GENERAL ITERATIVE APPROXIMATION METHODS FOR COMMON SOLUTIONS OF GENERALIZED EQUILIBRIUM AND FIXED POINT PROBLEMS IN HILBERT SPACES AND VISCOSITY APPROXIMATION METHODS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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ABSTRACT

In this research, we establish the following results. Firstly, we introduce new iterative algorithms with perturbations for finding a common element of the set of solutions of the system of generalized equilibrium problems and the set of common fixed points of two quasi-nonexpansive mappings in a Hilbert space. Secondly, we introduce a new general iterative scheme for finding a common element of $F(T) \cap (A+B)^{-1}0 \cap F^{-1}0$ which is a unique solution of a hierarchical variational inequality, where $F(T)$ is the set of fixed points of T, $(A+B)^{-1}0$ and $F^{-1}0$ are the sets of zero points of $A+B$ and F, respectively. Then, we prove a strong convergence theorem. Finally, we introduce the iterative schemes for finding a fixed point of an asymptotically nonexpansive mapping which is the unique solution of some variational inequalities in CAT(0) spaces.

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CHAPTER I

INTRODUCTION

One of the most important problems in nonlinear analysis is called equilibrium problem (abbreviated (EP)), which can be formulated as follows. Let C be a nonempty set and $f : C \times C \to \mathbb{R}$ be a given function. The problem consists on finding an element $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \text{ for all } y \in C. \tag{EP}$$

The element $\hat{x}$ satisfying (EP) is called equilibrium point of f on C.

(EP) has been extensively studied in recent years (e.g. [4, 5, 6, 7] and the references therein). A part from its theoretical interest, important problems arising from economics, mechanics, electricity and other practical sciences motivate the study of (EP). Equilibrium problems include, as particular cases, variational inequalities, Nash equilibria problems, complementarity problems, fixed point problems, etc.

In 1994, equilibrium problems were introduced by Blum and Oettli [7]. Since that time, the equilibrium problem has been wildly studied by many authors, for example, mixed equilibrium problem (MEP) [12, 13], generalized equilibrium problem (GEP) [54], generalized mixed equilibrium problem (GMEP) [26], and so on.

In 2000, the viscosity approximation method (VAM) for solving nonlinear operator equations has recently attracted much attention. One advantage of the VAM is that one can obtain a solution which satisfies some particular properties, for example, it solves a variational inequality. The VAM was introduced to nonexpansive mappings by Moudafi [43] in Hilbert spaces Recently, [41] Marino and Xu considered a general iterative method which is more general than VAM. They proved that the iteration converges strongly to a fixed point of a nonexpansive mapping which solves some variational inequalities in a Hilbert spaces.

In 2007, by using the viscosity approximation method, S. Takahashi and W. Takahashi [53] introduced another iterative scheme for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a nonexpansive mapping. In 2011, Yao and Shahzad [60] gave the iteration process for nonexpansive mappings with perturbation. On the other hand, very recently, Chuang, et al. [14] considered the iteration process for finding a common element of the set of solutions of the equilibrium problem and the set of common fixed points for a quasi-nonexpansive mapping with perturbation.

In 2010, Kocourek, et al. [31] introduced a class of nonlinear mappings, say generalized hybrid mappings. Recently, Maruyama, et al. [42] defined a more general class of nonlinear mappings than the class of generalized hybrid mappings. Such a mapping is a 2-generalized hybrid mapping. Moreover, Takahashi, et al. [56] proved a strong convergence theorem for finding a point of fixed points and the set of zero points. Manaka and Takahashi [40] proved the weakly convergence to a fixed point of nonspreading mapping and set zero points of α inverse strongly monotone and maximal monotone operator. Very recently, Liu, et al. [38] generalized the iterative for finding a common element of the set of fixed points of a nonspreading mapping and the set of zero points of a monotone operator $(A + B)$ (A is an α inverse strongly monotone and B is maximal monotone operator). On the other hand, Marino and Xu [41] introduced the following general iterative scheme based on the viscosity approximation method introduced by Moudafi [43]. They proved strongly convergence to the unique solution of the variational inequality. Very recently, Lin and Takahashi [36] obtained the strong convergence theorem for finding of fixed points of α-inverse strongly monotone mapping and maximal monotone operators which is a unique solution of a hierarchical variational inequality.

In CAT(0) spaces, fixed point theory was first studied by Kirk (see [27, 28]) in 2003-2004. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. In 2008, Kirk and Panyanak [29] specialized Lims concept [35] of

Δ-convergence in a general metric space to CAT(0) spaces and showed that many Banach space results which involve weak convergence have precise analogs in this setting; for instance, the Opial property, the Kadec-Klee property and the demiclosedness principle for LANE mappings.

In 2010, Saejung [46] studied the convergence theorems of the following Halpern's iterations for a nonexpansive mapping in a complete CAT(0) space. They proved stongly convergence which is nearest under certain appropriate conditions. Moreover, the author applied his result to find a common fixed point of a countable family of nonexpansive mappings. In 2012, Shi and Chen [47], studied the convergence theorems of the following Moudafi's viscosity iterations for a nonexpansive mapping for a contraction. They proved strongly convergence in the framework of CAT(0) space satisfying property $\mathcal{P}$, i.e., if for $x, u, y_1, y_2 \in X$,

$$d(x, P_{[x,y_1]}u)d(x, y_1) \leq d(x, P_{[x,y_2]}u)d(x, y_2) + d(x, u)d(y_1, y_2).$$

Furthermore, they also obtained that strongly convergence under certain appropriate conditions imposed. Recently, using the concept of quasilinearization, Wangkeeree and Preechasilp [58] studied the strong convergence theorems of the iterative schemes in CAT(0) spaces without the property $\mathcal{P}$. They proved the iterative schemes converges strongly which is the unique solution of the variational inequality (VIP) :

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T). \tag{1.0.1}$$

On the other hand, Shi, Chen and Wu [48] studied the Δ-convergence of the iteration sequence for asymptotically nonexpansive mappings in CAT(0) spaces. For more related works, see [33, 34, 49].

Motivated and inspired by the above works, the purposes of this thesis are to extend, to generalize and to improve the iteration schemes for finding the solutions of equilibrium problems, variational inequality problems and fixed point problems in Hilbert spaces and CAT(0) spaces.

This thesis is divided into 5 chapters. Chapter 1 is an introduction to the research problem. Chapter 2 is dealing with some preliminaries and give some useful results that will be deplicated in later Chapter.

Chapter 3 and 4 are the main results of this research. Precisely, in section 3.1, we introduce new iterative algorithms with perturbations for finding a common element of the set of solutions of the system of generalized equilibrium problems and the set of common fixed points of two quasi-nonexpansive mappings in a Hilbert space. Furthermore, we also consider the iterative algorithms with perturbations for finding a common element of the solution set of the systems of generalized equilibrium problems and the common fixed point set of the super hybrid mappings in Hilbert spaces. In section 3.2, let C be a closed and convex subset of a real Hilbert space H. Let T be a 2-generalized hybrid mapping of C into itself, let A be an α-inverse strongly monotone mapping of C into H, and let B and F be maximal monotone operators on $D(B) \subset C$ and $D(F) \subset C$ respectively. We introduce a general iterative scheme for finding a point of fixed points and the sets of zero points $F(T) \cap (A+B)^{-1} \cap F^{-1}0$ which is a unique solution of a hierarchical variational inequality, where $F(T)$ is the set of of $T, (A+B)^{-1}0$ and $F^{-1}0$ of $A+B$ and F, respectively. Further, we consider the problem for finding a common element of the set of solutions of a mathematical model related to mixed equilibrium problems and the set of fixed points of a 2-generalized hybrid mapping in a real Hilbert space. Section 4.1, we introduce the iterative schemes for finding a fixed point of an asymptotically nonexpansive mapping which is the unique solution of some variational inequalities in CAT(0) spaces. The strong convergence theorem of the proposed iterative schemes is established.

The conclusion of research is in Chapter 5.

CHAPTER II

PRELIMINARIES

In this chapter, we give some notations, definitions, and some useful results that will be used in the later chapter. Throughout this dissertation, we let $\mathbb{R}$ be the set of all real numbers, $\mathbb{N}$ be the set of all natural numbers, H be a Hilbert space.

2.1 Basic Concepts

Definition 2.1.1. [51] Let C be a nonempty set, and assume that each pair of elements x and y in C can be combined by a process called *addition* to yield an element z in C denote by $x+y$. Assume also that this operation of addition satisfies the following conditions $(V1) - (V4)$:

(V1) $(x+y)+z = x+(y+z)$;

(V2) $x+y = y+x$;

(V3) there exists a unique element in C, denote by 0 and called the *zero element*, or the *origin*, such that $x+0 = x$ for all $x \in C$;

(V4) for each $x \in C$ there corresponds a unique element in C, denote by $-x$ and called the *negative* of x, such that $x+(-x) = 0$.

We also assume that each scalar $\alpha \in \mathbb{R}$ and each element x in C can be combined by a process called *scalar multication* to yield an element y in C denoted by $y = \alpha x$ satisfying $(V5) - (V8)$:

(V5) $\alpha(\beta x) = (\alpha\beta)x$;

(V6) $1 \cdot x = x$;

(V7) $(\alpha+\beta)x = \alpha x + \beta x$;

(V8) $\alpha(x+y) = \alpha x + \alpha y$.

The algebraic system C defined by these operations and axims is called a *linear space.* A linear space is often called a *vector space,* and its elements are spoken of as vectors.

Remark 2.1.2. [51] Since we admit the real numbers as scalars, a linear space is also called a *real linear space.*

Remark 2.1.3. [51] We obtain a few simple facts which are easy to prove from the axioms:

(1) $x + z = y + z \Rightarrow x = y$;

(2) $\alpha \cdot 0 = 0$;

(3) $0 \cdot x = 0$;

(4) $(-1)x = -x$.

Definition 2.1.4. [30] Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$, satisfying the following condition for all x, y and z in X:

(M1) $d(x, y) = 0 \Leftrightarrow x = y$;

(M2) $d(x, y) = d(y, x)$;

(M3) $d(x, y) \leq d(x, z) + d(z, y)$.

The function d assigns to each pair (x, y) of element of X a nonnegative real number $d(x, y)$, which does not on the order of the elements; $d(x, y)$ is called the distance between x and y. The set X together with a metric, denoted by (X, d), is called a metric space. The conditions $(M1) - (M3)$ are usually called the metric axioms.

Definition 2.1.5. [30] A normed linear space is a vector space V over $\mathbb{R}$ (or $\mathbb{C}$) and a mapping $\|\cdot\| : V \to \mathbb{R}$, called norm, for all $x, y \in V$ and all $\alpha \in \mathbb{R}$ that satisfies:

(i) $\|x\| \geq 0$;

(ii)$\|x\| = 0 \Longleftrightarrow x = 0$;

(iii) $\|\alpha x\| = |\alpha|\|x\|$;

(iv) $\|x + y\| \leq \|x\| + \|y\|$.

From this norm we can define a metric, induced by the norm $\|\cdot\|$, by

$$d(x,y)=\|x-y\|,\quad x,y\in X.$$

A linear space X equipped with the norm $\|\cdot\|$ is called a normed linear space.

Definition 2.1.6. [30] Let $(X,\|\cdot\|)$ be a normed space.

1) A sequence $\{x_n\}\subset X$ is said to *converge strongly* in X if there exists $x\in X$ such that $\lim_{n\to\infty}\|x_n-x\|=0$. That is, if for any $\varepsilon>0$ there exists a positive integer N such that $\|x_n-x\|<\varepsilon,\forall n\geq N$. We often write $\lim_{n\to\infty}x_n=x$ or $x_n\to x$ to mean that x is the limit of the sequence $\{x_n\}$.

2) A sequence $\{x_n\}\subset X$ is said to be *a Cauchy sequence* if for any $\varepsilon>0$ there exists a positive integer N such that $\|x_m-x_n\|<\varepsilon,\forall\, m,n\geq N$. That is, $\{x_n\}$ is *a Cauchy sequence* in X if and only if $\|x_m-x_n\|\to 0$ as $m,n\to\infty$.

3) A sequence $\{x_n\}\subset X$ is said to be *a bounded sequence* if there exists $M>0$ such that $\|x_n\|\leqslant M$, $\forall n\in\mathbb{N}$.

Definition 2.1.7. [51] A subset C of a normed linear space X is said to be *convex subset* in X if $\lambda x+(1-\lambda)y\in C$ for each $x,y\in C$ and for each scalar $\lambda\in[0,1]$.

Definition 2.1.8. [30] A normed space X is called to be *complete* if every Cauchy sequence in X converges to an element in X.

Definition 2.1.9. [51] An inner product space is a complex linear space H which for any pair of elements x and y in H there corresponds a complex number, denoted by $\langle x,y\rangle$, and called the inner product of x and y, with the following properties:

(I1) $\langle x,x\rangle\geq 0$, and $\langle x,x\rangle=0\Leftrightarrow x=0$;

(I2)$\langle x+y,z\rangle=\langle x,z\rangle+\langle y,z\rangle$;

(I3) $\langle \alpha x,y\rangle=\alpha\langle x,y\rangle$;

(I4) $\langle x,y\rangle=\overline{\langle y,x\rangle}$.

Remark 2.1.10. [51] An inner product space is called a real inner product space for the case when the scalars are the real numbers and $\langle x, y\rangle$ is a real number. For the case, (I4) means

$$\langle x, y\rangle = \langle y, x\rangle.$$

If X is a linear space with an inner product $\langle\cdot,\cdot\rangle$, then we can define a norm on X by

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Thus, any inner product space is a norm space.

Definition 2.1.11. [51] A Hilbert spaces is an inner product space which is complete under the norm induced by its inner product.

Lemma 2.1.12. [30](Schwarz Inequality) Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space, then for all $x, y \in H$

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Lemma 2.1.13. [30](Triangle Inequality) Let H be an inner product space, then for all $x, y \in H$

$$\|x + y\| \leq \|x\| + \|y\|.$$

Lemma 2.1.14. [30](Parallelogram Law) Let H be an inner product space, then for all $x, y \in H$

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Lemma 2.1.15. [51] Let H be a real Hilbert space. Then the following inequalities hold:

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;

(ii) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$;

(iii) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$.

Definition 2.1.16. [30] The metric projection (or nearest point) from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_Cx \in C$ satisfying the property

$$\|x - P_Cx\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

Lemma 2.1.17. [51] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then

(i) $z = P_Cx \Leftrightarrow \langle z - x, y - x\rangle \geq 0,\ \forall y \in C;$

(ii) $\|P_Cx - P_Cy\| \leq \|x - y\|,\ \forall x, y \in H;$

(iii) $\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x - y\rangle,\ \forall x, y \in H;$

(iv) $\langle x - P_Cx, y - P_Cx\rangle \leq 0,\ \forall x \in H, y \in C;$

(v) $\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2,\ \forall x \in H, y \in C.$

Lemma 2.1.18. [51] Let $\{x_n\}$ be a sequence of a normed space $(X, \|\cdot\|), x \in X$ and let $x_n \to x$ if and only if, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converging to x.

Lemma 2.1.19. [51] Let X be an inner product space and $\{x_n\}$ be a bounded sequence of H such that $x_n \rightharpoonup x$. Then following inequality holds:

$$\|x\| \leq \lim_{n\to\infty} \inf \|x_n\|.$$

Lemma 2.1.20. [37, 59] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n,\ n \geq 0,$$

where $\{\alpha_n\} \subseteq (0, 1)$ and $\{\beta_n\} \subseteq \mathbb{R}$ such that:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty;$

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$ or $\sum_{n=0}^{\infty} |\alpha_n\beta_n| < \infty.$

Then, $\{a_n\}$ converges to zero as $n \to \infty$.

Next, we introduce basic concepts in Hilbert spaces. Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. We have the following are hold:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle,$$

and

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2,$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

Definition 2.1.21. [51] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$. Then, f is called *proper* if there exists $x \in C$ with $f(x) < \infty$, that is,

$$D(f) = \{x \in C : f(x) < \infty\} \neq \emptyset.$$

Definition 2.1.22. [50] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let f be a function of C into $(-\infty, \infty]$, where $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$. Then, f is called *lower semicontinuous* (short l.w.c) if for any $a \in \mathbb{R}$, the set $\{x \in C : f(x) \leq a\}$ is closed.

Theorem 2.1.23. [51] (Opial's theorem). Let H be a Hilbert space and suppose $x_n \rightharpoonup x$. Then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in H$ with $x \neq y$.

Lemma 2.1.24. [2](Demi-closedness Principle) Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a real Hilbert space H. If T has a fixed point, the $I - T$ is demi-closed; that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ (for short, $x_n \rightharpoonup x \in C$), and the sequence $\{(I - T)x_n\}$ converges strongly to some y (for short, $(I - T)x_n \to y$), it follows that $(I - T)x = y$. Here I is the identity operator of H.

2.2 The Classical of Fixed Point Theory

Definition 2.2.1. An element $x \in C$ is said to be a *fixed point* of a mapping $T : C \to C$. The set of all fixed points of T is denoted by $F(T) = \{x \in C : Tx = x\}$.

Definition 2.2.2. Let H be a Hilbert space and let C a nonempty bounded convex subset of H. A mapping $T : C \to C$ is called *nonexpansive* on C if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

Lemma 2.2.3. [50] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.

Theorem 2.2.4. [30] Let H be a Hilbert space and let C be a nonempty bounded closed convex subset of H. Let T be a nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.

Definition 2.2.5. [57] Let H be a Hilbert space and let C a nonempty bounded convex subset of H. A mapping $f : C \to C$ is called a *contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \ \forall x, y \in C.$$

2.3 Some Nonlinear Mappings in Hilbert Spaces

Let C be a closed convex subset of a real Hilbert space H with inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $T : C \to C$ a nonlinear mapping.

Definition 2.3.1. [2] Let C be a subset of an inner product space H. A mapping $A : C \to C$ is said to be *monotone* if for all $x, y \in C$,

$$\langle Ax - Ay, x - y \rangle \geq 0.$$

Definition 2.3.2. [2] Let C be a nonempty closed convex subset of H and let $A : C \longrightarrow H$. A mapping A is said to be $\alpha-$*inverse strongly monotone* if there exists a positive real number α such that for all $x, y \in C$,

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2.$$

Lemma 2.3.3. [2] Let $A : H \to H$ be a α-*inverse-strongly monotone* mapping. If $\lambda \leq 2\alpha$, for any $\lambda > 0$ then $I - \lambda A$ is a nonexpansive mapping from H into itself.

Proof Let $u, v \in H$ and $\lambda > 0$,

$$\begin{aligned}\|(I - \lambda A)u - (I - \lambda A)v\|^2 &= \|(u - v) - \lambda(Au - Av)\|^2 \\ &= \|u - v\|^2 - 2\lambda\langle u - v, Au - Av\rangle + \lambda^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 - 2\lambda(\lambda - 2\alpha)\|Au - Av\|^2.\end{aligned}$$

Definition 2.3.4. [2] A mapping $A : C \to C$ is called *L-Lipschitz-continuous* if there exists a positive real number L such that

$$\|Au - Av\| \leq L\|u - v\|, \forall u, v \in C.$$

Remark 2.3.5. It is easy to see that if A is an α-inverse strongly monotone mapping of C into H, then A is $\frac{1}{\alpha}$-Lipschitz continuous.

Definition 2.3.6. [2] A mapping $B : H \to H$ is called *strongly positive bounded linear operator* on H if there is a constant $\bar{\gamma} > 0$ with property

$$\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2,$$

for all $x \in H$.

Definition 2.3.7. A nonlinear operator $V : H \to H$ is called *strongly monotone* if there exists $\bar{\gamma} > 0$ such that

$$\langle x - y, Vx - Vy\rangle \geq \bar{\gamma}\|x - y\|^2,$$

for all $x, y \in H$.

