppose that u and v are two coincidence points of T and f. Therefore $Tu = fu$ and $Tv = fv$. We will show that $fu = fv$. Suppose that $fu \neq fv$. By (v), we have $\alpha(fu, fv) \geq 1$ or $\alpha(fv, fu) \geq 1$. Suppose that $\alpha(fu, fv) \geq 1$. By condition (4.2.1), we obtain that

$$\psi(d(fu, fv)) = \psi(d(Tu, Tv)) \leq \alpha(fu, fv)\psi(d(Tu, Tv)) \leq [\psi(R(u, v))]^{\lambda},$$

where

$$\begin{aligned} R(u, v) &= \max\{d(fu, fv), d(fu, Tu), d(fv, Tv)\} \\ &= d(fu, fv). \end{aligned}$$

This implies that

$$\psi(d(fu, fv)) \leq [\psi(d(fu, fv))]^{\lambda} < \psi(d(fu, fv)),$$

which is a contradiction. Thus $fu = fv$. This implies that T and f have a unique point of coincidence. Since the pair $\{T, f\}$ is weakly compatible and by Proposition 2.2.33, we have that T and f have a unique common fixed point. □

Theorem 4.2.4. *Let (X, d) be a BMS and $T, f : X \to X$ be such that $TX \subseteq fX$ where one of these two subsets of X being complete. Suppose that $\alpha : X \times X \to [0, \infty)$ and the following conditions hold :*

(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \ \text{ implies } \ \alpha(fx, fy) \cdot \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}, \qquad (4.2.14)$$

where

$$R(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\};$$

(ii) there exists $x_1 \in X$ *such that* $\alpha(fx_1, Tx_1) \geq 1$*;*

(iii) T *is a triangular* f*-*α*-admissible mapping;*

(iv) if $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x) \geq 1$ *for all* $k \in \mathbb{N}$*;*

(v) either $\alpha(fu, fv) \geq 1$ *or* $\alpha(fv, fu) \geq 1$ *whenever* $fu = Tu$ *and* $fv = Tv$*.*

Then T *and* f *have a unique point of coincidence. Moreover, if the pair* $\{T, f\}$ *is weakly compatible, then* T *and* f *have a unique common fixed point.*

Proof. As in the proof of Theorem 4.2.3, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_n = fx_{n+1} = Tx_n, \quad \text{for all } n \in \mathbb{N},$$

$\alpha(fx_n, fx_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} fx_n = fx$. By (iv), there exists a subsequence $\{fx_{n_k}\}$ of $\{fx_n\}$ such that $\alpha(fx_{n_k}, fx) \geq 1$ for all $k \in \mathbb{N}$. We can suppose that $fx_{n_k} \neq Tx$. Applying inequality (4.2.14), we obtain that

$$\begin{aligned}\psi(d(Tx_{n_k}, Tx)) &\leq \alpha(fx_{n_k}, fx)\psi(d(Tx_{n_k}, Tx)) \\ &\leq [\psi(R(x_{n_k}, x))]^{\lambda},\end{aligned}$$

where

$$\begin{aligned}R(x_{n_k}, x) &= \max\Big\{d(fx_{n_k}, fx), d(fx_{n_k}, Tx_{n_k}), d(fx, Tx)\Big\} \\ &= \max\Big\{d(y_{n_k-1}, fx), d(y_{n_k-1}, y_{n_k}), d(fx, Tx)\Big\}.\end{aligned}$$

Taking the limit as $k \to \infty$ and since ψ is continuous, we obtain that

$$\lim_{k\to\infty} R(x_{n_k}, x) = d(fx, Tx).$$

We will prove that $fx = Tx$. Suppose that $fx \neq Tx$. Therefore

$$d(fx, Tx) \leq d(fx, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k}) + d(Tx_{n_k}, Tx).$$

It follows that

$$d(fx, Tx) \leq \lim_{k\to\infty} d(Tx_{n_k}, Tx).$$

Since ψ is continuous and nondecreasing, we obtain that

$$\psi(d(fx, Tx)) \leq \lim_{k\to\infty} \psi(d(Tx_{n_k}, Tx)) \leq [\psi(d(fx, Tx))]^{\lambda} < \psi(d(fx, Tx)),$$

which is a contradiction. Thus $fx = Tx$. Let $z = fx = Tx$. Hence z is a point of coincidence for T and f. As in the proof of Theorem 4.2.3, we obtain that T and f have a unique point of coincidence. Since the pair $\{T, f\}$ is weakly compatible and by Proposition 2.2.33, then we have that T and f have a unique common fixed point. □

Let X be a nonempty set. If (X, d) is a BMS and $(X, \preceq)$ is a partially ordered set, then $(X, d, \preceq)$ is called a partially ordered BMS. We say that $x, y \in X$ are comparable if $x \preceq y$ or $y \preceq x$. Let $(X, \preceq)$ be a partially ordered set and $T, f : X \to X$. A mapping T is called an f-nondecreasing mapping if $Tx \preceq Ty$ whenever $fx \preceq fy$ for all $x, y \in X$.

Using Theorem 4.2.4, we obtain the following theorem in the setting of partially ordered BMS spaces.

Theorem 4.2.5. *Let $(X, d, \preceq)$ be a partially ordered BMS and let T and f be self-mappings on X such that $TX \subseteq fX$. Assume that (fX, d) is a complete BMS. Suppose that the following conditions hold :*

(i) there exist $\psi \in \Psi_2$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$ with $fx \preceq fy$,

$$d(Tx, Ty) \neq 0 \ \ \textit{implies} \ \ \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}, \tag{4.2.15}$$

where

$$R(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\};$$

(ii) T is f-nondecreasing;

(iii) there exists $x_1 \in X$ such that $fx_1 \preceq Tx_1$;

(iv) if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$;

(v) fu and fv are comparable whenever $fu = Tu$ and $fv = Tv$.

Then T and f have a unique point of coincidence. Moreover, if the pair $\{T, f\}$ is weakly compatible, then T and f have a unique common fixed point.

Proof. Define a mapping $\alpha : X \times X \to [0, \infty)$ such that

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x, y \in X \text{ and } x \preceq y; \\ 0 & \text{otherwise.} \end{cases}$$

We first show that T is f-α-admissible. Let $x, y \in X$ such that $\alpha(fx, fy) \geq 1$. Therefore $fx \preceq fy$. Since T is f-nondecreasing, we have $Tx \preceq Ty$ and then $\alpha(Tx, Ty) \geq 1$. We next prove that T is a triangular f-α-admissible. Let $x, y \in X$ such that $\alpha(fx, fy) \geq 1$ and $\alpha(fy, Ty) \geq 1$. Then we have $fx \preceq fy$ and $fy \preceq Ty$. This implies that $fx \preceq Ty$. So $\alpha(fx, Ty) \geq 1$. Therefore T is a triangular f-α-admissible mapping. Since there exists $x_1 \in X$ such that $fx_1 \preceq Tx_1$, we have $\alpha(fx_1, Tx_1) \geq 1$. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. By definition of α, we have $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. By (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \preceq x$ for all $k \in \mathbb{N}$ and hence $\alpha(x_{n_k}, x) \geq 1$ for all $k \in \mathbb{N}$. Let $u, v \in X$ such that $fu = Tu$ and $fv = Tv$. Since fu and fv are comparable, then we have $fu \preceq fv$ or $fv \preceq fu$. This implies that $\alpha(fu, fv) \geq 1$ or $\alpha(fv, fu) \geq 1$. Finally, we prove that 4.2.4 holds. Let $x, y \in X$ and $d(Tx, Ty) \neq 0$. If $\alpha(fx, fy) = 1$, then $fx \preceq fy$ and then 4.2.4 holds. If $\alpha(fx, fy) = 0$, then 4.2.4 holds. It follows that all assumptions of Theorem 4.2.4 hold. By Theorem 4.2.4, we obtain that T and f have a unique common fixed point. □

CHAPTER V

GENERALIZED CONTRACTIONS WITH TRIANGULAR α-ORBITAL ADMISSIBLE MAPPINGS WITH RESPECT TO η ON PARTIAL RECTANGULAR METRIC SPACES

In this chapter, we introduce a notion of generalized contractions in the setting of partial rectangular metric spaces. The existence of fixed point theorems for generalized contractions with triangular α-orbital admissible mappings with respect to η in the complete partial rectangular metric spaces is proven. Moreover, we also give the example for supporting our main result.

5.1 The fixed point theorems

We now prove the following lemma needed in proving our result. The idea comes from [18] but the proof is slightly different.

