WELL-POSEDNESS FOR SET OPTIMIZATION PROBLEMS INVOLVING SET ORDER RELATIONS

PHAM THI VUI

A Thesis Submitted to the Graduate School of Naresuan University
in Partial Fulfillment of the Requirements
for the Doctor of Philosophy Degree in Mathematics
March 2019

Copyright 2019 by Naresuan University

This thesis entitled " Well-posedness for set optimization problems involving set order relations"

by Pham Thi Vui

has been approved by the Graduate School as partial fulfillment of the requirements for the Doctor of Philosophy Degree in Mathematics of Naresuan University

Oral Defense Committee

..... Chair

(Professor Poom Kumam, Ph.D.)

..... Advisor

(Associate Professor Rabian Wangkeeree, Ph.D.)

..... Internal Examiner

(Professor Somyot Plubtieng, Ph.D.)

..... Internal Examiner

(Associate Professor Narin Petrot, Ph.D.)

..... External Examiner

(Associate Professor Lam Quoc Anh, Ph.D.)

Approved

.....

(Professor Paisarn Muneesawang, Ph.D.)

Dean of the Graduate School

2015

ACKNOWLEDGEMENT

I would like to express the deepest appreciation to my advisors, Associate Professor Rabian Wangkeeree and Associate Professor Dr. Lam Quoc Anh, for the continuous support of my studying and doing research. I am deeply thanks to their inspired guidances and encouragements which enable me to carry out my study successfully.

This work contains a number of improvement based on comments and suggestions provided by advisors, proposal and thesis commitees: Associate Professor Dr. Rabian Wangkeeree, Associate Professor Dr. Lam Quoc Anh, Professor Dr. Somyot Plubtieng, Professor Dr. Poom Kumam, Associate Professor Dr. Narin Petrot, Assistant Professor Dr. Anchalee Kaewcharoen, Assistant Professor Dr. Chakkrid Klin-eam, Assistant Professor Dr. Kasamsuk Ungchittrakool. I would like to express my sincere gratitude to all of them for their generous assistance.

It is my pleasure to express my great thanks to my teachers in Thailand and my former teachers in Vietnam for valuable lectures and guidances to build the background for this research.

I would like to thank my family for supporting me spiritually throughout studying this program and my life in general. I would also like to thank to my friends: Thanatporn Bantaojai, Panatda Boonman, Nithirat Sisarat, Thanatchaporn Sirichunwijit, Chanoksuda Khantree, Jutamas Kerdkaew, Anteneh Getachew Gebrie, Pornpimon Boriwan, Natthaphon Artsawang, Tran Quoc Duy, Tran Ngoc Tam, Nguyen Thanh Qui, Nguyen Hieu Thao for their helps, encouragements and their great friendship.

I express my special thanks to the scholars of International Student Scholarship, Naresuan University, Thailand and 911 Scholarship, Ministry of Education and Training, Vietnam for financial support.

Pham Thi Vui

Title	WELL-POSEDNESS FOR SET OPTIMIZATION PROBLEMS INVOLVING SET ORDER RELATIONS
Author	Pham Thi Vui
Advisor	Associate Professor Rabian Wangkeeree, Ph.D.
Academic Paper	Thesis Ph.D. in Mathematics, Naresuan University, 2018
Keywords	Set order relation, Set optimization problem, Existence condition, L-well-posedness, B-well-posedness, Levitin-Polyak well-posedness, Metrically well-posedness, Stability.

ABSTRACT

In this thesis, set optimization problems involving three kinds of set order relations, namely lower set less relation, upper set less relation and set less relation, are concerned. We mainly focus on three types of well-posedness for such problems. More precisely, we first aim to investigate well-posedness in the sense of Loridan (L-well-posedness in short). We next aim to study well-posedness in the sense of Bednarczuk (B-well-posedness in brief). We also aim to survey Levitin-Polyak well-posedness (LP well-posedness for shortly). For each types of these well-posednesses, we examine both pointwise and global ones. To achieve our goals, we introduce some new concepts related to well-posedness for set optimization problems and propose generalized form of notions related to convexity and semicontinuity involving cone of a set-valued map in the literature. After that, necessary and sufficient conditions for the above problems to be well-posed are given by employing useful tools in set-valued analysis, convex analysis. Moreover, characterizations of well-posedness for set optimization problems are obtained by using Kuratowski measure of noncompactness.

In many publications, the authors often assume that the solution set is nonempty when they just want to focus on studying well-posedness. To make our investigation on well-posedness is completed, we also survey solution existence con-

ditions of these problems. Moreover, to the best of our knowledge, conditions to guarantee the existence of solutions of set optimization problems involving the set less relation could not find out in the literature.

Furthermore, we also pay more attention to study links between well-posedness and stability of problems mentioned above because of the important role of stability and relationships of stability and well-posedness.

Many examples and comments are given to illustrate our results and compare them with existing ones in the literature.

LIST OF CONTENTS

Chapter Page

I INTRODUCTION 1

II PRELIMINARIES 8

Basic concepts 8

Order relation induced by cone and set order relations 14

Semicontinuity and cone semicontinuity of a set-valued map 19

Hausdorff set-convergence, KKM-map and useful tools 21

III L-WELL-POSEDNESS FOR SET OPTIMIZATION PROBLEMS 24

Set approach and the investigated problem (P_α) 25

Existence conditions of solutions 27

Global L-well-posedness 33

Pointwise L-well-posedness 42

Links between L-well-posedness and stability 44

IV B-WELL-POSEDNESS FOR SET OPTIMIZATION PROBLEMS 51

Global B-well-posedness 52

Pointwise B-well-posedness 56

Links between pointwise B-well-posedness and pointwise L-well-posedness 59

LIST OF CONTENTS (CONT.)

Chapter Page

V LEVITIN-POLYAK (LP) WELL-POSEDNESS FOR SET OPTIMIZATION PROBLEMS 61

Pointwise LP well-posedness and generalized pointwise LP well-posedness 62

Metrically LP well-posedness 74

Global LP well-posedness 77

Links between LP well-posedness and stability 83

VI CONCLUSION 89

REFERENCES 92

BIOGRAPHY 100

LIST OF FIGURES

Figure Page

1 Comparision of two vectors by cone 15

2 Illustration of lower set less relation $A \leq^l B$ 15

3 Illustration of $A \leq^l B$ but $A \not\leq^u B$ 16

4 Illustration of upper set less relation $A \leq^u B$ 16

5 Illustration of $A \leq^u B$ but $A \not\leq^l B$ 16

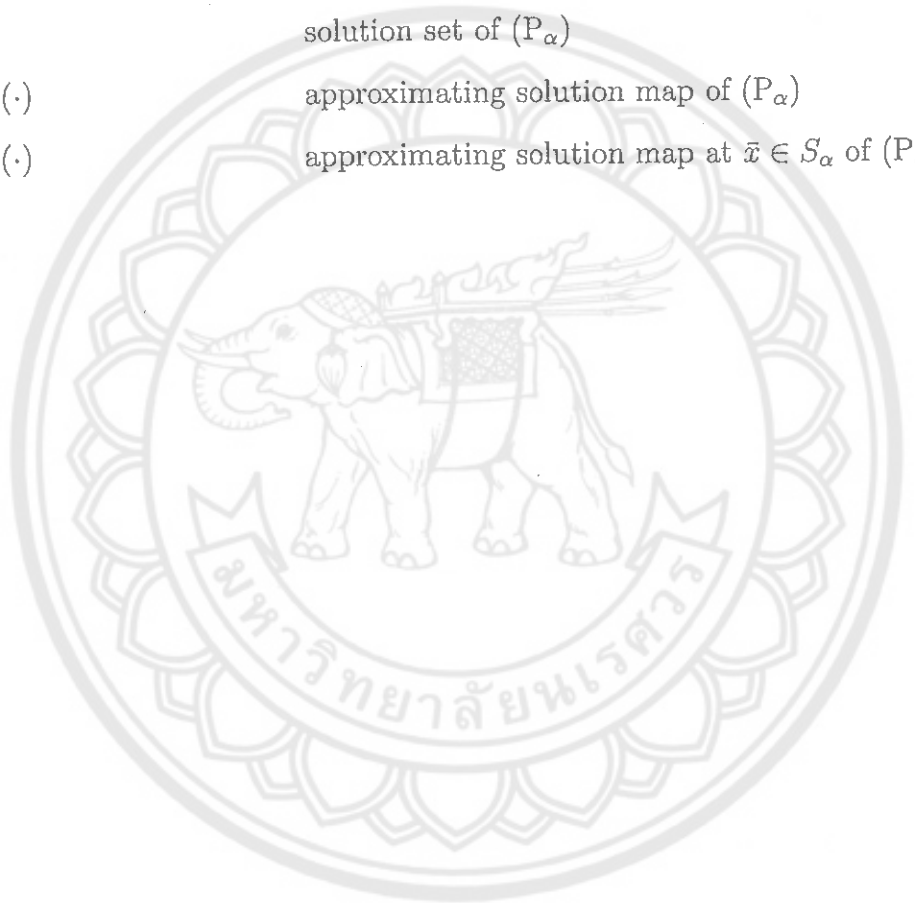
6 Illustration of set less relation $A \leq^s B$ 16

LIST OF SYMBOLS

$A + B$	algebraic sum of A and B
$A + B$	algebraic difference of A and B
$B(x_0, r)$	open ball centered at x_0 with radius r
$\overline{B}(x_0, r)$	closed ball centered at x_0 with radius r
$x \to \bar{x}$	x converges to $\bar{x}$
$\|\cdot\|$	norm
$\mathrm{int}A$	interior of A
$\mathrm{cl}A$	closure of A
$\mathrm{bd}A$	boundary of A
$A \leq^l B$	lower set less relation
$A \leq^u B$	upper set less relation
$A \leq^s B$	set less relation
$A \leq^\alpha B$	set order relations with $\alpha \in \{l, u, s\}$
$A \sim^\alpha B$	both $A \leq^\alpha B$ and $B \leq^\alpha A$ with $\alpha \in \{l, u, s\}$
$A \ll^l B$	weak lower set less relation
$A \ll^u B$	weak upper set less relation
$A \ll^s B$	weak set less relation
$H^*(A, B)$	Hausdorff distance from set A to set B
$H(A, B)$	Hausdorff distance between sets A and B
$A_n \multimap A$	lower Hausdorff set-convergence
$A_n \rightharpoonup A$	upper Hausdorff set-convergence
$A_n \to A$	Hausdorff set-convergence
$\mu(M)$	Kuratowski measure of noncompactness of set M
$F : X \rightrightarrows Y$	set-valued map
$f : X \to Y$	single-valued map

LIST OF SYMBOLS (CONT.)

$[a, b)$	half-open line segment starting at a, including a, excluding b
$[a, b]$	closed line segment
$]a, b[$	open line segment
(a, b)	point in space $\mathbb{R}^2$
$\bar{x}$	solution of (P_α)
S_α	solution set of (P_α)
$\widehat{S}_\alpha(\cdot)$	approximating solution map of (P_α)
$\widehat{S}_\alpha^{\bar{x}}(\cdot)$	approximating solution map at $\bar{x} \in S_\alpha$ of (P_α)



CHAPTER I

INTRODUCTION

Set-valued optimization plays an important role in optimization and has attracted a great deal of attention of mathematicians in recent years. It deals with optimization problems which the objective map and/or the constraint map(s) are set-valued maps. It is clear that single-valued maps work as a particular case of set-valued maps, and hence set-valued optimization provides an important generalization and unification of scalar optimization as well as vector optimization.

It is worth noting that there are two approaches to formulate optimality notions for set-valued optimization problems, namely the vector approach and the set approach. These criteria depend on the way that the notion of minimality is defined. In the *vector approach*, optimal solutions are defined as the efficient points of the union of all images of the set-valued objective map (see e.g.,[1, 2, 3, 4] and the references therein). Hence, the set-valued optimization problem is called set-valued vector optimization problem in this approach. On the other hand, the *set approach* is based on set order relations defined on the power set of the objective space, which was firstly introduced by Kuroiwa et al. [5] in 1997, and originally studied independently by Young [6] and Nishnianidze [7]. By set approach one compares all images of the set-valued objective map and hence in this approach the set-valued optimization problem is called *set optimization problem* (see e.g.,[8, 9, 10, 11, 12] and the references therein). Both of them may be considered as generalizations of the usual definition of minimality in vector optimization. In fact, when objective map is a vector single-valued map, they reduce to the known concept of efficiency. In this thesis, set approach is applied and set order relations are used to define optimality notions for set-valued optimization problems.

We would like to give a brief review of set order relations. The introduction of set order relations was presented by Kuroiwa et al. [5], by Young [6] and Nishnianidze

[7]. Kuroiwa [11] showed six relations among sets including lower set less relation, upper set less relation and set less relation (combination of the lower and the upper set less relation). Later on, Jahn and Ha [10] introduced some new set order relations. As pointed out in [10], the set less relation is generalized and more appropriate in practical problems than both the lower and upper set less relations. Furthermore, the set less relation plays a center role in relationships with other new order relations for sets proposed in [10] which are more useful in set optimization. Lower set less relation and upper set less relation were studied in some publications [13, 14] and the references therein. To the best of our knowledge, there has been no publication about well-posedness, solution existence condition and stability for optimization problems involving the set less relation and new types of set order relations proposed in [10]. Due to a prospective issue, it should study further about these topics.

The set approach seems to be more natural and interesting than vector approach whenever we need to consider preferences over sets. Set optimization problems play important roles and useful applications in the practical situations. Many important and significant applications of set order relations were studied and discussed. The possibly less order relation has been applied to interval arithmetic and Fortran compiler f95 of Sun Microsystems. order relations was presented by Neukel [15] in the project investigating relationship between noise disturbance and quality of life in the region surrounding the Frankfurt Airport in Germany. Another application of set order relations in the field of finance about measures of risk was found by Hamel and Heyde [16]. Some simple but very useful applications of set order relation can be found and applied in daily activities, for instance, comparing teams of football players, ranking group of students on the same academic level,... For further reading and references, we refer to studies [13, 14, 17, 18, 19, 20, 21, 22] and the reference therein.

With the first introduction was given two decades ago, set optimization can be seen as a very young direction in the field of optimization. However, in the role

of an important and prospective issuse, it has attracted a great deal of attention of mathematicians. As pointed out by Jahn in [23], *the set optimization criterion opens a new and wide field of research.* Many important and interesting results have been obtained in different topics in this area such as solution existence [24, 25, 26], optimality conditions [27, 28, 29], nonlinear scalarization [9, 30, 31], Ekeland variational principles [8, 32], duality theory [33], well-posedness [13, 34, 35], and stability [36, 37, 38, 39].

Well-posedness plays an important role in both theory results and numerical methods. Many mathematicians have paid much attention on this topic (see e.g., [40, 41, 42, 43] and the reference therein). In 1966, Tikhonov [44] introduced a definition of well-posedness for unconstrained optimization problems which is called *Tikhonov well-posedness*. This concept requires two conditions. The first condition is the existence and uniqueness of the solution and the second one is the convergence of each minimizing sequence to the unique solution. On this topic, many results have been devoted to a lot of important problems such as variational inequalities [45], equilibrium problems [46], inclusion problems [47] and the references therein. Later on, generalizations of Tikhonov well-posedness were introduced and studied widely. In 1995, Loridan [48] considered the vector optimization problem and introduced a definition of well-posedness based on the convergence of a subsequence of a minimizing sequence. One of extensions of Tikhonov well-posedness is the so-called *B-well-posedness* proposed by Bednarczuck for vector optimization problems in [49]. After that, B-well-posedness has been intensively considered for various problems related to optimization [47, 50, 51, 52, 53]. An initial concept of the *well-posedness by perturbations* (also named extended well-posedness) for a optimization problem is proposed by Zolezzi [54, 55]. This concept established a form of continuous dependence of the solutions upon a parameter. It should be noted that the fundamental requirement in the Tikhonov well-posedness is that every minimizing sequence must lie in the feasible set. Levitin-Polyak [40] introduced a new notion of well-

posedness that strengthened Tikhonov's concept as it required the convergence to the optimal solution of each sequence belonging to a larger set of minimizing sequences. More precisely, *the well-posedness in the sense of Levitin-Polyak* allows the minimizing sequences can be outside of the feasible set and requires the distance of the minimizing sequence from the feasible region to approach zero. There is a large amount of works related to well-posedness for different important optimization problems [41, 42, 46, 48, 50, 56, 57, 58, 59, 60].

In the field of set optimization problems, the concept of well-posedness was firstly given by Zhang et al. [61] in 2009. In this publication, the authors introduced three kinds of well-posedness for set optimization problems involving lower set less relation. Using the scalarization method [62], they established relationships between the well-posedness of a set optimization problem and the well-posedness of scalar optimization problems. Moreover, the authors provided some criteria for set optimization problems to be well-posed. Later, Gutiérrez et al. [34] extended some results of Zhang et al. [61] by using another form of the scalarization function due to [9]. Recently, Long and Peng [13] introduced three kinds of well-posedness in the sense of Bednarczuk, named B-well-posedness, for set optimization problems involving upper set less relation and established some relationships among them. They also gave characterizations of well-posedness for these problems. After that, Crespi et al. [63] introduced a definition of well-posedness which slightly generalizes the one in Zhang et al. [61], and investigated well-posedness properties of set optimization problems. Very recently, Khoshkhabar-Amiranloo and Khorram [64] studied pointwise Levitin-Polyak well-posedness for set optimization problems involving lower set less relation and obtained some criteria and characterizations of this issue. In a word, studying on well-posedness for set optimization problems is a prospective topic, and hence it should be study more on different types of well-posedness for these problems involving various kinds of set order relations. This thesis focuses mainly on three types of well-posedness, namely L-well-posedness, B-well-posedness, LP well-posedness, involving

three kinds of set order relations introduced by Kuroiwa under the names lower set less relation, upper set less relation and set less relation.

In a close relationship with well-posedness, the *stability* analysis of the solution maps for optimization problems is an important direction in studying optimization problems. As we know, the study of stability concerns on how the solution set maps vary with the changes in the parameters. Generally speaking, stability requires that for a small perturbations or a small measurement error of the parameters, the induced perturbations of solution set map is also very small. Investigating on stability of optimization problems can be understood as studying on some properties of the solution maps such as the upper semicontinuity, lower semicontinuity, continuity, Hölder/ Lipschitz continuity or (generalized) differentiability [36, 37, 38, 65, 66]. In this research, we aim to give connections between well-posedness and stability for concerned problems in the sense of upper semicontinuity of approximating solution map.

Solution existence is a crucial topic in studying any mathematics model. In set optimization, existence results for problems involving lower and/or upper set less relation were presented in [67] under assumptions on compactness of the domain of objective map and reported in [24, 25] with conditions about the cone-regularity and cone-semicompactness. On the other hand, using nonlinear scalarization method, existence conditions of optimal solutions were established in [9, 68]. However, there is no publications mentioned conditions to guarantee existence conditions of solutions of set optimization problems involving set less relation. Therefore, our goals include giving solution existence conditions for these problems instead of just focusing on well-posedness properties by a assumption that solution set is empty as many publications in the literature.

Due to the important role of set optimization problems and the high volume of active research in optimization, there is a wide prospect to do research in optimization theory. These works mentioned above motivate strongly us to study

well-posedness as well as stability and solution existence for such problems involving different kinds of set order relations. We aim to investigate set optimization problems involving different kinds of set order relations. We focus on the set less relation because it has more appropriate applications in real life and furthermore it is a generalization form of both lower set less relation and upper set less relation. To the best of our knowledge, there is no publication about well-posedness of set optimization problem with the set less relation.

In the following we give a description of the contents of this thesis.

Chapter II. This chapter includes some notations, definitions, and some useful results that will be deplicated in later chapters.

Chapter III. Studying on L-well-posedness for set optimization problems with three types of set order relations is the main goal of this chapter. Various kinds of well-posedness for these problems (pointwise L-well-posedness, global L-well-posedness, metrically α-well-posedness, weak metrically α-well-posedness) and their relationships are concerned. We give necessary and sufficient conditions for set optimization problems to be pointwise/global L-well-posed. Moreover, Kuratowski measure of noncompactness is applied to survey characterizations of L-well-posedness for set optimization problems. Furthermore, approximating solution maps and their stability are researched to propose links between stability of the approximating problem and well-posedness of the set optimization problem. In addition to studying well-posedness, existence conditions of such problems are given by employing properties of a KKM-map.

Chapter IV. B-well-posedness for set optimization problems involving three kinds of set order relations is presented in this chapter. We give characterizations and sufficient and/or necessary conditions of B-well-posedness. Moreover, relationships between pointwise L-well-posedness and pointwise B-well-posedness are studied.

Chapter V. This chapter investigates LP well-posedness for set optimization problems involving three types of set order relations. Necessary and sufficient

conditions for the reference problems to be LP well-posed are given. Furthermore, using the Kuratowski measure of noncompactness, we study characterizations of LP well-posedness for set optimization problems. Moreover, links between stability and LP well-posedness of such problems are established via the study on approximating solution maps. We also study metrically LP well-posedness and bridges between it and LP well-posedness.

Chapter VI. We give the concluding remarks. Some obtained results in this study are summarized. Moreover, we also give some comments about prospect remarks related to what we done to develop this research.

CHAPTER II

PRELIMINARIES

In this chapter, several of notations, definitions, and useful results used in what follows are recalled.

2.1 Basic concepts

We first recall the definition of a real vector space and present some other notations.

Definition 2.1.1. [23] Let X be a given set. Assume that an addition on X, i.e. a map from $X \times X$ into X, and a scalar multiplication on X, i.e. a map from $\mathbb{R} \times X$ into X, are defined. The set X is called a *real vector space* (or real linear space), if the following axioms are satisfied for arbitrary $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$:

(V1) $x + y = y + x$;

(V2) $(x + y) + z = x + (y + z)$;

(V3) there is an element $0_X \in X$, called the *zero vector* of X, such that $x + 0_X = x$ for all $x \in X$;

(V4) for every element $x \in X$, there exists an element $-x \in X$, called the *additive inverse* or the *negative* of x, such $x + (-x) = 0_X$;

(V5) $\alpha(x + y) = \alpha x + \alpha y$;

(V6) $(\alpha + \beta)x = \alpha x + \beta x$;

(V7) $(\alpha\beta)x = \alpha(\beta)x$;

(V8) $1x = x$.

Definition 2.1.2. [23] Let A and B be nonempty subsets of a real linear space X. Then, we define the algebraic sum of A and B as

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\},$$

and the algebraic difference of A and B as

$$A - B = \{a - b \mid a \in A \text{ and } b \in B\}.$$

For an arbitrary $\lambda \in \mathbb{R}$, the notation λA will be used as

$$\lambda A = \{\lambda a \mid a \in A\}.$$

Definition 2.1.3. [69] A *metric space* is an ordered pair (X, d) consiting of a set X together with a function $d : X \times X \to \mathbb{R}$ satisfying for all $x, y, z \in X$

(M1) $d(x, y) \geq 0$;

(M2) $d(x, y) = 0$ if and only if $x = y$;

(M3) $d(x, y) = d(y, x)$ (Symmetry);

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality).

The defined metric space is denoted by (X, d) or X and $d(x, y)$ is referred to as the distance between x and y.

Definition 2.1.4. [69] Given a point x_0 of the metric space X and a real number $r > 0$. Open ball and closed ball are defined respectively as the following:

(i) $B(x_0, r) = \{x \in X \mid d(x, x_0) < r\}$.

(ii) $\overline{B}(x_0, r) = \{x \in X \mid d(x, x_0) \leq r\}$.

The point x_0 is called the *center* and r is called the *radius* of these balls.

Definition 2.1.5. [70] A subset M of a metric space X is said to be

(i) *open* if and only if for each $x \in M$, there is $r > 0$ such that $B(x, r) \subset M$.

(ii) *closed* if and only if its complement (in X) is open, i.e., $M^c = X \setminus M$ is open, where $(\cdot)^c$ is the complement of $(\cdot)$.

Definition 2.1.6. [70] A sequence $\{x_n\}$ in a metric space (X, d) is said to be *convergent* if there is an element $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Then, x is called the *limit* of $\{x_n\}$ and we denote

$$\lim_{n\to\infty} x_n = x$$

or, simply, $x_n \to x$. We say that $\{x_n\}$ *converges* to x or has the limit x. If $\{x_n\}$ is not convergent, it is said to be *divergent.*

Lemma 2.1.7. [70]. *Let (X, d) be a metric space. Then,*

(i) a convergent sequence in X is bounded and its limit is unique.

(ii) if $x_n \to x$ and $y_n \to y$ in X, then $d(x_n, y_n) \to d(x, y)$.

Definition 2.1.8. [70] A sequence $\{x_n\}$ in a metric space (X, d) is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$d(x_m, x_n) < \varepsilon \qquad \text{for every} \quad m, n \geq N.$$

The space (X, d) is said to be *complete* if every Cauchy sequence in (X, d) converges (i.e., has a limit which is an element of X).

Theorem 2.1.9. [70] *Every convergent sequence in a metric space is a Cauchy sequence.*

Definition 2.1.10. [70] A norm on a vector space X is a real-valued function defined on X whose value at an $x \in X$ is denoted by $\|x\|$ such that for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$, we have:

(N1) $\|x\| \geq 0$;

(N2) $\|x\| = 0 \Leftrightarrow x = 0$;

(N3) $\|\alpha x\| = |\alpha|\|x\|$;

(N4) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality).

A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\| \qquad \text{for all } x, y \in X$$

and is called the *metric induced by the norm.* The normed space just defined is denoted by $(X, \|\cdot\|)$ or simply by X.

Definition 2.1.11. [70] Let M be a nonempty subset of a normed space X. Then M is *bounded* if and only if there is a positive number c such that $\|x\| \leq c$ for every $x \in M$.

Definition 2.1.12. [70] A metric space X is said to be *compact* if every sequence in X has a convergent subsequence. A subset M of X is said to be *compact* if M is compact considered as a subspace of X, i.e., if every sequence in M has a convergent subsequence whose limit is an element of M.

Lemma 2.1.13. [70] *A compact subset M of a metric space is closed and bounded.*

However, for a finite dimensional normed space we have the following result:

Theorem 2.1.14. [70] *In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.*

Definition 2.1.15. [69] A topology on a set X is a collection τ of subsets of X, called the open sets, satisfying the following conditions:

(T1) $\emptyset \in \tau$ and $X \in \tau$;

(T2) the union of any members of τ is a member of τ;

(T3) the intersection of finitely many members of τ is a member of τ.

