# GENERALIZATIONS OF FUZZY SUBSEMIGROUPS ON ORDERED SEMIGROUPS

RATSIRI SANGUANWONG

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**Oral Defense Committee**

A. Iampan ........................................ Chair

(Assistant Professor Aiyared Iampan, Ph.D.)

Manoj Siripitukdet ........................................ Advisor

(Associate Professor Manoj Siripitukdet, Ph.D.)

E. Laysirikul ........................................ Internal Examiner

(Ekkachai Laysirikul, Ph.D.)

Approved

........................................

(Panu Putthawong, Ph.D.)

Associate Dean for Administration and Planning

for Dean of the Graduate School

7 JUL 2017

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| **Author** | Ratsiri Sanguanwong |
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## ABSTRACT

In this research, two binary operations on a closed interval $[0, 1]$ of real numbers are defined. By using the binary operations, we introduce the concept of generalized $(s, t)$-fuzzy bi-ideals, which is a generalization of $(s, t)$-fuzzy bi-ideals of ordered semigroups. We define a fuzzy subset $f_{s,t}$ of an ordered semigroup, where $f$ is a fuzzy subset and investigate some interesting results. Then we apply this idea to generalized $(s, t)$-fuzzy interior ideals and generalized $(s, t)$-fuzzy left ideals. The characterization of completely regular, intra-regular, left regular, and simple ordered semigroups by their generalized $(s, t)$-fuzzy bi-ideals, generalized $(s, t)$-fuzzy interior ideal, and generalized $(s, t)$-fuzzy left ideals is given. In addition, we show that in regular, completely regular, intra-regular, and left regular ordered semigroups, the concepts of generalized $(s, t)$-fuzzy interior ideals and generalized $(s, t)$-fuzzy ideals coincide.

# LIST OF CONTENTS

Chapter Page

# CHAPTER I

# INTRODUCTION

In 1965, the concept of fuzzy subset was introduced by Zadeh [22]. The theory of fuzzy subsets can be applied in many scientific fields such as computer science, coding theory, decision theory, and social science. In algebra, Rosenfeld [18] applied the concept of fuzzy subsets to groupoids and groups. By applying Rosenfeld's idea, the concepts of fuzzy ideals, fuzzy bi-ideals, and fuzzy interior ideals in semigroups were studied by Kuroki [14, 15]. Since that day, the concepts of fuzzy subsets in semigroups have been studied by many authors. An ordered semigroup is a semigroup together with a partial order, which is compatible with the operation. This type of semigroup has been interested by some authors to study because every semigroup can be treated as an ordered semigroup.

By using "belong to" relation ($\in$) and "quasi-coincident with" relation ($q$), Bhakat and Das [1, 2] introduced the concept of $(\alpha, \beta)$-fuzzy subgroup, where $\alpha, \beta \in \{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha \neq \in\wedge q$. This concept of $(\alpha, \beta)$-fuzzy subgroups is a generalization of fuzzy subgroups. By the same idea, Shabir et. al. [19] introduced the concepts of $(\alpha, \beta)$-fuzzy bi-ideals in semigroups ($\alpha, \beta \in \{\in, q, \in\vee q, \in\wedge q\}$ and $\alpha \neq \in\wedge q$). The results of $(\in, \in\vee q)$-fuzzy bi-ideals were established. Shabir et. al. [20] also generalized the concept of $(\in, \in\vee q)$-fuzzy bi-ideals in semigroups to $(\in, \in\vee q_k)$-fuzzy bi-ideals and characterized some types of semigroups. By applying these ideas, the similar results hold in case of ordered semigroups [4, 11]. In 2012, Khan et. al. [13] introduced $(\lambda, \theta)$-fuzzy bi-ideals, where $\lambda, \theta \in [0, 1]$ in ordered semigroups. Some authors prefer to use the term fuzzy bi-ideals with thresholds $(\lambda, \theta]$ instead of $(\lambda, \theta)$-fuzzy bi-ideals. This concept of $(\lambda, \theta)$-fuzzy bi-ideals is a generalized concept of $(\in, \in\vee q)$-fuzzy bi-ideals (resp. $(\in, \in\vee q_k)$-fuzzy bi-ideals). The former work showed that the concept of $(\in, \in\vee q)$-fuzzy bi-ideals (resp. $(\in, \in\vee q_k)$-fuzzy bi-ideals) and $(0, 0.5)$-fuzzy bi-ideals (resp. $(0, \frac{1-k}{2})$-fuzzy bi-ideals) are the same. The characterization of completely regular ordered semigroups by their $(\lambda, \theta)$-fuzzy bi-ideals was also given. The similar works in cases of fuzzy interior ideals and fuzzy left ideals can be founded in [3, 5, 10, 12, 19, 20, 21]. Siripitukdet and Ruanon [21] also showed that in regular, completely regular, intra-regular, and left regular ordered semigroups, the concepts of fuzzy interior ideals with thresholds $(s, t]$ and fuzzy ideals with thresholds $(s, t]$ coincide.

In this research, two binary operations on a closed interval $[0, 1]$ of real numbers are defined. By using the binary operations, we introduce the concept of generalized $(s, t)$-fuzzy bi-ideals which is a generalization of $(s, t)$-fuzzy bi-ideals of ordered semigroups. We define a fuzzy subset $f_{s,t}$ of an ordered semigroup, where $f$ is a fuzzy subset and investigate some interesting results. Then we apply this idea to generalized $(s, t)$-fuzzy interior ideals and generalized $(s, t)$-fuzzy left ideals. The characterization of completely regular, intra-regular, and left regular ordered semigroups by their generalized $(s, t)$-fuzzy bi-ideals, generalized $(s, t)$-fuzzy interior ideal, and generalized $(s, t)$-fuzzy left ideals are given. Finally, we show that in regular, completely regular, intra-regular, and left regular ordered semigroups, the concepts of generalized $(s, t)$-fuzzy interior ideals and generalized $(s, t)$-fuzzy ideals coincide.

This thesis is organized into 4 chapters. The Chapter I is an introduction of this research. In Chapter II, we give some definitions, notations, examples, and basic results which are used in this research. This chapter is divided into two sections. The first section deals with ordered semigroups and subsemigroups. The next one is about fuzzy subsemigroups on ordered semigroups. In Chapter III, we introduce two binary operations. By using the operations, we generalized some types of fuzzy subsemigroups on ordered semigroups and investigated some basic results. The characterizations of some types of ordered semigroups are also given in this chapter. All main results of this thesis are concluded and listed in Chapter IV.

# CHAPTER II

# PRELIMINARIES

In this chapter, we give some definitions, notations, examples, and basic results which are used in this research. This chapter is divided into two sections. The first section deals with ordered semigroups and subsemigroups. The next one is about fuzzy subsemigroups on ordered semigroups.

## 2.1 Basic definitions in ordered semigroups

**Definition 2.1.1.** ([17]) A *binary relation* or simply a *relation* $R$ from a set $A$ into a set $B$ is a subset of $A \times B$. If $A = B$, then $R$ is called a *relation on* $A$.

**Definition 2.1.2.** ([17]) A relation $R$ on a set $S$ is called a *partial order* on $S$ if it satisfies the following conditions:

(1) $(\forall x \in S)((x, x) \in R)$,

(2) $(\forall x, y \in S)((x, y) \in R$ and $(y, x) \in R$ implies $x = y)$,

(3) $(\forall x, y, z \in S)((x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R)$.

A partial order on a set $S$ is usually denoted by $\leq$. We often write $x \leq y$ rather than $(x, y) \in \leq$.

**Definition 2.1.3.** ([17]) A nonempty set together with a partial order on it is called *partially ordered set.*

**Definition 2.1.4.** ([10]) By an *ordered semigroup*, we mean a structure $(S, \cdot, \leq)$ in which the following conditions are satisfied:

(1) $(S, \cdot)$ is a semigroup,

(2) $(S, \leq)$ is a partially ordered set,

(3) $(\forall a, b, c \in S)(a \leq b$ implies $ac \leq bc$ and $ca \leq cb)$.

For convenience, we write $S$ instead of $(S, \cdot, \leq)$ when there is no danger of ambiguity.

For any nonempty subsets $A, B$ of an ordered semigroup $S$, denote $AB := \{ab \mid a \in A, b \in B\}$.

**Definition 2.1.5.** ([4]) A nonempty subset $A$ of an ordered semigroup $S$ is called a *bi-ideal* of $S$ if it satisfies:

(1) $A^2 \subseteq A$,

(2) $ASA \subseteq A$,

(3) $(\forall a \in S)(\forall b \in A)(a \leq b$ implies $a \in A)$.

**Definition 2.1.6.** ([10]) A nonempty subset $A$ of an ordered semigroup $S$ is called an *interior ideal* of $S$ if it satisfies:

(1) $A^2 \subseteq A$,

(2) $SAS \subseteq A$,

(3) $(\forall a \in S)(\forall b \in A)(a \leq b$ implies $a \in A)$.

**Definition 2.1.7.** ([12]) A nonempty subset $A$ of an ordered semigroup $S$ is called a *left (resp. right) ideal* of $S$ if it satisfies:

(1) $SA \subseteq A$ (resp. $AS \subseteq A$),

(2) $(\forall a \in S)(\forall b \in A)(a \leq b$ implies $a \in A)$.

A nonempty subset $A$ of an ordered semigroup $S$ is called an *ideal* of $S$ if it is both a left ideal and a right ideal of $S$.

**Definition 2.1.8.** ([11]) An ordered semigroup $S$ is called *simple* if it does not contain any proper ideal.

Let $S$ be an ordered semigroup. For any nonempty subset $A$ of $S$, we denote $(A] := \{b \in S \mid b \leq a \text{ for some } a \in A\}$. If $A = \{a\}$, we write $(a]$ instead of $(\{a\}]$.

For any element $a$ of an ordered semigroup $S$, we denote by $I(a)$ (resp. $L(a), B(a)$) the ideal (resp. left ideal, bi-ideal) generated by $a$. It is well-known that $I(a) = (\{a\} \cup aS \cup Sa \cup SaS]$, $L(a) = (\{a\} \cup Sa]$, and $B(a) = (\{a\} \cup \{a^2\} \cup aSa]$.

**Definition 2.1.9.** ([11]) An ordered semigroup $S$ is called *regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$.

**Definition 2.1.10.** ([13]) An ordered semigroup $S$ is called *completely regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq a^2xa^2$.

**Definition 2.1.11.** ([10]) An ordered semigroup $S$ is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$.

**Definition 2.1.12.** ([11]) An ordered semigroup $S$ is called *left regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$.

## 2.2 Fuzzy subsets on ordered semigroups

**Definition 2.2.1.** ([17]) Let $A$ be a nonempty subset of a set $X$. The *characteristic function* of $A$ is a function $C_A$ of $X$ into $\{0,1\}$ defined by $C_A(x) = 1$ if $x \in A$ and $C_A(x) = 0$ if $x \notin A$.