Lemma 5.1.1. *Let (X,p) be a partial rectangular metric space and $\{x_n\}$ be a sequence in (X,p) such that $p(x_n,x)\to p(x,x)$ as $n\to\infty$ for some $x\in X$, $p(x,x)=0$ and $\lim_{n\to\infty} p(x_n,x_{n+1})=0$. Then $p(x_n,y)\to p(x,y)$ as $n\to\infty$ for all $y\in X$.*

Proof. Suppose that $x\neq y$. If $x_n=y$ for arbitrarily large n, then $p(y,x)=p(x,x)=p(y,y)$. Therefore $x=y$. Assume that $y\neq x_n$ for all $n\in\mathbb{N}$. We also suppose that $x_n\neq x$ for infinitely many n. Otherwise, the result is complete. It follows that we may assume that $x_n\neq x_m\neq x$ and $x_n\neq x_m\neq y$ for all $m,n\in\mathbb{N}$ with $m\neq n$. By the partial rectangular inequality, we have

$$\begin{aligned}p(y,x)&\leq p(y,x_n)+p(x_n,x_{n+1})+p(x_{n+1},x)-p(x_n,x_n)-p(x_{n+1},x_{n+1})\\&\leq p(y,x_n)+p(x_n,x_{n+1})+p(x_{n+1},x)\end{aligned}$$

and

$$\begin{aligned} p(y,x_n) &\le p(y,x)+p(x,x_{n+1})+p(x_{n+1},x_n)-p(x,x)-p(x_{n+1},x_{n+1}) \\ &\le p(y,x)+p(x,x_{n+1})+p(x_{n+1},x_n). \end{aligned}$$

Since $\lim_{n\to\infty} p(x_n,x_{n+1})=0$ and taking the limit as $n\to\infty$ in the above inequalities, we have

$$\limsup_{n\to\infty} p(y,x_n) \le p(y,x) \le \liminf_{n\to\infty} p(y,x_n).$$

Hence the proof is complete. □

Theorem 5.1.2. *Let (X,p) be a complete partial rectangular metric space, $T: X\to X$ be a mapping and let $\alpha,\eta : X\times X\to[0,\infty)$ be functions. Suppose that the following conditions hold :*

(1) there exist $\theta\in\Psi_1$, and $\lambda\in(0,1)$ such that for all $x,y\in X$,

$$p(Tx,Ty)>0 \text{ and } \alpha(x,y)\ge\eta(x,y) \text{ imply } \theta(p(Tx,Ty))\le[\theta(R(x,y))]^{\lambda}, \quad (5.1.1)$$

where

$$R(x,y)=\max\left\{p(x,y),p(x,Tx),p(y,Ty),\frac{p(x,Tx)p(y,Ty)}{1+p(x,y)}\right\};$$

(2) there exists $x_1\in X$ such that $\alpha(x_1,Tx_1)\ge\eta(x_1,Tx_1)$;

(3) T is a triangular α-orbital admissible mapping with respect to η;

(4) if $\{T^nx_1\}$ is a sequence in X such that $\alpha(T^nx_1,T^{n+1}x_1)\ge\eta(T^nx_1,T^{n+1}x_1)$ for all $n\in\mathbb{N}$ and $T^nx_1\to x\in X$ as $n\to\infty$, then there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^nx_1\}$ such that $\alpha(T^{n(k)}x_1,x)\ge\eta(T^{n(k)}x_1,x)$ for all $k\in\mathbb{N}$;

(5) θ is continuous;

(6) if z is a periodic point T, then $\alpha(z,Tz)\ge\eta(z,Tz)$.

Then T has a fixed point.

Proof. By (2), there exists $x_1\in X$ such that $\alpha(x_1,Tx_1)\ge\eta(x_1,Tx_1)$. Define the sequence $\{x_n\}$ in X by $x_n=Tx_{n-1}=T^nx_1$ for all $n\in\mathbb{N}$. By Lemma 2.2.29, we obtain that

$$\alpha(T^n x_1, T^{n+1}x_1) \geq \eta(T^n x_1, T^{n+1}x_1) \quad \text{for all } n \in \mathbb{N}. \tag{5.1.2}$$

If $T^n x_1 = T^{n+1}x_1$ for some $n \in \mathbb{N}$, then $T^n x_1$ is a fixed point of T. Thus we suppose that $T^n x_1 \neq T^{n+1}x_1$ for all $n \in \mathbb{N}$. That is $p(T^n x_1, T^{n+1}x_1) > 0$. Applying (5.1.1), for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\theta(p(T^n x_1, T^{n+1}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^n x_1))) \\ &\leq [\theta(R(T^{n-1}x_1, T^n x_1))]^\lambda,\end{aligned} \tag{5.1.3}$$

where

$$\begin{aligned}&R(T^{n-1}x_1, T^n x_1) \\ &= \max\Big\{p(T^{n-1}x_1, T^n x_1), p(T^{n-1}x_1, T(T^{n-1}x_1)), p(T^n x_1, T(T^n x_1)), \\ &\quad \frac{p(T^{n-1}x_1, T(T^{n-1}x_1))p(T^n x_1, T(T^n x_1))}{1 + p(T^{n-1}x_1, T^n x_1)}\Big\} \\ &= \max\Big\{p(T^{n-1}x_1, T^n x_1), p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1), \\ &\quad \frac{p(T^{n-1}x_1, T^n x_1)p(T^n x_1, T^{n+1}x_1)}{1 + p(T^{n-1}x_1, T^n x_1)}\Big\} \\ &= \max\{p(T^{n-1}x_1, T^n x_1), p(T^n x_1, T^{n+1}x_1)\}.\end{aligned}$$

If $R(T^{n-1}x_1, T^n x_1) = p(T^n x_1, T^{n+1}x_1)$. By (5.1.3), we have

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^n x_1, T^{n+1}x_1))]^\lambda.$$

This implies that

$$\ln[\theta(p(T^n x_1, T^{n+1}x_1))] \leq \lambda \ln[\theta(p(T^n x_1, T^{n+1}x_1))],$$

which is a contradiction with $\lambda \in (0,1)$. This implies that $R(T^{n-1}x_1, T^n x_1) = p(T^{n-1}x_1, T^n x_1)$ for all $n \in \mathbb{N}$. From (5.1.3), we obtain that

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^{n-1}x_1, T^n x_1))]^\lambda \quad \text{for all } n \in \mathbb{N}.$$

This implies that

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^{n-1}x_1, T^n x_1))]^\lambda \leq [\theta(p(T^{n-2}x_1, T^{n-1}x_1))]^{\lambda^2}$$

$$\leq \cdots \leq [\theta(p(x_1, Tx_1))]^{\lambda^n}.$$

It follows that

$$1 \leq \theta(p(T^n x_1, T^{n+1} x_1)) \leq [\theta(p(x_1, Tx_1))]^{\lambda^n} \quad \text{for all } n \in \mathbb{N}. \tag{5.1.4}$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+1} x_1)) = 1. \tag{5.1.5}$$

Since $\theta \in \Psi_1$, we have

$$\lim_{n\to\infty} p(T^n x_1, T^{n+1} x_1) = 0. \tag{5.1.6}$$

From $\theta \in \Psi_1$, there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(p(T^n x_1, T^{n+1} x_1)) - 1}{[p(T^n x_1, T^{n+1} x_1)]^r} = \ell.$$

Assume that $\ell < \infty$. Let $B = \frac{\ell}{2} > 0$. It follows that there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(p(T^n x_1, T^{n+1} x_1)) - 1}{[p(T^n x_1, T^{n+1} x_1)]^r} - \ell \right| \leq B \quad \text{for all } n \geq n_0.$$

This implies that

$$\frac{\theta(p(T^n x_1, T^{n+1} x_1)) - 1}{[p(T^n x_1, T^{n+1} x_1)]^r} \geq \ell - B = B \quad \text{for all } n \geq n_0.$$

Thus we have

$$n[p(T^n x_1, T^{n+1} x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1} x_1)) - 1] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Assume that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. It follows that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(p(T^n x_1, T^{n+1} x_1)) - 1}{[p(T^n x_1, T^{n+1} x_1)]^r} \geq B \quad \text{for all } n \geq n_0.$$

This implies that

$$n[p(T^n x_1, T^{n+1} x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1} x_1)) - 1] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. From the above two cases, there exist $A > 0$ and $n_0 \in N$ such that

$$n[p(T^n x_1, T^{n+1} x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1} x_1)) - 1] \quad \text{for all } n \geq n_0.$$

Using (5.1.4), we have

$$n[p(T^n x_1, T^{n+1} x_1)]^r \leq An([\theta(p(x_1, Tx_1))]^{\lambda^n} - 1) \quad \text{for all } n \geq n_0. \tag{5.1.7}$$

Taking the limit as $n \to \infty$ in the above inequality, we get that

$$\lim_{n\to\infty} n[p(T^n x_1, T^{n+1} x_1)]^r = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+1} x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_1. \tag{5.1.8}$$