Definition 2.3.8. [32] Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to H$ a mapping. A mapping T is called *firmly nonspreading* on C if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \ \forall x, y \in C.$$

Definition 2.3.9. [23] Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to H$ a mapping. A mapping T is called *nonspreading* on C if and only if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \ \forall x, y \in C.$$

Definition 2.3.10. Let C be a nonempty closed convex subset of a Hilbert space H. A mapping $T : C \to H$ and $F(T) \neq \emptyset$. A mapping T is called *quasi-nonexpansive* on C if

$$\|Tx - y\| \le \|x - y\|, \quad \forall x \in C, y \in F(T).$$

Definition 2.3.11. [9, 21] Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $F : C \to H$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \le \langle x - y, Fx - Fy\rangle,$$

for all $x, y \in C$.

Definition 2.3.12. [52] Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to H$ is said to be *hybrid* if

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2,$$

for all $x, y \in C$.

Definition 2.3.13. [31] Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping T is called *generalized hybrid* (or (α, β)-generalized hybrid) if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2, \quad (2.3.1)$$

for all $x, y \in C$.

For example, generalize hybrid mappings

(1) If $\alpha = 1, \beta = 0$ in (2.3.1), then T is nonexpansive mapping.

(2) If $\alpha = 2, \beta = 1$ in (2.3.1), then T is nonspreading mapping.

(3) If $\alpha = \frac{3}{2}, \beta = \frac{1}{2}$ in (2.3.1), then T is hybrid mapping.

Definition 2.3.14. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is called *2-generalized hybrid* (or $(\alpha_1, \alpha_2, \beta_1, \beta_2)$-generalized hybrid) if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1-\alpha_1-\alpha_2)\|x - Ty\|^2 \\ \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 \\ +(1-\beta_1-\beta_2)\|x - y\|^2, \quad (2.3.2)\end{aligned}$$

for all $x, y \in C$.

Remark 2.3.15. If T is a 2-generalized hybrid mapping and $x = Tx$, then for any $y \in C$,

$$\begin{aligned}\alpha_1\|x - Ty\|^2 + \alpha_2\|x - Ty\|^2 + (1-\alpha_1-\alpha_2)\|x - Ty\|^2 \\ \leq \beta_1\|x - y\|^2 + \beta_2\|x - y\|^2 \\ +(1-\beta_1-\beta_2)\|x - y\|^2.\end{aligned}$$

Hence $\|x - Ty\| \leq \|x - y\|$.

This means that a 2-generalized hybrid mapping with a fixed point is quasi-nonexpansive.

Remark 2.3.16. If $\alpha_1 = 0, \beta_1 = 0$ in (2.3.2), then T is (α_2, β_2)-generalized hybrid mapping.

Definition 2.3.17. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $S : C \to H$ is called ***super hybrid***(or (α, β, γ)-super hybrid) if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\|Sx - Sy\|^2 + (1 - \alpha + \gamma)\|x - Sy\|^2 \\ &\quad \leq (\beta + (\beta - \alpha)\gamma)\|Sx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \\ &\qquad +(\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2, \end{aligned} \tag{2.3.3}$$

for all $x, y \in C$.

We call such a mapping an (α, β, γ)-super hybrid mapping.

Remark 2.3.18. If $\gamma = 0$ in (2.3.3), then S is is (α, β)-generalized hybrid.

So, the class of super hybrid mappings contains the class of generalized hybrid mappings. A super hybrid mapping is not quasi-nonexpansive.

Lemma 2.3.19. [55] Let C be a nonempty subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let S and T be mappings of C into H such that $S = \frac{1}{1+\gamma}T + \frac{\gamma}{1+\gamma}I$. Then, T is (α, β, γ)-super hybrid if and only if S is (α, β)-generalized hybrid. In this case, $F(S) = F(T)$.

Lemma 2.3.20. [55] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $S : C \to H$ be a generalized hybrid mapping. Then S is demi-closed on C.

Lemma 2.3.21. [39] Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(1) \leq \tau(2) \leq \ldots$ and $\tau(n) \to \infty$;

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}, \quad \forall n \in \mathbb{N}$.

Lemma 2.3.22. [1] Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in [0, 1] with $\Sigma_{n=1}^{\infty}\alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\Sigma_{n=1}^{\infty}u_n < \infty$, $\{t_n\}$ a sequence of real numbers with $\limsup t_n \leq 0$. Suppose that

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n t_n + u_n, \text{ for all } n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} a_n = 0$.

Definition 2.3.23. Let B be a mapping of H into 2^H. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$.

Definition 2.3.24. A multi-valued mapping B on H is called *monotone* if for all $x, y \in D(B), u \in Bx$, and $v \in By$ imply $\langle x-y, u-v\rangle \geq 0$.

Definition 2.3.25. A monotone mapping $B : H \to 2^H$ is maximal if the graph of $G(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if for $(x, f) \in H \times H, \langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(B)$ implies $f \in Bx$.

Definition 2.3.26. Let H be a Hilbert space and let $A \subset H \times H$ be a set-valued mappoing. Then A is called *accretive* if for any $(x_1, y_1), (x_2, y_2) \in A$,

$$\langle x_1 - x_2, y_1, y_2\rangle \geq 0.$$

Definition 2.3.27. Let $A \subset H \times H$ be an accretive operator and for any $r > 0$ and $x \in H$, define the $J_r x$ by

$$J_r x = \{z \in H : x \in z + rAz\}.$$

Definition 2.3.28. Let the set-valued mapping $B : H \to 2^H$ be a maximal monotone. We define the resolvent operator J_r associate with B and $r > 0$ as follows:

$$J_r = (I + rB)^{-1}.$$

It is worth mentioning that the resolvent operator J_r is single-valued, nonexpansive and 1-inverse strongly monotone.

We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of B for r. We know [51] that

$$A_r x \in BJ_r x, \quad \forall x \in H, \ \ r > 0. \tag{2.3.4}$$

Definition 2.3.29. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. Then resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$, i.e.,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{2.3.5}$$

Lemma 2.3.30. Let H be a real Hilbert space, and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2, \tag{2.3.6}$$

for all $s, t > 0$ and $x \in H$.

From Lemma 2.3.30, we have that

$$\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu|/\lambda)\|x - J_\lambda x\|, \tag{2.3.7}$$

for all $\lambda, \mu > 0$ and $x \in H$; see also [20, 50]. To prove our main result, we need the following lemmas.

Remark 2.3.31. It is not hard to show that if A is an α-inverse strongly monotone mapping, then it is $\frac{1}{\alpha}$ -Lipschitzian,and hence uniformly continuous. Clearly, the class of monotone mappings include the class of α-inverse strongly monotone mappings.

Remark 2.3.32. It is well known that if $T : C \to C$ is a nonexpansive mapping, then $I - T$ is $\frac{1}{2}$ -inverse strongly monotone, where I is the identity mapping on H; see, for instance, [51]. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$.

Lemma 2.3.33. [40] Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$. Let A be an α-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for any $\lambda > 0$. Then, the following statements hold:

(i) if $u, v \in (A + B)^{-1}(0)$, then $Au = Av$;

(ii) for any $\lambda > 0, u \in (A + B)^{-1}(0)$ if and only if $u = J_\lambda(I - \lambda A)u$.

Lemma 2.3.34. [36] Let H be a Hilbert space, and let $g : H \to H$ be a k-contraction with $0 < k < 1$. Let V be a $\bar{\gamma}$ -strongly monotone and L-Lipschitzian continuous operator on H with $\bar{\gamma} > 0$ and $L > 0$. Let a real number γ satisfy $0 < \gamma < \frac{\bar{\gamma}}{k}$. Then $V - \gamma g : H \to H$ is a $(\bar{\gamma} - \gamma k)$-strongly monotone and $(L + \gamma k)$-Lipschitzian continuous mapping. Furthermore, let C be a nonempty closed convex subset of H. Then $P_C(I - V + \gamma g)$ has a unique fixed point z_0 in C. This point $z_0 \in C$ is also a unique solution of the variational inequality

$$\langle (V - \gamma f)z_0, q - z_0 \rangle \geq 0, \quad \forall q \in C.$$

Definition 2.3.35. Let H be a Hilbert space and let f be a proper lower semicontinuous convex function of H into $(-\infty, \infty]$. Then, the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \leq f(y), y \in H\},$$

for all $x \in H$; see, for instance, [51]. From Rockafellar [45], we know that ∂f is maximal monotone.

Definition 2.3.36. Let C be a nonempty closed convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & x \in C \\ \infty, & x \notin C. \end{cases}$$

Then, i_C is a proper lower semicontinuous convex function of H into $(-\infty, \infty]$ and then the subdifferential ∂_{i_C} of i_C is a maximal monotone operator.

So, we can define the resolvent J_λ of ∂_{i_C} for $\lambda > 0$, i.e.,

$$J_\lambda x = (I + \lambda \partial_{i_C})^{-1} x,$$

for all $x \in H$. We have that for any $x \in H$ and $u \in C$,

$$\begin{aligned} u = J_\lambda x \quad &\Leftrightarrow \quad x \in u + \lambda \partial_{i_C} u \\ &\Leftrightarrow \quad x \in u + \lambda N_C u \\ &\Leftrightarrow \quad x - u \in \lambda N_C u \\ &\Leftrightarrow \quad \frac{1}{\lambda}\langle x - u, v - u\rangle \leq 0, \quad \forall v \in C \\ &\Leftrightarrow \quad \langle x - u, v - u\rangle \leq 0, \quad \forall v \in C \\ &\Leftrightarrow \quad u = P_C x, \end{aligned}$$

where $N_C u$ is the normal cone to C at u, i.e.,

$$N_C u = \{x \in H : \langle x, v - u\rangle \leq 0, \forall v \in C\}.$$

2.4 Equilibrium Problems

Definition 2.4.1. Let G be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $G : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$G(x, y) \geq 0, \quad \forall y \in C. \tag{2.4.1}$$

The set of solutions of 2.4.1 is denoted by $EP(G)$, that is,

$$EP(G) = \{x \in C : G(x, y) \geq 0, \, \forall y \in C\}.$$

Definition 2.4.2. Let $G : C \times C \to \mathbb{R}$ be a bifuction and $\Psi : C \to H$ be a μ-inverse strongly monotone mapping. The generalized equilibrium problem (for short, GEP) for G and Ψ is to find $z \in C$ such that

$$G(z, y) + \langle \Psi z, y - z \rangle \geq 0, \, \forall y \in C. \tag{2.4.2}$$

The set of solutions of (2.4.2) is denoted by $GEP(G, \Psi)$, that is,

$$GEP(G, \Psi) = \{z \in C : G(z, y) + \langle \Psi z, y - z \rangle \geq 0, \, \forall y \in C\}.$$

Remark 2.4.3. If $\Psi \equiv 0$, in (2.4.2), then GEP reduces into to the classical equilibrium problem.

Remark 2.4.4. If $G \equiv 0$ in (2.4.2), then GEP reduces to the classical variational inequality and $GEP(0, \Psi)$ is denoted by $VI(\Psi, C)$, that is,

$$VI(\Psi, C) = \{z \in C : \langle \Psi z, y - z \rangle \geq 0, \, \forall y \in C\}.$$

Definition 2.4.5. Let $G : C \times C \to \mathbb{R}$ be a bifuction and let φ be a real valued function. The mixed equilibrium problem (for short, MEP) is to find $x \in C$ such that

$$G(x, y) + \varphi(y) - \varphi(x) \geq 0, \, \forall y \in C. \tag{2.4.3}$$

Remark 2.4.6. If $\varphi = 0$, in (2.4.3), then MEP reduces to the equilibrium problem.

Definition 2.4.7. [7] For solving the equilibrium problem, let us assume that the bifunction G satisfies the following conditions:

(A1) $G(x, x) = 0$, for all $x \in C$;

(A2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$, for any $x, y \in C$;

(A3) for each $x, y, z \in C$

$$\lim_{t \downarrow 0} G(tz + (1-t)x, y) \leq G(x, y);$$

(A4) for each $x \in C, G(x, \cdot)$ is convex and lower semicontinuous.

Definition 2.4.8. [7] For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction G, φ and the set C:

(A1) $G(x, x) = 0$, for all $x \in C$;

(A2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$, for any $x, y \in C$;

(A3) for each $x, y, z \in C$

$$\lim_{t \downarrow 0} G(tz + (1-t)x, y) \leq G(x, y);$$

(A4) for each $x \in C, G(x, \cdot)$ is convex and lower semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y \in C$ such that for any $z \in C \backslash D_x$,

$$G(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x \rangle < 0;$$

(B2) C is a bounded set.

Lemma 2.4.9. [7] Let C be a nonempty closed convex subset of H and let G be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r > 0$ and $x \in H$. Then, there exists a unique $z \in C$ such that

$$G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

Lemma 2.4.10. [15] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions $(A1)-(A4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \{z \in C : G(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\},$$

for all $x \in H$. Then the following statements hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for any $x, y \in C$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) $EP(G)$ is a closed convex subset of C;

(iv) $G(T_r) = EP(G)$.

Remark 2.4.11. For any $x \in H$ and $r > 0$, by Lemma 2.4.10 (i), there exists $u \in C$ such that

$$G(u,y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \forall y \in H. \tag{2.4.4}$$

Replacing x with $x - r\Psi x \in H$ in (2.4.4), we have

$$G(u,y) + \langle \Psi x, y - u\rangle + \frac{1}{r}\langle y - u, u - x\rangle \geq 0, \forall y \in H, \tag{2.4.5}$$

where $\Psi : H \to H$ is an inverse-strongly monotone mapping.

Lemma 2.4.12. [44] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions $(A1)-(A4)$ and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:

$$T_r(x) = \{z \in C : G(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \ \forall y \in C\},$$

for all $x \in H$. Then following conclusions hold:

(1) For each $x \in H, T_r(x) \neq \emptyset$;

(2) T_r is single-valued;

(3) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle;$$

(4) $F(T_r) = MEP(G, \varphi)$;

(5) $MEP(G, \varphi)$ is closed and convex.

We call such T_r the resolvent of f for $r > 0$. Using Lemmas 2.4.9 and 2.4.12, Takahashi, Takahashi and Toyoda [56] obtained the following lemma.

Lemma 2.4.13. [56] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $G : C \times C \to \mathbb{R}$ satisfy $(A1) - (A4)$. Let A_G be a set-valued mapping of H into itself defined by

$$A_G x = \begin{cases} \{z \in H : G(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \forall x \in C \\ \emptyset, & \forall x \notin C. \end{cases}$$

Then, $MEP(G) = A_G^{-1}0$ and A_G is maximal monotone operator with dom $A_G \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of G coincides with the resolvent of A_G, i.e.,

$$T_r x = (I + rA_G)^{-1}x.$$

Applying the idea of the proof in Lemma 2.4.13, we have the following results.

Lemma 2.4.14. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $G : C \times C \to \mathbb{R}$ satisfy $(A1)-(A4)$ and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. Let $A_{(G,\varphi)}$ be a set- valued mapping of H into itself defined by

$$A_{(G,\varphi)}x = \begin{cases} \{z \in H : G(x,y) + \varphi(y) - \varphi(x) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \forall x \in C \\ \emptyset, \quad \forall x \notin C. \end{cases} \tag{2.4.6}$$

Then, $MEP(G,\varphi) = A^{-1}_{(G,\varphi)}0$ and $A_{(G,\varphi)}$ is a maximal monotone operator with dom $A_{(G,\varphi)} \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r of G coincides with the resolvent of $A_{(G,\varphi)}$, i.e.,

$$T_r x = (I + rA_{(G,\varphi)})^{-1}x.$$

Proof. It is obvious that $MEP(G,\varphi) = A^{-1}_{(G,\varphi)}0$. In fact, we have that

$$\begin{aligned} z \in MEP(G,\varphi) &\Leftrightarrow G(z,y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C \\ &\Leftrightarrow G(z,y) + \varphi(y) - \varphi(z) \geq \langle y - z, 0\rangle, \quad \forall y \in C \\ &\Leftrightarrow 0 \in A_{(G,\varphi)}z \\ &\Leftrightarrow z \in A^{-1}_{(G,\varphi)}0. \end{aligned}$$

We show that $A_{(G,\varphi)}$ is monotone. Let $(x_1,z_1),(x_2,z_2) \in A_{(G,\varphi)}$ be given. Then, we have, for all $y \in C$,

$$G(x_1,y) + \varphi(y) - \varphi(x_1) \geq \langle y - x_1, z_1\rangle \quad \text{and} \quad G(x_2,y) + \varphi(y) - \varphi(x_2) \geq \langle y - x_2, z_2\rangle$$

and hence

$$G(x_1,x_2) + \varphi(x_2) - \varphi(x_1) \geq \langle x_2 - x_1, z_1\rangle \quad \text{and} \quad G(x_2,x_1) + \varphi(x_1) - \varphi(x_2) \geq \langle x_1 - x_2, z_2\rangle.$$

It follows from (A2) that

$$0 \geq G(x_1, x_2) + G(x_2, x_1) \geq \langle x_2 - x_1, z_1 \rangle + \langle x_1 - x_2, z_2 \rangle = -\langle x_1 - x_2, z_1 - z_2 \rangle.$$

This implies that $A_{(G,\varphi)}$ is monotone. We next prove that $A_{(G,\varphi)}$ is maximal monotone. To show that $A_{(G,\varphi)}$ is maximal monotone, it is sufficient to show from [50] that $R(I + rA_{(G,\varphi)}) = H$ for all $r > 0$, where $R(I + rA_{(G,\varphi)})$ is the range of $I + rA_{(G,\varphi)}$. Let $x \in H$ and $r > 0$. Then, from Lemma 2.4.12, there exists $z \in C$ such that

$$G(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

So, we have that

$$G(z, y) + \varphi(y) - \varphi(z) \geq \langle y - z, \frac{1}{r}(x - z) \rangle, \quad \forall y \in C.$$

By the definition of $A_{(G,\varphi)}$, we get

$$A_{(G,\varphi)}z \ni \frac{1}{r}(x - z)$$

and hence $x \in z + rA_{(G,\varphi)}z$.
Therefore, $H \subset R(I + rA_{(G,\varphi)})$ and $R(I + rA_{(G,\varphi)}) = H$. Also, $x \in z + rA_{(G,\varphi)}z$ implies that $T_r x = (I + rA_{(G,\varphi)})^{-1}x$ for all $x \in H$ and $r > 0$. □

2.5 CAT(0) spaces

In this section, we present the special metric space which has the geometry defined on it. We also introduce the concept of several types of convergence on it.

Definition 2.5.1. [8] A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that

(i) $c(0) = x, c(l) = y$;

(ii) $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$.

In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or metric) *segment* joining x and y.

When it is unique this geodesic segment is denoted by $[x, y]$.

Definition 2.5.2. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$.

Definition 2.5.3. A subset C of a CAT(0) space is convex if $[x, y] \subseteq C$ for all $x, y \in C$.

Definition 2.5.4. A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for the geodesic triangle $\overline{\triangle}(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for all $i, j \in 1, 2, 3$ (see Figure 1).

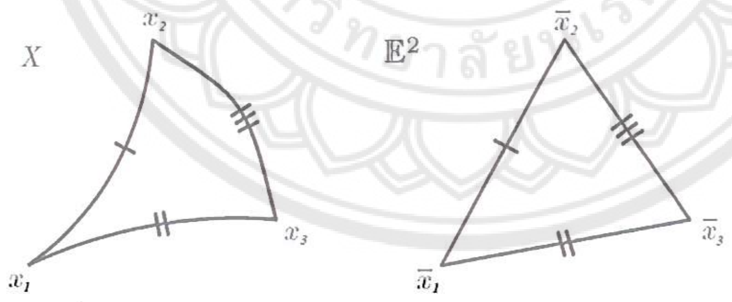

Figure 1 Comparison triangle

Definition 2.5.5. A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0) : Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \triangle$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})$$

(see Figure 2).