**Definition 2.2.2.** ([17]) A *fuzzy subset* $f$ of a nonempty set $X$ is a function $f$ from $X$ into the closed interval $[0,1]$.

For any $a, b \in [0,1]$, we denote $a \wedge b := \min\{a,b\}$ and $a \vee b = \max\{a,b\}$. Then $\vee$ and $\wedge$ are binary operations on $[0,1]$.

**Definition 2.2.3.** ([11]) A fuzzy subset $f$ of an ordered semigroup $S$ is called a *fuzzy subsemigroup* of $S$ if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$.

**Definition 2.2.4.** ([4]) A fuzzy subset $f$ of an ordered semigroup $S$ is called a *fuzzy bi-ideal* of $S$ if it satisfies:

(1) $(\forall x, y \in S)(f(xy) \geq f(x) \wedge f(y))$,

(2) $(\forall x, y, z \in S)(f(xyz) \geq f(x) \wedge f(z))$,

(3) $(\forall x, y \in S)(x \leq y$ implies $f(x) \geq f(y))$.

**Lemma 2.2.5.** ([4]) *A nonempty subset $A$ of an ordered semigroup $S$ is a bi-ideal of $S$ if and only if the characteristic function $C_A$ is a fuzzy bi-ideal of $S$.*

**Definition 2.2.6.** ([10]) A fuzzy subset $f$ of an ordered semigroup $S$ is called a *fuzzy interior ideal* of $S$ if it satisfies:

(1) $(\forall x, y \in S)(f(xy) \geq f(x) \wedge f(y))$,

(2) $(\forall x, y, z \in S)(f(xyz) \geq f(y))$,

(3) $(\forall x, y \in S)(x \leq y$ implies $f(x) \geq f(y))$.

**Lemma 2.2.7.** ([10]) *A nonempty subset $A$ of an ordered semigroup $S$ is an interior ideal of $S$ if and only if the characteristic function $C_A$ is a fuzzy interior ideal of $S$.*

**Definition 2.2.8.** ([10]) A fuzzy subset $f$ of an ordered semigroup $S$ is called a *fuzzy left ideal (resp. fuzzy right ideal)* of $S$ if it satisfies:

(1) $(\forall x, y \in S)(f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)))$,

(2) $(\forall x, y \in S)(x \leq y$ implies $f(x) \geq f(y))$.

A fuzzy subset $f$ of an ordered semigroup $S$ is called a *fuzzy ideal* of $S$ if it is both a fuzzy left ideal and a fuzzy right ideal of $S$.

**Lemma 2.2.9.** ([6]) *A nonempty subset $A$ of an ordered semigroup $S$ is a left ideal of $S$ if and only if the characteristic function $C_A$ is a fuzzy left ideal of $S$.*

**Remark 2.2.10.** Every fuzzy ideal of an ordered semigroup is a fuzzy bi-ideal and a fuzzy interior ideal, but a fuzzy bi-ideal or a fuzzy interior ideal does not necessarily to be a fuzzy ideal.

**Definition 2.2.11.** ([13]) Let $s, t \in [0,1]$ with $s < t$. A fuzzy subset $f$ of an ordered semigroup $S$ is called an $(s,t)$-*fuzzy bi-ideal* of $S$ if it satisfies:

(1) $(\forall x, y \in S)(f(xy) \vee s \geq f(x) \wedge f(y) \wedge t)$,

(2) $(\forall x, y, z \in S)(f(xyz) \vee s \geq f(x) \wedge f(z) \wedge t)$,

(3) $(\forall x, y \in S)(x \leq y$ implies $f(x) \vee s \geq f(y) \wedge t)$.

**Example 2.2.12.** ([13]) *Let $S = \{a, b, c, d, e\}$. Define a binary operation and a partial order on $S$ as follows:*

| $\cdot$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $d$ | $a$ | $d$ | $d$ |
| $b$ | $a$ | $b$ | $a$ | $d$ | $d$ |
| $c$ | $a$ | $d$ | $c$ | $d$ | $e$ |
| $d$ | $a$ | $d$ | $a$ | $d$ | $d$ |
| $e$ | $a$ | $d$ | $c$ | $d$ | $e$ |

$$\leq \, = \{(a,a), (a,c), (a,d), (a,e), (b,b), (b,d), (b,e), (c,c), (c,e), (d,d), (d,e), (e,e)\}.$$

*Let $f$ be a fuzzy subset of $S$ defined by*

$$f(x) = \begin{cases} 0.8, & \text{if } x = a, \\ 0.7, & \text{if } x = b, \\ 0.3, & \text{if } x = c, \\ 0.5, & \text{if } x = d, \\ 0.6, & \text{if } x = e. \end{cases}$$

*Then $S$ is an ordered semigroup and $f$ is a $(0.2, 0.4)$-fuzzy bi-ideal of $S$, but not a fuzzy bi-ideal of $S$ because $f(ab) = 0.5 < 0.7 = f(a) \wedge f(b)$.*

**Lemma 2.2.13.** ([13]) *Let $s, t \in [0, 1]$ with $s < t$. A nonempty subset $A$ of an ordered semigroup $S$ is a bi-ideal of $S$ if and only if the characteristic function $C_A$ is an $(s, t)$-fuzzy bi-ideal of $S$.*

**Definition 2.2.14.** ([21]) Let $s, t \in [0, 1]$ with $s < t$. A fuzzy subset $f$ of an ordered semigroup $S$ is called an *$(s, t)$-fuzzy interior ideal* of $S$ if it satisfies:

(1) $(\forall x, y \in S)(f(xy) \vee s \geq f(x) \wedge f(y) \wedge t)$,

(2) $(\forall x, y, z \in S)(f(xyz) \vee s \geq f(y) \wedge t)$,

(3) $(\forall x, y \in S)(x \leq y$ implies $f(x) \vee s \geq f(y) \wedge t)$.

**Example 2.2.15.** *Let $S = \{a, b, c, d\}$. Define a binary operation and a partial order on $S$ as follows:*

| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $d$ | $a$ |
| $c$ | $a$ | $a$ | $a$ | $a$ |
| $d$ | $a$ | $a$ | $a$ | $a$ |

$$\leq = \{(a, a), (a, c), (a, d), (b, b), (c, c), (d, d)\}.$$

*Let $f$ be a fuzzy subset of $S$ defined by*

$$f(x) = \begin{cases} 0.7, & \text{if } x = a, \\ 0.8, & \text{if } x = b, \\ 0.5, & \text{if } x = c, \\ 0.4, & \text{if } x = d. \end{cases}$$

*Then $S$ is an ordered semigroup and $f$ is a $(0.5, 0.7)-$fuzzy interior ideal of $S$, but not a fuzzy interior ideal of $S$ because $f(bc) = 0.4 < 0.5 = f(b) \wedge f(c)$.*

**Lemma 2.2.16.** ([21]) *Let $s, t \in [0, 1]$ with $s < t$. A nonempty subset $A$ of an ordered semigroup $S$ is an interior ideal of $S$ if and only if the characteristic function $C_A$ is a $(s, t)$-fuzzy interior ideal of $S$.*

**Definition 2.2.17.** ([3]) Let $s, t \in [0, 1]$ with $s < t$. A fuzzy subset $f$ of an ordered semigroup $S$ is called an *$(s, t)$-fuzzy left ideal (resp. $(s, t)$-fuzzy right ideal)* of $S$ if it satisfies:

(1) $(\forall x, y \in S)(f(xy) \vee s \geq f(y) \wedge t$ (resp. $f(xy) \vee s \geq f(x) \wedge t$)),

(2) $(\forall x, y \in S)(x \leq y$ implies $f(x) \vee s \geq f(y) \wedge t)$.

A fuzzy subset $f$ of an ordered semigroup $S$ is called an $(s, t)$-*fuzzy* ***ideal*** of $S$ if it is both an $(s, t)$-fuzzy left ideal and an $(s, t)$-fuzzy right ideal of $S$.

# CHAPTER III

# MAIN RESULTS

In this chapter, we introduce two binary operations. By using the operations, we generalized some types of fuzzy subsemigroups on ordered semigroups.

## 3.1 The binary operations $\wedge_f$ and $\vee_f$ on $[0,1]$

In this section, two binary operations on $[0,1]$ are given and some properties of them are investigated. Through the research, we write $a \wedge b$ and $a \vee b$ instead of $\min\{a,b\}$ and $\max\{a,b\}$, respectively.

For any fuzzy subset $f$ of a nonempty set $X$ and any $a, b \in [0,1]$, denote

$$\begin{aligned} \underline{a}_f &:= \{m \in \mathrm{Im}(f) | 0 < m \leq a\}, \\ \overline{a}_f &:= \{m \in \mathrm{Im}(f) | a \leq m < 1\}. \end{aligned}$$

Define two binary operations $\wedge_f$ and $\vee_f$ on $[0,1]$ as follows:

$$a \wedge_f b = \begin{cases} \sup \underline{a \wedge b}_f, & \text{if } \underline{a \wedge b}_f \neq \emptyset, \\ a \wedge b, & \text{otherwise} \end{cases}$$

and

$$a \vee_f b = \begin{cases} \inf \overline{a \vee b}_f, & \text{if } \overline{a \vee b}_f \neq \emptyset, \\ a \vee b, & \text{otherwise.} \end{cases}$$

**Example 3.1.1.** *Let $X = \{a,b,c,d\}$ and let $f$ be a fuzzy subset of $X$ defined as follows:*

$$f(x) = \begin{cases} 0.8, & \text{if } x = a, \\ 0.7, & \text{if } x = b, \\ 0.4, & \text{if } x = c, \\ 0.2, & \text{if } x = d. \end{cases}$$

*We can see that $\underline{0.5}_f = \{0.2, 0.4\}$ and $\overline{0.7}_f = \{0.7, 0.8\}$. Then $0.6 \wedge_f 0.5 = \sup \underline{0.5}_f = 0.4$ and $0.7 \vee_f 0.1 = \inf \overline{0.7}_f = 0.7$.*

The following propositions are obtained and will often be used in the next section.

**Proposition 3.1.2.** *Let $f$ be a fuzzy subset of a nonempty set $X$ and $a, b, c \in [0, 1]$. The following statements hold:*

(1) $a \wedge_f b = b \wedge_f a$,

(2) $a \vee_f b = b \vee_f a$,

(3) $a \wedge_f b \leq a \wedge b$,

(4) $a \vee_f b \geq a \vee b$,

(5) $a \wedge_f b = a \wedge b$ *if* $a, b \in \mathrm{Im}(f)$,

(6) $a \vee_f b = a \vee b$ *if* $a, b \in \mathrm{Im}(f)$,

(7) $a \wedge_f b \leq a \wedge_f c$ *if* $b \leq c$,

(8) $a \vee_f b \leq a \vee_f c$ *if* $b \leq c$,

(9) $a \wedge_f c \geq b \wedge_f c$ *if* $a \geq c$,

(10) $a \wedge_f c = b \wedge_f c$ *if* $a \wedge b \wedge c = c$,

(11) $a \vee_f c \leq b \vee_f c$ *if* $a \leq c$,

(12) $a \vee_f c = b \vee_f c$ *if* $a \vee b \vee c = c$.