We now prove that T has a periodic point. Suppose that T does not have periodic points. Thus $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. Using Lemma 4.2.5 and (5.1.1), we get that

$$\begin{aligned}\theta(p(T^n x_1, T^{n+2} x_1)) &= \theta(p(T(T^{n-1} x_1), T(T^{n+1} x_1))) \\ &\leq [\theta(R(T^{n-1} x_1, T^{n+1} x_1))]^\lambda,\end{aligned}$$

where

$$\begin{aligned}&R(T^{n-1} x_1, T^{n+1} x_1) \\ &= \max\Big\{p(T^{n-1} x_1, T^{n+1} x_1), p(T^{n-1} x_1, T(T^{n-1} x_1)), p(T^{n+1} x_1, T(T^{n+1} x_1)), \\ &\quad \frac{p(T^{n-1} x_1, T(T^{n-1} x_1)) p(T^{n+1} x_1, T(T^{n+1} x_1))}{1 + p(T^{n-1} x_1, T^{n+1} x_1)}\Big\} \\ &= \max\Big\{p(T^{n-1} x_1, T^{n+1} x_1), p(T^{n-1} x_1, T^n x_1), p(T^{n+1} x_1, T^{n+2} x_1), \\ &\quad \frac{p(T^{n-1} x_1, T^n x_1) p(T^{n+1} x_1, T^{n+2} x_1)}{1 + p(T^{n-1} x_1, T^{n+1} x_1)}\Big\} \\ &= \max\{p(T^{n-1} x_1, T^{n+1} x_1), p(T^{n-1} x_1, T^n x_1), p(T^{n+1} x_1, T^{n+2} x_1)\}.\end{aligned}$$

Thus we have

$$\theta(p(T^n x_1, T^{n+2} x_1))$$

$$\leq [\theta(\max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^n x_1), p(T^{n+1}x_1, T^{n+2}x_1)\})]^{\lambda}.$$

It follows that

$$\begin{aligned}&\theta(p(T^n x_1, T^{n+2}x_1)) \\ &\leq [\max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}]^{\lambda}.\end{aligned} \tag{5.1.9}$$

Let I be the set of $n \in \mathbb{N}$ such that

$$\begin{aligned}u_n &:= \max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\} \\ &= \theta(p(T^{n-1}x_1, T^{n+1}x_1)).\end{aligned}$$

If $|I| < \infty$, then there exists $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\begin{aligned}&\max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\} \\ &= \max\{\theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}.\end{aligned}$$

From (5.1.9), we obtain that

$$\begin{aligned}1 &\leq \theta(p(T^n x_1, T^{n+2}x_1)) \\ &\leq [\max\{\theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}]^{\lambda} \quad \text{for all } n \geq N.\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality and using (5.1.5), we get that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+2}x_1)) = 1.$$

If $|I| = \infty$, then we can find a subsequence of $\{u_n\}$, denoted by $\{u_n\}$, such that $u_n = \theta(p(T^{n-1}x_1, T^{n+1}x_1))$ for large n. From (5.1.9), we have

$$\begin{aligned}1 \leq \theta(p(T^n x_1, T^{n+2}x_1)) &\leq [\theta(p(T^{n-1}x_1, T^{n+1}x_1))]^{\lambda} \leq [\theta(p(T^{n-2}x_1, T^n x_1))]^{\lambda^2} \\ &\leq \cdots \leq [\theta(p(x_1, T^2 x_1))]^{\lambda^n},\end{aligned}$$

for large n. Taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+2}x_1)) = 1. \tag{5.1.10}$$

Then in all cases, we obtain that (5.1.10) holds. By using (5.1.10) and $\theta \in \Psi_1$, we get that

$$\lim_{n\to\infty} p(T^n x_1, T^{n+2} x_1) = 0.$$

As an analogous proof as above, from $\theta \in \Psi_1$, there exists $n_2 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+2} x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_2. \tag{5.1.11}$$

Let $M = \max\{n_1, n_2\}$. We consider the following two cases.

Case 1: If $m > 2$ is odd, then $m = 2L + 1$ for some $L \geq 1$. Using (5.1.8), for all $n \geq M$, we get that

$$\begin{aligned}
&p(T^n x_1, T^{n+m} x_1) \\
&\leq p(T^n x_1, T^{n+1} x_1) + p(T^{n+1} x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+2L+1} x_1) \\
&\qquad - p(T^{n+1} x_1, T^{n+1} x_1) - p(T^{n+2} x_1, T^{n+2} x_1) \\
&\leq p(T^n x_1, T^{n+1} x_1) + p(T^{n+1} x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+2L+1} x_1) \\
&\leq p(T^n x_1, T^{n+1} x_1) + p(T^{n+1} x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) \\
&\qquad + p(T^{n+3} x_1, T^{n+4} x_1) + p(T^{n+4} x_1, T^{n+2L+1} x_1) - p(T^{n+3} x_1, T^{n+3} x_1) \\
&\qquad - p(T^{n+4} x_1, T^{n+4} x_1) \\
&\leq p(T^n x_1, T^{n+1} x_1) + p(T^{n+1} x_1, T^{n+2} x_1) + p(T^{n+2} x_1, T^{n+3} x_1) \\
&\qquad + p(T^{n+3} x_1, T^{n+4} x_1) + p(T^{n+4} x_1, T^{n+2L+1} x_1) \\
&\;\;\vdots \\
&\leq p(T^n x_1, T^{n+1} x_1) + p(T^{n+1} x_1, T^{n+2} x_1) + \cdots + p(T^{n+2L} x_1, T^{n+2L+1} x_1) \\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \qquad\qquad (5.1.12) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}$$

Case 2: If $m > 2$ is even, then $m = 2L$ for some $L \geq 2$. Using (5.1.8) and (5.1.11), for all $n \geq M$, we get that

$$p(T^n x_1, T^{n+m} x_1)$$

$$
\begin{aligned}
&\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + p(T^{n+3}x_1, T^{n+2L}x_1) \\
&\quad - p(T^{n+2}x_1, T^{n+2}x_1) - p(T^{n+3}x_1, T^{n+3}x_1) \\
&\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + p(T^{n+3}x_1, T^{n+2L}x_1) \\
&\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + p(T^{n+3}x_1, T^{n+4}x_1) \\
&\quad + p(T^{n+4}x_1, T^{n+5}x_1) + p(T^{n+5}x_1, T^{2L}x_1) - p(T^{n+4}x_1, T^{n+4}x_1) \\
&\quad - p(T^{n+5}x_1, T^{n+5}x_1) \\
&\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + p(T^{n+3}x_1, T^{n+4}x_1) \\
&\quad + p(T^{n+4}x_1, T^{n+5}x_1) + p(T^{n+5}x_1, T^{2L}x_1) \\
&\ \vdots \\
&\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + \cdots + p(T^{n+2L-1}x_1, T^{n+2L}x_1) \\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \cdots + \frac{1}{(n+2L-1)^{1/r}} \qquad (5.1.13) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}
$$

From Case 1 and Case 2, we have

$$p(T^n x_1, T^{n+m}x_1) \leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \quad \text{for all } n \geq M. \qquad (5.1.14)$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$ is convergent (since $\frac{1}{r} > 1$) and (5.1.14), we have

$$\lim_{n,m\to\infty} p(T^n x_1, T^{n+m}x_1) = 0.$$

This implies that $\{T^n x_1\}$ is a Cauchy sequence in (X, p). By Lemma 2.2.11, we have $\{T^n x_1\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, then (X, d_p) is complete. This implies that there exists $z \in X$ such that $\lim_{n\to\infty} d_p(T^n x_1, z) = 0$. Using Lemma 2.2.10, we have $\lim_{n\to\infty} p(T^n x_1, z) = \lim_{n\to\infty} p(T^n x_1, T^n x_1) = p(z, z)$. By applying Proposition 2.2.8, we obtain that

$$
\begin{aligned}
2p(T^n x_1, z) &= d_p(T^n x_1, z) + p(T^n x_1, T^n x_1) + p(z, z) \\
&\leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1}x_1) + p(T^n x_1, z).
\end{aligned}
$$

Therefore $p(T^n x_1, z) \leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1}x_1)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain that $p(z, z) = \lim_{n\to\infty} p(T^n x_1, z) = 0$. We now suppose that

$p(z,Tz) > 0$. By condition (4), there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)}x_1, z) \geq \eta(T^{n(k)}x_1, z)$ for all $k \in \mathbb{N}$. Since $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ with $m \neq n$, without loss of generality, we can assume that $T^{n(k)+1}x_1 \neq Tz$. And applying the condition (5.1.1), we obtain that

$$\begin{aligned}\theta(p(T^{n(k)+1}x_1, Tz)) &= \theta(p(T(T^{n(k)}x_1), Tz)) \\ &\leq [\theta(R(T^{n(k)}x_1, z))]^{\lambda},\end{aligned}$$

where

$$\begin{aligned}R(T^{n(k)}x_1, z) &= \max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T(T^{n(k)}x_1)), p(z,Tz), \\ &\qquad \frac{p(T^{n(k)}x_1, T(T^{n(k)}x_1))p(z,Tz)}{1+p(T^{n(k)}x_1, z)}\Big\} \\ &= \max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z,Tz), \\ &\qquad \frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z,Tz)}{1+p(T^{n(k)}x_1, z)}\Big\}.\end{aligned}$$

Thus we have

$$\begin{aligned}\theta(p(T^{n(k)+1}x_1, Tz)) \leq \Big[\theta\Big(\max\Big\{&p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), \\ &p(z,Tz), \frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z,Tz)}{1+p(T^{n(k)}x_1, z)}\Big\}\Big)\Big]^{\lambda}.\end{aligned} \tag{5.1.15}$$