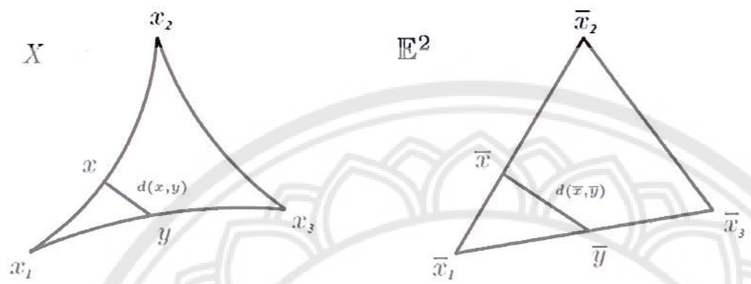

Figure 2 CAT(0) inequality

Definition 2.5.6. Let x, y_1, y_2 be the points in a CAT(0) space and y_0 be the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \qquad \text{(CN)}$$

This is the (CN) inequality of Bruhat and Tits [11]. In fact (cf. [8], p. 163), a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality.

It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces, R-trees (see [8]), Euclidean buildings (see [10]), the complex Hilbert ball with a hyperbolic metric (see [22]), and many others. Complete CAT(0) spaces are often called Hadamard spaces.

Next, we collect some useful lemmas in CAT(0) spaces.

Lemma 2.5.7. [8, Proposition 2.2] Let X be a CAT(0) space, $p, q, r, s \in X$ and $\lambda \in [0, 1]$. Then

$$d(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s) \leq \lambda d(p, r) + (1-\lambda)d(q, s).$$

Lemma 2.5.8. [18, Lemma 2.4] Let X be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0, 1]$. Then

$$d(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d(x, z) + (1-\lambda)d(y, z).$$

Lemma 2.5.9. [18, Lemma 2.5] Let X be a CAT(0) space, $x, y, z \in X$ and $\lambda \in [0, 1]$. Then

$$d^2(\lambda x \oplus (1-\lambda)y, z) \leq \lambda d^2(x, z) + (1-\lambda)d^2(y, z) - \lambda(1-\lambda)d^2(x, y).$$

We give the concept of Δ-convergence and collect some basic properties. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known from Proposition 7 of [17] that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point.

Definition 2.5.10. A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Uniqueness of asymptotic center implies that CAT(0) space X satisfies Opial's property, i.e., for given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$,

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Since it is not possible to formulate the concept of demiclosedness in a CAT(0) setting, as stated in linear spaces, let us formally say that "$I - T$ is demiclosed at zero"if the conditions, $\{x_n\} \subseteq C$ Δ- converges to x and $d(x_n, Tx_n) \to 0$ imply $x \in F(T)$.

Lemma 2.5.11. [29] Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.

Lemma 2.5.12. [16] If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.

Lemma 2.5.13. [16] If C is a closed convex subset of X and $T : C \to X$ is a asympototically nonexpansive mapping, then the conditions $\{x_n\}$ Δ-convergence to x and $d(x_n, Tx_n) \to 0$, and imply $x \in C$ and $Tx = x$.

Definition 2.5.14. [3] Let X be a CAT(0) space and $a, b, c, d \in X$. Then *quasi-linearization* is defined as a map $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right). \tag{2.5.1}$$

It is easily seen that $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ax}, \overrightarrow{cd} \rangle + \langle \overrightarrow{xb}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$ for all $a, b, c, d, x \in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d), \tag{2.5.2}$$

for all $a, b, c, d \in X$.

Having the notion of quasilinearization, Kakavandi and Amini [25] introduced the following notion of convergence.

Definition 2.5.15. A sequence $\{x_n\}$ in the complete CAT(0) space (X, d) w-converges to $x \in X$ if $\lim_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle = 0$, *i.e.*, $\lim_{n\to\infty}(d^2(x_n, x) - d^2(x_n, y) + d^2(x, y)) = 0$ for all $y \in X$.

Lemma 2.5.16. [58] Let X be a complete CAT(0) space. Then for all $u, x, y \in X$, the following inequality holds

$$d^2(x, u) \leq d^2(y, u) + 2\langle\overrightarrow{xy}, \overrightarrow{xu}\rangle.$$

Lemma 2.5.17. [58] Let X be a CAT(0) space. For any $t \in [0, 1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,

1. $\langle\overrightarrow{u_tx}, \overrightarrow{u_ty}\rangle \leq t\langle\overrightarrow{ux}, \overrightarrow{u_ty}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{u_ty}\rangle$;
2. $\langle\overrightarrow{u_tx}, \overrightarrow{uy}\rangle \leq t\langle\overrightarrow{ux}, \overrightarrow{uy}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{uy}\rangle$ and
   $\langle\overrightarrow{u_tx}, \overrightarrow{vy}\rangle \leq t\langle\overrightarrow{ux}, \overrightarrow{vy}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{vy}\rangle$.

It is obvious that convergence in the metric implies w-convergence, and it is easy to check that w-convergence implies Δ-convergence [25, Proposition 2.5], but it is showed in ([24, Example 4.7]) that the converse is not valid. However the following lemma shows another characterization of Δ-convergence as well as, more explicitly, a relation between w-convergence and Δ-convergence.

Lemma 2.5.18. [24, Theorem 2.6] Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if

$$\limsup_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0, \quad \text{for all } y \in X.$$

Theorem 2.5.19. [8] Let X be a complete CAT(0) space, and let C be a nonempty closed convex subset of X. Then,

(i) for every $x \in X$, there exists a unique point $y_0 \in C$, denoted by $P_C x$, such that $d(x, y_0) = d(x, C) := \inf_{y \in C} d(x, y)$;

(ii) if x' belongs to the geodesic segment $[x, y_0]$, then $P_C x' = P_C x$.

Theorem 2.5.20. [19] Let C be a nonempty convex subset of a complete CAT(0) space X, $x \in X$ and $u \in C$. Then

$$u = P_C x \quad \text{if and only if} \quad \langle \overrightarrow{yu}, \overrightarrow{ux} \rangle \geq 0, \quad \text{for all } y \in C.$$

CHAPTER III

ITERATIVE APPROXIMATION METHODS FOR GENERALIZED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

3.1 Iterative algorithms with perturbations for solving the systems of generalized equilibrium problems and the fixed point problems of two quasi-nonexpansive mappings

In this section, we present the iterative algorithms with perturbations for finding a common element of the set of solutions of the system of generalized equilibrium problems and the set of common fixed points of two quasi-nonexpansive mappings in a Hilbert space.

Theorem 3.1.1. Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \dots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and Ψ_i a μ_i-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be two quasi-nonexpansive mappings such that $I - T_j$ are demiclosed at zero with $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} GEP(G_i, \Psi_i)) \neq \emptyset$. Let the sequences$\{x_n\}$, $\{y_n\}$, and

$\{z_n\}$ be defined by

$$\begin{cases} x_1 \in H, \\ G_1(u_{n,1}, y) + \langle \Psi_1 x_n, y - u_{n,1} \rangle + \frac{1}{r_n} \langle y - u_{n,1}, u_{n,1} - x_n \rangle \geq 0, \ \ \forall y \in C, \\ G_2(u_{n,2}, y) + \langle \Psi_2 x_n, y - u_{n,2} \rangle + \frac{1}{r_n} \langle y - u_{n,2}, u_{n,2} - x_n \rangle \geq 0, \ \ \forall y \in C, \\ \cdots \\ \cdots \\ \cdots \\ G_k(u_{n,k}, y) + \langle \Psi_k x_n, y - u_{n,k} \rangle + \frac{1}{r_n} \langle y - u_{n,k}, u_{n,k} - x_n \rangle \geq 0, \ \ \forall y \in C, \\ \omega_n = \frac{1}{k} \sum_{i=1}^{k} u_{n,i}, \\ y_n = \gamma_n \omega_n + (1 - \gamma_n) T_1 \omega_n, \\ z_n = \beta_n y_n + (1 - \beta_n) T_2 \omega_n, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \ \ \forall n \in \mathbb{N}, \end{cases} \tag{3.1.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0, 1)$ and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Suppose the following conditions are satisfied.

- (C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;
- (C2) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$;
- (C3) $\liminf_{n\to\infty} \gamma_n(1 - \gamma_n) > 0$;
- (C4) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$.

Proof. We first have that for all $i = 1, 2, \ldots, k$, $I - r_n \Psi_i$ is a nonexpansive mapping. Indeed, for all $x, y \in C$, we obtain

$$\begin{aligned} \|(I - r_n \Psi_i)x - (I - r_n \Psi_i)y\|^2 &= \|(x - y) - r_n(\Psi_i x - \Psi_i y)\|^2 \\ &= \|x - y\|^2 - 2r_n \langle \Psi_i x - \Psi_i y, x - y \rangle \end{aligned}$$

$$
\begin{aligned}
& +r_n^2\|\Psi_i x - \Psi_i y\|^2 \\
\leq\ & \|x-y\|^2 - 2r_n\mu_i\|\Psi_i x - \Psi_i y\|^2 \\
& +r_n^2\|\Psi_i x - \Psi_i y\|^2 \\
=\ & \|x-y\|^2 - r_n(2\mu_i - r_n)\|\Psi_i x - \Psi_i y\|^2 \\
\leq\ & \|x-y\|^2.
\end{aligned}
$$

Thus $I - r_n\Psi_i$ is nonexpansive for each $i \in \{1, 2, \ldots, k\}$. Now, let $w \in \Omega$ be arbitrary. By (C4), $\{u_n\}$ is a bounded sequence, there exists $M > 0$ such that

$$\sup_{n\in\mathbb{N}} \|u_n - w\| \leq M.$$

For each $i = 1, 2, \ldots, k$ and $n \in \mathbb{N}$, we have from $u_{n,i} = T_{r_{n,i}}(x_n - r_n\Psi_i x_n)$ that

$$
\begin{aligned}
\|u_{n,i} - w\| &= \|T_{r_{n,i}}(x_n - r_n\Psi_i x_n) - T_{r_{n,i}}(w - r_n\Psi_i w)\| \\
&\leq \|(x_n - r_n\Psi_i x_n) - (w - r_n\Psi_i w)\| \\
&\leq \|x_n - w\|, \qquad (3.1.2)
\end{aligned}
$$

which gives also that

$$\|\omega_n - w\| \leq \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - w\| \leq \|x_n - w\| \ \ \forall w \in \Omega. \qquad (3.1.3)$$

Since T_1 is quasi-nonexpansive we have

$$
\begin{aligned}
\|y_n - w\| &= \|\gamma_n\omega_n + (1-\gamma_n)T_1\omega_n - w\| \\
&= \|\gamma_n(\omega_n - w) + (1-\gamma_n)(T_1\omega_n - w)\| \\
&\leq \gamma_n\|\omega_n - w\| + (1-\gamma_n)\|T_1\omega_n - w\| \\
&\leq \|\omega_n - w\|. \qquad (3.1.4)
\end{aligned}
$$

So, we have from (3.1.3) and (3.1.4) and the quasi-nonexpansiveness of T_2 that

$$\|x_{n+1} - w\| = \|\alpha_n(u_n - w) + (1-\alpha_n)(z_n - w)\|$$

$$
\begin{aligned}
&\leq \alpha_n\|u_n - w\| + (1-\alpha_n)\|z_n - w\| \\
&\leq \alpha_n\|u_n - w\| + (1-\alpha_n)\{\beta_n\|y_n - w\| + (1-\beta_n)\|T_2\omega_n - w\|\} \\
&\leq \alpha_n\|u_n - w\| + (1-\alpha_n)\{\beta_n\|\omega_n - w\| + (1-\beta_n)\|\omega_n - w\|\} \\
&\leq \alpha_n\|u_n - w\| + (1-\alpha_n)\|\omega_n - w\| \\
&\leq \alpha_n\|u_n - w\| + (1-\alpha_n)\|x_n - w\| \\
&\leq \max\{M, \|x_n - w\|\}.
\end{aligned}
$$

By Induction, we have that

$$\|x_n - w\| \leq \max\{\|x_1 - w\|, M\}, \ \forall n \in \mathbb{N}.$$

Thus we obtain that $\{\|x_n - w\|\}$ is bounded, so also $\{x_n\}, \{y_n\}, \{z_n\}, \{\omega_n\}, \{T_1\omega_n\}$ and $\{T_2\omega_n\}$ are bounded. Since Ω is closed and convex, we can take $x^* = P_\Omega u$. It follows that

$$
\begin{aligned}
\|y_n - x^*\|^2 &= \|\gamma_n(\omega_n - x^*) + (1-\gamma_n)(T_1\omega_n - x^*)\|^2 \\
&= \gamma_n\|\omega_n - x^*\|^2 + (1-\gamma_n)\|T_1\omega_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2 \\
&\leq \gamma_n\|\omega_n - x^*\|^2 + (1-\gamma_n)\|\omega_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2 \\
&= \|\omega_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2 \\
&\leq \|\omega_n - x^*\|^2. \qquad (3.1.5)
\end{aligned}
$$

From (3.1.5), we have

$$
\begin{aligned}
\|z_n - x^*\|^2 &= \|\beta_n(y_n - x^*) + (1-\beta_n)(T_2\omega_n - x^*)\|^2 \\
&= \beta_n\|y_n - x^*\|^2 + (1-\beta_n)\|T_2\omega_n - x^*\|^2 - \beta_n(1-\beta_n)\|y_n - T_2\omega_n\|^2 \\
&\leq \beta_n\|\omega_n - x^*\|^2 + (1-\beta_n)\|\omega_n - x^*\|^2 - \beta_n(1-\beta_n)\|y_n - T_2\omega_n\|^2 \\
&= \|\omega_n - x^*\|^2 - \beta_n(1-\beta_n)\|y_n - T_2\omega_n\|^2 \\
&\leq \|\omega_n - x^*\|^2. \qquad (3.1.6)
\end{aligned}
$$

Hence we have from (3.1.3), (3.1.5) and (3.1.6) that

$$\begin{aligned}
\|\omega_{n+1} - x^*\|^2 &\leq \|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n(u_n - x^*) + (1-\alpha_n)(z_n - x^*)\|^2 \\
&= \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|u_n - z_n\|^2 \\
&\leq \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2 \\
&= \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\{\beta_n\|y_n - x^*\|^2 + (1-\beta_n)\|T_2\omega_n - x^*\|^2 \\
&\quad -\beta_n(1-\beta_n)\|y_n - T_2\omega_n\|^2\} \\
&\leq \alpha_n\|u_n - x^*\|^2 + \beta_n\{\gamma_n\|\omega_n - x^*\|^2 + (1-\gamma_n)\|T_1\omega_n - x^*\|^2 \\
&\quad -\gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2\} + (1-\beta_n)\|T_2\omega_n - x^*\|^2 \\
&\quad -\beta_n(1-\beta_n)\|y_n - T_2\omega_n\|^2 \\
&\leq \alpha_n\|u_n - x^*\|^2 + \beta_n(\gamma_n\|\omega_n - x^*\|^2 + (1-\gamma_n)\|\omega_n - x^*\|^2 \\
&\quad -\gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2) + (1-\beta_n)\|\omega_n - x^*\|^2 \\
&\quad -\beta_n(1-\beta_n)\|y_n - T_2\omega_n\|^2 \\
&= \alpha_n\|u_n - x^*\|^2 + \|\omega_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2 \\
&\quad -\beta_n(1-\beta_n)\|y_n - T_2\omega_n\|^2. \qquad (3.1.7)
\end{aligned}$$

We also have that

$$\begin{aligned}
\gamma_n(1-\gamma_n)\|\omega_n - T_1\omega_n\|^2 &\leq \alpha_n\|u_n - x^*\|^2 + \|\omega_n - x^*\|^2 \\
&\quad -\|\omega_{n+1} - x^*\|^2, \qquad (3.1.8)
\end{aligned}$$

and

$$\begin{aligned}
\beta_n(1-\beta_n)\|y_n - T_2\omega_n\|^2 &\leq \alpha_n\|u_n - x^*\|^2 + \|\omega_n - x^*\|^2 \\
&\quad -\|\omega_{n+1} - x^*\|^2. \qquad (3.1.9)
\end{aligned}$$

Furthemore, we have from $y_n = \gamma_n\omega_n + (1-\gamma_n)T_1\omega_n$ that

$$\|\omega_n - T_2\omega_n\| \leq \|\omega_n - y_n\| + \|y_n - T_2\omega_n\|$$

$$
\begin{aligned}
&= \|\omega_n - \gamma_n\omega_n - (1-\gamma_n)T_1\omega_n\| + \|y_n - T_2x_n\| \\
&= (1-\gamma_n)\|\omega_n - T_1\omega_n\| + \|y_n - T_2\omega_n\|. \qquad (3.1.10)
\end{aligned}
$$

On the other hand, since $x_{n+1} - x^* = \alpha_n(u_n - x^*) + (1-\alpha_n)(z_n - x^*)$, we have

$$
\begin{aligned}
\|\omega_{n+1} - x^*\|^2 &\leq \|x_{n+1} - x^*\|^2 \\
&\leq (1-\alpha_n)\|z_n - x^*\|^2 + 2\alpha_n\langle u_n - x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)\|\omega_n - x^*\|^2 + 2\alpha_n\langle u_n - x^*, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n)\|\omega_n - x^*\|^2 + 2\alpha_n\langle u_n - u, x_{n+1} - x^*\rangle \\
&\quad +2\alpha_n\langle u - x^*, x_{n+1} - x^*\rangle \\
&= (1-\alpha_n)\|\omega_n - x^*\|^2 + 2\alpha_n\langle u_n - u, x_{n+1} - x^*\rangle \\
&\quad +2\alpha_n\langle u - x^*, x_{n+1} - \omega_n\rangle + 2\alpha_n\langle u - x^*, \omega_n - x^*\rangle. \qquad (3.1.11)
\end{aligned}
$$

We also have that

$$
\begin{aligned}
\|x_{n+1} - \omega_n\| &\leq \|x_{n+1} - y_n\| + \|y_n - \omega_n\| \\
&= \|\alpha_n(u_n - y_n) + (1-\alpha_n)(z_n - y_n)\| + \|(1-\gamma_n)(\omega_n - T_1\omega_n)\| \\
&\leq \alpha_n\|u_n - y_n\| + (1-\alpha_n)\|\beta_n y_n + (1-\beta_n)T_2\omega_n - y_n\| \\
&\quad +(1-\gamma_n)\|\omega_n - T_1\omega_n\| \\
&= \alpha_n\|u_n - y_n\| + (1-\alpha_n)(1-\beta_n)\|y_n - T_2\omega_n\| \\
&\quad +(1-\gamma_n)\|\omega_n - T_1\omega_n\|. \qquad (3.1.12)
\end{aligned}
$$

Moreover, for any $i \in \{1, 2, \ldots, k\}$, we have from $u_{n,i} = T_{r_{n,i}}(x_n - r_n\Psi_i x_n)$ that

$$
\begin{aligned}
\|u_{n,i} - x^*\|^2 &\leq \|(x_n - x^*) - r_n(\Psi_i x_n - \Psi_i x^*)\|^2 \\
&= \|x_n - x^*\|^2 - 2r_n\langle x_n - x^*, \Psi_i x_n - \Psi_i x^*\rangle + r_n^2\|\Psi_i x_n - \Psi_i x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2.
\end{aligned}
$$

It follows that

$$\begin{aligned}\|\omega_n - x^*\|^2 &= \|\sum_{i=1}^{k}\frac{1}{k}(u_{n,i} - x^*)\|^2\\ &\leq \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x^*\|^2\\ &\leq \|x_n - x^*\|^2 - \frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2. \qquad (3.1.13)\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(u_n - x^*) + (1-\alpha_n)(z_n - x^*)\|^2\\ &\leq \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\|\omega_n - x^*\|^2\\ &\leq \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2\\ &\quad -(1-\alpha_n)\frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2,\end{aligned}$$

and hence

$$\begin{aligned}&(1-\alpha_n)\frac{1}{k}\sum_{i=1}^{k} r_n(2\mu_i - r_n)\|\Psi_i x_n - \Psi_i x^*\|^2\\ \leq\ &\alpha_n\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \qquad (3.1.14)\end{aligned}$$

Furthermore, we have from Lemma 2.4.10 that for any $i \in 1, 2, \ldots, k$, we have

$$\begin{aligned}\|u_{n,i} - x^*\|^2 &\leq \langle (x_n - r_n\Psi_i x_n) - (x^* - r_n\Psi_i x^*), u_{n,i} - x^*\rangle\\ &= \frac{1}{2}\{\|(x_n - r_n\Psi_i x_n) - (x^* - r_n\Psi_i x^*)\|^2 + \|u_{n,i} - x^*\|^2\\ &\quad -\|(x_n - r_n\Psi_i x_n) - (x^* - r_n\Psi_i x^*) - (u_{n,i} - x^*)\|^2\}\\ &\leq \frac{1}{2}\{\|x_n - x^*\|^2 + \|u_{n,i} - x^*\|^2 - \|(x_n - u_{n,i}) - r_n(\Psi_i x_n - \Psi_i x^*)\|^2\}\\ &= \frac{1}{2}\{\|x_n - x^*\|^2 + \|u_{n,i} - x^*\|^2 - \|x_n - u_{n,i}\|^2 - r_n^2\|\Psi_i x_n - \Psi_i x^*\|^2\\ &\quad +2r_n\langle x_n - u_{n,i}, \Psi_i x_n - \Psi_i x^*\rangle\}.\end{aligned}$$

This implies that

$$\begin{aligned}\|u_{n,i} - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - u_{n,i}\|^2 \\ &\quad +2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\|. \qquad (3.1.15)\end{aligned}$$

Then we have from (3.1.15) that

$$\begin{aligned}\|\omega_n - x^*\|^2 &\leq \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 - \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x_n\|^2 \\ &\quad +\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\|. \qquad (3.1.16)\end{aligned}$$

Hence we have from (3.1.16) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\|\omega_n - x^*\|^2 \\ &\leq \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\left(\|x_n - x^*\|^2 - \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x_n\|^2\right) \\ &\quad +(1-\alpha_n)\left(\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\|\right).\end{aligned}$$

It follows that

$$\begin{aligned}(1-\alpha_n)\frac{1}{k}\sum_{i=1}^{k}\|u_{n,i} - x_n\|^2 &\leq \alpha_n\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\ &\quad +(1-\alpha_n)\left(\frac{1}{k}\sum_{i=1}^{k}2r_n\|x_n - u_{n,i}\|\|\Psi_i x_n - \Psi_i x^*\|\right). \qquad (3.1.17)\end{aligned}$$

Next, we shall consider the following two cases.