*Proof.* (1)-(8) are straightforward from the definitions of $\wedge_f$ and $\vee_f$.

(9) Assume that $a \geq c$. If $c \geq b$, then $a \wedge_f c \geq c \wedge_f c \geq b \wedge_f c$. Now, suppose that $c < b$. We will consider the following cases:

Case 1: $\underline{c}_f = \emptyset$. Then $\underline{a \wedge c}_f = \emptyset = \underline{b \wedge c}_f$. This means that $a \wedge_f c = a \wedge c = b \wedge c = b \wedge_f c$.

Case 2: $\underline{c}_f \neq \emptyset$. Then $a \wedge_f c = \sup \underline{a \wedge c}_f = \sup \underline{c}_f = \sup \underline{b \wedge c}_f = b \wedge_f c$.

Thus (9) holds.

(10) Suppose that $a \wedge b \wedge c = c$. Then $a \geq c$ and $b \geq c$. By applying (9), $a \wedge_f c = b \wedge_f c$.

(11) Assume that $a \leq c$. If $c \leq b$, then $a \vee_f c \leq c \vee_f c \leq b \vee_f c$. Now, suppose that $c > b$. We will consider the following cases:

Case 1: $\overline{c}_f = \emptyset$. Then $\overline{a \vee c}_f = \emptyset = \overline{b \vee c}_f$. This means that $a \vee_f c = a \vee c = b \vee c = b \vee_f c$.

Case 2: $\overline{c}_f \neq \emptyset$. Then $a \vee_f c = \inf \overline{a \vee c}_f = \inf \overline{c}_f = \inf \overline{b \vee c}_f = b \vee_f c$.

Thus (11) holds.

(12) Suppose that $a \vee b \vee c = c$. Then $a \leq c$ and $b \leq c$. By applying (11), $a \vee_f c = b \vee_f c$. □

**Proposition 3.1.3.** *Let $f$ be a fuzzy subset of a nonempty subset $X$ and $a, b \in [0, 1]$ such that $\underline{a}_f$ and $\underline{b}_f$ are nonempty. Then the following statements hold:*

(1) $\underline{a}_f \subseteq \underline{b}_f$ *if* $a \leq b$,

(2) $\underline{a}_f = \underline{\sup \underline{a}_f}_f$,

(3) $a \wedge_f b = a \wedge_f \sup \underline{b}_f = \sup \underline{a}_f \wedge_f \sup \underline{b}_f$.

*Proof.* (1) Obvious.

(2) Since $\sup \underline{a}_f \leq a$, we have $\underline{\sup \underline{a}_f}_f \subseteq \underline{a}_f$. On the other hand, let $m \in \underline{a}_f$. Then $m \leq \sup \underline{a}_f$. This implies that $m \in \underline{\sup \underline{a}_f}_f$. Thus $\underline{a}_f \subseteq \underline{\sup \underline{a}_f}_f$.

(3) Note that $a \geq \sup \underline{a}_f$ and $b \geq \sup \underline{b}_f$. Then $a \wedge_f b \geq a \wedge_f \sup \underline{b}_f \geq \sup \underline{a}_f \wedge_f \sup \underline{b}_f$. On the other hand, we have $\sup \underline{a}_f \wedge_f \sup \underline{b}_f \geq \sup \underline{a \wedge b}_f \wedge_f \sup \underline{a \wedge b}_f = \sup \underline{\sup \underline{a \wedge b}_f}_f = \sup \underline{a \wedge b}_f = a \wedge_f b$. □

**Proposition 3.1.4.** *Let $f$ be a fuzzy subset of a nonempty subset $X$ and $a, b \in [0,1]$ such that $\overline{a}_f$ and $\overline{b}_f$ are nonempty. Then the following statements hold:*

(1) $\overline{a}_f \subseteq \overline{b}_f$ *if* $b \leq a$,

(2) $\overline{a}_f = \overline{\inf \overline{a}_f}_f$,

(3) $a \vee_f b = a \vee_f \inf \overline{b}_f = \inf \overline{a}_f \vee_f \inf \overline{b}_f$.

*Proof.* (1) Obvious.

(2) Since $a \leq \inf \overline{a}_f$, we have $\overline{\inf \overline{a}_f}_f \subseteq \overline{a}_f$. On the other hand, let $m \in \overline{a}_f$. Then $m \geq \inf \overline{a}_f$. This implies that $m \in \overline{\inf \overline{a}_f}_f$. Thus $\overline{a}_f \subseteq \overline{\inf \overline{a}_f}_f$.

(3) Note that $a \leq \inf \overline{a}_f$ and $b \leq \inf \overline{b}_f$. Then $a \vee_f b \leq a \vee_f \inf \overline{b}_f \leq \inf \overline{a}_f \vee_f \inf \overline{b}_f$. On the other hand, we have $\inf \overline{a}_f \vee_f \inf \overline{b}_f \leq \inf \overline{a \vee b}_f \vee_f \inf \overline{a \vee b}_f = \inf \overline{\inf \overline{a \vee b}_f}_f = \inf \overline{a \vee b}_f = a \vee_f b$. □

**Proposition 3.1.5.** *Let $f$ be a fuzzy subset of a nonempty set $X$ and $a, b, c \in [0,1]$. The following statements hold:*

(1) $a \wedge_f (b \wedge_f b) = a \wedge_f b$,

(2) $a \vee_f (b \vee_f b) = a \vee_f b$,

(3) $(a \wedge_f b) \wedge_f c = a \wedge_f (b \wedge_f c)$,

(4) $(a \vee_f b) \vee_f c = a \vee_f (b \vee_f c)$.

*Proof.* (1) and (2) are straightforward from Proposition 3.1.3 and 3.1.4, respectively. To show that (3) is true, we may assume that $a \leq b \wedge c$. If $\underline{a}_f = \emptyset$, then $(a \wedge_f b) \wedge_f c = a = a \wedge_f (b \wedge_f c)$. Suppose that $\underline{a}_f \neq \emptyset$. By Proposition 3.1.3, we have $\underline{b}_f$ and $\underline{c}_f$ are nonempty. We can verify that $(a \wedge_f b) \wedge_f c = \underline{a}_f = a \wedge_f (b \wedge_f c)$. Thus (3) holds. By using a similar reasoning, (4) is true. □

## 3.2 Generalized $(s,t)$-fuzzy bi-ideals in ordered semigroups

In this section, we introduced a generalized $(s,t)$-fuzzy bi-ideal of an ordered semigroup and investigate some interesting properties. From now on, let $s,t \in [0,1]$ with $s < t$.

**Definition 3.2.1.** A fuzzy subset $f$ of an ordered semigroup $S$ is called a *generalized $(s,t)$-fuzzy bi-ideal* of $S$ if it satisfies:

(1) $(\forall x,y \in S)(f(xy) \vee_f s \geq f(x) \wedge_f f(y) \wedge_f t)$,

(2) $(\forall x,y,z \in S)(f(xyz) \vee_f s \geq f(x) \wedge_f f(z) \wedge_f t)$,

(3) $(\forall x,y \in S)(x \leq y$ implies $f(x) \vee_f s \geq f(y) \wedge_f t)$.

Every $(s,t)$-fuzzy bi-ideal of an ordered semigroup $S$ is a generalized $(s,t)$-fuzzy bi-ideal of $S$, but the converse is not true.

**Example 3.2.2.** *Let $S = \{a,b,c,d,e\}$. Define a binary operation and a partial order on $S$ as follows:*

| $\cdot$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $d$ | $a$ | $d$ | $d$ |
| $b$ | $a$ | $b$ | $a$ | $d$ | $d$ |
| $c$ | $a$ | $d$ | $c$ | $d$ | $e$ |
| $d$ | $a$ | $d$ | $a$ | $d$ | $d$ |
| $e$ | $a$ | $d$ | $c$ | $d$ | $e$ |

$$\leq \; = \{(a,a),(a,c),(a,d),(a,e),(b,b),(b,d),(b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\}.$$

*Let $f$ be a fuzzy subset of $S$ defined by*

$$f(x) = \begin{cases} 0.8, & \text{if } x = a, \\ 0.7, & \text{if } x = b, \\ 0.3, & \text{if } x = c \text{ or } x = d, \\ 0.6, & \text{if } x = e. \end{cases}$$

*Then $S$ is an ordered semigroup and $f$ is a generalized $(0.2, 0.4)$-fuzzy bi-ideal of $S$, but not a $(0.2, 0.4)$-fuzzy bi-ideal because $f(d) \vee 0.2 = 0.3 < 0.4 = f(e) \wedge 0.4$.*

The following theorem shows the relation between a nonempty subset of an ordered semigroup and its characteristic function.

**Theorem 3.2.3.** *A nonempty subset $A$ of an ordered semigroup $S$ is a bi-ideal of $S$ if and only if the characteristic function $C_A$ is a generalized $(s,t)$-fuzzy bi-ideal of $S$.*

*Proof.* Assume that $A$ is a bi-ideal of an ordered semigroup $S$. By Lemma 2.2.13, $C_A$ is an $(s,t)$-fuzzy bi-ideal of $S$. Thus $C_A$ is a generalized $(s,t)$-fuzzy bi-ideal of $S$. Conversely, suppose that $C_A$ is a generalized $(s,t)$-fuzzy bi-ideal of $S$. Since $\mathrm{Im}(C_A) \subseteq \{0,1\}$, we have $C_A(x) \wedge_{C_A} t = C_A(x) \wedge t$ and $C_A(x) \vee_{C_A} s = C_A(x) \vee s$ for all $x \in S$. Thus $C_A$ is also an $(s,t)$-fuzzy bi-ideal of $S$. By Lemma 2.2.13, $A$ is a bi-ideal of $S$. □

For any fuzzy subset $f$ of an ordered semigroup $S$ and any $p \in [0,1]$, we define a subset $U_s^t(f;p)$ of $S$ as follows:

$$U_s^t(f;p) = \{x \in S \mid f(x) \vee_f s \geq p \wedge_f t\}.$$

**Lemma 3.2.4.** *Let $f$ be a fuzzy subset of an ordered semigroup $S$. The following statements are equivalent:*

(1) $(\forall x, y \in S)(f(xy) \vee_f s \geq f(x) \wedge_f f(y) \wedge_f t)$,

(2) $(\forall p \in (s,t])(\forall x, y \in U_s^t(f;p))(xy \in U_s^t(f;p))$.

*Proof.* Assume that (1) holds. Let $p \in (s,t]$ and $x, y \in U_s^t(f;p)$. Without loss of generality, we may assume that $f(x) \leq f(y)$. Consider the following cases:

Case 1: $f(x) \geq s$. Then $f(xy) \vee_f s \geq f(x) \wedge_f f(y) \wedge_f t = f(x) \wedge_f t = (f(x) \vee_f s) \wedge_f t \geq (p \wedge_f t) \wedge_f t = p \wedge_f t$. Thus $xy \in U_s^t(f;p)$.