Taking the limit as $k \to \infty$ in (5.1.15), using the continuity of θ and Lemma 5.1.1, we obtain that

$$\theta(p(z,Tz)) \leq [\theta(p(z,Tz))]^{\lambda} < \theta(p(z,Tz)),$$

which is a contradiction. Thus we obtain that $p(z,Tz) = 0$. By Remark 2.2.6, we have $Tz = z$, which contradicts to the assumption that T does not have a periodic point. Thus T has a periodic point, say z of period q. Suppose that the set of fixed points of T is empty. Then we have $q > 1$ and $p(z,Tz) > 0$. By using (5.1.1) and condition (6), we get that

$$\theta(p(z,Tz)) = \theta(p(T^q z, T^{q+1}z)) \leq [\theta(p(z,Tz))]^{\lambda^q} < \theta(p(z,Tz)),$$

which is a contradiction. This implies that the set of fixed points of T is non-empty. Hence T has at least one fixed point. $\square$

Example 5.1.3. Let $X = \{0,1,2,3,4,5\}$ and define $p : X \times X \to [0,+\infty)$ such that

$$p(x,y) = \begin{cases} x & \text{if } x = y; \\ \frac{2x+y}{2} & \text{if } x,y \in \{0,1,2\},\ x \neq y; \\ \frac{2+x+2y}{2} & \text{otherwise.} \end{cases}$$

Then (X,p) is a complete partial rectangular metric space. Since, for all $x \in X$ and $x > 0$, then we have $p(x,x) = x > 0$. Therefore (X,p) is not a rectangular metric space.

Define a mapping $T : X \to X$ by

$$T0 = T1 = T4 = 0, T2 = T3 = 2, \text{ and } T5 = 4.$$

Let $\alpha, \eta : X \times X \to [0,+\infty)$ be functions defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in \{0,1,2,3\}; \\ 0 & \text{otherwise.} \end{cases}$$

$$\eta(x,y) = \begin{cases} \frac{1}{2} & \text{if } x,y \in \{0,1,2,3\}; \\ 1 & \text{otherwise.} \end{cases}$$

Also define $\psi : (0,\infty) \to (1,\infty)$ by $\psi(t) = e^{\sqrt{t}}$. We next illustrate that all conditions in Theorem 5.1.2 hold. Taking $x_1 = 1$, we have $\alpha(1,T1) = \alpha(1,0) = 1 \geq \frac{1}{2} = \eta(1,0) = \eta(1,T1)$. Next, we prove that T is α-orbital admissible with respect to η. Let $\alpha(x,Tx) \geq \eta(x,Tx)$. Thus $x, Tx \in \{0,1,2,3\}$. By the definitions of a, η, we obtain that

$$\alpha(T0,T^20) = \alpha(0,0) = 1 \geq \frac{1}{2} = \eta(0,0) = \eta(T0,T^20),$$

$$\alpha(T1,T^21) = \alpha(0,0) = 1 \geq \frac{1}{2} = \eta(0,0) = \eta(T1,T^21),$$

$$\alpha(T2,T^22) = \alpha(2,2) = 1 \geq \frac{1}{2} = \eta(2,2) = \eta(T2,T^22),$$

$$\alpha(T3, T^2 3) = \alpha(2,2) = 1 \geq \frac{1}{2} = \eta(2,2) = \eta(T3, T^2 3).$$

It follows that T is α-orbital admissible with respect to η. Let $\alpha(x,y) \geq \eta(x,y)$ and $\alpha(y,Ty) \geq \eta(y,Ty)$. By definitions of α, η, we have $x, y, Ty \in \{0,1,2,3\}$. This yields

$$\alpha(0,0) \geq \eta(0,0) \text{ and } \alpha(0,T0) \geq \eta(0,T0) \text{ imply } \alpha(0,T0) \geq \eta(0,T0),$$

$$\alpha(0,1) \geq \eta(0,1) \text{ and } \alpha(1,T1) \geq \eta(1,T1) \text{ imply } \alpha(0,T1) \geq \eta(0,T1),$$

$$\alpha(0,2) \geq \eta(0,2) \text{ and } \alpha(2,T2) \geq \eta(2,T2) \text{ imply } \alpha(0,T2) \geq \eta(0,T2),$$

$$\alpha(0,3) \geq \eta(0,3) \text{ and } \alpha(3,T3) \geq \eta(3,T3) \text{ imply } \alpha(0,T3) \geq \eta(0,T3),$$

$$\alpha(1,0) \geq \eta(1,0) \text{ and } \alpha(0,T0) \geq \eta(0,T0) \text{ imply } \alpha(1,T0) \geq \eta(1,T0),$$

$$\alpha(1,1) \geq \eta(1,1) \text{ and } \alpha(1,T1) \geq \eta(1,T1) \text{ imply } \alpha(1,T1) \geq \eta(1,T1),$$

$$\alpha(1,2) \geq \eta(1,2) \text{ and } \alpha(2,T2) \geq \eta(2,T2) \text{ imply } \alpha(1,T2) \geq \eta(1,T2),$$

$$\alpha(1,3) \geq \eta(1,3) \text{ and } \alpha(3,T3) \geq \eta(3,T3) \text{ imply } \alpha(1,T3) \geq \eta(1,T3),$$

$$\alpha(2,0) \geq \eta(2,0) \text{ and } \alpha(0,T0) \geq \eta(0,T0) \text{ imply } \alpha(2,T0) \geq \eta(2,T0),$$

$$\alpha(2,1) \geq \eta(2,1) \text{ and } \alpha(1,T1) \geq \eta(1,T1) \text{ imply } \alpha(2,T1) \geq \eta(2,T1),$$

$$\alpha(2,2) \geq \eta(2,2) \text{ and } \alpha(2,T2) \geq \eta(2,T2) \text{ imply } \alpha(2,T2) \geq \eta(2,T2),$$

$$\alpha(2,3) \geq \eta(2,3) \text{ and } \alpha(3,T3) \geq \eta(3,T3) \text{ imply } \alpha(2,T3) \geq \eta(2,T3),$$

$$\alpha(3,0) \geq \eta(3,0) \text{ and } \alpha(0,T0) \geq \eta(0,T0) \text{ imply } \alpha(3,T0) \geq \eta(3,T0),$$

$$\alpha(3,1) \geq \eta(3,1) \text{ and } \alpha(1,T1) \geq \eta(1,T1) \text{ imply } \alpha(3,T1) \geq \eta(3,T1),$$

$$\alpha(3,2) \geq \eta(3,2) \text{ and } \alpha(2,T2) \geq \eta(2,T2) \text{ imply } \alpha(3,T2) \geq \eta(3,T2),$$

$$\alpha(3,3) \geq \eta(3,3) \text{ and } \alpha(3,T3) \geq \eta(3,T3) \text{ imply } \alpha(3,T3) \geq \eta(3,T3).$$

This implies that T is triangular α-orbital admissible with respect to η. Afterward, let $\{T^n x_1\}$ be a sequence such that $\alpha(T^n x_1, T^{n+1} x_1) \geq \eta(T^n x_1, T^{n+1} x_1)$ for all $n \in \mathbb{N}$ and $T^n x_1 \to x$ as $n \to \infty$. By the definitions of α, η for each $n \in \mathbb{N}$, we get $T^n x_1 \in \{0,1,2,3\}$. We obtain that $x \in \{0,1,2,3\}$. Thus we have $\alpha(T^n x_1, x) \geq$

$\eta(T^n x_1, x)$ for each $n \in \mathbb{N}$. Let $x, y \in X$ be such that $p(Tx, Ty) > 0$. We could observe that if $x, y \in \{0, 1, 4\}$, then $Tx = Ty = 0$. This implies that $p(Tx, Ty) = 0$. So we consider the following cases:

- $x \in \{0, 1, 4\}$ and $y \in \{2, 3\}$ or
- $x \in \{0, 1, 4\}$ and $y = 5$ or
- $x = \{2, 3\}$ and $y = 5$.