Case A : Put $\Gamma_n = \|\omega_n - x^*\|^2$ for all $n \in \mathbb{N}$. Suppose that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \in \mathbb{N}$. In this case $\lim_{n\to\infty} \Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. By (C1), (C3) and (3.1.8), we have

$$\lim_{n\to\infty} \|\omega_n - T_1\omega_n\| = 0. \tag{3.1.18}$$

Similarly by (C1), (C2) and (3.1.9), we also have

$$\lim_{n\to\infty} \|y_n - T_2\omega_n\| = 0. \tag{3.1.19}$$

So, we have from (3.1.10), (3.1.18) and (3.1.19) that

$$\lim_{n\to\infty} \|\omega_n - T_2\omega_n\| = 0. \tag{3.1.20}$$

Since $\lim_{n\to\infty} \|\omega_n - x^*\|$ exists, we have from (3.1.7) and (3.1.18)

$$\lim_{n\to\infty} \|\omega_n - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|. \tag{3.1.21}$$

We also have from (C1), (3.1.12), (3.1.18) and (3.1.19) that

$$\lim_{n\to\infty} \|x_{n+1} - \omega_n\| = 0. \tag{3.1.22}$$

Since $\lim_{n\to\infty} \|x_n - x^*\|$ exists we have from (C1) and (3.1.14) that

$$\lim_{n\to\infty} \|\Psi_i x_n - \Psi_i x^*\| = 0, \quad \forall i = 1, 2, \ldots, k. \tag{3.1.23}$$

This together with (3.1.17) and the existence of $\lim_{n\to\infty} \|x_n - x^*\|$ implies that

$$\lim_{n\to\infty} \|u_{n,i} - x_n\| = 0, \quad \forall i = 1, 2, \ldots, k, \tag{3.1.24}$$

which gives that

$$\|\omega_n - x_n\| \leq \frac{1}{k}\sum_{i=1}^{k} \|u_{n,i} - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.1.25}$$

So, from (3.1.22), $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Furthermore, we have from (3.1.25) that

$$\|\omega_{n+1} - \omega_n\| \le \|\omega_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - \omega_n\| \to 0 \text{ as } n \to \infty;$$

that is

$$\lim_{n\to\infty} \|\omega_{n+1} - \omega_n\| = 0. \tag{3.1.26}$$

Now, since $\{\omega_n\}$ is a bounded sequence, there exists a subsequence $\{\omega_{n_j}\}$ of $\{\omega_n\}$ such that

$$\limsup_{n\to\infty} \langle u - x^*, \omega_n - x^* \rangle = \lim_{j\to\infty} \langle u - x^*, \omega_{n_j} - x^* \rangle. \tag{3.1.27}$$

Without loss of generality, we may assume that $\omega_{n_j} \rightharpoonup v$. Since T_1 is demiclosed at zero and by (3.1.18), we conclude that $v \in F(T_1)$. Similarly, since T_2 is demiclosed at zero and by (3.1.20), we have $v \in F(T_2)$. Therefore, we get that

$$v \in F(T_1) \cap F(T_2). \tag{3.1.28}$$

Next, we show that $v \in \cap_{i=1}^{k} GEP(G_i, \Psi_i)$. For each $i \in \{1, 2, \ldots, k\}$, since $u_{n,i} = T_{r_{n,i}}(x_n - r_n \Psi_i x_n)$, we have

$$G_i(u_{n,i}, y) + \langle \Psi_i x_n, y - u_{n,i} \rangle + \frac{1}{r_n} \langle y - u_{n,i}, u_{n,i} - x_n \rangle \ge 0, \quad \forall y \in C.$$

From $(A2)$, we also have

$$\langle \Psi_i x_n, y - u_{n,i} \rangle + \frac{1}{r_n} \langle y - u_{n,i}, u_{n,i} - x_n \rangle \ge G_i(y, u_{n,i}).$$

Replacing n by n_j, we have

$$\langle \Psi_i x_{n_j}, y - u_{n_j,i} \rangle + \langle y - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j}} \rangle \ge G_i(y, u_{n_j,i}). \tag{3.1.29}$$

Put $y_t = ty + (1-t)v$ for all $t \in (0,1]$ and $y \in C$. Since $v \in C$, then $y_t \in C$ and

$$\begin{aligned}
\langle y_t - u_{n_j,i}, \Psi_i y_t \rangle &\geq \langle y_t - u_{n_j,i}, \Psi_i y_t \rangle - \langle y_t - u_{n_j,i}, \Psi_i x_{n_j} \rangle \\
&\quad - \langle y_t - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j}} \rangle + G_i(y_t, u_{n_j,i}) \\
&= \langle y_t - u_{n_j,i}, \Psi_i y_t - \Psi_i u_{n_j,i} \rangle + \langle y_t - u_{n_j,i}, \Psi_i u_{n_j,i} - \Psi_i x_{n_j} \rangle \\
&\quad - \langle y_t - u_{n_j,i}, \frac{u_{n_j,i} - x_{n_j}}{r_{n_j}} \rangle + G_i(y_t, u_{n_j,i}). \qquad (3.1.30)
\end{aligned}$$

Since $\|u_{n_j,i} - x_{n_j}\| \to 0$ as $j \to \infty$, we obtain that $\|\Psi_i u_{n_j,i} - \Psi_i x_{n_j}\| \to 0$ as $j \to \infty$. Furthermore, by the monotonicity of Ψ_i, we obtain that

$$\langle y_t - u_{n_j,i}, \Psi_i y_t - \Psi_i u_{n_j,i} \rangle \geq 0.$$

Taking $j \to \infty$ in (3.1.30), we have from (A4) that

$$\langle y_t - v, \Psi_i y_t \rangle \geq G_i(y_t, v). \qquad (3.1.31)$$

Now, from $(A1)$, $(A4)$ and (3.1.31), we also have

$$\begin{aligned}
0 = G_i(y_t, y_t) &\leq tG_i(y_t, y) + (1-t)G_i(y_t, v) \\
&\leq tG_i(y_t, y) + (1-t)\langle y_t - v, \Psi_i y_t \rangle \\
&= tG_i(y_t, y) + (1-t)t\langle y - v, \Psi_i y_t \rangle,
\end{aligned}$$

which yields that

$$G_i(y_t, y) + (1-t)\langle y - v, \Psi_i y_t \rangle \geq 0.$$

Taking $t \to 0$, we have, for each $y \in C$

$$G_i(v, y) + \langle y - v, \Psi_i v \rangle \geq 0, \quad \text{for all } i \in \{1, 2, \ldots, k\}.$$

This shows $v \in GEP(G_i, \Psi_i)$, for all $i = 1, 2, \ldots, k$. Then, $v \in \cap_{i=1}^{k} GEP(G_i, \Psi_i)$. Hence we have $v \in F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} GEP(G_i, \Psi_i)) := \Omega$. So, we have from (3.1.27) that

$$\limsup_{n\to\infty} \langle u - x^*, \omega_n - x^* \rangle = \langle u - x^*, v - x^* \rangle \leq 0. \tag{3.1.32}$$

By (C1), (C4),(3.1.11), (3.1.22), (3.1.32) and Lemma 2.3.22, we obtain that $\lim_{n\to\infty} \|\omega_n - x^*\| = 0$. Hence we have from (3.1.21) that $\{x_n\}$ converges to x^*, where $x^* = P_\Omega u$.

Case B: Assume that there exists a subsequence $\{\Gamma_{n_i}\}_{i\geq 0}$ of $\{\Gamma_n\}_{n\geq 0}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, it follows from Lemma 2.3.21 that there exists a subsequence $\{\Gamma_{\tau(n)}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{\tau(n)+1} > \Gamma_{\tau(n)}$, where $\tau : \mathbb{N} \to \mathbb{N}$ is defined by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}, \text{ for all } n \in \mathbb{N}.$$

So, from (3.1.8), that

$$\|\omega_{\tau(n)+1} - x^*\|^2 - \|\omega_{\tau(n)} - x^*\|^2 + \gamma_{\tau(n)}(1 - \gamma_{\tau(n)})\|\omega_{\tau(n)} - T_1\omega_{\tau(n)}\|^2 \leq \alpha_{\tau(n)}\|u_{\tau(n)} - x^*\|^2.$$

Since $\|\omega_{\tau(n)} - x^*\|^2 := \Gamma_{\tau(n)} < \Gamma_{\tau(n)+1} := \|\omega_{\tau(n)+1} - x^*\|^2$, we have

$$\gamma_{\tau(n)}(1 - \gamma_{\tau(n)})\|\omega_{\tau(n)} - T_1\omega_{\tau(n)}\|^2 \leq \alpha_{\tau(n)}\|u_{\tau(n)} - x^*\|^2. \tag{3.1.33}$$

By (C1) and (C3), we have

$$\lim_{n\to\infty} \|\omega_{\tau(n)} - T_1\omega_{\tau(n)}\| = 0. \tag{3.1.34}$$

By (3.1.11), we have

$$\|\omega_{\tau(n)+1} - x^*\|^2 \leq (1 - \alpha_{\tau(n)})\|\omega_{\tau(n)} - x^*\|^2$$

$$+2\alpha_{\tau(n)}\langle u_{\tau(n)} - x^*, x_{\tau(n)+1} - x^*\rangle. \tag{3.1.35}$$

Now, in view of $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$, we see that

$$\begin{aligned} \|\omega_{\tau(n)} - x^*\|^2 &\leq 2\langle u_{\tau(n)} - x^*, x_{\tau(n)+1} - x^*\rangle \\ &= 2\langle u_{\tau(n)} - u, x_{\tau(n)+1} - x^*\rangle + 2\langle u - x^*, x_{\tau(n)+1} - \omega_{\tau(n)}\rangle \\ &\quad +2\langle u - x^*, \omega_{\tau(n)} - x^*\rangle. \end{aligned} \tag{3.1.36}$$

Furthermore, we also have from (3.1.9) that

$$\begin{aligned} \beta_{\tau(n)}(1-\beta_{\tau(n)})\|y_{\tau(n)} - T_2\omega_{\tau(n)}\|^2 &\leq \alpha_{\tau(n)}\|u_{\tau(n)} - x^*\|^2 + \|\omega_{\tau(n)} - x^*\|^2 \\ &\quad -\|\omega_{\tau(n)+1} - x^*\|^2 \\ &\leq \alpha_{\tau(n)}\|u_{\tau(n)} - x^*\|^2. \end{aligned} \tag{3.1.37}$$

Applying (C1) and (C2) to the last inequality, we get that

$$\lim_{n\to\infty} \|y_{\tau(n)} - T_2\omega_{\tau(n)}\| = 0. \tag{3.1.38}$$

By (C1), (3.1.12), (3.1.34) and (3.1.38), we have

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - \omega_{\tau(n)}\| = 0. \tag{3.1.39}$$

By (3.1.25), we have

$$\lim_{n\to\infty} \|\omega_{\tau(n)+1} - x_{\tau(n)+1}\| = 0. \tag{3.1.40}$$

It follows from (3.1.39) and (3.1.40) that

$$\lim_{n\to\infty} \|\omega_{\tau(n)+1} - \omega_{\tau(n)}\| = 0. \tag{3.1.41}$$

Since $\{\omega_{\tau(n)}\}$ is a bounded sequence, there exists a subsequence $\{\omega_{\tau(n_j)}\}$ such that

$$\limsup_{n\to\infty}\langle u - x^*, \omega_{\tau(n)} - x^*\rangle = \lim_{j\to\infty}\langle u - x^*, \omega_{\tau(n_j)} - x^*\rangle. \tag{3.1.42}$$

Following the same argument as the proof of Case A for $\{\omega_{\tau(n_j)}\}$, we have that

$$\limsup_{n\to\infty}\langle u - x^*, \omega_{\tau(n)} - x^*\rangle \leq 0. \tag{3.1.43}$$

Using (C4), (3.1.36), (3.1.39) and (3.1.43), we have that

$$\lim_{n\to\infty}\|\omega_{\tau(n)} - x^*\| = 0. \tag{3.1.44}$$

By (3.1.41) and (3.1.44), we have that

$$\lim_{n\to\infty}\|\omega_{\tau(n)+1} - x^*\| = 0. \tag{3.1.45}$$

By Lemma 2.3.21 (ii), we get $\lim_{n\to\infty}\Gamma_n = 0$; that is $\lim_{n\to\infty}\|\omega_n - x^*\| = 0$. We observe that

$$\|x_{n+1} - x^*\|^2 \leq \alpha_n\|u_n - x^*\|^2 + (1-\alpha_n)\|\omega_n - x^*\|^2.$$

Applying (C1), (C4) and $\lim_{n\to\infty}\|\omega_n - x^*\|^2 = 0$, we have immediately

$$\lim_{n\to\infty}\|x_n - x^*\| = 0;$$

that is $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$. □

Setting $\Psi_i \equiv 0$ for all $i = 1, 2, \ldots, k$ in Theorem 3.1.1, we obtain the following result.

Corollary 3.1.2. Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. For each $j = 1, 2$, let $T_j : C \to H$ be two quasi-nonexpansive mappings such that $I - T_j$ are demiclosed at zero with $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} EP(G_i)) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by

$$\begin{cases} x_1 \in H, \\ G_1(u_{n,1}, y) + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n \rangle \geq 0, \ \forall y \in C, \\ G_2(u_{n,2}, y) + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n \rangle \geq 0, \ \forall y \in C, \\ \cdots \\ \cdots \\ \cdots \\ G_k(u_{n,k}, y) + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n \rangle \geq 0, \ \forall y \in C, \\ \omega_n = \frac{1}{k}\sum_{i=1}^{k} u_{n,i}, \\ y_n = \gamma_n \omega_n + (1 - \gamma_n) T_1 \omega_n, \\ z_n = \beta_n y_n + (1 - \beta_n) T_2 \omega_n, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0, 1)$ and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Suppose the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$;

(C3) $\liminf_{n\to\infty} \gamma_n(1 - \gamma_n) > 0$;

(C4) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$.

In the next results, using Theorem 3.1.1, we have new strong convergence theorems for two nonexpansive mappings in a Hilbert space.

Corollary 3.1.3. Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and Ψ_i a μ_i-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be two nonexpansive mappings such that $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} GEP(G_i, \Psi_i)) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by (3.1.1), where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in (0, 1) and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Suppose the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$;

(C3) $\liminf_{n\to\infty} \gamma_n(1 - \gamma_n) > 0$;

(C4) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$.

In this section, we used super hybrid in proved strong convergence. Setting $S_j := \frac{1}{1+\gamma_j} T_j + \frac{\gamma_j}{1+\gamma_j} I$ in Theorem 3.1.1, where T_j is a super hybrid mapping and γ_j is a real number, we obtain the following result.

Theorem 3.1.4. Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and Ψ_i a μ_i-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be $(\alpha_j, \beta_j, \gamma_j)$-super hybrid mappings such that $\Omega := F(T_1) \cap F(T_2) \cap$

$(\cap_{i=1}^{k} GEP(G_i, \Psi_i)) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by

$$\begin{cases} x_1 \in H, \\ G_1(u_{n,1}, y) + \langle \Psi_1 x_n, y - u_{n,1} \rangle + \frac{1}{r_n} \langle y - u_{n,1}, u_{n,1} - x_n \rangle \geq 0, \ \ \forall y \in C, \\ G_2(u_{n,2}, y) + \langle \Psi_2 x_n, y - u_{n,2} \rangle + \frac{1}{r_n} \langle y - u_{n,2}, u_{n,2} - x_n \rangle \geq 0, \ \ \forall y \in C, \\ \cdots \\ \cdots \\ \cdots \\ G_k(u_{n,k}, y) + \langle \Psi_k x_n, y - u_{n,k} \rangle + \frac{1}{r_n} \langle y - u_{n,k}, u_{n,k} - x_n \rangle \geq 0, \ \ \forall y \in C, \\ \omega_n = \frac{1}{k} \sum_{i=1}^{k} u_{n,i}, \\ y_n = \gamma_n \omega_n + (1 - \gamma_n) \left(\frac{1}{1+\gamma_1} T_1 \omega_n + \frac{\gamma_1}{1+\gamma_1} \omega_n \right), \\ z_n = \beta_n y_n + (1 - \beta_n) \left(\frac{1}{1+\gamma_2} T_2 \omega_n + \frac{\gamma_2}{1+\gamma_2} \omega_n \right), \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \ \ \forall n \in \mathbb{N}, \end{cases} \tag{3.1.46}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in (0, 1) and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Suppose the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$;

(C3) $\liminf_{n\to\infty} \gamma_n(1 - \gamma_n) > 0$;

(C4) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$.

Proof. For each $j = 1, 2$, setting

$$S_j = \frac{1}{1+\gamma_j} T_j + \frac{\gamma_j}{1+\gamma_j} I,$$

we have from Lemma 2.3.19 that each S_j is a generalized hybrid mapping and $F(S_j) = F(T_j)$. Since $F(S_j) \neq \emptyset$, we have that each S_j is quasi-nonexpansive.

Following the proof of Theorem 3.1.1 and applying Lemma 2.3.20, we have the desired result. This completes the proof. $\square$

Setting $\Psi \equiv 0$ in Theorem 3.1.4, we obtains the following result.

