FIXED POINT THEOREMS FOR SOME NONLINEAR MAPPINGS IN GENERALIZED METRIC SPACES

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ABSTRACT

In this thesis, we establish the following results. Firstly, we prove fixed point theorems for generalized multi-valued mappings satisfying some inequalities in metric spaces. Moreover, we present some examples to illustrate and support our results. Secondly, we introduce generalized metric spaces and prove basic properties in such spaces. Moreover, we prove fixed point theorems for the generalized Banach contraction, generalized Kannan mapping and present some examples as a satisfying the theorems in such spaces. Finally, we prove fixed point theorems for generalized nonexpansive mappings and approximate a fixed point for such mappings in hyperbolic spaces. Furthermore, we prove some properties of the set of fixed point for generalized nonexpansive mappings in hyperbolic spaces.

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CHAPTER I

INTRODUCTION

In mathematics, a fixed point theorem is a result saying that a self-mapping T on a nonempty set X, will have at least one fixed point (a point x for which $Tx = x$), under some conditions on T that can be stated in general terms [1]. Moreover, fixed point theory have useful to science, technology and the daily life of humans as well as academic progress. When the science problem is converted to the image of Mathematical Model which describes in the form of equations, an inequalities or operators and the question that follows is

(1) such equations, an inequalities or operators are solution or not.

(2) how to find those solution.

The education of fixed point theory is divided by two main story. Firstly, mathematicians study existence theorems which is to solve the problem (1). Secondly, they study convergence theorems which is to solve the problem (2).

The Banach fixed point theorem is an important tool in the theory of metric spaces, it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points was introduced by Banach [2] in 1922. Important, the study of fixed point theory started from such theorem as follows: Let T be a self-mapping on metric spaces X. Then T is called a *contraction mapping* if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y), \qquad \text{for all } x, y \in X.$$

In 1969, Kannan [3] extended the concept of Banach [2] and obtained the same conclusion as in Banach's Theorem but with different sufficient conditions as follows: Let T be a self-mapping on a metric space X. Then T is called a *Kannan* *mapping* if there exists $r \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq rd(x, Tx) + rd(y, Ty), \qquad \text{for all } x, y \in X.$$

In 1969, Nadler [4] combined the ideas of set-valued mapping and proved some fixed point theorems about multi-valued contraction mappings which extended the concept of Banach [2] as theorem follows: Let T be a mapping from X into the family of all nonempty closed bounded subsets of a metric space X. Define

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\},$$

for $A, B \in CB(X) := \{C | C \text{ is a nonempty closed and bounded subset of } X\}$, where $d(x, B) = \inf_{y\in B} d(x, y)$. A mapping T is said to be a *multi-valued contraction* if there exists $r \in [0, 1)$ such that

$$H(Tx, Ty) \leq rd(x, y), \qquad \text{for all } x, y \in X.$$

In 1972, Bianchini [5] introduced generalized Kannan mapping which generalized the concept of Kannan [3] as follows: Let T be a self-mapping on a metric space X. Then T is called a *generalized Kannan mapping* if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq r \max\{d(x, Tx), d(y, Ty)\}, \qquad \text{for all } x, y \in X.$$

The same year, Taylor [6] proved fixed point theorems for nonexpansive mappings which extended the concept of Banach [2] as follows: Let T be a self-mapping on a metric space X. Then T is called a *nonexpansive mapping* if

$$d(Tx, Ty) \leq d(x, y), \qquad \text{for all } x, y \in X.$$

In 1973, Hardy and Rogers [7] introduced condition as follow: Let T be a self-mapping on a metric space X. Then there exists $r_i \geq 0, \quad (i = 1, 2, 3, 4, 5)$ such that

$$d(Tx, Ty) \leq r_1 d(x, y) + r_2 d(x, Tx) + r_3 d(y, Ty) + r_4 d(x, Ty) + r_5 d(y, Tx),$$

where $\Sigma_{i=1}^{5} r_i < 1$, for all $x, y \in X$, which generalized the concept of Banach [2] and Kannan [3].

In 1974, Ciric [8] introduced the following condition which extended the results of Bianchini [5] and Hardy, Rogers [7] as follows: Let T be a self-mapping on a metric space X. Then there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq r \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \text{ for all } x, y \in X.$$

Moreover, if X is complete and at least one of above conditions holds, then T has a unique fixed point (see [2, 3, 4, 5, 6, 7, 8]).

In 2001, Rhoades [9] introduced weakly contractive mappings and proved some fixed point theorems for such mappings which generalized the concept of Banach [2] as follows: A mapping $T : X \to X$ is said to be a *weakly contractive* if for all $x, y \in X$,

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

In 2003, Kirk, et al. [10] introduced cyclic contraction mappings and proved some fixed point theorems for such mappings which generalized the concept of Banach [2] as follows: Let A and B be nonempty subsets of a metric space X and let $T : A \cup B \to A \cup B$. Then T is called a *cyclic map* iff $T(A) \subseteq B$ and $T(B) \subseteq A$. A cyclic map T is said to be a *cyclic contraction* if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y), \quad \text{for all } x \in A \text{ and } y \in B.$$

Next, Bhaskar and Lakshmikantham [11] established a fixed point theorem for mixed monotone mappings in partially ordered metric spaces in 2006, as follows: A mapping $T : X \times X \to X$ is called a *coupled contraction mapping* if there exists $r \in [0,1)$ such that

$$d(T(x,y),T(u,v)) \leq \tfrac{r}{2}[d(x,u)+d(x,u)], \text{ for all } x,y,u,v \in X.$$

In 2008, Suzuki [12] introduced condition C and proved some fixed point theorems for such mappings as follows: Let T be a self-mapping on complete metric spaces X. Then T is said to satisfy *condition C* if

$$\tfrac{1}{2}d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq d(x,y), \quad \text{for all } x,y \in X.$$

It is obvious that every nonexpansive mapping satisfies condition C, but the converse is not true.

The same year, Kikkawa and Suzuki [13] extended the concept of Kannan [3] by used condition C as theorem follows: Let T be a self-mapping on complete metric space (X,d) and let φ be a non-increasing function from $[0,1)$ into $(\frac{1}{2},1]$ defined by

$$\varphi(r) = \begin{cases} 1, & if \ \ 0 \leq r < \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & if \ \ \frac{1}{\sqrt{2}} \leq r < \frac{1}{2}, \end{cases}$$

for all $\alpha \in [0,\frac{1}{2})$ and put $r = \frac{\alpha}{1-\alpha} \in [0,1)$, such that

$$\varphi(r)d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq \alpha d(x,Tx) + \alpha d(y,Ty), \text{ for all } x,y \in X.$$

In 2008, Kikkawa and Suzuki [13] introduced the result which of was a generalization of the result of Nadler [4] as theorem follows: Let (X,d) be a complete metric space and let T be a mapping from X into the family of all nonempty closed bounded subsets of a metric space X. Define a strictly decreasing function η from $[0,1)$ into $(\frac{1}{2},1]$ by $\eta(r)=\frac{r}{1+r}$ and assume that there exists $r\in[0,1)$ such that

$$\eta(r)d(x,Tx)\leq d(x,y) \text{ implies } H(Tx,Ty)\leq rd(x,y), \text{ for all } x,y\in X.$$

In 2010, Karapinar and Erhan [14] proved some fixed point theorems for Kannan type cyclic contraction mappings which generalized the concept of Kannan [3] as follows: A cyclic map $T: A\cup B\to A\cup B$ is called a *Kannan type cyclic contraction* if there exists $r\in[0,\frac{1}{2})$ such that

$$d(Tx,Ty)\leq r[d(x,Tx)+d(y,Ty)], \text{ for all } x\in A \text{ and } y\in B.$$

In 2011, Karapinar and Tas [15] stated some new definitions which are modifications of condition C, as follows: Let T be a self-mapping on a metric space X.

(i) A mapping T is said to satisfy *condition KSC* if

$$\tfrac{1}{2}d(x,Tx)\leq d(x,y) \text{ implies } d(Tx,Ty)\leq \tfrac{1}{2}[d(x,Tx)+d(y,Ty)], \text{ for each } x,y\in K.$$

(ii) A mapping T is said to satisfy *condition CSC* if

$$\tfrac{1}{2}d(x,Tx)\leq d(x,y) \text{ implies } d(Tx,Ty)\leq \tfrac{1}{2}[d(y,Tx)+d(x,Ty)], \text{for each } x,y\in K.$$

(iii) A mapping T is said to satisfy *condition SKC* if

$$\tfrac{1}{2}d(x,Tx)\leq d(x,y) \text{ implies } d(Tx,Ty)\leq N(x,y), \text{ for each } x,y\in K,$$

where $N(x,y)=\max\{d(x,y),\frac{1}{2}[d(x,Tx)+d(y,Ty)],\frac{1}{2}[d(y,Tx)+d(x,Ty)]\}$.

(iv) A mapping T is said to satisfy *condition SCC* if

$$\tfrac{1}{2}d(x,Tx)\leq d(x,y) \text{ implies } d(Tx,Ty)\leq M(x,y), \text{ for each } x,y\in K,$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx), d(x,Ty)\}$. Moreover, it is clear, every condition C satisfies condition SCC, but the converse is not true as follows: Define a self-mapping T on $[0,3]$ by

$$Tx = \begin{cases} 0, & if \;\; x \neq 3, \\ 2, & if \;\; x = 3. \end{cases}$$

Then T does not satisfy condition C, but T satisfies condition SCC (see [15]).

In 2011, Damjanović and Dorić [16] generalized result of Kikkawa [13] and Kannan [3] as theorem follows: Let (X,d) be a complete metric space and let T be a mapping from X into the family of all nonempty closed bounded subsets of a metric space X. Define a non-increasing function φ from $[0,1)$ into $(0,1]$ by

$$\varphi(r) = \begin{cases} 1, & if \;\; 0 \leq r < \frac{\sqrt{5}-1}{2}, \\ 1-r, & if \;\; \frac{\sqrt{5}-1}{2} \leq r < 1, \end{cases}$$

such that

$$\varphi(r)d(x,Tx) \leq d(x,y) \text{ implies } H(Tx,Ty) \leq r\max\{d(x,Tx), d(y,Ty)\},$$

for all $x, y \in X$.

In 2012, Dhompongsa [17] introduced condition C_λ, and E_μ and proved some fixed point theorems for such condition as follows: Let T be a self-mapping on a subset K of a metric space X and $\mu \geq 1$. T is said to satisfy *condition* E_μ if

$$d(x,Ty) \leq \mu d(x,Tx) + d(x,y), \quad \text{for all } x, y \in K.$$

Moreover, T is said to satisfy *condition* E, whenever T satisfies the condition E_μ for some $\mu \geq 1$. Therefore, if T satisfies by one of the conditions SKC, KSC, SCC and CSC, then T satisfies condition E_μ for $\mu = 5$. Let T be a self-mapping on a subset K of a metric space X and $\lambda \in (0,1)$. A mapping T is said to satisfy *condition C_λ* if

$$\lambda d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq d(x,y), \quad \text{for all } x,y \in K.$$

In 2013, Zoto [18] introduced d-cyclic-ϕ-contraction mappings and proved some fixed point theorems which generalized the concept of Banach [2] as follows: A cyclic map $T : A \cup B \to A \cup B$ is said to be a *d-cyclic-ϕ-contraction* if $\phi \in \Phi$ such that

$$d(Tx,Ty) \leq \phi(d(x,y)), \text{ for all } x \in A,\ y \in B,$$

where Φ the family of nondecreasing functions: $\phi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$, where n is the n-th iterate of ϕ.

In 2014, Ghoncheh and Razani [19] introduced the following definition and recall some other conditions which generalize the Suzuki and study fixed point for some generalized nonexpansive mappings as follow: Let X be a metric space and K be a subset of X. A mapping $T : K \to K$ is said to be *fundamentally nonexpansive* if

$$d(T^2x,Ty) \leq d(Tx,y), \quad \text{for all } x,y \in K.$$

Moreover, every mapping which satisfies condition C is fundamentally nonexpansive (see Lemma 3 in [19]), but the converse is not true as follow: Suppose $X = \{(0,0),(0,1),(1,1),(1,2)\}$. Define

$$d((x_1,y_1),(x_2,y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

Define T on X by $T(0,0) = (1,2)$, $T(0,1) = (0,0)$, $T(1,1) = (1,1)$, $T(1,2) = (0,1)$. Then T is fundamentally nonexpansive, but T is not condition C. (see [19]).

Next, we discusses the development of spaces. Conceptions of quasi-metric spaces and b-metric spaces were introduced by Wilson [20] and Bakhtin [21] in 1931, 1963 and 1989 as a generalization of metric spaces, respectively. Later, in 2000, Hitzler and Seda [22] introduced dislocated metric space as a generalization of metric space, Zeyada, Hassan and Ahmad [23] introduced the concept of dislocated quasi-metric space as a generalization the result of Hitzler, Seda and Wilson. Finally in many other generalized b-metric space, such as, quasi b-metric space [24], b-metric-like space [25], and quasi b-metric-like space [26] as follows: Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ such that constant $b \geq 1$ satisfies the following conditions:

(d_1) $d(x, x) = 0$, for all $x \in X$;

(d_2) $d(x, y) = d(y, x) = 0$ implies $x = y$, for all $x, y \in X$;

(d_3) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(d_4) $d(x, y) \leq b[d(x, z) + (z, y)]$, for all $x, y, z \in X$.

If d satisfies the conditions $(d_1) - (d_4)$, then d is called a *b-metric space* on X, if d satisfies the conditions (d_1), (d_2) and (d_4), then d is called a *b-quasi metric* on X. Next, if d satisfies the conditions (d_2), (d_3) and (d_4), then d is called a *b-dislocated metric* on X, if d satisfies the conditions (d_2) and (d_4) then d is called a *b-dislocated quasi metric* on X. Moreover, b-Metric spaces, b-Quasi metric spaces, b-Dislocated metric spaces and b-Dislocated quasi metric spaces are called *metric spaces, quasi metric spaces, dislocated metric spaces* and *Dislocated quasi metric spaces* with $b = 1$, respectively (see [20, 21, 22, 23, 24, 25, 26]).

On the other hand, Banach Contraction Principle gave the result for an approximation of fixed point as follows: Let x_0 be an arbitrary but fixed element in X. Define a sequence of iterates $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n = 0, 1, 2, \ldots$. Then $\{x_n\}$ converges to a fixed point of T. Now, fixed point iteration processes for approximating fixed point of nonexpansive mappings have been studied many mathematicians such as Krasnoselskij's iteration [27], Halpern's iteration [28], Mann's iteration [29]. Further, The convergence theorems are studied the development of fixed point iteration processes, it is studied the development of spaces such as Hilbert spaces, Banach spaces, Hyperbolic spaces and others for conditions of mappings above (see: [12, 15, 19])

The purpose of this research is to establish the following results. Firstly, we prove fixed point theorems for generalized multi-valued mappings satisfying some inequalities in metric spaces. Moreover, we present some examples to illustrate and support our results. Secondly, we introduce generalized metric spaces and prove basic properties in such spaces. Other than, we prove fixed point theorems for the generalized Banach contraction, generalized Kannan mapping and present some examples as a satisfying the theorems in such spaces. Finally, we prove fixed point theorems for generalized nonexpansive mappings and approximate a fixed point for such mappings in hyperbolic spaces. Furthermore, we prove some properties of the set of fixed points for generalized nonexpansive mappings in hyperbolic spaces.

This thesis is divided into 6 chapters. Chapter 1 is an introduction to the origin and significance of the research problems. Chapter 2 we present the basic definitions, examples and results concerning that will be applied in our main results of this research. Chapter 3, Chapter 4 and Chapter 5 are the main results of of this research. Precisely, in Section 3.1 we introduce some nonlinear mappings and prove fixed point theorems for generalized multi-valued mappings satisfying some inequalities in metric spaces. In Section 3.2, we introduce the notions of type multi-valued-coupled contraction, multi-valued-coupled Kannan mapping and prove coupled fixed point theorems on metric spaces. Next, in Section 4.1, we establish dislocated quasi-b-metric spaces and prove basic properties of dislocated quasi-b-metric spaces. Moreover, we present some examples to illustrate and support our results. In Section 4.2, we introduce the notions of type dqb-cyclic-Banach contraction, dqb-cyclic-Kannan mapping, type dqb-cyclic-weak Banach contraction and dqb-cyclic- contraction. Moreover, we derive the existence of fixed point theorems on dislocated quasi-b-metric spaces and present some examples to illustrate and support our results. Next, in Section 5.1, we prove some properties of a fundamentally nonexpansive self-mapping on a nonempty subset of a hyperbolic space and prove convergence and Δ-convergence theorems of the generalized Krasnoselskij-type iterative process to approximate a fixed point for fundamentally nonexpansive operators in a hyperbolic space. In Section 5.2, we prove fixed point theorems for some generalized nonexpansive self-mappings on a nonempty subset of a hyperbolic space and approximate a fixed point for such mappings in a hyperbolic space. Chapter 6 is the conclusion of this research.

CHAPTER II

PRELIMINARIES

The aim of this chapter is to introduce the some basic definitions, notations and some results that will be used in the later chapter. Throughout this thesis, we let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the set of all natural numbers, the set of all real numbers and complex numbers, respectively.

2.1 Metric Spaces and some properties

In this section, we discuss various forms of metric spaces and generalized metric spaces.

Definition 2.1.1. ([30]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies the following conditions:

($d1$) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

($d2$) $d(x, y) = d(y, x)$ for all $x, y \in X$;

($d3$) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *metric* on X. The value of metric d at (x, y) is called *distance* between x and y, and the pair (X, d) is then called a *metric space.*

Example 2.1.2. ([31]) Let $X = \mathbb{C}$ and defined $d(x, y) = |x - y|$ for $x, y \in \mathbb{C}$. Then d is a metric on $\mathbb{C}$. The d is often called the *usual metric.*

Example 2.1.3. ([31]) Let X be any nonempty set. For any $x, y \in X$, define

$$d(x, y) = \begin{cases} 0 & if \cdot x = y, \\ 1 & if \quad x \neq y. \end{cases}$$

Then d is a metric on X. The metric d is called *discrete metric* and the space (X, d) is is called *discrete metric space.*

Example 2.1.4. ([31]) Let $X = \mathbb{R}^n$, the set of ordered $n-$tuples of real numbers. For any $x = (x_1, x_2, ...x_n) \in X$ and $y = (y_1, y_2, ...y_n) \in X$, we define

(a) $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$;

(b) $d_2(x, y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{\frac{1}{2}}$; (called *usual metric*)

(c) $d_p(x, y) = (\sum_{i=1}^{n} (x_i - y_i)^p)^{\frac{1}{p}}, \quad p \geq 1$;

(d) $d_\infty(x, y) = \max_{1 \leq i \leq n}\{|x_i - y_i|\}$. (called *max metric*)

It is easy to verify that d_1, d_2, d_p, and d_∞ are metrics on X.

Definition 2.1.5. ([30]) Let (X, d) be a metric space. For any $r > 0$ and an element $x \in X$, we defined

$B_r(\bar{x}) := \{y \in X | d(\bar{x}, y) < r\}$, the open ball with center $\bar{x}$ and radius r;

$B_r[\bar{x}] := \{y \in X | d(\bar{x}, y) \leq r\}$, the closed ball with center $\bar{x}$ and radius r;

$\partial B_r(\bar{x}) := \{y \in X | d(\bar{x}, y) = r\}$, the boundary of ball with center $\bar{x}$ and radius r.

Definition 2.1.6. ([30]) Let $\{x_n\}$ be a sequence in a metric space X. A sequence $\{x_n\}$ *converges* to $x \in X$ if

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

In this case x is called a *limit point* of $\{x_n\}$ and we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$. A sequence which is not convergent is said to be *divergent*.

More preciously, a sequence $\{x_n\}$ in a metric space X converges to a point $x \in X$ if the sequence $\{d(x_n, x)\}$ of real numbers converges to 0 as $n \to \infty$. Every convergent sequence in a metric space has a unique limit point. If $\{x_{n_k}\}$ is a subsequence of a sequence $\{x_n\}$ in a metric space X, with $x_n \to x$, then $x_{n_k} \to x$. So, $x_n \to x$ if and only if, for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{k_r}}\}$ of $\{x_{n_k}\}$ which $x_{n_{k_r}} \to x$, (see [31]).

Definition 2.1.7. ([31]) Let (X, d) be a metric space and let K be a subset of X. The set K is *bounded* if there exists $\bar{x} \in X$, $M > 0$ such that $d(x, \bar{x}) \leq M$ for all $x \in K$.

Remark 2.1.8. ([31]) Let $\{x_n\}$ be a sequence in a metric space (X, d). A sequence $\{x_n\}$ is called a *bounded* if there exists $\bar{x} \in X$, $M > 0$ such that $d(x_n, \bar{x}) \leq M$ for all $n \in \mathbb{N}$.

In a metric space, every convergent sequence is bounded. (see [31])

Definition 2.1.9. ([31]) Let (X, d) be a metric space and let K be a subset of X. The set K is called *closed* if every convergent sequence of points of K has its limit in K.

Definition 2.1.10. ([31]) Let $\{x_n\}$ be a sequence in a metric space (X, d). A sequence $\{x_n\}$ is called a *Cauchy sequence* if

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Every convergent sequence in a metric space X is a Cauchy sequence, and every Cauchy sequence is bounded (see [31]).

Definition 2.1.11. ([31]) Let $\{x_n\}$ be a sequence in a metric space (X, d). A metric space (X, d) is complete if every Cauchy sequence in X is converges.

Let Y be a subspace of a complete metric space X. Then Y is complete if and only if Y is closed (see [31]).

Definition 2.1.12. ([31]) Let (X, d) be a metric space and let K be a subset of X. Then K is compact if and only if K is bounded and closed.

Definition 2.1.13. ([31]) Let (X, d_1) and (Y, d_2) be a metric spaces and let f be a mapping of X into Y. Then for $x_0 \in X$, f continuous at x_0 if, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(x, x_0) < \delta \Rightarrow d(f(x), f(x_0)) < \epsilon.$$

Definition 2.1.14. ([31]) Let (X, d_1) and (Y, d_2) be metric spaces and let f be a mapping of X into Y. Then f uniformly continuous on X if, for all $x, y \in X$, any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon.$$

Definition 2.1.15. Let (X, d) be metric spaces. A function $T : X \to X$ is said to be a *contraction mapping* if then

$$d(Tx, Ty) \leq \alpha d(x, y) \quad \text{for all } x, y \in X, \quad \text{where } 0 \leq \alpha < 1.$$

We see that a contraction mapping is continuous.

Let (X, d) be a metric space. We denote by $CB(X)$ the family of all nonempty closed bounded subsets of X. Let $H(\cdot, \cdot)$ be the Hausdorff metric, i.e.,

$$H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

for $A, B \in CB(X)$, where

$$d(x, B) = \inf_{y \in B} d(x, y).$$

Definition 2.1.16. ([32]) Let (X, d) be a metric space. A function $T : X \to CB(X)$ is said to be a *multi-valued contraction mapping* if then

$$H(Tx, Ty) \leq \alpha d(x, y) \quad \text{for all } x, y \in X, \quad \text{where } 0 \leq \alpha < 1.$$

We see that a multi-valued contraction mapping is continuous.

Definition 2.1.17. ([20, 22, 23]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies the following conditions:

(m_1) $d(x, x) = 0$, for all $x \in X$;

(m_2) $d(x, y) = d(y, x) = 0$ implies $x = y$, for all $x, y \in X$;

(m_3) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(m_4) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

The pair (X, d) is called a *metric space.* If d satisfies the conditions (m_1), (m_2) and (m_4) then, d is called a *quasi metric* on X. If d satisfies the conditions (m_2), (m_3) and (m_4), then d is called a *dislocated metric* on X. If d satisfies the conditions (m_2) , (m_4), then d is called a *dislocated quasi metric* on X.

It is evident that every metric on X is a dislocated metric on X, but the converse is not necessarily true as is clear from the following example.

Example 2.1.18. (Example of Dislocated Metric Spaces [33]) Let $X = [0, \infty)$ and define the distance function $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \max\{x, y\},$$

for all $x, y \in X$.

Next, we see that every metric on X is a quasi metric on X, but the converse is not true.

Example 2.1.19. (Example of Quasi Metric Spaces [34]) Let $X = [0, \infty)$ and $d : X \times X \to [0; \infty)$ be defined by

$$d(x, y) = \begin{cases} 1 \quad if \quad x > y, \\ 0 \quad if \quad x \leq y. \end{cases}$$

Then (X, d) is a quasi metric space.

Furthermore, from the following example one can say that a dislocated quasi metric on X needs not be a dislocated metric on X.

Example 2.1.20. (Example of Dislocated Quasi Metric Spaces [33]) Let $X = [0, 1]$ and define the distance function $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x - y| + |x|,$$

for all $x, y \in X$.

Definition 2.1.21. ([24, 25, 26]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ such that constant $b \geq 1$ satisfies the following conditions:

(b_1) $d(x, x) = 0$ for all $x \in X$;

(b_2) $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \in X$;

(b_3) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(b_4) $d(x, y) \leq b[d(x, z) + (z, y)]$, for all $x, y, z \in X$.

The pair (X, d) is then called a *b-metric space.* If d satisfies the conditions (b_1), (b_2) and (b_4), then X, d is called a *quasi b-metric* on X, and if d satisfies the conditions (b_2), (b_3) and (b_4), then d is called a *dislocated b-metric* on X.

Example 2.1.22. (Example of b-Metric Spaces [35]) The set of real numbers together with the functional

$$d(x, y) := |x - y|^2$$

for all $x, y \in \mathbb{R}$, is a b-metric space with constant $b = 2$. Also, we obtain that d is not a metric on X.

Example 2.1.23. (Example of b-Metric Spaces [36]) The set $l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) := \{\{x_n\} \subseteq \mathbb{R} : |x_n| < \infty\}$, together with the functional $d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}$,

$$d(x, y) := (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}},$$

for each $x = \{x_n\}, y = \{y_n\} \in l_p(\mathbb{R})$, is a b-metric space with coefficient $b = 2^{\frac{1}{p}} > 1$.

Example 2.1.24. (Example of Dislocated b-Metric Spaces [25]) Let $X = [0, \infty)$. Define the function $d : X \times X \to [0, \infty)$ by $d(x, y) = (x + y)^2$ or $d(x, y) = (\max\{x + y\})^2$. Then (X, d) is a dislocated b-metric space with constant $b = 2$. Clearly, (X, d) is not a b-metric or dislocated metric space.

Example 2.1.25. (Example of Quasi b-Metric Spaces [34]) Let $X = \mathbb{Q}$ be equipped with $d : X \times X \to [0,1)$ defined by

$$d(x,y) = \begin{cases} 1 & if \ \ x > y, \\ (y-x)^3 & if \ \ x \leq y. \end{cases}$$

Then (X,d) is a quasi b-metric space but not a quasi metric space.