If $x = 4$ and $y \in \{2, 3\}$ or $x \in \{0, 1, 4\}$ and $y = 5$ or $x = \{2, 3\}$ and $y = 5$, then we have $\alpha(x, y) \ngeq \eta(x, y)$. We divide the proof into four cases as follows:

(1) If $(x, y) \in \{(0, 2), (2, 0)\}$, then

$$\begin{aligned} R(0, 2) &= \max\left\{p(0, 2), p(0, 0), p(2, 2), \frac{p(0, 0)p(2, 2)}{1 + p(0, 2)}\right\} \\ &= \max\left\{1, 0, 2, 0\right\} \\ &= 2. \end{aligned}$$

This implies that

$$\psi(p(T0, T2)) = \psi(p(0, 2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{2}}]^{0.71} = [\psi(2)]^{0.71} \leq [\psi(R(0, 2))]^{0.71}.$$

Therefore

$$\psi(p(T0, T2)) \leq [\psi(R(0, 2))]^{0.71}.$$

Since $p(x, y) = p(y, x)$ for all $x, y \in X$, we also obtain that

$$\psi(p(T2, T0)) \leq [\psi(R(2, 0))]^{0.71}.$$

(2) If $(x, y) \in \{(1, 2), (2, 1)\}$, then

$$\begin{aligned} R(1, 2) &= \max\left\{p(1, 2), p(1, 0), p(2, 2), \frac{p(1, 0)p(2, 2)}{1 + p(1, 2)}\right\} \\ &= \max\left\{2, 1, 2, \frac{2}{3}\right\} \end{aligned}$$

$$= 2.$$

This implies that

$$\psi(p(T1,T2)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{2}}]^{0.71} = [\psi(2)]^{0.71} \leq [\psi(R(1,2))]^{0.71}.$$

Therefore

$$\psi(p(T1,T2)) \leq [\psi(R(1,2))]^{0.71}.$$

Since $p(x,y) = p(y,x)$ for all $x, y \in X$, we also obtain that

$$\psi(p(T2,T1)) \leq [\psi(R(2,1))]^{0.71}.$$

(3) If $(x,y) \in \{(0,3),(3,0)\}$, then

$$\begin{aligned} R(0,3) &= \max\left\{p(0,3), p(0,0), p(3,2), \frac{p(0,0)p(3,2)}{1+p(0,3)}\right\} \\ &= \max\left\{4, 0, \frac{9}{2}, 0\right\} \\ &= \frac{9}{2}. \end{aligned}$$

This implies that

$$\psi(p(T0,T3)) = \psi(p(0,2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{\frac{9}{2}}}]^{0.5} = [\psi(\frac{9}{2})]^{0.5} \leq [\psi(R(0,3))]^{0.5}.$$

Therefore

$$\psi(p(T0,T3)) \leq [\psi(R(0,3))]^{0.5}.$$

Since $p(x,y) = p(y,x)$ for all $x, y \in X$, we also obtain that

$$\psi(p(T3,T0)) \leq [\psi(R(3,0))]^{0.5}.$$

(4) If $(x,y) \in \{(1,3),(3,1)\}$, then

$$\begin{aligned} R(1,3) &= \max\left\{p(1,3), p(1,0), p(3,2), \frac{p(1,0)p(3,2)}{1+p(1,3)}\right\} \\ &= \max\left\{\frac{9}{2}, 1, \frac{9}{2}, \frac{9}{11}\right\} \end{aligned}$$

$$= \frac{9}{2}.$$

This implies that

$$\psi(p(T1, T3)) = \psi(p(0, 2)) = \psi(1) = e^{\sqrt{1}} \leq [e^{\sqrt{\frac{9}{2}}}]^{0.5} = [\psi(\frac{9}{2})]^{0.5} \leq [\psi(R(1, 3))]^{0.5}.$$

Therefore

$$\psi(p(T1, T3)) \leq [\psi(R(1, 3))]^{0.5}.$$

Since $p(x, y) = p(y, x)$ for all $x, y \in X$, we also obtain that

$$\psi(p(T3, T1)) \leq [\psi(R(3, 1))]^{0.5}.$$

It follows that if $x, y \in X$, $p(Tx, Ty) > 0$ and $\alpha(x, y) \geq \eta(x, y)$, Then $\psi(p(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda}$. Hence all assumptions in Theorem 5.1.2 are satisfied and thus T has a fixed point which are $x = 0$ and $x = 2$.

Theorem 5.1.4. *Let (X, p) be a complete partial rectangular metric space, $T : X \to X$ be a mapping and let $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Suppose that the following conditions hold :*

(1) there exist $\theta \in \Psi_1$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$p(Tx, Ty) > 0 \text{ and } \alpha(x, y) \geq \eta(x, y) \text{ imply } \theta(p(Tx, Ty)) \leq [\theta(R(x, y))]^{\lambda}, \quad (5.1.16)$$

where

$$R(x, y) = \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Tx)p(y, Ty)}{1 + p(x, y)}\right\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$ and $\alpha(x_1, T^2x_1) \geq \eta(x_1, T^2x_1)$;

(3) T is an α-orbital admissible mapping with respect to η;

(4) T is an α-orbital attractive mapping with respect to η;

(5) θ is continuous;

(6) if z is a periodic point of T, then $\alpha(z, Tz) \geq \eta(z, Tz)$.

Then T has a fixed point.

Proof. By (2), there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$ and $\alpha(x_1, T^2x_1) \geq \eta(x_1, T^2x_1)$. Define the iterative sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1} = T^nx_1$ for all $n \in \mathbb{N}$. Since T is an α-orbital admissible mapping with respect to η, we obtain that

$$\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1) \text{ implies } \alpha(Tx_1, T^2x_1) \geq \eta(Tx_1, T^2x_1)$$

and

$$\alpha(x_1, T^2x_1) \geq \eta(x_1, T^2x_1) \text{ implies } \alpha(Tx_1, T^3x_1) \geq \eta(Tx_1, T^3x_1).$$

By continuing this process, we get that

$$\alpha(T^nx_1, T^{n+1}x_1) \geq \eta(T^nx_1, T^{n+1}x_1) \quad \text{for all } n \in \mathbb{N} \tag{5.1.17}$$

and

$$\alpha(T^nx_1, T^{n+2}x_1) \geq \eta(T^nx_1, T^{n+2}x_1) \quad \text{for all } n \in \mathbb{N}. \tag{5.1.18}$$

If $T^nx_1 = T^{n+1}x_1$ for some $n \in \mathbb{N}$, then T^nx_1 is a fixed point of T. Thus we suppose that $T^nx_1 \neq T^{n+1}x_1$ for all $n \in \mathbb{N}$. That is $p(T^nx_1, T^{n+1}x_1) > 0$. Applying the condition (5.1.16) and (5.1.17), for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} \theta(p(T^nx_1, T^{n+1}x_1)) &= \theta(p(T(T^{n-1}x_1), T(T^nx_1))) \\ &\leq [\theta(R(T^{n-1}x_1, T^nx_1))]^{\lambda}, \end{aligned} \tag{5.1.19}$$

where

$$\begin{aligned} &R(T^{n-1}x_1, T^nx_1) \\ &= \max\Big\{p(T^{n-1}x_1, T^nx_1), p(T^{n-1}x_1, T(T^{n-1}x_1)), p(T^nx_1, T(T^nx_1)), \\ &\quad \frac{p(T^{n-1}x_1, T(T^{n-1}x_1))p(T^nx_1, T(T^nx_1))}{1 + p(T^{n-1}x_1, T^nx_1)}\Big\} \\ &= \max\Big\{p(T^{n-1}x_1, T^nx_1), p(T^{n-1}x_1, T^nx_1), p(T^nx_1, T^{n+1}x_1), \\ &\quad \frac{p(T^{n-1}x_1, T^nx_1)p(T^nx_1, T^{n+1}x_1)}{1 + p(T^{n-1}x_1, T^nx_1)}\Big\} \\ &= \max\{p(T^{n-1}x_1, T^nx_1), p(T^nx_1, T^{n+1}x_1)\}. \end{aligned}$$

If $R(T^{n-1}x_1, T^n x_1) = p(T^n x_1, T^{n+1}x_1)$. By using (5.1.19), we get that

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^n x_1, T^{n+1}x_1))]^{\lambda}.$$

This implies that

$$\ln[\theta(p(T^n x_1, T^{n+1}x_1))] \leq \lambda \ln[\theta(p(T^n x_1, T^{n+1}x_1))],$$

which is a contradiction with $\lambda \in (0,1)$. It follows that $R(T^{n-1}x_1, T^n x_1) = p(T^{n-1}x_1, T^n x_1)$ for all $n \in \mathbb{N}$. From (5.1.19), we obtain that

$$\theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(T^{n-1}x_1, T^n x_1))]^{\lambda} \quad \text{for all } n \in \mathbb{N}.$$

Therefore

$$\begin{aligned}\theta(p(T^n x_1, T^{n+1}x_1)) &\leq [\theta(p(T^{n-1}x_1, T^n x_1))]^{\lambda} \leq [\theta(p(T^{n-2}x_1, T^{n-1}x_1))]^{\lambda^2} \\ &\leq \cdots \leq [\theta(p(x_1, Tx_1))]^{\lambda^n}.\end{aligned}$$

It follows that

$$1 \leq \theta(p(T^n x_1, T^{n+1}x_1)) \leq [\theta(p(x_1, Tx_1))]^{\lambda^n} \quad \text{for all } n \in \mathbb{N}. \tag{5.1.20}$$

Taking the limit as $n \to \infty$, we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+1}x_1)) = 1. \tag{5.1.21}$$

Since $\theta \in \Psi_1$, we have

$$\lim_{n\to\infty} p(T^n x_1, T^{n+1}x_1) = 0.$$

From $\theta \in \Psi_1$, there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} = \ell.$$

Assume that $\ell < \infty$. Let $B = \frac{\ell}{2} > 0$. It follows that there exists $n_0 \in \mathbb{N}$ such that

$$\left|\frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} - \ell\right| \leq B \quad \text{for all } n \geq n_0.$$