Corollary 3.1.5. Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. For each $j = 1, 2$, let $T_j : C \to H$ be $(\alpha_j, \beta_j, \gamma_j)$-super hybrid mappings such that $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} EP(G_i)) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by

$$\begin{cases} x_1 \in H, \\ G_1(u_{n,1}, y) + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n\rangle \geq 0, \ \ \forall y \in C, \\ G_2(u_{n,2}, y) + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n\rangle \geq 0, \ \ \forall y \in C, \\ \cdots \\ \cdots \\ \cdots \\ G_k(u_{n,k}, y) + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n\rangle \geq 0, \ \forall y \in C, \\ \omega_n = \frac{1}{k}\sum_{i=1}^{k} u_{n,i}, \\ y_n = \gamma_n \omega_n + (1-\gamma_n)\left(\frac{1}{1+\gamma_1}T_1\omega_n + \frac{\gamma_1}{1+\gamma_1}\omega_n\right), \\ z_n = \beta_n y_n + (1-\beta_n)\left(\frac{1}{1+\gamma_2}T_2\omega_n + \frac{\gamma_2}{1+\gamma_2}\omega_n\right), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n) z_n, \ \ \forall n \in \mathbb{N}, \end{cases} \tag{3.1.47}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in (0, 1) and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Suppose the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

(C3) $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$;

(C4) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$.

In Corollary 3.1.5, put $G_i(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$. Then we have that $u_{n,i} = x_n$ for all $i = 1, 2, \ldots, k$, which gives that $\omega_n = \frac{1}{k}\sum_{i=1}^{k} u_{n,i} = x_n$. Thus we obtain the following results from Corollary 3.1.5.

Corollary 3.1.6. Let C be a nonempty closed convex subset of a Hilbert space H. For each $j = 1, 2$, let $T_j : C \to H$ be $(\alpha_j, \beta_j, \gamma_j)$-super hybrid mappings such that $F(T_1) \cap F(T_2) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by

$$\begin{cases} x_1 \in H, \\ y_n = \gamma_n x_n + (1 - \gamma_n)\left(\frac{1}{1+\gamma_1} T_1 x_n + \frac{\gamma_1}{1+\gamma_1} x_n\right), \\ z_n = \beta_n y_n + (1 - \beta_n)\left(\frac{1}{1+\gamma_2} T_2 x_n + \frac{\gamma_2}{1+\gamma_2} x_n\right), \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \ \forall n \in \mathbb{N}, \end{cases} \tag{3.1.48}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in (0, 1) and $\{u_n\} \subset H$ is a sequence. Suppose the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$;

(C3) $\liminf_{n\to\infty} \gamma_n(1 - \gamma_n) > 0$;

(C4) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_{F(T_1)\cap F(T_2)} u$.

In Corollary 3.1.6, put $T_1 = I$, the identity mapping, and $T_2 := T$, an (α, β, γ)-super hybrid mapping. Thus we obtain the following results.

Corollary 3.1.7. Let C be a nonempty closed convex subset of a Hilbert space H. Let T be an (α, β, γ)-super hybrid mapping such that $F(T) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by

$$\begin{cases} x_1 \in H, \\ z_n = \beta_n x_n + (1 - \beta_n)\left(\frac{1}{1+\gamma} T x_n + \frac{\gamma}{1+\gamma} x_n\right), \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \;\; \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0, 1) and $\{u_n\} \subset H$ is a sequence. Suppose the following conditions are satisfied.

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$;

(C3) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_{F(T)} u$.

3.2 A general iterative method for two maximal monotone operators and 2-generalized hybrid mappings in Hilbert spaces

In this section, we are a position to propose the new general iterative sequence for 2-generalized hybrid mappings and establish the strong convergence theorem for the proposed sequence.

Theorem 3.2.1. Let H be a real Hilbert space and let C a nonempty, closed and convex subset of H. Let $\alpha > 0$ and A an α-inverse-strongly monotone mapping of C into H. Let the set-valued maps $B : D(B) \subset C \to 2^H$ and $F : D(F) \subset C \to 2^H$ be maximal monotone. Let $J_\lambda = (I + \lambda B)^{-1}$ and $T_r = (I + rF)^{-1}$ be the resolvent of B for $\lambda > 0$ and F for $r > 0$, respectively. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $T : C \to C$ be a 2-generalized hybrid mapping such that $\Omega := F(T) \cap (A + B)^{-1}0 \cap F^{-1}0 \neq \emptyset$. Take $\mu, \gamma \in \mathbb{R}$ as follows:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\bar{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let the sequence $\{x_n\} \subset H$ be generated by

$$\begin{cases} x_1 = x \in H, \text{arbitrarily}, \\ z_n = J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n, \\ y_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k z_n, \\ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)y_n, \quad \forall\, n = 1, 2, \dots, \end{cases} \tag{3.2.1}$$

where the sequences $\{\alpha_n\}, \{\lambda_n\}$ and $\{r_n\}$ satisfy the following restrictions :

(i) $\{\alpha_n\} \subset [0,1]$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) there exist constants a and b such that $0 < a \leq \lambda_n \leq b < 2\alpha$ for all $n \in \mathbb{N}$;

(iii) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to a point p_0 of Ω, where p_0 is a unique fixed point of $P_\Omega(I - V + \gamma g)$. This point $p_0 \in \Omega$ is also a unique solution of the hierarchical variational inequality

$$\langle (V - \gamma g)p_0, q - p_0 \rangle \geq 0, \forall q \in \Omega. \tag{3.2.2}$$

Proof. First we prove that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \Omega$. Let $p \in \Omega$, we have that $p = J_{\lambda_n}(I - \lambda_n A)p$ and $p = T_{r_n}p$. Putting $u_n = T_{r_n}x_n$, we have that

$$\begin{aligned} \|z_n - p\|^2 &= \|J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n - J_{\lambda_n}(I - \lambda_n A)p\|^2 \\ &\leq \|(T_{r_n}x_n - T_{r_n}p) - \lambda_n(AT_{r_n}x_n - AT_{r_n}p)\|^2 \\ &= \|T_{r_n}x_n - T_{r_n}p\|^2 - 2\lambda_n\langle u_n - p, Au_n - Ap\rangle + \lambda_n^2\|Au_n - Ap\|^2 \end{aligned}$$

$$\begin{aligned} &\leq \|u_n - p\|^2 - 2\lambda_n\alpha\|Au_n - Ap\|^2 + \lambda_n^2\|Au_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 - \lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.2.3}$$

This together with quasi-nonexpansiveness of T implies that

$$\begin{aligned} \|y_n - p\| &= \left\|\frac{1}{n}\sum_{k=0}^{n-1} T^k z_n - p\right\| \\ &\leq \frac{1}{n}\sum_{k=0}^{n-1}\|T^k z_n - p\| \\ &\leq \frac{1}{n}\sum_{k=0}^{n-1}\|z_n - p\| \\ &= \|z_n - p\| \leq \|x_n - p\|. \end{aligned} \tag{3.2.4}$$

Therefore, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\gamma g(x_n) - Vp) + (I - \alpha_n V)y_n - (I - \alpha_n V)p\| \\ &\leq \alpha_n\|\gamma g(x_n) - Vp\| + \|(I - \alpha_n V)y_n - (I - \alpha_n V)p\| \\ &\leq \alpha_n\gamma k\|x_n - p\| + \alpha_n\|\gamma g(p) - Vp\| \\ &\quad + \|(I - \alpha_n V)y_n - (I - \alpha_n V)p\|. \end{aligned} \tag{3.2.5}$$

Putting $\tau = \bar{\gamma} - \frac{L^2\mu}{2}$, we can calculate the following,

$$\begin{aligned} \|(I - \alpha_n V)y_n - (I - \alpha_n V)p)\|^2 &= \|(y_n - p) - \alpha_n(Vy_n - Vp)\|^2 \\ &= \|y_n - p\|^2 - 2\alpha_n\langle y_n - p, Vy_n - Vp\rangle \\ &\quad + \alpha_n^2\|Vy_n - Vp\|^2 \\ &\leq \|y_n - p\|^2 - 2\alpha_n\bar{\gamma}\|y_n - p\|^2 + \alpha_n^2 L^2\|y_n - p\|^2 \\ &= (1 - 2\alpha_n\bar{\gamma} + \alpha_n^2 L^2)\|y_n - p\|^2 \\ &= (1 - 2\alpha_n\tau - \alpha_n L^2\mu + \alpha_n^2 L^2)\|y_n - p\|^2 \\ &\leq (1 - 2\alpha_n\tau - \alpha_n(L^2\mu - \alpha_n L^2) \end{aligned}$$

$$
\begin{aligned}
&+\alpha_n^2\tau^2)\|y_n - p\|^2 \\
&\leq \ (1 - 2\alpha_n\tau + \alpha_n^2\tau^2)\|y_n - p\|^2 \\
&= \ (1 - \alpha_n\tau)^2\|y_n - p\|^2. \qquad (3.2.6)
\end{aligned}
$$

Since $1 - \alpha_n\tau > 0$, we obtain that

$$\|(I - \alpha_n V)y_n - (I - \alpha_n V)p)\| \ \leq \ (1 - \alpha_n\tau)\|y_n - p\|.$$

Therefore, by (3.2.5), we have

$$
\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n\gamma k\|x_n - p\| + \alpha_n\|\gamma g(p) - Vp\| + (1 - \alpha_n\tau)\|y_n - p\| \\
&\leq \alpha_n\gamma k\|x_n - p\| + \alpha_n\|\gamma g(p) - Vp\| + (1 - \alpha_n\tau)\|x_n - p\| \\
&= (1 - \alpha_n(\tau - \gamma k))\|x_n - p\| + \alpha_n\|\gamma g(p) - Vp\| \\
&= (1 - \alpha_n(\tau - \gamma k))\|x_n - p\| + \alpha_n(\tau - \gamma k)\frac{\|\gamma g(p) - Vp\|}{\tau - \gamma k} \\
&\leq \max\left\{\|x_n - p\|, \frac{\|\gamma g(p) - Vp\|}{\tau - \gamma k}\right\}, \text{ for all } n \in \mathbb{N},
\end{aligned}
$$

which arrives that the sequence $\{\|x_n - p\|\}$ is bounded, so are $\{x_n\}$, $\{y_n\}$, $\{Vy_n\}$, $\{g(x_n)\}$ and $\{T^k z_n\}$. Using Lemma 2.3.34, we can take a unique $p_0 \in \Omega$ of the hierarchical variational inequality

$$\langle (V - \gamma g)p_0, q - p_0\rangle \geq 0, \quad \forall q \in \Omega. \qquad (3.2.7)$$

We show that $\limsup_{n\to\infty}\langle (V - \gamma g)p_0, x_n - p_0\rangle \geq 0$. We may assume without loss of generality that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to $w \in C$, as $k \to \infty$, such that

$$\limsup_{n\to\infty}\langle (V - \gamma g)p_0, x_n - p_0\rangle = \lim_{k\to\infty}\langle (V - \gamma g)p_0, x_{n_k} - p_0)\rangle.$$

Since $\{\|x_{n_k} - p\|\}$ is bounded, there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $\lim_{i\to\infty}\|x_{n_{k_i}} - p\|$ exists. Now we shall prove that $w \in \Omega$.

(a) We first prove $w \in F(T)$. We notice that

$$\|x_{n+1} - y_n\| = \|\alpha_n \gamma g(x_n) + (I - \alpha_n V)y_n - y_n\| = \alpha_n \|\gamma g(x_n) - V y_n\|.$$

In particular, replacing n by n_{k_i} and taking $i \to \infty$ in the last equality, we have

$$\lim_{i\to\infty} \|x_{n_{k_i}+1} - y_{n_{k_i}}\| \;=\; 0,$$

so we have $y_{n_{k_i}} \rightharpoonup w$. Since T is 2-generalized hybrid, there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 \\ +(1 - \beta_1 - \beta_2)\|x - y\|^2,\end{aligned}$$

for all $x, y \in C$. For any $n \in \mathbb{N}$ and $k = 0, 1, 2, \ldots, n-1$, we compute the following

$$\begin{aligned}
0 \;\leq\;& \beta_1\|T^2T^kz_n - y\|^2 + \beta_2\|TT^kz_n - y\|^2 + (1 - \beta_1 - \beta_2)\|T^kz_n - y\|^2 \\
& -\alpha_1\|T^2T^kz_n - Ty\|^2 - \alpha_2\|TT^kz_n - Ty\|^2 - (1 - \alpha_1 - \alpha_2)\|T^kz_n - Ty\|^2 \\
=\;& \beta_1\|T^{k+2}z_n - y\|^2 + \beta_2\|T^{k+1}z_n - y\|^2 + (1 - \beta_1 - \beta_2)\|T^kz_n - y\|^2 \\
& -\alpha_1\|T^{k+2}z_n - Ty\|^2 - \alpha_2\|T^{k+1}z_n - Ty\|^2 - (1 - \alpha_1 - \alpha_2)\|T^kz_n - Ty\|^2 \\
\leq\;& \beta_1\{\|T^{k+2}z_n - Ty\|^2 + \|Ty - y\|^2\} + \beta_2\{\|T^{k+1}z_n - Ty\|^2 + \|Ty - y\|^2\} \\
& +(1 - \beta_1 - \beta_2)\{\|T^kz_n - Ty\|^2 + \|Ty - y\|^2\} - \alpha_1\|T^{k+2}z_n - Ty\|^2 \\
& -\alpha_2\|T^{k+1}z_n - Ty\|^2 - (1 - \alpha_1 - \alpha_2)\|T^kz_n - Ty\|^2 \\
=\;& \beta_1\{\|T^{k+2}z_n - Ty\|^2 + \|Ty - y\|^2 + 2\langle T^{k+2}z_n - Ty, Ty - y\rangle\} \\
& +\beta_2\{\|T^{k+1}z_n - Ty\|^2 + \|Ty - y\|^2 + 2\langle T^{k+1}z_n - Ty, Ty - y\rangle\} \\
& +(1 - \beta_1 - \beta_2)\{\|T^kz_n - Ty\|^2 + \|Ty - y\|^2 + 2\langle T^kz_n - Ty, Ty - y\rangle\} \\
& -\alpha_1\|T^{k+2}z_n - Ty\|^2 - \alpha_2\|T^{k+1}z_n - Ty\|^2 - (1 - \alpha_1 - \alpha_2)\|T^kz_n - Ty\|^2 \\
=\;& (\beta_1 - \alpha_1)\|T^{k+2}z_n - Ty\|^2 + (\beta_2 - \alpha_2)\|T^{k+1}z_n - Ty\|^2
\end{aligned}$$

$$
\begin{aligned}
&+(\alpha_1+\alpha_2-\beta_1-\beta_2)\|T^k z_n - Ty\|^2 + (\beta_1+\beta_2+1-\beta_1-\beta_2)\|Ty-y\|^2 \\
&+2\langle \beta_1 T^{k+2} z_n - \beta_1 Ty + \beta_2 T^{k+1} z_n - \beta_2 Ty \\
&+(1-\beta_1-\beta_2)T^k z_n - (1-\beta_1-\beta_2)Ty, Ty-y\rangle \\
= \;& (\beta_1-\alpha_1)\|T^{k+2} z_n - Ty\|^2 + (\beta_2-\alpha_2)\|T^{k+1} z_n - Ty\|^2 \\
&-((\beta_1-\alpha_1)+(\alpha_2-\beta_2))\|T^k z_n - Ty\|^2 + \|Ty-y\|^2 \\
&+2\langle \beta_1 T^{k+2} z_n + \beta_2 T^{k+1} z_n + (1-\beta_1-\beta_2)T^k z_n - Ty, Ty-y\rangle \\
= \;& (\beta_1-\alpha_1)(\|T^{k+2} z_n - Ty\|^2 - \|T^k z_n - Ty\|^2) + (\beta_2-\alpha_2)(\|T^{k+1} z_n - Ty\|^2 \\
&-\|T^k z_n - Ty\|^2) + \|Ty-y\|^2 + 2\langle \beta_1 T^{k+2} z_n + \beta_2 T^{k+1} z_n \\
&+(1-\beta_1-\beta_2)T^k z_n - Ty, Ty-y\rangle \\
= \;& \|Ty-y\|^2 + 2\langle T^k z_n - Ty, Ty-y\rangle + 2\langle \beta_1(T^{k+2} z_n - T^k x_n) \\
&+\beta_2(T^{k+1} z_n - T^k z_n), Ty-y\rangle \\
&+(\beta_1-\alpha_1)(\|T^{k+2} z_n - Ty\|^2 - \|T^k z_n - Ty\|^2) \\
&+(\beta_2-\alpha_2)(\|T^{k+1} z_n - Ty\|^2 - \|T^k z_n - Ty\|^2).
\end{aligned}
$$

Summing up these inequalities from $k=0$ to $n-1$,

$$
\begin{aligned}
0 \;\le\; & \sum_{k=0}^{n-1}\|Ty-y\|^2 + 2\langle \sum_{k=0}^{n-1}(T^k z_n - Ty), Ty-y\rangle \\
&+2\langle \beta_1 \sum_{k=0}^{n-1}(T^{k+2} z_n - T^k z_n) + \beta_2 \sum_{k=0}^{n-1}(T^{k+1} z_n - T^k z_n), Ty-y\rangle \\
&+(\beta_1-\alpha_1)\sum_{k=0}^{n-1}(\|T^{k+2} z_n - Ty\|^2 - \|T^k z_n - Ty\|^2) \\
&+(\beta_2-\alpha_2)\sum_{k=0}^{n-1}(\|T^{k+1} z_n - Ty\|^2 - \|T^k z_n - Ty\|^2) \\
= \;& n\|Ty-y\|^2 + 2\langle \sum_{k=0}^{n-1} T^k z_n - nTy, Ty-y\rangle \\
&+2\langle \beta_1(T^{n+1} z_n - T^n z_n - z_n - Tz_n) + \beta_2(T^n z_n - z_n), Ty-y\rangle \\
&+(\beta_1-\alpha_1)(\|T^{n+1} z_n - Ty\|^2 + \|T^n z_n - Ty\|^2 - \|z_n - Ty\|^2 - \|Tz_n - Ty\|^2) \\
&+(\beta_2-\alpha_2)(\|T^n z_n - Ty\|^2 - \|z_n - Ty\|^2).
\end{aligned}
$$

Dividing this inequality by n, we get

$$\begin{aligned}
0 &\le \|Ty-y\|^2+2\langle y_n-Ty,Ty-y\rangle \\
&\quad +2\langle \frac{1}{n}\beta_1(T^{n+1}z_n-T^nz_n-z_n-Tz_n)+\frac{1}{n}\beta_2(T^nz_n-z_n),Ty-y\rangle \\
&\quad +\frac{1}{n}(\beta_1-\alpha_1)(\|T^{n+1}z_n-Ty\|^2+\|T^nz_n-Ty\|^2-\|z_n-Ty\|^2-\|Tz_n-Ty\|^2) \\
&\quad +\frac{1}{n}(\beta_2-\alpha_2)(\|T^nz_n-Ty\|^2-\|z_n-Ty\|^2).
\end{aligned}$$

Replacing n by n_{k_i} and letting $i\to\infty$ in the last inequality, we have

$$0 \le \|Ty-y\|^2+2\langle w-Ty,Ty-y\rangle, \text{ for all } y\in C. \tag{3.2.8}$$

In particular, replacing y by w in (3.2.8), we obtain that

$$0 \le \|Tw-w\|^2+2\langle w-Tw,Tw-w\rangle=-\|Tw-w\|^2,$$

which ensure that $w\in F(T)$.