The ordered pair (X, τ) is called *a topological space*, sometimes abbreviated "X is topological space" when no confusion can result about τ.

Definition 2.1.16. [69] If X is topological space and $E \subset X$, we say E is closed if and only if $M^c = X \setminus M$ is open.

Definition 2.1.17. [69] Let X be a topological space, let U be a subset of X and let x be a given element of X.

(i) The set U is called *a neighborhood of* x if and only if there is an open set V with $x \in V \subset U$.

(ii) x is called *an interior element of* U if and only if there is a neighborhood V of x contained in U.

(iii) The set of all interior elements of U is called the interior of U and it is denoted by int$U$.

Definition 2.1.18. [69] Let F be a subset of a topological space X. Then the *closure* of F is the smallest closed set containing F. The closure of F is denoted by cl(F).

Definition 2.1.19. [69] A sequence $\{x_n\}$ in a topological space X is said to *converge* to $x \in X$, and we write $x_n \to x$, if and only if for each neighborhood U of x, there is some positive integer n_0 such that $n \geq n_0$ implies $x_n \in U$.

Definition 2.1.20. [69] A set Λ is a directed set if and only if there is a relation $\leq$ on Λ satisfying:

(i) $\lambda \leq \lambda$ for each $\lambda \in \Lambda$;

(ii) if $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$, then $\lambda_1 \leq \lambda_3$;

(iii) if $\lambda_1, \lambda_2 \in \Lambda$, then there is some $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$, $\lambda_1 \leq \lambda_3$.

Definition 2.1.21. [69] A net in a set X is a function $P : \Lambda \to X$ where Λ is a directed set. The point $P(\lambda)$ is usually denoted by x_λ.

Definition 2.1.22. [69] Let $\{x_\lambda\}$ be a net in space X. Then, $\{x_\lambda\}$ converges to $x \in X$ if and only if for each neighborhood U of x, there is some $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $x_\lambda \in U$.

Definition 2.1.23. [69] A topological space (X, τ) is said to be a *Hausdorff topological space* (or a T_2-space) if for each pair of distinct points a and b in X, there exist open sets U and V such that $a \in U, b \in V$, and $U \cap V = \emptyset$.

Theorem 2.1.24. [69] *In a Hausdorff topological space, every convergent sequence has a unique limit.*

Definition 2.1.25. [69] A topological space X is said to be *compact* if every open cover has a finite subcover, i.e., if whenever $X = \bigcup_{i \in I} G_i$, where G_i is an open set, then $X = \bigcup_{i \in J} G_i$ for some finite subset J of I.

Definition 2.1.26. [69] A subset M of a topological space X is said to be *compact* if every open cover has a finite open subcover, i.e., if whenever $M \subset \bigcup_{i \in I} G_i$, where G_i is an open set, then $M \subset \bigcup_{i \in J} G_i$ for some finite subset J of I.

Theorem 2.1.27. [69] *A compact subset of a Hausdorff topological space is closed.*

Definition 2.1.28. [23] Let X be a real vector space and let τ be a topology on X. (X, τ) is called a real topological vector space, if addition and scalar multiplication are continuous maps, i.e. the maps

$$(x, y) \mapsto x + y \quad \text{with } x, y \in X,$$

$$(\alpha, x) \mapsto \alpha x \quad \text{with } \alpha \in \mathbb{R} \text{ and } x \in X$$

are continuous on $X \times X$ and $\mathbb{R} \times X$, respectively. In many situations we use, for simplicity, the notation X instead of (X, τ) for a real topological vector space.

2.2 Order relation induced by cone and set order relations

Definition 2.2.1. [23] Let C be a nonempty subset of a real vector space X.

(i) The set C is called a *cone*, if

$$x \in C, \lambda \geq 0 \Longrightarrow \lambda x \in C.$$

(ii) A cone C is called *pointed*, if

$$C \cap (-C) = \{0_X\}.$$

Definition 2.2.2. [23] Let M be a subset of a real vector space X. The set M is called *convex* if and only if for every $x, y \in M$ we have

$$\lambda x + (1 - \lambda)y \in M \quad \text{for all} \quad \lambda \in [0, 1].$$

Lemma 2.2.3. [23] *A cone C in a real vector space is convex if and only if*

$$C + C \subset C.$$

The space Y is endowed with an order relation induced by cone K in the following way

$$x \leq_K y \Leftrightarrow y - x \in K,$$

$$x <_K y \Leftrightarrow y - x \in \text{int}K.$$

Example 2.2.4. Let $X = \mathbb{R}^2, K = \mathbb{R}^2_+, a = (-2, 0.5), b = (0, 2), c = (-1, 2.5)$. Then, $b - a = (2, 1.5) \in K, b - c = (1, -0.5) \notin K$. We have $a \leq_K b$ and $c \not\leq_K b$.

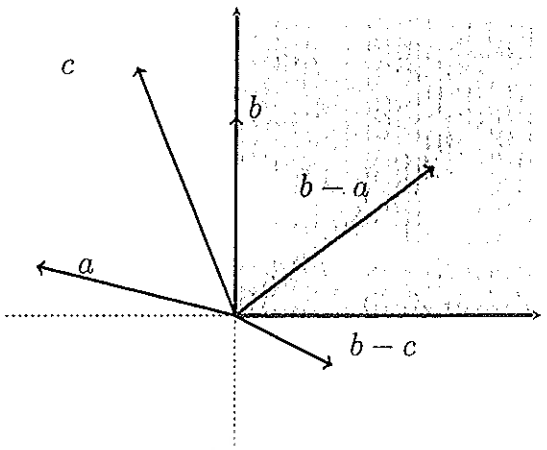


Figure 1 Comparision of two vectors by cone.

The cone K induces various set orderings in Y. These such orderings, given below, were introduced in [6, 7, 10, 71]. Consider $\mathcal{P}(Y)$, the family of all nonempty subsets of Y, for $A, B \in \mathcal{P}(Y)$ *lower set less relation*, *upper set less relation* and *set less relation*, firstly introduced by Kuroiwa [5] in 1997, respectively, are defined by

$$A \leq^l B \Leftrightarrow B \subset A + K,$$

$$A \leq^u B \Leftrightarrow A \subset B - K,$$

$$A \leq^s B \Leftrightarrow A \subset B - K \text{ and } B \subset A + K.$$

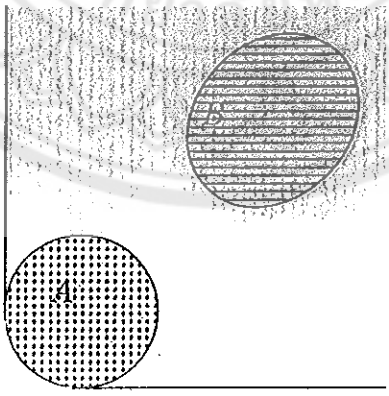


Figure 2 Illustration of lower set less relation $A \leq^l B$.

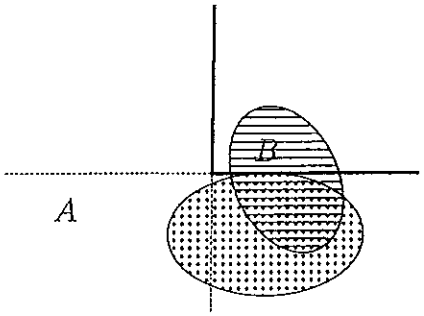


Figure 3 Illustration of $A \leq^l B$ but $A \not\leq^u B$.

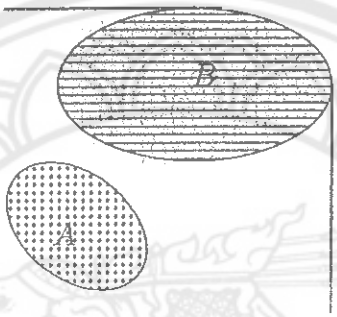


Figure 4 Illustration of upper set less relation $A \leq^u B$.

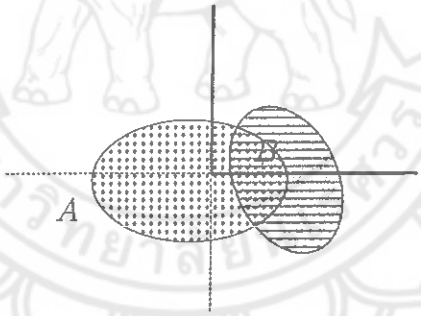


Figure 5 Illustration of $A \leq^u B$ but $A \not\leq^l B$.

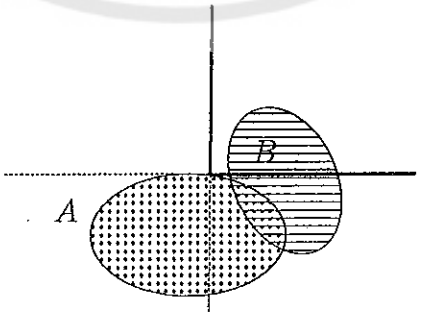


Figure 6 Illustration of set less relation $A \leq^s B$.

We also have weak lower set less relation, weak upper set less relation, weak set less relation defined respectively as the following

$$A \ll^l B \Leftrightarrow B \subset A + \text{int}K,$$

$$A \ll^u B \Leftrightarrow A \subset B - \text{int}K,$$

$$A \ll^s B \Leftrightarrow A \subset B - \text{int}K \text{ and } B \subset A + \text{int}K.$$

The notation $A \sim^\alpha B$ with $\alpha \in \{l, u, s\}$ means both $A \leq^\alpha B$ and $B \leq^\alpha A$ are satisfied.

We observe from the definitions of set order relations that $\leq^s$ is a combination of $\leq^l$ and $\leq^u$. Relationships between $\leq^l$ and $\leq^u$ were given in Remark 2.6.10 of [19] as the following

$$A \leq^l B \Leftrightarrow -B \leq^u -A.$$

Some properties about these set order relations are demonstrated in next results.

Proposition 2.2.5. *The following statements are true:*

(i) *if* $A \leq^\alpha B$, *then* $\lambda A \leq^\alpha \lambda B$, $\forall \lambda > 0$;

(ii) $A \leq^l B \Leftrightarrow \lambda A \geq^u \lambda B$, $\forall \lambda < 0$;

(iii) $A \leq^s B \Leftrightarrow \lambda A \geq^s \lambda B$, $\forall \lambda < 0$.

Proof. (i) We give the proof of the assertion (i) for the case $\alpha = s$, proofs of this assertion for other cases $\alpha = l$ and $\alpha = u$ are similar. Since $A \leq^s B$, $B \subset A + K$ and $A \subset B - K$. Clearly, for any $\lambda > 0$, we get $\lambda B \subset \lambda A + K$ and $\lambda A \subset \lambda B - K$. Hence, $\lambda A \leq^s \lambda B$.

(ii) We have $B \subset A + K$ as $A \leq^l B$. For any $\lambda < 0$, this yields $\lambda B \subset \lambda A - K$, i.e., $\lambda B \leq^u \lambda A$.

(iii) Since $A \leq^s B$, $B \subset A + K$ and $A \subset B - K$. For any $\lambda < 0$, this implies that $\lambda B \subset \lambda A - K$ and $\lambda A \subset \lambda B + K$. So, $\lambda A \geq^s \lambda B$. □

The *certainly less relation* and *possibly less relation*, respectively, are stated as the following

$$A \leq^c B \Leftrightarrow (A = B) \text{ or } (A \neq B, B - A \subset K),$$

$$A \leq^p B \Leftrightarrow A \cap (B - K) \neq \emptyset,$$

$$\Leftrightarrow B \cap (A + K) \neq \emptyset.$$

As can be seen in [10], from a practical point of view, the order relation $\leq^s$ seems to be more appropriate in applications.

The *minmax less order relation*, *minmax certainly less order relation* and *minmax certainly nondominated order relation*, firstly introduced by Jahn and Ha [10] in 2011, respectively, are defined by

$$A \leq^m B \Leftrightarrow \min A \leq^s \min B \text{ and } \max A \leq^s \max B,$$

$$A \leq^{mc} B \Leftrightarrow (A = B) \text{ or } (A \neq B, \min A \leq^c \min B \text{ and } \max A \leq^c \max B),$$

$$A \leq^{mn} B \Leftrightarrow (A = B) \text{ or } (A \neq B, \max A \leq^s \min B).$$

Definition 2.2.6. [10] The binary relation $\leq$ is said to be

(i) compatible with the addition if and only if $A \leq B$ and $D \leq E$ imply $A + D \leq B + E$ for all $A, B, D, E \in \mathcal{P}(Y)$;

(ii) compatible with the multiplication with a nonnegative real number if and only if $A \leq B$ implies $\lambda A \leq \lambda B$ for all scalars $\lambda \geq 0$ and all $A, B \in \mathcal{P}(Y)$;

(iii) compatible with the conlinear structure of $\mathcal{P}(Y)$ if and only if it is compatible with both the addition and the multiplication with a nonnegative real number.

Proposition 2.2.7. [10]

(i) *The order relations $\leq^l$, $\leq^u$ and $\leq^s$ are pre-order (i.e., these relations are reflexive and transitive).*

(ii) *The order relations* $\leq^l$, $\leq^u$ *and* $\leq^s$ *are compatible with the conlinear structure of* $\mathcal{P}(Y)$.

(iii) *In general, the order relations* $\leq^l$, $\leq^u$ *and* $\leq^s$ *are not antisymmetric; more precisely, for arbitrary sets* $A, B \in \mathcal{P}(Y)$ *we have*

$$(A \leq^l B \quad \text{and} \quad B \leq^l A) \Leftrightarrow A + K = B + K,$$

$$(A \leq^u B \quad \text{and} \quad B \leq^u A) \Leftrightarrow A - K = B - K,$$

$$(A \leq^s B \quad \text{and} \quad B \leq^s A) \Leftrightarrow (A + K = B + K \quad \text{and} \quad A - K = B - K).$$

2.3 Semicontinuity and cone semicontinuity of a set-valued map

We recall definitions of semicontinuity for a set-valued map and their properties used in the sequel.

Definition 2.3.1. [72] A set-valued map $F : X \rightrightarrows Y$ is said to be

(i) upper semicontinuous at $x_0 \in \mathrm{Dom}F$ if and only if for any open subset V of Y with $F(x_0) \subset V$ there is a neighborhood U of x_0 such that $F(x) \subset V$ for all $x \in U$.

(ii) lower semicontinuous at $x_0 \in \mathrm{Dom}F$ if and only if for any open subset V of Y with $F(x_0) \cap V \neq \emptyset$ there is a neighborhood U of x_0 such that $F(x) \cap V \neq \emptyset$ for all $x \in U$.

(iii) lower (upper) semicontinuous on a subset D of X if and only if it is lower (upper) semicontinuous at every $x \in D$;

where $\mathrm{Dom}F = \{x \in X \mid F(x) \neq \emptyset\}$.

Lemma 2.3.2. (See e.g., [52, 56, 72]) *Let* $F : X \rightrightarrows Y$ *be a set-valued map.*

(i) F *is lower semicontinuous at* $x_0 \in \mathrm{Dom}F$ *if and only if for any net* $\{x_\alpha\} \subset X$ *converging to* x_0 *and for any* $y \in F(x_0)$, *there exist* $y_\alpha \in F(x_\alpha)$ *such that* $\{y_\alpha\}$ *converges to* y.

(ii) *If $F(x_0)$ is compact, then F is upper semicontinuous at $x_0 \in \mathrm{Dom}F$ if and only if for any net $\{x_\alpha\}$ converging to x_0 and for any $y_\alpha \in F(x_\alpha)$, there exist $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $\{y_\beta\}$ converges to y_0. If, in addition, $F(x_0) = \{y_0\}$ is a singleton, then for the above nets, $\{y_\beta\}$ converges to y_0.*

Definition 2.3.3. [56] A set-valued map $F : X \rightrightarrows Y$ is said to be

(i) Hausdorff upper semicontinuous at $x_0 \in \mathrm{Dom}F$ if and only if for each neighborhood V of the origin in Y, there exists a neighborhood U of x_0 such that $F(x) \subset F(x_0) + V$ for all $x \in U$.

(ii) Hausdorff lower semicontinuous at $x_0 \in \mathrm{Dom}F$ if and only if for any neighborhood V of the origin in Y, there exists a neighborhood U of x_0 such that $F(x_0) \subset F(x) + V$ for all $x \in U$.

(iii) Hausdorff lower (upper) semicontinuous on a subset D of X if and only if F is Hausdorff lower (upper) semicontinuous at every point of D.

Remark 2.3.4. [73] If F is upper semicontinuous at $x_0 \in \mathrm{Dom}F$, then F is Hausdorff upper semicontinuous at x_0; the converse implication is true when $F(x_0)$ is compact.

The following concepts about cone-semicontinuity were given in [73].

Definition 2.3.5. [73] Let K be a convex cone of Y. A set-valued map $F : X \rightrightarrows Y$ is said to be K-upper continuous (resp. K-lower continuous) at $x_0 \in \mathrm{dom}F$ if and only if for any open subset U of Y with $F(x_0) \subset U$ (resp. $F(x_0) \cap U \neq \emptyset$) there is a neighborhood N of x_0 such that $F(x) \subset U + K$ for every $x \in N$ (resp. $F(x) \cap (U - K) \neq \emptyset$ for all $x \in N$).

If $K = \{0\}$, then concepts of K-upper continuity and K-lower continuity coincide with concepts of upper semicontinuity and lower semicontinuity, respectively.

Concepts related to cone-convexity of a set-valued map can be found in [74].

Definition 2.3.6. [74] Let D be a nonempty convex set of X and K be a convex cone of a vector space Y. A set-valued map $F : X \rightrightarrows Y$ with nonempty values is said to be properly K-quasiconvex (resp. properly K-quasiconcave) on D if and only if for every $x_1, x_2 \in D$ and $t \in [0,1]$, either $F(x_t) \subset F(x_1) - K$ or $F(x_t) \subset F(x_2) - K$ (resp. either $y_1 \in F(x_t) - K$ or $y_2 \in F(x_t) - K$ with $y_i \in F(x_i)$) where $x_t = tx_1 + (1-t)x_2$.

Definition 2.3.7. [74] Let D be a nonempty convex subset of X. A set-valued map $F : X \rightrightarrows Y$ is said to be strictly lower K-convex (resp. strictly upper K-convex) on D if and only if for any $x_1, x_2 \in D$ with $x_1 \neq x_2$ and for any $t \in]0,1[$ we have $tF(x_1) + (1-t)F(x_2) \subset F(x_t) + \text{int}K$ (resp. $F(x_t) \subset tF(x_1) + (1-t)F(x_2) - \text{int}K$) where $x_t = tx_1 + (1-t)x_2$.

2.4 Hausdorff set-convergence, KKM-map and useful tools

We recall the notion of KKM-map and a well-known lemma about this map [75].

Definition 2.4.1. [75] Let $F : X \rightrightarrows Y$ be a set-valued map and M be a nonempty subset of X. F is called a KKM-map on M if and only if for any finite subset $\{x_1, \ldots, x_n\}$ of M, $\text{conv}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$ where "conv" denotes for the convex hull.

Lemma 2.4.2. (The KKM-Fan Lemma) *Let M be a nonempty subset of a Hausdorff topological vector space X, and $F : X \rightrightarrows Y$ be a KKM-map with closed values on M. If $F(x)$ is compact for at least one $x \in M$, then $\cap_{x \in M} F(x) \neq \emptyset$.*

In this research, beside the convergence of a sequence of points in a norm/metric space, we also need to examine the convergence of sequence which elements are sets. Now we recall the concepts of Hausdorff distance between two sets and Hausdorff convergence of sequence of sets.

If S is a nonempty subset of X and $x \in X$, then the distance d between x and S is defined by

$$d(x, S) = \inf_{u \in S} \|x - u\|.$$

If S_1 and S_2 are two nonempty subsets of X, then Hausdorff distance between S_1 and S_2, denoted by $H(S_1, S_2)$, is defined by

$$H(S_1, S_2) = \max\{H^*(S_1, S_2), H^*(S_2, S_1)\},$$

where $H^*(S_1, S_2) = \sup_{x \in S_1} d(x, S_2)$.

Definition 2.4.3. [76] Let $\{A_n\}$ be a sequence of subsets of X. We say that

(i) A_n converge to $A \subset X$ in the sense of the upper Hausdorff set-convergence, denoted by $A_n \rightharpoonup A$, if and only if $H^*(A_n, A) \to 0$.

(ii) A_n converge to $A \subset X$ in the sense of the lower Hausdorff set-convergence, denoted by $A_n \rightharpoondown A$, if and only if $H^*(A, A_n) \to 0$.

(iii) A_n converge to $A \subset X$ in the sense of the Hausdorff set-convergence, denoted by $A_n \to A$, if and only if $H(A_n, A) \to 0$.

Kuratowski measure of noncompactness is an important tool to survey characterizations of well-posedness and it will be used more to obtain main results of this study. We now recall the concept of Kuratowski measure of noncompactness and it's properties used in the sequel.

Definition 2.4.4. [77] Let M be a nonempty subset of X. The Kuratowski measure of noncompactness μ of the set M is defined by

$$\mu(M) = \inf\left\{\varepsilon > 0 \mid M \subset \bigcup_{i=1}^{n} M_i, \operatorname{diam} M_i < \varepsilon, i = 1, \cdots, n \text{ for some } n \in \mathbb{N}\right\},$$

where $\operatorname{diam} M_i = \sup\{d(x, y) \mid x, y \in M_i\}$ is the diameter of M_i.

Lemma 2.4.5. [77] *The following assertions are true:*

(i) $\mu(M) = 0$ *if* M *is compact;*

(ii) $\mu(M) \leq \mu(N)$ *whenever* $M \subset N$;

(iii) *if* $\{M_n\}$ *is a sequence of closed subsets in* X *satisfying* $M_{n+1} \subset M_n$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \mu(M_n) = 0$, *then* $K = \bigcap_{n\in\mathbb{N}} M_n$ *is nonempty compact and* $\lim_{n\to\infty} H(M_n, K) = 0$.

In the detail of proofs in later chapters, we need these following results. We here establish and give their proofs.

Lemma 2.4.6. *Let* X *be a normed space and* A, B *be subsets of* X. *If* A *is compact and* B *is closed, then* $A + B$ *is closed.*

Proof. Assume that $\{a_n + b_n\}$, $a_n \in A, b_n \in B$, converges to c for some $c \in X$. We show that $c \in A + B$. In fact, since A is compact, there exist a subsequence $\{a_{n_k}\}$ of sequence $\{a_n\}$ and $a \in A$ such that $\{a_{n_k}\}$ converges to a. We have

$$\|b_{n_k} - c + a\| = \|(b_{n_k} + a_{n_k}) - c + (a - a_{n_k})\| \leq \|b_{n_k} + a_{n_k} - c\| + \|a - a_{n_k}\|.$$

We obtain that $\{b_{n_k}\}$ converges to $c - a$. Since B is closed, we get $c - a \in B$. Hence, there exists $b \in B$ such that $b = c - a$. Then, $c = a + b \in A + B$. So, $A + B$ is closed. □

Lemma 2.4.7. *Let* M *be a nonempty subset of a normed space* X. *Then, for every* $x, y \in X$, $|d(x, M) - d(y, M)| \leq \|x - y\|$.

Proof. Let $x, y \in X$, we have $\|x - y\| + d(y, M) = \|x - y\| + \inf_{z\in M} \|y - z\| = \inf_{z\in M}\{\|x - y\| + \|y - z\|\} \geq \inf_{z\in M} \|x - z\| = d(x, M)$. Hence, $\|x - y\| \geq d(x, M) - d(y, M)$. Similarly, we also get $\|x - y\| \geq d(y, M) - d(x, M)$. We conclude that $|d(x, M) - d(y, M)| \leq \|x - y\|$. □

CHAPTER III

L-WELL-POSEDNESS FOR SET OPTIMIZATION PROBLEMS

In 2009, Zhang et al. [61] firstly introduced three kinds of well-posedness including one pointwise well-posedness and two global ones. The authors obtained some sufficient and necessary conditions for set optimization problems involving the relation $\leq^l$ to be well-posed. Moreover, criteria and characterizations of L-well-posedness for this problem were established by using the scalarization method. L-well-posedness properties for such problems with a class of generalized convex set-valued maps were obtained by Crespi et al. [63]. Using assumptions on cone properness, Gutiérrez et al. [34] investigated pointwise well-posedness for set optimization problems involving the relation $\leq^l$. Recently, Dhingra and Lalitha [14] introduced a concept of well-setness and proved that it is an extension of generalized well-posedness which was considered in [61]. Furthermore, they gave sufficient conditions of well-setness for set optimization problems involving the relation $\leq^l$ and obtained characterizations of well-setness for them by the scalarization method. As mentioned in [10, 15, 16, 19], the relation $\leq^s$ plays an important role in real-life situations. To the best of our knowledge, there is no work devoted to well-posedness for set optimization problems involving the relation $\leq^s$. Hence, studying on well-posedness for problems involving these relations is significant.

Motivated and inspired by the above works, in this chapter we aim to investigate various types of L-well-posedness for set optimization problems involving different kinds of set order relations. We introduce many kinds of L-well-posedness for such problems and study relationships between them as well as sufficient conditions of these types of well-posedness. Moreover, Kuratowski measure of noncompactness is applied to survey characterizations of L-well-posedness for set optimization problems.

Finally, approximating solution maps and their stability properties are researched to propose links between stability of the approximating problem and L-well-posedness of set optimization problems.

In this chapter, let X be a metric space and Y be a Hausdorff topological vector space. Let $F : X \rightrightarrows Y$ be a set-valued map with nonempty values on X. Let K be a closed convex pointed cone in Y with $\text{int}K \neq \emptyset$. The notation $F(M) = \bigcup_{x \in M} F(x)$ stands for the image of set M under set-valued map F.

3.1 Set approach and the investigated problem (P_α)

Firstly, set-valued optimization problem and vector approach are recalled. After that, set approach and set optimization problem investigated in this thesis are presented.

The set-valued optimization problem defined as the following has been studied widely in the literature

$$\begin{aligned} (\text{P}) \qquad & \text{Min } F(x) \\ & \text{subject to} \quad x \in M, \end{aligned}$$

where M is a nonempty subset of X.