This implies that

$$\frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} \geq \ell - B = B \quad \text{for all } n \geq n_0.$$

Thus we have

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. Assume that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. It follows that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(p(T^n x_1, T^{n+1}x_1)) - 1}{[p(T^n x_1, T^{n+1}x_1)]^r} \geq B \quad \text{for all } n \geq n_0.$$

This implies that

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0,$$

where $A = \frac{1}{B}$. From the above two cases, there exist $A > 0$ and $n_0 \in N$ such that

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An[\theta(p(T^n x_1, T^{n+1}x_1)) - 1] \quad \text{for all } n \geq n_0.$$

By using (5.1.20), we get that

$$n[p(T^n x_1, T^{n+1}x_1)]^r \leq An([\theta(p(x_1, Tx_1))]^{\lambda^n} - 1) \quad \text{for all } n \geq n_0. \tag{5.1.22}$$

Taking the limit as $n \to \infty$ in the inequality (5.1.22), we obtain that

$$\lim_{n\to\infty} n[p(T^n x_1, T^{n+1}x_1)]^r = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+1}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_1. \tag{5.1.23}$$

We now prove that T has a periodic point. Suppose that T does not have periodic points. Thus $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ such that $n \neq m$. Using condition (5.1.16) and (5.1.18), we get that

$$\theta(p(T^n x_1, T^{n+2}x_1)) = \theta(p(T(T^{n-1}x_1), T(T^{n+1}x_1)))$$

$$\leq [\theta(R(T^{n-1}x_1, T^{n+1}x_1))]^{\lambda},$$

where

$$\begin{aligned}
&R(T^{n-1}x_1, T^{n+1}x_1)\\
&= \max\Big\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T(T^{n-1}x_1)), p(T^{n+1}x_1, T(T^{n+1}x_1)),\\
&\quad \frac{p(T^{n-1}x_1, T(T^{n-1}x_1))p(T^{n+1}x_1, T(T^{n+1}x_1))}{1 + p(T^{n-1}x_1, T^{n+1}x_1)}\Big\}\\
&= \max\Big\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^{n}x_1), p(T^{n+1}x_1, T^{n+2}x_1),\\
&\quad \frac{p(T^{n-1}x_1, T^{n}x_1)p(T^{n+1}x_1, T^{n+2}x_1)}{1 + p(T^{n-1}x_1, T^{n+1}x_1)}\Big\}\\
&= \max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^{n}x_1), p(T^{n+1}x_1, T^{n+2}x_1)\}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\theta(p(T^{n}x_1, T^{n+2}x_1))\\
&\leq [\theta(\max\{p(T^{n-1}x_1, T^{n+1}x_1), p(T^{n-1}x_1, T^{n}x_1), p(T^{n+1}x_1, T^{n+2}x_1)\})]^{\lambda}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\theta(p(T^{n}x_1, T^{n+2}x_1))\\
&\leq [\max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^{n}x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}]^{\lambda}.
\end{aligned} \tag{5.1.24}$$

Let I be the set of $n \in \mathbb{N}$ such that

$$\begin{aligned}
u_n &:= \max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^{n}x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}\\
&= \theta(p(T^{n-1}x_1, T^{n+1}x_1)).
\end{aligned}$$

If $|I| < \infty$, then there exists $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\begin{aligned}
&\max\{\theta(p(T^{n-1}x_1, T^{n+1}x_1)), \theta(p(T^{n-1}x_1, T^{n}x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}\\
&= \max\{\theta(p(T^{n-1}x_1, T^{n}x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}.
\end{aligned}$$

From (5.1.24), we obtain that

$$1 \leq \theta(p(T^{n}x_1, T^{n+2}x_1))$$

$$\leq [\max\{\theta(p(T^{n-1}x_1, T^n x_1)), \theta(p(T^{n+1}x_1, T^{n+2}x_1))\}]^{\lambda} \quad \text{for all } n \geq N.$$

Taking the limit as $n \to \infty$ in the above inequality and using (5.1.21), we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+2}x_1)) = 1.$$

If $|I| = \infty$, then we can find a subsequence of $\{u_n\}$, denoted by $\{u_n\}$, such that $u_n = \theta(p(T^{n-1}x_1, T^{n+1}x_1))$ for large n. From (5.1.24), we have

$$1 \leq \theta(p(T^n x_1, T^{n+2}x_1)) \leq [\theta(p(T^{n-1}x_1, T^{n+1}x_1))]^{\lambda} \leq [\theta(p(T^{n-2}x_1, T^n x_1))]^{\lambda^2}$$
$$\leq \cdots \leq [\theta(p(x_1, T^2 x_1))]^{\lambda^n},$$

for large n. Taking the limit as $n \to \infty$ in the above inequality, we obtain that

$$\lim_{n\to\infty} \theta(p(T^n x_1, T^{n+2}x_1)) = 1. \tag{5.1.25}$$

Then in all cases, (5.1.25) holds. By using (5.1.25) and $\theta \in \Psi_1$, we get that

$$\lim_{n\to\infty} p(T^n x_1, T^{n+2}x_1) = 0.$$

As an analogous proof as above. Since $\theta \in \Psi_1$ there exists $n_2 \in \mathbb{N}$ such that

$$p(T^n x_1, T^{n+2}x_1) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_2. \tag{5.1.26}$$

Let $h = \max\{n_1, n_2\}$. We consider the following two cases.

Case 1: If $m > 2$ is odd, then $m = 2L + 1$ for some $L \geq 1$. Using (5.1.23), for all $n \geq h$, we obtain that

$$\begin{aligned}
&p(T^n x_1, T^{n+m}x_1)\\
&\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+2L+1}x_1)\\
&\qquad - p(T^{n+1}x_1, T^{n+1}x_1) - p(T^{n+2}x_1, T^{n+2}x_1)\\
&\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+2L+1}x_1)\\
&\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1)
\end{aligned}$$

$$
\begin{aligned}
&+ p(T^{n+3}x_1, T^{n+4}x_1) + p(T^{n+4}x_1, T^{n+2L+1}x_1) - p(T^{n+3}x_1, T^{n+3}x_1)\\
&- p(T^{n+4}x_1, T^{n+4}x_1)\\
&\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1)\\
&+ p(T^{n+3}x_1, T^{n+4}x_1) + p(T^{n+4}x_1, T^{n+2L+1}x_1)\\
&\vdots\\
&\leq p(T^n x_1, T^{n+1}x_1) + p(T^{n+1}x_1, T^{n+2}x_1) + \cdots + p(T^{n+2L}x_1, T^{n+2L+1}x_1)\\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}}\\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}
$$

Case 2: If $m > 2$ is even, then $m = 2L$ for some $L \geq 2$. Using (5.1.23) and (5.1.26), for all $n \geq h$, we get that

$$
\begin{aligned}
&p(T^n x_1, T^{n+m}x_1)\\
&\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + p(T^{n+3}x_1, T^{n+2L}x_1)\\
&- p(T^{n+2}x_1, T^{n+2}x_1) - p(T^{n+3}x_1, T^{n+3}x_1)\\
&\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + p(T^{n+3}x_1, T^{n+2L}x_1)\\
&\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + p(T^{n+3}x_1, T^{n+4}x_1)\\
&+ p(T^{n+4}x_1, T^{n+5}x_1) + p(T^{n+5}x_1, T^{2L}x_1) - p(T^{n+4}x_1, T^{n+4}x_1)\\
&- p(T^{n+5}x_1, T^{n+5}x_1)\\
&\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + p(T^{n+3}x_1, T^{n+4}x_1)\\
&+ p(T^{n+4}x_1, T^{n+5}x_1) + p(T^{n+5}x_1, T^{2L}x_1)\\
&\vdots\\
&\leq p(T^n x_1, T^{n+2}x_1) + p(T^{n+2}x_1, T^{n+3}x_1) + \cdots + p(T^{n+2L-1}x_1, T^{n+2L}x_1)\\
&\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}}\\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}.
\end{aligned}
$$

From Case 1 and Case 2, we obtain that

$$p(T^n x_1, T^{n+m} x_1) \leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \quad \text{for all } n \geq h. \tag{5.1.27}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/r}}$ is convergent (since $\frac{1}{r} > 1$) and (5.1.27), we have

$$\lim_{n,m\to\infty} p(T^n x_1, T^{n+m} x_1) = 0.$$

This implies that $\{T^n x_1\}$ is a Cauchy sequence in (X, p). By Lemma 2.2.11, we have $\{T^n x_1\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, then (X, d_p) is complete. This implies that there exists $z \in X$ such that $\lim_{n\to\infty} d_p(T^n x_1, z) = 0$. Using Lemma 2.2.10, we have $\lim_{n\to\infty} p(T^n x_1, z) = \lim_{n\to\infty} p(T^n x_1, T^n x_1) = p(z, z)$. By applying Proposition 2.2.8, we obtain that

$$\begin{aligned} 2p(T^n x_1, z) &= d_p(T^n x_1, z) + p(T^n x_1, T^n x_1) + p(z, z) \\ &\leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1) + p(T^n x_1, z). \end{aligned}$$

Therefore $p(T^n x_1, z) \leq d_p(T^n x_1, z) + p(T^n x_1, T^{n+1} x_1)$ for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain that $p(z, z) = \lim_{n\to\infty} p(T^n x_1, z) = 0$. We now prove that $z = Tz$. Suppose that $z \neq Tz$. Since T is α-orbital attractive with respect to η, we obtain that for all $n \in \mathbb{N}$,

$$\alpha(T^n x_1, z) \geq \eta(T^n x_1, z) \text{ or } \alpha(z, T^{n+1} x_1) \geq \eta(z, T^{n+1} x_1).$$

We divide the proof in two cases as follows.