(b) We prove that $w\in(A+B)^{-1}0$. From (3.2.3), (3.2.4) and (3.2.6),

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \|(I-\alpha_nV)y_n-(I-\alpha_nV)p\|^2+2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle \\
&\le (1-\alpha_n\tau)^2\|y_n-p\|^2+2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle \\
&\le (1-\alpha_n\tau)^2\|z_n-p\|^2+2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle \\
&\le (1-\alpha_n\tau)^2\{\|x_n-p\|^2-\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\} \\
&\quad +2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle \\
&= (1-2\alpha_n\tau+\alpha_n^2\tau^2)\|x_n-p\|^2-(1-\alpha_n\tau)^2\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2 \\
&\quad +2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle \\
&\le \|x_n-p\|^2+\alpha_n^2\tau^2\|x_n-p\|^2-(1-\alpha_n\tau)^2\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2 \\
&\quad +2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle, \qquad (3.2.9)
\end{aligned}$$

and hence

$$(1-\alpha_n\tau)^2\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n^2\tau^2\|x_n-p\|^2 + 2\alpha_n\langle \gamma g(x_n)-Vp, x_{n+1}-p\rangle. \quad (3.2.10)$$

Replacing n by n_{k_i} in (3.2.10), we have

$$\begin{aligned}(1-\alpha_{n_{k_i}}\tau)^2\lambda_{n_{k_i}}(2\alpha-\lambda_{n_{k_i}})\|Au_{n_{k_i}}-Ap\|^2 \le\ & \|x_{n_{k_i}}-p\|^2-\|x_{n_{k_i}+1}-p\|^2 \\ & +\alpha_{n_{k_i}}^2\tau^2\|x_{n_{k_i}}-p\|^2 \\ & +2\alpha_{n_{k_i}}\langle \gamma g(x_n)-Vp, x_{n_{k_i}+1}-p\rangle.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n=0$, $0<a\le\lambda_n\le b<2\alpha$ and the existence of $\lim_{i\to\infty}\|x_{n_{k_i}}-p\|$, we have

$$\lim_{i\to\infty}\|Au_{n_{k_i}}-Ap\|=0. \quad (3.2.11)$$

We also have from (2.3.5) that

$$\begin{aligned}2\|u_n-p\|^2 &= 2\|T_{r_n}x_n-T_{r_n}p\|^2 \\ &\le 2\langle x_n-p, u_n-p\rangle \\ &= \|x_n-p\|^2+\|u_n-p\|^2-\|u_n-x_n\|^2,\end{aligned}$$

and hence

$$\|u_n-p\|^2 \le \|x_n-p\|^2-\|u_n-x_n\|^2. \quad (3.2.12)$$

From (3.2.3), (3.2.4), (3.2.6) and (3.2.12), we obtain the following,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|(I-\alpha_nV)y_n-(I-\alpha_nV)p\|^2+2\alpha_n\langle \gamma g(x_n)-Vp, x_{n+1}-p\rangle \\ &\le (1-\alpha_n\tau)^2\|y_n-p\|^2+2\alpha_n\langle \gamma g(x_n)-Vp, x_{n+1}-p\rangle\end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n\tau)^2\|z_n-p\|^2+2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle\\
&\le (1-\alpha_n\tau)^2\{\|u_n-p\|^2-\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\}\\
&\quad +2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle\\
&\le (1-\alpha_n\tau)^2\{\|x_n-p\|^2-\|u_n-x_n\|^2\}\\
&\quad -(1-\alpha_n\tau)^2\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\\
&\quad +2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle\\
&\le (1-2\alpha_n\tau+\alpha_n^2\tau^2)\|x_n-p\|^2-(1-\alpha_n\tau)^2\|u_n-x_n\|^2\\
&\quad -(1-\alpha_n\tau)^2\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\\
&\quad +2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle\\
&\le \|x_n-p\|^2+\alpha_n^2\tau^2\|x_n-p\|^2-(1-\alpha_n\tau)^2\|u_n-x_n\|^2\\
&\quad -(1-\alpha_n\tau)^2\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\\
&\quad +2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle,
\end{aligned}
$$

and hence

$$
\begin{aligned}
(1-\alpha_n\tau)^2\|u_n-x_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n^2\tau^2\|x_n-p\|^2\\
&\quad -(1-\alpha_n\tau)^2\lambda_n(2\alpha-\lambda_n)\|Au_n-Ap\|^2\\
&\quad +2\alpha_n\langle\gamma g(x_n)-Vp,x_{n+1}-p\rangle. \qquad (3.2.13)
\end{aligned}
$$

Replacing n by n_{k_i} in (3.2.13), we have

$$
\begin{aligned}
(1-\alpha_{n_{k_i}}\tau)^2\|u_{n_{k_i}}-x_{n_{k_i}}\|^2 &\le \|x_{n_{k_i}}-p\|^2-\|x_{n_{k_i}+1}-p\|^2+\alpha_{n_{k_i}}^2\tau^2\|x_{n_{k_i}}-p\|^2\\
&\quad -(1-\alpha_{n_{k_i}}\tau)^2\lambda_{n_{k_i}}(2\alpha-\lambda_{n_{k_i}})\|Au_{n_{k_i}}-Ap\|^2\\
&\quad +2\alpha_{n_{k_i}}\langle\gamma g(x_{n_{k_i}})-Vp,x_{n_{k_i}+1}-p\rangle.
\end{aligned}
$$

From (3.2.11), $\lim_{n\to\infty}\alpha_n=0$ and the existence of $\lim_{i\to\infty}\|x_{n_{k_i}}-p\|$, we have

$$
\lim_{i\to\infty}\|u_{n_{k_i}}-x_{n_{k_i}}\|=0. \qquad (3.2.14)
$$

On the other hand, since J_{λ_n} is firmly nonexpansive and $u_n = T_{r_n}x_n$, we have that

$$\begin{aligned}
\|z_n - p\|^2 &= \|J_{\lambda_n}(I - \lambda_n A)u_n - J_{\lambda_n}(I - \lambda_n A)p\|^2 \\
&\leq \langle z_n - p, (I - \lambda_n A)u_n - (I - \lambda_n A)p\rangle \\
&= \frac{1}{2}(\|z_n - p\|^2 + \|(I - \lambda_n A)u_n - (I - \lambda_n A)p\|^2 \\
&\quad -\|z_n - p - (I - \lambda_n A)u_n + (I - \lambda_n A)p\|^2) \\
&\leq \frac{1}{2}\{\|z_n - p\|^2 + \|u_n - p\|^2 - \|z_n - p - (I - \lambda_n A)u_n + (I - \lambda_n A)p\|^2\} \\
&\leq \frac{1}{2}(\|z_n - p\|^2 + \|x_n - p\|^2 - \|z_n - u_n\|^2 - 2\lambda_n\langle z_n - u_n, Au_n - Ap\rangle \\
&\quad -\lambda_n^2\|Au_n - Ap\|^2),
\end{aligned}$$

and hence

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|x_n - p\|^2 - \|z_n - u_n\|^2 - 2\lambda_n\langle z_n - u_n, Au_n - Ap\rangle \\
&\quad -\lambda_n^2\|Au_n - Ap\|^2. \qquad (3.2.15)
\end{aligned}$$

From (3.2.3), (3.2.4), (3.2.6) and (3.2.15), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|(I - \alpha_n V)y_n - (I - \alpha_n V)p\|^2 + 2\alpha_n\langle\gamma g(x_n) - Vp, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n\tau)^2\|y_n - p\|^2 + 2\alpha_n\langle\gamma g(x_n) - Vp, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n\tau)^2\|z_n - p\|^2 + 2\alpha_n\langle\gamma g(x_n) - Vp, x_{n+1} - p\rangle \\
&\leq (1 - \alpha_n\tau)^2(\|x_n - z\|^2 - \|z_n - u_n\|^2 - 2\lambda_n\langle z_n - u_n, Au_n - Ap\rangle \\
&\quad -\lambda_n^2\|Au_n - Ap\|^2) + 2\alpha_n\langle\gamma g(x_n) - Vp, x_{n+1} - p\rangle \\
&\leq \|x_n - p\|^2 + \alpha_n^2\tau^2\|x_n - p\|^2 - (1 - \alpha_n\tau)^2\|z_n - u_n\| \\
&\quad -2(1 - \alpha_n\tau)^2\lambda_n(\lambda_n - 2\alpha)\|z_n - u_n\|\|Au_n - Ap\| \\
&\quad -(1 - \alpha_n\tau)^2\lambda_n^2\|Au_n - Ap\|^2 + 2\alpha_n\langle\gamma g(x_n) - Vp, x_{n+1} - p\rangle,
\end{aligned}$$

and hence

$$(1 - \alpha_n\tau)^2\|z_n - u_n\| \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2\tau^2\|x_n - p\|^2$$

$$-2(1-\alpha_n\tau)^2\lambda_n(\lambda_n-2\alpha)\|z_n-u_n\|\|Au_n-Ap\|$$
$$-(1-\alpha_n\tau)^2\lambda_n^2\|Au_n-Ap\|^2+2\alpha_n\langle\gamma g(x_n)$$
$$-Vp, x_{n+1}-p\rangle. \quad (3.2.16)$$

Replacing n by n_{k_i} in (3.2.16), we have

$$\begin{aligned}(1-\alpha_{n_{k_i}}\tau)^2\|z_{n_{k_i}}-u_{n_{k_i}}\|^2 \;\leq\;& \|x_{n_{k_i}}-p\|^2-\|x_{n_{k_i}+1}-p\|^2+\alpha_{n_{k_i}}^2\tau^2\|x_{n_{k_i}}-p\|^2\\ &-2(1-\alpha_{n_{k_i}}\tau)^2\lambda_{n_{k_i}}(\lambda_{n_{k_i}}-2\alpha)\|z_{n_{k_i}}-u_{n_{k_i}}\|\|Au_{n_{k_i}}-Ap\|\\ &-(1-\alpha_{n_{k_i}}\tau)^2\lambda_{n_{k_i}}^2\|Au_{n_{k_i}}-Ap\|^2\langle\gamma g(x_{n_{k_i}})\\ &+2\alpha_{n_{k_i}}-Vp, x_{n_{k_i}+1}-p\rangle.\end{aligned}$$

From (3.2.16), $\lim_{n\to\infty}\alpha_n=0$ and the existence of $\lim_{i\to\infty}\|x_{n_{k_i}}-p\|$, we obtain that

$$\lim_{i\to\infty}\|z_{n_{k_i}}-u_{n_{k_i}}\|=0. \quad (3.2.17)$$

Since $\|z_{n_{k_i}}-x_{n_{k_i}}\| \leq \|z_{n_{k_i}}-u_{n_{k_i}}\|+\|u_{n_{k_i}}-x_{n_{k_i}}\|$, by (3.2.14) and (3.2.17), we obtain that

$$\lim_{i\to\infty}\|z_{n_{k_i}}-x_{n_{k_i}}\|=0. \quad (3.2.18)$$

Since A is Lipschitz continuous, we also obtain

$$\lim_{i\to\infty}\|Az_{n_{k_i}}-Ax_{n_{k_i}}\|=0. \quad (3.2.19)$$

Since $z_n=J_\lambda(I-\lambda A)u_n$, we have that

$$\begin{aligned} z_n &= (I+\lambda_nB)^{-1}(I-\lambda_nA)u_n\\ &\Leftrightarrow (I-\lambda_nA)u_n\in(I+\lambda_nB)z_n=z_n+\lambda_nBz_n\\ &\Leftrightarrow u_n-z_n-\lambda_nAu_n\in\lambda_nBz_n\end{aligned}$$

$$\Leftrightarrow \quad \frac{1}{\lambda_n}(u_n - z_n - \lambda_n A u_n) \in B z_n.$$

Since B is monotone, we have that for $(u, v) \in B$,

$$\langle z_n - u, \frac{1}{\lambda_n}(u_n - z_n - \lambda_n A u_n) - v \rangle \geq 0,$$

and hence

$$\langle z_n - u, u_n - z_n - \lambda_n(A u_n + v) \rangle \geq 0. \quad (3.2.20)$$

Replacing n by n_{k_i} in (3.2.20), we have that

$$\langle z_{n_{k_i}} - u, u_{n_{k_i}} - z_{n_{k_i}} - \lambda_{n_{k_i}}(A u_{n_{k_i}} + v) \rangle \geq 0. \quad (3.2.21)$$

Since $x_{n_{k_i}} \rightharpoonup w$, and $x_{n_{k_i}} - u_{n_{k_i}} \to 0$, so $u_{n_{k_i}} \rightharpoonup w$. From (3.2.21), we get that $z_{n_{k_i}} \rightharpoonup w$, together with (3.2.17), we have that

$$\langle w - u, -Aw - v \rangle \geq 0.$$

Since B is maximal monotone, $(-Aw) \in Bw$. That is, $w \in (A + B)^{-1}0$.

(c) Next, we show that $w \in F^{-1}0$. Since F is a maximal monotone operator, we have from (2.3.4) that $A_{r_{n_{k_i}}} x_{n_{k_i}} \in F T_{r_{n_{k_i}}} x_{n_{k_i}}$, where A_r is the Yosida approximation of F for $r > 0$. Furthermore, we have that for any $(u, v) \in F$,

$$\left\langle u - u_{n_{k_i}}, v - \frac{x_{n_{k_i}} - u_{n_{k_i}}}{r_{n_{k_i}}} \right\rangle \geq 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, $u_{n_{k_i}} \rightharpoonup w$ and $x_{n_{k_i}} - u_{n_{k_i}} \to 0$, we have

$$\langle u - w, v \rangle \geq 0.$$

Since F is a maximal monotone operator, we have $0 \in Fw$, that is $w \in F^{-1}0$. By (a), (b) and (c), we conclude that

$$w \in F(T) \cap (A+B)^{-1}0 \cap F^{-1}0.$$

Using (3.2.7), we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle (V-\gamma g)p_0, x_n - p_0\rangle &= \lim_{k\to\infty}\langle (V-\gamma g)p_0, x_{n_k} - p_0)\rangle \\ &= \langle (V-\gamma g)p_0, w - p_0)\rangle \geq 0.\end{aligned}$$

Finally, we prove that $x_n \to p_0$. Notice that

$$x_{n+1} - p_0 = \alpha_n(\gamma g(x_n) - p_0) + (I - \alpha_n V)y_n - (I - \alpha_n V)p_0,$$

we have

$$\begin{aligned}\|x_{n+1} - p_0\|^2 &\leq (1-\alpha_n\tau)^2\|y_n - p_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vp_0, x_{n+1} - p_0\rangle \\ &\leq (1-\alpha_n\tau)^2\|x_n - p_0\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vp_0, x_{n+1} - p_0\rangle \\ &\leq (1-\alpha_n\tau)^2\|x_n - p_0\|^2 + 2\alpha_n\gamma k\|x_n - p_0\|\|x_{n+1} - p_0\| \\ &\quad +2\alpha_n\langle \gamma g(p_0) - Vp_0, x_{n+1} - p_0\rangle \\ &\leq (1-\alpha_n\tau)^2\|x_n - p_0\|^2 + \alpha_n\gamma k(\|x_n - p_0\|^2 + \|x_{n+1} - p_0\|^2) \\ &\quad +2\alpha_n\langle \gamma g(p_0) - Vp_0, x_{n+1} - p_0\rangle \\ &\leq \{(1-\alpha_n\tau)^2 + \alpha_n\gamma k\}\|x_n - p_0\|^2 + \alpha_n\gamma k\|x_{n+1} - p_0\|^2 \\ &\quad +2\alpha_n\langle \gamma g(p_0) - Vp_0, x_{n+1} - p_0\rangle,\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1} - p_0\|^2 &\leq \frac{1 - 2\alpha_n\tau + (\alpha_n\tau)^2 + \alpha_n\gamma k}{1 - \alpha_n\gamma k}\|x_n - p_0\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\gamma k}\langle \gamma g(p_0) - Vp_0, x_{n+1} - p_0\rangle\end{aligned}$$

$$
\begin{aligned}
&= \left\{1 - \frac{2(\tau - \gamma k)\alpha_n}{1 - \alpha_n \gamma k}\right\} \|x_n - p_0\|^2 + \frac{(\alpha_n \tau)^2}{1 - \alpha_n \gamma k}\|x_n - p_0\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n \gamma k}\langle \gamma g(p_0) - V p_0, x_{n+1} - p_0 \rangle \\
&= \left\{1 - \frac{2(\tau - \gamma k)\alpha_n}{1 - \alpha_n \gamma k}\right\} \|x_n - p_0\|^2 + \frac{\alpha_n \cdot \alpha_n \tau^2}{1 - \alpha_n \gamma k}\|x_n - p_0\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n \gamma k}\langle \gamma g(p_0) - V p_0, x_{n+1} - p_0 \rangle \\
&= (1 - \beta_n)\|x_n - p_0\|^2 \\
&\quad + \beta_n \left\{ \frac{\alpha_n \tau^2 \|x_n - p_0\|^2}{2(\tau - \gamma k)} + \frac{1}{\tau - \gamma k}\langle \gamma g(p_0) - V p_0, x_{n+1} - p_0 \rangle \right\},
\end{aligned}
\tag{3.2.22}
$$

where $\beta_n = \frac{2(\tau - \gamma k)\alpha_n}{1 - \alpha_n \gamma k}$. Since $\sum_{n=1}^{\infty} \beta_n = \infty$, we have from Lemma 2.1.20 and (3.2.22) that $x_n \to p_0$. This completes the proof. □

Next, using Theorem 3.2.1, we obtain the following results for an inverse-strongly monotone mapping.

Theorem 3.2.2. Let H be a real Hilbert space and let C a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $T : C \to C$ be a 2-generalized hybrid mapping such that $\Gamma := F(T) \cap VI(C, A) \neq \emptyset$. Take $\mu, \gamma \in \mathbb{R}$ as follows:

$$
0 < \mu < \frac{2\bar{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\bar{\gamma} - \frac{L^2 \mu}{2}}{k}.
$$

Let $\{x_n\} \subset H$ be a sequence generated by

$$
\begin{cases}
x_1 = x \in H, \text{arbitrarily}, \\
z_n = P_C(I - \lambda_n A)P_C x_n, \\
y_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k z_n, \quad \forall\, n = 1, 2, \dots, \\
x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)y_n, \text{ for all } n \in \mathbb{N},
\end{cases}
\tag{3.2.23}
$$

where $\{\alpha_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then $\{x_n\}$ converges strongly to a point p_0 of Γ, where p_0 is a unique fixed point of $P_\Gamma(I - V + \gamma g)$. This point $p_0 \in \Gamma$ is also a unique solution of the hierarchical variational inequality

$$\langle (V - \gamma g)p_0, q - p_0 \rangle \geq 0, \forall q \in VI(C, A). \tag{3.2.24}$$

Proof. Put $B = F = \partial i_C$ in Theorem 3.2.1. Then for $\lambda_n > 0$ and $r_n > 0$, we have that

$$J_{\lambda_n} = T_{r_n} = P_C.$$

Furthermore we have, from the proof of [36, Theorem 12], that

$$(\partial i_C)^{-1}0 = C \text{ and } (A + \partial i_C)^{-1}0 = VI(C, A).$$

Thus we obtained the desired results by Theorem 3.2.1. □

Using Theorem 3.2.1, we finally prove a strong convergence theorem for inverse-strongly monotone operators and equilibrium problems in a Hilbert space.