Case 2: $f(x) < s$. Then $f(xy) \vee_f s \geq f(x) \vee_f s \geq p \wedge_f t$. Thus $xy \in U_s^t(f;p)$.

Conversely, assume that (2) is true. Let $x, y \in S$. Without loss of generality, we may assume that $f(x) \leq f(y)$. Consider the following cases:

Case 1: $f(x) \leq s$. We have $f(xy) \vee_f s \geq s \geq f(x) \geq f(x) \wedge_f f(y) \wedge_f t$.

Case 2: $t < f(x)$. Then $f(y) \wedge_f t = f(x) \wedge_f t = t \wedge_f t$. This implies that $x, y \in U_s^t(f;t)$. By the assumption, $xy \in U_s^t(f;t)$. Thus $f(xy) \vee_f s \geq t \wedge_f t = f(x) \wedge_f t = f(x) \wedge_f f(y) \wedge_f t$.

Case 3: $s < f(x) \leq t$. Since $f(y) \vee_f s \geq f(x) \vee_f s = f(x) = f(x) \wedge_f t$, we have $x, y \in U_s^t(f; f(x))$. By the assumption, $xy \in U_s^t(f; f(x))$. Thus $f(xy) \vee_f s \geq f(x) \wedge_f t = f(x) \wedge_f f(y) \wedge_f t$. □

**Lemma 3.2.5.** *Let $f$ be a fuzzy subset of an ordered semigroup $S$. The following statements are equivalent:*

(1) $(\forall x, y, z \in S)(f(xyz) \vee_f s \geq f(x) \wedge_f f(z) \wedge_f t)$,

(2) $(\forall p \in (s,t])(\forall x, z \in U_s^t(f;p))(\forall y \in S)(xyz \in U_s^t(f;p))$.

*Proof.* Assume that (1) holds. Let $p \in (s,t]$ and $x, y, z \in S$ with $x, z \in U_s^t(f;p)$. Without loss of generality, we may assume that $f(x) \leq f(z)$. Consider the following cases:

Case 1: $f(x) \geq s$. Then $f(xyz) \vee_f s \geq f(x) \wedge_f f(z) \wedge_f t = f(x) \wedge_f t = (f(x) \vee_f s) \wedge_f t \geq (p \wedge_f t) \wedge_f t = p \wedge_f t$. Thus $xyz \in U_s^t(f;p)$.

Case 2: $f(x) < s$. Then $f(xyz) \vee_f s \geq f(x) \vee_f s \geq p \wedge_f t$. Thus $xyz \in U_s^t(f;p)$.

Conversely, assume that (2) is true. Let $x, y, z \in S$. Without loss of generality, we may assume that $f(x) \leq f(z)$. Consider the following cases:

Case 1: $f(x) \leq s$. We have $f(xyz) \vee_f s \geq s \geq f(x) \geq f(x) \wedge_f f(z) \wedge_f t$.

Case 2: $t < f(x)$. Then $f(z) \wedge_f t = f(x) \wedge_f t = t \wedge_f t$. This implies that $x, z \in U_s^t(f; t)$. By the assumption, $xyz \in U_s^t(f; t)$. Thus $f(xyz) \vee_f s \geq t \wedge_f t = f(x) \wedge_f t = f(x) \wedge_f f(z) \wedge_f t$.

Case 3: $s < f(x) \leq t$. Since $f(z) \vee_f s \geq f(x) \vee_f s = f(x) = f(x) \wedge_f t$, we have $x, z \in U_s^t(f; f(x))$. By the assumption, $xyz \in U_s^t(f; f(x))$. Thus $f(xyz) \vee_f s \geq f(x) \wedge_f t = f(x) \wedge_f f(z) \wedge_f t$. □

**Lemma 3.2.6.** *Let $f$ be a fuzzy subset of an ordered semigroup $S$. The following statements are equivalent:*

(1) $(\forall x, y \in S)(x \leq y$ *implies* $f(x) \vee_f s \geq f(y) \wedge_f t)$,

(2) $(\forall p \in (s, t])(\forall x \in S)(\forall y \in U_s^t(f; p))(x \leq y$ *implies* $x \in U_s^t(f; p))$.

*Proof.* Assume that (1) holds. Let $p \in (s, t]$ and $x, y \in S$ with $y \in U_s^t(f; p)$ and $x \leq y$. Consider the following cases:

Case 1: $f(y) \geq s$. Then $f(x) \vee_f s \geq f(y) \wedge_f t = (f(y) \vee_f s) \wedge_f t \geq (p \wedge_f t) \wedge_f t = p \wedge_f t$. Thus $x \in U_s^t(f; p)$.

Case 2: $f(y) < s$. Then $f(x) \vee_f s \geq f(y) \vee_f s \geq p \wedge_f t$. Thus $x \in U_s^t(f; p)$.

Conversely, assume that (2) is true. Let $x, y \in S$ with $x \leq y$. Consider the following cases:

Case 1: $f(y) \leq s$. We have $f(x) \vee_f s \geq s \geq f(y) \geq f(y) \wedge_f t$.

Case 2: $t < f(y)$. Then $f(y) \wedge_f t = t \wedge_f t$. This implies that $y \in U_s^t(f; t)$. By the assumption, $x \in U_s^t(f; t)$. Thus $f(x) \vee_f s \geq t \wedge_f t = f(y) \wedge_f t$.

Case 3: $s < f(y) \leq t$. Since $f(y) \vee_f s = f(y) = f(y) \wedge_f t$, we have $y \in U_s^t(f; f(y))$. By the assumption, $x \in U_s^t(f; f(y))$. Thus $f(x) \vee_f s \geq f(y) \wedge_f t$. □

The following theorem is derived from Lemma 3.2.4 - 3.2.6. This theorem shows a relation between a generalized $(s,t)$-fuzzy bi-ideal of an ordered semigroup and the subset $U_s^t(f;p)$.

**Theorem 3.2.7.** *A fuzzy subset $f$ of an ordered semigroup $S$ is a generalized $(s,t)$-fuzzy bi-ideal of $S$ if and only if each nonempty subset $U_s^t(f;p)$ is a bi-ideal of $S$, where $p \in (s,t]$.*

**Definition 3.2.8.** Let $f$ be a fuzzy subset of an ordered semigroup $S$. We define a fuzzy subset $f_{s,t}$ of $S$ by $f_{s,t}(x) = (f(x) \vee_f s) \wedge_f t$ for all $x \in S$.

The following lemma is straightforward from Definition 3.2.8.

**Lemma 3.2.9.** *Let $A$ be a subset of an ordered semigroup $S$. Then*

$$(C_A)_{s,t}(x) = \begin{cases} t, & \text{if } x \in A; \\ s, & \text{if } x \notin A. \end{cases}$$

**Lemma 3.2.10.** *Let $f$ be a fuzzy subset of an ordered semigroup $S$ and $x, y \in S$. If $f(xy) \vee_f s \geq f(x) \wedge_f f(y) \wedge_f t$, then $f_{s,t}(xy) \geq f_{s,t}(x) \wedge f_{s,t}(y)$.*

*Proof.* Suppose that $f(xy) \vee_f s \geq f(x) \wedge_f f(y) \wedge_f t$. Without loss of generality, we may assume that $f(x) \leq f(y)$. Then $f_{s,t}(x) \leq f_{s,t}(y)$. Consider the following cases:

Case 1: $f(x) < s$. Then $f(xy) \vee_f s \geq f(x) \vee_f s$. This implies that $f_{s,t}(xy) = (f(xy) \vee_f s) \wedge_f t \geq (f(x) \vee_f s) \wedge_f t = f_{s,t}(x) = f_{s,t}(x) \wedge f_{s,t}(y)$.

Case 2: $s \leq f(x)$. By the assumption, we have $f_{s,t}(xy) = (f(xy) \vee_f s) \wedge_f t \geq (f(x) \wedge_f f(y) \wedge_f t) \wedge_f t = f(x) \wedge_f t = (f(x) \vee_f s) \wedge_f t = f_{s,t}(x) = f_{s,t}(x) \wedge f_{s,t}(y)$. □

**Lemma 3.2.11.** *Let $f$ be a fuzzy subset of an ordered semigroup $S$ and $x, y, z \in S$. If $f(xyz) \vee_f s \geq f(x) \wedge_f f(z) \wedge_f t$, then $f_{s,t}(xyz) \geq f_{s,t}(x) \wedge f_{s,t}(z)$.*

*Proof.* Suppose that $f(xyz) \vee_f s \geq f(x) \wedge_f f(z) \wedge_f t$. Without loss of generality, we may assume that $f(x) \leq f(z)$. Then $f_{s,t}(x) \leq f_{s,t}(z)$. Consider the following cases:

Case 1: $f(x) < s$. Then $f(xyz) \vee_f s \geq f(x) \vee_f s$. This implies that $f_{s,t}(xyz) = (f(xyz) \vee_f s) \wedge_f t \geq (f(x) \vee_f s) \wedge_f t = f_{s,t}(x) = f_{s,t}(x) \wedge f_{s,t}(z)$.

Case 2: $s \leq f(x)$. By the assumption, we have $f_{s,t}(xyz) = (f(xyz) \vee_f s) \wedge_f t \geq (f(x) \wedge_f f(z) \wedge_f t) \wedge_f t = f(x) \wedge_f t = (f(x) \vee_f s) \wedge_f t = f_{s,t}(x) = f_{s,t}(x) \wedge f_{s,t}(z)$. □

**Lemma 3.2.12.** *Let $f$ be a fuzzy subset of an ordered semigroup $S$ and $x, y \in S$ with $x \leq y$. If $f(x) \vee_f s \geq f(y) \wedge_f t$, then $f_{s,t}(x) \geq f_{s,t}(y)$.*

*Proof.* Suppose that $f(x) \vee_f s \geq f(y) \wedge_f t$. Consider the following cases:

Case 1: $f(y) < s$. Then $f(x) \vee_f s \geq f(y) \vee_f s$. This implies that $f_{s,t}(x) = (f(x) \vee_f s) \wedge_f t \geq (f(y) \vee_f s) \wedge_f t = f_{s,t}(y)$.

Case 2: $s \leq f(y)$. By the assumption, we have $f_{s,t}(x) = (f(x) \vee_f s) \wedge_f t \geq (f(y) \wedge_f t) \wedge_f t = f(y) \wedge_f t = (f(y) \vee_f s) \wedge_f t = f_{s,t}(y)$. □

The following theorem is derived from Lemma 3.2.10 - 3.2.12.