(1) There exists an infinite subset J of $\mathbb{N}$ such that $\alpha(T^{n(k)} x_1, z) \geq \eta(T^{n(k)} x_1, z)$ for every $k \in J$.

(2) There exists an infinite subset L of $\mathbb{N}$ such that $\alpha(z, T^{n(k)+1} x_1) \geq \eta(z, T^{n(k)+1} x_1)$ for every $k \in L$.

For the case (1), since $T^n x_1 \neq T^m x_1$ for all $n, m \in \mathbb{N}$ with $n \neq m$, without loss of the generality, we can assume that $T^{n(k)+1} x_1 \neq z$ for all $k \in J$. Applying the condition (5.1.16), we get that

$$\theta(p(T^{n(k)+1} x_1, Tz)) = \theta(p(T(T^{n(k)} x_1), Tz))$$

$$\leq [\theta(R(T^{n(k)}x_1, z))]^{\lambda},$$

where

$$R(T^{n(k)}x_1, z) = \max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T(T^{n(k)}x_1)), p(z, Tz), \frac{p(T^{n(k)}x_1, T(T^{n(k)}x_1))p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}$$
$$= \max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z, Tz), \frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}.$$

Then we have

$$\theta(p(T^{n(k)+1}x_1, Tz)) \leq \Big[\theta\Big(\max\Big\{p(T^{n(k)}x_1, z), p(T^{n(k)}x_1, T^{n(k)+1}x_1), p(z, Tz), \frac{p(T^{n(k)}x_1, T^{n(k)+1}x_1)p(z, Tz)}{1 + p(T^{n(k)}x_1, z)}\Big\}\Big)\Big]^{\lambda}.$$

Taking the limit as $k \to \infty$ in the above equality, using the continuity of θ and Lemma 5.1.1, we obtain that

$$\theta(p(z, Tz)) \leq [\theta(p(z, Tz))]^{\lambda} < \theta(p(z, Tz)),$$

which is a contradiction. For the case (2), the proof is similar. Therefore $z = Tz$, which is a contradiction with the assumption that T does not have a periodic point. Thus T has a periodic point, say z of period q. Suppose that the set of fixed points of T is empty, Then we have $q > 1$ and $p(z, Tz) > 0$. Applying (5.1.16) and condition (6), we get that

$$\theta(p(z, Tz)) = \theta(p(T^q z, T^{q+1}z)) \leq [\theta(p(z, Tz))]^{\lambda} < \theta(p(z, Tz)),$$

which is a contradiction. Thus the set of fixed points of T is non-empty. Hence T has at least one fixed point. □

Since a rectangular metric space is a partial rectangular metric space, we immediately obtain the following results by applying Theorem 5.1.2 and Theorem 5.1.4.

Corollary 5.1.5. [20] *Let (X, d) be a complete rectangular metric space, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Suppose that the following conditions hold :*

(1) there exist $\theta \in \Psi_1$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \ \text{implies} \ \alpha(x, y)\theta(d(Tx, Ty)) \leq [\theta(R(x, y))]^{\lambda},$$

where

$$R(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\right\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;

(3) T is a triangular α-orbital admissible mapping;

(4) if $\{T^n x_1\}$ is a sequence in X such that $\alpha(T^n x_1, T^{n+1} x_1) \geq 1$ for all $n \in \mathbb{N}$ and $T^n x_1 \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{T^{n(k)} x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)} x_1, x) \geq 1$ for all $k \in \mathbb{N}$;

(5) θ is continuous.

Then T has a fixed point z in X and $\{T^n x_1\}$ converges to z.

Corollary 5.1.6. [20] *Let (X, d) be a complete rectangular metric space, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Suppose that the following conditions hold :*

(1) there exist $\theta \in \Psi_1$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \ \text{implies} \ \alpha(x, y)\theta(d(Tx, Ty)) \leq [\theta(R(x, y))]^{\lambda},$$

where

$$R(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\right\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$ and $\alpha(x_1, T^2 x_1) \geq 1$;

(3) T is an α-orbital admissible mapping;

(4) T is an α-orbital attractive mapping;

(5) θ is continuous.

Then T has a fixed point z in X and $\{T^n x_1\}$ converges to z.

CHAPTER VI

CONCLUSION

The following results are all results of this thesis:

6.1 Fixed point theorems for generalized Geraghty contractions in complete partial b-metric spaces

Theorem 6.1.1. *Let (X, p) be a complete partial b-metric space with $s \geq 1$ and let $\alpha, \eta : X \times X \to [0, \infty)$ be functions. Let $f : X \to X$ be a mapping. Suppose that there exist $\beta \in S$ and $\psi \in \Psi$ such that for all $x, y \in X$, $\alpha(x, fx) \geq \eta(x, fx)$ and $\alpha(y, fy) \geq \eta(y, fy)$ imply that*

$$\psi(sp(fx, fy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)),$$

where

$$M(x, y) = \max\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2s}[p(x, fy) + p(fx, y)]\}.$$

Assume that

(i) f is α-orbital admissible with respect to η;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, fx_n) \geq \eta(x_n, fx_n)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ converges to x, then $\alpha(x, fx) \geq \eta(x, fx)$.

Then f has a fixed point in X.

Theorem 6.1.2. *Let $(X, \preceq, p)$ be a complete partially ordered partial b-metric space with $s \geq 1$. Let $f : X \to X$ be a nondecreasing mapping. Suppose that there exist $\beta \in S$ and $\psi \in \Psi$ such that for all comparable $x, y \in X$,*

$$\psi(sp(fx, fy)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)),$$

where

$$M(x, y) = \max\{p(x, y), p(x, fx), p(y, fy), \frac{1}{2s}[p(x, fy) + p(fx, y)]\}.$$

Assume that

(i) there exists $x_0 \in X$ *such that* $x_0 \preceq fx_0$;

(ii) if $\{x_n\}$ *is a nondecreasing sequence that converges to* x *such that* $x_n \preceq fx_n$ *for all* $n \in \mathbb{N} \cup \{0\}$, *then* $x \preceq fx$.

Then f *has a fixed point* z *in* X.

In Theorem 6.1.1 and Theorem 6.1.2, if we put $\eta(x,y) = 1$ and $\psi(t) = t$, then we obtain the following result proved by Sastry [10].

Corollary 6.1.3. [10] *Let* (X,p) *be a complete partial b-metric space with* $s \geq 1$ *and let* $\alpha : X \times X \to [0,\infty)$ *be a function. Let* $f : X \to X$ *be a mapping. Suppose that there exists* $\beta \in S$ *such that for all* $x, y \in X$,

$$\alpha(x, fx)\alpha(y, fy)sp(fx, fy) \leq \beta(M(x,y))M(x,y),$$

where

$$M(x,y) = \max\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2s}[p(x,fy) + p(fx,y)]\}.$$

Assume that

(i) f *is* α*-orbital admissible;*

(ii) there exists $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$;

(iii) if $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, fx_n) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\{x_n\}$ *converges to* x, *then* $\alpha(x, fx) \geq 1$.

Then f *has a fixed point in* X.

Corollary 6.1.4. [10] *Let* $(X, \preceq, p)$ *be a complete partially ordered partial b-metric space with* $s \geq 1$. *Let* $f : X \to X$ *be a nondecreasing mapping. Suppose that there exists* $\beta \in S$ *such that for all comparable* $x, y \in X$,

$$sp(fx, fy) \leq \beta(M(x,y))M(x,y),$$

where

$$M(x,y) = \max\{p(x,y), p(x,fx), p(y,fy), \frac{1}{2s}[p(x,fy) + p(fx,y)]\}.$$

Assume that

(i) there exists $x_0 \in X$ *such that* $x_0 \preceq fx_0$;

(ii) if $\{x_n\}$ *is a nondecreasing sequence that converges to* x *such that* $x_n \preceq fx_n$ *for all* $n \in \mathbb{N} \cup \{0\}$, *then* $x \preceq fx$.

Then f *has a fixed point* z *in* X.