Theorem 3.2.3. Let H be a real Hilbert space and let C a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $B : D(B) \subset C \to 2^H$ be a maximal monotone Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1) - (A4)$ and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$

be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. Let $T : C \to C$ be a 2-generalized hybrid mapping with $\Theta := F(T) \cap (A+B)^{-1}0 \cap MEP(G, \varphi) \neq \emptyset$. Take $\mu, \gamma \in \mathbb{R}$ as follows:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\bar{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $\{x_n\} \subset H$ be a sequence generated by

$$\begin{cases} x_1 = x \in H, \text{arbitrarily}, \\ G(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C, \\ z_n = J_{\lambda_n}(I - \lambda_n A)u_n, \\ y_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k z_n, \quad \forall\, n = 1, 2, \ldots, \\ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)y_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.2.25}$$

where $\{\alpha_n\} \subset (0,1)$, and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then $\{x_n\}$ converges strongly to a point p_0 of Θ, where p_0 is a unique fixed point of $P_\Theta(I - V + \gamma g)$. This point $p_0 \in \Theta$ is also a unique solution of the hierarchical variational inequality

$$\langle (V - \gamma g)p_0, q - p_0\rangle \geq 0, \forall q \in \Theta. \tag{3.2.26}$$

Proof. Since G is a bifunction of $C \times C$ into $\mathbb{R}$ satisfying the conditions $(A1) - (A4)$ and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous and convex function, we have the mapping $A_{G,\varphi}$ defined by (2.4.6) is a maximal monotone operator with dom $A_{G,\varphi} \subset C$. Put $F = A_{G,\varphi}$ in Theorem 3.2.1. Then we obtain that $u_n = T_{r_n} x_n$. Therefore, we arrive the desired results. □

CHAPTER IV

VISCOSITY APPROXIMATION METHODS FOR FIXED POINT PROBLEMS IN CAT(0) SPACES

4.1 Viscosity approximation methods for asymptotically nonexpansive mappings in CAT(0) spaces

Let C be a nonempty subset of a complete CAT(0) space X. A mapping T of C into itself is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $d(T^n x, T^n y) \leq k_n d(x, y)$ for all integers $n \geq 1$ and all $x, y \in C$. A mapping f of C into itself is called *contraction* with coefficient $\alpha \in (0, 1)$ iff $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in C$.

In this section, we present strong convergence theorems of Moudafi's viscosity methods in CAT(0) spaces.

Theorem 4.1.1. Let C be a closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_n = 1$ such that $F(T) \neq \emptyset$. Let f be a contraction on C with coefficient $0 < \alpha < 1$. Let $\{\alpha_n\}$ be a sequence of real numbers with $0 < \alpha_n < 1$. Then the following statements hold:

(i) For each $n \in \mathbb{N}$, if $\frac{k_n - 1}{\alpha_n} < 1 - \alpha$, then there exists y_n such that

$$y_n = \alpha_n f(y_n) \oplus (1 - \alpha_n) T^n y_n. \tag{4.1.1}$$

(ii) If $\alpha_n \to 0$ and $\frac{k_n - 1}{\alpha_n} \to 0$ as $n \to \infty$, then $\{y_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{F(T)} f(\tilde{x})$ which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T). \tag{4.1.2}$$

Proof. For each integer $n \leq 1$, define a mapping $G_n : C \to C$ by

$$G_n(x) = \alpha_n f(x) \oplus (1-\alpha_n)T^n x, \quad \forall x \in C.$$

We shall show that G_n is a contraction mapping. For any $x, y \in C$

$$\begin{aligned} d(G_n(x), G_n(y)) &= d(\alpha_n f(x) \oplus (1-\alpha_n)T^n x, \alpha_n f(y) \oplus (1-\alpha_n)T^n y) \\ &\leq \alpha_n d(f(x), f(y)) + (1-\alpha_n) d(T^n x, T^n y) \\ &\leq \alpha_n \alpha d(x,y) + (1-\alpha_n) k_n d(x,y) \\ &= (k_n - \alpha_n k_n + \alpha \alpha_n) d(x,y). \end{aligned}$$

Since $0 < \frac{k_n - 1}{\alpha_n} < 1 - \alpha$, we have

$$0 < \frac{k_n - 1}{\alpha_n} < 1 - \alpha \leq \alpha_n k_n - \alpha \alpha_n.$$

It follows that $0 < k_n - \alpha_n k_n + \alpha_n \alpha < 1$. We have G_n is a contraction map with coefficient $(k_n - \alpha_n k_n + \alpha_n \alpha)$. For each integer $n \leq 1$, there exists a unique $y_n \in C$ such that $G_n(y_n) = y_n$, that is

$$y_n = \alpha_n f(y_n) \oplus (1-\alpha_n) T^n y_n.$$

Next, we show that $\{y_n\}$ is bounded. For any $p \in F(T)$, we have that

$$\begin{aligned} d(y_n, p) &= d(\alpha_n f(y_n) \oplus (1-\alpha_n) T^n y_n, p) \\ &\leq \alpha_n d(f(y_n), f(p)) + \alpha_n d(f(p), p) + (1-\alpha_n) d(T^n y_n, p) \\ &\leq \alpha \alpha_n d(y_n, p) + \alpha_n d(f(p), p) + k_n (1-\alpha_n) d(y_n, p) \\ &= \{k_n - (k_n - \alpha)\alpha_n\} d(y_n, p) + \alpha_n d(f(p), p). \end{aligned}$$

Then

$$d(y_n, p) \leq \frac{\alpha_n}{(k_n - \alpha)\alpha_n - (k_n - 1)} d(f(p), p)$$

$$\leq \frac{1}{1-\alpha} d(f(p), p).$$

Hence $\{y_n\}$ is bounded, so are $\{Ty_n\}$ and $\{f(y_n)\}$. We get that

$$\begin{aligned}
\lim_{n\to\infty} d(y_n, T^n y_n) &= \lim_{n\to\infty} d(\alpha_n f(y_n) \oplus (1-\alpha_n)T^n y_n, T^n y_n) \\
&\leq \lim_{n\to\infty} [\alpha_n d(f(y_n), T^n y_n) + (1-\alpha_n) d(T^n y_n, T^n y_n)] \\
&\leq \lim_{n\to\infty} \alpha_n d(f(y_n), T^n y_n).
\end{aligned}$$

Thus

$$\lim_{n\to\infty} d(y_n, T^n y_n) = 0. \tag{4.1.3}$$

Let $L = \sup_n k_n$, then we have

$$d(T^n y_n, p) \leq k_n d(y_n, p) \leq L d(y_n, p).$$

It follows that the sequence $\{T^n y_n\}$ is bounded. We claim that $\lim_{n\to\infty} d(y_n, Ty_n) = 0$. Indeed, we have that

$$\begin{aligned}
d(y_n, T^{n-1} y_n) &= d(\alpha_n f(y_n) \oplus (1-\alpha_n)T^n y_n, T^{n-1} y_n) \\
&\leq \alpha_n d(f(y_n), T^{n-1} y_n) + (1-\alpha_n)(T^n y_n, T^{n-1} y_n) \\
&= \alpha_n d(f(y_n), T^{n-1} y_n) + (1-\alpha_n)(T^{n-1} T y_n, T^{n-1} y_n) \\
&\leq \alpha_n d(f(y_n), T^{n-1} y_n) + (1-\alpha_n) k_{n-1}(T y_n, y_n). \qquad (4.1.4)
\end{aligned}$$

By (4.1.4), we have

$$\begin{aligned}
d(y_n, Ty_n) &= d(\alpha_n f(y_n) \oplus (1-\alpha_n)T^n y_n, Ty_n) \\
&\leq \alpha_n d(f(y_n), Ty_n) + (1-\alpha_n) d(T^n y_n, Ty_n) \\
&= \alpha_n d(f(y_n), Ty_n) + (1-\alpha_n) d(T(T^{n-1} y_n), Ty_n) \\
&\leq \alpha_n d(f(y_n), Ty_n) + (1-\alpha_n) k_1 d(T^{n-1} y_n, y_n) \\
&\leq \alpha_n d(f(y_n), Ty_n) + (1-\alpha_n) k_1 [\alpha_n d(f(y_n), T^{n-1} y_n)
\end{aligned}$$

$$
\begin{aligned}
& +(1-\alpha_n)k_{n-1}(Ty_n, y_n)] \\
= \;& \alpha_n d(f(y_n), Ty_n) + (1-\alpha_n)\alpha_n k_1 d(f(y_n), T^{n-1}y_n) \\
& +(1-\alpha_n)^2 k_1 k_{n-1}(Ty_n, y_n) \\
\leq \;& \alpha_n d(f(y_n), Ty_n) + (1-\alpha_n)\alpha_n L d(f(y_n), T^{n-1}y_n) \\
& +(1-\alpha_n)^2 L^2 (Ty_n, y_n).
\end{aligned}
$$

This implies that

$$
d(y_n, Ty_n) \leq \frac{\alpha_n}{1-(1-\alpha_n)^2 L^2} d(f(y_n, Ty_n)) + \frac{(1-\alpha_n)\alpha_n L}{1-(1-\alpha_n)^2 L^2} d(f(y_n), T^{n-1}y_n).
$$

Since $\alpha_n \to 0$, we get

$$
\lim_{n\to\infty} d(y_n, Ty_n) = 0.
$$

Next, we will show that $\{y_n\}$ contains a subsequence converging strongly to $\tilde{x}$ such that $\tilde{x} = P_{F(T)} f(\tilde{x})$ which is equivalent to the following variational inequality:

$$
\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \ \forall x \in F(T).
$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ which Δ-converges to $\tilde{x}$. By Lemma 2.5.11, 2.5.13, we may assume that $\{y_{n_j}\}$ Δ-converges to a point $\tilde{x}$ and $\tilde{x} \in F(T)$. It follows from Lemma 2.5.17 (i) that

$$
\begin{aligned}
d^2(y_{n_j}, \tilde{x}) \;=\;& \langle \overrightarrow{y_{n_j}\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}} \rangle \\
\leq \;& \alpha_{n_j} \langle \overrightarrow{f(y_{n_j})\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}} \rangle + (1-\alpha_{n_j}) \langle \overrightarrow{T^{n_j} y_{n_j}\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}} \rangle \\
\leq \;& \alpha_{n_j} \langle \overrightarrow{f(y_{n_j})\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}} \rangle + (1-\alpha_{n_j}) d(T^{n_j} y_{n_j}, \tilde{x}) d(y_{n_j}, \tilde{x}) \\
\leq \;& \alpha_{n_j} \langle \overrightarrow{f(y_{n_j})\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}} \rangle + (1-\alpha_{n_j}) k_{n_j} d(y_{n_j}, \tilde{x}) d(y_{n_j}, \tilde{x}) \\
= \;& \alpha_{n_j} \langle \overrightarrow{f(y_{n_j})\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}} \rangle + (1-\alpha_{n_j}) k_{n_j} d^2(y_{n_j}, \tilde{x}).
\end{aligned}
$$

It follows that

$$
d^2(y_{n_j}, \tilde{x}) \;\leq\; \frac{\alpha_{n_j}}{(1-(1-\alpha_{n_j})k_{n_j})} \langle \overrightarrow{f(y_{n_j})\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}} \rangle
$$

$$
\begin{aligned}
&= \frac{\alpha_{n_j}}{(1-(1-\alpha_{n_j})k_{n_j})}\left[\langle\overrightarrow{f(y_{n_j})f(\tilde{x})}, \overrightarrow{y_{n_j}\tilde{x}}\rangle + \langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}}\rangle\right] \\
&\leq \frac{\alpha_{n_j}}{(1-(1-\alpha_{n_j})k_{n_j})}\left[d(f(y_{n_j}), f(\tilde{x}))d(y_{n_j}, \tilde{x}) + \langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}}\rangle\right] \\
&\leq \frac{\alpha_{n_j}}{(1-(1-\alpha_{n_j})k_{n_j})}\left[\alpha d^2(y_{n_j}, \tilde{x}) + \langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}}\rangle\right],
\end{aligned}
$$

and hence

$$
\begin{aligned}
d^2(y_{n_j}, \tilde{x}) &\leq \frac{\alpha_{n_j}}{\alpha_{n_j}(k_{n_j}-\alpha)-(k_{n_j}-1)}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}}\rangle \\
&\leq \frac{1}{1-\alpha}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}}\rangle. \qquad (4.1.5)
\end{aligned}
$$

Since $\{y_{n_j}\}$ Δ-converges to $\tilde{x}$, by Lemma 2.5.18, we have

$$
\limsup_{n\to\infty}\langle\overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{y_{n_j}\tilde{x}}\rangle \leq 0. \qquad (4.1.6)
$$

It follows from (4.1.5) that $\{y_{n_j}\}$ converges strongly to $\tilde{x}$. Next, we show that $\tilde{x}$ solves the variational inequality (4.1.2). Applying Lemma 2.5.9, for any $q \in F(T)$,

$$
\begin{aligned}
d^2(y_{n_j}, q) &= d^2(\alpha_{n_j} f(y_{n_j}) \oplus (1-\alpha_{n_j})T^{n_j}y_{n_j}, q) \\
&\leq \alpha_{n_j} d^2(f(y_{n_j}), q) + (1-\alpha_{n_j})d^2(T^{n_j}y_{n_j}, q) \\
&\quad -\alpha_{n_j}(1-\alpha_{n_j})d^2(f(y_{n_j}), T^{n_j}y_{n_j}) \\
&\leq \alpha_{n_j} d^2(f(y_{n_j}), q) + (1-\alpha_{n_j})k_{n_j}^2 d^2(y_{n_j}, q) \\
&\quad -\alpha_{n_j}(1-\alpha_{n_j})d^2(f(y_{n_j}), T^{n_j}y_{n_j}),
\end{aligned}
$$

and hence

$$
(1-\alpha_{n_j})d^2(f(y_{n_j}), T^{n_j}y_{n_j}) + k_{n_j}^2 d(y_{n_j}, q) \leq d^2(f(y_{n_j}), q) + \frac{k_{n_j}^2 - 1}{\alpha_{n_j}} d^2(y_{n_j}, q).
$$

We then have

$$
(1-\alpha_{n_j})d^2(f(y_{n_j}), T^{n_j}y_{n_j}) + k_{n_j}^2 d(y_{n_j}, q) \leq d^2(f(y_{n_j}), q) + \frac{k_{n_j} - 1}{\alpha_{n_j}}\overline{M},
$$

where $\overline{M} = (k_{n_j}+1)d^2(y_{n_j}, q)$. Since $y_{n_j} \to \tilde{x}$ and by (4.1.3), we have $T^{n_j}y_{n_j} \to \tilde{x}$.

It follows from $\alpha_{n_j} \to 0$, $k_{n_j} \to 1$, $\frac{k_{n_j}-1}{\alpha_{n_j}} \to 0$ and continuity of metric distance d that

$$d^2(f(\tilde{x}), \tilde{x}) + d^2(\tilde{x}, q) \leq d^2(f(\tilde{x}), q).$$

Hence

$$0 \leq \frac{1}{2}\left[d^2(\tilde{x}, \tilde{x}) + d^2(f(\tilde{x}), q) - d^2(\tilde{x}, q) - d^2(f(\tilde{x}), \tilde{x})\right] = \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{q\tilde{x}} \rangle, \quad \forall q \in F(T).$$

This is, $\tilde{x}$ solves the inequality (4.1.2). Assume there exists subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which Δ-converges to $\hat{x}$ by the same argument, we get that $\hat{x} \in F(T)$ and solves the variational inequality (4.1.2), i.e.,

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle \leq 0, \tag{4.1.7}$$

and

$$\langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\tilde{x}\hat{x}} \rangle \leq 0. \tag{4.1.8}$$

Adding up (4.1.7) and (4.1.8), we obtain that

$$\begin{aligned}
0 &\geq \langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle - \langle \overrightarrow{\hat{x}f(\hat{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle \\
&= \langle \overrightarrow{\tilde{x}f(\hat{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle + \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle - \langle \overrightarrow{\hat{x}\tilde{x}}, \overrightarrow{\hat{x}\tilde{x}} \rangle - \langle \overrightarrow{\tilde{x}f(\hat{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle \\
&= \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - \langle \overrightarrow{f(\hat{x})f(\tilde{x})}, \overrightarrow{\hat{x}\tilde{x}} \rangle \\
&\geq \langle \overrightarrow{\tilde{x}\hat{x}}, \overrightarrow{\tilde{x}\hat{x}} \rangle - d(f(\hat{x}), f(\tilde{x}))d(\hat{x}, \tilde{x}) \\
&\geq d^2(\tilde{x}, \hat{x}) - \alpha d(\hat{x}, \tilde{x})d(\hat{x}, \tilde{x}) \\
&= d^2(\tilde{x}, \hat{x}) - \alpha d^2(\hat{x}, \tilde{x}) \\
&= (1 - \alpha)d^2(\tilde{x}, \hat{x}).
\end{aligned}$$

Since $0 < \alpha < 1$, we have that $d(\tilde{x}, \hat{x}) = 0$, and so $\tilde{x} = \hat{x}$. Hence $\{y_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ which solves the variational inequality (4.1.2). □

Now, we present a strong convergence theorem for asymptotically nonexpansive mappings.

Theorem 4.1.2. Let C be a closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_n = 1$ such that $F(T) \neq \emptyset$. Let f be a contraction on C with coefficient $0 < \alpha < 1$. For the arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T^n x_n, \ \ \forall n \geq 0, \tag{4.1.9}$$

where $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$;

(iv) T satisfies the asymptotically regularity $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{F(T)} f(\tilde{x})$ which is equivalent to the variational inequality (4.1.2).

Proof. We first show that the sequence $\{x_n\}$ is bounded. By condition (iv), for any $0 < \varepsilon < 1 - \alpha$ and sufficient large $n \geq 0$, we have $k_n - 1 \leq \varepsilon \alpha_n$. For any $p \in F(T)$, we have that

$$\begin{aligned}
d(x_{n+1}, p) &= d(\alpha_n f(x_n) \oplus (1 - \alpha_n)T^n x_n, p) \\
&\leq \alpha_n d(f(x_n), p) + (1 - \alpha_n) d(T^n x_n, p) \\
&\leq \alpha_n \left(d(f(x_n), f(p)) + d(f(p), p) \right) + (1 - \alpha_n) d(T^n x_n, p) \\
&\leq \alpha_n \alpha d(x_n, p) + \alpha_n d(f(p), p) + (1 - \alpha_n) k_n d(x_n, p) \\
&= (k_n(1 - \alpha_n) + \alpha \alpha_n) d(x_n, p) + \alpha_n d(f(p), p)
\end{aligned}$$

$$\begin{aligned}
&= (1+(k_n-1)-(k_n-\alpha)\alpha_n)d(x_n,p)+\alpha_n d(f(p),p)\\
&\le (1+\varepsilon\alpha_n-(k_n-\alpha)\alpha_n)d(x_n,p)+\alpha_n d(f(p),p)\\
&= (1-(k_n-\alpha-\varepsilon)\alpha_n)d(x_n,p)+\alpha_n d(f(p),p)\\
&\le (1-(1-\alpha-\varepsilon)\alpha_n)d(x_n,p)+\alpha_n d(f(p),p)\\
&\le \max\left\{d(x_n,p),\frac{1}{(1-\alpha-\varepsilon)}d(f(p),p)\right\},
\end{aligned}$$

for all $n\in\mathbb{N}$. This implies that $\{x_n\}$ is bounded, so are $\{T^nx_n\}$ and $\{f(x_n)\}$. Next, we claim that $\lim_{n\to\infty} d(x_n,Tx_n)=0$. Indeed we have

$$\begin{aligned}
d(x_{n+1},x_n) &\le d(x_{n+1},T^nx_n)+d(T^nx_n,x_n)\\
&= d(\alpha_n f(x_n)\oplus(1-\alpha_n)T^nx_n,T^nx_n)+d(T^nx_n,x_n)\\
&\le \alpha_n d(f(x_n),T^nx_n)+d(T^nx_n,x_n) \to 0.
\end{aligned}$$

$$\begin{aligned}
d(x_n,T^{n-1}x_n) &= d(\alpha_{n-1}f(x_{n-1})\oplus(1-\alpha_{n-1})T^{n-1}x_{n-1},T^{n-1}x_n)\\
&\le \alpha_{n-1}d(f(x_{n-1}),T^{n-1}x_n)+(1-\alpha_{n-1})d(T^{n-1}x_{n-1},T^{n-1}x_n)\\
&\le \alpha_{n-1}d(f(x_{n-1}),T^{n-1}x_n)+(1-\alpha_{n-1})k_{n-1}d(x_{n-1},x_n) \to 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
d(x_n,Tx_n) &\le d(x_n,T^nx_n)+d(T^nx_n,Tx_n)\\
&= d(x_n,T^nx_n)+k_n d(T^{n-1}x_n,x_n) \to 0.
\end{aligned}$$