2.2 Hyperbolic Spaces and some properties

In this section, we discuss various forms of hyperbolic spaces. Throughout this thesis, we work in the setting of hyperbolic spaces introduced by Kohlenbach [37].

Definition 2.2.1. ([37]) A hyperbolic space is a metric space (X,d) with a mapping $W : X^2 \times [0,1] \to X$ satisfying the following conditions.

(i) $d(u, W(x,y,\alpha)) \leq (1-\alpha)d(u,x) + \alpha d(u,y)$;

(ii) $d(W(x,y,\alpha), W(x,y,\beta)) = |\alpha - \beta| d(x,y)$;

(iii) $W(x,y,\alpha) = W(y,x,1-\alpha)$;

(iv) $d(W(x,z,\alpha), W(y,w,\alpha)) \leq (1-\alpha)d(x,y) + \alpha d(z,w)$.

for all $x,y,z,w \in X$ and $\alpha, \beta \in [0,1]$.

Example 2.2.2. Let X be a real Banach space (see [38]). Define the function $d : X^2 \to [0,\infty)$ by

$$d(x,y) = ||x-y||$$

as a metric on X. We see that (X,d) is a hyperbolic space with the mapping $W : X^2 \times [0,1] \to X$ defined by $W(x,y,\alpha) = (1-\alpha)x + \alpha y, \ \ \forall (x,y) \in X^2$ and $\alpha \in [0,1]$.

Definition 2.2.3. ([37]) Let X be a hyperbolic space with a mapping $W : X^2 \times [0,1] \to X$. A nonempty subset $K \subseteq X$ is said to be *convex* if $W(x,y,\alpha) \in K$ for all $x,y \in K$ and $\alpha \in [0,1]$. A hyperbolic space is said to be *strictly convex* if for any $r > 0$, and $\alpha \in (0, 1]$ such that for all $u, x, y \in X$,

$$d(W(x, y, \alpha), u) \leq r,$$

provided $d(x, u) \leq r$ and $d(y, u) \leq r$. A hyperbolic space is said to be *uniformly convex* if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $u, x, y \in X$,

$$d(W(x, y, \frac{1}{2}), u) \leq (1 - \delta)r,$$

provided $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \epsilon r$. A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$, is known as a modulus of uniform convexity of X. η is said to be monotone, if it decreases with r (for a fixed ϵ), i.e., $\forall \epsilon > 0$, $\forall r_1 \geq r_2 > 0$ $[\eta(r_2, \epsilon) \leq \eta(r_1, \epsilon)]$.

Definition 2.2.4. ([37]) Let $\{x_n\}$ be a bounded sequence in a hyperbolic space (X, d). For $x \in X$, we define a continuous functional $r(\cdot, x_n) : X \to [0, \infty)$ by

$$r(x, x_n) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, x_n) : x \in X\}.$$

The asymptotic center $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $K \subseteq X$ is the set

$$A_K(\{x_n\}) = \{x \in X : r(x, x_n) \leq r(y, x_n), \quad \forall y \in K\}.$$

This implies that the asymptotic center is the set of minimizer of the functional $r(\cdot, x_n)$ in K. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and CAT(0) spaces enjoy the property that bounded sequences have unique asymptotic centers with respect to closed convex subsets.

Definition 2.2.5. A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic centers of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$. Moreover, if $x_n \to x$, then $\Delta\text{-}\lim_{n\to\infty} x_n = x$ (see [37, 39]).

Lemma 2.2.6. ([39]) *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center in X.*

Lemma 2.2.7. ([39]) *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \leq c, \quad \limsup_{n\to\infty} d(y_n, x) \leq c \text{ and } \limsup_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = c,$$

for some $c \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 2.2.8. ([19]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a hyperbolic space (X, d, W) such that $x_{n+1} = W(y_n, x_n, \alpha)$ and $d(y_n, y_{n+1}) \leq d(x_n, x_{n+1})$ for all $n \in N$, where $\alpha \in (0, 1)$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Remark 2.2.9. ([40]) Let K be a nonempty subset of a metric space X, and $T : K \to K$. If T is a fundamentally nonexpansive mapping and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.

Lemma 2.2.10. ([19]) *Let K be a nonempty subset of metric space X. If $T : K \to K$ is a fundamentally nonexpansive mapping, then*

$$d(x, Ty) \leq 3d(x, Tx) + d(x, y) \tag{2.2.1}$$

for all $x, y \in K$.

Lemma 2.2.11. ([19]) *Let K be a nonempty subset of metric space X. If $T : K \to K$ satisfies condition SCC, then*

$$d(x, Ty) \leq 3d(x, Tx) + d(x, y) \tag{2.2.2}$$

for all $x, y \in K$.

Remark 2.2.12. Lemma 2.2.11 holds if one replaces condition SCC by the condition C.

Lemma 2.2.13. ([19]) *Let T be a mapping on a closed subset K of a metric space X and T satisfies condition SKC, then*

$$d(x, Ty) \leq 5d(x, Tx) + d(x, y) \tag{2.2.3}$$

for all $x, y \in K$.

Remark 2.2.14. ([19]) Lemma 2.2.13 holds if one replaces condition SKC by one of the conditions KSC, SCC, and CSC.

Lemma 2.2.15. ([41]) *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \zeta$. If $\{y_m\}$ is another sequence in K such that $\lim_{n\to\infty} r(y_m, \{x_n\}) = \zeta$, then $\lim_{n\to\infty} y_m = y$.*

2.3 Classical Fixed Point Theorems

We will consider both real vector spaces and complex vector spaces.

Definition 2.3.1. ([31, 37]) Let (X, d) be a metric space and let K be a nonempty subset of X. A point $x \in X$ is said to be a *fixed point* of T provided $x = Tx$. We will denote the fixed point set of a mapping T by

$$F(T) = \{x \in K \mid Tx = x\}.$$

Theorem 2.3.2. ([2]) *(Banach Fixed Point Theorem) Let T be a self-mapping on complete metric spaces X. If T satisfies contraction mapping (i.e., there exists $r \in [0, 1)$ such that $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$), then T has a unique fixed point.*

Theorem 2.3.3. ([3]) *Let T be a self-mapping on a complete metric space X. If T satisfies Kannan mapping (i.e., there exists $r \in [0, \frac{1}{2})$ such that $d(Tx, Ty) \leq rd(x, Tx) + rd(y, Ty)$, for all $x, y \in X$), then T has a unique fixed point.*

Theorem 2.3.4. ([5]) *Let T be a self-mapping on a complete metric space X. If T satisfies generalized Kannan mapping (i.e., there exists $r \in [0,1)$ such that $d(Tx,Ty) \leq r\max\{d(x,Tx),d(y,Ty)\}$, for all $x,y \in X$), then T has a unique fixed point.*

Theorem 2.3.5. ([7]) *Let T be a self-mapping on a complete metric space X. If there exists $r_i, \geq 0 \quad (i = 1,2,3,4,5)$ such that*

$$d(Tx,Ty) \leq r_1 d(x,y) + r_2 d(x,Tx) + r_3 d(y,Ty) + r_4 d(x,Ty) + r_5 d(y,Tx),$$

where $\Sigma_{i=1}^{5} r_i < 1$, for all $x,y \in X$, then T has a unique fixed point.

Theorem 2.3.6. ([8]) *Let T be a self-mapping on a complete metric space X. If there exists $r \in [0,1)$ $d(Tx,Ty) \leq r\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$, for all $x,y \in X$, then T has a unique fixed point.*

Definition 2.3.7. ([32]) Let T be a multi-valued mapping on a metric space X. A point $x \in X$ is said to be a *fixed point* of T provided $x \in Tx$.

Theorem 2.3.8. ([4]) *Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $r \in [0,1)$ such that $H(Tx,Ty) \leq rd(x,y)$, for all $x,y \in X$. Then there exists $z \in X$ such that $z \in Tz$.*

Example 2.3.9. ([32]) Let $I = [0,1]$ denote the unit interval of real numbers(with the usual metric) and let $f : I \to I$ be given by

$$f(x) = \begin{cases} \frac{x}{2} + \frac{1}{2},\ if\ \ 0 \leq x \leq \frac{1}{2} \\ \frac{-x}{2} + 1,\ if\ \ \frac{1}{2} < x \leq 1. \end{cases} \tag{2.3.1}$$

Define $F :\to I \to 2^I$ by $f(x) = \{0\} \bigcup \{fx\}$ for each $x \in I$. We observe that:

(1) F is a multi-valued contraction mapping,

(2) the set of fixed point of F is $\{0, \frac{2}{3}\}$.

Theorem 2.3.10. ([9]) *Let T be a self-mapping on a complete metric space X. If T satisfies weakly contractive (i.e., if for all $x, y \in X$, $d(Tx,Ty) \leq d(x,y) - \phi(d(x,y))$, where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$), then T has a unique fixed point.*

Definition 2.3.11. ([11]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping is said to has the mixed monotone property if $F(x_1, y_1) \preceq F(x_2, y_2)$ for all $x_1, x_2, y_1, y_2 \in X$ with $x_1 \preceq x_2$ and $y_2 \preceq y_1$.

Theorem 2.3.12. ([11]) *Let $T : X \times X \to X$ be a continuous mapping with the mixed monotone property on a metric space X. Assume that there exists a $r \in [0, 1)$ with*

$$d(T(x,y), T(u,v)) \leq \frac{r}{2}[d(x,u) + d(x,u)], \tag{2.3.2}$$

or all $x \preceq u$, $y \preceq v$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq T(x_0, y_0)$ and $T(y_0, x_0) \preceq y_0$, Then T has a coupled fixed point in X, (i.e., there exist $x, y \in X$ such that $T(x,y) = x$ and $T(y,x) = y$.) Moreover, we call a mapping $T : X \times X \to X$ is continuous provided that $T(x_n, y_n) \to T(x,y)$ whenever $x_n \to x$ and $y_n \to y$, where $x, y, x_n, y_n \in X$, for all $n \in \mathbb{N}$.

Theorem 2.3.13. ([13]) *Let T be a mapping on a complete metric space (X, d) and let φ be a non-increasing function from $[0, 1)$ into $(\frac{1}{2}, 1]$ defined by*

$$\varphi(r) = \begin{cases} 1, & if \ \ 0 \leq r < \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & if \ \ \frac{1}{\sqrt{2}} \leq r < \frac{1}{2}. \end{cases}$$

Let $\alpha \in [0, \frac{1}{2})$ and put $r = \frac{\alpha}{1-\alpha} \in [0, 1)$. Suppose that

$$\varphi(r)d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq \alpha d(x,Tx) + \alpha d(y,Ty),$$

for all $x, y \in X$. Then T has a unique fixed point z and $\lim_{n\to\infty} T^n x = z$ holds for every $x \in X$.

Theorem 2.3.14. ([13]) *Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$. Define a strictly decreasing function η from $[0,1)$ onto $(\frac{1}{2},1]$ by $\eta(r)=\frac{r}{1+r}$ and assume that there exists $r\in[0,1)$ such that*

$$\eta(r)d(x,Tx)\leq d(x,y) \text{ implies } H(Tx,Ty)\leq rd(x,y),$$

for all $x,y\in X$. Then there exists $z\in X$ such that $z\in Tz$.

Definition 2.3.15. ([14]) Let A and B be nonempty subsets of a metric space X, and let $T: A\cup B\to A\cup B$ be a mapping. A mapping T is called a *cyclic map* if $T(A)\subseteq B$ and $T(B)\subseteq A$.

Theorem 2.3.16. ([14]) *Let (X,d) be a complete metric space and let T be a cyclic mapping from $A\cup B$ into $A\cup B$. If T satisfies Kannan type cyclic contraction (i.e., there exists $r\in[0,\frac{1}{2})$ such that $d(Tx,Ty)\leq r[d(x,Tx)+d(y,Ty)]$, for all $x\in A$ and $y\in B$.), then T has a unique fixed point.*

Theorem 2.3.17. ([16]) *Define a non-increasing function φ from $[0,1)$ into $(0,1]$ by*

$$\varphi(r)=\begin{cases}1, & if \;\; 0\leq r<\frac{\sqrt{5}-1}{2},\\ 1-r, & if \;\; \frac{\sqrt{5}-1}{2}\leq r<1.\end{cases}$$

Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that

$$\varphi(r)d(x,Tx)\leq d(x,y) \text{ implies } H(Tx,Ty)\leq r\max\{d(x,Tx),d(y,Ty)\},$$

for all $x,y\in X$. Then there exists $z\in X$ such that $z\in Tz$.

Theorem 2.3.18. ([18]) *Let (X,d) be a complete metric space and let T be a cyclic mapping from $A\cup B$ into $A\cup B$. If T satisfies d-cyclic-ϕ-contraction (i.e., if $\phi\in\Phi$*

such that $d(Tx, Ty) \leq \phi(d(x, y))$, *for all* $x \in A$, $y \in B$, *where* Φ *the family of non-decreasing functions:* $\phi : [0, \infty) \to [0, \infty)$ *such that* $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ *for each* $t > 0$, *where* n *is the* n*-th iterate of* ϕ*), then* T *has a unique fixed point.*

CHAPTER III

FIXED POINT THEOREMS IN METRIC SPACES

3.1 Fixed Point Theorems for Generalized Multivalued Mappings in Metric Spaces

In this section, we prove fixed point theorems for generalized multi-valued mappings satisfying some inequalities in metric spaces. These results improve those of Damjanović and Dorić [16]; see [42] more details.

Theorem 3.1.1. *Define a non-increasing function φ from $[0, \frac{1}{2})$ into $(0, 1]$ by*

$$\varphi(r) = \begin{cases} 1 & if \;\; 0 \le r < \frac{\sqrt{5}-1}{\sqrt{5}+1}, \\ \frac{1-2r}{1-r} & if \;\; \frac{\sqrt{5}-1}{\sqrt{5}+1} \le r < \frac{1}{2}. \end{cases}$$

Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that Tx is compact for all $x \in X$. Suppose that there exists $r \in [0, \frac{1}{2})$ such that

$$\varphi(r)d(x, Tx) \le d(x, y) \;\; implies \;\; H(Tx, Ty) \le rM(x, y) \tag{3.1.1}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$, for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Proof. Let r_1 be a real number such that $0 \le r < r_1 < \frac{1}{2}$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, then $d(u_2, Tu_2) \le H(Tu_1, Tu_2)$ and

$$\varphi(r)d(u_1, Tu_1) \le d(u_1, Tu_1) \le d(u_1, u_2).$$

Thus from the assumption (3.1.1), we obtain that

$$d(u_2, Tu_2) \le H(Tu_1, Tu_2) \le rM(u_1, u_2)$$

where $M(u_1, u_2) = \max\{d(u_1, u_2), d(u_1, Tu_1), d(u_2, Tu_2), d(u_1, Tu_2), d(u_2, Tu_1)\}$.
We consider

$$d(u_2, Tu_2) \le r \max\{d(u_1, u_2), d(u_1, Tu_1), d(u_2, Tu_2), d(u_1, Tu_2), d(u_2, Tu_1)\}$$

$$= r\max\{d(u_1,u_2), d(u_1,Tu_2)\}.$$

If $\max\{d(u_1,u_2), d(u_1,Tu_2)\} = d(u_1,Tu_2)$, then

$$\begin{aligned} d(u_2,Tu_2) &\leq rd(u_1,Tu_2) \\ &\leq rd(u_1,u_2) + rd(u_2,Tu_2) \end{aligned}$$

so,

$$d(u_2,Tu_2) \leq (\frac{r}{1-r})d(u_1,u_2).$$

If $\max\{d(u_1,u_2), d(u_1,Tu_2)\} = d(u_1,u_2)$, then

$$d(u_2,Tu_2) \leq rd(u_1,u_2) \leq (\frac{r}{1-r})d(u_1,u_2).$$

We obtain that,

$$d(u_2,Tu_2) \leq (\frac{r}{1-r})d(u_1,u_2).$$

So, there exists $u_3 \in Tu_2$ such that $d(u_2,u_3) \leq (\frac{r_1}{1-r_1})d(u_1,u_2)$. Thus, we can construct a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1},u_{n+2}) \leq (\frac{r_1}{1-r_1})d(u_n,u_{n+1}).$$

Hence, by induction,

$$d(u_n,u_{n+1}) \leq (\frac{r_1}{1-r_1})^{n-1}d(u_1,u_2).$$

Let $n,m \in \mathbb{N}$ with $m > n > n(\epsilon)$, using the triangular inequality, we have:

$$\begin{aligned} d(u_m,u_n) &\leq d(u_m,u_{m-1}) + d(u_{m-1},u_{m-2}) + \ldots + d(u_{n+1},u_n) \\ &\leq (\frac{r_1}{1-r_1})^{m-1}d(u_1,u_2) + (\frac{r_1}{1-r_1})^{m-2}d(u_1,u_2) + \ldots + (\frac{r_1}{1-r_1})^{n}d(u_1,u_2) \\ &= \sum_{i=n}^{m-1}(\frac{r_1}{1-r_1})^{i}d(u_1,u_2) \\ &\leq \sum_{n=1}^{\infty}(\frac{r_1}{1-r_1})^{n-1}d(u_1,u_2) < \infty. \end{aligned} \tag{3.1.2}$$

Thus, $d(u_m,u_n) \to 0$ as $n,m \to \infty$. Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n\to\infty} u_n = z.$$

Now, we will show that $d(z,Tx) \leq rd(x,Tx)$ for all $x \in X \setminus \{z\}$. Let $x \in X \setminus \{z\}$. Since $u_n \to z$, there exists $n_0 \in N$ such that $d(z,u_n) \leq \frac{1}{3}d(z,x)$ for all $n \geq n_0$. Then, we have

$$\begin{aligned}
\varphi(r)d(u_n,Tu_n) &\leq d(u_n,Tu_n) \\
&\leq d(u_n,u_{n+1}) \\
&\leq d(u_n,z) + d(z,u_{n+1}) \\
&\leq (\frac{2}{3})d(z,x) \\
&= d(z,x) - \frac{1}{3}d(z,x) \\
&\leq d(z,x) - d(z,u_n) \\
&\leq d(x,u_n). \qquad (3.1.3)
\end{aligned}$$

Then from (3.1.1), we have

$$H(Tu_n,Tx) \leq r\max\{d(u_n,x), d(u_n,Tu_n), d(x,Tx), d(u_n,Tx), d(x,Tu_n)\}.$$

Since $u_{n+1} \in Tu_n$, then $d(u_{n+1},Tx) \leq H(Tu_n,Tx)$, so that

$$d(u_{n+1},Tx) \leq r\max\{d(u_n,x), d(u_n,u_{n+1}), d(x,Tx), d(u_n,Tx), d(x,u_{n+1})\}$$

for all $n \geq n_0$. Letting $n \to \infty$, we obtain that

$$d(z,Tx) \leq r\max\{d(z,x), d(x,Tx), d(z,Tx)\}.$$

It follows that

$$d(z,Tx) \leq (\frac{r}{1-r})d(x,Tx) \qquad (3.1.4)$$

for all $x \in X \setminus \{z\}$. Next, we show that $z \in Tz$. Suppose that z is not an element in Tz.

Case i; $0 \leq r < \frac{\sqrt{5}-1}{\sqrt{5}+1}$. Let $a \in Tz$. Then $a \neq z$ and so by (3.1.4), we have

$$d(z,Ta) \leq (\frac{r}{1-r})d(a,Ta).$$

On the other hand, since $\varphi(r)d(z,Tz) = d(z,Tz) \leq d(z,a)$, from (3.1.1) we have

$$H(Tz,Ta) \leq r\max\{d(z,a), d(z,Tz), d(a,Ta), d(z,Ta), d(a,Tz)\}.$$

So,

$$d(a,Ta) \leq H(Tz,Ta) \leq r\max\{d(z,a), d(z,Tz), d(z,Ta)\}. \tag{3.1.5}$$

It implies that

$$d(a,Ta) \leq r\max\{d(z,a), d(z,Tz), d(z,Ta)\}.$$

Since $d(z,a) \leq d(z,Tz) + d(Tz,a) = d(z,Tz)$, we have

$$d(a,Ta) \leq (\frac{r}{1-r})d(z,Tz). \tag{3.1.6}$$

Using, (3.1.4)-(3.1.6), we have

$$\begin{aligned}
d(z,Tz) &\leq d(z,Ta) + H(Ta,Tz)\\
&\leq (\frac{r}{1-r})d(a,Ta) + r\max\{d(z,a), d(z,Tz), d(z,Ta)\}\\
&\leq (\frac{r}{1-r})d(a,Ta) + r\max\{d(z,a), d(z,Tz), (\frac{r}{1-r})d(a,Ta)\}\\
&\leq (\frac{r}{1-r})d(a,Ta) + r\max\{d(z,a), d(z,Tz)\}\\
&\leq (\frac{r}{1-r})d(a,Ta) + rd(z,Tz)\\
&\leq (\frac{r}{1-r})^2 d(z,Tz) + rd(z,Tz)\\
&\leq (\frac{r}{1-r})^2 d(z,Tz) + (\frac{r}{1-r})d(z,Tz)\\
&\leq [(\frac{r}{1-r})^2 + (\frac{r}{1-r})]d(z,Tz)\\
&\leq [k^2 + k]d(z,Tz),
\end{aligned}$$

where $k = \frac{r}{1-r}$. Since $r < \frac{\sqrt{5}-1}{\sqrt{5}+1}$, we have $k^2 + k < 1$ and so, $d(z,Tz) < d(z,Tz)$, which is a contradiction. Thus $z \in Tz$.

Case ii; $\frac{\sqrt{5}-1}{\sqrt{5}+1} \leq r < \frac{1}{2}$. Let $x \in X$.

If $x = z$, then $H(Tx,Tz) \leq r\max\{d(x,z), d(x,Tx), d(z,Tz), d(x,Tz), d(z,Tx)\}$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z,y_n) \leq d(z,Tx) + (\frac{1}{n})d(x,z).$$

We consider

$$\begin{aligned} d(x,Tx) &\le d(x,y_n) \\ &\le d(x,z)+d(z,y_n) \\ &\le d(x,z)+d(z,Tx)+(\frac{1}{n})d(x,z) \\ &\le d(x,z)+(\frac{r}{1-r})d(x,Tx)+(\frac{1}{n})d(x,z). \end{aligned}$$

Thus, $(\frac{1-2r}{1-r})d(x,Tx) \le (1+\frac{1}{n})d(x,z)$. Take $n \to \infty$, we obtain that

$$(\frac{1-2r}{1-r})d(x,Tx) \le d(x,z).$$

By using (3.1.1), this implies that

$$H(Tx,Tz) \le r\max\{d(x,z),d(x,Tx),d(z,Tz),d(x,Tz),d(z,Tx)\}.$$

Hence, as $u_{n+1} \in Tu_n$, it follows that with $x = u_n$, this yields

$$\begin{aligned} d(z,Tz) &= \lim_{n\to\infty} d(u_{n+1},Tz) \\ &\le \lim_{n\to\infty} H(Tu_n,Tz) \\ &\le \lim_{n\to\infty} r\max\{d(u_n,z),d(u_n,Tu_n),d(z,Tz),d(u_n,Tz),d(z,Tu_n)\} \\ &\le \lim_{n\to\infty} r\max\{d(u_n,z),d(u_n,u_{n+1}),d(z,Tz),d(u_n,Tz),d(z,u_{n+1})\} \\ &\le rd(z,Tz). \end{aligned}$$

Therefore, $(1-r)d(z,Tz) \le 0$, which implies $d(z,Tz) = 0$. Since Tz is closed, we have $z \in Tz$. This completes the proof. □

Example 3.1.2. Let $X = [0,\infty)$ be endowed with the usual metric d. Define $T : X \to CB(X)$ by

$$T(x) = \begin{cases} [0,x^2], & 0 \le x \le \frac{1}{2}, \\ [0,\frac{x}{3}], & \frac{1}{2} < x < 1, \\ [0,\log(x)], & 1 \le x. \end{cases} \tag{3.1.7}$$

Proof. We show that T satisfies (3.1.1). Let $x, y \in X$. We divide the proof by cases.

Case i: Suppose that $x, y \in [0, \frac{1}{2}]$. Thus, if $x^2 \leq y$, then

$$\varphi(\frac{1}{4})d(x, Tx) = |x - x^2| \geq |x - y| = d(x, y).$$

But, if $x^2 > y$, then

$$\varphi(\frac{1}{4})d(x, Tx) = |x - x^2| \leq |x - y| = d(x, y)$$

and

$$\begin{aligned} H(Tx, Ty) &= |x^2 - y^2| \\ &\leq \frac{1}{4}|(2x)^2 - (2y)^2| \\ &\leq \frac{1}{4}|x - 2y^2| \\ &\leq \frac{1}{4}|x - y^2| \\ &= \frac{1}{4}\max\{|x - y|, |x - x^2|, |y - y^2|, |x - y^2|, |y - x^2|\} \\ &= \frac{1}{4}\max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ &= rM(x, y), \end{aligned} \tag{3.1.8}$$

where $r = \frac{1}{4}$. Hence T satisfies (3.1.1).

Case ii: Suppose that $x, y \in (\frac{1}{2}, 1)$. Thus, if $\frac{x}{3} \leq y$, then

$$\varphi(\frac{1}{3})d(x, Tx) = |x - \frac{x}{3}| \geq |x - y| = d(x, y).$$

But, if $\frac{x}{3} > y$, then

$$\varphi(\frac{1}{3})d(x, Tx) = |x - \frac{x}{3}| \leq |x - y| = d(x, y)$$

and

$$\begin{aligned} H(Tx, Ty) &= \frac{1}{3}|x - y| \\ &\leq \frac{1}{3}|x - \frac{y}{3}| \end{aligned}$$

$$= \frac{1}{3}\max\{|x-y|, |x-\frac{x}{3}|, |y-\frac{y}{3}|, |x-\frac{y}{3}|, |y-\frac{x}{3}|\}$$
$$= \frac{1}{3}\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$
$$= rM(x,y), \tag{3.1.9}$$

where $r = \frac{1}{3}$. Hence T satisfies (3.1.1).

Case *iii*: Suppose that $x, y \in [1, \infty]$. Thus, if $\log(x) \leq y$, then

$$\varphi(\frac{1}{3})d(x,Tx) = |x - \log(x)| \geq |x-y| = d(x,y).$$

But, if $\log(x) > y$, then

$$\varphi(\frac{1}{3})d(x,Tx) = |x - \log(x)| \leq |x-y| = d(x,y)$$

and

$$H(Tx,Ty) = |\log(x) - \log(y)|$$
$$= \frac{1}{3}(3\log(x) - 3\log(y))$$
$$\leq \frac{1}{3}|x - \log(y)|$$
$$= \frac{1}{3}\max\{|x-y|, |x-\log(x)|, |y-\log(y)|, |x-\log(y)|, |y-\log(x)|\}$$
$$= \frac{1}{3}\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$
$$= rM(x,y), \tag{3.1.10}$$

where $r = \frac{1}{3}$. Hence T satisfies (3.1.1).