Definition 3.1.1. Given a nonempty set $A \subset Y$. An element $\bar{a} \in \text{Min}A$ is called a Edgeworth-Pareto minimal point of set A with respect to cone K, where $\text{Min}A = \{\bar{a} \in A \mid A \cap (\bar{a} - K) = \{\bar{a}\}\}$.

If F is a set-valued map, then there are many distinct values $y \in Y$ such that $y \in F(x)$ for every $x \in \text{Dom}F$. Hence, in vector approach, when studying minimizers of a set-valued map, we fix one element $y \in F(x)$ and formulate the following solution concept based on the concept of Edgeworth-Pareto minimality.

Definition 3.1.2. Let $\bar{x} \in M$. A pair $(\bar{x}, \bar{y}) \in \text{graph}F$ is called a minimizer of the set-valued optimization problem (P) if $\bar{y} \in \text{Min}(F(M), K)$.

This concept is important and has been studied widely in spite of some drawback in practical application. As pointed out in [19], a minimizer $(\bar{x}, \bar{y})$ depends on only certain special element $\bar{y}$ of $F(\bar{x})$ and other elements of $F(\bar{x})$ are ignored. In other words, an element $\bar{x}$ become a solution of the set-valued optimization problem (P) if there exists at least one element $\bar{y} \in F(\bar{x})$ which is a Edgeworth-Pareto minimal point of the image set of F even if there exist many bad elements in $F(\bar{x})$. For this reason, the solution concepts introduced in vector approach are sometimes improper. In order to avoid this disadvantage it is necessary to work with practically relevant order relations for sets. This leads to solution concepts for set-valued optimization problems based on comparisons among values of the set-valued objective map F, and it is so called solution concepts based on set approach.

As presented by Kuroiwa in [67], the following example illustrates the difference between solution concepts based on vector approach and set approach. Let $X = \mathbb{R}, Y = \mathbb{R}^2, K = \mathbb{R}^2_+, M = [0, 1]$ and $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [(1,0),(0,1)], & \text{if} \quad x = 0, \\ [(1-x,x),(1,1)], & \text{otherwise}, \end{cases}$$

where $[(a, b), (c, d)]$ is a closed line segment starting at point (a, b), ending at (c, d) and including these points in $\mathbb{R}^2$.

It is clear that the element $x = 0$ is a minimal solution with respect to set order relation $\leq^l$ ($F(0) \leq^l F(x)$ for all $x \in M$). However, all elements $(\bar{x}, \bar{y}) \in \text{graph}F$ with $\bar{y} = (1 - \bar{x}, \bar{x})$ if $\bar{x} \neq 0$ and $\bar{y} = (1, 0)$ if $\bar{x} = 0$ are minimizers in the sense of Definition 3.1.2. This illustrates that the solution concept based on set approach with respect to $\leq^l$ is more natural and useful than one based on vector approach. In what follows, we use set approach with the solution concept defined on the comparision of all images of the set-valued objective map by using set order relations. We mainly focus on (P_α) defined as the following.

Let $F : X \rightrightarrows Y$ be a set-valued map with nonempty values on X. For each $\alpha \in \{l, u, s\}$, we consider the following set optimization problem

$$(\mathrm{P}_\alpha) \qquad \alpha\text{-Min } F(x)$$
$$\text{subject to} \quad x \in M,$$

where M is a nonempty closed subset of X. A point $\bar{x} \in M$ is said to be an α-minimal solution of (P_α) if and only if for any $x \in M$ such that $F(x) \leq^\alpha F(\bar{x})$ then $F(\bar{x}) \leq^\alpha F(x)$. The set of all α-minimal solutions of (P_α) is denoted by S_α.

It can be seen that if $\bar{x} \in S_\alpha$ and $F(\bar{x}) \sim^\alpha F(\hat{x})$ for some $\hat{x} \in M$, then $\hat{x} \in S_\alpha$.

3.2 Existence conditions of solutions

This section focuses on existence conditions of solutions for set optimization problems involving three types of set order relations (P_α). Some solution existence conditions for set optimization problems with upper set relation or upper set relation can be found in literature [25, 26, 67]. To the best of our knowledge, there have been no publication devoted solution existence conditions for set optimization problems with set less relation $\leq^s$. It urges us to study this topic. In addition, to investigate well-posedness of optimization problems, some researchers assume that the solution set is nonempty and then they pay more attention on necessary, sufficient conditions for the problems to be well-posed. Therefore, it is better to point out conditions to guarantee that the solution set is nonempty.

Motivated by the study on cone-semicontinuity in [73], we introduce a concept so-called K-lower pseudocontinuity for a set-valued map. After that, we investigate some properties of this notion and apply them to establish the existence conditions for (P_α). In this section, we assume that M is a nonempty closed convex subset of X.

Let $F : X \rightrightarrows Y$ be a set-valued map and $A \in \mathcal{P}(Y)$ be given. We denote $L_{\leq^\alpha A}F = \{x \in X \mid F(x) \leq^\alpha A\}$. Next, we give sufficient conditions for closedness of these level sets.

Lemma 3.2.1. *Let $F : X \rightrightarrows Y$ be a set-valued map.*

(i) *If F is K-upper continuous and compact-valued, then $L_{\leq^l A}F$ is closed for every subset A of Y.*

(ii) *If F is $(-K)$-lower continuous, then $L_{\leq^u A}F$ is closed for every compact subset A of Y.*

(iii) *If F is K-upper continuous, $(-K)$-lower continuous and compact-valued, then $L_{\leq^s A}F$ is closed for every compact subset A of Y.*

Proof. (i) Let $\{x_\beta\} \subset L_{\leq^l A}F$ converge to x_0, we get $F(x_\beta) \leq^l A$, i.e., $A \subset F(x_\beta) + K$. Let $a \in A$ be arbitrary, there exist $y_\beta \in F(x_\beta)$ such that $a \in y_\beta + K$. Let V be a neighborhood of 0_Y, we have $F(x_0) \subset F(x_0) + V$. Because F is K-upper continuous and $\{x_\beta\}$ converges to x_0, $F(x_\beta) \subset F(x_0) + V + K$. So, $a \in y_\beta + K \subset F(x_0) + V + K$. Since $F(x_0)$ is compact, this implies that $a \in F(x_0) + K$. Therefore, $A \subset F(x_0) + K$. This leads to the closedness of $L_{\leq^l A}F$.

(ii, iii) These statements can be proved by similar arguments. □

In [78], concepts of semi pseudocontinuity for extended real-valued maps were proposed. These proposed concepts in [78] are defined by limit inferior of a sequence. Therefore, this type of limit can not be applied for a class of set-valued maps involving cone. Using closedness of level sets, we generalize these concepts for a set-valued map.

Definition 3.2.2. A set-valued map $F : X \rightrightarrows Y$ is said to be K-lower pseudocontinuous on X if and only if for every $b \in X$, $L_{\leq^\alpha F(b)}F = \{x \in X \mid F(x) \leq^\alpha F(b)\}$ is closed.

Applying Lemma 3.2.1, the sufficient conditions for the closedness of level sets are given in the following result.

Proposition 3.2.3. *Assume that F is compact-valued. Then,*

(i) *If F is K-upper continuous, then $L_{\leq^l F(b)}F$ is closed for every $b \in X$.*

(ii) *If F is $(-K)$-lower continuous, then $L_{\leq^u F(b)}F$ is closed for every $b \in X$.*

(iii) *If F is K-upper continuous and $(-K)$-lower continuous, then $L_{\leq^s F(b)}F$ is closed for every $b \in X$.*

Remark 3.2.4. (i) The concept in Definition 3.2.2 is more generalized than the concept about cone-semicontinuity in [73].

(ii) When the objective map is a single-valued map, $f : X \to Y$, the concept in Definition 3.2.2 can be written as the following: f is said to be K-lower pseudocontinuous if $L_{\leq\beta}f = \{x \in X : f(x) \leq_K \beta\}$ is closed for all $\beta \in f(X)$. Furthermore, when $Y = \mathbb{R}$ and $K = [0, +\infty)$, by Proposition 2.1 in [78], this concept reduces to the notion so-called sequentially lower pseudocontinuity in Definition 2.4 in [78].

Proposition 3.2.5. *F is K-lower pseudocontinuous if and only if for every $b \in X$ and $x_0 \in X$ with $F(x_0) \not\leq^\alpha F(b)$, there exists a neighborhood N of x_0 such that $F(N) \not\leq^\alpha F(b)$.*

Proof. Let $x_\beta \to \hat{x}$ with $F(x_\beta) \leq^\alpha F(b)$, we need to prove that $F(\hat{x}) \leq^\alpha F(b)$. By contradiction, suppose that $F(\hat{x}) \not\leq^\alpha F(b)$. It follows from the assumption that there is a neighborhood N of $\bar{x}$ such that $F(x) \not\leq^\alpha F(b)$ for all $x \in N$. Since $x_\beta \to \hat{x}$, we can assume that $x_\beta \in N$, therefore $F(x_\beta) \not\leq^\alpha F(b)$ which is a contradiction.

Conversely, suppose F is K-lower pseudocontinuous and there exists $x_0 \in X$ with $F(x_0) \not\leq^\alpha F(b)$ but for any neighborhood N of x_0, there exists $x \in N$ such that $F(x) \leq^\alpha F(b)$, i.e., there is a net $\{x_\beta\}$ converging to x_0 with $F(x_\beta) \leq^\alpha F(b)$.

Then, since F is K-lower pseudocontinuous, by the closedness of $L_{\leq^\alpha F(b)}F$ we get $F(x_0) \leq^\alpha F(b)$ which is a contradiction. □

Level sets $L_{\leq^\alpha F(b)}F = \{x \in X \mid F(x) \leq^\alpha F(b)\}$ can be considered as a set-valued map by setting $L_\alpha : X \rightrightarrows X$, $L_\alpha(b) = \{x \in X \mid F(x) \leq^\alpha F(b)\}$. The next result gives a sufficient condition for this map to be a KKM-map.

Proposition 3.2.6. *If for every $x \in X$, $\{b \in X \mid F(x) \not\leq^\alpha F(b)\}$ is convex, then L_α is a KKM-map.*

Proof. Let $\{x_1, \ldots, x_n\} \subset X$ and $x \in \text{conv}\{x_1, \ldots, x_n\}$ be arbitrary. Suppose that $x \notin \bigcup_{i=1}^n L_\alpha(x_i)$, i.e, $x \notin L_\alpha(x_i)$ for all $i = 1, \ldots, n$. Therefore, $x_i \in \{b \in X \mid F(x) \not\leq^\alpha F(b)\}$ for all $i = 1, \ldots, n$. It follows from the convexity of this set that $x \in \{b \in X \mid F(x) \not\leq^\alpha F(b)\}$, i.e., $F(x) \not\leq^\alpha F(x)$ which is a contradiction. □

Employing the KKM-Fan Lemma, existence conditions for (P_α) are presented in the next result.

Theorem 3.2.7. *Suppose that the following conditions hold:*

(i) *F is K-lower pseudocontinuous;*

(ii) *for every $x \in M$, $\{b \in M \mid F(x) \not\leq^\alpha F(b)\}$ is convex;*

(iii) (Coercivity condition) *There exists a nonempty compact set $A \subset M$ such that for each finite set B of points $b_i \in M$, $i = 1, \ldots, n$, there is a nonempty compact convex set $C \subset M$ satisfying $C \supset B$ and, for all $x \in C \setminus A$, there exists $b \in B$ with $F(x) \not\leq^\alpha F(b)$.*

Then, $S_\alpha \neq \emptyset$.

Proof. Let $B = \{b_1, b_2, \ldots, b_n\}$ with $b_i \in M$, $i = 1, \ldots, n$. By the assumption (iii), there exists a nonempty compact subset A of M such that there is a compact convex

subset C of M satisfying $C \supset B$ and for all $x \in C \setminus A$, there exists $b \in B$ with $F(x) \not\leq^{\alpha} F(b)$.

Define the set-valued map $L : M \rightrightarrows M$ by $L(b) = \{x \in M \mid F(x) \leq^{\alpha} F(b)\}$. Let Q be the restriction of L on C, i.e., $Q : C \rightrightarrows M$, $Q(b) = C \cap L(b)$ for all $b \in C$. We first prove that Q is a KKM-map on C. It is clear that $Q(b) \neq \emptyset$ for each $b \in C$. Let $x_j \in C$, $j = 1, \ldots, m$, and $x \in \text{conv}\{x_1, \ldots, x_m\}$. It implies from the convexity of C that $x \in C$. Combining Proposition 3.2.6 and assumption (ii), we obtain that L is a KKM-map on M. Therefore, there exists $j \in \{1, \ldots, m\}$ such that $x \in L(x_j)$. Also, $x \in Q(x_j)$. We conclude that Q is a KKM-map on C.

On the other hand, $Q(b)$ is compact since $Q(b) = C \cap L(b)$. Applying Lemma 2.4.2, we get $\bigcap_{b \in C} Q(b) \neq \emptyset$. Therefore, there exist $x \in C$ and $x \in \bigcap_{b \in C} L(b)$, i.e., $F(x) \leq^{\alpha} F(b)$ for all $b \in C$, and hence $x \in A$ by (iii). Furthermore, $x \in Q(b_i)$ for all $i = 1, \ldots n$ as $x \in L(b_i)$. Hence, $x \in \bigcap_{i=1}^{n} Q(b_i)$ and $A \cap (\bigcap_{i=1}^{n} Q(b_i)) \neq \emptyset$. We also have $A \cap (\bigcap_{i=1}^{n} L(b_i)) \neq \emptyset$. Since A is compact and $L(b)$ is closed for each $b \in M$, $A \cap (\bigcap_{b \in M} L(b)) \neq \emptyset$ is closed. Therefore, $\bigcap_{b \in M} L(b) \neq \emptyset$. This means that $x \in S_{\alpha}$. □

Remark 3.2.8. In Theorem 3.2.7, the sufficient conditions for the condition (i) are studied in Proposition 3.2.3. The last condition, coercivity condition, is satisfied if M is compact. Furthermore, it was studied in some publications [79, 80, 81]. We next give sufficient conditions for the condition (ii) in this theorem.

Proposition 3.2.9. *Let $x \in M$ be given.*

(i) *If F is properly $(-K)$-quasiconcave on M, then $\{b \in M \mid F(x) \not\leq^{l} F(b)\}$ is convex.*

(ii) *If F is properly K-quasiconvex on M, then $\{b \in M \mid F(x) \not\leq^{u} F(b)\}$ is convex.*

(iii) *If F is properly $(-K)$-quasiconcave and properly K-quasiconvex on M, then $\{b \in M \mid F(x) \not\leq^{s} F(b)\}$ is convex.*

Proof. (i) Denote $G(x) = \{b \in M \mid F(x) \not\leq^l F(b)\}$. Let $b_i \in G(x)$, $i = 1, 2$ and $b = tb_1 + (1-t)b_2$ for any $t \in [0, 1]$. Suppose that $b \notin G(x)$, hence $F(b) \subset F(x) + K$. Since $b_i \in G(x)$, $i = 1, 2$, there exist $y_i \in F(b_i)$ such that $y_i \notin F(x) + K$, $i = 1, 2$. It follows from the properly $(-K)$-quasiconcavity on M of F that there exists $j \in \{1, 2\}$ such that $y_j \in F(b) + K$. Therefore, $y_j \in F(x) + K$ which is a contradiction.

(ii, iii) By the similar arguments, we can show that these assertions are true. □

Next, as applications of Theorem 3.2.7, we discuss results for particular cases which the objective map is a single-valued map and a real-valued map, respectively, as examples.

Corollary 3.2.10. *Let $f : X \to Y$ be a single-valued map. Suppose that the following conditions hold:*

(i) *f is K-lower pseudocontinuous;*

(ii) *for every $x \in M$, $\{b \in M \mid f(x) \not\leq_K \beta\}$ is convex;*

(iii) (Coercivity condition) *M is compact or there exists a nonempty compact set $A \subset M$ such that for each finite set B of points $b_i \in M$, $i = 1, \ldots, n$, there is a nonempty compact convex set $C \subset M$ satisfying $C \supset B$ and, for all $x \in C \setminus A$, there exists $b \in B$ with $f(x) \not\leq_K \beta$.*

Then, the solution set is nonempty.

Corollary 3.2.11. *Let $f : X \to \mathbb{R}$ be a real-valued map. Suppose that the following conditions hold*

(i) *f is sequentially lower pseudocontinuity;*

(ii) *for every $x \in M$, $\{b \in M \mid f(x) > \alpha\}$ is convex;*

(iii) (Coercivity condition) *M is compact or there exists a nonempty compact set $A \subset M$ such that for each finite set B of points $b_i \in M$, $i = 1, \ldots, n$, there is a*

nonempty compact convex set $C \subset M$ satisfying $C \supset B$ and, for all $x \in C \setminus A$, there exists $b \in B$ with $f(x) > \alpha$.

Then, the solution set is a nonempty set.

Conditions of existence of solutions are given in the previous section. In what follows, we assume that conditions of existence of solutions are satisfied and we next focus on well-posedness properties.

3.3 Global L-well-posedness

Motivated by the study in [61], we introduce concepts of generalized minimizing sequence and employ them to study several types of well-posedness for (P_α). Let e be a fixed element of $\mathrm{int}K$.

Definition 3.3.1. A sequence $\{x_n\} \subset M$ is called a generalized minimizing sequence of (P_α) if and only if there exist sequences $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 and $\{z_n\} \subset S_\alpha$ satisfying $F(x_n) \leq^\alpha F(z_n) + \varepsilon_n e$ for all n.

Definition 3.3.2. Problem (P_α) is said to be e-well-posed in the sense of Loridan (shortly, L-well-posed) if and only if for every generalized minimizing sequence $\{x_n\}$ of (P_α) there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\bar{x} \in S_\alpha$ such that $\{x_{n_k}\}$ converges to $\bar{x}$.

Remark 3.3.3. When $\alpha = l$, Definitions 3.3.1 and 3.3.2 reduce to Definition 2.2 in [61] and corresponding concepts in [14].

Consider the problem (P_α), we define a set-valued map $L_\alpha : M \times \mathbb{R}^+ \rightrightarrows M$ as, for all $(x, \varepsilon) \in M \times \mathbb{R}^+$,

$$L_\alpha(x, \varepsilon) = \{\hat{x} \in M \mid F(\hat{x}) \leq^\alpha F(x) + \varepsilon e\}.$$

We refer to the set $L_\alpha(x, \varepsilon)$ as level set at x with level ε and L_α as level set-valued map. It is clear that $\{x_n\}$ is a generalized minimizing sequence of (P_α) if there exist $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 and $\{z_n\} \subset S_\alpha$ such that $x_n \in L_\alpha(z_n, \varepsilon_n)$.

The following proposition plays an important role in our work.

Proposition 3.3.4. *Let L_α be a level set-valued map. Then the following statements hold:*

(i) $x \in L_\alpha(x, \varepsilon)$ *for all* $x \in M$;

(ii) *if* $\varepsilon_1 < \varepsilon_2$, *then* $L_\alpha(x, \varepsilon_1) \subset L_\alpha(x, \varepsilon_2)$;

(iii) $\bigcup_{z \in S_\alpha} L_\alpha(z, 0) = S_\alpha$.

Proof. We only prove the assertions (i)-(iii) for the case $\alpha = s$; the proofs of these assertions for the cases $\alpha = l$ and $\alpha = u$ are similar.

(i) For any $\varepsilon > 0$ and $x \in M$, because $\varepsilon e \in \text{int}K \subset K$ we have $F(x) + \varepsilon e \subset F(x) + K$, i.e., $F(x) \leq^l F(x) + \varepsilon e$.

On the other hand, by the convexity of K, one gets $-K \subset \varepsilon e - K$. Therefore,

$$F(x) \subset F(x) - K \subset F(x) + \varepsilon e - K,$$

i.e., $F(x) \leq^u F(x) + \varepsilon e$. So, $x \in L_s(x, \varepsilon)$.

(ii) For $\varepsilon_1 < \varepsilon_2$ and $\hat{x} \in L_s(x, \varepsilon_1)$, we have $F(\hat{x}) \leq^s F(x) + \varepsilon_1 e$, i.e.,

$$F(x) + \varepsilon_1 e \subset F(\hat{x}) + K \quad \text{and} \quad F(\hat{x}) \subset F(x) + \varepsilon_1 e - K.$$

Obviously,

$$F(x) + \varepsilon_2 e = F(x) + \varepsilon_1 e + (\varepsilon_2 - \varepsilon_1)e,$$

and

$$F(x) + \varepsilon_2 e - K = F(x) + \varepsilon_1 e - K + (\varepsilon_2 - \varepsilon_1)e.$$

Combining the convexity of cone K with Proposition 2.2.7(ii), we get

$$F(x) + \varepsilon_2 e \subset F(\hat{x}) + K, \quad F(\hat{x}) \subset F(x) + \varepsilon_2 e - K,$$

and hence $F(\hat{x}) \leq^s F(x) + \varepsilon_2 e$. So, $L_s(x, \varepsilon_1) \subset L_s(x, \varepsilon_2)$.

(iii) Let $\bar{x} \in S_s$, we always get $F(\bar{x}) \leq^s F(\bar{x})$ because $\leq^s$ is reflexive. We have $\bar{x} \in L_s(\bar{x}, 0)$. So, $\bar{x} \in \bigcup_{z \in S_s} L_s(z, 0)$, and hence $S_s \subset \bigcup_{z \in S_s} L_s(z, 0)$.

Conversely, let $\bar{x} \in \bigcup_{z \in S_s} L_s(z, 0)$ and suppose that there exists $x \in M$ satisfying $F(x) \leq^s F(\bar{x})$, we need to prove that $F(\bar{x}) \leq^s F(x)$. Since $\bar{x} \in \bigcup_{z \in S_s} L_s(z, 0)$, there exists $z \in S_s$ such that $\bar{x} \in L_s(z, 0)$. Equivalently, $F(\bar{x}) \leq^s F(z)$. Since $F(x) \leq^s F(\bar{x})$, by the transitivity property of $\leq^s$, we get $F(x) \leq^s F(z)$. This implies $F(z) \leq^s F(x)$ as $z \in S_s$. Using the transitivity property, we conclude that $F(\bar{x}) \leq^s F(x)$. So, $\bar{x} \in S_s$, and hence $\bigcup_{z \in S_s} L_s(z, 0) \subset S_s$. □

Remark 3.3.5. When $\alpha = l$, Proposition 3.3.4 reduces to Proposition 3.1 (without proof) in [14], and Proposition 3.3.4 is new for cases where $\alpha = u$ and $\alpha = s$.

Inspired by [14], we next introduce notions of metrically α-well-posedness for (P_α) by using the Hausdorff distance.

Definition 3.3.6. Problem (P_α) is said to be metrically α-well-posed if and only if S_α is nonempty, and for every generalized minimizing sequence $\{x_n\}$ of (P_α),

$$H^*(L_\alpha(x_n, \varepsilon_n), S_\alpha) \to 0,$$

where $\{\varepsilon_n\} \subset \mathbb{R}^+$ is the sequence corresponding to $\{x_n\}$.

Next, we propose a new kind of well-posedness for (P_α) which is a relaxed form of metrically α-well-posedness and useful to improve some known results.

Definition 3.3.7. Problem (P_α) is said to be weak metrically α-well-posed if and only if S_α is nonempty, and for every generalized minimizing sequence $\{x_n\}$ of (P_α),

$$d(x_n, S_\alpha) \to 0.$$

Remark 3.3.8. By Proposition 3.3.4(i), it is clear that if the problem (P_α) is metrically α-well-posed, then it is weak metrically α-well-posed.

We now study a necessary condition of L-well-posedness for (P_α).

Theorem 3.3.9. *If* (P_α) *is L-well-posed, then* S_α *is compact.*

Proof. For $\{x_n\} \subset S_\alpha$ and $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0, for each n, we have $F(x_n) \leq^\alpha F(x_n) + \varepsilon_n e$ as $e \in \mathrm{int}K$. So, $\{x_n\}$ is a generalized minimizing sequence of (P_α). By the L-well-posedness of (P_α), there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\bar{x} \in S_\alpha$ such that $\{x_{n_k}\}$ converges to $\bar{x}$. Hence, S_α is compact. □

Combining Theorem 3.3.9 with Theorem 3.1 in [14], we get relationships between L-well-posedness and (weak) metrically α-well-posedness for (P_α).

Corollary 3.3.10. *If* (P_α) *is L-well-posed, then* (P_α) *is metrically* α*-well-posed.*

Remark 3.3.11. Corollary 3.3.10 improves Theorem 3.1 in [14] by removing the closedness of S_α.

The below example illustrates that the converse of Corollary 3.3.10 is not true.

Example 3.3.12. Let $X = Y = \mathbb{R}$, $M = [-1, 1]$, $K = \mathbb{R}_+$, $e = 1$ and $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases}]0, 1[, & \text{if } x \in]0, 1[, \\ ]0, 2[, & \text{otherwise.} \end{cases}$$

Clearly, $S_\alpha =]0, 1[$ and (P_α) is metrically α-well-posed. However, (P_α) is not L-well-posed by Theorem 3.3.9.

Note that Example 3.3.12 also shows that Theorem 3.2 in [14] is not true. The following result is a correction version of this theorem.

Theorem 3.3.13. *If* S_α *is compact and* (P_α) *is weak metrically* α*-well-posed, then* (P_α) *is L-well-posed.*

Proof. Let $\{x_n\}$ be a generalized minimizing sequence of (P_α), we have $d(x_n, S_\alpha) \to 0$ as (P_α) is weak metrically α-well-posed. By the compactness of S_α, there exists a

sequence $\{\bar{x}_n\} \subset S_\alpha$ such that

$$d(x_n, \bar{x}_n) = d(x_n, S_\alpha) \to 0.$$

Then, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ converging to some $\bar{x} \in S_\alpha$ as S_α is compact. Due to

$$d(x_{n_k}, \bar{x}) \leq d(x_{n_k}, \bar{x}_{n_k}) + d(\bar{x}_{n_k}, \bar{x}),$$

$\{x_{n_k}\}$ converges to $\bar{x}$. We conclude that (P_α) is L-well-posed. □

Next, we now give sufficient conditions for (P_α) to be L-well-posed.

Theorem 3.3.14. *Suppose that the following conditions hold:*

(i) *M and S_α are compact;*

(ii) *F is continuous and compact-valued on M.*

Then, (P_α) is L-well-posed.