By Theorem 4.1.1, we have

$$\limsup_{n\to\infty}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_n\tilde{x}}\rangle\le 0. \tag{4.1.10}$$

Finally, we prove that $x_n\to\tilde{x}$ as $n\to\infty$. For any $n\in\mathbb{N}$, we set $y_n=\alpha_n\tilde{x}\oplus(1-\alpha_n)T^nx_n$. It follows from Lemma 2.5.16 and 2.5.17 (i), (ii) that

$$d^2(x_{n+1},\tilde{x}) = d^2(\alpha_n f(x_n)\oplus(1-\alpha_n)T^nx_n,\tilde{x})$$

$$
\begin{aligned}
&\le d^2(y_n,\tilde{x}) + 2\langle \overrightarrow{x_{n+1}y_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\le (\alpha_n d(\tilde{x},\tilde{x}) + (1-\alpha_n)d(T^n x_n,\tilde{x}))^2 \\
&\quad +2[\alpha_n\langle \overrightarrow{f(x_n)y_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle + (1-\alpha_n)\langle \overrightarrow{T^n x_n y_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle] \\
&\le (1-\alpha_n)^2 k_n^2 d^2(x_n,\tilde{x}) + 2[\alpha_n\alpha_n\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad +\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)T^n x_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad +\alpha_n(1-\alpha_n)\langle \overrightarrow{T^n x_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad +(1-\alpha_n)^2\langle \overrightarrow{T^n x_n T^n x_n}, \overrightarrow{x_{n+1}\tilde{x}}] \\
&\le (1-\alpha_n)^2 k_n^2 d^2(x_n,\tilde{x}) + 2[\alpha_n\alpha_n\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad +\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)T^n x_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad +\alpha_n(1-\alpha_n)\langle \overrightarrow{T^n x_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad +(1-\alpha_n)^2 d(T^n x_n, T^n x_n)d(x_{n+1},\tilde{x})] \\
&= (1-\alpha_n)^2 k_n^2 d^2(x_n,\tilde{x}) + 2[\alpha_n\alpha_n\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad +\alpha_n(1-\alpha_n)\langle \overrightarrow{f(x_n)T^n x_n}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad +\alpha_n(1-\alpha_n)\langle \overrightarrow{T^n x_n\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle] \\
&= (1-\alpha_n)^2 k_n^2 d^2(x_n,\tilde{x}) + 2\alpha_n\langle \overrightarrow{f(x_n)\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&= (1-\alpha_n)^2 k_n^2 d^2(x_n,\tilde{x}) + 2\alpha_n\langle \overrightarrow{f(x_n)f(\tilde{x})}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\quad +2\alpha_n\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\le (1-\alpha_n)^2 k_n^2 d^2(x_n,\tilde{x}) + 2\alpha_n\alpha d(x_n,\tilde{x})d(x_{n+1},\tilde{x}) \\
&\quad +2\alpha_n\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle \\
&\le (1-\alpha_n)^2 k_n^2 d^2(x_n,\tilde{x}) + \alpha_n\alpha(d^2(x_n,\tilde{x}) + d^2(x_{n+1},\tilde{x})) \\
&\quad +2\alpha_n\langle \overrightarrow{f(\tilde{x})\tilde{x}}, \overrightarrow{x_{n+1}\tilde{x}}\rangle.
\end{aligned}
$$

Since $\{\alpha_n\}$ and $\{x_n\}$ are bounded, there exists $M > 0$ such that

$$\frac{1}{1-\alpha\alpha_n}k_n^2 d^2(x_n,\tilde{x}) \le M.$$

It follows that

$$\begin{aligned} d^2(x_{n+1},\tilde{x}) &\leq \frac{(1-\alpha_n)^2k_n^2+\alpha\alpha_n}{1-\alpha\alpha_n}d^2(x_n,\tilde{x}) \\ &\quad +\frac{2\alpha_n}{1-\alpha\alpha_n}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle \\ &\leq \frac{(1-2\alpha_n)k_n^2+\alpha\alpha_n}{1-\alpha\alpha_n}d^2(x_n,\tilde{x}) \\ &\quad +\frac{2\alpha_n}{1-\alpha\alpha_n}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle+\alpha_n^2M \\ &\leq \left(1-\frac{1-2\alpha\alpha_n-(1-2\alpha_n)k_n^2}{1-\alpha\alpha_n}\right)d^2(x_n,\tilde{x}) \\ &\quad +\frac{2\alpha_n}{1-\alpha\alpha_n}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle+\alpha_n^2M \\ &\leq \left(1-\frac{1-2\alpha\alpha_n-(1-2\alpha_n)k_n^2}{1-\alpha\alpha_n}\right)d^2(x_n,\tilde{x}) \\ &\quad +\alpha_n\left(\frac{2}{1-\alpha\alpha_n}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle+\alpha_n^2M\right). \end{aligned}$$

Now, taking $\gamma_n=\frac{1-2\alpha\alpha_n-(1-2\alpha_n)k_n^2}{1-\alpha\alpha_n}, \delta_n=\alpha_n\left(\frac{2}{1-\alpha\alpha_n}\langle\overrightarrow{f(\tilde{x})\tilde{x}},\overrightarrow{x_{n+1}\tilde{x}}\rangle+\alpha_n^2M\right)$. Applying Lemma 2.1.20 and (4.1.10), we can conclude that $x_n\to\tilde{x}$. □

If $T:C\to C$ in Theorem 4.1.2 is a nonexpansive mapping, we can obtain the following result immediately.

Corollary 4.1.3. [58, Theorem 3.4] Let C be a closed convex subset of a complete CAT(0) space X, and let $T:C\to C$ be a nonexpansive mapping with $F(T)\neq\emptyset$. Let f be a contraction on C with coefficient $0<\alpha<1$. For the arbitrary initial point $x_0\in C$, let $\{x_n\}$ be generated by

$$x_{n+1}=\alpha_nf(x_n)\oplus(1-\alpha_n)Tx_n,\ \ \forall n\geq 0, \tag{4.1.11}$$

where $\{\alpha_n\}\subset(0,1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ which is equivalent to the variational inequality (4.1.2).

CHAPTER V

CONCLUSION

The following results are all main theorems of this dissertation:

1. Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$ and Ψ_i a μ_i-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be two quasi-nonexpansive mappings such that $I - T_j$ are demiclosed at zero with $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} GEP(G_i, \Psi_i)) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by

$$\begin{cases} x_1 \in H, \\ G_1(u_{n,1}, y) + \langle \Psi_1 x_n, y - u_{n,1} \rangle + \frac{1}{r_n} \langle y - u_{n,1}, u_{n,1} - x_n \rangle \geq 0, \ \ \forall y \in C, \\ G_2(u_{n,2}, y) + \langle \Psi_2 x_n, y - u_{n,2} \rangle + \frac{1}{r_n} \langle y - u_{n,2}, u_{n,2} - x_n \rangle \geq 0, \ \ \forall y \in C, \\ \cdots \\ \cdots \\ \cdots \\ G_k(u_{n,k}, y) + \langle \Psi_k x_n, y - u_{n,k} \rangle + \frac{1}{r_n} \langle y - u_{n,k}, u_{n,k} - x_n \rangle \geq 0, \ \ \forall y \in C, \\ \omega_n = \frac{1}{k} \sum_{i=1}^{k} u_{n,i}, \\ y_n = \gamma_n \omega_n + (1 - \gamma_n) T_1 \omega_n, \\ z_n = \beta_n y_n + (1 - \beta_n) T_2 \omega_n, \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n) z_n, \ \ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0, 1)$ and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Suppose the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

(C3) $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$;

(C4) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$.

2. Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. For each $j = 1, 2$, let $T_j : C \to H$ be two quasi-nonexpansive mappings such that $I - T_j$ are demiclosed at zero with $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} EP(G_i)) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by

$$\begin{cases} x_1 \in H, \\ G_1(u_{n,1}, y) + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n\rangle \geq 0, \ \ \forall y \in C, \\ G_2(u_{n,2}, y) + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n\rangle \geq 0, \ \ \forall y \in C, \\ \cdots \\ \cdots \\ \cdots \\ G_k(u_{n,k}, y) + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n\rangle \geq 0, \ \ \forall y \in C, \\ \omega_n = \frac{1}{k}\sum_{i=1}^{k} u_{n,i}, \\ y_n = \gamma_n\omega_n + (1-\gamma_n)T_1\omega_n, \\ z_n = \beta_n y_n + (1-\beta_n)T_2\omega_n, \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)z_n, \ \ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in (0, 1) and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Suppose the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

(C3) $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$;

(C4) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$.

3. Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)-(A4)$ and Ψ_i a μ_i-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be two nonexpansive mappings such that $\Omega := F(T_1)\cap F(T_2)\cap(\cap_{i=1}^{k} GEP(G_i, \Psi_i)) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by (3.1.1), where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in (0, 1) and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Suppose the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

(C3) $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$;

(C4) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$.

4. Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1)-(A4)$ and Ψ_i a μ_i-inverse strongly monotone mapping. For each $j = 1, 2$, let $T_j : C \to H$ be $(\alpha_j, \beta_j, \gamma_j)$-super hybrid mappings such that $\Omega := F(T_1) \cap F(T_2) \cap$

$(\cap_{i=1}^{k} GEP(G_i, \Psi_i)) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by

$$\begin{cases} x_1 \in H, \\ G_1(u_{n,1}, y) + \langle \Psi_1 x_n, y - u_{n,1}\rangle + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n\rangle \geq 0, \ \ \forall y \in C, \\ G_2(u_{n,2}, y) + \langle \Psi_2 x_n, y - u_{n,2}\rangle + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n\rangle \geq 0, \ \ \forall y \in C, \\ \cdots \\ \cdots \\ \cdots \\ G_k(u_{n,k}, y) + \langle \Psi_k x_n, y - u_{n,k}\rangle + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n\rangle \geq 0, \ \ \forall y \in C, \\ \omega_n = \frac{1}{k}\sum_{i=1}^{k} u_{n,i}, \\ y_n = \gamma_n \omega_n + (1-\gamma_n)\left(\frac{1}{1+\gamma_1} T_1 \omega_n + \frac{\gamma_1}{1+\gamma_1}\omega_n\right), \\ z_n = \beta_n y_n + (1-\beta_n)\left(\frac{1}{1+\gamma_2} T_2 \omega_n + \frac{\gamma_2}{1+\gamma_2}\omega_n\right), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n) z_n, \ \ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in (0, 1) and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, 2\mu_i)$ for some $a > 0$ and for all $i \in \{1, 2, \ldots, k\}$. Suppose the following conditions are satisfied.

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

(C3) $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$;

(C4) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$.

5. Let C be a nonempty closed convex subset of a Hilbert space H. For each $i = 1, 2, \ldots, k$, let $G_i : C \times C \to \mathbb{R}$ be a bifunction satisfying $(A1) - (A4)$. For each $j = 1, 2$, let $T_j : C \to H$ be $(\alpha_j, \beta_j, \gamma_j)$-super hybrid mappings such that $\Omega := F(T_1) \cap F(T_2) \cap (\cap_{i=1}^{k} EP(G_i)) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$

be defined by

$$\begin{cases} x_1 \in H, \\ G_1(u_{n,1}, y) + \frac{1}{r_n}\langle y - u_{n,1}, u_{n,1} - x_n\rangle \geq 0, \ \forall y \in C, \\ G_2(u_{n,2}, y) + \frac{1}{r_n}\langle y - u_{n,2}, u_{n,2} - x_n\rangle \geq 0, \ \forall y \in C, \\ \cdots \\ \cdots \\ \cdots \\ G_k(u_{n,k}, y) + \frac{1}{r_n}\langle y - u_{n,k}, u_{n,k} - x_n\rangle \geq 0, \ \forall y \in C, \\ \omega_n = \frac{1}{k}\sum_{i=1}^{k} u_{n,i}, \\ y_n = \gamma_n \omega_n + (1-\gamma_n)\left(\frac{1}{1+\gamma_1}T_1\omega_n + \frac{\gamma_1}{1+\gamma_1}\omega_n\right), \\ z_n = \beta_n y_n + (1-\beta_n)\left(\frac{1}{1+\gamma_2}T_2\omega_n + \frac{\gamma_2}{1+\gamma_2}\omega_n\right), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)z_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $(0, 1)$ and $\{u_n\} \subset H$ is a sequence and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Suppose the following conditions are satisfied.

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\liminf\limits_{n\to\infty} \beta_n(1-\beta_n) > 0$;

(C3) $\liminf\limits_{n\to\infty} \gamma_n(1-\gamma_n) > 0$;

(C4) $\lim\limits_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_\Omega u$.

6. Let C be a nonempty closed convex subset of a Hilbert space H. For each $j = 1, 2$, let $T_j : C \to H$ be $(\alpha_j, \beta_j, \gamma_j)$-super hybrid mappings such that

$F(T_1) \cap F(T_2) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by

$$\begin{cases} x_1 \in H, \\ y_n = \gamma_n x_n + (1-\gamma_n)\left(\frac{1}{1+\gamma_1}T_1 x_n + \frac{\gamma_1}{1+\gamma_1}x_n\right), \\ z_n = \beta_n y_n + (1-\beta_n)\left(\frac{1}{1+\gamma_2}T_2 x_n + \frac{\gamma_2}{1+\gamma_2}x_n\right), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)z_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in (0, 1) and $\{u_n\} \subset H$ is a sequence. Suppose the following conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

(C3) $\liminf_{n\to\infty} \gamma_n(1-\gamma_n) > 0$;

(C4) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_{F(T_1)\cap F(T_2)}u$.

7. Let C be a nonempty closed convex subset of a Hilbert space H. Let T be an (α, β, γ)-super hybrid mapping such that $F(T) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be defined by

$$\begin{cases} x_1 \in H, \\ z_n = \beta_n x_n + (1-\beta_n)\left(\frac{1}{1+\gamma}T x_n + \frac{\gamma}{1+\gamma}x_n\right), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)z_n, \ \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0, 1) and $\{u_n\} \subset H$ is a sequence. Suppose the following conditions are satisfied.

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$;

(C3) $\lim_{n\to\infty} u_n = u$ for some $u \in H$.

Then $\{x_n\}$ converges strongly to x^*, where $x^* = P_{F(T)}u$.

8. Let H be a real Hilbert space and let C a nonempty, closed and convex subset of H. Let $\alpha > 0$ and A an α-inverse-strongly monotone mapping of C into H. Let the set-valued maps $B : D(B) \subset C \to 2^H$ and $F : D(F) \subset C \to 2^H$ be maximal monotone. Let $J_\lambda = (I+\lambda B)^{-1}$ and $T_r = (I+rF)^{-1}$ be the resolvent of B for $\lambda > 0$ and F for $r > 0$, respectively. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $T : C \to C$ be a 2-generalized hybrid mapping such that $\Omega := F(T) \cap (A+B)^{-1}0 \cap F^{-1}0 \neq \emptyset$. Take $\mu, \gamma \in \mathbb{R}$ as follows:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\bar{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let the sequence $\{x_n\} \subset H$ be generated by

$$\begin{cases} x_1 = x \in H, \text{arbitrarily}, \\ z_n = J_{\lambda_n}(I - \lambda_n A)T_{r_n}x_n, \\ y_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k z_n, \\ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)y_n, \quad \forall\, n = 1, 2, \ldots, \end{cases}$$

where the sequences $\{\alpha_n\}, \{\lambda_n\}$ and $\{r_n\}$ satisfy the following restrictions :

(i) $\{\alpha_n\} \subset [0,1]$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) there exist constants a and b such that $0 < a \leq \lambda_n \leq b < 2\alpha$ for all $n \in \mathbb{N}$;

(iii) $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to a point p_0 of Ω, where p_0 is a unique fixed point of $P_\Omega(I - V + \gamma g)$. This point $p_0 \in \Omega$ is also a unique solution of the hierarchical variational inequality

$$\langle (V - \gamma g)p_0, q - p_0 \rangle \geq 0, \forall q \in \Omega.$$

9. Let H be a real Hilbert space and let C a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $T : C \to C$ be a 2-generalized hybrid mapping such that $\Gamma := F(T) \cap VI(C, A) \neq \emptyset$. Take $\mu, \gamma \in \mathbb{R}$ as follows:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\bar{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $\{x_n\} \subset H$ be a sequence generated by

$$\begin{cases} x_1 = x \in H, \text{arbitrarily}, \\ z_n = P_C(I - \lambda_n A)P_C x_n, \\ y_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k z_n, \quad \forall\, n = 1, 2, \dots, \\ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)y_n, \text{ for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset (0, 1)$, and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then $\{x_n\}$ converges strongly to a point p_0 of Γ, where p_0 is a unique fixed point of $P_\Gamma(I - V + \gamma g)$. This point $p_0 \in \Gamma$ is also a unique solution of the hierarchical variational inequality

$$\langle (V - \gamma g)p_0, q - p_0 \rangle \geq 0, \forall q \in VI(C, A).$$

10. Let H be a real Hilbert space and let C a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $B : D(B) \subset C \to 2^H$ be a maximal monotone Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $0 < k < 1$ and let g be a k-contraction of H into itself. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator

with $\bar{\gamma} > 0$ and $L > 0$. Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(A1) - (A4)$ and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either $(B1)$ or $(B2)$ holds. Let $T : C \to C$ be a 2-generalized hybrid mapping with $\Theta := F(T) \cap (A+B)^{-1}0 \cap MEP(G, \varphi) \neq \emptyset$. Take $\mu, \gamma \in \mathbb{R}$ as follows:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\bar{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $\{x_n\} \subset H$ be a sequence generated by

$$\begin{cases} x_1 = x \in H, \text{arbitrarily}, \\ G(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ z_n = J_{\lambda_n}(I - \lambda_n A)u_n, \\ y_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k z_n, \quad \forall\, n = 1, 2, \ldots, \\ x_{n+1} = \alpha_n \gamma g(x_n) + (I - \alpha_n V)y_n, \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$, and $\{r_n\} \subset (0, \infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Then $\{x_n\}$ converges strongly to a point p_0 of Θ, where p_0 is a unique fixed point of $P_\Theta(I - V + \gamma g)$. This point $p_0 \in \Theta$ is also a unique solution of the hierarchical variational inequality

$$\langle (V - \gamma g)p_0, q - p_0 \rangle \geq 0, \forall q \in \Theta.$$

11. Let C be a closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_n = 1$ such that $F(T) \neq \emptyset$. Let f be a contraction on C with coefficient $0 < \alpha < 1$. Let $\{\alpha_n\}$ be a sequence of real numbers with $0 < \alpha_n < 1$. Then the following statement hold:

(i) For each $n \in \mathbb{N}$, if $\frac{k_n-1}{\alpha_n} < 1-\alpha$, then there exists y_n such that

$$y_n = \alpha_n f(y_n) \oplus (1-\alpha_n)T^n y_n.$$

(ii) If $\alpha_n \to 0$ and $\frac{k_n-1}{\alpha_n} \to 0$ as $n \to \infty$, then $\{y_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ which is equivalent to the following variational inequality:

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T).$$

12. Let C be a closed convex subset of a complete CAT(0) space X, and let $T: C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, +\infty)$ and $\lim_{n\to\infty} k_n = 1$ such that $F(T) \neq \emptyset$. Let f be a contraction on C with coefficient $0 < \alpha < 1$. For the arbitrary initial point $x_0 \in C$, let $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0$;

(iv) T satisfies the asymptotically regularity $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$.

Then $\{x_n\}$ converges strongly as $n \to \infty$ to $\tilde{x}$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ which is equivalent to the variational inequality:

$$\langle \overrightarrow{\tilde{x}f(\tilde{x})}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T).$$

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