Case *iv*: Suppose that $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1)$. Then $x^2 < x < y$. Thus, $\varphi(\frac{1}{3})d(x,Tx) = |x - x^2| \geq |x-y| = d(x,y)$. Hence T satisfies (3.1.1).

Case *v*: Suppose that $x \in (\frac{1}{2}, 1)$. and $y \in [0, \frac{1}{2}]$. So $x > y$. Thus, if $\frac{x}{3} \leq y$, then

$$\varphi(\frac{1}{3})d(x,Tx) = |x - \frac{x}{3}| \geq |x-y| = d(x,y).$$

But, if $\frac{x}{3} > y$, then

$$\varphi(\frac{1}{3})d(x,Tx) = |x - \frac{x}{3}| \leq |x-y| = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= |\frac{x}{3} - y^2| \\
&\leq \frac{1}{3}|x - 3y^2| \\
&\leq \frac{1}{3}|x - y^2| \\
&= \frac{1}{3}\max\{|x-y|, |x - \frac{x}{3}|, |y - y^2|, |x - y^2|, |y - \frac{x}{3}|\} \\
&= \frac{1}{3}\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \\
&= rM(x,y), \qquad (3.1.11)
\end{aligned}$$

where $r = \frac{1}{3}$. Hence T satisfies (3.1.1).

Case *vi*: Suppose that $x \in [0, \frac{1}{2}]$ and $y \in [1, \infty]$.

$$\varphi(\frac{1}{3})d(x,Tx) = |x - x^2| \leq |x - y| = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= |x^2 - \log(y)| \\
&= \frac{1}{3}|3x^2 - 3\log(y)| = \frac{1}{3}|3\log(y) - 3x^2| \\
&\leq \frac{1}{3}\max\{|y - \log(y)|, |y - x^2|\} \\
&= \frac{1}{3}\max\{|x-y|, |x - x^2|, |y - \log(y)|, |x - \log(y)|, |y - x^2|\} \\
&= \frac{1}{3}\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \\
&= rM(x,y), \qquad (3.1.12)
\end{aligned}$$

where $r = \frac{1}{3}$. Hence T satisfies (3.1.1).

Case *vii*: Suppose that $x \in [1, \infty]$ and $y \in [0, \frac{1}{2}]$. Thus, if $\log(x) \leq y$, then

$$\varphi(\frac{1}{4})d(x,Tx) = |x - \log(x)| \geq |x - y| = d(x,y).$$

But, if $\log(x) > y$, then

$$\varphi(\frac{1}{4})d(x,Tx) = |x - \log(x) \leq |x - y| = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= |\log(x) - y^2| \\
&= \frac{1}{4}|4\log(x) - 4y^2| \\
&\leq \frac{1}{4}|x - y^2| \\
&= \frac{1}{4}\max\{|x-y|, |x-\log(x)|, |y-y^2|, |x-y^2|, |y-\log(x)|\} \\
&= \frac{1}{4}\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \\
&= rM(x,y), \qquad (3.1.13)
\end{aligned}$$

where $r = \frac{1}{4}$. Hence T satisfies (3.1.1).

Case *viii*: Suppose that $x \in (\frac{1}{2}, 1)$ and $y \in [1, \infty]$.

$$\varphi(\frac{1}{3})d(x,Tx) = |x - \frac{x}{3}| \leq |x-y| = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= |\frac{x}{3} - \log(y)| \\
&= \frac{1}{3}|x - 3\log(y)| = \frac{1}{3}|3\log(y) - x| \\
&\leq \frac{1}{3}\max\{|y-\log(y)|, |y - \frac{x}{3}|\} \\
&= \frac{1}{3}\max\{|x-y|, |x-\frac{x}{3}|, |y-\log(y)|, |x-\log(y)|, |y-\frac{x}{3}|\} \\
&= \frac{1}{3}\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \\
&= rM(x,y), \qquad (3.1.14)
\end{aligned}$$

where $r = \frac{1}{3}$. Hence T satisfies (3.1.1).

Case *ix*: Suppose that $x \in [1, \infty]$ and $y \in (\frac{1}{2}, 1)$. Thus, if $\log(x) \leq y$, then

$$\varphi(\frac{1}{3})d(x,Tx) = |x - \log(x)| \geq |x-y| = d(x,y).$$

But, if $\log(x) > y$, then

$$\varphi(\frac{1}{3})d(x,Tx) = |x - \log(x) \leq |x-y| = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= |\log(x) - \frac{y}{3}| \\
&= \frac{1}{3}|3\log(x) - y| \\
&\le \frac{1}{3}|x - y| \\
&= \frac{1}{3}\max\{|x-y|, |x-\log(x)|, |y-\frac{y}{3}|, |x-\frac{y}{3}|, |y-\log(x)|\} \\
&= \frac{1}{3}\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \\
&= rM(x,y), \qquad (3.1.15)
\end{aligned}$$

where $r = \frac{1}{3}$. Thus we see that T satisfies condition (3.1.1) and satisfies all conditions in theorem 3.1.1. So, there exists $z \in X$ such that $z \in Tz$. Moreover, $0 \in T(0)$. □

Theorem 3.1.3. *Define a non-increasing function φ from $[0, \frac{1}{5})$ into $(0, 1]$ by*

$$\varphi(r) = \begin{cases} 1 & if \;\; 0 \le r < \frac{\sqrt{5}-1}{4+2\sqrt{5}}, \\ \frac{1-5r}{1-2r} & if \;\; \frac{\sqrt{5}-1}{4+2\sqrt{5}} \le r < \frac{1}{5}. \end{cases}$$

Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that Tx is compact for all $x \in X$. Suppose that there exists $r \in [0, \frac{1}{5})$ such that

$$\varphi(r)d(x,Tx) \le d(x,y) \;\; implies \;\; H(Tx,Ty) \le S(x,y) \qquad (3.1.16)$$

where $S(x,y) = rd(x,y) + rd(x,Tx) + rd(y,Ty) + rd(x,Ty) + rd(y,Tx)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

Proof. Let r_1 be a real number such that $0 \le r < r_1 < 1$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, then $d(u_2, Tu_2) \le H(Tu_1, Tu_2)$ and

$$\varphi(r)d(u_1, Tu_1) \le d(u_1, Tu_1) \le d(u_1, u_2).$$

Thus, from the assumption (3.1.16), we obtain that

$$d(u_2, Tu_2) \le H(Tu_1, Tu_2) \le S(u_1, u_2)$$

where $S(u_1, u_2) = rd(u_1, u_2) + rd(u_1, Tu_1) + rd(u_2, Tu_2) + rd(u_1, Tu_2) + rd(u_2, Tu_1)$.
Consider,

$$\begin{aligned} d(u_2, Tu_2) &\leq rd(u_1, u_2) + rd(u_1, Tu_1) + rd(u_2, Tu_2) + rd(u_1, Tu_2) + rd(u_2, Tu_1) \\ &\leq 3rd(u_1, u_2) + 2rd(u_2, Tu_2). \end{aligned}$$

So,

$$d(u_2, Tu_2) \leq (\frac{3r}{1-2r})d(u_1, u_2).$$

So, there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) \leq (\frac{3r_1}{1-2r_1})d(u_1, u_2)$. Thus, we can construct a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1}, u_{n+2}) \leq (\frac{3r_1}{1-2r_1})d(u_n, u_{n+1}).$$

Hence, by induction,

$$d(u_n, u_{n+1}) \leq (\frac{3r_1}{1-2r_1})^{n-1}d(u_1, u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty}(\frac{3r_1}{1-2r_1})^{n-1}d(u_1, u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is a point $z \in X$ such that

$$\lim_{n\to\infty} u_n = z.$$

Now, we will show that $d(z, Tx) \leq (\frac{3r}{1-2r})d(x, Tx)$ for all $x \in X \setminus \{z\}$.
Let $x \in X \setminus \{z\}$. Since $u_n \to z$, there exists $n_0 \in N$ such that $d(z, u_n) \leq \frac{1}{3}d(z, x)$ for all $n \geq n_0$.
By using (3.1.3), we get

$$\varphi(r)d(u_n, Tu_n) \leq d(x, u_n).$$

Then from (3.1.16), we have

$$H(Tu_n, Tx) \leq r[d(u_n, x) + d(u_n, Tu_n) + d(x, Tx) + d(u_n, Tx) + d(x, Tu_n)].$$

Since $u_{n+1} \in Tu_n$, then $d(u_{n+1}, Tx) \leq H(Tu_n, Tx)$, so that

$$d(u_{n+1}, Tx) \leq r[d(u_n, x) + d(u_n, u_{n+1}) + d(x, Tx) + d(u_n, Tx) + d(x, u_{n+1})]$$

for all $n \geq n_0$. Letting $n \to \infty$, we obtain that

$$\begin{aligned} d(z, Tx) &\leq r[2d(z, x) + d(x, Tx) + d(z, Tx)] \\ &\leq r3d(z, x) + r2d(z, Tx). \end{aligned}$$

It follows that

$$d(z, Tx) \leq (\frac{3r}{1-2r})d(x, Tx), \quad \forall x \in X \setminus \{z\}. \tag{3.1.17}$$

Next, we show that $z \in Tz$. Suppose that z is not an element in Tz.

Case i; $0 \leq r < \frac{\sqrt{5}-1}{4+2\sqrt{5}}$. Let $a \in Tz$. Then $a \neq z$ and so by (3.1.17), we have

$$d(z, Ta) \leq (\frac{3r}{1-2r})d(a, Ta).$$

On the other hand, since $\varphi(r)d(z, Tz) = d(z, Tz) \leq d(z, a)$, from (3.1.16) we have

$$H(Tz, Ta) \leq r[d(z, a) + d(z, Tz) + d(a, Ta) + d(z, Ta) + d(a, Tz)].$$

So,

$$\begin{aligned} d(a, Ta) \leq H(Tz, Ta) &\leq r[2d(z, a) + d(a, Ta) + d(z, Ta)] \\ &\leq r[3d(z, a) + 2d(a, Ta)]. \end{aligned} \tag{3.1.18}$$

Since $d(z, a) \leq d(z, Tz) + d(Tz, a) = d(z, Tz)$, we have

$$d(a, Ta) \leq (\frac{3r}{1-2r})d(z, Tz).$$

Using (3.1.16)-(3.1.18), we have

$$\begin{aligned} d(z, Tz) &\leq d(z, Ta) + H(Ta, Tz) \\ &\leq (\frac{3r}{1-2r})d(a, Ta) + S(a, z) \\ &\leq (\frac{3r}{1-2r})d(a, Ta) + r[d(z, a) + d(z, Tz) + d(a, Ta) + d(z, Ta) + d(a, Tz)] \end{aligned}$$

$$\leq (\frac{3r}{1-2r})d(a,Ta) + 3rd(z,a)$$
$$\leq (\frac{3r}{1-2r})^2 d(z,Tz) + (\frac{3r}{1-2r})d(z,Tz)$$
$$\leq (k^2+k)d(z,Tz),$$

where $k = \frac{3r}{1-2r}$. Since $0 \leq r < \frac{\sqrt{5}-1}{4+2\sqrt{5}}$, we have $0 \leq k^2 + k < 1$ and so, $d(z,Tz) < d(z,Tz)$, which is a contradiction. Thus $z \in Tz$.

Case ii; $\frac{\sqrt{5}-1}{4+2\sqrt{5}} \leq r < \frac{1}{5}$. Let $x \in X$. If $x = z$, then $H(Tx,Tz) \leq r[d(x,z) + d(x,Tx) + d(z,Tz) + d(x,Tz) + d(z,Tx)]$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z,y_n) \leq d(z,Tx) + (\frac{1}{n})d(x,z).$$

We consider

$$\begin{aligned} d(x,Tx) &\leq d(x,y_n) \\ &\leq d(x,z) + d(z,y_n) \\ &\leq d(x,z) + d(z,Tx) + (\frac{1}{n})d(x,z) \\ &\leq d(x,z) + (\frac{3r}{1-2r})d(x,Tx) + (\frac{1}{n})d(x,z). \end{aligned}$$

Thus, $(\frac{1-5r}{1-2r})d(x,Tx) \leq (1+\frac{1}{n})d(x,z)$. Take $n \to \infty$, we obtain that

$$(\frac{1-5r}{1-2r})d(x,Tx) \leq d(x,z).$$

By using (3.1.16), this implies that $H(Tx,Tz) \leq S(x,z)$,
where $S(x,z) = r[d(x,z) + d(x,Tx) + d(z,Tz) + d(x,Tz) + d(z,Tx)]$.
Hence, as $u_{n+1} \in Tu_n$, it follows that with $x = u_n$, this yields

$$\begin{aligned} d(z,Tz) &= \lim_{n\to\infty} d(u_{n+1},Tz) \\ &\leq \lim_{n\to\infty} H(Tu_n,Tz) \\ &\leq \lim_{n\to\infty} r[d(u_n,z) + d(u_n,Tu_n) + d(z,Tz) + d(u_n,Tz) + d(z,Tu_n)] \\ &\leq \lim_{n\to\infty} [rd(u_n,z) + rd(u_n,u_{n+1}) + rd(z,Tz) + rd(u_n,Tz) + rd(z,u_{n+1})] \end{aligned}$$

$$\leq (2r)d(z, Tz). \tag{3.1.19}$$

Using (3.1.19), we have $(1-2r)d(z,Tz) \leq 0$, which implies $d(z,Tz) = 0$. Since Tz is closed, we have $z \in Tz$. This completes the proof. □

Example 3.1.4. Let $X = [0, \frac{1}{2}]$ with the metric $d(x,y) = \frac{|x-y|}{|x-y|+1}$ for all $x, y \in X$. Define $T : X \to CB(X)$ by

$$T(x) = [0, x^2].$$

Proof. We show that T satisfies (3.1.16). Let $x, y \in X$. Thus, if $x^2 \leq y$, then

$$\varphi(\frac{1}{6})d(x,Tx) = \frac{|x-x^2|}{|x-x^2|+1} \geq \frac{|x-y|}{|x-y|+1} = d(x,y).$$

But, if $x^2 > y$, then

$$\varphi(\frac{1}{6})d(x,Tx) = \frac{|x-x^2|}{|x-x^2|+1} \leq \frac{|x-y|}{|x-y|+1} = d(x,y)$$

and

$$\begin{aligned}
H(Tx,Ty) &= \frac{|x^2-y^2|}{|x^2-y^2|+1} \\
&= \frac{1}{6}\frac{6|x^2-y^2|}{|x^2-y^2|+1} \\
&= \frac{1}{6}\{\frac{|x^2-y^2|}{|x^2-y^2|+1} + \frac{|x^2-y^2|}{|x^2-y^2|+1} + \frac{|x^2-y^2|}{|x^2-y^2|+1} + \frac{2|x^2-y^2|}{|x^2-y^2|+1} \\
&\quad + \frac{|x^2-y^2|}{|x^2-y^2|+1}\} \\
&< \frac{1}{6}\{\frac{|x-y|}{|x-y|+1} + \frac{|x-x^2|}{|x-x^2|+1} + \frac{|y-y^2|}{|y-y^2|+1} + \frac{|x-y^2|}{|x-y^2|+1} \\
&\quad + \frac{|y-x^2|}{|y-x^2|+1}\} \\
&= \frac{1}{6}\{d(x,y) + d(x,Tx) + d(y,Ty) + d(x,Ty) + d(y,Tx)\} \\
&= \frac{1}{6}S(x,y),
\end{aligned} \tag{3.1.20}$$

where $r = \frac{1}{6}$. Thus we see that T satisfies condition (3.1.16) and satisfies all assumptions theorem 3.1.3. So, there exists $z \in X$ such that $z \in Tz$. Moreover, $0 \in T(0)$. □

Theorem 3.1.5. *Define a non-increasing function* φ *from* $[0,1)$ *into* $(0,1]$ *by*

$$\varphi(r) = \begin{cases} 1 & \text{if } \ 0 \le r < \frac{\sqrt{5}-1}{2}, \\ 1-r & \text{if } \ \frac{\sqrt{5}-1}{2} \le r < 1. \end{cases}$$

Let (X,d) *be a complete metric space and let* T *be a mapping from* X *into* $CB(X)$ *such that* Tx *is compact for all* $x \in X$. *Assume that there exists* $\alpha \in [0, \frac{1}{2})$ *such that*

$$\varphi(r)d(x,Tx) \le d(x,y) \ \text{ implies } \ H(Tx,Ty) \le \alpha M(x,y) \tag{3.1.21}$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$, *for all* $x, y \in X$, *and* $r = \frac{\alpha}{1-\alpha}$. *Then there exists* $z \in X$ *such that* $z \in Tz$.

Proof. Let α_1 be a real number such that $0 \le \alpha < \alpha_1 < \frac{1}{2}$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, then $d(u_2, Tu_2) \le H(Tu_1, Tu_2)$ and

$$\varphi(r)d(u_1, Tu_1) \le d(u_1, Tu_1) \le d(u_1, u_2).$$

Thus, from the assumption (3.1.21), we obtain that

$$d(u_2, Tu_2) \le H(Tu_1, Tu_2) \le \alpha M(u_1, u_2),$$

where $M(u_1,u_2) = max\{d(u_1,u_2), d(u_1,Tu_1), d(u_2,Tu_2), d(u_1,Tu_2), d(u_2,Tu_1)\}$. Consider,

$$\begin{aligned} d(u_2, Tu_2) &\le \alpha \max\{d(u_1,u_2), d(u_1,Tu_1), d(u_2,Tu_2), d(u_1,Tu_2), d(u_2,Tu_1)\} \\ &= \alpha \max\{d(u_1,u_2), d(u_1,Tu_2)\}. \end{aligned}$$

If $\max\{d(u_1,u_2), d(u_1,Tu_2)\} = d(u_1,Tu_2)$, then

$$\begin{aligned} d(u_2, Tu_2) &\le \alpha d(u_1, Tu_2) \\ &\le \alpha d(u_1,u_2) + \alpha d(u_2, Tu_2) \end{aligned}$$

and then

$$d(u_2, Tu_2) \leq (\frac{\alpha}{1-\alpha})d(u_1, u_2) = rd(u_1, u_2),$$

where $r = \frac{\alpha}{1-\alpha}$.

So, there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) < r_1 d(u_1, u_2)$, where $r = \frac{\alpha_1}{1-\alpha_1}$. Thus, we can construct a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1}, u_{n+2}) \leq r_1 d(u_n, u_{n+1}).$$

Hence, by induction

$$d(u_n, u_{n+1}) \leq (r_1)^{n-1} d(u_1, u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} (r_1)^{n-1} d(u_1, u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n\to\infty} u_n = z.$$

Now, we will show that $d(z, Tx) \leq rd(x, Tx)$ for all $x \in X \setminus \{z\}$.

Let $x \in X \setminus \{z\}$.

Since $u_n \to z$, there exists $n_0 \in N$ such that $d(z, u_n) \leq (\frac{1}{3})d(z, x)$ for all $n \geq n_0$.

By using form (3.1.3), we get

$$\varphi(r) d(u_n, Tu_n) \leq d(x, u_n).$$

Then from (3.1.21), we have

$$H(Tu_n, T_x) \leq \alpha \max\{d(u_n, x), d(u_n, Tu_n), d(x, Tx), d(u_n, Tx), d(x, Tu_n)\}.$$

Since $u_{n+1} \in Tu_n$, then $d(u_{n+1}, T_x) \leq H(Tu_n, T_x)$, so that

$$d(u_{n+1}, Tx) \leq \alpha \max\{d(u_n, x), d(u_n, u_{n+1}), d(x, Tx), d(u_n, Tx), d(x, u_{n+1})\}$$

for all $n \geq n_0$. Letting $n \to \infty$, we obtain

$$d(z,Tx) \leq \alpha \max\{d(z,x), d(x,Tx), d(z,Tx)\}.$$

We obtain that

$$d(z,Tx) \leq (\frac{\alpha}{1-\alpha})d(x,Tx) = rd(x,Tx), \quad \forall x \in X \setminus \{z\}. \tag{3.1.22}$$

Next, to show that $z \in Tz$. Suppose that z is not an element in Tz.

Case i; $0 \leq r < \frac{\sqrt{5}-1}{2}$. Let $a \in Tz$. Then $a \neq z$ and so by (3.1.22), we have

$$d(z,Ta) \leq rd(a,Ta).$$

On the other hand, since $\varphi(r)d(z,Tz) = d(z,Tz) \leq d(z,a)$, from (3.1.21) we have

$$H(Tz,Ta) \leq \alpha \max\{d(z,a), d(z,Tz), d(a,Ta), d(z,Ta), d(a,Tz)\}.$$

So,

$$d(a,Ta) \leq H(Tz,Ta) \leq \alpha \max\{d(z,a), d(z,Tz), d(z,Ta)\}. \tag{3.1.23}$$

It implies that

$$d(a,Ta) \leq \alpha \max\{d(z,a), d(z,Tz), d(z,Ta)\}.$$

Since $d(z,a) \leq d(z,Tz) + d(Tz,a) = d(z,Tz)$, we have

$$d(a,Ta) \leq rd(z,Tz). \tag{3.1.24}$$

Using (2.20), (2.21), (2.22) and (2.23), we have

$$\begin{aligned}
d(z,Tz) &\leq d(z,Ta) + H(Ta,Tz)\\
&\leq rd(a,Ta) + \alpha \max\{d(z,a), d(z,Tz), d(z,Ta)\}\\
&\leq rd(a,Ta) + \alpha \max\{d(z,a), d(z,Tz), rd(a,Ta)\}\\
&\leq rd(a,Ta) + \alpha \max\{d(z,a), d(z,Tz)\}\\
&\leq rd(a,Ta) + \alpha d(z,Tz)
\end{aligned}$$

$$\leq (r)^2 d(z,Tz) + rd(z,Tz)$$
$$\leq (r^2 + r)d(z,Tz),$$

where $r = \frac{\alpha}{1-\alpha}$. Since $r < \frac{\sqrt{5}-1}{2}$, we have $r^2 + r < 1$ and so $d(z,Tz) < d(z,Tz)$, which is contradiction. Thus $z \in Tz$.

Case ii; $\frac{\sqrt{5}-1}{2} \leq r < 1$. Let $x \in X$.
If $x = z$, then $H(Tx,Tz) \leq \alpha \max\{d(x,z), d(x,Tx), d(z,Tz), d(x,Tz), d(z,Tx)\}$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z,y_n) \leq d(z,Tx) + (\frac{1}{n})d(x,z).$$

We consider

$$\begin{aligned} d(x,Tx) &\leq d(x,y_n) \\ &\leq d(x,z) + d(z,y_n) \\ &\leq d(x,z) + d(z,Tx) + (\frac{1}{n})d(x,z) \\ &\leq d(x,z) + rd(x,Tx) + (\frac{1}{n})d(x,z). \end{aligned}$$

Thus, $(1-r)d(x,Tx) \leq (1+\frac{1}{n})d(x,z)$. Take $n \to \infty$, we obtain that

$$(1-r)d(x,Tx) \leq d(x,z).$$

By using (3.1.21), implies $H(Tx,Tz) \leq \alpha \max\{d(x,z), d(x,Tx), d(z,Tz), d(x,Tz), d(z,Tx)\}$. Hence, as $u_{n+1} \in Tu_n$, it follows that with $x = u_n$, this yields

$$\begin{aligned} d(z,Tz) &= \lim_{n\to\infty} d(u_{n+1},Tz) \\ &\leq \lim_{n\to\infty} H(Tu_n,Tz) \\ &\leq \lim_{n\to\infty} \alpha \max\{d(u_n,z), d(u_n,Tu_n), d(z,Tz), d(u_n,Tz), d(z,Tu_n)\} \\ &\leq \lim_{n\to\infty} \alpha \max\{d(u_n,z), d(u_n,u_{n+1}), d(z,Tz), d(u_n,Tz), d(z,u_{n+1})\} \\ &\leq \alpha d(z,Tz). \end{aligned}$$

Therefore, $(1-\alpha)d(z,Tz) \leq 0$, which implies $d(z,Tz) = 0$. Since Tz is closed, we have $z \in Tz$. This completes the proof. □

Corollary 3.1.6. *Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that Tx is compact for all $x \in X$, with the function φ is defined as Theorem 3.1.5. Assume that there exists $\alpha \in [0, \frac{1}{5})$ such that*

$$\varphi(r)d(x,Tx) \leq d(x,y) \text{ implies } H(Tx,Ty) \leq S(x,y)$$

where $S(x,y) = \alpha d(x,y) + \alpha d(x,Tx) + \alpha d(y,Ty) + \alpha d(x,Ty) + \alpha d(y,Tx)$ for all $x, y \in X$, and $r = 5\alpha$. Then there exists $z \in X$ such that $z \in Tz$.

Theorem 3.1.7. *Define a non-increasing function φ from $[0,1)$ into $(0,1]$ by*

$$\varphi(r) = \begin{cases} 1, & \text{if } 0 \leq r < \frac{1}{2}, \\ 1 - r, & \text{if } \frac{1}{2} \leq r < 1. \end{cases}$$

Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that Tx is compact for all $x \in X$. Assume that there exists $\alpha \in [0, \frac{1}{5})$ such that

$$\varphi(r)d(x,Tx) \leq d(x,y) \text{ implies } H(Tx,Ty) \leq S(x,y) \tag{3.1.25}$$

where $S(x,y) = \alpha d(x,y) + \alpha d(x,Tx) + \alpha d(y,Ty) + \alpha d(x,Ty) + \alpha d(y,Tx)$ for all $x, y \in X$, and $r = \frac{3\alpha}{1-2\alpha}$. Then there exists $z \in X$ such that $z \in Tz$.