**Theorem 3.2.13.** *If $f$ is a generalized $(s, t)$-fuzzy bi-ideal of an ordered semigroup $S$, then $f_{s,t}$ is a fuzzy bi-ideal of $S$.*

**Theorem 3.2.14.** *Let $A$ be a nonempty subset of an ordered semigroup $S$. Then $A$ is a bi-ideal of $S$ if and only if $(C_A)_{s,t}$ is a fuzzy bi-ideal of $S$.*

*Proof.* Assume that $A$ is a bi-ideal of $S$. By Theorem 3.2.3 and 3.2.13, $(C_A)_{s,t}$ is a fuzzy bi-ideal of $S$. Conversely, suppose that $(C_A)_{s,t}$ is a fuzzy bi-ideal of $S$. For any $x \in S$, we have $(C_A)_{s,t}(x) = t$ if and only if $C_A(x) = 1$. Then $C_A$ is a fuzzy bi-ideal of $S$ and also a generalized $(s,t)$-fuzzy bi-ideal of $S$. By Theorem 3.2.3, $A$ is a bi-ideal of $S$. □

**Theorem 3.2.15.** *An ordered semigroup $S$ is completely regular if and only if each generalized $(s,t)$-fuzzy bi-ideal $f$ of $S$, we have $f_{s,t}(a) = f_{s,t}(a^2)$ for all $a \in S$.*

*Proof.* Suppose that $S$ is a completely regular ordered semigroup. Let $f$ be a generalized $(s,t)$-fuzzy bi-ideal of $S$ and $a \in S$. Since $S$ is completely regular, there exists $x \in S$ such that $a \leq a^2xa^2$. Note that $f_{s,t}$ is a fuzzy bi-ideal of $S$ by theorem 3.2.13. This implies that $f_{s,t}(a^2) \geq f_{s,t}(a) \wedge f_{s,t}(a) = f_{s,t}(a)$. On the other hand, we have $f_{s,t}(a) \geq f_{s,t}(a^2xa^2) \geq f_{s,t}(a^2) \wedge f_{s,t}(a^2) = f_{s,t}(a^2)$. Thus $f_{s,t}(a) = f_{s,t}(a^2)$. Conversely, suppose that each generalized $(s,t)$-fuzzy bi-ideal $f$ of $S$, we have $f_{s,t}(a) = f_{s,t}(a^2)$ for all $a \in S$. Let $a \in S$. We will consider $B(a^2)$, which is a bi-ideal of $S$ generated by $a^2$. By Theorem 3.2.3, $C_{B(a^2)}$ is a generalized $(s,t)$-fuzzy bi-ideal of $S$. By the assumption, $(C_{B(a^2)})_{s,t}(a) = (C_{B(a^2)})_{s,t}(a^2) = t$. This means that $a \in B(a^2)$. Then $a \leq b$ for some $b \in a^2 \cup a^4 \cup a^2Sa^2$. If $b = a^2$, then $a \leq a^2 = aa \leq a^2a^2 = aaa^2 \leq a^2aa^2$. Similarly, we have $a \leq a^4 = aaa^2 \leq a^4aa^2 = a^2a^3a^2$ if $b = a^4$. In case $b \in a^2Sa^2$, there exists $x \in S$ such that $a \leq a^2xa^2$. Hence $S$ is completely regular. □

## 3.3 Generalized $(s,t)$-fuzzy interior ideals in ordered semigroups

**Definition 3.3.1.** A fuzzy subset $f$ of an ordered semigroup $S$ is called a *generalized $(s,t)$-fuzzy interior ideal* of $S$ if it satisfies:

(1) $(\forall x, y \in S)(f(xy) \vee_f s \geq f(x) \wedge_f f(y) \wedge_f t)$,

(2) $(\forall x, y, z \in S)(f(xyz) \vee_f s \geq f(y) \wedge_f t)$,

(3) $(\forall x, y \in S)(x \leq y$ implies $f(x) \vee_f s \geq f(y) \wedge_f t)$.

Every $(s,t)$-fuzzy interior ideal of an ordered semigroup $S$ is a generalized $(s,t)$-fuzzy interior ideal of $S$, but the converse is not true.

**Example 3.3.2.** *Let $S = \{a,b,c,d\}$. Define a binary operation and a partial order on $S$ as follows:*

| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $d$ | $a$ |
| $c$ | $a$ | $a$ | $a$ | $a$ |
| $d$ | $a$ | $a$ | $a$ | $a$ |

$$\leq \; = \{(a,a),(a,c),(a,d),(b,b),(c,c),(d,d)\}.$$

*Let $f$ be a fuzzy subset of $S$ defined by*

$$f(x) = \begin{cases} 0.6, & \text{if } x = a, \\ 0.8, & \text{if } x = b, \\ 0.2, & \text{if } x = c, \\ 0, & \text{if } x = d. \end{cases}$$

*Then $S$ is an ordered semigroup and $f$ is a generalized $(0.1, 0.5)$-fuzzy interior ideal of $S$, but not a $(0.1, 0.5)$-fuzzy interior ideal because $f(bc) \vee 0.1 = f(d) \vee 0.1 = 0.1 < 0.2 = f(b) \wedge f(c) \wedge 0.5$.*

The following theorem shows the relation between a nonempty subset of an ordered semigroup and its characteristic function.

**Theorem 3.3.3.** *A nonempty subset $A$ of an ordered semigroup $S$ is an interior ideal of $S$ if and only if the characteristic function $C_A$ is a generalized $(s,t)$-fuzzy interior ideal of $S$.*

*Proof.* Assume that $A$ is an interior ideal of an ordered semigroup $S$. By Lemma 2.2.16, $C_A$ is a $(s,t)$-fuzzy interior ideal of $S$. Thus $C_A$ is a generalized $(s,t)$-fuzzy interior ideal of $S$. Conversely, suppose that $C_A$ is a generalized $(s,t)$-fuzzy interior ideal of $S$. Since $\text{Im}(C_A) \subseteq \{0,1\}$, we have $C_A(x) \wedge_{C_A} t = C_A(x) \wedge t$ and $C_A(x) \vee_{C_A} s = C_A(x) \vee s$ for all $x \in S$. Thus $C_A$ is also a $(s,t)$-fuzzy interior ideal of $S$. By Lemma 2.2.16, $A$ is an interior ideal of $S$. □

**Lemma 3.3.4.** *Let $f$ be a fuzzy subset of an ordered semigroup $S$. The following statements are equivalent:*

(1) $(\forall x, y, z \in S)(f(xyz) \vee_f s \geq f(y) \wedge_f t)$,

(2) $(\forall p \in (s,t])(\forall y \in U_s^t(f;p))(\forall x, z \in S)(xyz \in U_s^t(f;p))$.

*Proof.* Assume that (1) holds. Let $p \in (s,t]$ and $x, y, z \in S$ with $y \in U_s^t(f;p)$. Consider the following cases:

Case 1: $f(y) \geq s$. Then $f(xyz) \vee_f s \geq f(y) \wedge_f t = (f(y) \vee_f s) \wedge_f t \geq (p \wedge_f t) \wedge_f t = p \wedge_f t$. Thus $xyz \in U_s^t(f;p)$.

Case 2: $f(y) < s$. Then $f(xyz) \vee_f s \geq f(y) \vee_f s \geq p \wedge_f t$. Thus $xyz \in U_s^t(f;p)$.

Conversely, assume that (2) is true. Let $x, y, z \in S$. Consider the following cases:

Case 1: $f(y) \leq s$. We have $f(xyz) \vee_f s \geq s \geq f(y) \geq f(y) \wedge_f t$.

Case 2: $t < f(y)$. Then $f(y) \wedge_f t = t \wedge_f t$. This implies that $y \in U_s^t(f;t)$. By the assumption, $xyz \in U_s^t(f;t)$. Thus $f(xyz) \vee_f s \geq t \wedge_f t = f(y) \wedge_f t$.

Case 3: $s < f(y) \leq t$. Since $f(y) \vee_f s = f(y) = f(y) \wedge_f t$, we have $y \in U_s^t(f;f(y))$. By the assumption, $xyz \in U_s^t(f;f(y))$. Thus $f(xyz) \vee_f s \geq f(y) \wedge_f t$. □

The following theorem is derived from Lemma 3.2.4, 3.2.6, and 3.3.4. This theorem shows a relation between a generalized $(s,t)$-fuzzy interior ideal of an ordered semigroup and the subset $U_s^t(f;p)$.

**Theorem 3.3.5.** *A fuzzy subset $f$ of an ordered semigroup $S$ is a generalized $(s,t)$-fuzzy interior ideal of $S$ if and only if each nonempty subset $U_s^t(f;p)$ is an interior ideal of $S$, where $p \in (s,t]$.*

**Lemma 3.3.6.** *Let $f$ be a fuzzy subset of an ordered semigroup $S$ and $x,y,z \in S$. If $f(xyz) \vee_f s \geq f(y) \wedge_f t$, then $f_{s,t}(xyz) \geq f_{s,t}(y)$.*

*Proof.* Suppose that $f(xyz) \vee_f s \geq f(y) \wedge_f t$. Consider the following cases:

Case 1: $f(y) < s$. Then $f(xyz) \vee_f s \geq f(y) \vee_f s$. This implies that $f_{s,t}(xyz) = (f(xyz) \vee_f s) \wedge_f t \geq (f(y) \vee_f s) \wedge_f t = f_{s,t}(y)$.

Case 2: $s \leq f(y)$. By the assumption, we have $f_{s,t}(xyz) = (f(xyz) \vee_f s) \wedge_f t \geq (f(y) \wedge_f t) \wedge_f t = (f(y) \vee_f s) \wedge_f t = f_{s,t}(y)$. □

The following theorem is derived from Lemma 3.2.10, 3.2.12, and 3.3.6.