Proof. We only demonstrate the proof of the statement for the case $\alpha = u$ since the technique to prove the statement for the cases $\alpha = l$ and $\alpha = s$ is similar. Suppose that (P_u) is not L-well-posed, it follows from Theorem 3.3.13 that (P_u) is not weak metrically u-well-posed. Then, there exists a generalized minimizing sequence $\{x_n\}$ of (P_u) such that

$$d(x_n, S_u) \not\to 0. \tag{3.3.1}$$

Since $\{x_n\}$ is a generalized minimizing sequence, there exists $\{z_n\} \subset S_u$ such that

$$F(x_n) \leq^u F(z_n) + \varepsilon_n e. \tag{3.3.2}$$

Since M is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some $\bar{x} \in M$. It follows from (3.3.2) that

$$F(x_{n_k}) \subset F(z_{n_k}) + \varepsilon_{n_k} e - K. \tag{3.3.3}$$

By the compactness of S_u, we can assume that $\{z_{n_k}\}$ converges to some $\bar{z} \in S_u$. Let $\bar{v} \in F(\bar{x})$ be arbitrary, there exists $\{v_{n_k}\}$ with $v_{n_k} \in F(x_{n_k})$ converging to $\bar{v}$ because of the lower semicontinuity of F at $\bar{x}$. By (3.3.3), we get $v_{n_k} \in F(z_{n_k}) + \varepsilon_{n_k} e - K$, and hence there exist $u_{n_k} \in F(z_{n_k})$ such that

$$v_{n_k} \in u_{n_k} + \varepsilon_{n_k} e - K. \tag{3.3.4}$$

Since F is upper semicontinuous and compact-valued at $\bar{z}$, there exist $\bar{u} \in F(\bar{z})$ and a subsequence of $\{u_{n_k}\}$, denoted by the same indexes, such that $\{u_{n_k}\}$ converges to $\bar{u}$. Taking limit as $n \to \infty$ in (3.3.4), we get $\bar{v} \in \bar{u} - K$. Thus, $\bar{v} \in F(\bar{z}) - K$. By the arbitrariness of $\bar{v}$, we conclude that $F(\bar{x}) \subset F(\bar{z}) - K$, i.e., $F(\bar{x}) \leq^u F(\bar{z})$. Since $\bar{z} \in S_u$, we have $F(\bar{x}) \sim^u F(\bar{z})$, and hence $\bar{x} \in S_u$ which contradicts (3.3.1). So, (P_α) is L-well-posed. □

Theorem 3.3.14 gives sufficient conditions of the L-well-posedness for (P_α) in the case the constraint set is compact. The following result devotes to the non-compactness case of this set.

Theorem 3.3.15. *Suppose that the following conditions hold:*

(i) *X is locally compact and S_α is compact;*

(ii) *F is compact-valued on M;*

(iii) *there exists $\delta > 0$ such that the level set $L_\alpha(x, \varepsilon)$ is connected for every $x \in S_\alpha$ and for every $\varepsilon \in (0, \delta)$. Then,*

  (a) (P_l) *is L-well-posed if F is lower semicontinuous on S_l and F is upper semicontinuous on M.*

  (b) (P_u) *is L-well-posed if F is upper semicontinuous on S_u and F is lower semicontinuous on M.*

  (c) (P_s) *is L-well-posed if F is continuous on M.*

Proof. By the similarity, we here only demonstrate the proof for the statement (b). Suppose that (P_u) is not L-well-posed. By the assumption (i) and Theorem 3.3.13, (P_u) is also not metrically well-posed. Then, there exists a generalized minimizing sequence $\{x_n\}$ of (P_u) such that

$$H^*(L_u(x_n, \varepsilon_n), S_u) \not\to 0,$$

where $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 is the sequence corresponding to $\{x_n\}$. Because $\{x_n\}$ is a generalized minimizing sequence, for each $n \in \mathbb{N}$ there exists $z_n \in S_u$ such that $F(x_n) \leq^u F(z_n) + \varepsilon_n e$. Since $H^*(L_u(x_n, \varepsilon_n), S_u) \not\to 0$, we can assume that there is $\beta > 0$ satisfying $H^*(L^u(x_n, \varepsilon_n), S_u) \geq \beta$ for all n (take a subsequence if necessary). By (i), there exists an open neighborhood U of S_u such that its closure, $\mathrm{cl}(U)$, is compact and $L^u(x_n, \varepsilon_n) \not\subset \mathrm{cl}(U)$. Hence, for each $n \in \mathbb{N}$ there exists $\hat{x}_n \in L_u(x_n, \varepsilon_n)$ such that

$$\hat{x}_n \notin \mathrm{cl}(U). \tag{3.3.5}$$

Since $\{x_n\}$ is a generalized minimizing sequence and $\hat{x}_n \in L_u(x_n, \varepsilon_n)$, i.e., $F(\hat{x}_n) \leq^u F(x_n) + \varepsilon_n e$, we conclude that $F(\hat{x}_n) \leq^u F(z_n) + 2\varepsilon_n e$. Hence, $\hat{x}_n \in L_u(z_n, 2\varepsilon_n)$. Combining this with (3.3.5), we get

$$L_u(z_n, 2\varepsilon_n) \cap (\mathrm{cl}(U))^c \neq \emptyset, \tag{3.3.6}$$

where $(\mathrm{cl}(U))^c$ denotes the complement of $\mathrm{cl}(U)$ in X. Also, we obtain

$$z_n \in L_u(z_n, 2\varepsilon_n) \cap \mathrm{int}\,\mathrm{cl}(U). \tag{3.3.7}$$

We next claim that $L_u(z_n, 2\varepsilon_n) \cap \mathrm{bd}(\mathrm{cl}(U)) \neq \emptyset$ for every $n \in \mathbb{N}$ satisfying $2\varepsilon_n < \delta$. Suppose on the contrary that there exists $\overline{m}_n \in \mathbb{N}$ such that $L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \subset \mathrm{int}\,\mathrm{cl}(U) \cup \mathrm{int}(\mathrm{cl}(U))^c$. This leads to

$$L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) = (L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \cap \mathrm{int}\,\mathrm{cl}(U)) \cup (L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \cap \mathrm{int}(\mathrm{cl}(U))^c). \tag{3.3.8}$$

We note that $L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \cap \operatorname{int}\operatorname{cl}(U)$ and $L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \cap \operatorname{int}(\operatorname{cl}(U))^c$ are separated because $\operatorname{cl}(\operatorname{int}\operatorname{cl}(U)) \cap \operatorname{int}(\operatorname{cl}(U))^c = \emptyset$ and $\operatorname{int}\operatorname{cl}(U) \cap \operatorname{cl}(\operatorname{int}(\operatorname{cl}(U))^c) = \emptyset$. Employing (3.3.6)-(3.3.8) and $L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \cap \operatorname{int}\operatorname{cl}(U) \cap (\operatorname{cl}(U))^c = \emptyset$, we arrive at a contradiction of the fact that $L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n})$ is a connected set. Therefore, there exists a sequence $\{w_n\}$ such that

$$w_n \in L_u(z_n, 2\varepsilon_n) \cap \operatorname{bd}(\operatorname{cl}(U)). \tag{3.3.9}$$

By the compactness of $\operatorname{cl}(U)$, there exists a subsequence of $\{w_n\}$ which is still denoted by $\{w_n\}$ converging to some $\overline{w} \in \operatorname{cl}(U)$. Since $w_n \in L_u(z_n, 2\varepsilon_n)$, $F(w_n) \subset F(z_n) + 2\varepsilon_n e - K$. Due to the compactness of S_u, there is a subsequence of $\{z_n\}$ which is still denoted by $\{z_n\}$ converging to some $\bar{z} \in S_u$. Now, we show that $F(\overline{w}) \subset F(\bar{z}) - K$. Let $\bar{v} \in F(\overline{w})$ be arbitrary, by the lower semicontinuity of F at $\overline{w}$, there exists a sequence $\{v_n\}$ converging to $\bar{v}$ where $v_n \in F(w_n)$ for all n. We get $v_n \in F(z_n) + 2\varepsilon_n e - K$. Thus, there exists $u_n \in F(z_n)$ such that

$$v_n \in u_n + 2\varepsilon_n e - K. \tag{3.3.10}$$

Since F is upper semicontinuous and compact-valued at $\bar{z}$, there exist $\bar{u} \in F(\bar{z})$ and a subsequence of $\{u_n\}$, denoted by the same indexes, converging to $\bar{u}$. Taking limit as $n \to \infty$ in (3.3.10), we get $\bar{v} \in \bar{u} - K$. Therefore, $\bar{v} \in F(\bar{z}) - K$. By the arbitrariness of $\bar{v}$, we have $F(\overline{w}) \subset F(\bar{z}) - K$, i.e., $F(\bar{w}) \leq^u F(\bar{z})$. Since $\bar{z} \in S_u$, we have $F(\overline{w}) \sim^u F(\bar{z})$, and hence $\overline{w} \in S_u$ which contradicts (3.3.9). □

The following examples show that Theorems 3.3.14 and 3.3.15 are not comparable.

Example 3.3.16. Let $X = Y = \mathbb{R}$, $M = [0, 1]$, $K = \mathbb{R}^+$, $e = 1$ and $F : X \rightrightarrows Y$ be defined by $F(x) = [-x^2 + x, -2x^2 + 2x]$. Clearly, all conditions of Theorem 3.3.14 are satisfied but the condition (iii) of Theorem 3.3.15 does not hold. Indeed, let $\delta = \frac{1}{4}$, direct cacullations give us $S_\alpha = \{0, 1\}$ and the level set $L_\alpha(x, \varepsilon) = [0, \frac{1-\sqrt{1-4\varepsilon}}{2}] \cup [\frac{1+\sqrt{1-4\varepsilon}}{2}, 1]$ is not connected for every $x \in S_\alpha$ and every $\varepsilon \in]0, \delta[$.

Theorem 3.3.15 does not require the compactness of constraint set M, and hence when M is not compact, Theorem 3.3.14 does not work while Theorem 3.3.15 can apply. Furthermore, the below example shows that even in the case M is compact, they are also not comparable.

Example 3.3.17. Let $X = Y = \mathbb{R}$, $M = [-2, 2]$, $K = \mathbb{R}^+$, $e = 1$ and $F, G : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [x^2, 2x^2], & \text{if } -1 \leq x \leq 1, \\ [0, x^2 + 4], & \text{otherwise.} \end{cases}$$

$$G(x) = \begin{cases} [x^2 - 1, x^2 + 1], & \text{if } x = 0, \\ [\frac{x^2}{4}, \frac{x^2}{2}], & \text{otherwise.} \end{cases}$$

Then, solution set of (P_u) with objective map F and solution set of (P_l) with objective map G are $\{0\}$, and hence they are compact. For $\delta = 1$ and for each $\varepsilon \in]0, \delta[$, we have $L_u(0, \varepsilon) = [-\sqrt{\frac{\varepsilon}{2}}, \sqrt{\frac{\varepsilon}{2}}]$, $L_l(0, \varepsilon) = [-\sqrt{\varepsilon}, \sqrt{\varepsilon}]$, and thus they are connected. Therefore, all conditions of Theorem 3.3.15 are satisfied. Employing this theorem, we conclude that the problems (P_u) and (P_l) with respect to F and G, respectively, are L-well-posed. However, the Theorem 3.3.14 does not work as F and G are not continuous on M.

The next example illustrates that the assumption (i) of Theorem 3.3.15 cannot be dropped.

Example 3.3.18. Let $X = l^\infty$ be the space of all bounded sequences of real numbers with the sup norm, $\|x\|_\infty = \sup |x_n|$ for all $x = (x_1, x_2, \ldots, x_n, \ldots)$. Let $Y = \mathbb{R}$, $M = X$, $K = \mathbb{R}^+$, $e = 1$ and $F : X \rightrightarrows Y$ be defined by

$$F(x) = [\,|\,\|x\| - 1|, |\,\|x\| - 1| + 1].$$

Then, X is a metric space with the metric $d_\infty(x, y) = \|x - y\|$. Since X is a infinite dimensional space, we conclude that (X, d_∞) is not locally compact, and hence the

assumption (i) of Theorem 3.3.15 is not satisfied. It is obvious that $S_\alpha = \{x \in X \mid \|x\| = 1\}$, and F is continuous and compact-valued on M. Moreover, for $\delta = 1$, the level set $L_\alpha(x, \varepsilon) = \{x \in X \mid 1-\varepsilon \leq \|x\| \leq 1+\varepsilon\}$ is connected for every $x \in S_\alpha$ and for every $\varepsilon \in]0, \delta[$. Let $\{x^i\} \subset M$, x^i here is the sequence which is zero everywhere except for a 1 at the ith position. Then, $\{x^i\}$ is a minimizing sequence, but it has no convergent subsequence. So, the problem (P_α) is not L-well-posedness and the assumption (i) is crucial.

3.4 Pointwise L-well-posedness

Motivated by the studies in [35, 61], we introduce the concept of pointwise L-well-posedness for the problem (P_α).

Definition 3.4.1. A sequence $\{x_n\} \subset M$ is said to be an x_0-minimizing sequence of (P_α) where $x_0 \in S_\alpha$ if and only if there exists $\{k_n\} \subset K$ converging to 0 such that

$$F(x_n) \leq^\alpha F(x_0) + k_n.$$

Definition 3.4.2. Problem (P_α) is said to be L-well-posed at $x_0 \in S_\alpha$ if and only if every x_0-minimizing sequence of (P_α) has a subsequence converging to some element $\bar{x} \in S_\alpha$.

Remark 3.4.3. Definition 3.4.1 and Definition 3.4.2 reduce respectively to Definition 3.3 in [35] when $\alpha = u$ and Definition 2.1 in [61] when $\alpha = l$.

At a reference point $x_0 \in M$, we define a corresponding set-valued map $Q_\alpha^{x_0} : K \rightrightarrows M$ as follows

$$Q_\alpha^{x_0}(k) = \{x \in M \mid F(x) \leq^\alpha F(x_0) + k\}.$$

Proposition 3.4.4. *$Q_\alpha^{x_0}(0) \subset S_\alpha$ with $x_0 \in S_\alpha$.*

Proof. By the similarity, we prove the assertion for the case $\alpha = s$. Let $x \in Q_s^{x_0}(0)$ and $y \in M$ such that

$$F(y) \leq^s F(x), \tag{3.4.1}$$

we need to show that $F(x) \leq^s F(y)$. Since $x \in Q_s^{x_0}(0)$,

$$F(x) \leq^s F(x_0). \tag{3.4.2}$$

Combining (3.4.1) and (3.4.2), we get $F(y) \leq^s F(x_0)$. Because $x_0 \in S_\alpha$, $F(x_0) \leq^s F(y)$. Therefore, $F(x) \leq^s F(y)$. □

We are going to study sufficient and necessary conditions of pointwise L-well-posedness for (P_α) via upper semicontinuity of map $Q_\alpha^{x_0}$ which will be studied in next chapter to survey B-well-posedness and relationships between pointwise L-well-posedness and pointwise B-well-posedness.

Theorem 3.4.5. *Let $x_0 \in S_\alpha$ be given.*

(i) *If (P_α) is L-well-posed at x_0 and $Q_\alpha^{x_0}(0) = S_\alpha$, then $Q_\alpha^{x_0}$ is upper semicontinuous and compact-valued at 0.*

(ii) *If $Q_\alpha^{x_0}$ is upper semicontinuous and compact-valued at 0, then (P_α) is L-well-posed at x_0.*

Proof. (i) Assume that (P_α) is L-well-posed at x_0. We now show that $Q_\alpha^{x_0}$ is upper semicontinuous at 0. By contradiction, suppose that $Q_\alpha^{x_0}$ is not upper semicontinuous at 0. Then, there exist a neighborhood U of $Q_\alpha^{x_0}(0)$ and $\{k_n\} \subset K$ converging to 0 such that for each $n \in \mathbb{N}$, there exists $x_n \in Q_\alpha^{x_0}(0) \setminus U$, i.e.,

$$x_n \notin U \tag{3.4.3}$$

and

$$F(x_n) \leq^\alpha F(x_0) + k_n. \tag{3.4.4}$$

It follows from (3.4.4) that $\{x_n\}$ is an x_0-minimizing sequence of (P_α). Because (P_α) is L-well-posed at x_0, there exists a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$, converging to some element $\bar{x} \in S_\alpha$, and thus we get $\bar{x} \in Q_\alpha^{x_0}(0)$. Therefore, $\bar{x} \in U$ which contradicts (3.4.3). So, $Q_\alpha^{x_0}$ is upper semicontinuous at 0.

Next, we prove that $Q_\alpha^{x_0}(0)$ is compact. Indeed, for every sequence $\{x_n\} \subset Q_\alpha^{x_0}(0)$, we have $F(x_n) \leq^\alpha F(x_0) + k_n$ where $\{k_n\} \subset K$ converges to 0. This means that $\{x_n\}$ is an x_0-minimizing sequence of (P_α). By the L-well-posedness of (P_α) at x_0, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to an element $\bar{x} \in S_\alpha$. Therefore, $\bar{x} \in Q_\alpha^{x_0}(0)$. This leads to the compactness of $Q_\alpha^{x_0}(0)$.

(ii) Let $\{x_n\} \subset M$ be an x_0-minimizing sequence of (P_α), there exists $\{k_n\} \subset K$ converging to 0 such that $F(x_n) \leq^\alpha F(x_0) + k_n$. Hence, $x_n \in Q_\alpha^{x_0}(k_n)$. Since $Q_\alpha^{x_0}$ is upper semicontinuous and compact-valued at 0, there exists a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$, converging to some $\bar{x} \in Q_\alpha^{x_0}(0)$. Combining this with Proposition 3.4.4, we get $\bar{x} \in S_\alpha$. So, (P_α) is L-well-posed at x_0. □

3.5 Links between L-well-posedness and stability

In this section, we study some characterizations and criteria of L-well-posedness for (P_α). This type of well-posedness can be characterized by the behavior of $\widehat{S}_\alpha(\varepsilon)$ as $\varepsilon \to 0$, which is given by

$$\widehat{S}_\alpha(\varepsilon) = \{x \in M \mid \exists z \in S_\alpha, F(x) \leq^\alpha F(z) + \varepsilon e\}.$$

The set-valued map $\widehat{S}_\alpha : \mathbb{R}^+ \rightrightarrows M$ is considered as approximating solution map of (P_α), and it has a close relationship with the level set, more precisely, $S_\alpha(\varepsilon) = \bigcap_{x \in S_\alpha} L_F^\alpha(x, \varepsilon)$.

The following proposition gives some properties of the map S_α which are useful in the sequel.

Proposition 3.5.1. *The following statements hold:*

(i) $\widehat{S}_\alpha(0) = S_\alpha$;

(ii) *if* $\varepsilon_1 \leq \varepsilon_2$, *then* $\widehat{S}_\alpha(\varepsilon_1) \subset \widehat{S}_\alpha(\varepsilon_2)$;

(iii) *if* F *is compact-valued on* M, *then* $\bigcap_{\varepsilon > 0} \widehat{S}_\alpha(\varepsilon) = S_\alpha$.

Proof. We only prove the assertions (i)-(iii) for the case $\alpha = s$; proofs of these assertions for cases $\alpha = l$ and $\alpha = u$ are given by similar arguments.

(i) Obviously, $S_s \subset \widehat{S}_s(0)$. Conversely, let $x \in \widehat{S}_s(0)$, there exists $z \in S_s$ such that $F(x) \leq^s F(z)$. Taking $y \in M$ satisfying $F(y) \leq^s F(x)$, we show that $F(x) \leq^s F(y)$. By the transitivity property of $\leq^s$, we have $F(y) \leq^s F(z)$, and hence $F(z) \leq^s F(y)$ as $z \in S_s$. Again by the transitivity property, one gets $F(x) \leq^s F(y)$. So, $x \in S_s$.

(ii) Assume $\varepsilon_1 \leq \varepsilon_2$. Let $x \in \widehat{S}_s(\varepsilon_1)$, there exists $z \in S_s$ such that $F(x) \leq^s F(z) + \varepsilon_1 e$, i.e.,

$$F(z) + \varepsilon_1 e \subset F(x) + K, \quad \text{and} \quad F(x) \subset F(z) + \varepsilon_1 e - K.$$

We have

$$F(z) + \varepsilon_2 e = F(z) + \varepsilon_1 e + (\varepsilon_2 - \varepsilon_1)e,$$

and

$$F(z) + \varepsilon_2 e - K = F(z) + \varepsilon_1 e - K + (\varepsilon_2 - \varepsilon_1)e.$$

Combining the convexity of K with Proposition 2.2.7, we obtain that

$$F(z) + \varepsilon_2 e \subset F(x) + K, \quad \text{and} \quad F(x) \subset F(z) + \varepsilon_2 e - K,$$

i.e., $F(x) \leq^s F(z) + \varepsilon_2 e$. Consequently, $x \in \widehat{S}_s(\varepsilon_2)$ as $z \in S_s$. Hence, $\widehat{S}_s(\varepsilon_1) \subset \widehat{S}_s(\varepsilon_2)$.

(iii) Let $x \in S_s$ and $\varepsilon > 0$ be arbitrary. We have $x \in \widehat{S}_s(\varepsilon)$ for any $\varepsilon > 0$, and so $x \in \bigcap_{\varepsilon>0} \widehat{S}_s(\varepsilon)$. Conversely, let $x \in \bigcap_{\varepsilon>0} \widehat{S}_s(\varepsilon)$, we get $x \in \widehat{S}_s(\varepsilon)$ for any $\varepsilon > 0$. By the definition of $\widehat{S}_s(\varepsilon)$, there exists $z \in S_s$ such that $F(x) \leq^s F(z) + \varepsilon e$, i.e.,

$$F(z) + \varepsilon e \subset F(x) + K, \quad \text{and} \quad F(x) \subset F(z) + \varepsilon e - K. \tag{3.5.1}$$

Since F is compact-valued, $F(x) - K$ and $F(z) - K$ are closed by Lemma 2.4.6. From (3.5.1), let $\varepsilon \to 0$, we have

$$F(z) \subset F(x) + K, \quad \text{and} \quad F(x) \subset F(z) - K.$$

Equivalently, $F(x) \leq^s F(z)$. This together with $z \in S_s$ implies that $x \in S_s$. Indeed, suppose that $F(y) \leq^s F(x)$ for some $y \in M$. Then, $F(y) \leq^s F(z)$ because $F(x) \leq^s$

$F(z)$. Moreover, as $z \in S_s$, we get $F(z) \leq^s F(y)$. It yields that $F(x) \leq^s F(y)$, and hence $x \in S_s$. So, $\bigcap_{\varepsilon>0} \widehat{S}_s(\varepsilon) \subset S_s$. □

Using the Kuratowski measure of noncompactness of approximate solution sets, we now establish a metric characterization of the L-well-posedness for (P_α).

Theorem 3.5.2.

(i) *If the problem* (P_α) *is L-well-posed, then* $\mu(\widehat{S}_\alpha(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$.

(ii) *If the following conditions hold:*

(a) F *is compact-valued on* M;

(b) $\widehat{S}_\alpha(\varepsilon)$ *is closed for all* $\varepsilon > 0$;

(c) $\mu(\widehat{S}_\alpha(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$.

Then, the problem (P_α) *is L-well-posed.*

Proof. (i) Suppose that (P_α) is L-well-posed. Notice that S_α is compact by Theorem 3.3.9. Hence, for any $\varepsilon > 0$, there are M_i $(i = 1, \ldots, n)$ with $\mathrm{diam} M_i \leq \varepsilon$ and $S_\alpha \subset \bigcup_{i=1}^n M_i$. For each $i \in \{1, \ldots, n\}$, we denote

$$N_i = \{x \in X \mid d(x, M_i) \leq H(\widehat{S}_\alpha(\varepsilon), S_\alpha)\}.$$

Firstly, we show that $\widehat{S}_\alpha(\varepsilon) \subset \bigcup_{i=1}^n N_i$. Letting $x \in \widehat{S}_\alpha(\varepsilon)$, we have

$$d(x, S_\alpha) \leq H(\widehat{S}_\alpha(\varepsilon), S_\alpha).$$

Since $S_\alpha \subset \bigcup_{i=1}^n M_i$, we conclude that

$$d(x, \cup_{i=1}^n M_i) \leq d(x, S_\alpha) \leq H(\widehat{S}_\alpha(\varepsilon), S_\alpha).$$

So, there is $k_0 \in \{1, \ldots, n\}$ such that

$$d(x, M_{k_0}) \leq H(\widehat{S}_\alpha(\varepsilon), S_\alpha),$$

i.e., $x \in N_{k_0}$. Hence, $\widehat{S}_\alpha(\varepsilon) \subset \bigcup_{i=1}^{n} N_i$. Notice further that

$$\text{diam}N_i = \text{diam}M_i + 2H(\widehat{S}_\alpha(\varepsilon), S_\alpha) \leq \varepsilon + 2H(\widehat{S}_\alpha(\varepsilon), S_\alpha).$$

Therefore,

$$\mu(\widehat{S}_\alpha(\varepsilon)) \leq \mu(S_\alpha) + 2H(\widehat{S}_\alpha(\varepsilon), S_\alpha).$$

Since S_α is compact, $\mu(S_\alpha) = 0$. Hence,

$$\mu(\widehat{S}_\alpha(\varepsilon)) \leq 2H(\widehat{S}_\alpha(\varepsilon), S_\alpha).$$

Finally, we show that $H(\widehat{S}_\alpha(\varepsilon), S_\alpha) \to 0$ as $\varepsilon \to 0$. Since $S_\alpha \subset \widehat{S}_\alpha(\varepsilon)$, $H^*(S_\alpha, \widehat{S}_\alpha(\varepsilon)) = 0$, and hence we only need to prove that $H^*(\widehat{S}_\alpha(\varepsilon), S_\alpha) \to 0$ as $\varepsilon \to 0$. Assume, by contradiction, that there exist a real number $r > 0$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging 0 such that for each n, there exists $x_n \in \widehat{S}_\alpha(\varepsilon_n)$ satisfying

$$d(x_n, S_\alpha) \geq r. \tag{3.5.2}$$

Because $x_n \in \widehat{S}_\alpha(\varepsilon_n)$, for each n, there exists $z_n \in S_\alpha$ such that $F(x_n) \leq^\alpha F(z_n) + \varepsilon_n e$. This means that $\{x_n\}$ is a generalized mimimizing sequence of (P_α). So, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some $x \in S_\alpha$. Therefore, for n_k sufficiently large, we have $d(x_{n_k}, x) < r$ which contradicts (3.5.2). Hence, $\mu(\widehat{S}_\alpha(\varepsilon)) \to 0$ as $\varepsilon \to 0$.