Proof. Let α_1 be a real number such that $0 \leq \alpha < \alpha_1 < \frac{1}{5}$. Let $u_1 \in X$ and $u_2 \in Tu_1$ be arbitrary. Since $u_2 \in Tu_1$, then $d(u_2, Tu_2) \leq H(Tu_1, Tu_2)$ and

$$\varphi(r)d(u_1, Tu_1) \leq d(u_1, Tu_1) \leq d(u_1, u_2).$$

Thus, from the assumption (3.1.25), we obtain that

$$d(u_2, Tu_2) \leq H(Tu_1, Tu_2) \leq S(u_1, u_2)$$

where $S(u_1, u_2) = \alpha d(u_1, u_2) + \alpha d(u_1, Tu_1) + \alpha d(u_2, Tu_2) + \alpha d(u_1, Tu_2) + \alpha d(u_2, Tu_1)$. Consider,

$$d(u_2, Tu_2) \leq \alpha d(u_1, u_2) + \alpha d(u_1, Tu_1) + \alpha d(u_2, Tu_2) + \alpha d(u_1, Tu_2) + \alpha d(u_2, Tu_1)$$

$$\leq 3\alpha d(u_1, u_2) + 2\alpha d(u_2, Tu_2).$$

Then,

$$d(u_2, Tu_2) \leq (\frac{3\alpha}{1-2\alpha})d(u_1, u_2) = rd(u_1, u_2),$$

where $r = \frac{3\alpha}{1-2\alpha}$.
So, there exists $u_3 \in Tu_2$ such that $d(u_2, u_3) \leq r_1 d(u_1, u_2)$, where $r_1 = \frac{3\alpha_1}{1-2\alpha_1}$. Thus, we can construct a sequence $\{u_n\}$ in X such that $u_{n+1} \in Tu_n$ and

$$d(u_{n+1}, u_{n+2}) \leq r_1 d(u_n, u_{n+1}).$$

By induction, we obtain that

$$d(u_n, u_{n+1}) \leq (r_1)^{n-1} d(u_1, u_2).$$

Then by the triangle inequality, we have

$$\sum_{n=1}^{\infty} d(u_n, u_{n+1}) \leq \sum_{n=1}^{\infty} (r_1)^{n-1} d(u_1, u_2) < \infty.$$

Hence we conclude that $\{u_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that

$$\lim_{n\to\infty} u_n = z.$$

Now, we will show that $d(z, Tx) \leq rd(x, Tx)$ for all $x \in X \setminus \{z\}$.
Let $x \in X \setminus \{z\}$.
Since $u_n \to z$, there exists $n_0 \in N$ such that $d(z, u_n) \leq (\frac{1}{3})d(z, x)$ for all $n \geq n_0$.
By using form (3.1.3), we get that

$$\varphi(r)d(u_n, Tu_n) \leq d(x, u_n).$$

Then from (3.1.1), we have

$$H(Tu_n, T_x) \leq \alpha[d(u_n, x) + d(u_n, Tu_n) + d(x, Tx) + d(u_n, Tx) + d(x, Tu_n)].$$

Since $u_{n+1} \in Tu_n$, then $d(u_{n+1}, T_x) \leq H(Tu_n, T_x)$, so that

$$d(u_{n+1}, Tx) \leq \alpha[d(u_n, x) + d(u_n, u_{n+1}) + d(x, Tx) + d(u_n, Tx) + d(x, u_{n+1})]$$

for all $n \geq n_0$. Letting $n \to \infty$, we obtain that

$$\begin{aligned} d(z,Tx) &\leq \alpha[2d(z,x) + d(x,Tx) + d(z,Tx)] \\ &\leq \alpha 3d(z,x) + \alpha 2d(z,Tx). \end{aligned}$$

It follows that

$$d(z,Tx) \leq (\frac{3\alpha}{1-2\alpha})d(x,Tx) = rd(x,Tx), \quad \forall x \in X \setminus \{z\}. \tag{3.1.26}$$

Next, we show that $z \in Tz$. Suppose that z is not an element in Tz.

Case i; $0 \leq r < \frac{1}{2}$. Let $a \in Tz$. Then $a \neq z$ and so by (3.1.26), we have

$$d(z,Ta) \leq rd(a,Ta).$$

On the other hand, since $\varphi(r)d(z,Tz) = d(z,Tz) \leq d(z,a)$, from (3.1.25) we have

$$H(Tz,Ta) \leq \alpha[d(z,a) + d(z,Tz) + d(a,Ta) + d(z,Ta) + d(a,Tz)].$$

So,

$$\begin{aligned} d(a,Ta) \leq H(Tz,Ta) &\leq \alpha[2d(z,a) + d(a,Ta) + d(z,Ta)] \\ &\leq \alpha[3d(z,a) + 2d(a,Ta)]. \end{aligned} \tag{3.1.27}$$

Since $d(z,a) \leq d(z,Tz) + d(Tz,a) = d(z,Tz)$, we have

$$d(a,Ta) \leq (\frac{3\alpha}{1-2\alpha})d(z,Tz) = rd(z,Tz). \tag{3.1.28}$$

Using, (3.1.25)-(3.1.28), we have

$$\begin{aligned} d(z,Tz) &\leq d(z,Ta) + H(Ta,Tz) \\ &\leq rd(a,Ta) + S(a,z) \\ &\leq rd(a,Ta) + \alpha[d(z,a) + d(z,Tz) + d(a,Ta) + d(z,Ta) + d(a,Tz)] \\ &\leq (r+2\alpha)d(a,Ta) + 3\alpha d(z,a) \\ &\leq (r+2\alpha)rd(z,Tz) + 3\alpha d(z,Tz) \\ &\leq (r+r)rd(z,Tz) + rd(z,Tz) \end{aligned}$$

$$\leq (2r^2 + r)d(z, Tz).$$

Since $0 \leq r < \frac{1}{2}$, we have $0 \leq 2r^2 + r < 1$ and so, $d(z, Tz) < d(z, Tz)$, a contradiction. Thus $z \in Tz$.

Case ii; $\frac{1}{2} \leq r < 1$. Let $x \in X$.
If $x = z$, then $H(Tx, Tz) \leq \alpha[d(x, z) + d(x, Tx) + d(z, Tz) + d(x, Tz) + d(z, Tx)]$ holds. If $x \neq z$, then for all $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z, y_n) \leq d(z, Tx) + (\frac{1}{n})d(x, z).$$

We consider

$$\begin{aligned} d(x, Tx) &\leq d(x, y_n) \\ &\leq d(x, z) + d(z, y_n) \\ &\leq d(x, z) + d(z, Tx) + (\frac{1}{n})d(x, z) \\ &\leq d(x, z) + rd(x, Tx) + (\frac{1}{n})d(x, z). \end{aligned}$$

Thus, $(1 - r)d(x, Tx) \leq (1 + \frac{1}{n})d(x, z)$. Take $n \to \infty$, we obtain that

$$(1 - r)d(x, Tx) \leq d(x, z).$$

By using (3.1.25), implies $H(Tx, Tz) \leq S(x, z)$,
where $S(x, z) = \alpha[d(x, z) + d(x, Tx) + d(z, Tz) + d(x, Tz) + d(z, Tx)]$.
Hence, as $u_{n+1} \in Tu_n$. It follows that with $x = u_n$, this yields

$$\begin{aligned} d(z, Tz) &= \lim_{n\to\infty} d(u_{n+1}, Tz) \\ &\leq \lim_{n\to\infty} H(Tu_n, Tz) \\ &\leq \lim_{n\to\infty} \alpha[d(u_n, z) + d(u_n, Tu_n) + d(z, Tz) + d(u_n, Tz) + d(z, Tu_n)] \\ &\leq \lim_{n\to\infty} [\alpha d(u_n, z) + \alpha d(u_n, u_{n+1}) + \alpha d(z, Tz) + \alpha d(u_n, Tz) + \alpha d(z, u_{n+1})] \\ &\leq (2\alpha)d(z, Tz). \end{aligned} \tag{3.1.29}$$

Therefore, $(1 - 2\alpha)d(z, Tz) \leq 0$, which implies $d(z, Tz) = 0$. Since Tz is closed, we have $z \in Tz$. This completes the proof. □

3.2 Coupled Fixed Point Theorems for Multivalued Mappings in Metric Spaces

In this section, we introduce the notions of type multi-valued-coupled contraction mappings and multi-valued-coupled Kannan mappings and prove coupled fixed point theorems on metric spaces. Moreover, we present some examples to illustrate and support our results. Let (X, d) be a metric spaces. A map $T : X \times X \to CB(X)$ has a property coupled fixed point in X, if there exist $x, y \in X$ such that $x \in T(x, y)$ and $y \in T(y, x)$. Moreover, we call a mapping T is *continuous* provided that $T(x_n, y_n) \to T(x, y)$ whenever $x_n \to x$ and $y_n \to y$, where $x, y, x_n, y_n \in X$, for $n = 1, 2, 3, \dots$. The family of all nonempty compact subset of X, denoted by 2^X.

Definition 3.2.1. Let (X, d) be a metric space and let $\{A_n\} \subseteq 2^X$. Then $A_n \to A$ if and only if $H(A_n, A) \to 0$, where $A \in 2^X$.

Definition 3.2.2. Let (X, d) be a metric space. A map $T : X \times X \to 2^X$ is said to be a *multi-valued-coupled contraction mapping* of X if and only if there exists $k \in [0, 1)$ such that

$$H(T(x, y), T(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \tag{3.2.1}$$

for all $x, y, u, v \in X$.

Lemma 3.2.3. *Let (X, d) be a metric space and let $T : X \times X \to 2^X$ be a multi-valued-coupled contraction mapping with constant $k \in [0, 1)$. Then T is continuous on X.*

Proof. Suppose that $x_n \to x$ and $y_n \to y$. Since

$$H(T(x_n, y_n), T(x, y)) \leq \frac{k}{2}[d(x_n, x) + d(y_n, y)] \to 0 \tag{3.2.2}$$

as $n \to \infty$. Hence $T(x_n, y_n) \to T(x, y)$. Therefore T is continuous on X. □

Theorem 3.2.4. *Let (X,d) be a complete metric space and let $T : X \times X \to 2^X$ be a multi-valued-coupled contraction mapping with constant $k \in [0,1)$. Suppose that $x_0, y_0 \in X$. If there exist $x_1, y_1 \in X$ such that $x_1 \in T(x_0, y_0)$ and $y_1 \in T(y_0, x_0)$, then T has coupled fixed points in X.*

Proof. Let $x_0, y_0 \in X$. Choose $x_1 \in T(x_0, y_0)$ and $y_1 \in T(y_0, x_0)$. Since $T(x_0, y_0), T(y_0, x_0) \in CB(X)$ and $x_1 \in T(x_0, y_0)$, $y_1 \in T(y_0, x_0)$, there are points $x_2 \in T(x_1, y_1)$ and $y_2 \in T(y_1, x_1)$ such that

$$d(x_1, x_2) \leq H(T(x_0, y_0), T(x_1, y_1)) \tag{3.2.3}$$

and

$$d(y_1, y_2) \leq H(T(y_0, x_0), T(y_1, x_1)). \tag{3.2.4}$$

Since $T(x_1, y_1), T(y_1, x_1) \in CB(X)$ and $x_2 \in T(x_1, y_1)$, $y_2 \in T(y_1, x_1)$, there are points $x_3 \in T(x_2, y_2)$ and $y_3 \in T(y_2, x_2)$ such that

$$d(x_2, x_3) \leq H(T(x_1, y_1), T(x_2, y_2)) \tag{3.2.5}$$

and

$$d(y_2, y_3) \leq H(T(y_1, x_1), T(y_2, x_2)). \tag{3.2.6}$$

Continuing in this previous producers, we obtain sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ of points of X such that $x_{i+1} \in T(x_i, y_i)$, $y_{i+1} \in T(y_i, x_i)$ and

$$d(x_i, x_{i+1}) \leq H(T(x_{i-1}, y_{i-1}), T(x_i, y_i)), \tag{3.2.7}$$

$$d(y_i, y_{i+1}) \leq H(T(y_{i-1}, x_{i-1}), T(y_i, x_i)) \tag{3.2.8}$$

for all $i \geq 1$, respectively. We consider

$$\begin{aligned} d(x_i, x_{i+1}) &\leq H(T(x_{i-1}, y_{i-1}), T(x_i, y_i)) \\ &\leq \frac{k}{2} d(x_{i-1}, x_i) + \frac{k}{2} d(y_{i-1}, y_i) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{k}{2}H(T(x_{i-2}, y_{i-2}), T(x_{i-1}, y_{i-1})) + \frac{k}{2}H(T(y_{i-2}, x_{i-2}), T(y_{i-1}, x_{i-1}))\\
&\le \frac{k^2}{2}d(x_{i-2}, x_{i-1}) + \frac{k^2}{2}d(x_{i-2}, x_{i-1})\\
&\vdots\\
&\le \frac{k^i}{2}[d(x_0, x_1) + d(x_1, x_0)] \qquad (3.2.9)
\end{aligned}
$$

for all $i \geq 1$. Similarly,

$$
d(y_i, y_{i+1}) \le \frac{k^i}{2}[d(x_0, x_1) + d(x_1, x_0)]. \qquad (3.2.10)
$$

Take $i \to \infty$ in (3.2.9) and (3.2.10), we get $d(x_i, x_{i+1}) \to 0$ and $d(y_i, y_{i+1}) \to 0$. Let $n, m \in \mathbb{N}$ with $m > n$, using the triangular inequality, we conclude that

$$
\begin{aligned}
d(x_m, x_n) &\le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n)\\
&\le H(T(x_{m-1}), T(x_{m-2})) + H(T(x_{m-2}), T(x_{m-3})) + ... + H(T(x_n), T(x_{n-1}))\\
&\le \frac{k^{m-1}}{2}[d(x_0, x_1) + d(x_1, x_0)] + \frac{k^{m-2}}{2}[d(x_0, x_1) + d(x_1, x_0)] + ...\\
&\quad + \frac{k^{n-1}}{2}[d(x_0, x_1) + d(x_1, x_0)]\\
&\le \frac{k^{n-1}}{2}[1 + k + ... + k^{m-n} + ...][d(x_0, x_1) + d(x_1, x_0)]\\
&= \frac{k^{n-1}}{2-2k}[d(x_0, x_1) + d(x_1, x_0)] \to 0 \qquad (3.2.11)
\end{aligned}
$$

as $n \to \infty$. Similarly, $d(y_m, y_n) \to 0$. Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete, we have $x_n \to x$ and $y_n \to y$. By the continuity of T, it follows that $T(x_n, y_n) \to T(x, y)$ and $T(y_n, x_n) \to T(y, x)$. Since $x_n \in T(x_{n-1}, y_{n-1})$ and $y_n \in T(y_{n-1}, x_{n-1})$ for all n, we get $x \in T(x, y)$ and $y \in T(y, x)$. That is, T has coupled a fixed point. □

Example 3.2.5. Let $X = \mathbb{R}$. Define $d : X \times X \to \mathbb{R}$ by

$$
d(x, y) = |x - y|,
$$

for every $x, y \in \mathbb{R}$ and $T : X \times X \to 2^X$ be defined by $T(x, y) = [\frac{\min\{x,y\}}{3}, \frac{\max\{x,y\}}{3}]$ for all $x, y \in X$. Indeed, we see that d is a complete metric on X. Next, we will show that T satisfies (3.2.1). Let $x, y, u, v \in X$. If $x \geq y$ and $u \geq v$, then

$$d(T(x,y),T(u,v)) = d([\frac{y}{3},\frac{x}{3}],[\frac{v}{3},\frac{u}{3}]) = \max\{|\frac{x}{3}-\frac{u}{3}|,|\frac{y}{3}-\frac{v}{3}|\}$$
$$\leq \frac{1}{3}(|x-u|+|y-v|) = \frac{k}{2}(d(x,u)+d(y,v)). \quad (3.2.12)$$

If $x \geq y$ and $u < v$, then

$$d(T(x,y),T(u,v)) = d([\frac{y}{3},\frac{x}{3}],[\frac{u}{3},\frac{v}{3}]) = \max\{|\frac{x}{3}-\frac{v}{3}|,|\frac{y}{3}-\frac{u}{3}|\}$$
$$\leq \frac{1}{3}(|x-u|+|y-v|) = \frac{k}{2}(d(x,u)+d(y,v)). \quad (3.2.13)$$

If $x < y$ and $u \geq v$, then

$$d(T(x,y),T(u,v)) = d([\frac{x}{3},\frac{y}{3}],[\frac{v}{3},\frac{u}{3}]) = \max\{|\frac{y}{3}-\frac{u}{3}|,|\frac{x}{3}-\frac{v}{3}|\}$$
$$\leq \frac{1}{3}[|x-u|+|y-v|] = \frac{k}{2}(d(x,u)+d(y,v)). \quad (3.2.14)$$

If $x < y$ and $u < v$, then

$$d(T(x,y),T(u,v)) = d([\frac{x}{3},\frac{y}{3}],[\frac{u}{3},\frac{v}{3}]) = \max\{|\frac{y}{3}-\frac{v}{3}|,|\frac{x}{3}-\frac{u}{3}|\}$$
$$\leq \frac{1}{3}[|x-u|+|y-v|] = \frac{k}{2}(d(x,u)+d(y,v)), \quad (3.2.15)$$

where $k = \frac{2}{3} < 1$. By Theorem 3.2.4, we obtain that $(0,0) \in T(0,0)$.

Definition 3.2.6. Let (X,d) be a metric space. A map $T : X \times X \to 2^X$ is said to be a *multi-valued-coupled Kannan mapping* of X if and only if there exists $r \in [0, \frac{1}{2})$ such that

$$H(T(x,y),T(u,v)) \leq r[d(x,T(x,y)) + d(u,T(u,v))], \quad (3.2.16)$$

for all $x, y, u, v \in X$.

Theorem 3.2.7. *Let (X,d) be a complete metric space, and $T : X \times X \to 2^X$ be a multi-valued-coupled Kannan mapping with constant $k \in [0,1)$. Suppose that $x_0, y_0 \in X$. If there exist $x_1, y_1 \in X$ such that $x_1 \in T(x_0,y_0)$ and $y_1 \in T(y_0,x_0)$, then T has coupled fixed points in X.*

Proof. Let $x_0, y_0 \in X$. Choose $x_1 \in T(x_0, y_0)$ and $y_1 \in T(y_0, x_0)$. Since $T(x_0, y_0), T(y_0, x_0) \in CB(X)$ and $x_1 \in T(x_0, y_0)$, $y_1 \in T(y_0, x_0)$, there are points $x_2 \in T(x_1, y_1)$ and $y_2 \in T(y_1, x_1)$ such that

$$d(x_1, x_2) \leq H(T(x_0, y_0), T(x_1, y_1)) \leq rd(x_0, T(x_0, y_0)) + rd(x_1, T(x_1, y_1)) \leq rd(x_0, x_1) + rd(x_1, x_2) \quad (3.2.17)$$

and

$$d(y_1, y_2) \leq H(T(y_0, x_0), T(y_1, x_1)) \leq rd(y_0, T(y_0, x_0)) + rd(y_1, T(y_1, x_1)) \leq rd(y_0, y_1) + rd(y_1, y_2). \quad (3.2.18)$$

So,

$$d(x_1, x_2) \leq \frac{r}{1-r} d(x_0, x_1), \quad d(y_1, y_2) \leq \frac{r}{1-r} d(y_0, y_1). \quad (3.2.19)$$

Since $T(x_1, y_1), T(y_1, x_1) \in CB(X)$ and $x_2 \in T(x_1, y_1)$, $y_2 \in T(y_1, x_1)$, there are points $x_3 \in T(x_2, y_2)$ and $y_3 \in T(y_2, x_2)$ such that

$$d(x_2, x_3) \leq H(T(x_1, y_1), T(x_2, y_2)) \leq rd(x_1, T(x_1, y_1)) + rd(x_2, T(x_2, y_2)) \leq rd(x_1, x_2) + rd(x_2, x_3) \quad (3.2.20)$$

and

$$d(y_2, y_3) \leq H(T(y_1, x_1), T(y_2, x_2)) \leq rd(y_1, T(y_1, x_1)) + rd(y_2, T(y_2, x_2)) \leq rd(y_1, y_2) + rd(y_2, y_3). \quad (3.2.21)$$

Thus

$$d(x_2, x_3) \leq \frac{r}{1-r} d(x_1, x_2), \quad d(y_2, y_3) \leq \frac{r}{1-r} d(y_1, y_2). \quad (3.2.22)$$

Continuing in this previous producers, we obtain sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$ of points of X such that $x_{i+1} \in T(x_i, y_i)$, $y_{i+1} \in T(y_i, x_i)$ and

$$d(x_i, x_{i+1}) \leq \frac{r}{1-r} d(x_{i-1}, x_i), \quad (3.2.23)$$

$$d(y_i, y_{i+1}) \leq \frac{r}{1-r} d(y_{i-1}, y_i) \tag{3.2.24}$$

for all $i \geq 1$, respectively. We note that

$$\begin{aligned} d(x_i, x_{i+1}) &\leq \frac{r}{1-r} d(x_{i-1}, x_i) \\ &\leq (\frac{r}{1-r})^2 d(x_{i-2}, x_{i-1}) \\ &\leq (\frac{r}{1-r})^3 d(x_{i-3}, x_{i-2}) \\ &\vdots \\ &\leq (\frac{r}{1-r})^i d(x_0, x_1) \end{aligned} \tag{3.2.25}$$

for all $i \geq 1$. Similarly,

$$d(y_i, y_{i+1}) \leq (\frac{r}{1-r})^i d(x_1, x_0). \tag{3.2.26}$$

Take $i \to \infty$ in (3.2.25) and (3.2.26), we get $d(x_i, x_{i+1}) \to 0$ and $d(y_i, y_{i+1}) \to 0$. Let $n, m \in \mathbb{N}$ with $m > n > n(\epsilon)$, using the triangular inequality, we obtain that

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + ... + d(x_{n+1}, x_n) \\ &\leq (\frac{r}{1-r})m - 1d(x_0, x_1) + (\frac{r}{1-r})m - 2d(x_0, x_1) + ... \\ &\quad + (\frac{r}{1-r})n - 1d(x_0, x_1) \\ &\leq \frac{k^{n-1}}{2}[1 + k + ... + k^{m-n} + ...][d(x_0, x_1) + d(x_1, x_0)] \\ &= \frac{k^{n-1}}{2-2k}[d(x_0, x_1) + d(x_1, x_0)] \to 0 \end{aligned} \tag{3.2.27}$$

as $n \to \infty$. Similarly, $d(y_m, y_n) \to 0$. Thus $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete, we have that $\{x_n\}$ and $\{y_n\}$ converges to some $z_1, z_2 \in X$, respectively. Next, we will show that $d(z_1, T(z_1, z_2)) = 0$ and $d(z_2, T(z_2, z_1)) = 0$. Consider

$$\begin{aligned} d(z_1, T(z_1, z_2)) &\leq d(z_1, x_n) + d(x_n, T(z_1, z_2)) \\ &\leq d(z_1, x_n) + H(T(x_{n-1}, y_{n-1}), T(z_1, z_2)) \\ &\leq d(z_1, x_n) + rd(x_{n-1}, T(x_{n-1}, y_{n-1})) + rd(z_1, T(z_1, z_2)) \end{aligned}$$

$$\leq d(z_1, x_n) + rd(x_{n-1}, x_n) + rd(z_1, T(z_1, z_2)). \tag{3.2.28}$$

It follows that,

$$\begin{aligned} d(z_1, T(z_1, z_2)) &\leq d(z_1, x_n) + d(x_n, T(z_1, z_2)) \\ &\leq \frac{1}{1-r} d(z_1, x_n) + \frac{r}{1-r} d(x_{n-1}, x_n) \to 0 \end{aligned} \tag{3.2.29}$$

as $n \to \infty$. Next, consider

$$\begin{aligned} d(z_2, T(z_2, z_1)) &\leq d(z_2, y_n) + d(y_n, T(z_2, z_1)) \\ &\leq d(z_2, y_n) + H(T(y_{n-1}, x_{n-1}), T(z_2, z_1)) \\ &\leq d(z_2, y_n) + rd(y_{n-1}, T(y_{n-1}, x_{n-1})) + rd(z_2, T(z_2, z_1)) \\ &\leq d(z_2, y_n) + rd(y_{n-1}, y_n) + rd(z_2, T(z_2, z_1)). \end{aligned} \tag{3.2.30}$$

We get,

$$\begin{aligned} d(z_2, T(z_2, z_1)) &\leq d(z_2, y_n) + d(y_n, T(z_2, z_1)) \\ &\leq \frac{1}{1-r} d(z_2, y_n) + \frac{r}{1-r} d(y_{n-1}, y_n) \to 0 \end{aligned} \tag{3.2.31}$$

as $n \to \infty$. Hence $z_1 \in T(z_1, z_2)$ and $z_2 \in T(z_2, z_1)$. Therefore T has a coupled fixed points. □

Example 3.2.8. Let $X = [0, \infty)$. Define a function $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x - y|,$$

for every $x, y \in \mathbb{R}$ and $T : X \times X \to 2^X$ defined by $T(x, y) = [0, \frac{\max\{x,y\}}{4}]$ for all $x, y \in X$. Indeed, we see that d is a complete metric on X. Next, we will show that T satisfies (3.2.1). Let $x, y, u, v \in X$. If $x \geq y$ and $u \geq v$, then

$$\begin{aligned} d(T(x, y), T(u, v)) &= d([0, \frac{x}{4}], [0, \frac{u}{4}]) = |\frac{x}{4} - \frac{u}{4}| \\ &\leq \frac{1}{3}(|\frac{3x}{4}| + |\frac{3u}{4}|) = \frac{1}{3}(|x - \frac{x}{4}| + |u - \frac{u}{4}|) \end{aligned}$$

$$= r[d(x, T(x,y)) + d(u, T(u,v))]. \tag{3.2.32}$$

If $x \geq y$ and $u < v$, then

$$\begin{aligned} d(T(x,y), T(u,v)) &= d([0, \frac{x}{4}], [0, \frac{v}{4}] = |\frac{x}{4} - \frac{v}{4}| \\ &\leq \frac{1}{3}(|\frac{3x}{4}| + |\frac{3v}{4}|) = \frac{1}{3}(|\frac{3x}{4}| + |\frac{v}{4} - v|) \\ &\leq \frac{1}{3}(|x - \frac{x}{4}| + |\frac{v}{4} - u|) = r[d(x, T(x,y)) + d(u, T(u,v))]. \end{aligned} \tag{3.2.33}$$

If $x < y$ and $u \geq v$, then

$$\begin{aligned} d(T(x,y), T(u,v)) &= d([0, \frac{y}{4}], [0, \frac{u}{4}]) = |\frac{y}{4} - \frac{u}{4}| \\ &\leq \frac{1}{3}(|\frac{y}{4} - y| + |\frac{3u}{4}|) \leq \frac{1}{3}(|\frac{y}{4} - x| + |\frac{3u}{4}|) \\ &= r[d(x, T(x,y)) + d(u, T(u,v))]. \end{aligned} \tag{3.2.34}$$

If $x < y$ and $u < v$, then

$$\begin{aligned} d(T(x,y), T(u,v)) &= d([0, \frac{y}{4}], [0, \frac{v}{4}]) = |\frac{y}{4} - \frac{v}{4}| \\ &\leq \frac{1}{3}(|\frac{y}{4} - y| + |\frac{v}{4} - v|) \leq \frac{1}{3}(|\frac{y}{4} - x| + |\frac{v}{4} - u|) \\ &= r[d(x, T(x,y)) + d(u, T(u,v))], \end{aligned} \tag{3.2.35}$$

where $k = \frac{1}{3} < 1$. By Theorem 3.2.4, it follows that $(0,0) \in T(0,0)$.