**Theorem 3.3.7.** *If $f$ is a generalized $(s,t)$-fuzzy interior ideal of an ordered semigroup $S$, then $f_{s,t}$ is a fuzzy interior ideal of $S$.*

**Theorem 3.3.8.** *Let $A$ be a nonempty subset of an ordered semigroup $S$. Then $A$ is an interior ideal of $S$ if and only if $(C_A)_{s,t}$ is a fuzzy interior ideal of $S$.*

*Proof.* Assume that $A$ is an interior ideal of $S$. By Theorem 3.3.3 and Theorem 3.3.7, $(C_A)_{s,t}$ is a fuzzy interior ideal of $S$. Conversely, suppose that $(C_A)_{s,t}$ is a fuzzy interior ideal of $S$. For any $x \in S$, we have $(C_A)_{s,t}(x) = t$ if and only if $C_A(x) = 1$. Then $C_A$ is a fuzzy interior ideal of $S$ and also a generalized $(s,t)$-fuzzy interior ideal of $S$. By Theorem 3.3.3, $A$ is an interior ideal of $S$. □

**Theorem 3.3.9.** *An ordered semigroup $S$ is intra-regular if and only if for each generalized $(s,t)$-fuzzy interior ideal $f$ of $S$, we have $f_{s,t}(a) = f_{s,t}(a^2)$ for all $a \in S$.*

*Proof.* Suppose that $S$ is a intra-regular ordered semigroup. Let $f$ be a generalized $(s,t)$-fuzzy interior ideal of $S$ and $a \in S$. Since $S$ is intra-regular, there exist

$x, y \in S$ such that $a \leq xa^2y$. Note that $f_{s,t}$ is a fuzzy interior ideal of $S$ by theorem 3.3.7. This implies that $f_{s,t}(a^2) \geq f_{s,t}(a) \wedge f_{s,t}(a) = f_{s,t}(a)$. On the other hand, we have $f_{s,t}(a) \geq f_{s,t}(xa^2y) \geq f_{s,t}(a^2)$. Thus $f_{s,t}(a) = f_{s,t}(a^2)$. Conversely, suppose that each generalized $(s,t)$-fuzzy interior ideal $f$ of $S$, we have $f_{s,t}(a) = f_{s,t}(a^2)$ for all $a \in S$. Let $a \in S$. We will consider $I(a^2)$, which is an ideal of $S$ generated by $a^2$. By Theorem 3.3.3, $C_{I(a^2)}$ is a generalized $(s,t)$-fuzzy interior ideal of $S$. By the assumption, $(C_{I(a^2)})_{s,t}(a) = (C_{I(a^2)})_{s,t}(a^2) = t$. This means that $a \in I(a^2)$. Then $a \leq b$ for some $b \in a^2 \cup a^2S \cup Sa^2 \cup Sa^2S$. If $b = a^2$, then $a \leq a^2 = aa \leq a^2a^2 = aa^2a$. Similarly, we have $a \leq xa^2 = xaa \leq x(xa^2)a = x^2a^2a$ for some $x \in S$ if $b \in Sa^2$. In case $b \in a^2S$, there exists $y \in S$ such that $a \leq aa^2y^2$. If $b \in Sa^2S$, then $a \leq xa^2y$ for some $x, y \in S$. Hence $S$ is intra-regular. □

**Theorem 3.3.10.** *Let $S$ be an ordered semigroup. Then $S$ is simple if and only if for each generalized $(s,t)$-fuzzy interior ideal $f$ of $S$, we have $f_{s,t}$ is a constant function.*

*Proof.* Assume that $S$ is simple. Let $f$ be a generalized $(s,t)$-fuzzy bi-ideal of $S$ and $x, y \in S$. Note that $(SyS]$ is an ideal of $S$. By the assumption, $(SyS] = S$. There exists $a, b \in S$ such that $x \leq ayb$. By Proposition 3.3.7, $f_{s,t}$ is an interior ideal of $S$. Then $f_{s,t}(x) \geq f_{s,t}(ayb) \geq f_{s,t}(y)$. By a similar reasoning, we can verify that $f_{s,t}(y) \geq f_{s,t}(x)$. Thus $f_{s,t}$ is constant. Conversely, suppose that for each generalized $(s,t)$-fuzzy interior ideal $f$ of $S$, we have $f_{s,t}$ is a constant function. Let $A$ be an ideal of $S$. Then $A$ is also an interior ideal of $S$. By Theorem 3.3.3, $C_A$ is a generalized $(s,t)$-fuzzy interior ideal of $S$. Since $A$ is not empty, there exists $a \in S$ such that $(C_A)_{s,t}(a) = t$. By the assumption, $(C_A)_{s,t}(x) = t$ for all $x \in S$. This implies that $x \in A$ for all $x \in S$. Thus $A = S$. Hence $S$ is simple. □

## 3.4 Generalized $(s,t)$-fuzzy ideals in ordered semigroups

**Definition 3.4.1.** A fuzzy subset $f$ of an ordered semigroup $S$ is called a *generalized $(s,t)$-fuzzy left ideal (resp. generalized $(s,t)$-fuzzy right ideal)* of $S$ if it satisfies:

(1) $(\forall x, y \in S)(f(xy) \vee_f s \geq f(y) \wedge_f t$ [resp. $f(xy) \vee_f s \geq f(x) \wedge_f t$]),

(2) $(\forall x, y \in S)(x \leq y$ implies $f(x) \vee_f s \geq f(y) \wedge_f t)$.

A fuzzy subset $f$ of an ordered semigroup $S$ is called a *generalized* $(s,t)$*-fuzzy ideal* of $S$ if it is both a generalized $(s,t)$-fuzzy left ideal and a generalized $(s,t)$-fuzzy right ideal of $S$.

Every $(s,t)$-fuzzy ideal of an ordered semigroup $S$ is a generalized $(s,t)$-fuzzy ideal of $S$, but the converse is not true.

**Example 3.4.2.** *Let $S = \{a, b, c, d\}$. Define a binary operation and a partial order on $S$ as follows:*

| $\cdot$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $a$ | $b$ | $c$ |
| $c$ | $a$ | $b$ | $c$ |

$$\leq = \{(a,a), (a,b), (a,c), (b,b), (b,c), (c,c)\}.$$

*Let $f$ be a fuzzy subset of $S$ defined by*

$$f(x) = \begin{cases} 0.3, & \text{if } x = a, \\ 0.4, & \text{if } x = b, \\ 0.8, & \text{if } x = c. \end{cases}$$

*Then $S$ is an ordered semigroup and $f$ is a generalized $(0.4, 0.7)$-fuzzy ideal of $S$, but not a $(0.4, 0.7)$-fuzzy ideal because $f(ca) \vee 0.4 = f(a) \vee 0.4 = 0.4 < 0.7 = f(c) \wedge 0.7$.*

The following theorem shows the relation between a nonempty subset of an ordered semigroup and its characteristic function.

**Theorem 3.4.3.** *A nonempty subset $A$ of an ordered semigroup $S$ is an left ideal of $S$ if and only if the characteristic function $C_A$ is a generalized $(s,t)$-fuzzy left ideal of $S$.*

*Proof.* Assume that $A$ is a left ideal of an ordered semigroup $S$. By Lemma 2.2.9, $C_A$ is a fuzzy left ideal of $S$. Thus $C_A$ is a generalized $(s,t)$-fuzzy left ideal of $S$. Conversely, suppose that $C_A$ is a generalized $(s,t)$-fuzzy left ideal of $S$. Since $\text{Im}(C_A) \subseteq \{0,1\}$, we have $C_A(x) \wedge_{C_A} t = C_A(x) \wedge t$ and $C_A(x) \vee_{C_A} s = C_A(x) \vee s$ for all $x \in S$. Let $x, y \in S$. By the assumption, $C_A(xy) \vee s \geq C_A(x) \wedge C_A(y) \wedge t$. We can verify that $C_A(xy) \geq C_A(x) \wedge C_A(y)$. By the similar reasoning, we have $C_A(x) \vee s \geq C_A(y) \wedge t$ if $x \leq y$ and $y \in A$. Then $C_A$ is a fuzzy left ideal of $S$. By Lemma 2.2.9, $A$ is a left ideal of $S$. □

**Lemma 3.4.4.** *Let $f$ be a fuzzy subset of an ordered semigroup $S$. The following statements are equivalent:*

(1) $(\forall x, y, z \in S)(f(xy) \vee_f s \geq f(y) \wedge_f t)$,

(2) $(\forall p \in (s,t])(\forall y \in U_s^t(f;p))(\forall x \in S)(xyz \in U_s^t(f;p))$.

*Proof.* Assume that (1) holds. Let $p \in (s,t]$ and $x, y \in S$ with $y \in U_s^t(f;p)$. Consider the following cases:

Case 1: $f(y) \geq s$. Then $f(xy) \vee_f s \geq f(y) \wedge_f t = (f(y) \vee_f s) \wedge_f t \geq (p \wedge_f t) \wedge_f t = p \wedge_f t$. Thus $xy \in U_s^t(f;p)$.

Case 2: $f(y) < s$. Then $f(xy) \vee_f s \geq f(y) \vee_f s \geq p \wedge_f t$. Thus $xy \in U_s^t(f;p)$.

Conversely, assume that (2) is true. Let $x, y \in S$. Consider the following cases:

Case 1: $f(y) \leq s$. We have $f(xy) \vee_f s \geq s \geq f(y) \geq f(y) \wedge_f t$.

Case 2: $t < f(y)$. Then $f(y) \wedge_f t = t \wedge_f t$. This implies that $y \in U_s^t(f;t)$. By the assumption, $xy \in U_s^t(f;t)$. Thus $f(xy) \vee_f s \geq t \wedge_f t = f(y) \wedge_f t$.

Case 3: $s < f(y) \leq t$. Since $f(y) \vee_f s = f(y) = f(y) \wedge_f t$, we have $y \in U_s^t(f; f(y))$. By the assumption, $xy \in U_s^t(f; f(y))$. Thus $f(xy) \vee_f s \geq f(y) \wedge_f t$. □

The following theorem is derived from Lemma 3.2.6, and 3.4.4. This theorem shows a relation between a generalized $(s,t)$-fuzzy left ideal of an ordered semigroup and the subset $U_s^t(f;p)$.

**Theorem 3.4.5.** *A fuzzy subset $f$ of an ordered semigroup $S$ is a generalized $(s,t)$-fuzzy left ideal of $S$ if and only if each nonempty subset $U_s^t(f;p)$ is a left ideal of $S$, where $p \in (s,t]$.*

**Lemma 3.4.6.** *Let $f$ be a fuzzy subset of an ordered semigroup $S$ and $x, y \in S$. If $f(xy) \vee_f s \geq f(y) \wedge_f t$, then $f_{s,t}(xy) \geq f_{s,t}(y)$.*

*Proof.* Suppose that $f(xy) \vee_f s \geq f(y) \wedge_f t$. Consider the following cases:

Case 1: $f(y) < s$. Then $f(xy) \vee_f s \geq f(y) \vee_f s$. This implies that $f_{s,t}(xy) = (f(xy) \vee_f s) \wedge_f t \geq (f(y) \vee_f s) \wedge_f t = f_{s,t}(y)$.

Case 2: $s \leq f(y)$. By the assumption, we have $f_{s,t}(xy) = (f(xy) \vee_f s) \wedge_f t \geq (f(y) \wedge_f t) \wedge_f t = (f(y) \vee_f s) \wedge_f t = f_{s,t}(y)$. □

The following theorem is derived from Lemma 3.2.12, and 3.4.6.