(ii) Let $\{x_n\}$ be a generalized minimizing sequence of (P_α), then there exist $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 and $\{z_n\} \subset S_\alpha$ satisfying $F(x_n) \leq^\alpha F(z_n) + \varepsilon_n e$. Thus, $x_n \in \widehat{S}_\alpha(\varepsilon_n)$. Using Proposition 3.5.1, we have S_α is compact and $H(\widehat{S}_\alpha(\varepsilon_n), S_\alpha) \to 0$. We get $d(x_n, S_\alpha) \to 0$ as $d(x_n, S_\alpha) \leq H(\widehat{S}_\alpha(\varepsilon_n), S_\alpha)$. Therefore, there exists a sequence $\{\bar{x}_n\} \subset S_\alpha$ such that $d(x_n, \bar{x}_n) \to 0$ as $n \to \infty$. By the compactness of S_α, there is a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ converging to some $\bar{x} \in S_\alpha$. Then, the corresponding subsequence $\{x_{n_k}\}$ converges to $\bar{x}$. So, (P_α) is L-well-posed. □

Remark 3.5.3. For the necessary conditions of L-well-posedness, our result in Theorem 3.5.2(i) improves Proposition 4.1 of [61]. More precisely, the L-well-posedness of (P_α) is obtained without using compactness of solution set, closedness of approximating solution set and compact values of F imposed in Proposition 4.1 of [61].

In Theorem 3.5.2(ii), we use assumption about the closedness of approximating solution sets. In the next result, we give sufficient conditions for this assumption.

Proposition 3.5.4. *Assume that the following conditions are satisfied:*

(i) *F is continuous and compact-valued on M;*

(ii) *S_α is compact.*

Then, $\widehat{S}_\alpha(\varepsilon)$ is closed for each $\varepsilon \geq 0$.

Proof. By the similarity, we only prove the assertion for the case $\alpha = u$. For each $\varepsilon \geq 0$, let $\{x_n\} \subset \widehat{S}_u(\varepsilon)$ be a sequence converging to some $x \in M$, we need to prove that $x \in \widehat{S}_u(\varepsilon)$. Since $\{x_n\} \subset \widehat{S}_u(\varepsilon)$, there exists $\{z_n\} \subset S_u$ such that

$$F(x_n) \leq^u F(z_n) + \varepsilon e. \tag{3.5.3}$$

By the compactness of S_u, there exist a subsequence of $\{z_n\}$ which is still denoted by $\{z_n\}$ and $z \in S_u$ such that $\{z_n\}$ converges to z. Next, we show that $F(x) \leq^u F(z)+\varepsilon e$. Indeed, from (3.5.3), we get

$$F(x_n) \subset F(z_n) + \varepsilon e - K. \tag{3.5.4}$$

Let $y \in F(x)$, by the lower semicontinuity of F, there exists $\{y_n\}$ converging to y where $y_n \in F(x_n)$. Combining this with (3.5.4), for each n, there exists $w_n \in F(z_n)$ such that

$$y_n \in w_n + \varepsilon e - K. \tag{3.5.5}$$

Since F is upper semicontinuous and compact-valued at z, there exist $w \in F(z)$ and a subsequence of $\{w_n\}$ which is still denoted by $\{w_n\}$ such that $\{w_n\}$ converges to w. From (3.5.5), there exist $k_n \in K$ satisfying $y_n = w_n + \varepsilon e - k_n$, i.e., $k_n = w_n + \varepsilon e - y_n$. Then, k_n converge to $w + \varepsilon e - y \in K$ as K is closed. Therefore, there exists $k \in K$ such that $y = w + \varepsilon e - k \in w + \varepsilon e - K \subset F(z) + \varepsilon e - K$ as $w \in F(z)$. We arrive at the fact that $F(x) \subset F(z) + \varepsilon e - K$. It means that $F(x) \leq^u F(z) + \varepsilon e$. So, $x \in \widehat{S}_u(\varepsilon)$ as $z \in S_u$. □

The next result give sufficient and necessary conditions for L-well-posedness of (P_α) via upper semicontinuity of approximating solution map of (P_α). When $\alpha = l$, this result coincides with Proposition 4.3(i) in [61].

Theorem 3.5.5. *$\widehat{S}_\alpha$ is upper semicontinuous and compact-valued at 0 if and only if the problem (P_α) is L-well-posed.*

Proof. Suppose that (P_α) is L-well-posed. By Theorem 3.3.9, S_α is compact. By contradiction, suppose that $\widehat{S}_\alpha$ is not upper semicontinuous at 0. Then, there exists a neighborhood N of $\widehat{S}_\alpha(0)$ such that for any neighborhood U of 0, $\widehat{S}_\alpha(U) \not\subset N$. It means that there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 such that for each n, $\widehat{S}_\alpha(\varepsilon_n) \not\subset N$. Then, there exist $x_n \in \widehat{S}_\alpha(\varepsilon_n) \setminus N$ for all n, and hence there exist $z_n \in S_\alpha$ such that $F(x_n) \leq^\alpha F(z_n) + \varepsilon_n e$. This implies that $\{x_n\}$ is a generalized minimizing sequence of (P_α). Because (P_α) is L-well-posed, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in S_\alpha$ such that $\{x_{n_k}\}$ converges to z which is impossible since $x_{n_k} \notin N$ for all n_k.

Conversely, let $\{x_n\} \subset M$ be a generalized minimizing sequence of (P_α), there are $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 and $z_n \in S_\alpha$ satisfying $F(x_n) \leq^\alpha F(z_n) + \varepsilon_n e$. This means that $x_n \in \widehat{S}_\alpha(\varepsilon_n)$. Since $\widehat{S}_\alpha$ is upper semicontinuous at 0 and N is a neighborhood of $\widehat{S}_\alpha(0)$, $x_n \in N$ for n sufficiently large. Equivalently, for every neighborhood W of 0, there exists $n_0 \in \mathbb{N}$ such that $x_n \in S_\alpha + W$ for any $n \geq n_0$. By the compactness of S_α, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\bar{x} \in S_\alpha$ such that $\{x_{n_k}\}$ converges to $\bar{x}$. So, (P_α) is L-well-posed. □

To end up this section, we give the sufficient conditions for the upper semicontinuity of approximating solution map of (P_α) used in the previous result.

Proposition 3.5.6. *If M is compact and the conditions of Proposition 3.5.4 are satisfied, then $\widehat{S}_\alpha$ is upper semicontinuous at 0.*

Proof. We only give the proof of the assertion for the case $\alpha = u$. Suppose to the contrary that $\widehat{S}_u$ is not upper semicontinuous at 0. Then, there exist an open

neighborhood W_0 of $\widehat{S}_u(0)$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 such that $\widehat{S}_u(\varepsilon_n) \not\subset W_0$ for all $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, there exists

$$x_n \in \widehat{S}_u(\varepsilon_n) \setminus W_0. \tag{3.5.6}$$

Since M is compact, we can assume that $\{x_n\}$ converges to an element $x_0 \in M$. Moreover, by (3.5.6), for each $n \in \mathbb{N}$, there exists $z_n \in S_u$ such that $F(x_n) \leq^u F(z_n) + \varepsilon_n e$. Hence,

$$F(x_n) \subset F(z_n) + \varepsilon_n e - K. \tag{3.5.7}$$

By the compactness of S_u, we can assume that $\{z_n\}$ converges to some $z_0 \in S_u$. Next, we prove that $F(x_0) \subset F(z_0) - K$. Indeed, for any $u_0 \in F(x_0)$, by the lower semicontinuity of F on M and Lemma 2.3.2, there exists a sequence $\{u_n\}$, $u_n \in F(x_n)$, such that $\{u_n\}$ converges to u_0. It follows from (3.5.7) that, for each $n \in \mathbb{N}$, there exists $v_n \in F(z_n)$ such that

$$u_n - v_n - \varepsilon_n e \in -K. \tag{3.5.8}$$

Since F is upper semicontinuous and compact-valued at z_0, we can assume that $\{v_n\}$ converges to $v_0 \in F(z_0)$. By (3.5.8) and the closedness of K, $u_0 - v_0 \in -K$. Therefore, $u_0 \in F(z_0) - K$. Since $u_0 \in F(x_0)$ is arbitrary, $F(x_0) \subset F(z_0) - K$, and hence $x_0 \in \widehat{S}_u(0) \subset W_0$ which is impossible as $x_n \notin W_0$ for all n. So, $\widehat{S}_u$ is upper semicontinuous at 0. □

CHAPTER IV

B-WELL-POSEDNESS FOR SET OPTIMIZATION PROBLEMS

In the previous chapter, we studied L-well-posedness which is defined base on the convergence of a subsequence of a minimizing sequence. In this chapter, we investigate another kind of well-posedness, so-called B-well-posedness. This concept is defined by (Hausdorff)semicontinuity of a set-valued map.

In 2013, as the first authors concerned B-well-posedness for set optimization problems, Long and Peng [13] introduced three types of B-well-posedness for set optimization problems involving upper set less relations $\leq^u$ and established some relations among these kinds of B-well-posedness. Moreover, the authors also provided necessary and sufficient conditions of these notions for set optimization problems. To extend the research in [13], Han and Huang [36] studied B-well-posedness for set optimization problems involving set order relations $\leq^l$ and $\leq^u$. They gave characterizations for the generalized l-B-well-posedness and the generalized u-B-well-posedness and provided necessary, sufficient conditions for the semicontinuity of solution map.

As mentioned in [10, 23] that among three kinds of set order relations introduced in [5], the set less relation $\leq^s$ is generalized and more appropriate in practical problems than both the lower and upper set less relations; and it also occupies an important role in relationships with other new order relations for sets proposed in [10] which are more useful in real world. Moreover, to the best of our knowledge, there is no paper devoted to well-posedness for set optimization problems involving the set less relation, and hence well-posedness properties for such problems are deserved to study more. Consequently, we aim to survey both pointwise and global B-well-posedness for set optimization problems involving three kinds of set order relations.

This chapter investigates well-posedness in the sense of Bednarczuk for the

problem (P_α). Both global and pointwise of this kind of well-posedness are studied. Necessary and sufficient conditions for (P_α) to be well-posed are given. Moreover, relationships between B-well-posedness and L-well-posedness are demonstrated.

Let X and Y be normed spaces and let M be a nonempty subset of X.

4.1 Global B-well-posedness

In this section, two kinds of global B-well-posedness for the problem (P_α) are considered and their relationships are discussed. Moreover, we also provide characterizations and sufficient conditions of B-well-posedness for such problems.

We define a set-valued map $Q : K \rightrightarrows M$ as follows

$$Q_\alpha(k) = \bigcup_{y \in S_\alpha} \{x \in M \mid F(x) \leq^\alpha F(y) + k\}.$$

The following results provide some properties of this map.

Proposition 4.1.1. *These assertions hold:*

(i) *If* $k_1 \leq_K k_2$, *then* $Q_\alpha(k_1) \subset Q_\alpha(k_2)$;

(ii) $S_\alpha \subset Q_\alpha(0)$;

(iii) $Q_\alpha(0) = \cap_{k \in K} Q_\alpha(k)$.

Proof. (i) We only demonstrate the proof of the above assertion for the case $\alpha = s$, proofs of this assertion for other cases are proved similarly. Let $x \in Q_s(k_1)$ be given, then there exists $y \in S_s$ such that $F(x) \leq^s F(y) + k_1$, i.e., $F(y) + k_1 \subset F(x) + K$ and $F(x) \subset F(y) + k_1 - K$. Combining this with $k_1 \leq_K k_2$, we get $F(y) + k_2 = F(y)+k_1+(k_2-k_1) \subset F(x)+K$ and $F(x) \subset F(y)+k_1-K = F(y)+k_2+(k_1-k_2)-K \subset F(y) + k_2 - K$. This means that $F(x) \leq^l F(y) + k_2$ and $F(x) \leq^u F(y) + k_2$. Hence, $x \in Q_s(k_2)$.

(ii) It is clear that for every $x \in S_\alpha$, we have $F(x) \leq^\alpha F(x)$, and hence $x \in Q_\alpha(0)$. Therefore, $S_\alpha \subset Q_\alpha(0)$.

(iii) It is obvious that $Q_\alpha(0) \subset Q_\alpha(k)$ for all $k \in K$, and thus $Q_\alpha(0) \subset \cap_{k\in K} Q_\alpha(k)$. Conversely, suppose that there exists $x \in \cap_{k\in K} Q_\alpha(k)$ but $x \notin Q_\alpha(0)$, i.e., $x \notin \cup_{y\in S_\alpha}\{z \in M | F(z) \leq^\alpha F(y)\}$. Then, $F(x) \not\leq^\alpha F(y)$ for any $y \in S_\alpha$. On the other hand, since $x \in \cap_{k\in K} Q_\alpha(k)$, $x \in Q_\alpha(k)$ for all $k \in K$. Therefore, there is $y \in S_\alpha$ such that $F(x) \leq^\alpha F(y) + k$ for all $k \in K$. Particularly, for $k = 0$, there exists $y \in S_\alpha$ such that $F(x) \leq^\alpha F(y)$ which is a contradiction. □

Next, we give two concepts related to global B-well-posedness for (P_α).

Definition 4.1.2. Problem (P_α) is said to be

(i) B-well-posed if and only if $S_\alpha \neq \emptyset$ and Q_α is upper semicontinuous at $k = 0$.

(ii) generalized B-well-posed if and only if $S_\alpha \neq \emptyset$ and Q_α is Hausdorff upper semicontinuous at $k = 0$.

Remark 4.1.3. Clearly, if the problem (P_α) is B-well-posed, then it is generalized B-well-posed. It follows from Remark 2.3.4 that the converse holds if S_α is compact. In the sequel, we focus on generalized B-well-posedness.

Definition 4.1.4. A sequence $\{x_n\} \subset M$ is said to be a generalized B-minimizing sequence of (P_α) if and only if there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_\alpha$ such that $F(x_n) \leq^\alpha F(y_n) + k_n$.

Equivalently, $\{x_n\}$ is a generalized B-minimizing sequence of (P_α) if and only if there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_\alpha$ such that $x_n \in Q_\alpha(k_n)$.

Remark 4.1.5. When $\alpha = u$, concepts in Definitions 4.1.2 and 4.1.4 reduce to ones in Definitions 3.1-3.3 of [13], respectively.

Characterizations of B-well-posedness for (P_α) are provided in the next result through the B-minimizing sequence.

Theorem 4.1.6. *Problem* (P_α) *is generalized B-well-posed if and only if these following conditions are satisfied*

(i) $S_\alpha \neq \emptyset$;

(ii) *for every generalized B-minimizing sequence* $\{x_n\} \subset M$ *and for every neighborhood* U *of the origin in* X, *there exists* $n_0 \in \mathbb{N}$ *such that* $x_n \in Q_\alpha(0) + U$ *for all* $n \geq n_0$.

Proof. Suppose that (P_α) is generalized B-well-posed. Let $\{x_n\} \subset M$ be a generalized B-minimizing sequence of (P_α), then there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_\alpha$ such that $x_n \in Q_\alpha(k_n)$. Since (P_α) is generalized B-well-posed, Q_α is Hausdorff upper semicontinuous at 0. Let U be a neighborhood of the origin in X, there exists $n_0 \in \mathbb{N}$ such that $Q_\alpha(k_n) \subset Q_\alpha(0) + U$ for all $n \geq n_0$. Therefore, we get $x_n \in Q_\alpha(0) + U$ for all $n \geq n_0$.

Conversely, suppose on the contrary that (P_α) is not generalized B-well-posed. Thus, Q_α is not Hausdorff upper semicontinuous at 0. Then, there exists a neighborhood U of the origin in X such that $Q_\alpha(k) \not\subset Q_\alpha(0)+U$ for some k belongs to a neighborhood of 0. So, we can build a sequence $\{k_n\} \subset K$ converging to 0 such that $Q_\alpha(k_n) \not\subset Q_\alpha(0) + U$. It leads to the existence of a sequence $\{x_n\}$ with $x_n \in Q_\alpha(k_n)$ satisfying $x_n \notin Q_\alpha(0) + U$ which contradicts the assumption (b). This completes the proof. □

We now give sufficient conditions for (P_α) to be generalized B-well-posed.

Theorem 4.1.7. *Suppose that* $S_\alpha \neq \emptyset$ *and for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$(F(M) - F(S_\alpha)) \cap (\delta \overline{B}(0,1) - K) \subset \varepsilon \overline{B}(0,1), \tag{4.1.1}$$

where $\overline{B}(0,1)$ *is the closed unit ball of* Y. *Then,*

(i) (P_l) *is generalized B-well-posed if each sequence of sets* $\{A_n\} \subset M$ *with* $F(A_n) \rightharpoonup F(S_l)$ *satisfies* $A_n \rightharpoonup S_l$.

(ii) (P_u) *is generalized B-well-posed if each sequence of sets* $\{A_n\} \subset M$ *with* $F(A_n) \rightharpoonup F(S_u)$ *satisfies* $A_n \rightharpoonup S_u$.

(iii) (P_s) *is generalized B-well-posed if each sequence of sets* $\{A_n\} \subset M$ *with* $F(A_n) \to F(S_s)$ *satisfies* $A_n \to S_s$.

Proof. (i) By contradiction, suppose that (P_l) is not generalized B-well-posed. It follows from Theorem 4.1.6 that there exist a generalized B-minimizing sequence $\{x_n\}$ and a neighborhood U of the origin in X such that for some $n_0 \in \mathbb{N}$, $x_n \notin Q_l(0) + U$ for all $n \geq n_0$. Combining this with Proposition 4.1.1(ii), we get

$$x_n \notin S_l + U, \ \forall n \geq n_0. \tag{4.1.2}$$

Since $\{x_n\}$ is a generalized B-minimizing sequence, there exist $\{k_n\} \subset K$ converging to 0 and $\{y_n\} \subset S_l$ such that

$$F(x_n) \leq^l F(y_n) + k_n. \tag{4.1.3}$$

We consider two following cases:

Case 1: If $F(x_n) \rightharpoonup F(S_l)$, then choosing $A_n = \{x_n\}$. It implies from the hypothesis that $\{x_n\} \rightharpoonup S_l$, and hence $H^*(S_l, \{x_n\}) \to 0$. Therefore, $d(x_n, S_l) \to 0$ which contradicts (4.1.2).

Case 2: If $F(x_n) \not\rightharpoonup F(S_l)$, then $\sup_{x \in S_l} d(x, F(x_n)) \not\to 0$. So, there exists $x \in S_l$ such that $d(x, F(x_n)) \not\to 0$, i.e., there exist $n_1 \in \mathbb{N}$ and a neighborhood V of the origin in Y such that

$$x \notin F(x_n) + V, \ \forall n \geq n_1. \tag{4.1.4}$$

Take ε such that $\varepsilon\overline{B}(0,1) \subset V$. For δ satisfying (4.1.1), since $\{k_n\} \subset K$ converges to 0, there exists $n_2 \in \mathbb{N}$ such that for $n \geq n_2$, we have

$$k_n \in \delta\overline{B}(0,1). \tag{4.1.5}$$

By (4.1.3), we get $F(y_n) + k_n \subset F(x_n) + K$. Therefore, $F(y_n) \subset F(x_n) - k_n + K \subset F(x_n) + \delta\overline{B}(0,1) + K$. This implies that for an arbitrary $z_n \in F(y_n)$, there exists $\bar{z}_n \in F(x_n)$ such that $z_n \in \bar{z}_n + \delta\overline{B}(0,1) + K$, and thus $z_n - \bar{z}_n \in \delta\overline{B}(0,1) + K$. So,

$\bar{z}_n - z_n \in \delta\overline{B}(0,1) - K$. On the other hand, we have $\bar{z}_n - z_n \in F(x_n) - F(y_n) \subset F(M) - F(S_l)$. It derives from (4.1.1) that $\bar{z}_n - z_n \in \varepsilon\overline{B}(0,1) \subset V$. So, now we get $z_n \in \bar{z}_n + V \subset F(x_n) + V$ which contradicts (4.1.4).

For (ii) (iii), the proofs of these assertions are technically similar to that of the assertion (i). □

4.2 Pointwise B-well-posedness

In this section, we consider a notion of pointwise B-well-posedness for the problem (P_α). For convenience in reading, we recall map $Q_\alpha^{x_0}$. At a reference point $x_0 \in M$, $Q_\alpha^{x_0}(k) = \{x \in M \mid F(x) \leq^\alpha F(x_0)+k\}$. Clearly, for each $x_0 \in S_\alpha$, $\{x_n\} \subset M$ is an x_0-minimizing sequence if and only if there exists $\{k_n\} \subset K$ converging to 0 such that $x_n \in Q_\alpha^{x_0}(k_n)$.

Definition 4.2.1. Problem (P_α) is said to be B-well-posed at $x_0 \in S_\alpha$ if and only if $Q_\alpha^{x_0}$ is upper semicontinuous at $k = 0$.

We observe that $Q_\alpha^{x_0}(0) = \{x \in M \mid F(x) \sim^\alpha F(x_0)\}$ for each $x_0 \in S_\alpha$.

The next results give some properties of the map $Q_\alpha^{x_0}$.

Proposition 4.2.2. *The following statements are true:*

(i) *if $k_1 \leq_K k_2$, then $Q_\alpha^{x_0}(k_1) \subset Q_\alpha^{x_0}(k_2)$;*

(ii) *for $x_0 \in S_\alpha$, $S_\alpha \subset Q_\alpha^{x_0}(0)$ if and only if $F(x) \sim^\alpha F(y)$ for all $x, y \in S_\alpha$.*

(iii) *$x_0 \in Q_\alpha^{x_0}(0) \subset Q_\alpha^{x_0}(k)$ for every $k \in K$;*

(iv) *$Q_\alpha(0) = \cup_{x_0 \in S_\alpha} Q_\alpha^{x_0}(0)$.*

Proof. (i) The statement is proved by a similar argument in Proposition 4.1.1(i).

For (ii), (iii) and (iv), these assertions were implied by definitions of maps Q_α and $Q_\alpha^{x_0}$. □

The next result illustrates the relationship between the pointwise B-well-posedness and global B-well-posedness for (P_α).

Theorem 4.2.3. *If S_α is a finite set and (P_α) is pointwise B-well-posed at every $x \in S_\alpha$, then it is B-well-posed.*

Proof. Let $S_\alpha = \{x_1, \ldots, x_n\}$ and V be an open set in X such that $Q_\alpha(0) \subset V$. By Proposition 4.2.2(iv), $Q_\alpha^{x_i}(0) \subset V$ for all $i = 1, \ldots, n$. Since (P_α) is pointwise B-well-posed at every x_i, $Q_\alpha^{x_i}$ is upper semicontinuous at $k = 0$, and hence for each $i \in \{1, \ldots, n\}$, there exists a neighborhood U_{x_i} of 0 such that $Q_\alpha^{x_i}(U_{x_i}) \subset V$. Let $U = \cap_{i=1}^n U_{x_i}$, this finite intersection of neighborhoods U_{x_i} is also a neighborhood of 0. Obviously, we have $Q_\alpha^{x_i}(U) \subset V$ for all $i = 1, \ldots, n$. By definitions of maps $Q_\alpha^{x_i}$ and Q_α, we also get $Q_\alpha(U) \subset V$. It leads to the upper semicontinuity at $k = 0$ of Q_α. We conclude that (P_α) is B-well-posed. □

We next investigate characterizations of pointwise B-well-posedness for (P_α).

Theorem 4.2.4. *Problem (P_α) is pointwise B-well-posed at $x_0 \in S_\alpha$ if and only if for a given $e \in intK$, the set-valued map $(Q_\alpha^{x_0})^+ : \mathbb{R}_+ \rightrightarrows M$ defined as*

$$(Q_\alpha^{x_0})^+(t) = \{x \in M \mid F(x) \leq^\alpha F(x_0) + te\}$$

is upper semicontinuous at $t = 0$.

Proof. Assume that (P_α) is pointwise B-well-posed at $x_0 \in S_\alpha$, then Q_{x_0} is upper semicontinuous at $k = 0$. Let V be an open set in X such that $(Q_\alpha^{x_0})^+(0) \subset V$, we get $Q_\alpha^{x_0}(0) \subset V$. By the upper semicontinuity of $Q_\alpha^{x_0}$, there is a positive number r such that $Q_\alpha^{x_0}(k) \subset V$ for all $k \in B(0,r) \cap K$, where $B(0,r)$ is the open ball centered at the origin in Y with radius r. Then, there exists a positive number β such that $[0, \beta e) \subset B(0,r)$, where $[0, \beta e) = \{te \mid t \in [0, \beta[\}$. For $t \in [0, \beta[$ and $x \in (Q_\alpha^{x_0})^+(t)$, we have $F(x) \leq^\alpha F(x_0) + te$, which implies that $x \in Q_\alpha^{x_0}(te)$. This fact together with $te \in B(0,r)$ yields $x \in V$, and so $(Q_\alpha^{x_0})^+(t) \subset V$. We conclude that $(Q_\alpha^{x_0})^+$ is upper semicontinuous at $t = 0$.