CHAPTER IV

FIXED POINT THEOREMS IN DISLOCATED QUASI-B-METRIC SPACES

4.1 Basic Properties of Dislocated Quasi-b-Metric Spaces

In this section, we establish dislocated quasi-b-metric spaces and prove basic properties of dislocated quasi-b-metric spaces. Moreover, we give examples as a satisfying the such spaces. Every dislocated quasi-b-metric space X can be considered as a topological space on which the topology is introduced by taking, for any $x \in X$, the collection $\{B_r(x) | r > 0\}$ as a base of the neighbourhood filter of the point x. Here the ball $B_r(x)$ is defined by the equality $B_r(x) = \{y \in X | \max\{d(x,y), d(y,x)\} < r\}$; see [43] for more details.

Definition 4.1.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ such that constant $s \geq 1$ satisfies the following conditions:

(d1) $d(x,y) = d(y,x) = 0$ implies $x = y$ for all $x, y \in X$;

(d2) $d(x,y) \leq s[d(x,z) + d(z,y)]$, for all $x, y, z \in X$.

The pair (X, d) is then called a *dislocated quasi b-metric space (or simply dqb-metric)*. The number s is called to be the *coefficient* of d.

Remark 4.1.2. It is obvious that b-metric spaces, quasi b-metric space and b-metric-like spaces is dislocated quasi b-metric space but conversely is not true.

Example 4.1.3. Let $X = \mathbb{R}$ and let

$$d(x,y) = |x-y|^2 + \frac{|x|}{n} + \frac{|y|}{m},$$

where $n, m \in \mathbb{N} \setminus \{1\}$ with $n \neq m$. Then (X, d) is a dislocated quasi b-metric space with the coefficient $s = 2$.

Proof. Let $x, y, z \in X$. Suppose that $d(x,y) = 0$.

Then

$$|x-y|^2 + \frac{|x|}{n} + \frac{|y|}{m} = 0.$$

It implies that $|x-y|^2 = 0$ and so, $x = y$.

Next, consider

$$\begin{aligned}
d(x,y) &= |x-y|^2 + \frac{|x|}{n} + \frac{|y|}{m} \\
&\leq (|x-z| + |z-y|)^2 + \frac{|x|}{n} + \frac{|y|}{m} \\
&\leq |x-z|^2 + 2|x-z|\cdot|z-y| + |z-y|^2 + \frac{|x|}{n} + \frac{|y|}{m} \\
&\leq 2(|x-z|^2 + |z-y|^2) + \frac{|x|}{n} + \frac{|z|}{m} + \frac{|z|}{n} + \frac{|y|}{m} \\
&\leq s[d(x,z) + d(z,y)],
\end{aligned}$$

where $s = 2$.

Then (X,d) is a dislocated quasi b-metric space with the coefficient $s = 2$, but since $d(1,1) \neq 0$, we have (X,b) is not a quasi b-metric space and since $d(1,2) \neq d(2,1)$, we have (X,b) is not a b-metric-like space. And, (X,b) is not a dislocated quasi-metric space because

$$\begin{aligned}
d(\frac{1}{2},\frac{1}{4}) &= |\frac{1}{2} - \frac{1}{4}|^2 + \frac{|\frac{1}{2}|}{n} + \frac{|\frac{1}{4}|}{m} \\
&= \frac{1}{16} + \frac{1}{2n} + \frac{1}{4m} \\
&= \frac{324}{5184} + \frac{3}{6n} + \frac{4}{12m} \\
&> \frac{180}{5184} + \frac{5}{6n} + \frac{7}{12m} \\
&= \frac{1}{36} + \frac{1}{2n} + \frac{1}{3m} + \frac{1}{144} + \frac{1}{3n} + \frac{1}{4m} \\
&= |\frac{1}{2} - \frac{1}{3}|^2 + \frac{|\frac{1}{2}|}{n} + \frac{|\frac{1}{3}|}{m} + |\frac{1}{3} - \frac{1}{4}|^2 + \frac{|\frac{1}{3}|}{n} + \frac{|\frac{1}{4}|}{m} \\
&= d(\frac{1}{2},\frac{1}{3}) + d(\frac{1}{3},\frac{1}{4}),
\end{aligned}$$

where $n, m > 42$. □

Example 4.1.4. ([44]) Let $X = 0, 1, 2$, and let

$$d(x,y) = \begin{cases} 2 & x = y = 0, \\ \frac{1}{2} & x = 0,\ y = 1, \\ 2 & x = 1\ y = 0, \\ \frac{1}{2} & \textit{otherwise}. \end{cases}$$

Then (X, d) is a dislocated quasi b-metric space with the coefficient $s = 2$, but since $d(1, 1) \neq 0$, we have (X, b) is not a quasi b-metric space and since $d(1, 2) \neq d(2, 1)$, we have (X, b) is not a b-metric-like space. It is obvious that (X, b) is not a dislocated quasi-metric space.

Example 4.1.5. Let $X = \mathbb{R}$ and let

$$d(x, y) = |x - y|^2 + 3|x|^2 + 2|y|^2.$$

Then (X, d) is a dislocated quasi b-metric space with the coefficient $s = 2$, but since $d(0, 1) \neq d(1, 0)$, we have (X, b) is not a b-metric-like space, since $d(1, 1) \neq 0$, we have (X, b) is not a quasi b-metric space. It is obvious that (X, b) is not a dislocated quasi-metric space.

Example 4.1.6. Let $X = \mathbb{R}$ and let

$$d(x, y) = |2x - y|^2 + |2x + y|^2.$$

Then (X, d) is a dislocated quasi b-metric space with the coefficient $s = 2$, but since $d(1, 1) \neq 0$, we have (X, b) is not a quasi b-metric space. It is obvious that (X, b) is not a dislocated quasi-metric space.

We will introduce dqb-convergent sequence, dqb-Cauchy sequence and complete of spaces according to Zoto, Kumari and Hoxha [18].

Definition 4.1.7. Let (X, d) be a dqb-metric space.

(1) A sequence $\{x_n\}$ in X, *converges* (for short, dqb-converges) to $x \in X$

if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

In this case x is called a *limit point* (for short, dqb-limit point) of $\{x_n\}$ and we write $x_n \to x$.

(2) A sequence $\{x_n\}$ in X, is call *Cauchy* (for short, dqb-Cauchy) if

$$\lim_{n,m\to\infty} d(x_n, x_m) = \lim_{n,m\to\infty} d(x_m, x_n) = 0.$$

(3) A dqb-metric space (X, d) is complete if every dqb-Cauchy sequence in it is dqb-convergent in X.

Definition 4.1.8. ([45]) Let X be a topological space. Then X is said to be *Hausdorff topological space* if for any distinct points $x, y \in X$, there exists two open sets K_1 and K_2 such that $x \in K_1$, and $y \in K_2$ and $K_1 \cap K_2 = \emptyset$.

Proposition 4.1.9. *Every dqb-metric space is a Hausdorff topological space.*

Proof. Let x and y be two distinct points in X. Then $d(x, y) > 0$ and $d(y, x) > 0$. Choose $\delta = \frac{d(x,y)}{2s}$. Then, we have

$$B_\delta(x) = \{z \in X | \max\{d(x, z), d(z, x)\} < \delta\}$$

and

$$B_\delta(y) = \{z \in X | \max\{d(y, z), d(z, y)\} < \delta\}$$

such that $x \in B_\delta(x)$ and $y \in B_\delta(y)$.

To show that $B_\delta(x) \cap B_\delta(y) = \emptyset$, suppose that $B_\delta(x) \cap B_\delta(y) \neq \emptyset$. Then, there exists $z \in B_\delta(x) \cap B_\delta(y)$. We have

$$\begin{aligned} d(x, y) &\leq sd(x, z) + sd(z, y) \\ &\leq s\max\{d(x, z), d(z, x)\} + s\max\{d(y, z), d(z, y)\} \\ &< s\delta + s\delta \end{aligned}$$

$$= d(x, y)$$

So, $d(x, y) < d(x, y)$ which is a contradiction. Therefore $B_\delta(x) \cap B_\delta(y) = \emptyset$. □

Proposition 4.1.10. *Every dqb-convergent sequence in a dqb-metric space (X, d) is a dqb-Cauchy sequence.*

Proof. Suppose that $\{x_n\}$ is dqb-convergent. Then there exists $x \in X$ such that $x_n \to x$, that is

$$\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n).$$

Consider, for any $n, m \in \mathbb{N}$,

$$d(x_n, x_m) \leq sd(x_n, x) + sd(x, x_m).$$

Taking $n, m \to \infty$ we obtain

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Similarly,

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0.$$

Therefore $\{x_n\}$ is dqb-Cauchy. □

Definition 4.1.11. A subset K of a dqb-metric space (X, d) is bounded if there exists $\bar{x}$, $M \in (0, \infty)$ such that $\max\{d(x, \bar{x}), d(\bar{x}, x)\} \leq M$ for all $x \in K$.

Proposition 4.1.12. *Every dqb-convergent sequence in a dqb-metric space (X, d) is a bounded sequence.*

Proof. Suppose that $\{x_n\}$ is dqb-convergent. Then there exists $x \in X$ such that $x_n \to x$, that is

$$\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n).$$

Let $\epsilon = 1$. Then there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < 1$ and $d(x, x_n) < 1$, for all $n \geq n_0$. Choose

$$K = \max\{d(x_1, x), d(x_2, x), ..., d(x_{n_0-1}, x), d(x, x_1), d(x, x_2), ..., d(x, x_{n_0-1}), 1\}.$$

Thus, $\max\{d(x_n, x), d(x, x_n)\} \leq K$, for all $n \in \mathbb{N}$, and so $\{x_n\}$ is a bounded sequence. □

Proposition 4.1.13. *Every dqb-Cauchy sequence in a dqb-metric space (X, d) is a bounded sequence.*

Proof. Suppose that $\{x_n\}$ is dqb-Cauchy. Then

$$\lim_{n\to\infty} d(x_n, x_m) = 0 = \lim_{n\to\infty} d(x_m, x_n).$$

Let $\epsilon = 1$. Then there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ and $d(x_m, x_n) < 1$ for all $n, m \geq n_0$. Let p be any point in the space, and let

$$k = \max_{i \leq m} d(x_i, p).$$

The maximum exists, since $\{x_i : i \leq m\}$ is a finite set. If $n \leq m$, then $d(x_n, p) \leq k$. If $n > m$, then $d(x_n, p) \leq d(x_n, x_m) + d(x_m, p) \leq 1 + k$ for all $n \in \mathbb{N}$. Therefore $\{x_n\}$ is a bounded sequence. □

For the subsequence of dqb-convergent sequence, we have the following important results:

Proposition 4.1.14. *Every subsequence of dqb-convergent sequence in a dqb-metric space (X, d) is a dqb-convergent sequence.*

For the subsequence of dqb-Cauchy sequence, we have the following important results:

Proposition 4.1.15. *Every subsequence of dqb-Cauchy sequence in a dqb-metric space (X, d) is a dqb-Cauchy sequence.*

Proposition 4.1.16. *Let $\{x_n\}$ be sequence in a dqb-metric space (X, d). If $x_n \to x$ and $x_n \to y$, then $x = y$.*

Proof. Suppose that $x_n \to x$ and $x_n \to y$. Then

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = \lim_{n\to\infty} d(x_n, y) = \lim_{n\to\infty} d(y, x_n) = 0.$$

Consider,

$$0 \leq d(x,y) \leq sd(x,x_n) + sd(x_n,y)$$

and

$$0 \leq d(y,x) \leq sd(y,x_n) + sd(x_n,x).$$

Taking limit as $n, m \to \infty$ we obtain

$$d(x,y) = d(y,x) = 0.$$

Therefore $x = y$. □

For the dqb-convergent sequence of dqb-metric space, we have the following important results:

Proposition 4.1.17. *Let $\{x_n\}$ be a sequence in a dqb-metric space (X, d). Then $x_n \to x$ if and only if $d(x_n, x) \to 0$ and $d(x, x_n) \to 0$.*

Now, we begin with introducing the property of a continuous functions.

Definition 4.1.18. Suppose that (X, d_X) and (Y, d_Y) are dislocated quasi b-metric spaces, $K \subset X$, $f : K \to Y$ and $p \in K$. Then f is continuous at p iff for all $\epsilon > 0$ there exists $\delta > 0$ such that $\max\{d_Y(fx, fp), d_Y(fp, fx)\} < \epsilon$ for all $x \in K$, when $\max\{d_X(x,p), d_X(p,x)\} < \delta$.

Theorem 4.1.19. *Let (X, d_X) and (X, d_Y) be dislocated quasi b-metric spaces, $K \subset X$, $f : K \to Y$ and $p \in K$. Then f is continuous at p if and only if for every dqb-converges sequence $\{x_n\}$ in X, $\lim_{n\to\infty} fx_n = fx$.*

Proof. Suppose that f is continuous at p and $\{x_n\}$ converges to p.
Let $\epsilon > 0$. Then there exists $\delta > 0$ such that $\max\{d_Y(fx, fp), d_Y(fp, fx)\} < \epsilon$,

when $\max\{d_X(x,p), d_X(p,x)\} < \delta$, for all $x \in K$. Since $\{x_n\}$ converges to p, there exists $N \in \mathbb{N}$ such that $\max\{d_X(x_n,p), d_Y(p,x_n)\} < \delta$, for all $n \geq N$. Since f is continuous at p, we have $\max\{d_Y(fx_n, fp), d_Y(fp, fx_n)\} < \epsilon$, for all $n \geq N$. Hence $\lim_n fx_n = fx$.

Conversely, let $x \in X$ and assume in the contrary that

$$\exists \epsilon > 0\ \forall \delta > 0 :\ \max\{d_X(x,p), d_X(p,x)\} < \delta,\ \max\{d_Y(fx,fp), d_Y(fp,fx)\} \geq \epsilon.$$

Applying these successively for all $\delta = \frac{1}{k}$, we find a sequence $\{x_k\}$ such that $\max\{d_X(x_k,p), d_X(p,x_k)\} < \frac{1}{k}$ and $\max\{d_Y(fx_k,fp), d_Y(fp,fx_k)\} \geq \epsilon'$. Thus

$$\lim_{k\to\infty} x_k = p.$$

By assumption, we have

$$\lim_{k\to\infty} fx_k = fp.$$

Hence, there exists a k_0 such that for all $k > k_0$, $\max\{d_Y(fx_k,fp), d_Y(fp,fx_k)\} < \epsilon$ which is a contradiction. □

Definition 4.1.20. Suppose that (X, d_X) and (Y, d_Y) are dislocated quasi-b-metric spaces, $K \subset X$, $f : K \to Y$ and $p \in K$. Then f is continuous on K iff f is continuous at p for all $p \in K$.

4.2 Fixed Point Theorems for Cyclic Contractions and Cyclic Weakly Contractions in Dislocated Quasi-b-Metric Spaces

In this section, we introduce the notions of type dqb-cyclic-Banach contraction, dqb-cyclic-Kannan mapping, type dqb-cyclic-weak Banach contraction and dqb-cyclic- contraction. Moreover, we prove fixed point theorems for some nonlinear mappings and give examples which satisfy the theorems in such spaces; see [43, 46] for more details.

Now, we begin with introducing the concept of a dqb-cyclic-Banach contraction.

Definition 4.2.1. Let A and B be nonempty subsets of a dislocated quasi-b-metric spaces (X, d), with constant $s \in [1, \infty)$. A cyclic map $T : A \cup B \to A \cup B$ is said to be a ***dqb-cyclic-Banach contraction*** and if there exists $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y), \tag{4.2.1}$$

for all $x \in A$, $y \in B$ and $sk \leq 1$.

Theorem 4.2.2. *Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X, d). Let T be a cyclic mapping. If T satisfies the condition a dqb-cyclic-Banach contraction with constant $k \in [0, 1)$, then T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A$, and using contractive condition of theorem, we have

$$\begin{aligned} d(T^2x, Tx) &= d(T(Tx), Tx) \\ &\leq kd(Tx, x), \end{aligned}$$

and

$$\begin{aligned} d(Tx, T^2x) &= d(Tx, T(Tx)) \\ &\leq kd(x, Tx). \end{aligned}$$

So,

$$d(T^2x, Tx) \leq k\alpha, \tag{4.2.2}$$

and

$$d(Tx, T^2x) \leq k\alpha, \tag{4.2.3}$$

where $\alpha = \max\{d(Tx, x), d(x, Tx)\}$.
By using (4.2.2) and (4.2.2), we have $d(T^3x, T^2x) \leq k^2\alpha$, and $d(T^2x, T^3x) \leq k^2\alpha$.
For all $n \in \mathbb{N}$, we get

$$d(T^{n+1}x, T^nx) \leq k^n\alpha,$$

and

$$d(T^n x, T^{n+1}x) \leq k^n \alpha.$$

Let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have,

$$\begin{aligned} d(T^m x, T^n x) &\leq s^{m-n} d(T^m x, T^{m-1}x) + s^{m-n-1} d(T^{m-1}x, T^{m-2}x) + ... + sd(T^{n+1}x, T^n x) \\ &\leq (s^{m-n}k^{m-1} + s^{m-n-1}k^{m-2} + s^{m-n-2}k^{m-3} + ... + s^2 k^{n+1} + sk^n)\alpha \\ &\leq (k^{n-1} + k^{n-1} + k^{n-1} + ... + k^{n-1} + k^{n-1})\alpha \\ &= (k^{n-1})(m - n + 1)\alpha \\ &\leq (k^{n-1})\xi\alpha, \end{aligned}$$

for some $\xi > m - n + 1$. Take $n \to \infty$, we get $d(T^m x, T^n x) \to 0$.
Similarly, let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have,

$$d(T^n x, T^m x) = (k^{n-1})\xi\alpha.$$

Take $n \to \infty$, we get $d(T^n x, T^m x) \to 0$. Thus $\{T^n x\}$ is a Cauchy sequence.
Since X is complete, we have $\{T^n x\}$ converges to some $z \in X$.
We note, that $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n-1}x\}$ is a sequence in B in a way that both sequences tend to the same limit z. Since A and B are closed, we have $z \in A \cap B$, and then $A \cap B \neq \emptyset$.

Now, we will show that $Tz = z$.
By using (4.2.1), we consider

$$\begin{aligned} d(T^n x, Tz) &= d(T(T^{n-1}x), Tz) \\ &\leq kd(T^{n-1}x, z) \\ &\leq d(T^{n-1}x, z). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(z, Tz) \leq kd(z, Tz) \leq d(z, Tz).$$

And so, $d(z, Tz) = kd(z, Tz)$, where $0 \leq k < 1$. This implies that $d(z, Tz) = 0$. Similarly considering from (4.2.1), we get

$$\begin{aligned} d(Tz, T^n x) &= d(Tz, T(T^{n-1}x)) \\ &\leq kd(z, T^{n-1}x) \\ &\leq d(z, T^{n-1}x). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(Tz, z) \leq kd(Tz, z) \leq d(Tz, z).$$

And so, $d(Tz, z) = kd(Tz, z)$, where $0 \leq k < 1$. This implies that $d(Tz, z) = 0$. Hence $d(z, Tz) = d(Tz, z) = 0$, which implies that $Tz = z$ that is z is a fixed point of T.

Finally, to prove the uniqueness of fixed point, let $z^* \in X$ be another fixed point of T. Thus $Tz^* = z^*$. Then, we have

$$d(z, z^*) = d(Tz, Tz^*) \leq kd(z, z^*). \tag{4.2.4}$$

On the other hand,

$$d(z^*, z) = d(Tz^*, Tz) \leq kd(z^*, z). \tag{4.2.5}$$

By (4.2.4) and (4.2.5), we obtain that $d(z, z^*) = d(z^*, z) = 0$, which implies that $z^* = z$. Therefore z is a unique fixed point of T. The proof is complete.. □

Example 4.2.3. Let $X = [-1, 1]$ and $T : A \cup B \to A \cup B$ defined by $Tx = \frac{-x}{5}$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Define the function $d : X^2 \to [0, \infty)$ by

$$d(x, y) = |x - y|^2 + \frac{|x|}{10} + \frac{|y|}{11}.$$

We see that d is a dislocated quasi-b-metric on X.

Now, let $x \in A$. Then $-1 \leq x \leq 0$. So, $0 \leq \frac{-x}{5} \leq \frac{1}{5}$. Thus, $Tx \in B$.

On the other hand, let $x \in B$. Then $0 \leq x \leq 1$. So, $\frac{-1}{5} \leq \frac{-x}{5} \leq 0$. Thus, $Tx \in A$.

Hence the map T is cyclic on X, because $T(A) \subset B$ and $T(B) \subset A$.

Next, we consider

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty|^2 + 3|Tx| + 2|Ty| \\ &= |\frac{-x}{5} - \frac{-y}{5}|^2 + \frac{1}{10}|\frac{-x}{5}| + \frac{1}{11}|\frac{-y}{5}| \\ &= \frac{1}{25}|x-y|^2 + \frac{1}{50}|x| + \frac{1}{55}|y| \\ &\leq \frac{1}{5}[|x-y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|] \\ &\leq kd(x,y), \end{aligned}$$

for $\frac{1}{5} \leq k < 1$. Thus T satisfies dqb-cyclic-Banach contraction of Theorem 4.2.2 and 0 is the unique fixed point of T.

Next, we begin with introducing the concept of a dqb-cyclic-Kannan mapping.

Definition 4.2.4. Let A and B be nonempty subsets of a dislocated quasi-b-metric space (X, d), with constant $s \geq 1$. A cyclic map $T : A \cup B \to A \cup B$ is called a *dqb-cyclic-Kannan mapping* if there exists $r \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq r(d(x, Tx) + d(x, Ty)). \tag{4.2.6}$$

for all $x \in A$, $y \in B$ and $sr \leq \frac{1}{2}$.

Theorem 4.2.5. *Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X, d). Let T be a cyclic mapping that satisfies the condition a **dqb-cyclic-Kannan mapping** Then, T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A$, and using contractive condition of theorem, we have

$$\begin{aligned} d(Tx, T^2x) &= d(Tx, T(Tx)) \\ &\leq rd(x, Tx) + rd(Tx, T^2x), \end{aligned}$$

so,

$$d(T^2x, Tx) \leq \frac{r}{1-r} d(Tx, x). \tag{4.2.7}$$

And from (4.2.7),

$$\begin{aligned} d(T^2x, Tx) &= d(T(Tx), Tx) \\ &\leq rd(Tx, T^2x) + rd(x, Tx) \\ &\leq \frac{r}{1-r} d(Tx, x) + rd(x, Tx) \\ &\leq \frac{r}{1-r} d(Tx, x) + \frac{r}{1-r} d(x, Tx) \\ &= \frac{r}{1-r} [d(Tx, x) + d(x, Tx)], \end{aligned}$$

so,

$$d(Tx, T^2x) \leq \frac{r}{1-r} \beta, \tag{4.2.8}$$

where $\beta = d(Tx, x) + d(x, Tx)$. By using (4.2.7) and (4.2.8), we have

$$d(T^3x, T^2x) \leq (\frac{r}{1-r})^2 \beta,$$

and

$$d(T^2x, T^3x) \leq (\frac{r}{1-r})^2 \beta.$$

For all $n \in \mathbb{N}$, we get that

$$d(T^{n+1}x, T^n x) \leq (\frac{r}{1-r})^n \beta,$$

and

$$d(T^n x, T^{n+1}x) \leq (\frac{r}{1-r})^n \beta.$$

Let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have,

$$\begin{aligned} d(T^m x, T^n x) &\leq s^{m-n} d(T^m x, T^{m-1}x) + s^{m-n-1} d(T^{m-1}x, T^{m-2}x) + \dots + sd(T^{n+1}x, T^n x) \\ &\leq (s^{m-n} k^{m-1} + s^{m-n-1} k^{m-2} + s^{m-n-2} k^{m-3} + \dots + s^2 k^{n+1} + sk^n) \beta \\ &\leq ((\frac{r}{1-r})^{n-1} + (\frac{r}{1-r})^{n-1} + (\frac{r}{1-r})^{n-1} + \dots + (\frac{r}{1-r})^{n-1} + (\frac{r}{1-r})^{n-1}) \beta \\ &= (\frac{r}{1-r})^{n-1} (m - n + 1) \beta \end{aligned}$$

$$< (\frac{r}{1-r})^{n-1}\xi\beta$$

for some $\xi > m - n + 1$. Take $n \to \infty$, we get $d(T^m x, T^n x) \to 0$.

Similarly, let $n, m \in \mathbb{N}$ with $m > n$, by using the triangular inequality, we have,

$$d(T^n x, T^m x) < (\frac{r}{1-r})^{n-1}\xi\beta.$$

Take $n \to \infty$, we get $d(T^n x, T^m x) \to 0$. Thus $\{T^n x\}$ is a Cauchy sequence. Since X is complete, we have $\{T^n x\}$ converges to some $z \in X$.
We note that $\{T^{2n} x\}$ is a sequence in A and $\{T^{2n-1} x\}$ is a sequence in B in a way that both sequences tend to same limit z.
Since A and B are closed, we have $z \in A \cap B$, and then $A \cap B \neq \emptyset$.

Now, we will show that $Tz = z$.
By using (4.2.6), consider

$$\begin{aligned} d(T^n x, Tz) &= d(T(T^{n-1}x), Tz) \\ &\leq rd(T^{n-1}x, T^n x) + rd(z, Tz). \end{aligned}$$

Taking limit as $n \to \infty$ in above inequality, we have

$$d(z, Tz) \leq rd(z, Tz)$$

Since $0 \leq r < \frac{1}{2}$, we have $d(z, Tz) = 0$.
Similarly considering from (4.2.6), we get

$$\begin{aligned} d(Tz, T^n x) &= d(Tz, T(T^{n-1}x)) \\ &\leq rd(z, Tz) + rd(T^{n-1}x, T^n x). \end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$d(Tz, z) \leq rd(z, Tz)$$

Since $d(z, Tz) = 0$, we have $d(z, Tz) = 0$.
Hence $d(z, Tz) = d(Tz, z) = 0$, and then $Tz = z$ and z is a fixed point of T.