**Theorem 3.4.7.** *If $f$ is a generalized $(s,t)$-fuzzy left ideal of an ordered semigroup $S$, then $f_{s,t}$ is a fuzzy left ideal of $S$.*

**Theorem 3.4.8.** *Let $A$ be a nonempty subset of an ordered semigroup $S$. Then $A$ is a left ideal of $S$ if and only if $(C_A)_{s,t}$ is a fuzzy left ideal of $S$.*

*Proof.* Assume that $A$ is a left ideal of $S$. By Theorem 3.4.3 and Theorem 3.4.7, $(C_A)_{s,t}$ is a fuzzy left ideal of $S$. Conversely, suppose that $(C_A)_{s,t}$ is a fuzzy left ideal of $S$. For any $x \in S$, we have $(C_A)_{s,t}(x) = t$ if and only if $C_A(x) = 1$. Then $C_A$ is a fuzzy left ideal of $S$ and also a generalized $(s,t)$-fuzzy left ideal of $S$. By Theorem 3.4.3, $A$ is a left ideal of $S$. □

**Theorem 3.4.9.** *An ordered semigroup $S$ is left regular if and only if each generalized $(s,t)$-fuzzy left ideal $f$ of $S$, we have $f_{s,t}(a) = f_{s,t}(a^2)$ for all $a \in S$.*

*Proof.* Suppose that $S$ is a left regular ordered semigroup. Let $f$ be a generalized $(s,t)$-fuzzy left ideal of $S$ and $a \in S$. Since $S$ is left regular, there exists $x \in S$ such that $a \le xa^2$. Note that $f_{s,t}$ is a fuzzy left ideal of $S$ by theorem 3.4.7. This implies that $f_{s,t}(a^2) \ge f_{s,t}(a) \wedge f_{s,t}(a) = f_{s,t}(a)$. On the other hand, we have $f_{s,t}(a) \ge f_{s,t}(xa^2) \ge f_{s,t}(a^2)$. Thus $f_{s,t}(a) = f_{s,t}(a^2)$. Conversely, suppose that each generalized $(s,t)$-fuzzy left ideal $f$ of $S$, we have $f_{s,t}(a) = f_{s,t}(a^2)$ for all $a \in S$. Let $a \in S$. We will consider $L(a^2)$, which is a left ideal of $S$ generated by $a^2$. By Theorem 3.4.3, $C_{L(a^2)}$ is a generalized $(s,t)$-fuzzy interior ideal of $S$. By the assumption, $(C_{L(a^2)})_{s,t}(a) = (C_{L(a^2)})_{s,t}(a^2) = t$. This means that $a \in L(a^2)$. Then $a \le b$ for some $b \in a^2 \cup Sa^2$. If $b = a^2$, then $a \le a^2 = aa \le aa^2$. Similarly, we have $a \le xa^2$ for some $x \in S$ if $b \in Sa^2$. Hence $S$ is left-regular. □

**Proposition 3.4.10.** *Every generalized $(s,t)$-fuzzy ideal of an ordered semigroup $S$ is generalized $(s,t)$-fuzzy interior ideal of $S$.*

*Proof.* Let $f$ be a generalized $(s,t)$-fuzzy ideal of $S$ and $x, y, z \in S$. Consider the following cases:

Case 1: $f(xy) \le s$. Then $f(xyz) \vee_f s \ge f(xy) \vee_f s \ge f(y) \wedge_f t$.

Case 2: $f(xy) > s$. Then $f(xyz) \vee_f s \ge f(xy) \wedge_f t = (f(xy) \vee_f s) \wedge_f t = (f(xy) \wedge_f t) \wedge_f t = f(xy) \wedge_f t$.

Thus $f$ is a generalized $(s,t)$-fuzzy interior ideal of $S$. □

**Example 3.4.11.** *Let $S = \{a, b, c, d\}$. Define a binary operation and a partial* *order on $S$ as follows:*

| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | $d$ | $a$ |
| $c$ | $a$ | $a$ | $a$ | $a$ |
| $d$ | $a$ | $a$ | $a$ | $a$ |

$$\leq = \{(a,a),(a,c),(a,d),(b,b),(c,c),(d,d)\}.$$

*Let $f$ be a fuzzy subset of $S$ defined by*

$$f(x) = \begin{cases} 0.9, & \text{if } x = a, \\ 0.2, & \text{if } x = b, \\ 0.6, & \text{if } x = c, \\ 0.4, & \text{if } x = d. \end{cases}$$

*Then $S$ is an ordered semigroup and $f$ is a generalized $(0.3, 0.7)$-fuzzy interior ideal of $S$, but not a generalized $(0.3, 0.7)$-fuzzy ideal because $f(bc) \vee_f 0.3 = f(d) \vee_f 0.3 = 0.4 < 0.6 = f(c) \wedge_f 0.7$.*

**Theorem 3.4.12.** *If $S$ is a regular ordered semigroup, then the concepts of generalized $(s,t)$-fuzzy interior ideal and a generalized $(s,t)$-fuzzy ideal of $S$ coincide.*

*Proof.* Assume that an ordered semigroup $S$ is regular. By Proposition 3.4.10, it is remaining to show that every generalized $(s,t)$-fuzzy interior ideal of $S$ is generalized $(s,t)$-fuzzy ideal of $S$. Let $f$ be a generalized $(s,t)$-fuzzy interior ideal of $S$ and $x, y \in S$. By hypothesis, there exists $a \in S$ such that $x \leq xax$. Then $xy \leq xaxy$. Consider the following cases:

Case 1: $f(xaxy) \leq s$. Then $f(xy) \vee_f s \geq f(xaxy) \vee_f s \geq f(x) \wedge_f t$.

Case 2: $f(xaxy) > s$. Then $f(xy) \vee_f s \geq f(xaxy) \wedge_f t = (f(xaxy) \vee_f s) \wedge_f t \geq (f(x) \wedge_f t) \wedge_f t = f(x) \wedge_f t$.

Thus $f$ is a generalized $(s,t)$-fuzzy right ideal of $S$. By using a similar reasoning, we can verify that $f$ is a generalized $(s,t)$-fuzzy left ideal of $S$. Thus $f$ is a generalized $(s,t)$-fuzzy ideal of $S$. □

**Corollary 3.4.13.** *If $S$ is a completely regular ordered semigroup, then the concepts of generalized $(s,t)$-fuzzy interior ideal and a generalized $(s,t)$-fuzzy ideal of $S$ coincide.*

**Theorem 3.4.14.** *If $S$ is a intra-regular ordered semigroup, then the concepts of generalized $(s,t)$-fuzzy interior ideal and a generalized $(s,t)$-fuzzy ideal of $S$ coincide.*

*Proof.* Assume that an ordered semigroup $S$ is intra-regular. By Proposition 3.4.10, it is remaining to show that every generalized $(s,t)$-fuzzy interior ideal of $S$ is generalized $(s,t)$-fuzzy ideal of $S$. Let $f$ be a generalized $(s,t)$-fuzzy interior ideal of $S$ and $x,y \in S$. By hypothesis, there exists $a,b \in S$ such that $x \leq ax^2b$. Then $xy \leq ax^2by$. Consider the following cases:

Case 1: $f(ax^2by) \leq s$. Then $f(xy) \vee_f s \geq f(ax^2by) \vee_f s \geq f(x) \wedge_f t$.

Case 2: $f(ax^2by) > s$. Then $f(xy) \vee_f s \geq f(ax^2by) \wedge_f t = (f(ax^2by) \vee_f s) \wedge_f t \geq (f(x) \wedge_f t) \wedge_f t = f(x) \wedge_f t$.

Thus $f$ is a generalized $(s,t)$-fuzzy right ideal of $S$. By using a similar reasoning, we can verify that $f$ is a generalized $(s,t)$-fuzzy left ideal of $S$. Thus $f$ is a generalized $(s,t)$-fuzzy ideal of $S$. □

**Theorem 3.4.15.** *If $S$ is a left regular ordered semigroup, then the concepts of generalized $(s,t)$-fuzzy interior ideal and a generalized $(s,t)$-fuzzy ideal of $S$ coincide.*

*Proof.* Assume that an ordered semigroup $S$ is left regular. By Proposition 3.4.10, it is remaining to show that every generalized $(s,t)$-fuzzy interior ideal of $S$ is generalized $(s,t)$-fuzzy ideal of $S$. Let $f$ be a generalized $(s,t)$-fuzzy interior ideal of $S$ and $x, y \in S$. By hypothesis, there exists $a \in S$ such that $x \leq ax^2$. Then $xy \leq ax^2y$. Consider the following cases:

Case 1: $f(ax^2y) \leq s$. Then $f(xy) \vee_f s \geq f(ax^2y) \vee_f s \geq f(x) \wedge_f t$.

Case 2: $f(ax^2y) > s$. Then $f(xy) \vee_f s \geq f(ax^2y) \wedge_f t = (f(ax^2y) \vee_f s) \wedge_f t \geq (f(x) \wedge_f t) \wedge_f t = f(x) \wedge_f t$.

Thus $f$ is a generalized $(s,t)$-fuzzy right ideal of $S$. By using a similar reasoning, we can verify that $f$ is a generalized $(s,t)$-fuzzy left ideal of $S$. Thus $f$ is a generalized $(s,t)$-fuzzy ideal of $S$. □

# CHAPTER IV

# CONCLUSIONS

In this chapter, we list all main results of this thesis below.

**1.** Let $f$ be a fuzzy subset of a nonempty set $X$ and $a, b, c \in [0, 1]$. The following statements hold:

(1) $a \wedge_f b = b \wedge_f a$,

(2) $a \vee_f b = b \vee_f a$,

(3) $a \wedge_f b \leq a \wedge b$,

(4) $a \vee_f b \geq a \vee b$,

(5) $a \wedge_f b = a \wedge b$ if $a, b \in \text{Im}(f)$,

(6) $a \vee_f b = a \vee b$ if $a, b \in \text{Im}(f)$,

(7) $a \wedge_f b \leq a \wedge_f c$ if $b \leq c$,

(8) $a \vee_f b \leq a \vee_f c$ if $b \leq c$,

(9) $a \wedge_f c \geq b \wedge_f c$ if $a \geq c$,

(10) $a \wedge_f c = b \wedge_f c$ if $a \wedge b \wedge c = c$,

(11) $a \vee_f c \leq b \vee_f c$ if $a \leq c$,

(12) $a \vee_f c = b \vee_f c$ if $a \vee b \vee c = c$.

**2.** Let $f$ be a fuzzy subset of a nonempty subset $X$ and $a, b \in [0, 1]$ such that $\underline{a}_f$ and $\underline{b}_f$ are nonempty. Then the following statements hold:

(1) $\underline{a}_f \subseteq \underline{b}_f$ if $a \leq b$,

(2) $\underline{a}_f = \underline{\sup \underline{a}_f}_f$,

(3) $a \wedge_f b = a \wedge_f \sup \underline{b}_f = \sup \underline{a}_f \wedge_f \sup \underline{b}_f$.

**3**. Let $f$ be a fuzzy subset of a nonempty subset $X$ and $a, b \in [0,1]$ such that $\overline{a}_f$ and $\overline{b}_f$ are nonempty. Then the following statements hold:

(1) $\overline{a}_f \subseteq \overline{b}_f$ if $b \leq a$,

(2) $\overline{a}_f = \overline{\inf \overline{a}_f}_f$,

(3) $a \vee_f b = a \vee_f \inf \overline{b}_f = \inf \overline{a}_f \vee_f \inf \overline{b}_f$.