Conversely, suppose that $(Q_\alpha^{x_0})^+$ is upper semicontinuous at $t = 0$. Let V be an open set in X such that $Q_\alpha^{x_0}(0) \subset V$, then $(Q_\alpha^{x_0})^+(0) \subset V$. It follows from the upper semicontinuity of $(Q_\alpha^{x_0})^+$ that there exists a positive number β such that $(Q_\alpha^{x_0})^+(t) \subset V$ for every $t \in [0, \beta[$. For convenience in writing, we only prove the assertion for case $\alpha = s$ because proofs of this assertion for other cases $\alpha = l$ and $\alpha = u$ are similar. Let $\gamma \in [0, \beta[$, there exists a positive number r such that $B(0,r) \subset \gamma e - K$ and $B(0,r) \subset -\gamma e + K$. Let $k \in B(0,r) \cap K$ and $x \in Q_s^{x_0}(k)$, it follows from the definition of $Q_s^{x_0}$ that

$$F(x) \leq^u F(x_0) + k, \tag{4.2.1}$$

and

$$F(x) \leq^l F(x_0) + k. \tag{4.2.2}$$

It yields from (4.2.1) that $F(x) \subset F(x_0) + k - K \subset F(x_0) + \gamma e - K$ as $k \in \gamma e - K$. On other hand, by (4.2.2), we get $F(x_0) + k \subset F(x) + K$, and hence $F(x_0) \subset F(x) - k + K$. Since $-k \in B(0,r) \subset -\gamma e + K$, we have $F(x_0) \subset F(x) - \gamma e + K$, i.e., $F(x_0) + \gamma e \subset F(x) + K$. So, we now get $F(x) \leq^s F(x_0) + \gamma e$. It implies that $x \in (Q_s^{x_0})^+(\gamma)$, and thus $x \in V$ and $Q_s^{x_0}(k) \subset V$. The proof is complete. □

Theorem 4.2.5. *If S_α is closed and (P_α) is B-well-posed at $x_0 \in S_\alpha$, then for every x_0-minimizing sequence $\{x_n\} \subset M \backslash S_\alpha$, one can extract a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to some $\bar{x} \in S_\alpha$.*

Proof. Assume that (P_α) is B-well-posed at $x_0 \in S_\alpha$. Then, $Q_\alpha^{x_0}$ is upper semicontinuous at $k = 0$. By contradiction, suppose that there exists an x_0-minimizing sequence $\{x_n\} \subset M \backslash S_\alpha$ which admits no subsequence $\{x_{n_k}\}$ converging to some $\bar{x} \in S_\alpha$. By the closedness of S_α, we may find an open set $V \subset X$ such that $S_\alpha \subset V$ and $x_n \notin V$. By Proposition 3.4.4, we have $Q_\alpha^{x_0}(0) \subset S_\alpha$, and hence we get $Q_\alpha^{x_0}(0) \subset V$. Since $\{x_n\}$ is an x_0-minimizing sequence, there exists $\{k_n\} \subset K$ converging to 0 such that $x_n \in Q_\alpha^{x_0}(k_n)$. It follows from the upper semicontinuity of $Q_\alpha^{x_0}$ at $k = 0$ that

$Q_\alpha^{x_0}(k_n) \subset V$. Hence, $x_n \in V$ which is a contradiction. So, we get the desired result. □

Remark 4.2.6. Our results extend the corresponding results of Long and Peng [13]. More precisely, when $\alpha = l$ or $\alpha = s$, our results here are new. To the best of our knowledge, there is no paper devoted to this type of well-posedness for set optimization problem involving the set less relation $\leq^s$.

4.3 Links between pointwise B-well-posedness and pointwise L-well-posedness

Relationships between the pointwise L-well-posedness and pointwise B-well-posedness for the problem (P_α) are illustrated in the following results.

Theorem 4.3.1. *Let $x_0 \in S_\alpha$ be given.*

(i) *If (P_α) is L-well-posed at x_0 and $Q_\alpha^{x_0}(0) = S_\alpha$, then it is B-well-posed at x_0.*

(ii) *If (P_α) is B-well-posed at x_0 and S_α is compact, then it is L-well-posed at x_0.*

Proof. (i) By contradiction, suppose that (P_α) is not B-well-posed at x_0. We get that $Q_\alpha^{x_0}$ is not upper semicontinuous at $k = 0$. Hence, there exist a neighborhood V of $Q_\alpha^{x_0}(0)$ and $\{k_n\} \subset K$ converging to 0 such that for each $n \in \mathbb{N}$, there exists $x_n \in Q_\alpha^{x_0}(k_n) \setminus V$. By definition of $Q_\alpha^{x_0}$, we have $F(x_n) \leq^\alpha F(x_0) + k_n$, i.e., $\{x_n\}$ is an x_0-minimizing sequence of (P_α). It follows from the L-well-posedness at x_0 of (P_α) that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to an element $\bar{x} \in S_\alpha$. Therefore, $\bar{x} \in Q_\alpha^{x_0}(0)$, and hence we now get

$$\bar{x} \in Q_\alpha^{x_0}(0) \subset V. \tag{4.3.1}$$

On the other hand, since $x_n \notin V$, $x_n \in X \setminus V$. By the closedness of $X \setminus V$, we have $\bar{x} \in X \setminus V$ which contradicts (4.3.1). So, (P_α) is B-well-posed at x_0.

(ii) Suppose that (P_α) is B-well-posed at x_0. Let $\{x_n\}$ be an x_0-minimizing sequence of (P_α), we consider two cases as follows:

Case 1: $\{x_n\}$ has infinite elements which belong to S_α. Since S_α is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to some $\bar{x} \in S_\alpha$. Hence, (P_α) is L-well-posed at x_0.

Case 2: $\{x_n\}$ has infinite elements which do not belong to S_α. Without lost of generality, we can assume that $\{x_n\} \subset M \backslash S_\alpha$. By Theorem 4.2.5, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some $\bar{x} \in S_\alpha$. Therefore, (P_α) is L-well-posed at x_0. □

CHAPTER V

LEVITIN-POLYAK (LP) WELL-POSEDNESS FOR SET OPTIMIZATION PROBLEMS

The well-posednesses studied in Chapters III and IV are defined by the convergence of minimizing sequences, and we note that these sequences belong to constrain set. As mentioned in the introduction that removing this condition is significant, and hence we next investigate LP well-posedness for set optimization problems in this chapter.

As we can see that the first introduction of well-posedness for set optimization problems was presented by Zhang et al. [61] in 2009. Then, this research was generalized by Gutiérrez et al. [34] under assumptions of cone properness. After that, Dhingra and Lalitha [14] introduced a concept of well-setness for such problems and proved that it is an extension of the generalized well-posedness which was considered in [61]. Recently, well-posedness of set optimization problems involving not only the lower but also the upper set less relation have been discussed in [35, 36, 64]. In all of these works, minimizing sequences (generalized minimizing sequences, B-minimizing sequences) belong to the constrain set. On the other hand, for LP well-posedness, a minimizing sequences does not need lie in this set. For this type of well-posedness, to the best of our knowledge there is only the paper of Khoshkhabar-amiranloo and Khorram [64] which studied set optimization problem involving the lower set less relation. Of course, such an important topic as LP well-posedness for set optimization problems must be the aim of many works. Moreover, the scalarization method which is the main tool used in the above paper [64] investigates difficultly set optimization problems involving different set order relations, therefore some other approaches to study well-posedness for such problems should be considered.

Motivated and inspired by works mentioned above, in this research, without

using the scalarization method, we investigate different types of LP well-posedness for set optimization problems involving several kinds of set order relations. More precisely, we concern set optimization problems involving three types of set order relations. Then, we introduce concepts of LP well-posedness for such problems and discuss relationships among them. Moreover, necessary and/or sufficient conditions for these concepts of well-posedness are investigated. Applying Kuratowski measure of noncompactness, we study characterizations of such concepts. Finally, approximating solution maps and their stability are studied to build the connection between stability of approximating problem and LP well-posedness of the set optimization problem.

In this chapter, let X be a normed space and Y be a real Hausdorff topological linear space.

5.1 Pointwise LP well-posedness and generalized pointwise LP well-posedness

Motivated by the study [64] on the pointwise LP well-posedness for (P_l), we are going to establish characterizations of this type of well-posedness for (P_α) without using the scalarization method. Let $e \in \mathrm{int}K$. Consider the problem (P_α), for a given $\bar{x} \in S_\alpha$, the LP approximating solution map at $\bar{x}$, $\widehat{S}_\alpha^{\bar{x}} : \{\bar{x}\} \times \mathbb{R}_+ \rightrightarrows X$ is defined by

$$\widehat{S}_\alpha^{\bar{x}}(\varepsilon) = \{x \in X \mid d(x, M) \leq \varepsilon, F(x) \leq^\alpha F(\bar{x}) + \varepsilon e\},$$

for each $\varepsilon \in \mathbb{R}_+$.

Inspired by ideas in [64] (Definition 5.1), we extend some notions for the problem (P_l) in [64] to the problem (P_α) and propose some new concepts for the problem (P_α).

Definition 5.1.1. Let $\bar{x} \in S_\alpha$ be given. A sequence $\{x_n\} \subset X$ is said to be a LP-minimizing sequence for the problem (P_α) at $\bar{x}$ if and only if there exists a sequence

$\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that

$$d(x_n, M) \leq \varepsilon_n, \quad F(x_n) \leq^{\alpha} F(\bar{x}) + \varepsilon_n e.$$

The equivalence of this concept is given by the following result.

Proposition 5.1.2. *$\{x_n\} \subset X$ is a LP-minimizing sequence for the problem (P_α) at $\bar{x} \in S_\alpha$ if and only if there exists sequence $\{d_n\} \subset K \setminus \{0\}$ converging to 0 such that*

$$d(x_n, M) \to 0, \quad F(x_n) \leq^{\alpha} F(\bar{x}) + d_n.$$

Proof. We only prove the assertion for the case $\alpha = s$; the proofs of the assertion for the cases $\alpha = l$ and $\alpha = u$ are similar. Take $\{d_n\} \subset K \setminus \{0\}$ converge to 0 and $\{x_n\} \subset X$ such that $d(x_n, M) \to 0$ and $F(x_n) \leq^s F(\bar{x}) + d_n$, i.e.,

$$F(\bar{x}) + d_n \subset F(x_n) + K\,, \quad F(x_n) \subset F(\bar{x}) + d_n - K. \tag{5.1.1}$$

Since $e - K$ is a neighborhood of the origin 0 in Y, there exists $\varepsilon > 0$ such that $\varepsilon \overline{B}(0,1) \subset e - K$ where $\overline{B}(x,r)$ is the closed ball centered x with radius r. For a given $n \in \mathbb{N}$, we have $d_n \in \|d_n\| \overline{B}(0,1) \subset \|d_n\| \varepsilon^{-1}(e - K) = \|d_n\| \varepsilon^{-1} e - K$. For each $n \in \mathbb{N}$, taking $\varepsilon_n = \|d_n\| \varepsilon^{-1}$, then $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converges to 0 and $\varepsilon_n e - d_n \in K$. It follows from (5.1.1) that

$$F(\bar{x}) + \varepsilon_n e \subset F(x_n) + K\,, \quad F(x_n) \subset F(\bar{x}) + \varepsilon_n e - K,$$

i.e., $F(x_n) \leq^s F(\bar{x}) + \varepsilon_n e$. So, $\{x_n\}$ is a LP-minimizing sequence for $(\mathrm{P_s})$ at $\bar{x}$. Conversely, it is clear that if $\{x_n\} \subset X$ is a LP-minimizing sequence for $(\mathrm{P_s})$ at $\bar{x} \in S_\alpha$, then the assertion is satisfied by setting $d_n = \varepsilon_n e$. □

Definition 5.1.3. The problem (P_α) is said to be

(i) LP well-posed at $\bar{x} \in S_\alpha$ if and only if any LP-minimizing sequence for (P_α) at $\bar{x}$ converges to $\bar{x}$.

(ii) generalized LP well-posed at $\bar{x} \in S_\alpha$ if and only if any LP-minimizing sequence for (P_α) at $\bar{x}$ has a subsequence converging to an element $\hat{x} \in S_\alpha(\bar{x}, 0)$.

Remark 5.1.4. When $\alpha = l$, the concept of pointwise well-posedness becomes the corresponding concepts studied in [64] (Definitions 5.1 and 5.2, respectively), even for this special case, the concept of generalized well-posedness is a new one.

The following examples illustrate the above-introduced concepts.

Example 5.1.5. Let $X = Y = \mathbb{R}$, $M = \mathbb{R}$, $K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by $F(x) = [0, 1]$ for all $x \in X$. Let $e = 1 \in \text{int}K$ and $\bar{x} = 0$. We have $\widehat{S}_\alpha^{\bar{x}}(0) = S_\alpha = \mathbb{R}$. Setting $x_n = n$, $\{x_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x} = 0$. Since $\{x_n\}$ admits no convergent subsequence, (P_α) is not both LP well-posed and generalized LP well-posed at 0.

Example 5.1.6. Let $X = Y = \mathbb{R}$, $M = \mathbb{R}$, $K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [0, 1], & \text{if} \quad -1 \leq x \leq 1, \\ [1, 1 + x^2], & \text{otherwise.} \end{cases}$$

Let $e = 1 \in \text{int}K$ and $\bar{x} = 0$. By direct computations, we have $\widehat{S}_\alpha^{\bar{x}}(0) = [-1, 1]$. Setting $x_n = 1 - \frac{1}{n}$, $\{x_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x} = 0$ and converges to 1. Hence, (P_α) is not LP well-posed at 0, but it is generalized LP well-posed at 0. Indeed, if $\{\hat{x}_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x} = 0$, then there is a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that $d(\hat{x}_n, M) \leq \varepsilon_n$ and $F(\hat{x}_n) \leq^\alpha F(0) + \varepsilon_n e = [\varepsilon_n, 1 + \varepsilon_n]$. This implies that $-1 \leq \hat{x}_n \leq 1$ for n sufficiently large, and hence there exists a subsequence of $\{\hat{x}_n\}$ converging to some point of $\widehat{S}_\alpha^{\bar{x}}(0)$.

Example 5.1.7. Let $X = \mathbb{R}$, $M = \mathbb{R}$, $K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by $F(x) = [x^2, 2x^2]$ for all $x \in X$. Let $e = 1 \in \text{int}K$ and $\bar{x} = 0$. Direct cacullations give us $\widehat{S}_\alpha^{\bar{x}}(0) = \{0\}$. Let $\{x_n\}$ be a LP-minimizing sequence for (P_α) at $\bar{x} = 0$. Then, there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(0) + \varepsilon_n e = \{\varepsilon_n\}$. It leads to $x_n^2 \leq \varepsilon_n$, so $\{x_n\}$ converges to 0. Therefore, (P_α) is LP well-posed at 0.

Lemma 5.1.8. *If* P_α *is generalized LP well-posed at* $\bar{x} \in S_\alpha$, *then* $\widehat{S}_\alpha^{\bar{x}}(0)$ *is compact.*

Proof. For every sequence $\{x_n\} \subset \widehat{S}_\alpha^{\bar{x}}(0)$, we always have $d(x_n, M) = 0$ and

$$F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$$

for any $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0. This means that $\{x_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x}$. By the generalized LP well-posedness of P_α at $\bar{x}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to an element $\hat{x} \in \widehat{S}_\alpha^{\bar{x}}(0)$. This leads to the compactness of $\widehat{S}_\alpha^{\bar{x}}(0)$. □

The next results give some properties of the map $\widehat{S}_\alpha^{\bar{x}}$ which are useful in the sequel.

Proposition 5.1.9. *Let $\bar{x} \in S_\alpha$ be given. Then, the following statements are true:*

(i) $S_\alpha = \bigcup_{z \in S_\alpha} \widehat{S}_\alpha^z(0)$;

(ii) *if $\varepsilon_1 \leq \varepsilon_2$, then $\widehat{S}_\alpha^{\bar{x}}(\varepsilon_1) \subset \widehat{S}_\alpha^{\bar{x}}(\varepsilon_2)$;*

(iii) *if F is compact-valued on M, then $\bigcap_{\varepsilon>0} \widehat{S}_\alpha^{\bar{x}}(\varepsilon) = \widehat{S}_\alpha^{\bar{x}}(0)$.*

Proof. We only demonstrate the proof of the assertions (i)-(iii) for the case $\alpha = s$; the proofs of these assertions for the cases $\alpha = l$ and $\alpha = u$ are similar.

(i) Let $z \in S_s$ be given. Since $z \in \widehat{S}_s^z(0)$, $S_s \subset \bigcup_{z \in S_s} \widehat{S}_s^z(0)$. Moreover, let $x \in \bigcup_{z \in S_s} \widehat{S}_s^z(0)$, there exists $z \in S_s$ such that $x \in \widehat{S}_s^z(0)$. Therefore, $d(x, M) = 0$ and $F(x) \leq^s F(z)$. Since $z \in S_s$, $x \in S_s$. It implies that $\bigcup_{z \in S_s} \widehat{S}_s^z(0) \subset S_s$.

(ii) Assume $\varepsilon_1 \leq \varepsilon_2$. Let $x \in \widehat{S}_s^z(\varepsilon_1)$, then $d(x, M) \leq \varepsilon_1$ and $F(x) \leq^s F(\bar{x}) + \varepsilon_1 e$. It follows from the definition of set less relation $\leq^s$ that

$$F(x) \leq^l F(\bar{x}) + \varepsilon_1 e \quad \text{and} \quad F(x) \leq^u F(\bar{x}) + \varepsilon_1 e,$$

i.e.,

$$F(\bar{x}) + \varepsilon_1 e \subset F(x) + K \quad \text{and} \quad F(x) \subset F(\bar{x}) + \varepsilon_1 e - K.$$

We observe that

$$F(\bar{x}) + \varepsilon_2 e - K = F(\bar{x}) + \varepsilon_1 e - K + (\varepsilon_2 - \varepsilon_1)e.$$

Combining the convexity of K with Proposition 2.2.7, we obtain that

$$F(\bar{x}) + \varepsilon_2 e \subset F(x) + K,$$

and

$$F(x) \subset F(\bar{x}) + \varepsilon_1 e - K \subset F(\bar{x}) + \varepsilon_2 e - K.$$

Thus, $F(x) \leq^l F(\bar{x}) + \varepsilon_2 e$ and $F(x) \leq^u F(\bar{x}) + \varepsilon_2 e$. Therefore, $F(x) \leq^s F(\bar{x}) + \varepsilon_2 e$. Moreover, $d(x, M) \leq \varepsilon_2$ as $d(x, M) \leq \varepsilon_1$. Since $\bar{x} \in S_s$, $x \in \widehat{S}_s^{\bar{x}}(\varepsilon_2)$. We conclude that $\widehat{S}_s^{\bar{x}}(\varepsilon_1) \subset \widehat{S}_s^{\bar{x}}(\varepsilon_2)$.

(iii) Let $x \in \widehat{S}_s^{\bar{x}}(0)$. It is clear that $x \in \widehat{S}_s^{\bar{x}}(\varepsilon)$ for any $\varepsilon > 0$. Therefore, $x \in \bigcap_{\varepsilon>0} \widehat{S}_s^{\bar{x}}(\varepsilon)$. For the converse, let $x \in \bigcap_{\varepsilon>0} \widehat{S}_s^{\bar{x}}(\varepsilon)$, we have $x \in \widehat{S}_s^{\bar{x}}(\varepsilon)$ for any $\varepsilon > 0$. It follows from definition of $\widehat{S}_s^{\bar{x}}(\varepsilon)$ that $d(x, M) \leq \varepsilon$ and $F(x) \leq^s F(\bar{x}) + \varepsilon e$, i.e.,

$$d(x, M) \leq \varepsilon,\ F(\bar{x}) + \varepsilon e \subset F(x) + K \quad \text{and} \quad F(x) \subset F(\bar{x}) + \varepsilon e - K. \tag{5.1.2}$$

By the compact-valuedness of F and Lemma 2.4.6, $F(x) - K$ and $F(\bar{x}) - K$ are closed. From (5.1.2), let $\varepsilon \to 0$, we obtain that

$$d(x, M) = 0,\ F(\bar{x}) \subset F(x) + K \quad \text{and} \quad F(x) \subset F(\bar{x}) - K,$$

i.e.,

$$d(x, M) = 0,\ F(x) \leq^s F(\bar{x}).$$

Hence, $x \in \widehat{S}_s^{\bar{x}}(0)$. We get $\bigcap_{\varepsilon>0} \widehat{S}_s^{\bar{x}}(\varepsilon) \subset \widehat{S}_s^{\bar{x}}(0)$. □

Next, using the Kuratowski measure of noncompactness of LP approximating solution sets, we establish metric characterizations of two types of pointwise LP well-posedness for (P_α).

Theorem 5.1.10.

(i) *If* (P_α) *is generalized LP well-posed at* $\bar{x} \in S_\alpha$, *then* $\mu(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$.

(ii) *If* (P_α) *is LP well-posed at* $\bar{x} \in S_\alpha$, *then* $\operatorname{diam}(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$.

Proof. (i) Suppose that (P_α) is generalized LP well-posed at $\bar{x} \in S_\alpha$. First of all, we show that $H(\widehat{S}_\alpha^{\bar{x}}(\varepsilon), \widehat{S}_\alpha^{\bar{x}}(0)) \to 0$ as $\varepsilon \to 0$. Indeed, we observe that, for each $\varepsilon > 0$, $\widehat{S}_\alpha^{\bar{x}}(0) \subset \widehat{S}_\alpha^{\bar{x}}(\varepsilon)$, and hence

$$H^*(\widehat{S}_\alpha^{\bar{x}}(0), \widehat{S}_\alpha^{\bar{x}}(\varepsilon)) = 0.$$

It is sufficient to show that $H^*(\widehat{S}_\alpha^{\bar{x}}(\varepsilon), \widehat{S}_\alpha^{\bar{x}}(0)) \to 0$ as $\varepsilon \to 0$. Suppose by contrary that there exist a real number $r > 0$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0, and for each $n \in \mathbb{N}$ there exists $x_n \in \widehat{S}_\alpha^{\bar{x}}(\varepsilon_n)$ such that $d(x_n, \widehat{S}_\alpha^{\bar{x}}(0)) \geq r$. We have $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$. This means that $\{x_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x}$, and hence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to some point $\hat{x} \in \widehat{S}_\alpha^{\bar{x}}(0)$. Therefore, for n_k sufficiently large, we have $\|x_{n_k} - \hat{x}\| < r$ which is a contradiction.

Next, we prove that $\mu(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \to 0$ as $\varepsilon \to 0$. By Lemma 5.1.8, $\widehat{S}_\alpha^{\bar{x}}(0)$ is compact. Now, for any $\varepsilon > 0$, there are sets $M_1, M_2, \ldots, M_n$ for some $n \in \mathbb{N}$ such that $\widehat{S}_\alpha^{\bar{x}}(0) \subset \cup_{i=1}^n M_i$ with $\operatorname{diam} M_i \leq \varepsilon$ for all $i = 1, \ldots, n$. For each $i \in \{1, \ldots, n\}$, denote

$$N_i = \{x \in X \mid d(x, M_i) \leq H(\widehat{S}_\alpha^{\bar{x}}(\varepsilon), \widehat{S}_\alpha^{\bar{x}}(0))\}.$$

We claim that $\widehat{S}_\alpha^{\bar{x}}(\varepsilon) \subset \cup_{i=1}^n N_i$. Indeed, let $x \in \widehat{S}_\alpha^{\bar{x}}(\varepsilon)$ be arbitrary, we have

$$d(x, \widehat{S}_\alpha^{\bar{x}}(0)) \leq H(\widehat{S}_\alpha^{\bar{x}}(\varepsilon), \widehat{S}_\alpha^{\bar{x}}(0)).$$

Since $\widehat{S}_\alpha^{\bar{x}}(0) \subset \cup_{i=1}^n M_i$, we conclude that

$$d(x, \cup_{i=1}^n M_i) \leq H(\widehat{S}_\alpha^{\bar{x}}(\varepsilon), \widehat{S}_\alpha^{\bar{x}}(0)).$$

So, there is $k_0 \in \{1, 2, \ldots, n\}$ such that

$$d(x, M_{k_0}) \leq H(\widehat{S}_\alpha^{\bar{x}}(\varepsilon), \widehat{S}_\alpha^{\bar{x}}(0)).$$

It means that $x \in N_{k_0}$. Therefore, $\widehat{S}_\alpha^{\bar{x}}(\varepsilon) \subset \cup_{i=1}^n N_i$. Notice further that

$$\begin{aligned} \mathrm{diam} N_i &= \mathrm{diam} M_i + 2H(\widehat{S}_\alpha^{\bar{x}}(\varepsilon), \widehat{S}_\alpha^{\bar{x}}(0)) \\ &\leq \varepsilon + 2H(\widehat{S}_\alpha^{\bar{x}}(\varepsilon), \widehat{S}_\alpha^{\bar{x}}(0)). \end{aligned}$$

Hence, we get

$$\mu(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \leq \mu(\widehat{S}_\alpha^{\bar{x}}(0)) + 2H(\widehat{S}_\alpha^{\bar{x}}(\varepsilon), \widehat{S}_\alpha^{\bar{x}}(0)).$$

Since $\widehat{S}_\alpha^{\bar{x}}(0)$ is compact, we have $\mu(\widehat{S}_\alpha^{\bar{x}}(0)) = 0$. Therefore,

$$\mu(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \leq 2H(\widehat{S}_\alpha^{\bar{x}}(\varepsilon), \widehat{S}_\alpha^{\bar{x}}(0)).$$

It follows that $\mu(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \to 0$ as $\varepsilon \to 0$.

(ii) Suppose, to the contrary, that there exist a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ and a positive real number r such that $\mathrm{diam}(\widehat{S}_\alpha^{\bar{x}}(\varepsilon_n)) > r$. Because $\bar{x} \in \widehat{S}_\alpha^{\bar{x}}(\varepsilon_n)$, for each n, there exists $x_n \in S_\alpha$ such that $\|x_n - \bar{x}\| > \frac{r}{2}$. However, since $\{x_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x}$, $\{x_n\}$ converges to $\bar{x}$ which is a contradiction. □

The next result gives sufficient conditions for the closedness of approximating solution set.

Proposition 5.1.11. *If F is continuous and compact-valued on M, then $\widehat{S}_\alpha^{\bar{x}}(\varepsilon)$ is closed for each $\varepsilon \geq 0$.*

Proof. We only prove the assertion for the case $\alpha = s$. Taking $\varepsilon \geq 0$, let $\{x_n\} \subset \widehat{S}_s^{\bar{x}}(\varepsilon)$ converge to x, we need to prove that $x \in \widehat{S}_s^{\bar{x}}(\varepsilon)$. Since $x_n \in \widehat{S}_s^{\bar{x}}(\varepsilon)$, $d(x_n, M) \leq \varepsilon$ and

$$F(x_n) \leq^s F(\bar{x}) + \varepsilon e. \tag{5.1.3}$$

By the continuity of the function $d(\cdot, M)$, $d(x, M) \leq \varepsilon$. Next, we show that $F(x) \leq^s F(\bar{x}) + \varepsilon e$. Indeed, from (5.1.3), we have

$$F(\bar{x}) + \varepsilon e \subset F(x_n) + K, \tag{5.1.4}$$

and

$$F(x_n) \subset F(\bar{x}) + \varepsilon e - K. \tag{5.1.5}$$

Let $y \in F(x)$ be arbitrary. Since F is lower semicontinuous and $\{x_n\}$ converges to x, there exist $y_n \in F(x_n)$ such that $\{y_n\}$ converges to y. Combining this with (5.1.5), there exist $w_n \in F(\bar{x})$ such that

$$y_n \in w_n + \varepsilon e - K. \tag{5.1.6}$$

Since $F(\bar{x})$ is compact, we can assume that $\{w_n\}$ converges to some $w \in F(\bar{x})$. By (5.1.6), there exist $k_n \in K$ such that $y_n = w_n + \varepsilon e - k_n$. This leads to $\lim_{n\to\infty} k_n = w + \varepsilon e - y$. Moreover, we get $w + \varepsilon e - y \in K$ as K is closed. Therefore, there exists $k \in K$ such that $w + \varepsilon e - y = k$. We have $y = w + \varepsilon e - k \in w + \varepsilon e - K$. It yields that $y \in F(\bar{x}) + \varepsilon e - K$ as $w \in F(\bar{x})$. We arrive at the fact that $F(x) \subset F(\bar{x}) + \varepsilon e - K$, i.e., $F(x) \leq^u F(\bar{x}) + \varepsilon e$.