Finally, we prove the uniqueness of fixed point. Let $z^* \in X$ be another fixed point of T. Thus $Tz^* = z^*$. Then, we have $d(z,z) = d(z^*,z^*) = 0$, by assumption. we get that

$$\begin{aligned} d(z,z^*) &= d(Tz,Tz^*) \\ &\leq rd(z,Tz) + rd(z^*,Tz^*) \\ &= rd(z,z) + rd(z^*,z^*) \\ &= 0. \end{aligned} \tag{4.2.9}$$

On the other hand, we have

$$\begin{aligned} d(z^*,z) &= d(Tz^*,Tz) \\ &\leq rd(z^*,Tz^*) + rd(z,Tz) \\ &= rd(z^*,z^*) + rd(z,z) \\ &= 0. \end{aligned} \tag{4.2.10}$$

By (4.2.9) and (4.2.10), we obtain that $d(z,z^*) = d(z^*,z) = 0$, and so $z^* = z$ Therefore z is a unique fixed point of T. This proof is completes. □

Example 4.2.6. Let $X = [-1,1]$ and $T : X \to X$ defined by $Tx = \frac{-x}{7}$. Suppose that $A = [-1,0]$ and $B = [0,1]$. Define the function $d : X^2 \to [0,\infty)$ by

$$d(x,y) = |x-y|^2 + 3|x| + 2|y|.$$

We see that d is a dislocated quasi-b-metric on X.
Now, let $x \in A$. Then $-1 \leq x \leq 0$. So, $0 \leq \frac{-x}{7} \leq \frac{1}{7}$. Thus, $Tx \in B$.
On the other hand, let $x \in B$. Then $0 \leq x \leq 1$. So, $\frac{-1}{7} \leq \frac{-x}{7} \leq 0$. Thus, $Tx \in A$.
Hence the map T is cyclic on X, because $T(A) \subset B$ and $T(B) \subset A$.
Next, we consider

$$\begin{aligned} d(Tx,Ty) &= |Tx-Ty|^2 + 3|Tx| + 2|Ty| \\ &= |\frac{-x}{7} - \frac{-y}{7}|^2 + 3|\frac{-x}{7}| + 2|\frac{-y}{7}| \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{49}|x-y|^2 + \frac{3}{7}|x| + \frac{2}{7}|y| \\
&\leq \frac{1}{49}(|x|+|y|)^2 + \frac{3}{7}|x| + \frac{2}{7}|y| \\
&\leq \frac{2}{49}|x|^2 + \frac{2}{49}|y|^2 + \frac{3}{7}|x| + \frac{2}{7}|y| \\
&\leq \frac{2}{23}([\frac{64}{49}|x|^2 + \frac{23}{7}|x|] + [\frac{64}{49}|y|^2 + \frac{23}{7}|y|]) \\
&= \frac{2}{23}([\frac{64}{49}|x|^2 + \frac{23}{7}|x|] + [\frac{64}{49}|y|^2 + \frac{23}{7}|y|]) \\
&= \frac{2}{23}([|x+\frac{1}{7}x|^2 + 3|x| + 2|\frac{1}{7}x|] + [|y+\frac{1}{7}y|^2 + 3|y| + 2|\frac{1}{7}y|]) \\
&= \frac{2}{23}([|x-Tx|^2 + 3|x| + 2|Tx|] + [|y-Ty|^2 + 3|y| + 2|Ty|]) \\
&= r(d(x,Tx) + d(y,Ty)),
\end{aligned}
$$

for $\frac{2}{23} \leq r < \frac{1}{2}$. Thus T satisfies dqb-cyclic-Banach contraction of Theorem 4.2.5 and 0 is the unique fixed point of T.

Furthermore, we begin with proving a fixed point theorems for dqb-cyclic-weak contractions.

Definition 4.2.7. Let A and B be nonempty closed subsets of a dislocated quasi-b-metric spaces (X,d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a *dqb-cyclic-weak contraction* if, for all $x \in A$, $y \in B$ such that

$$sd(Tx,Ty) \leq d(x,y) - \psi(d(x,y)), \tag{4.2.11}$$

where $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$.

Lemma 4.2.8. *Let (X,d_X) and (Y,d_Y) be dislocated quasi-b-metric spaces, A and B be nonempty closed subsets of a dislocated quasi-b-metric spaces (X,d). If T is a dqb-cyclic-weak contraction, then T is continuous.*

Proof. Let $\epsilon > 0$, all $x,p \in A \cup B$. Suppose that $\max\{d_X(x,p), d_C(p,x)\} < \delta$. Choose $\epsilon = \frac{\delta}{s}$. Since T is a dqb-cyclic-weak contraction, we have

$$sd(Tx,Tp) \leq d(x,p) - \psi(d(x,p))$$

$$\leq d(x,p) < \delta$$

and

$$\begin{aligned} sd(Tp,Tx) &\leq d(p,x) - \psi(d(p,x)) \\ &\leq d(p,x) < \delta. \end{aligned}$$

So, $d(Tx,Tp) < \epsilon$ and $d(Tp,Tx) < \epsilon$. Thus T is continuous at p, and hence T is continuous on $A \cup B$. □

Now, we present a fixed point theorem related to dqb-cyclic-weak contractions.

Theorem 4.2.9. *Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X,d). Let T be a dqb-cyclic-weak contraction. Then, T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A$ be fixed. Using contractive condition in assumptions, we have

$$\begin{aligned} d(T^2x,Tx) &\leq sd(T^2x,Tx) \\ &= sd(T(Tx),Tx) \\ &\leq d(Tx,x) - \psi(d(Tx,x)), \\ &\leq d(Tx,x), \end{aligned} \tag{4.2.12}$$

and

$$\begin{aligned} d(Tx,T^2x) &\leq sd(Tx,T^2x) \\ &= sd(Tx,T(Tx)) \\ &\leq d(x,Tx) - \psi((x,Tx)), \\ &\leq d(x,Tx). \end{aligned} \tag{4.2.13}$$

So,

$$d(T^3x,T^2x) \leq d(T^2x,Tx) - \psi(d(T^2x,Tx)), \tag{4.2.14}$$

and

$$d(T^2x, T^3x) \leq d(Tx, T^2x) - \psi(d(Tx, T^2x)). \tag{4.2.15}$$

For all $n \in \mathbb{N}$, we get

$$d(T^{n+2}x, T^{n+1}x) \leq d(T^{n+1}x, T^n x) - \psi(d(T^{n+1}x, T^n x)) \tag{4.2.16}$$

and

$$d(T^{n+1}x, T^{n+2}x) \leq d(T^n x, T^{n+1}x) - \psi(d(T^n x, T^{n+1}x)). \tag{4.2.17}$$

Set $\varsigma_n = d(T^{n+1}x, T^n x)$ and $\tau_n = d(T^n x, T^{n+1}x)$.
By inequalies (4.2.16) and (4.2.17), we get

$$\varsigma_{n+1} \leq \varsigma_n - \psi(\varsigma_n) \leq \varsigma_n \tag{4.2.18}$$

and

$$\tau_{n+1} \leq \tau_n - \psi(\tau_n) \leq \tau_n. \tag{4.2.19}$$

Thus $\{\varsigma_n\}$ and $\{\tau_n\}$ are decreasing sequences of non-negative real numbers, and hence possess a $\lim_{n\to\infty} \varsigma_n = \varsigma \geq 0$ and $\lim_{n\to\infty} \tau_n = \tau \geq 0$. Suppose that $\varsigma > 0$. Since ψ is nondecreasing, $\psi(\varsigma_n) \geq \psi(\varsigma) > 0$. By inequality (4.2.18), we have $\varsigma_{n+1} \leq \varsigma_n - \psi(\varsigma)$. Thus $\varsigma_{N+m} \leq \varsigma_m - N\psi(\varsigma)$, a contradiction for N large enough. Therefore $\varsigma = 0$. Similarly, we have $\tau = 0$. Next, we prove that the sequence $\{T^n x\}$ is Cauchy. Suppose that $\{T^n x\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and subsequence $\{T^{m_k}x\}$ and $\{T^{n_k}x\}$ with $m_k > n_k \geq n$ such that $d(T^{m_k}x, T^{n_k}x) \geq \epsilon$ and $d(T^{m_k-1}x, T^{n_k}x) < \epsilon$. Now, we consider

$$sd(T^{m_k}x, T^{n_k}x) \leq d(T^{m_k-1}x, T^{n_k-1}x) - \psi(d(T^{m_k-1}x, T^{n_k-1}x)) \tag{4.2.20}$$

$$\leq d(T^{m_k-1}x, T^{n_k-1}x) \tag{4.2.21}$$

which implies that

$$s\epsilon \leq d(T^{m_k-1}x, T^{n_k-1}x). \tag{4.2.22}$$

Take limit inferior in (4.2.22) as $k \to \infty$, we get

$$\epsilon s \leq \liminf_{k\to\infty} d(T^{m_k-1}x, T^{n_k-1}x). \tag{4.2.23}$$

We have

$$\begin{aligned} d(T^{m_k-1}x, T^{n_k-1}x) &\leq sd(T^{m_k-1}x, T^{n_k}x) + sd(T^{n_k}x, T^{n_k-1}x) \\ &< s\epsilon + sd(T^{n_k}x, T^{n_k-1}x). \end{aligned} \tag{4.2.24}$$

Take limit superior in (4.2.24) as $k \to \infty$, we get

$$\limsup_{k\to\infty} d(T^{m_k-1}x, T^{n_k-1}x) \leq s\epsilon. \tag{4.2.25}$$

By (4.2.23) and (4.2.25), we get

$$\lim_{k\to\infty} d(T^{m_k-1}x, T^{n_k-1}x) = s\epsilon. \tag{4.2.26}$$

Letting $k \to \infty$ in (4.2.20), by property of ψ and (4.2.26), we get

$$s\epsilon \leq s\epsilon - \psi(s\epsilon) < s\epsilon \tag{4.2.27}$$

which is a contradiction. Hence $\{T^n x\}$ is a dqb-Cauchy sequence. Since (X, d) is complete, we have $\{T^n x\}$ converges to some $z \in X$. We note, that $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n-1}x\}$ is a sequence in B in a way that both sequences tend to same limit z. Since A and B are closed, we have $z \in A \cap B$, and hence $A \cap B \neq \emptyset$. The continuity of T implies that the limit is a fixed point. Finally, to prove the uniqueness of fixed point, let $z^* \in X$ be another fixed point of T. Therefore $Tz^* = z^*$.

Then, we have

$$d(z, z^*) = d(Tz, Tz^*) \leq sd(Tz, Tz^*) \leq d(z, z^*) - \psi(d(z, z^*)) \leq d(z, z^*). \tag{4.2.28}$$

On the other hand,

$$d(z^*, z) = d(Tz^*, Tz) \leq sd(Tz^*, Tz) \leq d(z^*, z) - \psi(d(z, z^*)) \leq d(z^*, z). \tag{4.2.29}$$

By (4.2.32) and (4.2.29), we obtain that $d(z, z^*) = d(z^*, z) = 0$, this implies that $z^* = z$. Therefore z is a unique fixed point of T. This completes the proof. $\square$

Example 4.2.10. Let $X = [-1, 1]$ and $T : A \cup B \to A \cup B$ be defined by $Tx = \frac{-x}{3}$, and $\psi(t) = \frac{t}{50}$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Define the function $d : X^2 \to [0, \infty)$ by

$$d(x, y) = |x - y|^2 + \frac{|x|}{10} + \frac{|y|}{11}.$$

We see that d is a dislocated quasi-b-metric on X (see Example 4.1.3).

Let $x \in A$. Then $-1 \leq x \leq 0$. So, $0 \leq \frac{-x}{3} \leq \frac{1}{3}$. Thus, $Tx \in B$. On the other hand, let $x \in B$. Then $0 \leq x \leq 1$. So, $\frac{-1}{3} \leq \frac{-x}{3} \leq 0$. Thus, $Tx \in A$.

Hence the map T is cyclic on X, because $T(A) \subset B$ and $T(B) \subset A$.

Next, we consider

$$\begin{aligned}
2d(Tx, Ty) &= 2(|Tx - Ty|^2 + \frac{1}{10}|Tx| + \frac{1}{11}|Ty|) \\
&= 2(|\frac{-x}{3} - \frac{-y}{3}|^2 + \frac{1}{10}|\frac{-x}{3}| + \frac{1}{11}|\frac{-y}{3}|) \\
&= \frac{49}{50}(\frac{100}{441}|x - y|^2 + \frac{50}{1470}|x| + \frac{100}{539}|y|) \\
&\leq \frac{49}{50}(|x - y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|) \\
&= |x - y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y| - \psi(|x - y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|) \\
&= d(x, y) - \psi(d(x, y)).
\end{aligned}$$

Thus T satisfies dqb-cyclic-weak contraction of Theorem 4.2.9 and 0 is the unique fixed point of T.

Definition 4.2.11. Let A and B be nonempty subsets of a dislocated quasi-b-metric spaces (X, d), with constant $s \geq 1$. A cyclic map $T : A \cup B \to A \cup B$ is said to be a *dqb-cyclic-ϕ-contraction* if there exists $k \in [0, 1)$ such that

$$sd(Tx, Ty) \leq \phi(d(x, y)), \tag{4.2.30}$$

for all $x \in A$, $y \in B$, where Φ is the family of non-decreasing functions: $\phi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ and $\phi(t) < (t)$ for each $t > 0$, where n is the n-th iterate of ϕ.

Theorem 4.2.12. *Let A and B be nonempty closed subsets of a complete dislocated quasi-b-metric space (X, d). Let T be a a* dqb-cyclic-ϕ-contraction. *Then, T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A$, then using contractive condition of theorem, we have

$$\begin{aligned} sd(T^2x, Tx) &= sd(T(Tx), Tx) \\ &\leq \phi(d(Tx, x)), \end{aligned}$$

and

$$\begin{aligned} sd(Tx, T^2x) &= sd(Tx, T(Tx)) \\ &\leq \phi(d(x, Tx)). \end{aligned}$$

Inductively, we have for all $n \in \mathbb{N}$, we get

$$s^n d(T^{n+1}x, T^n x) \leq \phi^n(d(Tx, x)),$$

and

$$s^n d(T^n x, T^{n+1}x) \leq \phi^n(d(x, Tx)).$$

Let $\epsilon > 0$ be fixed and $n(\epsilon) \in \mathbb{N}$, such that

$$\Sigma_{n \geq n(\epsilon)} \phi^n(d(Tx, x)) < \epsilon,$$

and

$$\Sigma_{n \geq n(\epsilon)} \phi^n(d(x, Tx)) < \epsilon.$$

Let $n, m \in \mathbb{N}$ with $m > n > n(\epsilon)$, using the triangular inequality, we have:

$$\begin{aligned} d(T^m x, T^n x) &\leq s^{m-n} d(T^m x, T^{m-1}x) + s^{m-n-1} d(T^{m-1}x, T^{m-2}x) + ... + sd(T^{n+1}x, T^n x) \\ &\leq s^{m-1} d(T^m x, T^{m-1}x) + s^{m-2} d(T^{m-1}x, T^{m-2}x) + ... + s^n d(T^{n+1}x, T^n x) \\ &\leq \phi^{m-1}(d(Tx, x)) + \phi^{m-2}(d(Tx, x)) + \phi^{m-3}(d(Tx, x)) + ... + \phi^n(d(Tx, x)) \\ &= \Sigma_{k=n}^{m-1} \phi^k(d(x, Tx)) \end{aligned}$$

$$\leq \Sigma_{n \geq n(\epsilon)} \phi^n(d(x, Tx)) < \epsilon.$$

Similarly, we obtain that

$$d(T^n x, T^m x) < \epsilon.$$

Thus $\{T^n x\}$ is a Cauchy sequence. Since(X, d) is complete, we have $\{T^n x\}$ converges to some $z \in X$. We note that $\{T^{2n} x\}$ is a sequence in A and $\{T^{2n-1} x\}$ is a sequence in B in a way that both sequences tend to same limit z. Since A and B are closed, we have $z \in A \cap B$, and then $A \cap B \neq \emptyset$. Now, we will show that $Tz = z$. By using (4.2.30), consider

$$\begin{aligned} d(z, Tz) &\leq sd(z, T^{2n} x) + sd(T^{2n} x, Tz) \\ &\leq sd(z, T^{2n} x) + d(T^{2n-1} x, z). \end{aligned}$$

Taking limit as $n \longrightarrow \infty$ in above inequality, we have

$$d(z, Tz) = 0.$$

Similarly considering from (4.2.30), we get that

$$\begin{aligned} d(Tz, z) &\leq sd(Tz, T^{2n} x) + sd(T^{2n} x, z) \\ &\leq d(z, T^{2n-1} x) + sd(T^{2n} x, z). \end{aligned}$$

Taking limit as $n \longrightarrow \infty$ in above inequality, we have

$$d(Tz, z) = 0.$$

Hence $d(z, Tz) = d(Tz, z) = 0$, this implies that $Tz = z$ that is z is a fixed point of T. Finally, to prove the uniqueness of fixed point, let $z^* \in X$ be another fixed point of T such that $Tz^* = z^*$. Then, we have

$$d(z^*, z) \leq sd(Tz^*, T^n x) + sd(T^n x, Tz) \leq \phi(d(Tz^*, T^n x)) + \phi(d(T^n x, Tz)), \tag{4.2.31}$$

and on the other hand,

$$d(z, z^*) \leq sd(Tz, T^n x) + sd(T^n x, Tz^*) \leq \phi(d(Tz, T^n x)) + \phi(d(T^n x, Tz^*)). \tag{4.2.32}$$

Letting $n \to \infty$, we obtain that $d(z, z^*) = d(z^*, z) = 0$, this implies that $z^* = z$. Therefore z is a unique fixed point of T. This completes the proof. □

Example 4.2.13. Let $X = [-1, 1]$ and $T : A \cup B \to A \cup B$ be defined by $Tx = \frac{-x}{5}$. Suppose that $A = [-1, 0]$ and $B = [0, 1]$. Defined the function $d : X^2 \to [0, \infty)$ by

$$d(x, y) = |x - y|^2 + \frac{|x|}{10} + \frac{|y|}{11}.$$

We see that d is a dislocated quasi-b-metric on X, where $s = 2$. Let $x \in A$. Then $-1 \leq x \leq 0$. So, $0 \leq \frac{-x}{5} \leq \frac{1}{5}$. Thus, $Tx \in B$. On the other hand, let $x \in B$. Then $0 \leq x \leq 1$. So, $\frac{-1}{5} \leq \frac{-x}{5} \leq 0$. Thus, $Tx \in A$.
Hence the map T is cyclic on X, because $T(A) \subset B$ and $T(B) \subset A$.
Next, we consider

$$\begin{aligned}
sd(Tx, Ty) &= 2d(Tx, Ty) \\
&= 2(|Tx - Ty|^2 + \frac{1}{10}|Tx| + \frac{1}{11}|Ty|) \\
&= 2(|\frac{-x}{5} - \frac{-y}{5}|^2 + \frac{1}{10}|\frac{-x}{5}| + \frac{1}{11}|\frac{-y}{5}|) \\
&= \frac{2}{3}(\frac{3}{25}|x - y|^2 + \frac{3}{50}|x| + \frac{3}{55}|y|) \\
&\leq \frac{2}{3}(|x - y|^2 + \frac{5}{50}|x| + \frac{5}{55}|y|) \\
&= \frac{2}{3}(|x - y|^2 + \frac{1}{10}|x| + \frac{1}{11}|y|) \\
&= \phi(d(x, y)),
\end{aligned}$$

where the function $\phi \in \Phi$ is $\phi(t) = \frac{2t}{3}$. Clearly, 0 is the unique fixed point of T.

The following corollary can be taken as a particular case of theorem 4.2.12 if we take $\phi(t) = kt$ for all $t \geq 0$ and some $k \in [0, 1)$. That is the dqb-cyclic-Banach contraction, in the setting of dislocated quasi-b-metric spaces.

Corollary 4.2.14. *Let A and B be nonempty closed subsets of a complete dislocated quasi-b-metric space (X, d), with constant $s \geq 1$. Let T be a dqb-cyclic-Banach contraction; that is, if there exists $k \in [0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y). \tag{4.2.33}$$

for all $x \in A$, $y \in B$ and $sk \leq 1$. Then T has a unique fixed point in $A \cap B$.

CHAPTER V

FIXED POINT THEOREMS IN HYPERBOLIC SPACES

5.1 Fixed Point Theorems for Fundamentally Nonexpansive Mappings in Hyperbolic Spaces

In this section, we prove some properties of the fixed point set of fundamentally nonexpansive mappings and derive the existence of fixed point theorems as follows results of Salahifard, et al. [47] in hyperbolic spaces. Moreover, we prove convergence and Δ-convergence theorems of the generalized Krasnoselskij-type iterative process to approximate a fixed point for fundamentally nonexpansive operators in a hyperbolic space and show that if the hyperbolic space that satisfies the Δ-Opial condition, then the fixed points set of such a mapping with the convex range is nonempty; see [48] for more details.

Lemma 5.1.1. *Let K be a nonempty bounded closed convex subset of a strictly convex complete hyperbolic space X. Let $T : K \to K$ be fundamentally nonexpansive and $F(T) \neq \emptyset$, then $F(T)$ is Δ-closed and convex.*

Proof. Suppose that $\{x_n\}$ is a sequence in $F(T)$ which Δ-converges to some $y \in K$. We show that $y \in F(T)$. Since

$$d(x_n, Ty) = d(T^2x_n, Ty) \leq d(Tx_n, y) = d(x_n, y),$$

we have

$$\limsup_{n\to\infty} d(x_n, Ty) \leq \limsup_{n\to\infty} d(x_n, y).$$

Thus $Ty = y$. Hence $F(T)$ is closed.

Next, we will show that $F(T)$ is convex, let $x, y \in F(T)$ and $\alpha \in [0, 1]$. Then,

$$d(x, Tz) = d(T^2x, Tz) \leq d(Tx, z) = d(x, z)$$

and

$$d(y, Tz) = d(T^2y, Tz) \leq d(Ty, z) = d(y, z)$$

For $z = W(x,y,\alpha)$, we have

$$\begin{aligned} d(x,y) &\le d(x,Tz) + d(Tz,y) \\ &\le d(x,z) + d(z,y) \\ &= d(x, W(x,y,\alpha)) + d(W(x,y,\alpha),y) \\ &\le (1-\alpha)d(x,x) + \alpha d(x,y) + (1-\alpha)d(x,y) + \alpha d(y,y) \\ &= d(x,y). \end{aligned} \tag{5.1.1}$$

Thus $d(x,Tz) = d(x,z)$ and $d(Tz,y) = d(z,y)$, because if $d(x,Tz) < d(x,z)$ or $d(Tz,y) < d(z,y)$, then it contradiction to $d(x,y) < d(x,y)$. Therefore $Tz = W(x,y,\beta)$ for some $\beta \in [0,1]$ But, Hence

$$(1-\beta)d(x,y) = d(x,Tz) = d(x,z) = (1-\alpha)d(x,y)$$

$$(1-\beta)d(x,y) = d(y,Tz) = d(y,z) = (1-\alpha)d(x,y)$$

Therefor $Tz = z$, and then $W(x,y,\alpha) \in F(T)$. Hence $F(T)$ is convex.

□

Lemma 5.1.2. *Let K be a nonempty bounded closed subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{x_n\}$ be a sequence in K. If $T : K \to K$ is fundamentally nonexpansive, and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $F(T)$ is nonempty.*

Proof. By Lemma 2.2.6, the asymptotic center of any bounded sequence in K, particularly, the asymptotic center of approximate fixed point sequence for T is in K. Let $A(\{x_n\}) = \{y\}$, we want to show that y is a fixed point of T. By Lemma 2.2.10, we get

$$d(x_n, Ty) \le 3d(x_n, Tx_n) + d(x_n, y),$$

hence

$$\limsup_{n\to\infty} d(x_n, Ty) \le \limsup_{n\to\infty} d(x_n, y).$$

By the uniqueness of the asymptotic center $Ty = y$. □

Theorem 5.1.3. *Let K be a nonempty bounded closed strictly convex subset of complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{x_n\}$ be a sequence in K. If $T : K \to K$ is fundamentally nonexpansive and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $F(T)$ is nonempty, $\triangle$-closed and convex.*

Proof. By Lemma 5.1.1 and 5.1.2, we get $F(T)$ is nonempty $\triangle$-closed and convex. □

Definition 5.1.4. Let K be a nonempty subset of a metric space X and let T be a self-mapping of K. A sequence $\{x_n\}$ in K is called an *approximate fixed point sequence* for T if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Lemma 5.1.5. *Let T be a fundamentally nonexpansive self-mapping on a nonempty subset K of a complete hyperbolic space X, and let $T(K)$ be bounded and convex. Define a sequence $\{Tx_n\}$ in $T(K)$ by $x_1 \in K$ and $Tx_{n+1} = W(T^2x_n, Tx_n, \alpha)$ for all $n \in \mathbb{N}$, where $\alpha \in (0,1)$. Then $\{Tx_n\}$ is an approximate fixed point sequence for T.*

Proof. For any $n \in \mathbb{N}$, we have

$$d(T^2x_{n+1}, T^2x_n) \leq (Tx_{n+1}, Tx_n)$$

because T is fundamentally nonexpansive. By Lemma 2.2.8, we have

$$\lim_{n\to\infty} d(Tx_n, T^2x_n) = 0.$$

Hence $\{Tx_n\}$ is an approximate fixed point sequence for T. □

Proposition 5.1.6. *Let $T : K \to K$ be a fundamentally nonexpansive mapping, where K is a nonempty subset of a complete hyperbolic space X. Then $F(T)$ is closed. Moreover, if X is strictly convex, and K or $T(K)$ is convex, then $F(T)$ is also convex.*

Proof. We first show that $F(T)$ is closed. Suppose that $F(T)$ is not closed. Then there is an element $x \in cl(F(T))$ but $x \notin F(T)$. Set $r = \frac{d(x,Tx)}{3}$. Since $x \in cl(F(T))$, we have $B(x,r) \cap F(T)$ is nonempty. Let $u \in B(x,r) \cap F(T)$. Then $d(x,u) < r$ and $Tu = u$ hold. By Remark 2.2.9, we get that

$$d(x,Tx) \leq d(x,u) + d(u,Tx) \leq d(x,u) + d(u,x) < 2r = \frac{2d(x,Tx)}{3}, \quad (5.1.2)$$

which is a contradiction. Thus $F(T)$ is closed. Assume that X is strictly convex. To show that $F(T)$ is convex. Let $\lambda \in (0,1)$ and $x, y \in F(T)$ with $x \neq y$. By definition of hyperbolic spaces, we have

$$\begin{aligned} d(x,y) &\leq d(x,T(W(x,y,\lambda))) + d(T(W(x,y,\lambda)),y) \\ &\leq d(x,W(x,y,\lambda)) + d(W(x,y,\lambda),y) \\ &\leq (1-\lambda)d(x,x) + \lambda d(x,y) + (1-\lambda)d(x,y) + \lambda d(y,y) \\ &= d(x,y), \end{aligned} \quad (5.1.3)$$

so $d(x,T(W(x,y,\lambda))) + d(T(W(x,y,\lambda)),y) = d(x,W(x,y,\lambda)) + d(W(x,y,\lambda),y)$. Thus $d(x,T(W(x,y,\lambda))) = d(x,W(x,y,\lambda))$ and $d(T(W(x,y,\lambda)),y) = d(W(x,y,\lambda),y)$, because if $d(x,T(W(x,y,\lambda))) < d(x,W(x,y,\lambda))$ or $d(T(W(x,y,\lambda)),y) < d(W(x,y,\lambda),y)$, then it contradiction to $d(x,y) < d(x,y)$. Therefore $T(W(x,y,\alpha)) = W(x,y,\alpha)$ $W(x,y,\alpha) \in F(T)$. Hence $F(T)$ is convex. □

Proposition 5.1.7. *Let K be a nonempty subset of a uniformly convex hyperbolic space X, and let T be a fundamentally nonexpansive self-mapping from K onto K. If $T(K)$ is bounded and closed, then $F(T)$ is nonempty.*

Proof. Define a sequence $\{Tx_n\}$ in $T(K)$ by $x_1 \in K$ and $Tx_{n+1} = W(T^2x_n, T^2x_n, \frac{1}{2})$ for $n = 1, 2, 3, ...$ By Lemma 2.2.6, the asymptotic center of any bounded sequence in $T(K)$, particularly, the asymptotic center of approximate fixed point sequence for T is in K. Let $A(\{Tx_n\}) = \{y\}$. Then there exists $x \in X$ such that $Tx = y$, we want to show that y is a fixed point of T. We consider

$$d(Tx_n, T^2x) \leq 3d(Tx_n, T^2x_n) + d(Tx_n, Tx),$$

hence

$$\limsup_{n\to\infty} d(Tx_n, T^2x) \leq \limsup_{n\to\infty} d(Tx_n, Tx).$$

By the uniqueness of the asymptotic center $T^2x = Tx$. So, y is a fixed point of T. Therefore $F(T)$ is nonempty. □

By Proposition 5.1.6 and 5.1.7, we obtain the desired result.