6.2 Fixed point theorems for generalized contractions with triangular α-orbital admissible mappings on Branciari metric spaces

Theorem 6.2.1. *Let* (X, d) *be a complete BMS,* $T : X \to X$ *and* $\alpha : X \times X \to [0, \infty)$. *Suppose that the following conditions hold:*

(i) there exist $\psi \in \Psi_2$ *and* $\lambda \in (0, 1)$ *such that for all* $x, y \in X$,

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(x, y)\psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda},$$

where

$$R(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\};$$

(ii) there exists $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$;

(iii) T *is a triangular* α*-orbital admissible mapping;*

(iv) T *is continuous;*

Then T *has a fixed point.*

Theorem 6.2.2. *Let* (X, d) *be a complete BMS,* $T : X \to X$ *and* $\alpha : X \times X \to [0, \infty)$. *Suppose that the following conditions hold :*

(i) there exist $\psi \in \Psi_2$ *and* $\lambda \in (0, 1)$ *such that for all* $x, y \in X$,

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(x, y) \cdot \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda},$$

where

$$R(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\};$$

(ii) there exists $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$;

(iii) T *is a triangular* α*-orbital admissible mapping;*

(iv) if $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x) \geq 1$ *for all* $k \in \mathbb{N}$.

Then T *has a fixed point.*

Theorem 6.2.3. *Let* (X, d) *be a BMS and* $T, f : X \to X$ *be such that* $TX \subseteq fX$ *where one of these two subsets of* X *being complete. Assume that* $\alpha : X \times X \to [0, \infty)$ *and suppose that the following conditions hold:*

(i) there exist $\psi \in \Psi_2$ *and* $\lambda \in (0, 1)$ *such that for all* $x, y \in X$,

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(fx, fy) \cdot \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^\lambda,$$

where

$$R(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\};$$

(ii) there exists $x_1 \in X$ *such that* $\alpha(fx_1, Tx_1) \geq 1$;

(iii) T *is a triangular* f*-*α*-admissible mapping;*

(iv) T *is continuous with respect to* f;

(v) either $\alpha(fu, fv) \geq 1$ *or* $\alpha(fv, fu) \geq 1$ *whenever* $fu = Tu$ *and* $fv = Tv$.

Then T *and* f *have a unique point of coincidence. Moreover, if the pair* $\{T, f\}$ *is weakly compatible, then* T *and* f *have a unique common fixed point.*

Theorem 6.2.4. *Let* (X, d) *be a BMS and* $T, f : X \to X$ *be such that* $TX \subseteq fX$ *where one of these two subsets of* X *being complete. Suppose that* $\alpha : X \times X \to [0, \infty)$ *and the following conditions hold :*

(i) there exist $\psi \in \Psi_2$ *and* $\lambda \in (0, 1)$ *such that for all* $x, y \in X$,

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(fx, fy) \cdot \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^\lambda,$$

where

$$R(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\};$$

(ii) there exists $x_1 \in X$ *such that* $\alpha(fx_1, Tx_1) \geq 1$*;*

(iii) T *is a triangular* f*-*α*-admissible mapping;*

(iv) if $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n_k}, x) \geq 1$ *for all* $k \in \mathbb{N}$*;*

(v) either $\alpha(fu, fv) \geq 1$ *or* $\alpha(fv, fu) \geq 1$ *whenever* $fu = Tu$ *and* $fv = Tv$*.*

Then T *and* f *have a unique point of coincidence. Moreover, if the pair* $\{T, f\}$ *is weakly compatible, then* T *and* f *have a unique common fixed point.*

Using Theorem 6.2.4, we obtain the following theorem in the setting of partially ordered BMS spaces.

Theorem 6.2.5. *Let* $(X, d, \preceq)$ *be a partially ordered BMS and let* T *and* f *be self-mappings on* X *such that* $TX \subseteq fX$*. Assume that* (fX, d) *is a complete BMS. Suppose that the following conditions hold :*

(i) there exist $\psi \in \Psi_2$ *and* $\lambda \in (0, 1)$ *such that for all* $x, y \in X$ *with* $fx \preceq fy$*,*

$$d(Tx, Ty) \neq 0 \ \text{ implies } \ \psi(d(Tx, Ty)) \leq [\psi(R(x, y))]^{\lambda},$$

where

$$R(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\};$$

(ii) T *is* f*-nondecreasing;*

(iii) there exists $x_1 \in X$ *such that* $fx_1 \preceq Tx_1$*;*

(iv) if $\{x_n\}$ *is a sequence in* X *such that* $x_n \preceq x_{n+1}$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$*, then there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $x_{n_k} \preceq x$ *for all* $k \in \mathbb{N}$*;*

(v) fu *and* fv *are comparable whenever* $fu = Tu$ *and* $fv = Tv$*.*

Then T *and* f *have a unique point of coincidence. Moreover, if the pair* $\{T, f\}$ *is weakly compatible, then* T *and* f *have a unique common fixed point.*

6.3 Generalized contractions with triangular α-orbital admissible mappings with respect to η on partial rectangular metric spaces

Theorem 6.3.1. *Let (X,p) be a complete partial rectangular metric space, $T : X \to X$ be a mapping and let $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Suppose that the following conditions hold :*

(1) there exist $\theta \in \Psi_1$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$p(Tx,Ty) > 0 \text{ and } \alpha(x,y) \geq \eta(x,y) \text{ imply } \theta(p(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda},$$

where

$$R(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Tx)p(y,Ty)}{1+p(x,y)}\right\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$;

(3) T is a triangular α-orbital admissible mapping with respect to η;

(4) if $\{T^n x_1\}$ is a sequence in X such that $\alpha(T^n x_1, T^{n+1}x_1) \geq \eta(T^n x_1, T^{n+1}x_1)$ for all $n \in \mathbb{N}$ and $T^n x_1 \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{T^{n(k)}x_1\}$ of $\{T^n x_1\}$ such that $\alpha(T^{n(k)}x_1, x) \geq \eta(T^{n(k)}x_1, x)$ for all $k \in \mathbb{N}$;

(5) θ is continuous;

(6) if z is a periodic point T, then $\alpha(z,Tz) \geq \eta(z,Tz)$.

Then T has a fixed point.

Theorem 6.3.2. *Let (X,p) be a complete partial rectangular metric space, $T : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0,\infty)$ be functions. Suppose that the following conditions hold :*

(1) there exist $\theta \in \Psi_1$ and $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$p(Tx,Ty) > 0 \text{ and } \alpha(x,y) \geq \eta(x,y) \text{ imply } \theta(p(Tx,Ty)) \leq [\theta(R(x,y))]^{\lambda},$$

where

$$R(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Tx)p(y,Ty)}{1+p(x,y)}\right\};$$

(2) there exists $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$ *and* $\alpha(x_1, T^2x_1) \geq \eta(x_1, T^2x_1)$*;*

(3) T *is an* α*-orbital admissible mapping with respect to* η*;*

(4) T *is an* α*-orbital attractive mapping with respect to* η*;*

(5) θ *is continuous;*

(6) if z *is a periodic point of* T*, then* $\alpha(z, Tz) \geq \eta(z, Tz)$*.*

Then T *has a fixed point.*

Since a rectangular metric space is a partial rectangular metric space, we immediately obtain the following results by applying Theorem 6.3.1 and Theorem 6.3.2.

Corollary 6.3.3. [20] *Let* (X, d) *be a complete rectangular metric space,* $T : X \to X$ *be a mapping and* $\alpha : X \times X \to [0, \infty)$ *be a function. Suppose that the following conditions hold :*

(1) there exist $\theta \in \Psi_1$ *and* $\lambda \in (0, 1)$ *such that for all* $x, y \in X$*,*

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(x, y)\theta(d(Tx, Ty)) \leq [\theta(R(x, y))]^\lambda,$$

where

$$R(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\right\};$$

(2) there exists $x_1 \in X$ *such that* $\alpha(x_1, Tx_1) \geq 1$*;*

(3) T *is a triangular* α*-orbital admissible mapping;*

(4) if $\{T^nx_1\}$ *is a sequence in* X *such that* $\alpha(T^nx_1, T^{n+1}x_1) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $T^nx_1 \to x \in X$ *as* $n \to \infty$*, then there exists a subsequence* $\{T^{n(k)}x_1\}$ *of* $\{T^nx_1\}$ *such that* $\alpha(T^{n(k)}x_1, x) \geq 1$ *for all* $k \in \mathbb{N}$*;*

(5) θ *is continuous.*

(6) if z *is a periodic point of* T*, then* $\alpha(z, Tz) \geq \eta(z, Tz)$*.*

Then T *has a fixed point.*

Corollary 6.3.4. [20] *Let (X, d) be a complete rectangular metric space, $T : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$ be a function. Suppose that the following conditions hold :*

(1) there exist $\theta \in \Psi_1$ and $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \neq 0 \text{ implies } \alpha(x, y)\theta(d(Tx, Ty)) \leq [\theta(R(x, y))]^{\lambda},$$

where

$$R(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}\right\};$$

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$ and $\alpha(x_1, T^2x_1) \geq 1$;

(3) T is an α-orbital admissible mapping;

(4) T is an α-orbital attractive mapping;

(5) θ is continuous.

(6) if z is a periodic point of T, then $\alpha(z, Tz) \geq \eta(z, Tz)$.

Then T has a fixed point.

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