Similarly, let $t \in F(\bar{x})$ be arbitrary, it follows from (5.1.4) that, for each $n \in \mathbb{N}$, there exists $v_n \in F(x_n)$ such that

$$t \in v_n - \varepsilon e + K. \tag{5.1.7}$$

Since F is upper semicontinuous and compact-valued at x, we can assume that $\{v_n\}$ converges to some element $v \in F(x)$. It implies from (5.1.7) that there exist $k_n \in K$ such that $t = v_n - \varepsilon e + k_n$. Hence, $k_n = t + \varepsilon e - v_n$. This leads to $\lim_{n\to\infty} k_n = t + \varepsilon e - v$. Since K is closed, there exists $k \in K$ such that $t + \varepsilon e - v = k$. We get $t = v - \varepsilon e + k \in v - \varepsilon e + K$. It yields that $t \in F(x) - \varepsilon e + K$ as $v \in F(x)$. We have $F(\bar{x}) \subset F(x) - \varepsilon e + K$. It means that $F(x) \leq^l F(\bar{x}) + \varepsilon e$. So, we obtain $F(x) \leq^s F(\bar{x}) + \varepsilon e$. The proof is complete. □

Theorem 5.1.12. *Suppose that F is continuous and compact-valued on M. Then,*

(i) *If $\mu(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \to 0$ as $\varepsilon \to 0$, then (P_α) is generalized LP well-posed at $\bar{x} \in S_\alpha$.*

(ii) *If* $\operatorname{diam}(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$, *then* (P_α) *is LP well-posed at* $\bar{x} \in S_\alpha$.

Proof. (i) Suppose that $\mu(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Let $\{x_n\}$ be a LP-minimizing sequence for (P_α) at $\bar{x}$. Therefore, there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$. This means that $x_n \in \widehat{S}_\alpha^{\bar{x}}(\varepsilon_n)$. It is clear that $\mu(\widehat{S}_\alpha^{\bar{x}}(\varepsilon_n)) \to 0$ as $n \to \infty$, and hence by Lemma 2.4.5(iii), we have $\cap_{n\in\mathbb{N}} \widehat{S}_\alpha^{\bar{x}}(\varepsilon_n)$ is a nonempty compact set and

$$H(\widehat{S}_\alpha^{\bar{x}}(\varepsilon_n), \cap_{n\in\mathbb{N}} \widehat{S}_\alpha^{\bar{x}}(\varepsilon_n)) \to 0$$

as $n \to \infty$. Note further from Proposition 5.1.9(iii) that

$$\widehat{S}_\alpha(\bar{x}, 0) = \cap_{n\in\mathbb{N}} \widehat{S}_\alpha(\bar{x}, \varepsilon_n).$$

Hence, we conclude that $\widehat{S}_\alpha^{\bar{x}}(0)$ is compact and

$$H(\widehat{S}_\alpha^{\bar{x}}(\varepsilon_n), \widehat{S}_\alpha^{\bar{x}}(0)) \to 0$$

as $n \to \infty$. Thus,

$$d(x_n, \widehat{S}_\alpha^{\bar{x}}(0)) \to 0.$$

Therefore, there exists a sequence $\{\hat{x}_n\} \subset \widehat{S}_\alpha^{\bar{x}}(0)$ such that $d(x_n, \hat{x}_n) \to 0$ as $n \to \infty$. Since $\widehat{S}_\alpha^{\bar{x}}(0)$ is compact, there is a subsequence $\{\hat{x}_{n_k}\}$ of $\{\hat{x}_n\}$ converging to some $\hat{x} \in S_\alpha$. This implies that $\{x_n\}$ has a corresponding subsequence $\{x_{n_k}\}$ converging to $\hat{x}$. Hence, (P_α) is generalized LP well-posed at $\bar{x}$.

(ii) Assume that $\operatorname{diam}(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Then, $\mu(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \to 0$ as $\varepsilon \to 0$, and hence (P_α) is generalized LP well-posed at $\bar{x}$. By Proposition 5.1.9, $\widehat{S}_\alpha^{\bar{x}}(0)$ is a singleton. By Lemma 2.3.2(ii), (P_α) is LP well-posed at $\bar{x}$. □

The below example shows that Theorem 5.1.12 is applicable.

Example 5.1.13. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $M = [0, 1]$, $K = \mathbb{R}^2_+$. Let $F : X \rightrightarrows Y$ be defined by

$$F(x) = [x, x + 1] \times [x, x + 1], \forall x \in X.$$

Let $e = (1,1) \in \text{int}K$, $\bar{x} = 0$. Clearly, all assumptions of Theorem 5.1.12 hold. By direct cacullations, we get $\widehat{S}_\alpha^{\bar{x}}(\varepsilon) = [0, \varepsilon]$ and $S_\alpha = \{0\}$. So, $\text{diam}(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \to 0$ as $\varepsilon \to 0$. Applying Theorem 5.1.12, the problem (P_α) is LP well-posed at $\bar{x} = 0$. In fact, if $\{x_n\}$ is a LP minimizing sequence for (P_α) at $\bar{x}$, then there is a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(0) + \varepsilon_n e = [\varepsilon_n, \varepsilon_n + 1] \times [\varepsilon_n, \varepsilon_n + 1]$. We get $0 \leq x_n \leq \varepsilon_n$, and hence $\{x_n\}$ converges to 0. So, (P_α) is LP well-posed at 0.

The following example shows that the continuity of F in Theorem 5.1.12 is crucial.

Example 5.1.14. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $M = [-1, 1]$, $K = \mathbb{R}^2_+$. Let $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [0,1] \times [0,1], & \text{if} \quad x < 0, \\ [1,2] \times [1,2], & \text{if} \quad x \geq 0. \end{cases}$$

Let $e = (1,1) \in \text{int}K$, $\bar{x} = -\frac{1}{2}$. Then, F is compact-valued on M. Direct computations give us that $\widehat{S}_\alpha^{\bar{x}}(\varepsilon) = [-1, 0[$, and hence $\mu(\widehat{S}_\alpha^{\bar{x}}(\varepsilon)) \to 0$ as $\varepsilon \to 0$. However, the problem (P_α) is not generalized LP well-posed at $\bar{x}$. Indeed, setting $x_n = -\frac{1}{n}$, we have $\{x_n\}$ is a LP minimizing sequence for (P_α) at $\bar{x}$ but $\{x_n\}$ converges to $0 \notin \widehat{S}_\alpha^{\bar{x}}(0)$. The reason here is that F is not continuous.

Next, employing properties of the approximating solution map of (P_α), the connection between LP well-posedness of (P_α) and stability of approximating problem is established.

Theorem 5.1.15. *Let $\bar{x} \in S_\alpha$.*

(i) *Problem (P_α) is generalized LP well-posed at $\bar{x}$ if and only if $\widehat{S}_\alpha^{\bar{x}}$ is upper semicontinuous and compact-valued at 0.*

(ii) *Problem (P_α) is LP well-posed at $\bar{x}$ if and only if $\widehat{S}_\alpha^{\bar{x}}$ is upper semicontinuous at 0 and $\widehat{S}_\alpha^{\bar{x}}(0) = \{\bar{x}\}$.*

Proof. (i) Suppose that (P_α) is generalized LP well-posed at $\bar{x}$. By Lemma 5.1.8, $\widehat{S}^{\bar{x}}_\alpha(0)$ is compact. Suppose by contrary that $\widehat{S}^{\bar{x}}_\alpha$ is not upper semicontinuous at 0. Then, there exists an open set $N \supset \widehat{S}^{\bar{x}}_\alpha(0)$ such that for any $\delta > 0$, there exists $\varepsilon \in [0, \delta[$, $\widehat{S}^{\bar{x}}_\alpha(\varepsilon) \not\subset N$. It means that there exists a sequence $\{\varepsilon_n\}$ converging to 0 such that for each $n \in \mathbb{N}$, we have $\widehat{S}^{\bar{x}}_\alpha(\varepsilon_n) \not\subset N$. Thus, for each $n \in \mathbb{N}$, there is $x_n \in \widehat{S}^{\bar{x}}_\alpha(\varepsilon_n)$, $x_n \notin N$. Then, $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$, which imply that $\{x_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x}$. Because (P_α) is generalized LP well-posed at $\bar{x}$, there is a subsequence of $\{x_n\}$, denoted by $\{x_{n_k}\}$, converging to an element $\hat{x} \in \widehat{S}^{\bar{x}}_\alpha(0) \subset N$. This is impossible as $x_{n_k} \notin N$ for all k.

Conversely, let $\{x_n\} \subset X$ be a LP-minimizing sequence for (P_α) at $\bar{x}$. Then, there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that

$$d(x_n, M) \leq \varepsilon_n, \quad F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e.$$

So, $x_n \in \widehat{S}^{\bar{x}}_\alpha(\varepsilon_n)$. It follows from the upper semicontinuity, compact-valuedness of $\widehat{S}^{\bar{x}}_\alpha$ at 0 and Lemma 2.3.2(ii) that there exist an element $\hat{x} \in \widehat{S}^{\bar{x}}_\alpha(0)$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to $\hat{x}$. So, (P_α) is generalized LP well-posed at $\bar{x}$.

(ii) Let $\{x_n\} \subset X$ be a LP-minimizing sequence for (P_α) at $\bar{x}$, then there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^\alpha F(\bar{x}) + \varepsilon_n e$. This means that, for each $n \in \mathbb{N}$,

$$x_n \in \widehat{S}^{\bar{x}}_\alpha(\varepsilon_n). \tag{5.1.8}$$

Since $\widehat{S}^{\bar{x}}_\alpha$ is upper semicontinuous at 0, for any open set N, $\widehat{S}^{\bar{x}}_\alpha(0) \subset N$, there is a neighborhood U of 0 such that for all $t \in U, t \geq 0$, we have $\widehat{S}^{\bar{x}}_\alpha(t) \subset N$. Since $\{\varepsilon_n\}$ converges to 0, there exists $n_0 \in \mathbb{N}$ such that $\varepsilon_n \in B(0, \frac{1}{n_0})$ for all $n \geq n_0$. Combining this with (5.1.8), we obtain $x_n \in \widehat{S}^{\bar{x}}_\alpha(\varepsilon_n) \subset N$ for all $n \geq n_0$. Therefore, for every neighborhood W of 0, $x_n \in \widehat{S}^{\bar{x}}_\alpha(0) + W$ for all $n \geq n_0$. Since $\widehat{S}^{\bar{x}}_\alpha(0) = \{\bar{x}\}$, $\{x_n\}$ converges to $\bar{x}$. So, (P_α) is LP well-posed at $\bar{x}$.

For the converse, suppose that (P_α) is LP well-posed at $\bar{x}$. Using (i), $\widehat{S}_\alpha^{\bar{x}}$ is upper semicontinuous and compact-valued at 0. We show that $\widehat{S}_\alpha^{\bar{x}}(0)$ is a singleton. Suppose by the contrary that there exist $x_1, x_2 \in \widehat{S}_\alpha^{\bar{x}}(0)$ with $x_1 \neq x_2$. Putting $x_{2n+k} = x_k$ where $k = 1$ or $k = 2$. Clearly, $\{x_n\}$ is a LP-minimizing sequence for (P_α) at $\bar{x}$. However, $\{x_n\}$ is not convergent. This is a contradiction. Therefore, $\widehat{S}_\alpha^{\bar{x}}(0)$ is a singleton. Moreover, it is obvious that $\bar{x} \in \widehat{S}_\alpha^{\bar{x}}(0)$. So, $\widehat{S}_\alpha^{\bar{x}}(0) = \{\bar{x}\}$. □

The assumption about the upper semicontinuity of approximating solution map of (P_α) is used in Theorem 5.1.15. Next, we give the sufficient conditions for this assumption.

Proposition 5.1.16. *Suppose that the following conditions hold:*

(i) *M is compact;*

(ii) *F is continuous and compact-valued on M.*

Then, $\widehat{S}_\alpha^{\bar{x}}$ is upper semicontinuous at 0.

Proof. By the similarity, we only focus on the proof of the assertion for the case $\alpha = u$. By contradiction, suppose that $\widehat{S}_u^{\bar{x}}$ is not upper semicontinuous at 0. Then, there exist an open set $N \supset \widehat{S}_u^{\bar{x}}(0)$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+ \setminus \{0\}$ converging to 0 such that for each n, there exists x_n satisfying

$$x_n \in \widehat{S}_u^{\bar{x}}(\varepsilon_n) \setminus N. \tag{5.1.9}$$

Since $x_n \in \widehat{S}_u^{\bar{x}}(\varepsilon_n)$,

$$d(x_n, M) \leq \varepsilon_n \tag{5.1.10}$$

and

$$F(x_n) \subset F(\bar{x}) + \varepsilon_n e - K. \tag{5.1.11}$$

It implies from (5.1.10) that there exist $\hat{x}_n \in M$ such that $d(x_n, \hat{x}_n) \leq \varepsilon_n$. By the compactness of M, we can assume that $\{\hat{x}_n\}$ converges to an element $x_0 \in M$. Hence, $\{x_n\}$ converges to x_0. Next, we prove that

$$F(x_0) \subset F(\bar{x}) - K. \tag{5.1.12}$$

Indeed, by the compact-valuedness of F, the closedness of K and Lemma 2.4.6, $F(\bar{x}) - K$ is closed. From (5.1.11), taking $n \to \infty$, we obtain (5.1.12). It means that $x_0 \in \widehat{S}_u^{\bar{x}}(0)$. Combining this, (5.1.9) and the convergence to x_0 of $\{x_n\}$, we get a contradiction. Therefore, $\widehat{S}_u^{\bar{x}}$ is upper semicontinuous at 0. □

Corollary 5.1.17. *Suppose that the following conditions hold:*

(i) *M is compact;*

(ii) *F is continuous and compact-valued on M.*

Then,

(a) *(P_α) is generalized LP well-posed at $\bar{x} \in S_\alpha$ if $\widehat{S}_\alpha^{\bar{x}}(0)$ is closed.*

(b) *(P_α) is LP well-posed at $\bar{x} \in S_\alpha$ if $\widehat{S}_\alpha^{\bar{x}}(0) = \{\bar{x}\}$.*

5.2 Metrically LP well-posedness

Picking up the ideas in [64], we introduce the following new concepts of LP well-posedness related to metrically approach for the problem (P_α).

Definition 5.2.1. A sequence $\{x_n\} \subset X$ is said to be a

(i) metrically LP-minimizing sequence for problem (P_l) at $\bar{x} \in S_l$ if and only if $H^*(F(\bar{x}), F(x_n)) \to 0$ and $d(x_n, M) \to 0$ as $n \to \infty$.

(ii) metrically LP-minimizing sequence for problem (P_u) at $\bar{x} \in S_u$ if and only if $H^*(F(x_n), F(\bar{x})) \to 0$ and $d(x_n, M) \to 0$ as $n \to \infty$.

(iii) metrically LP-minimizing sequence for problem (P_s) at $\bar{x} \in S_s$ if and only if $H(F(\bar{x}), F(x_n)) \to 0$ and $d(x_n, M) \to 0$ as $n \to \infty$.

Definition 5.2.2. The problem (P_α) is said to be metrically LP well-posed if and only if $S_\alpha \neq \emptyset$ and for any metrically LP-minimizing sequence $\{x_n\}$ for problem (P_α) at some $\bar{x} \in S_\alpha$, we have $d(x_n, S_\alpha) \to 0$ as $n \to \infty$.

Remark 5.2.3. When $\alpha = l$, concepts introduced in Definitions 5.2.1(i) and 5.2.2 are similar to the corresponding ones studied in Definitions 4.5(ii) and 4.6(ii) in [64].

Example 5.2.4. (a) Let $X = Y = \mathbb{R}$, $M = [0, 1]$, $K = \mathbb{R}_+$, and let $F : X \rightrightarrows Y$ be defined by $F(x) = [1, 2]$ for all $x \in X$. Obviously, $S_\alpha = [0, 1] = M$, and the problem (P_α) is metrically LP well-posed.

(b) Let $X = Y = \mathbb{R}$, $M = K = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by $F(x) = [x^2, 3x^2]$ for all $x \in X$. Direct cacullations give us $S_\alpha = \{0\}$ and the problem (P_α) is metrically LP well-posed. Indeed, let $\{x_n\}$ be a metrically LP-minimizing sequence for (P_α) at $\bar{x} = 0$, it implies from definition of metrically LP-minimizing sequence for (P_α) at $\bar{x} = 0$ that $\{x_n\}$ converges to $\bar{x}$. Therefore, $d(x_n, S_\alpha) \to 0$.

Example 5.2.5. Let $X = Y = \mathbb{R}$, $M = [-1, 1]$, $K = \mathbb{R}_+$, and $F : X \rightrightarrows Y$ is defined by

$$F(x) = \begin{cases} [0, 1[, & \text{if} \quad x \leq 0, \\ ]0, 1], & \text{if} \quad x > 0. \end{cases}$$

By direct computations, we get $S_\alpha = [-1, 0]$. Taking $x_n = 1 + \frac{1}{n}$, then $\{x_n\}$ is a metrically LP-minimizing sequence for problem (P_α) at $\bar{x} = 0 \in S_\alpha$, but $d(x_n, S_\alpha) \to 1$. Therefore, the problem (P_α) is not metrically LP well-posed.

Next, we introduce a generalized form of the above concept.

Definition 5.2.6. The problem (P_α) is said to be generalized metrically LP well-posed if and only if $S_\alpha \neq \emptyset$ and for any metrically LP-minimizing sequence $\{x_n\}$

for (P_α) at some $\bar{x} \in S_\alpha$, $\{x_n\}$ has a subsequence, denoted by $\{x_{n_k}\}$, such that $d(x_{n_k}, S_\alpha) \to 0$ as $k \to \infty$.

It is clear that if (P_α) is metrically LP well-posed, then it is generalized metrically LP well-posed.

These following results give the relationships between these kinds of LP well-posedness considered in this study.

Theorem 5.2.7.

(i) *If (P_α) is LP well-posed at all $\bar{x} \in S_\alpha$, then (P_α) is metrically LP well-posed.*

(ii) *If (P_α) is generalized LP well-posed at all $\bar{x} \in S_\alpha$, then (P_α) is generalized metrically LP well-posed.*

Proof. By the similarity we verify the assertions (i), (ii) for the case $\alpha = s$ as an example.

(i) Let $\{x_n\}$ be a metrically LP-minimizing sequence for problem (P_s) at some $\bar{x} \in S_s$. We need to prove that $d(x_n, S_s) \to 0$. In fact, since $\{x_n\}$ is a metrically LP-minimizing sequence for problem (P_s) at some $\bar{x} \in S_s$, $d(x_n, M) \to 0$ and

$$H(F(\bar{x}), F(x_n)) \to 0. \tag{5.2.1}$$

Observe that we can choose a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}$ converging to 0 such that $d(x_n, M) \leq \varepsilon_n$, both $-\varepsilon_n e + K$ and $\varepsilon_n e - K$ are neighborhoods of the origin in Y. By (5.2.1), there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have

$$F(\bar{x}) \subset F(x_n) - \varepsilon_n e + K \text{ and } F(x_n) \subset F(\bar{x}) + \varepsilon_n e - K.$$

This implies that $F(x_n) \leq^s F(\bar{x}) + \varepsilon_n e$. Hence, $\{x_n\}$ is a LP-minimizing sequence for (P_s) at $\bar{x}$. By the LP well-posedness of (P_s) at $\bar{x}$, $\{x_n\}$ converges to $\bar{x}$. Moreover, since $\bar{x} \in S_s$, $d(x_n, S_s) \leq \|x_n - \bar{x}\| \to 0$. So, (P_s) is metrically LP well-posed.

(ii) Using a similar argument with one above, we can prove that the statement (ii) is satisfied. □

Remark 5.2.8. When $\alpha = l$, (P_α) reduces to (P_l) studied in [64]. To obtain the metrically LP well-posedness for (P_l), the authors used an important assumption about the K-closed values of F on M, i.e., $F(x) + K$ is closed for all $x \in M$. Using another approach, as in Theorem 5.2.7, we can remove this assumption but also obtain the metrically LP well-posedness for (P_α).

Combining Theorem 5.2.7 and Corollary 5.1.17, we obtain the following results.

Theorem 5.2.9. *Suppose that the following conditions are satisfied:*

(i) *M is compact;*

(ii) *F is continuous and compact-valued on M.*

Then,

(a) *(P_α) is metrically LP well-posed if $\widehat{S}_\alpha^{\bar{x}}(0) = \{\bar{x}\}$ for every $\bar{x} \in S_\alpha$.*

(b) *(P_α) is generalized metrically LP well-posed if $\widehat{S}_\alpha^{\bar{x}}(0)$ is closed for every $\bar{x} \in S_\alpha$.*

Remark 5.2.10. Very recently, in [82], the authors studied several kinds of well-posedness for set optimization problems via the lower set less relation, including B-well-posedness, L-well-posedness, DH-well-posedness, and they obtained many interesting results related to this topic. In this work, we consider the Levitin-Polyak well-posedness and the generalized Levitin-Polyak well-posedness for set optimization problems involving various kinds of set less relations, and hence the concepts of well-posedness investigated in this work are different from those in [82]. Therefore, it could not compare our results with theirs.

5.3 Global LP well-posedness

In this section we propose new concepts related to global LP well-posedness for (P_α). Necessary and sufficient conditions of LP well-posedness for (P_α) are studied.

Definition 5.3.1. A sequence $\{x_n\} \subset X$ is said to be a LP-minimizing sequence for (P_α) if and only if there exist $\{\varepsilon_n\} \subset \mathbb{R}^+ \setminus \{0\}$ converging to 0 and $\{z_n\} \subset S_\alpha$ such that $d(x_n, M) \leq \varepsilon_n$, $F(x_n) \leq^\alpha F(z_n) + \varepsilon_n e$.

Lemma 5.3.2. *$\{x_n\}$ is a LP-minimizing sequence for (P_α) if and only if there exist $\{d_n\} \subset K \setminus \{0\}$ converging to 0 and $\{z_n\} \subset S_\alpha$ such that $d(x_n, M) \to 0$, $F(x_n) \leq^\alpha F(z_n) + d_n$.*

Proof. By the similarity, we only demonstrate the proof of the statement for the case $\alpha = s$. Clearly, if $\{x_n\}$ is a LP-minimizing sequence for (P_s), then by letting $d_n = \varepsilon_n e$, we get a desired result. Conversely, for $\{x_n\} \subset X$, suppose that there exist $\{d_n\} \subset K \setminus \{0\}$ converging to 0 and $\{z_n\} \subset S_s$ such that $d(x_n, M) \to 0$ and $F(x_n) \leq^s F(z_n) + d_n$, i.e.,

$$F(z_n) + d_n \subset F(x_n) + K\,, \quad F(x_n) \subset F(z_n) + d_n - K. \tag{5.3.1}$$

Since $e - K$ is a neighborhood of the origin 0 in Y, there is $\varepsilon > 0$ such that $\varepsilon B(0,1) \subset e - K$ where $B(x,r)$ is the closed ball centered x with radius r. For a given $n \in \mathbb{N}$, $d_n \in \|d_n\| B(0,1) \subset \|d_n\| \varepsilon^{-1}(e - K) = \|d_n\| \varepsilon^{-1} e - K$. For each n, taking $\varepsilon_n = \|d_n\| \varepsilon^{-1}$, then $\{\varepsilon_n\} \subset \mathbb{R}^+ \setminus \{0\}$ converges to 0 and $\varepsilon_n e - d_n \in K$. Therefore, it follows from (5.3.1) that $F(z_n) + \varepsilon_n e \subset F(x_n) + K$ and $F(x_n) \subset F(z_n) + \varepsilon_n e - K$, i.e., $F(z_n) \leq^s F(\bar{x}) + \varepsilon_n e$. So, $\{x_n\}$ is a LP-minimizing sequence for (P_s). □

Definition 5.3.3. Problem (P_α) is said to be LP well-posed if and only if any LP-minimizing sequence for (P_α) has a subsequence converging to an element $\bar{x} \in S_\alpha$.

The necessary condition of LP well-posedness for (P_α) is provided in the next result.

Lemma 5.3.4. *If P_α is LP well-posed, then S_α is compact.*

Proof. For every sequence $\{x_n\} \subset S_\alpha$, we have $F(x_n) \leq^\alpha F(x_n) + \varepsilon_n e$ for any sequence $\{\varepsilon_n\} \subset \mathbb{R}^+ \setminus \{0\}$ converging to 0 as $e \in \mathrm{int}K$. This means that $\{x_n\}$ is a LP minimizing

sequence for (P_α). By the LP well-posedness of (P_α), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some $\bar{x} \in S_\alpha$. We conclude that S_α is compact. □

Sufficient conditions of LP well-posedness for (P_α) under the assumption of compactness of the constraint set are given as the following.

Theorem 5.3.5. *Suppose that the following conditions hold:*

(i) *M is convex, compact and S_α is compact;*

(ii) *F is continuous and compact-valued on M.*

Then, (P_α) is LP well-posed.