Theorem 5.1.8. *Let K be a nonempty subset of a uniformly convex hyperbolic space X, and let T be a fundamentally nonexpansive self-mapping of K and onto. If $T(K)$ is bounded and closed, then $F(T)$ is nonempty and closed. Moreover, if K or $T(K)$ is convex, then $F(T)$ is also convex.*

Example 5.1.9. Let X be a linear space over the field $\mathbb{R}$. Define the function $d : X^2 \to [0, \infty)$ by

$$d(x, y) = ||x - y||$$

such that $|| \cdot || : X \to [0, \infty)$ be a uniformly convex Banach space. Let K be a nonempty bounded closed convex subset of a Banach space. We see that d is a hyperbolic space on X, where

$$W(x, y, \alpha) = (1 - \alpha)x + \alpha y.$$

Define $T : K \to K$ by $T(x) = \frac{x}{2}$. Then T is fundamentally nonexpansive. By previous theorem, we get $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.

Theorem 5.1.10. *Let K be a nonempty compact subset complete hyperbolic space of X. Assume that $T : K \to K$ is fundamentally nonexpansive and $T(K)$ is convex. Then $\{Tx_n\}$ in $T(K)$ defined by $x_1 \in K$ and*

$$Tx_{n+1} = W(T^2x_n, Tx_n, \alpha) \tag{5.1.4}$$

for $n = 1, 2, 3, \ldots$, where $\alpha \in (0, 1)$, converges to a fixed point of T.

Proof. Since K is compact, there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\lim_{k\to\infty} Tx_{n_k} = z$ for some $z \in K$. By (5.1.4), we have $\lim_{k\to\infty} T^2x_{n_k} = z$. Since T is fundamentally nonexpansive, we have $d(T^2x_{n_k}, Tz) \leq d(Tx_{n_k}, z)$ for $k = 1, 2, 3...$. This implies that $\lim_{k\to\infty} T^2x_{n_k} = Tz$, hence z is a fixed point of T. On the other hand, we have

$$\begin{aligned} d(Tx_{n+1}, z) &= d(W(T^2x_n, Tx_n, \alpha), Tz) \\ &\leq \alpha d(T^2x_n, Tz) + (1-\alpha)d(Tx_n, z) \\ &\leq d(Tx_n, z) \end{aligned} \tag{5.1.5}$$

for all $n \in \mathbb{N}$. It follows that the sequence $\{d(Tx_n, z)\}$ is bounded and decreasing, hence it is convergent. Since $\lim_{n\to\infty} Tx_{n_k} = z$, we have $\lim_{n\to\infty} Tx_n = z$. □

By definition of weak compac in a Banach space, we introduce Δ-compact in a hyperbolic space.

Definition 5.1.11. Let K be a nonempty subset of a hyperbolic space X. Then K is said to be Δ-*compact* if for each sequence $\{x_n\}$ in K, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which Δ-converges to an element in K.

By definition of Opial condition [49] in a Banach space, we introduce Δ-Opial condition in a hyperbolic space.

Definition 5.1.12. Let X be a complete hyperbolic space. Then X is said to satisfy the Δ-*Opial condition* if whenever a sequence $\{x_n\}$ in X Δ-converges to $x \in X$, then

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y)$$

holds for all $y \in X$ with $x \neq y$. Moreover, for any uniformly convex complete hyperbolic space has the Δ-Opial condition.

Theorem 5.1.13. *Let K be a nonempty Δ-compact subset of a complete hyperbolic space X with the Δ-Opial condition. Assume that $T : K \to K$ is a fundamentally* *nonexpansive mapping and $T(K)$ is convex. Then $\{Tx_n\}$ in $T(K)$ defined by $x_1 \in K$ and*

$$Tx_{n+1} = W(T^2x_n, Tx_n, \alpha)$$

for $n = 1, 2, 3, \ldots$, where $\alpha \in (0, 1)$, Δ-converges to a fixed point of T.

Proof. Since K is Δ−compact, there is a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} Tx_{n_k} = z \in K$. We next show that z is a fixed point of T. Suppose that $Tz \neq z$. By Lemma 2.2.10, we obtain that $d(Tx_{n_k}, Tz) \leq 3d(Tx_{n_k}, T^2x_{n_k}) + d(T_{n_k}, z)$ for $n = 1, 2, 3, \ldots$. The above inequality implies that

$$\limsup_{k\to\infty} d(Tx_{n_k}, Tz) \leq \limsup_{k\to\infty} d(Tx_{n_k}, z),$$

which is a contradiction the Δ-Opial condition. Hence $Tz = z$. Next, we will show that $\Delta\text{-}\lim_{n\to\infty} Tx_{n_k} = z \in K$. Suppose, on the contrary. So, there is a subsequence sequence $\{Tx_{m_j}\}$ of $\{Tx_n\}$ such that $\Delta\text{-}\lim_{j\to\infty} x_{m_j} = y$ with $z \neq y$. Similarly, one can show that y is a fixed fixed of T. Consider

$$\begin{aligned}
\lim_{n\to\infty} d(Tx_n, z) &= \limsup_{k\to\infty} d(Tx_{n_k}, z) \\
&< \limsup_{k\to\infty} d(Tx_{n_k}, y) \\
&= \limsup_{j\to\infty} d(Tx_{m_j}, y) \\
&< \limsup_{j\to\infty} d(Tx_{m_j}, z) \\
&= \lim_{n\to\infty} d(Tx_n, z), \qquad (5.1.6)
\end{aligned}$$

which is a contradiction. Therefore, $\Delta\text{-}\lim_{n\to\infty} Tx_n = z$. □

Theorem 5.1.14. *Let K be a nonempty Δ-compact subset of a uniformly convex complete hyperbolic space X. Assume that $T : K \to K$ is fundamentally nonexpansive mapping, and $T(K)$ is convex. Then $\{Tx_n\}$ in $T(K)$ defined by $x_1 \in K$ and*

$$Tx_{n+1} = W(T^2x_n, Tx_n, \alpha)$$

for $n = 1, 2, 3, \ldots$, where $\alpha \in (0, 1)$, Δ-converges to a fixed point of T.

Proof. Since K is Δ-compact, there is a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} Tx_{n_k} = z \in K$. We next show that z is a fixed point of T. Suppose that $Tz \neq z$. Using Lemma 2.2.10, we obtain $d(Tx_{n_k}, Tz) \leq 3d(Tx_{n_k}, T^2x_{n_k}) + d(T_{n_k}, z)$ for $n = 1, 2, 3, \ldots$. The above inequality implies

$$\limsup_{k\to\infty} d(Tx_{n_k}, Tz) \leq \limsup_{k\to\infty} d(Tx_{n_k}, z).$$

By the uniqueness of the asymptotic center, $Tz = z$. Next, we will show that $\Delta\text{-}\lim_{n\to\infty} Tx_{n_k} = z \in K$. Suppose, for a contradiction, that there is a subsequence sequence $\{Tx_{m_j}\}$ of $\{Tx_n\}$ such that $\Delta\text{-}\lim_{j\to\infty} x_{m_j} = y$ with $z \neq y$. Similarly, one can show that y is a fixed fixed of T. Consider

$$\begin{aligned}
\lim_{n\to\infty} d(Tx_n, z) &= \limsup_{k\to\infty} d(Tx_{n_k}, z) \\
&< \limsup_{k\to\infty} d(Tx_{n_k}, y) \\
&= \limsup_{j\to\infty} d(Tx_{m_j}, y) \\
&< \limsup_{j\to\infty} d(Tx_{m_j}, z) \\
&= \lim_{n\to\infty} d(Tx_n, z),
\end{aligned} \tag{5.1.7}$$

which is a contradiction. Therefore, $\Delta\text{-}\lim_{n\to\infty} Tx_n = z$. □

5.2 Fixed Point Theorems for Generalized Nonexpansive Mappings in Hyperbolic Spaces

In this section, we prove fixed point theorems for generalized nonexpansive mappings and approximate a fixed point for such mappings in hyperbolic spaces. Furthermore, we prove some properties of a generalized nonexpansive mapping in hyperbolic spaces.

Proposition 5.2.1. *Let K be a nonempty and convex subset of a strictly convex hyperbolic space X. If $T : K \to K$ satisfies condition C, then $F(T)$ is closed and convex.*

Proof. Suppose that $\{x_n\}$ is a sequence in $F(T)$ which converges to some $x \in K$. We will show that $x \in F(T)$ By Remark 2.2.12, we get that

$$d(x_n, Tx) \leq 3d(x_n, Tx_n) + d(x_n, x), \tag{5.2.1}$$

so

$$\limsup_{n\to\infty} d(x_n, Tx) \leq \limsup_{n\to\infty} 3d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, x). \tag{5.2.2}$$

Since $\{x_n\} \subseteq F(T)$, we have $\limsup_{n\to\infty} d(x_n, Tx) \leq \limsup_{n\to\infty} d(x_n, x)$. By the uniqueness of the limit point implies $Tx = x$, and then $F(T)$ is closed. Next, we will show that $F(T)$ is convex. Let $x, y \in F(T)$, and $\alpha \in (0, 1)$. We get that

$$\begin{aligned} d(x, y) &\leq d(x, T(W(x, y, \alpha))) + d(T(W(x, y, \alpha)), y) \\ &\leq d(x, W(x, y, \alpha)) + d(W(x, y, \alpha), y) \\ &\leq d(x, y). \end{aligned} \tag{5.2.3}$$

We consider

$$\begin{aligned} d(x, T(W(x, y, \alpha))) &\leq 3d(x, Tx) + d(x, W(x, y, \alpha)) \\ &\leq d(x, W(x, y, \alpha)) \end{aligned} \tag{5.2.4}$$

and

$$\begin{aligned} d(y, T(W(x, y, \alpha))) &\leq 3d(y, Ty) + d(y, W(x, y, \alpha)) \\ &\leq d(y, W(x, y, \alpha)), \end{aligned} \tag{5.2.5}$$

we obtain that

$d(x, T(W(x, y, \alpha))) = d(x, W(x, y, \alpha))$ and $d(T(W(x, y, \alpha)), y) = d(W(x, y, \alpha), y)$, because if $d(x, T(W(x, y, \alpha))) < d(x, W(x, y, \alpha))$ or $d(T(W(x, y, \alpha)), y) < d(W(x, y, \alpha), y)$, then which is contradiction to $d(x, y) < d(x, y)$. Since K is strictly convex, we have $T(W(x, y, \alpha)) = W(x, y, \alpha)$, so $W(x, y, \alpha) \in F(T)$. Therefore $F(T)$ is convex. $\square$

Theorem 5.2.2. *Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity*

η. Suppose that $\{x_n\}$ is a sequence in K, with $d(x_n, Tx_n) \to 0$, and $T : K \to K$ satisfies condition C. If $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.

Proof. Assume that there exists the approximate fixed point sequence $\{x_n\}$. By Lemma 2.2.6, the asymptotic center of any bounded sequence is in K has a unique asymptotic center in K. Let $A(\{x_n\}) = \{x\}$. We will show that $x = Tx$. We writes

$$d(x_n, Tx) \leq 3d(x_n, Tx_n) + d(x_n, x), \tag{5.2.6}$$

by Remark 2.2.12. So,

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, Tx) &\leq \limsup_{n\to\infty} 3d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, x) \\ &= \limsup_{n\to\infty} d(x_n, x). \end{aligned} \tag{5.2.7}$$

The uniqueness of the asymptotic center implies $Tx = x$. Moreover, $F(T)$ closed and convex, by the proof in Proposition 5.2.1. □

Theorem 5.2.3. *Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. If $T : K \to K$ satisfies condition C, then $F(T)$ is nonempty closed and convex.*

Proof. Suppose that $\{x_n\}$ is a sequence in K defined by $x_1 \in K$ and $x_{n+1} = W(Tx_n, x_n, \alpha)$ for all $n \in \mathbb{N}$, where $\alpha \in [\frac{1}{2}, 1)$. By the assumption, we have

$$\frac{1}{2}d(x_n, Tx_n) \leq \alpha d(x_n, Tx_n) = d(x_n, x_{n+1})$$

for $x_1 \in K$. Thus,

$$d(Tx_n, Tx_{n+1}) \leq d(x_n, x_{n+1}).$$

So, by Lemma 2.2.8, we obtain that

$$d(x_n, Tx_n) \to 0.$$

By Lemma 2.2.6, we let $A(\{x_n\}) = \{x\}$. So,

$$d(x_n, Tx) \leq 3d(x_n, Tx_n) + d(x_n, x). \tag{5.2.8}$$

by Remark 2.2.12. Thus,

$$\begin{aligned} \limsup_{n\to\infty} d(x_n, Tx) &\leq \limsup_{n\to\infty} 3d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, x) \\ &= \limsup_{n\to\infty} d(x_n, x). \end{aligned} \tag{5.2.9}$$

The uniqueness of the asymptotic center implies $Tx = x$. Moreover, $F(T)$ closed and convex, by the proof in Proposition 5.2.1. □

If we replace the property of a bounded set by bounded sequence, we have corollarys as follows:

Corollary 5.2.4. *Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a bounded sequence in K with $d(x_n, Tx_n) \to 0$. If $T : K \to K$ satisfies condition C and $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.*

Corollary 5.2.5. *Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. If $T : K \to K$ satisfies condition C, then $F(T)$ is nonempty closed and convex.*

Proposition 5.2.6. *Let K be a nonempty and convex subset of a strictly convex hyperbolic space X. Suppose that $T : K \to K$ satisfies by one of the conditions SKC, KSC, SCC and CSC. Then $F(T)$ is closed and convex.*

Proof. Suppose that $\{x_n\} \subseteq F(T)$ which converges to some $x \in K$. We show that $x \in F(T)$ By Lemma 2.2.13, we get that

$$d(x_n, Tx) \leq 5d(x_n, Tx_n) + d(x_n, x), \tag{5.2.10}$$

and so

$$\limsup_{n\to\infty} d(x_n, Tx) \le \limsup_{n\to\infty} 5d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, x). \tag{5.2.11}$$

Thus, $\limsup_{n\to\infty} d(x_n, Tx) \le \limsup_{n\to\infty} d(x_n, x)$. The uniqueness of the limit point of K, implies $Tx = x$. Therefore $F(T)$ is closed. Now, we will show that $F(T)$ is convex. Let $x, y \in F(T)$ and $\alpha \in [0, 1]$, we get

$$\begin{aligned} d(x, T(W(x, y, \alpha))) &\le 5d(x, Tx) + d(x, W(x, y, \alpha)) \\ &\le d(x, W(x, y, \alpha)) \end{aligned} \tag{5.2.12}$$

and

$$\begin{aligned} d(y, T(W(x, y, \alpha))) &\le 5d(y, Ty) + d(y, W(x, y, \alpha)) \\ &\le d(y, W(x, y, \alpha)). \end{aligned} \tag{5.2.13}$$

We consider

$$\begin{aligned} d(x, y) &\le d(x, T(W(x, y, \alpha))) + d(T(W(x, y, \alpha)), y) \\ &\le d(x, W(x, y, \alpha)) + d(W(x, y, \alpha), y) \\ &\le d(x, y). \end{aligned} \tag{5.2.14}$$

Hence, $d(x, T(W(x, y, \alpha))) = d(x, W(x, y, \alpha))$ and $d(T(W(x, y, \alpha)), y) = d(W(x, y, \alpha), y)$, because if $d(x, T(W(x, y, \alpha))) < d(x, W(x, y, \alpha))$ or $d(T(W(x, y, \alpha)), y) < d(W(x, y, \alpha), y)$, then which is contradiction to $d(x, y) < d(x, y)$. Since K is strictly convex, we have $T(W(x, y, \alpha)) = W(x, y, \alpha)$, and so $W(x, y, \alpha) \in F(T)$. Therefore $F(T)$ is convex. □

Theorem 5.2.7. *Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a sequence in K with $d(x_n, Tx_n) \to 0$, and $T : K \to K$ satisfies by one of the conditions SKC, KSC, SCC and CSC. If $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.*

Proof. By Lemma 2.2.6, the asymptotic center of any bounded sequence is in K. Let $A(\{x_n\}) = \{x\}$. We show that x is a fixed point of T. By Lemma 2.2.13, we get

$$d(x_n, Tx) \leq 5d(x_n, Tx_n) + d(x_n, x). \tag{5.2.15}$$

Thus,

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, Tx) &\leq \limsup_{n\to\infty} 5d(x_n, Tx_n) + \limsup_{n\to\infty} d(x_n, x) \\ &= \limsup_{n\to\infty} d(x_n, x). \end{aligned} \tag{5.2.16}$$

The uniqueness of the asymptotic center implies $Tx = x$. Moreover, $F(T)$ is closed and convex, by the proof in Proposition 5.2.6. □

If we replace the property of a bounded set by bounded sequence, we have corollary as follows:

Corollary 5.2.8. *Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a bounded sequence in K with $d(x_n, Tx_n) \to 0$, and $T : K \to K$ satisfies by one of the conditions SKC, KSC, SCC and CSC. If $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.*

Corollary 5.2.9. *Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a sequence in K with $d(x_n, Tx_n) \to 0$, and $T : K \to K$ satisfies conditions E_μ. If $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.*

Corollary 5.2.10. *Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a sequence in K with $d(x_n, Tx_n) \to 0$, and $T : K \to K$ satisfies conditions C_λ. If $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.*

Definition 5.2.11. Let The $\{x_n\}$ be a bounded sequence in a hyperbolic space X. A selfmap T on a nonempty subset K of X, is said to be a *nonincreasing-asymptotic-mapping* for a sequence $\{x_n\}$, if

$$d(x, Tx_n) \leq d(x, x_n),$$

where $A_K(\{x_n\}) = \{x\}$.

Theorem 5.2.12. *Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K$ and $S_i : K \to K$, $i = 1, 2$ satisfies the condition SKC. Assume that $F := \bigcap_{i=1}^{n=2} F(T_i) \cap F(S_i) \neq \emptyset$, for arbitrarily chosen $x_i \in K$, such that $\{x_n\}$ is defined as follows*

$$\begin{aligned} x_{n+1} &= W(S_1x_n, T_1y_n, \alpha_n), \\ y_n &= W(S_2x_n, T_2x_n, \beta_n), \forall n \geq 1 \end{aligned} \tag{5.2.17}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following condition:
(i) Suppose that T_i and S_i, $i = 1, 2$ are nonincreasing-asymptotic-mapping for a sequence $\{x_n\}$.
(ii) Suppose that T_i and S_i, $i = 1, 2$ are continuous on K.
(ii) $d(x, T_iy) \leq d(S_ix, T_iy)$ for all $x, y \in K$ and $i = 1, 2$
Then the sequence $\{x_n\}$ defined by (5.2.17) Δ-converges to a common fixed point in F.

Proof. Step1: we prove that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Let $p \in F$. Since T_i and S_i, $i = 1; 2$, are SKC condition type, $\frac{1}{2}d(x_n, S_2p) \leq S_2d(x_n, p)$ and $\frac{1}{2}d(x_n, T_2p) \leq d(x_n, p)$, we consider

$$\begin{aligned} d(y_n, p) &= d(W(S_2x_n, T_2z_n, \beta_n), p) \\ &\leq (1 - \beta_n)d(S_2x_n, p) + \beta_n d(T_2x_n, p) \\ &= (1 - \beta_n)d(S_2x_n, S_2p) + \beta_n d(T_2x_n, T_2p) \end{aligned}$$

$$\leq (1-\beta_n)\max\{d(x_n,p), \frac{d(S_2x_n,x_n)+d(S_2p,p)}{2}, \frac{d(S_2x_n,p)+d(S_2p,x_n)}{2}\}$$
$$+\beta_n\max\{d(x_n,p), \frac{d(T_2x_n,x_n)+d(T_2p,p)}{2}, \frac{d(T_2x_n,p)+d(T_2p,x_n)}{2}\}$$

$$= (1-\beta_n)\max\{d(x_n,p), \frac{d(S_2x_n,x_n)}{2}, \frac{d(S_2x_n,p)+d(S_2p,x_n)}{2}\}$$
$$+\beta_n\max\{d(x_n,p), \frac{d(T_2x_n,x_n)}{2}, \frac{d(T_2x_n,p)+d(T_2p,x_n)}{2}\}$$
$$= (1-\beta_n)\max\{d(x_n,p), \frac{d(S_2x_n,p)+d(S_2p,x_n)}{2}\}$$
$$+\beta_n\max\{d(x_n,p), \frac{d(T_2x_n,p)+d(T_2p,x_n)}{2}\}$$
$$\leq (1-\beta_n)d(x_n,p)+\beta_n d(x_n,p)$$
$$= d(x_n,p). \tag{5.2.18}$$

By (5.2.18), $\frac{1}{2}d(x_n,S_1p) \leq d(x_n,p)$ and $\frac{1}{2}d(y_n,p) \leq T_1d(y_n,p)$, we get that

$$d(x_{n+1},p) = d(W(S_1x_n,T_1y_n,\alpha_n),p)$$
$$\leq (1-\alpha_n)d(S_1x_n,p)+\alpha_n d(T_1y_n,p)$$
$$= (1-\alpha_n)d(S_1x_n,S_1p)+\alpha_n d(T_1y_n,T_1p)$$
$$\leq (1-\alpha_n)\max\{d(x_n,p), \frac{d(S_1x_n,x_n)+d(S_1p,p)}{2}, \frac{d(S_1x_n,p)+d(S_1p,x_n)}{2}\}$$
$$+\alpha_n\max\{d(y_n,p), \frac{d(T_1y_n,y_n)+d(T_1p,p)}{2}, \frac{d(T_1y_n,p)+d(T_1p,y_n)}{2}\}$$
$$= (1-\alpha_n)\max\{d(x_n,p), \frac{d(S_1x_n,x_n)}{2}, \frac{d(S_1x_n,p)+d(p,x_n)}{2}\}$$
$$+\alpha_n\max\{d(y_n,p), \frac{d(T_1y_n,y_n)}{2}, \frac{d(T_1y_n,p)+d(T_1p,y_n)}{2}\}$$
$$= (1-\alpha_n)\max\{d(x_n,p), \frac{d(S_1x_n,p)+d(p,x_n)}{2}\}$$
$$+\alpha_n\max\{d(y_n,p), \frac{d(T_1y_n,p)+d(T_1p,y_n)}{2}\}$$
$$\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(y_n,p)$$
$$\leq d(x_n,p). \tag{5.2.19}$$

So, $\lim_{n\to\infty} d(x_n,p)$ exists for each $p \in F$.

Step2: We will show that $\lim_{n\to\infty} d(x_n,T_ix_n) = 0 = \lim_{n\to\infty} d(x_n,S_ix_n),\ i=1,2.$

Suppose that $\lim_{n\to\infty} d(x_n, p) = c \geq 0$. If $c = 0$, then

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0 = \lim_{n\to\infty} d(x_n, S_i x_n),\ i = 1, 2.$$

Next, we consider $c > 0$. By (5.2.18), we obtain that

$$d(y_n, p) \leq d(x_n, p) \tag{5.2.20}$$

Taking limsup in (5.2.20), we get that

$$\limsup_{n\to\infty} d(y_n, p) \leq c. \tag{5.2.21}$$

Since $d(y_n, T_1 p) \leq d(y_n, p)$, it follows that

$$\begin{aligned} d(T_1 y_n, p) &= d(T_1 y_n, T_1 p) \\ &\leq \max\{d(y_n, p), \frac{d(T_1 y_n, y_n) + d(T_1 p, p)}{2}, \frac{d(T_1 y_n, p) + d(T_1 p, y_n)}{2}\} \\ &= \max\{d(y_n, p), \frac{d(T_1 y_n, y_n)}{2}, \frac{d(T_1 y_n, p) + d(p, y_n)}{2}\} \\ &= \max\{d(y_n, p), \frac{d(T_1 y_n, p) + d(T_1 p, y_n)}{2}\} \\ &\leq d(y_n, p). \end{aligned} \tag{5.2.22}$$

Since $d(x_n, S_1 p) \leq d(x_n, p)$, we have

$$\begin{aligned} d(S_1 x_n, p) &= d(S_1 x_n, S_1 p) \\ &\leq \max\{d(x_n, p), \frac{d(S_1 x_n, x_n) + d(S_1 p, p)}{2}, \frac{d(S_1 x_n, p) + d(S_1 p, x_n)}{2}\} \\ &= \max\{d(x_n, p), \frac{d(S_1 x_n, x_n)}{2}, \frac{d(S_1 x_n, p) + d(p, x_n)}{2}\} \\ &= \max\{d(x_n, p), \frac{d(S_1 x_n, p) + d(S_1 p, x_n)}{2}\} \\ &\leq d(x_n, p). \end{aligned} \tag{5.2.23}$$

Thus,

$$\limsup_{n\to\infty} d(T_1 y_n, p) \leq c, \quad \limsup_{n\to\infty} d(S_1 x_n, p) \leq c. \tag{5.2.24}$$

It is easy to prove that

$$\limsup_{n\to\infty} d(W(S_1 x_n, T_1 y_n, \alpha_n), p) = \limsup_{n\to\infty} d(x_{n+1}, p) = c.$$

By Lemma 2.2.7, we obtain that

$$\lim_{n\to\infty} d(S_1x_n, T_1y_n) = 0. \tag{5.2.25}$$

On the other hand, we can also prove that

$$\lim_{n\to\infty} d(S_2x_n, T_2x_n) = 0. \tag{5.2.26}$$

By hypothesis, we get

$$\lim_{n\to\infty} d(x_n, T_1y_n) \le \lim_{n\to\infty} d(S_1x_n, T_1y_n) = 0 \tag{5.2.27}$$

and

$$\lim_{n\to\infty} d(x_n, T_2x_n) \le \lim_{n\to\infty} d(S_2x_n, T_2x_n) = 0. \tag{5.2.28}$$

$$\begin{aligned} d(y_n, T_2x_n) &= d(W(S_2x_n, T_2x_n), T_2x_n) \\ &\le d(S_2x_n, T_2x_n) \end{aligned} \tag{5.2.29}$$

$$\begin{aligned} d(y_n, S_2x_n) &= d(W(S_2x_n, T_2x_n), S_2x_n) \\ &\le d(T_2x_n, S_2x_n) \end{aligned} \tag{5.2.30}$$

By (5.2.26), (5.2.28) and (5.2.30), we obtain that

$$d(x_n, y_n) \le d(x_n, T_2x_n) + d(T_2x_n, S_2x_n) + d(S_2x_n, y_n) \to 0 \tag{5.2.31}$$

as $n \to \infty$.