**4**. Let $f$ be a fuzzy subset of a nonempty set $X$ and $a, b, c \in [0,1]$. The following statements hold:

(1) $a \wedge_f (b \wedge_f b) = a \wedge_f b$,

(2) $a \vee_f (b \vee_f b) = a \vee_f b$,

(3) $(a \wedge_f b) \wedge_f c = a \wedge_f (b \wedge_f c)$,

(4) $(a \vee_f b) \vee_f c = a \vee_f (b \vee_f c)$.

**5**. A nonempty subset $A$ of an ordered semigroup $S$ is a bi-ideal of $S$ if and only if the characteristic function $C_A$ is a generalized $(s,t)$-fuzzy bi-ideal of $S$.

**6**. Let $f$ be a fuzzy subset of an ordered semigroup $S$. The following statements are equivalent:

(1) $(\forall x, y \in S)(f(xy) \vee_f s \geq f(x) \wedge_f f(y) \wedge_f t)$,

(2) $(\forall p \in (s,t])(\forall x, y \in U_s^t(f;p))(xy \in U_s^t(f;p))$.

**7**. Let $f$ be a fuzzy subset of an ordered semigroup $S$. The following statements are equivalent:

(1) $(\forall x, y, z \in S)(f(xyz) \vee_f s \geq f(x) \wedge_f f(z) \wedge_f t)$,

(2) $(\forall p \in (s,t])(\forall x, z \in U_s^t(f;p))(\forall y \in S)(xyz \in U_s^t(f;p))$.

8. Let $f$ be a fuzzy subset of an ordered semigroup $S$. The following statements are equivalent:

(1) $(\forall x, y \in S)(x \leq y$ implies $f(x) \vee_f s \geq f(y) \wedge_f t)$,

(2) $(\forall p \in (s,t])(\forall x \in S)(\forall y \in U_s^t(f;p))(x \leq y$ implies $x \in U_s^t(f;p))$.

9. A fuzzy subset $f$ of an ordered semigroup $S$ is a generalized $(s,t)$-fuzzy bi-ideal of $S$ if and only if each nonempty subset $U_s^t(f;p)$ is a bi-ideal of $S$, where $p \in (s,t]$.

10. Let $A$ be a subset of an ordered semigroup $S$. Then

$$(C_A)_{s,t}(x) = \begin{cases} t, & \text{if } x \in A; \\ s, & \text{if } x \notin A. \end{cases}$$

11. Let $f$ be a fuzzy subset of an ordered semigroup $S$ and $x, y \in S$. If $f(xy) \vee_f s \geq f(x) \wedge_f f(y) \wedge_f t$, then $f_{s,t}(xy) \geq f_{s,t}(x) \wedge f_{s,t}(y)$.

12. Let $f$ be a fuzzy subset of an ordered semigroup $S$ and $x, y, z \in S$. If $f(xyz) \vee_f s \geq f(x) \wedge_f f(z) \wedge_f t$, then $f_{s,t}(xyz) \geq f_{s,t}(x) \wedge f_{s,t}(z)$.

13. Let $f$ be a fuzzy subset of an ordered semigroup $S$ and $x, y \in S$ with $x \leq y$. If $f(x) \vee_f s \geq f(y) \wedge_f t$, then $f_{s,t}(x) \geq f_{s,t}(y)$.

14. If $f$ is a generalized $(s,t)$-fuzzy bi-ideal of an ordered semigroup $S$, then $f_{s,t}$ is a fuzzy bi-ideal of $S$.

15. Let $A$ be a nonempty subset of an ordered semigroup $S$. Then $A$ is a bi-ideal of $S$ if and only if $(C_A)_{s,t}$ is a fuzzy bi-ideal of $S$.

16. An ordered semigroup $S$ is completely regular if and only if each generalized $(s,t)$-fuzzy bi-ideal $f$ of $S$, we have $f_{s,t}(a) = f_{s,t}(a^2)$ for all $a \in S$.

**17**. A nonempty subset $A$ of an ordered semigroup $S$ is an interior ideal of $S$ if and only if the characteristic function $C_A$ is a generalized $(s,t)$-fuzzy interior ideal of $S$.

**18**. Let $f$ be a fuzzy subset of an ordered semigroup $S$. The following statements are equivalent:

(1) $(\forall x, y, z \in S)(f(xyz) \vee_f s \geq f(y) \wedge_f t)$,

(2) $(\forall p \in (s,t])(\forall y \in U_s^t(f;p))(\forall x, z \in S)(xyz \in U_s^t(f;p))$.

**19**. A fuzzy subset $f$ of an ordered semigroup $S$ is a generalized $(s,t)$-fuzzy interior ideal of $S$ if and only if each nonempty subset $U_s^t(f;p)$ is an interior ideal of $S$, where $p \in (s,t]$.

**20**. Let $f$ be a fuzzy subset of an ordered semigroup $S$ and $x, y, z \in S$. If $f(xyz) \vee_f s \geq f(y) \wedge_f t$, then $f_{s,t}(xyz) \geq f_{s,t}(y)$.

**21**. If $f$ is a generalized $(s,t)$-fuzzy interior ideal of an ordered semigroup $S$, then $f_{s,t}$ is a fuzzy interior ideal of $S$.

**22**. Let $A$ be a nonempty subset of an ordered semigroup $S$. Then $A$ is an interior ideal of $S$ if and only if $(C_A)_{s,t}$ is a fuzzy interior ideal of $S$.

**23**. An ordered semigroup $S$ is intra-regular if and only if each generalized $(s,t)$-fuzzy interior ideal $f$ of $S$, we have $f_{s,t}(a) = f_{s,t}(a^2)$ for all $a \in S$.

**24**. Let $S$ be an ordered semigroup. Then $S$ is simple if and only if for each generalized $(s,t)$-fuzzy interior ideal $f$ of $S$, we have $f_{s,t}$ is a constant function.

**25**. A nonempty subset $A$ of an ordered semigroup $S$ is an left ideal of $S$ if and only if the characteristic function $C_A$ is a generalized $(s,t)$-fuzzy left ideal of $S$.

**26**. Let $f$ be a fuzzy subset of an ordered semigroup $S$. The following statements are equivalent:

(1) $(\forall x, y, z \in S)(f(xy) \vee_f s \geq f(y) \wedge_f t)$,

(2) $(\forall p \in (s,t])(\forall y \in U_s^t(f;p))(\forall x \in S)(xyz \in U_s^t(f;p))$.

**27**. A fuzzy subset $f$ of an ordered semigroup $S$ is a generalized $(s,t)$-fuzzy left ideal of $S$ if and only if each nonempty subset $U_s^t(f;p)$ is a left ideal of $S$, where $p \in (s,t]$.

**28**. Let $f$ be a fuzzy subset of an ordered semigroup $S$ and $x, y \in S$. If $f(xy) \vee_f s \geq f(y) \wedge_f t$, then $f_{s,t}(xy) \geq f_{s,t}(y)$.

**29**. If $f$ is a generalized $(s,t)$-fuzzy left ideal of an ordered semigroup $S$, then $f_{s,t}$ is a fuzzy left ideal of $S$.

**30**. Let $A$ be a nonempty subset of an ordered semigroup $S$. Then $A$ is a left ideal of $S$ if and only if $(C_A)_{s,t}$ is a fuzzy left ideal of $S$.

**31**. An ordered semigroup $S$ is left regular if and only if each generalized $(s,t)$-fuzzy left ideal $f$ of $S$, we have $f_{s,t}(a) = f_{s,t}(a^2)$ for all $a \in S$.

**32**. Every generalized $(s,t)$-fuzzy ideal of an ordered semigroup $S$ is generalized $(s,t)$-fuzzy interior ideal of $S$.

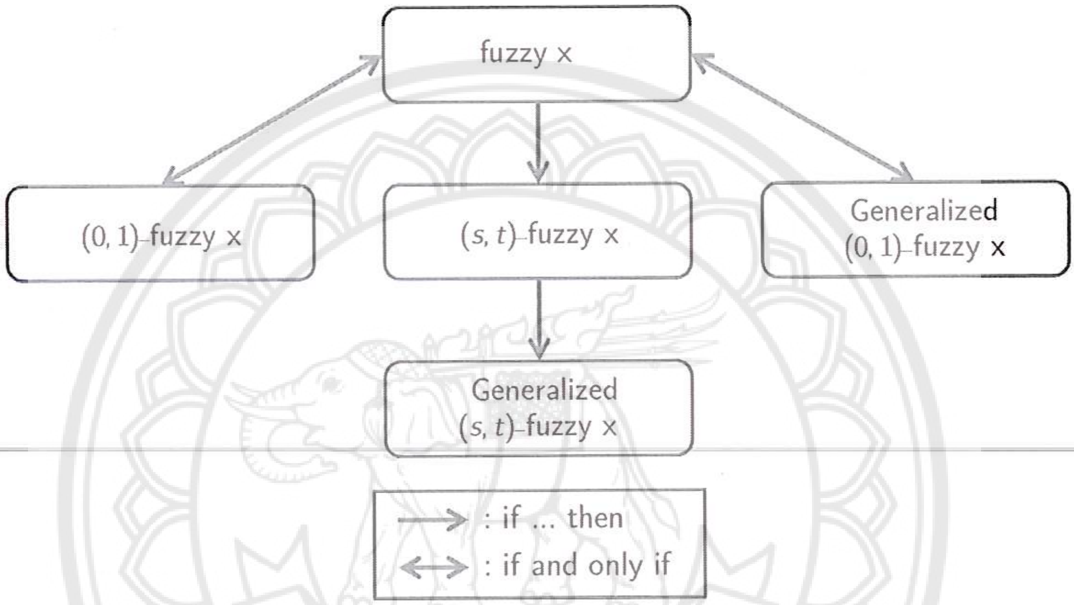
**33**. If $S$ is a regular ordered semigroup, then the concepts of generalized $(s,t)$-fuzzy interior ideal and a generalized $(s,t)$-fuzzy ideal of $S$ coincide.

**34**. If $S$ is a completely regular ordered semigroup, then the concepts of generalized $(s,t)$-fuzzy interior ideal and a generalized $(s,t)$-fuzzy ideal of $S$ coincide.

**35**. If $S$ is a intra-regular ordered semigroup, then the concepts of generalized $(s,t)$-fuzzy interior ideal and a generalized $(s,t)$-fuzzy ideal of $S$ coincide.

**36**. If $S$ is a left regular ordered semigroup, then the concepts of generalized $(s,t)$-fuzzy interior ideal and a generalized $(s,t)$-fuzzy ideal of $S$ coincide.

Let $x \in$ {bi-ideal, interior ideal, ideal}. The following diagram shows relationships of fuzzy bi-ideal (resp. fuzzy interior ideal, fuzzy ideal), $(s,t)$-fuzzy bi-ideal (resp. $(s,t)$-fuzzy interior ideal, $(s,t)$-fuzzy ideal) and generalized $(s,t)$-fuzzy bi-ideal (resp. generalized $(s,t)$-fuzzy interior ideal, generalized $(s,t)$-fuzzy ideal) on an ordered semigroup.



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