Since

$$d(x_n, T_2x_n) \le d(x_n, y_n) + d(y_n, T_2x_n) \tag{5.2.32}$$

and

$$d(x_n, S_2x_n) \le d(x_n, y_n) + d(y_n, S_2x_n), \tag{5.2.33}$$

we have

$$\lim_{n\to\infty} d(x_n, T_2x_n) = 0 = \lim_{n\to\infty} d(x_n, S_2x_n). \tag{5.2.34}$$

So,

$$d(x_n, S_1x_n) \leq d(x_n, T_1y_n) + d(T_1y_n, S_1x_n) \to 0 \tag{5.2.35}$$

as $n \to \infty$.

Consider

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, W(S_1x_n, T_1y_n, \alpha_n)) \\ &\leq (1-\alpha_n)d(x_n, S_1x_n) + \alpha_n d(x_n, T_1y_n) \\ &\leq (1-\alpha_n)d(x_n, T_1y_n) + (1-\alpha_n)d(T_1y_n, S_1x_n) + \alpha_n d(x_n, T_1y_n) \to 0 \end{aligned} \tag{5.2.36}$$

as $n \to \infty$.

Therefore,

$$\begin{aligned} d(x_n, T_1x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_1x_n) \\ &\leq d(x_n, x_{n+1}) + (1-\alpha_n)d(S_1x_n, T_1x_n) + \alpha_n d(T_1y_n, T_1x_n) \to 0 \end{aligned} \tag{5.2.37}$$

as $n \to \infty$.

Hence $\lim_{n\to\infty} d(x_n, T_ix_n) = 0 = \lim_{n\to\infty} d(x_n, S_ix_n),\ i = 1, 2$.

Step3: Next, we will show that the sequence $\{x_n\}$ Δ-converges to a common fixed point of F. We have $\lim_{n\to\infty} d(x_n, p)$ exist. So, $\{d(x_n, p)\}$ is bounded. By Lemma 2.2.6, we obtain that $\{x_n\}$ has a unique asymptotic center, say $A_k(\{x_n\}) = \{x\}$. Let $\{u_n\} \subseteq \{x_n\}$ with $A_k(\{u_n\}) = \{u\}$. It follows that $d(u_n, T_iu_n) \to 0$ as $n \to \infty$. Now, we show that $u \in F(T_i)$. Define a sequence $\{z_n\}$ in K by $z_j = T_i^j u$. So, we consider

$$\begin{aligned} d(z_j, u_n) &\leq d(T_i^j u, T_i^j u_n) + d(T_i^j u_n, T_i^{j-1} u_n) + \ldots + d(Tu_n, u_n) \\ &= d(T_i^j u, T_i^j u_n) + \sum_{k=1}^{j} d(T_i^k u_n, T_i^{k-1} u_n). \end{aligned} \tag{5.2.38}$$

Since $d(T_iu_n, T_iu_n) \leq d(T_iu_n, u_n)$,

$$
\begin{aligned}
d(T_i^2u_n, T_iu_n) &\leq \max\{d(T_iu_n, u_n), \frac{d(T_i^2u_n, T_iu_n) + d(T_iu_n, u_n)}{2}, \\
&\quad \frac{d(T_i^2u_n, u_n) + d(T_iu_n, T_iu_n)}{2}\} \\
&= \max\{d(T_iu_n, u_n), \frac{d(T_i^2u_n, T_iu_n) + d(T_iu_n, u_n)}{2}\} \\
&\leq d(T_iu_n, u_n). \qquad (5.2.39)
\end{aligned}
$$

Since $d(T_i^2u_n, T^2u_n) \leq d(T_i^2u_n, Tu_n)$,

$$
\begin{aligned}
d(T_i^3u_n, T_i^2u_n) &\leq \max\{d(T_i^2u_n, Tu_n), \frac{d(T_i^3u_n, T_i^2u_n) + d(T_i^2u_n, T_iu_n)}{2}, \\
&\quad \frac{d(T_i^3u_n, T_iu_n) + d(T_i^2u_n, T_i^2u_n)}{2}\} \\
&= \max\{d(T_i^2u_n, Tu_n), \frac{d(T_i^3u_n, T_i^2u_n) + d(T_i^2u_n, T_iu_n)}{2}\} \\
&\leq d(T_i^2u_n, Tu_n). \qquad (5.2.40)
\end{aligned}
$$

Thus, $d(T_i^ju_n, T_i^{j-1}u_n) \leq d(T_i^{j-1}u_n, T^{j-2}u_n) \leq d(T_i^{j-2}u_n, T^{j-3}u_n) \leq \ldots \leq d(T_iu_n, u_n)$.
Likewise,

$$
\begin{aligned}
d(T_iu, T_iu_n) \leq \max\{d(u, u_n), &\frac{d(T_iu, u) + d(T_iu_n, u_n)}{2}, \\
&\frac{d(T_iu, u_n) + d(T_iu_n, u)}{2}\},
\end{aligned}
$$

by (i). If

$$
\begin{aligned}
&\max\{d(u, u_n), \frac{d(T_iu, u) + d(T_iu_n, u_n)}{2}, \frac{d(T_iu, u_n) + d(T_iu_n, u)}{2}\} \\
&= \frac{d(T_iu, u) + d(T_iu_n, u_n)}{2}, \qquad (5.2.41)
\end{aligned}
$$

then

$$
\begin{aligned}
d(T_iu, T_iu_n) &\leq \frac{d(T_iu, u) + d(T_iu_n, u_n)}{2} \\
&\leq \frac{d(T_iu, T_iu_n) + d(T_iu_n, u_n) + d(u_n, u) + d(T_iu_n, u_n)}{2},
\end{aligned}
$$

and so

$$d(T_iu, T_iu_n) \le 2d(T_iu_n, u_n) + d(u_n, u). \tag{5.2.42}$$

If

$$\max\{d(u,u_n), \frac{d(T_iu,u) + d(T_iu_n,u_n)}{2}, \frac{d(T_iu,u_n) + d(T_iu_n,u)}{2}\}$$
$$= \frac{d(T_iu,u_n) + d(T_iu_n,u)}{2}, \tag{5.2.43}$$

then

$$d(T_iu, T_iu_n) \le \frac{d(T_iu,u_n) + d(T_iu_n,u)}{2}$$
$$\le \frac{d(T_iu,T_iu_n) + d(T_iu_n,u_n) + d(T_iu_n,u_n) + d(u_n,u)}{2},$$

and so

$$d(T_iu, T_iu_n) \le 2d(T_iu_n, u_n) + d(u_n, u). \tag{5.2.44}$$

Thus,

$$d(T_iu, T_iu_n) \le 2d(T_iu_n, u_n) + d(u_n, u). \tag{5.2.45}$$

Next,

$$d(T_i^2u, T_i^2u_n) \le \max\{d(T_iu, T_iu_n), \frac{d(T_i^2u, T_iu) + d(T_i^2u_n, T_iu_n)}{2},$$
$$\frac{d(T_i^2u, T_iu_n) + d(T_i^2u_n, T_iu)}{2}\}$$

by (i). If

$$\max\{d(T_iu, T_iu_n), \frac{d(T_i^2u, T_iu) + d(T_i^2u_n, T_iu_n)}{2}, \frac{d(T_i^2u, T_iu_n) + d(T_i^2u_n, T_iu)}{2}\}$$
$$= \frac{d(T_i^2u, T_iu) + d(T_i^2u_n, T_iu_n)}{2}, \tag{5.2.46}$$

then

$$d(T_i^2u, T_i^2u_n) \leq \frac{d(T_i^2u, T_iu) + d(T_i^2u_n, T_iu_n)}{2}$$
$$\leq \frac{d(T_i^2u, T_i^2u_n) + d(T_i^2u_n, T_iu_n) + d(T_iu_n, T_iu) + d(T_i^2u_n, T_iu_n)}{2},$$

thus

$$\begin{aligned}
d(T_i^2u, T_i^2u_n) &\leq d(T_iu, T_iu_n) + 2d(T_iu_n, T_i^2u_n) \\
&\leq 2d(T_iu_n, u_n) + d(u_n, u) \\
&\quad + 2\max\{d(u_n, T_iu_n) + \frac{d(T_iu_n, u_n) + d(T_i^2u_n, T_iu_n)}{2} \\
&\quad + \frac{d(T_iu_n, T_iu_n) + d(T_i^2u_n, u_n)}{2}\} \\
&\leq 2d(T_iu_n, u_n) + d(u_n, u) \\
&\quad + 2\max\{d(u_n, T_iu_n) + \frac{d(T_iu_n, u_n) + d(T_i^2u_n, T_iu_n)}{2}\} \\
&\leq 2d(T_iu_n, u_n) + d(u_n, u) + 2d(u_n, T_iu_n) \\
&\leq 2^2d(T_iu_n, u_n) + d(u_n, u). \qquad (5.2.47)
\end{aligned}$$

If

$$\max\{d(T_iu, T_iu_n), \frac{d(T_i^2u, T_iu) + d(T_i^2u_n, T_iu_n)}{2}, \frac{d(T_i^2u, T_iu_n) + d(T_i^2u_n, T_iu)}{2}\}$$
$$= \frac{d(T_i^2u, T_iu_n) + d(T_i^2u_n, T_iu)}{2}, \qquad (5.2.48)$$

then

$$d(T_i^2u, T_i^2u_n) \leq \frac{d(T_i^2u, T_iu_n) + d(T_i^2u_n, T_iu)}{2}$$
$$\leq \frac{d(T_i^2u, T_i^2u_n) + d(T_i^2u_n, T_iu_n) + d(T_i^2u_n, T_iu_n) + d(T_iu_n, T_iu)}{2},$$

thus

$$d(T_i^2u, T_i^2u_n) \leq 2d(T_i^2u_n, T_iu_n) + d(T_iu_n, T_iu)$$

$$\leq 2d(T_i^2 u_n, T_i u_n) + d(T_i u_n, T_i u)$$

$$\leq 2^2 d(T_i u_n, u_n) + d(u_n, u). \tag{5.2.49}$$

So,

$$d(T_i^2 u, T_i^2 u_n) \leq 2^2 d(T_i u_n, u_n) + d(u_n, u). \tag{5.2.50}$$

By above process, we obtain that

$$d(T_i^j u, T_i^j u_n) \leq 2^j d(T_i^{j-1} u_n, u_n) + d(u_n, u). \tag{5.2.51}$$

Hence

$$\begin{aligned} d(z_j, u_n) &\leq d(T_i^j u, T_i^j u_n) + \sum_{k=1}^{j} d(T_i^k u_n, T_i^{k-1} u_n) \\ &\leq 2^j d(T_i u_n, u_n) + d(u_n, u) + j d(T_i u_n, u_n) \\ &= (2^j + j) d(T_i u_n, u_n) + d(u_n, u). \end{aligned} \tag{5.2.52}$$

Therefore

$$r(z_j, , \{u_n\} = \limsup_{n\to\infty} d(z_j, u_n) \leq \limsup_{n\to\infty} d(u_n, u) = r(u, \{u_n\}). \tag{5.2.53}$$

Since $A_K(\{u_n\}) = \{u\}$, we have $r(u, \{u_n\}) \leq r(y, \{u_n\})$, for all $y \in K$. This implies that $\liminf_{j\to\infty} r(z_j, \{u_n\}) \leq r(u, \{u_n\})$. So, we have $\lim_{j\to\infty} r(z_j, \{u_n\}) \leq r(u, \{u_n\})$. It follow from Lemma 2.2.15 that $\lim_{j\to\infty} T_i u = u$. As T_i is uniformly continuous, so that $T_i u = T_i(\lim_{j\to\infty} T_i^j u) = \lim_{j\to\infty} T_i^{j+1} u = u$. That is $u \in F(T_i)$. Similarly, we also can show that $u \in F(S_i)$. Hence $u \in F$. Moreover, $\lim_{n\to\infty} d(x_n, u)$ exists by Step 1. Assume $x \neq u$. By the uniqueness of asymptotic centers, we have

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u), \end{aligned} \tag{5.2.54}$$

which is contradiction. Thus $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. Hence $\{x_n\}$ Δ-converges to a common fixed point in F. $\square$

CHAPTER VI

CONCLUSION

In this thesis, we establish the following results. Firstly, we introduce some nonlinear mappings and prove fixed point theorems for generalized multi-valued mappings satisfying some inequalities in metric spaces. Secondly, we introduce the notions of type multi-valued-coupled contraction, multi-valued-coupled Kannan mapping and prove coupled fixed point theorems on metric spaces. Thirdly, we establish dislocated quasi-b-metric spaces and prove basic properties of dislocated quasi-b-metric spaces. Fourthly, we introduce the notions of type dqb-cyclic-Banach contraction, dqb-cyclic-Kannan mapping, type dqb-cyclic-weak Banach contraction and dqb-cyclic- contraction. Other than, we present some examples to illustrate and support our results. Fifthly, we proved some properties of a fundamentally nonexpansive self-mapping on a nonempty subset of a hyperbolic space and and prove convergence and Δ-convergence theorems of the generalized Krasnoselskij-type iterative process to approximating a fixed point for fundamentally nonexpansive operator in a hyperbolic space. Moreover, we show that if the hyperbolic space is having the Δ-Opial condition, then the fixed points set of such a mapping with the convex range is nonempty. Finally, we prove fixed point theorems for some generalized nonexpansive self-mapping on a nonempty subset of a hyperbolic space and approximating a fixed point for such mappings in a a hyperbolic space. Furthermore, we obtain some properties of fixed point set of generalized nonexpansive mappings in hyperbolic spaces The following results are all main theorems of this thesis:

1. Define a non-increasing function φ from $[0, \frac{1}{2})$ into $(0, 1]$ by

$$\varphi(r) = \begin{cases} 1, & if \ \ 0 \leq r < \frac{\sqrt{5}-1}{\sqrt{5}+1}, \\ \frac{1-2r}{1-r}, & if \ \ \frac{\sqrt{5}-1}{\sqrt{5}+1} \leq r < \frac{1}{2}. \end{cases}$$

Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that Tx is compact for all $x \in X$. Suppose that there exists $r \in [0, \frac{1}{2})$ such that

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq rM(x, y)$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$, for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

2. Define a non-increasing function φ from $[0, \frac{1}{5})$ into $(0, 1]$ by

$$\varphi(r) = \begin{cases} 1, & if \;\; 0 \leq r < \frac{\sqrt{5}-1}{4+2\sqrt{5}}, \\ \frac{1-5r}{1-2r}, & if \;\; \frac{\sqrt{5}-1}{4+2\sqrt{5}} \leq r < \frac{1}{5}. \end{cases}$$

Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that Tx is compact for all $x \in X$. Suppose that there exists $r \in [0, \frac{1}{5})$ such that

$$\varphi(r)d(x, Tx) \leq d(x, y) \quad \text{implies} \quad H(Tx, Ty) \leq S(x, y)$$

where $S(x, y) = rd(x, y) + rd(x, Tx) + rd(y, Ty) + rd(x, Ty) + rd(y, Tx)$ for all $x, y \in X$. Then there exists $z \in X$ such that $z \in Tz$.

3. Define a non-increasing function φ from $[0, 1)$ into $(0, 1]$ by

$$\varphi(r) = \begin{cases} 1, & if \;\; 0 \leq r < \frac{\sqrt{5}-1}{2}, \\ 1 - r, & if \;\; \frac{\sqrt{5}-1}{2} \leq r < 1. \end{cases}$$

Let (X, d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that Tx is compact for all $x \in X$. Assume that there exists $\alpha \in [0, \frac{1}{2})$ such that

$$\varphi(r)d(x,Tx) \leq d(x,y) \quad \text{implies} \quad H(Tx,Ty) \leq \alpha M(x,y)$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$, for all $x, y \in X$, and $r = \frac{\alpha}{1-\alpha}$. Then there exists $z \in X$ such that $z \in Tz$.

4. Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$ such that Tx is compact for all $x \in X$, with the function φ is defined as Theorem 3.1.5. Assume that there exists $\alpha \in [0, \frac{1}{5})$ such that

$$\varphi(r)d(x,Tx) \leq d(x,y) \quad \text{implies} \quad H(Tx,Ty) \leq S(x,y)$$

where $S(x,y) = \alpha d(x,y) + \alpha d(x,Tx) + \alpha d(y,Ty) + \alpha d(x,Ty) + \alpha d(y,Tx)$ for all $x, y \in X$, and $r = 5\alpha$. Then there exists $z \in X$ such that $z \in Tz$.

5. Define a non-increasing function φ from $[0,1)$ into $(0,1]$ by

$$\varphi(r) = \begin{cases} 1, & if \;\; 0 \leq r < \frac{1}{2}, \\ 1-r, & if \;\; \frac{1}{2} \leq r < 1. \end{cases}$$

Let be (X,d) a complete metric space and let T be a mapping from X into $CB(X)$ such that Tx is compact for all $x \in X$. Assume that there exists $\alpha \in [0, \frac{1}{5})$ such that

$$\varphi(r)d(x,Tx) \leq d(x,y) \quad \text{implies} \quad H(Tx,Ty) \leq S(x,y)$$

where $S(x,y) = \alpha d(x,y) + \alpha d(x,Tx) + \alpha d(y,Ty) + \alpha d(x,Ty) + \alpha d(y,Tx)$ for all $x, y \in X$, and $r = \frac{3\alpha}{1-2\alpha}$. Then there exists $z \in X$ such that $z \in Tz$.

6. Let (X,d) be a complete metric space and let $T : X \times X \to 2^X$ be a multi-valued-coupled contraction mapping, (i.e., there exists $k \in [0,1)$ such that

$$H(T(x,y), T(u,v)) \leq \tfrac{k}{2}[d(x,u) + d(y,v)], \quad \text{for all } x, y, u, v \in X).$$

with constant $k \in [0,1)$. Suppose that $x_0, y_0 \in X$. If there exist $x_1, y_1 \in X$ such that $x_1 \in T(x_0, y_0)$ and $y_1 \in T(y_0, x_0)$, then T has coupled fixed points in X.

7. Let (X,d) be a complete metric space, and $T : X \times X \to CB(X)$ be a multi-valued-coupled Kannan mapping, (i.e., there exists $r \in [0, \frac{1}{2})$ such that

$$H(T(x,y),T(u,v)) \leq r[d(x,T(x,y)) + d(u,T(u,v))], \quad \text{for all } x,y,u,v \in X).$$

with $x_0, y_0 \in X$. If there exist $x_1, y_1 \in X$ such that $x_1 \in T(x_0, y_0)$ and $y_1 \in T(y_0, x_0)$, then T has coupled fixed points in X.

8. Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X,d). Let T be a cyclic self-mapping that satisfies the condition a dqb-cyclic-Banach Contraction, (i.e., there exists $k \in [0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y) \quad \text{for all } x \in A,\ y \in B \text{ and } sk \leq 1).$$

Then T has a unique fixed point in $A \cap B$.

9. Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X,d). Let T satisfies the condition a dqb-cyclic-Kannan mapping, (i.e., there exists $r \in [0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leq r(d(x,Tx) + d(x,Ty)), \quad \text{for all } x \in A,\ y \in B \text{ and } sr \leq \tfrac{1}{2}).$$

Then T has a unique fixed point in $A \cap B$.

10. Let A and B be nonempty subsets of a complete dislocated quasi-b-metric space (X,d). Let T satisfies the condition a dqb-cyclic-weak contraction, (i.e., for all $x \in A$, $y \in B$ such that $sd(Tx,Ty) \leq d(x,y) - \psi(d(x,y))$, where $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\psi(t) = 0$ if and only if $t = 0$). Then, T has a unique fixed point in $A \cap B$.

11. Let A and B be nonempty closed subsets of a complete dislocated quasi-b-metric space (X, d). Let T satisfies the condition a dqb-cyclic-ϕ-contraction, (i.e., there exists $k \in [0,1)$ such that $sd(Tx, Ty) \leq \phi(d(x, y))$, for all $x \in A$, $y \in B$, where Φ the family of non-decreasing functions: $\phi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ and $\phi(t) < t$ for each $t > 0$, where n is the n-th iterate of ϕ). Then, T has a unique fixed point in $A \cap B$.

12. Let K be a nonempty bounded closed convex subset of complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{x_n\}$ be a sequence in K. If $T : K \to K$ be fundamentally nonexpansive, and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $F(T)$ is nonempty Δ-closed and convex.

13. Let T be a fundamentally nonexpansive self-mapping on a nonempty subset K of a complete hyperbolic space X, and let $T(K)$ be bounded and convex. Define a sequence $\{Tx_n\}$ in $T(K)$ by $x_1 \in K$ and $Tx_{n+1} = W(T^2x_n, Tx_n, \alpha)$ for all $n \in \mathbb{N}$, where $\alpha \in (0,1)$. Then $\{Tx_n\}$ is an approximate fixed point sequence for T.

14. Let $T : K \to K$ be a fundamentally nonexpansive mapping, where K is a nonempty subset of a complete hyperbolic space X. Then $F(T)$ is closed. Moreover, if X is strictly convex, and K or $T(K)$ is convex, then $F(T)$ is also convex.

15. Let K be a nonempty subset of a uniformly convex hyperbolic space X, and let T be a fundamentally nonexpansive self-mapping from K onto K. If $T(K)$ is bounded, closed, then $F(T)$ is nonempty.

16. Let K be a nonempty subset of a uniformly convex hyperbolic space X, and let T be a fundamentally nonexpansive self-mapping from K onto K. If $T(K)$ is bounded, closed, then $F(T)$ is nonempty and closed. Moreover, if K or $T(K)$ is convex, then $F(T)$ is also convex.

17. Let K be a a nonempty compact subset complete hyperbolic space of X. Assume that $T : K \to K$ is fundamentally nonexpansive and $T(K)$ is convex. Then $\{Tx_n\}$ in $T(K)$ defined by $x_1 \in K$ and

$$Tx_{n+1} = W(T^2x_n, Tx_n, \alpha)$$

for $n = 1, 2, 3, \ldots$, where $\alpha \in (0, 1)$, converges to a fixed point of T.

18. Let K be a a nonempty Δ-compact subset complete hyperbolic space of X with the Δ-Opial condition. Assume that $T : K \to K$ is fundamentally nonexpansive mapping and $T(K)$ is convex. Then $\{Tx_n\}$ in $T(K)$ defined by $x_1 \in K$ and $Tx_{n+1} = W(T^2x_n, Tx_n, \alpha)$ for $n = 1, 2, 3, \ldots$, where $\alpha \in (0, 1)$, Δ-converges to a fixed point of T.

19. Let K be a a nonempty Δ-compact subset uniformly convex complete hyperbolic space of X. Assume that $T : K \to K$ is fundamentally nonexpansive mapping, and $T(K)$ is convex. Then $\{Tx_n\}$ in $T(K)$ defined by $x_1 \in K$ and $Tx_{n+1} = W(T^2x_n, Tx_n, \alpha)$ for $n = 1, 2, 3, \ldots$, where $\alpha \in (0, 1)$, Δ-converges to a fixed point of T.

20. Let K be a nonempty and convex subset of a strictly convex hyperbolic space X. If $T : K \to K$ satisfies condition C, Then$F(T)$ is closed and convex.

21. Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a sequence in K, with $d(x_n, Tx_n) \to 0$, and $T : K \to K$ satisfies condition C. If $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.

22. Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. If $T : K \to K$ satisfies condition C, then $F(T)$ is nonempty closed and convex.

23. Let K be a nonempty closed and convex subset of a complete uni-

formly convex hyperbolic space X, with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a bounded sequence in K with $d(x_n, Tx_n) \to 0$. If $T : K \to K$ satisfies condition C and $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.

24. Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. If $T : K \to K$ satisfies condition C, then $F(T)$ is nonempty closed and convex.

25. Let K be a nonempty and convex subset of a strictly convex hyperbolic space X. Suppose that $T : K \to K$ satisfies by one of the conditions SKC, KSC, SCC and CSC. Then $F(T)$ is closed and convex.

26. Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a sequence in K with $d(x_n, Tx_n) \to 0$, and $T : K \to K$ satisfies by one of the conditions SKC, KSC, SCC and CSC. If $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.

27. Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a bounded sequence in K with $d(x_n, Tx_n) \to 0$, and $T : K \to K$ satisfies by one of the conditions SKC, KSC, SCC and CSC. If $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.

28. Let K be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X, with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a sequence in K with $d(x_n, Tx_n) \to 0$, and $T : K \to K$ satisfies conditions E_μ. If $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.

29. Let K be a nonempty closed and convex subset of a complete uni-

formly convex hyperbolic space X, with monotone modulus of uniform convexity η. Suppose that $\{x_n\}$ is a sequence in K with $d(x_n, Tx_n) \to 0$, and $T : K \to K$ satisfies conditions C_λ. If $A(\{x_n\}) = \{x\}$, then x is a fixed point of T. Moreover, $F(T)$ is closed and convex.

30. Let K be a nonempty closed convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T_i : K \to K$ and $S_i : K \to K$, $i = 1, 2$ satisfies the condition SKC. Assume that $F := \bigcap_{i=1}^{n=2} F(T_i) \cap F(S_i) \neq \emptyset$, for arbitrarily chosen $x_i \in K$, such that $\{x_n\}$ is defined as follows

$$\begin{aligned} x_{n+1} &= W(S_1 x_n, T_1 y_n, \alpha_n), \\ y_n &= W(S_2 x_n, T_2 x_n, \beta_n), \forall n \geq 1 \end{aligned} \tag{5.2.55}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following condition:

(i) Suppose that T_i and S_i, $i = 1, 2$ are nonincreasing-asymptotic-mapping for a sequence $\{x_n\}$.

(ii) Suppose that T_i and S_i, $i = 1, 2$ are continuous on K.

(ii) $d(x, T_i y) \leq d(S_i x, T_i y)$ for all $x, y \in K$ and $i = 1, 2$

Then the sequence $\{x_n\}$ defined by (5.2.55) Δ-converges to a common fixed point in F.

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