Proof. We prove the assertion for the case $\alpha = u$, the proof of the assertion for other cases can be given by a similar argument. Let $\{x_n\} \subset X$ be a LP-minimizing sequence for (P_u), then there exist $\{\varepsilon_n\} \subset \mathbb{R}^+ \setminus \{0\}$ converging to 0 and $\{z_n\} \subset S_u$ such that

$$d(x_n, M) \leq \varepsilon_n, \tag{5.3.2}$$

and

$$F(x_n) \leq^u F(z_n) + \varepsilon_n e. \tag{5.3.3}$$

By (5.3.2), there exists $\{\hat{x}_n\} \subset M$ such that $d(x_n, \hat{x}_n) \leq \varepsilon_n$. It follows from the compactness of M that $\{\hat{x}_n\}$ has a subsequence, denoted by $\{\hat{x}_{n_k}\}$, converging to some $\bar{x} \in M$. By (5.3.3), one also has

$$F(x_{n_k}) \subset F(z_{n_k}) + \varepsilon_{n_k} e - K. \tag{5.3.4}$$

Since S_u is compact, we can assume that $\{z_{n_k}\}$ converges to some $\bar{z} \in S_u$.

Let $\bar{v} \in F(\bar{x})$ be arbitrary, there is a sequence $\{v_{n_k}\}$ with $v_{n_k} \in F(x_{n_k})$ converging to $\bar{v}$ because F is lsc at $\bar{x}$. By (5.3.4), there exist $u_{n_k} \in F(z_{n_k})$ satisfying

$$v_{n_k} \in u_{n_k} + \varepsilon_{n_k} e - K. \tag{5.3.5}$$

Since F is upper semicontinuous and compact-valued at $\bar{z}$, there exist $\bar{u} \in F(\bar{z})$ and a subsequence of $\{u_{n_k}\}$ which denoted by the same indexes such that $\{u_{n_k}\}$ converges to $\bar{u}$. From (5.3.5), taking $n \to \infty$, one obtains $\bar{v} \in \bar{u} - K$, therefore $\bar{v} \in F(\bar{z}) - K$. By the arbitrariness of $\bar{v}$, one gets $F(\bar{x}) \subset F(\bar{z}) - K$, i.e., $F(\bar{x}) \leq^u F(\bar{z})$, and hence $\bar{x} \in S_u$ as $\bar{z} \in S_u$. We conclude that (P_u) is LP well-posed. □

We define a set-valued map $L_\alpha : M \times \mathbb{R}_+ \rightrightarrows X$ as, for all $(x, \varepsilon) \in M \times \mathbb{R}_+$,

$$L_\alpha(x, \varepsilon) = \{\hat{x} \in X \mid d(\hat{x}, M) \leq \varepsilon_n \text{ and } F(\hat{x}) \leq^\alpha F(x) + \varepsilon e\}.$$

We refer to the set $L_\alpha(x, \varepsilon)$ as level set at x with level ε and L_α as level set-valued map. It is clear that $\{x_n\}$ is a LP minimizing sequence of (P_α), if there exist $\{\varepsilon_n\} \subset \mathbb{R}_+$ converging to 0 and $\{z_n\} \subset S_\alpha$ such that $x_n \in L_\alpha(z_n, \varepsilon_n)$.

Next, we give sufficient conditions of LP well-posedness for (P_α) in the case of noncompactness of the constraint set.

Theorem 5.3.6. *Suppose that the following conditions hold:*

(i) *X is locally compact and S_α is compact;*

(ii) *F is compact-valued on M;*

(iii) *there exists $\delta > 0$ such that the level set $L_\alpha(x, \varepsilon)$ is connected for every $x \in S_\alpha$ and for every $\varepsilon \in (0, \delta)$. Then,*

  (a) *(P_l) is LP well-posed if F is lower semicontinuous on S_l and upper semicontinuous on M.*

  (b) *(P_u) is LP well-posed if F is upper semicontinuous on S_u and lower semicontinuous on M.*

  (c) *(P_s) is LP well-posed if F is continuous on M.*

Proof. By the similarity, we here only demonstrate the proof for the statement (b). Suppose that (P_u) is not LP well-posed, then there exists a LP minimizing sequence

$\{x_n\}$ of (P_u) such that $\{x_n\}$ admits no subsequence converging to any point $\bar{x} \in S_u$. Because $\{x_n\}$ is a generalized minimizing sequence, there exist $\{\varepsilon_n\} \subset \mathbb{R}_+$ converging to 0 and $\{z_n\} \subset S_u$ satisfying $d(x_n, M) \leq \varepsilon_n$ and $F(x_n) \leq^u F(z_n) + \varepsilon_n e$. Since no subsequence of $\{x_n\}$ and $\{x_n\}$ are not convergent to any point $\bar{x} \in S_u$, there exists an open neighborhood N of S_u such that with some $n_0 \in \mathbb{N}$ we have $x_n \notin N$ for all $n \geq n_0$. By (i), there exists an open neighborhood U of S_u such that its closure, $\mathrm{cl}(U)$, is compact and $S_u \subset \mathrm{cl}(U) \subset N$. Therefore, $x_n \in (\mathrm{cl}(U))^c$ for all $n \geq n_0$. Since $\{x_n\}$ is a LP minimizing sequence, $x_n \in L_F^u(z_n, \varepsilon_n)$. Hence,

$$L_u(z_n, 2\varepsilon_n) \cap (\mathrm{cl}(U))^c \neq \emptyset. \tag{5.3.6}$$

Also, we obtain

$$z_n \in L_u(z_n, 2\varepsilon_n) \cap \mathrm{int}\,\mathrm{cl}(U). \tag{5.3.7}$$

We next claim that $L_u(z_n, 2\varepsilon_n) \cap \mathrm{bd}(\mathrm{cl}(U)) \neq \emptyset$ for every $n \in \mathbb{N}$ satisfying $2\varepsilon_n < \delta$. Suppose on the contrary that there exists $\overline{m}_n \in \mathbb{N}$ such that $L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \subset \mathrm{int}\,\mathrm{cl}(U) \cup \mathrm{int}(\mathrm{cl}(U))^c$. This leads to

$$L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) = (L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \cap \mathrm{int}\,\mathrm{cl}(U)) \cup (L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \cap \mathrm{int}(\mathrm{cl}(U))^c). \tag{5.3.8}$$

We note that $L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \cap \mathrm{int}\,\mathrm{cl}(U)$ and $L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \cap \mathrm{int}(\mathrm{cl}(U))^c$ are separated since $\mathrm{cl}(\mathrm{int}\,\mathrm{cl}(U)) \cap \mathrm{int}(\mathrm{cl}(U))^c = \emptyset$ and $\mathrm{int}\,\mathrm{cl}(U) \cap \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U))^c) = \emptyset$. Employing (5.3.6)-(5.3.8) and $L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n}) \cap \mathrm{int}\,\mathrm{cl}(U) \cap (\mathrm{cl}(U))^c = \emptyset$, we arrive at a contradiction of the fact that $L_u(z_{\overline{m}_n}, 2\varepsilon_{\overline{m}_n})$ is a connected set. Therefore, there exists a sequence $\{w_n\}$ such that

$$w_n \in L_u(z_n, 2\varepsilon_n) \cap \mathrm{bd}(\mathrm{cl}(U)), \tag{5.3.9}$$

where $\mathrm{bd}(A)$ denotes the boundary of a given set A. By the compactness of $\mathrm{cl}(U)$, there exists a subsequence of $\{w_n\}$ which is still denoted by $\{w_n\}$ converging to some $\overline{w} \in \mathrm{cl}(U)$. Since $w_n \in L_u(z_n, 2\varepsilon_n)$, $F(w_n) \subset F(z_n) + 2\varepsilon_n e - K$. Due to the

compactness of S_α, there is a subsequence of $\{z_n\}$ which is still denoted by $\{z_n\}$ converging to some $\bar{z} \in S_\alpha$. Now, we show that $F(\overline{w}) \subset F(\bar{z}) - K$. Let $\bar{v} \in F(\overline{w})$ be arbitrary, by the lower semicontinuity of F at $\bar{w}$, there exists a sequence $\{v_n\}$ converging to $\bar{v}$ where $v_n \in F(w_n)$ for all n. We get $v_n \in F(z_n) + 2\varepsilon_n e - K$. Thus, there exists $u_n \in F(z_n)$ such that

$$v_n \in u_n + 2\varepsilon_n e - K. \tag{5.3.10}$$

Since F is upper semicontinuous and compact-valued at $\bar{z}$, there exist $\bar{u} \in F(\bar{z})$ and a subsequence of $\{u_n\}$, denoted by the same indexes, converging to $\bar{u}$. Taking limit as $n \to \infty$ in (5.3.10), we get $\bar{v} \in \bar{u} - K$. Therefore, $\bar{v} \in F(\bar{z}) - K$. By the arbitrariness of $\bar{v}$, we have $F(\overline{w}) \subset F(\bar{z}) - K$, i.e., $F(\overline{w}) \leq^u F(\bar{z})$. Since $\bar{z} \in S_u$, we have $F(\overline{w}) \sim^u F(\bar{z})$, and hence $\overline{w} \in S_u$ which contradicts (5.3.9). □

Example 5.3.7. Let $X = Y = \mathbb{R}$, $M = [0, 2]$, $K = \mathbb{R}^+$, $e = 1$ and $F : X \rightrightarrows Y$ be defined by $F(x) = [-x^2 + 2x, -4x^2 + 8x]$. Clearly, all conditions except the condition (iii) of Theorem 5.3.6 are satisfied. Indeed, let $\delta = 1$, direct cacullations give us $S_\alpha = \{0, 1\}$ and the level set $L_F^\alpha(x, \varepsilon)$ is not connected for every $x \in S_\alpha$ and every ε with $0 < \varepsilon < \delta$.

Example 5.3.8. Let $X = Y = \mathbb{R}$, $M = [-2, 2]$, $K = \mathbb{R}^+$, $e = 1$ and $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [x^2, 2x^2], & \text{if } -1 \leq x \leq 1, \\ [0, x^2 + 4], & \text{otherwise.} \end{cases}$$

Then, $S_\alpha = \{0\}$, and hence this set is compact. For $\delta = 1$ and for each ε with $0 < \varepsilon < \delta$, we have $L_\alpha(0, \varepsilon) = [-\sqrt{\frac{\varepsilon}{2}}, \sqrt{\frac{\varepsilon}{2}}]$, and thus it is connected. Therefore, all conditions of Theorem 5.3.6 are satisfied. We can check that (P_α) is LP well-posed.

Remark 5.3.9. In [64], the authors used the scalarization method to investigate the metrically LP well-setness and the pointwise LP well-posedness for set optimization problems involving lower set less relation (corresponding the case $\alpha = l$) and their

relations. Note that the approach and these kinds of well-posedness studied in [64] are different from that of our study. More precisely, we do not use the scalarization method because it is not effective for set optimization problems involving set less relation (corresponding the case $\alpha = s$). The concept of global LP well-posedness in this research is a generalization of the B-well-posedness for vector optimization while the metrically LP well-setness in [64] is a generalization of the M-well-posedness for vector optimization (see, e.g., [50]). Hence, our work and that of Khoshkhabar-amiranloo and Khorram in [64] are not compariable in both the solution approach and scope of investigation. Moreover, to the best of our knowledge, there is no paper devoted to global LP well-posedness for the other cases, $\alpha = u$ and $\alpha = s$.

5.4 Links between LP well-posedness and stability

Consider (P_α), we define a set-valued map $\widehat{S}_\alpha : \mathbb{R}^+ \rightrightarrows M$ by

$$\widehat{S}_\alpha(\varepsilon) = \{x \in X \mid d(x, M) \leq \varepsilon, \exists z \in S_\alpha, F(x) \leq^\alpha F(z) + \varepsilon e\}.$$

It can be considered as LP approximating solution map for P_α. The following properties can directly prove.

Proposition 5.4.1. *The following statements are true:*

(i) $\widehat{S}_\alpha(0) = S_\alpha$;

(ii) *if* $\varepsilon_1 \leq \varepsilon_2$, *then* $\widehat{S}_\alpha(\varepsilon_1) \subset \widehat{S}_\alpha(\varepsilon_2)$;

(iii) *if* F *is compact-valued on* M, *then* $\bigcap_{\varepsilon>0} \widehat{S}_\alpha(\varepsilon) = S_\alpha$.

Theorem 5.4.2. *Suppose that the following conditions hold:*

(i) M *is compact*;

(ii) F *is continuous and compact-valued on* M;

(iii) S_α *is closed.*

Then, $\widehat{S}_\alpha(\varepsilon)$ is closed for each $\varepsilon > 0$.

Proof. We prove for the case $\alpha = s$. Taking $\varepsilon > 0$, let $\{x_n\} \subset \widehat{S}_s(\varepsilon)$ converge to x. Since $x_n \in \widehat{S}_s(\varepsilon)$, there exist $\{z_n\} \subset S_s$ such that $d(x_n, M) \leq \varepsilon$, $F(x_n) \leq^s F(z_n) + \varepsilon e$, i.e.,

$$F(z_n) + \varepsilon e \subset F(x_n) + K, \tag{5.4.1}$$

and

$$F(x_n) \subset F(z_n) + \varepsilon e - K. \tag{5.4.2}$$

Since M is compact and S_s is closed, we can assume that $\{z_n\}$ converges to z for some $z \in S_s$. Next, we show that $F(x) \leq^s F(z) + \varepsilon e$.

Let $y \in F(x)$. Since F is lower semicontinuous, there exist $y_n \in F(x_n)$ such that $\{y_n\}$ converges to y. Combining this with (5.4.2), there exist $w_n \in F(z_n)$ such that

$$y_n \in w_n + \varepsilon e - K. \tag{5.4.3}$$

Since F is upper semicontinuous and compact-valued at z, we can assume that $\{w_n\}$ converges to $w \in F(z)$. By (5.4.3), there are $k_n \in K$ such that $y_n = w_n + \varepsilon e - k_n$. We get $\|k_n - (w + \varepsilon e - y)\| \leq \|w_n - w\| + \|y - y_n\|$. So, $w + \varepsilon e - y \in K$ as K is closed. Therefore, there is $k \in K$ such that $w + \varepsilon e - y = k$. It follows that $y \in F(z) + \varepsilon e - K$ as $w \in F(z)$. We arrive at the fact that $F(x) \subset F(z) + \varepsilon e - K$. Similarly, combining the lower semicontinuity of F and (5.4.1), we can show that $F(z) + \varepsilon e \subset F(x) + K$. We obtain $F(x) \leq^s F(z) + \varepsilon e$. Moreover, $d(x, M) \leq \varepsilon$ as $d(x_n, M) \leq \varepsilon$ and $\{x_n\}$ converges to x. We conclude that $x \in \widehat{S}_s(\varepsilon)$. □

Next, sufficient conditions for the upper semicontinuity of LP approximating solution map for (P_α) are given.

Theorem 5.4.3. *Suppose that the following conditions hold:*

(i) *M is compact;*

(ii) *F is continuous and compact-valued on M;*

(iii) *S_α is closed.*

Then, $\widehat{S}_\alpha$ is upper semicontinuous and compact-valued at 0.

Proof. By the similarity we demonstrate only the proof for the case $\alpha = u$. Suppose on the contrary that $\widehat{S}_u$ is not upper semicontinuous at 0. Then there exist an open set $W_0 \supset \widehat{S}_u(0)$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 such that $\widehat{S}_u(\varepsilon_n) \not\subset W_0$ for all $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$ there exists $x_n \in \widehat{S}_u(\varepsilon_n) \setminus W_0$. Therefore, $d(x_n, M) \leq \varepsilon_n$ and there exists $\{z_n\} \subset S_u$ such that

$$F(x_n) \subset F(z_n) + \varepsilon_n e - K. \tag{5.4.4}$$

Since $d(x_n, M) \leq \varepsilon_n$ and M is compact, we can assume that $\{x_n\}$ converges to $x_0 \in M$. Also, by the compactness of M and the closedness of S_u, we can assume that $\{z_n\}$ converges to z_0 for some $z_0 \in S_u$. Next, we prove that $F(x_0) \subset F(z_0) - K$. Indeed, let $u_0 \in F(x_0)$ be arbitrary, by the lower semicontinuity of F on M and Lemma 2.3.2, there exists a sequence $\{u_n\}$, $u_n \in F(x_n)$, such that $\{u_n\}$ converges to u_0. It follows from (5.4.4) that for each $n \in \mathbb{N}$, there exists $v_n \in F(z_n)$ such that $u_n - v_n - \varepsilon_n e \in -K$. Since F is upper semicontinuous and compact-valued at z_0, we can assume that $\{v_n\}$ converges to v_0 for some $v_0 \in F(z_0)$. Combining this with the closedness of K, we have $u_0 - v_0 \in -K$. Since $u_0 \in F(x_0)$ is arbitrary, $F(x_0) \subset F(z_0) - K$. We conclude that $x_0 \in \widehat{S}_u(0)$ which is a contradiction.

Obviously, $\widehat{S}_u(0) = S_u$ is compact by assumptions (i) and (iii). □

Picking up ideas from [36], we propose new concepts about convexity which are weakendness of corresponding ones imposed in [36]. After that, sufficient conditions for closedness of solution set are given.

Definition 5.4.4. Let D be a nonempty convex subset of X. A set-valued map $F : X \rightrightarrows Y$ is said to be strictly K-convexlike with respect to α on D if and only if for any $x, y \in D$ with $x \neq y$, there exists $z \in D$ such that $F(z) \ll^{\alpha} tF(x)+(1-t)F(y)$ for any $t \in]0,1[$.

Proposition 5.4.5. *Suppose that M is closed. Then,*

(i) *S_l is closed if F is usc with nonempty compact values and strictly K-convexlike with respect to l on M;*

(ii) *S_u is closed if F is lsc with nonempty compact convex values and strictly K-convexlike with respect to u on M;*

(iii) *S_s is closed if F is continuous with nonempty compact convex values and strictly K-convexlike with respect to s on M.*

Proof. We prove for the assertion (iii), the other assertions can be proved by a similar argument. Let $\{x_n\} \subset S_s$ converge to x. We need to prove that $x \in S_s$. By the closedness of M, we have $x \in M$. Suppose that there exists $y \in M$ such that $F(y) \leq^{s} F(x)$, i.e.,

$$F(x) \subset F(y) + K, \tag{5.4.5}$$

and

$$F(y) \subset F(x) - K, \tag{5.4.6}$$

we will show that $x = y$. By contracdiction, assume that $x \neq y$. Since F is strictly K-convexlike with respect to s on M, there exist $z, z' \in M$ such that for all $t \in]0,1[$, we have

$$tF(x) + (1-t)F(y) \subset F(z) + \operatorname{int}K \text{ for all } t \in]0,1[, \tag{5.4.7}$$

and

$$F(z') \subset tF(x) + (1-t)F(y) - \operatorname{int}K \text{ for all } t \in]0,1[. \tag{5.4.8}$$

It implies from (5.4.5) and (5.4.7) that $F(x) \subset F(z) + \text{int}K$. Indeed, if not, then there is $\beta \in F(x)$ such that

$$\beta \notin F(z) + \text{int}K. \tag{5.4.9}$$

From (5.4.5), there exist $\eta \in F(y)$ and $c \in K$ such that $\beta = \eta + c$. By (5.4.7), $t\beta + (1-t)\eta \in F(z) + \text{int}K$, i.e., $tc + \eta \in F(z) + \text{int}K$, therefore $\beta \in F(z) + \text{int}K$ which contradicts (5.4.9). Moreover, combining (5.4.6) and (5.4.8), we get $F(z') \subset F(x) - \text{int}K$. Indeed, assume that there is $\beta \in F(z')$ such that

$$\beta \notin F(x) - \text{int}K. \tag{5.4.10}$$

From (5.4.8), there exist $\gamma \in F(x), \eta \in F(y)$ and $c_0 \in \text{int}K$ such that $\beta = t\gamma + (1-t)\eta - c_0$. By (5.4.6), there exist $\xi \in F(x)$ and $c \in K$ such that $\eta = \xi - c$. Hence, $\beta = t\gamma + (1-t)\xi - (1-t)c - c_0$. By the convexity of $F(x)$, $t\gamma + (1-t)\xi \in F(x)$. Therefore, we have $\beta \in F(x) - \text{int}K$ which contradicts (5.4.10). So, we get

$$F(x) \subset F(z) + \text{int}K, \tag{5.4.11}$$

and

$$F(z') \subset F(x) - \text{int}K. \tag{5.4.12}$$

Next, we prove that for all $n \in \mathbb{N}$,

$$F(x_n) \subset F(z) + \text{int}K, \tag{5.4.13}$$

and

$$F(z') \subset F(x_n) - \text{int}K. \tag{5.4.14}$$

Indeed, assume that (5.4.13) is not satisfied. Then, for each $n \in \mathbb{N}$ there exists $u_n \in F(x_n)$ such that

$$u_n \notin F(z) + \text{int}K. \tag{5.4.15}$$

Since F is upper semicontinuous and compact-valued at x, we can assume that u_n converge to u for some $u \in F(x)$. By (5.4.11), there exists $v \in F(z)$ such that $u \in v + \text{int}K$. This leads to $u_n \in v + \text{int}K$ for n sufficiently large, which contradicts (5.4.15). Furthermore, if (5.4.14) is not fullfilled, then there is $h \in F(z')$ satisfying

$$h \notin F(x_n) - \text{int}K. \tag{5.4.16}$$

From (5.4.12) there exists $k \in F(x)$ such that $h \in k - \text{int}K$, i.e.,

$$-k \in -h - \text{int}K. \tag{5.4.17}$$

Combining the lower semicontinuity of F and Lemma 2.3.2(i), there exist $k_n \in F(x_n)$ such that $\{k_n\}$ converges to k. By (5.4.17), $-k_n \in -h - \text{int}K$ for n sufficiently large, i.e., $h \in k_n - \text{int}K$. We conclude that $h \in F(x_n) - \text{int}K$ which contradicts (5.4.16). So, we get $F(x_n) \subset F(z) + \text{int}K$ and $F(z') \subset F(x_n) - \text{int}K$. This leads to a contradiction as $x_n \in S_s$. □

CHAPTER VI

CONCLUSION

In this chapter, we firstly summarize main results obtained in this study. Then, we give diagrams to illustrate relationships between different kinds of investigated well-posedness. Furthermore, we suggest some important comments on problems related to the work in this dissertation as a suggestion to develop this topic in future.

This thesis includes three types of well-posedness for set optimization problems involving three kinds of set order relations. More precisely, these types of well-posedness are well-posedness in the sense of Loridan, well-posedness in the sense of Bednarczuk and Levitin-Polyak well-posedness are investigated. For each types of these kinds of well-posedness, we examine both pointwise and global ones.

Employing useful tools in set-valued analysis, convex analysis and effective techniques in set-valued optimization, set order relations in particularly, we gave existence conditions of solutions, surveyed characterizations of well-posedness and proposed links between well-posedness and stability of solution maps for set optimization problems involving several types of set order relations. More precisely, we

1. surveyed set optimization problems involving various kinds of set order relations;

2. studied well-posedness in the sense of Loridan and connections with stability for set optimization problems involving different types of set order relations;

3. investigated well-posedness in the sense of Bednarczuk and bridges with stability for set optimization problems involving different types of set order relations;

4. studied Levitin-Polyak well-posedness and links with stability for set optimization problems involving different types of set order relations;

5. introduced some new and reasonable kinds of well-posedness. Relationships between these types of well-posedness in the field of set optimization are built;

6. obtained characterizations, criteria of well-posedness as well as sufficient and/or necessary conditions of semicontinuity of solution map for set optimization problems involving set order relations.

We summarize relationships between various types of well-posedness studied in this thesis as the following

1. *for pointwise well-posedness*

$$\text{L-well-posed at } x_0 \xrightarrow{Q_{x_0}(0)=S_\alpha} \text{B-well-posed at } x_0$$

$$\text{L-well-posed at } x_0 \xleftarrow{S_\alpha \text{compact}} \text{B-well-posed at } x_0$$

2. *for global well-posedness*

$$\text{weak metrically well-posed} \xrightarrow{S_\alpha \text{compact}} \text{L-well-posed}$$

$$\text{weak metrically well-posed} \leftarrow \text{metrically well-posed} \leftarrow \text{L-well-posed}$$

3. *for relationships between pointwise and global well-posedness*

$$\text{B-well-posed at every } \bar{x} \in S_\alpha \xrightarrow{S_\alpha \text{finite}} \text{B-well-posed} \rightarrow \text{generalized B-well-posed}$$

$$\text{B-well-posed} \xleftarrow{S_\alpha \text{compact}} \text{generalized B-well-posed}$$

$$\begin{array}{ccc} \text{LP well-posed at all } \bar{x} \in S_\alpha & \rightarrow & \text{generalized LP well-posed at all } \bar{x} \in S_\alpha \\ \downarrow & & \downarrow \\ \text{metrically LP well-posed} & \rightarrow & \text{generalized metrically LP well-posed} \end{array}$$

The study has a great significance as new concepts are introduced and new results are obtained in the field of set optimization. Our contribution is meaningful with generalization and development some corresponding results in literature.

Using techniques and the approach used in this research, we believe that it can be applied and successful in studying the following issues:

1. study these kinds of well-posedness for set optimization problems involving other types of set order relations such as $\leq^c, \leq^p, \leq^m, \leq^{mc}$ and $\leq^{mn}$; and investigate these issues for problems perturbated by parameters;

2. study stability for set optimization problems involving various types of set order relations. There are just a few publications devoted to stability of set

optimization problems in the literature. Due to the important role of this topic and the high volume of active research in set-valued optimization, it is worthwhile to do research on this topic to highlight the recent advances in this field. Therefore, properties on Hölder/Lipschitz continuity, lower semicontinuity, upper semicontinuity and continuity of such problems should be studied more;

3. study algorithm to point out solutions of set optimization problems involving other types of set order relations;

4. study relationships between set optimization problems and equilibrium problems, optimization problems, robust set optimization problems.

REFERENCES

REFERENCES

1. Chen GY, Jahn J. Optimality conditions for set-valued optimization problems. Math Meth Oper Res. 1998;48:187–200.
2. Ha TXD. Optimality conditions for several types of efficient solutions of set-valued optimization problems. In: Pardalos P, Rassias T, Khan A, editors. Nonlinear Analysis and Variational Problems. New York: Springer; 2010. p. 305–324.
3. Luc DT. Theory of Vector Optimization. Berlin: Springer; 1989.
4. Anh LQ, Khanh PQ. Semicontinuity of the solution set of parametric multivalued vector quasiequilibrium problems. J Math Anal Appl. 2004;294:699–711.
5. Kuroiwa D, Tanaka T, Ha TXD. On cone convexity of set-valued maps. Nonlinear Anal. 1997;30:1487–1496.
6. Young RC. The algebra of many-valued quantities. Math Ann. 1931;104:260–290.
7. Nishnianidze ZG. Fixed points of monotone multivalued operators. Soobshch Akad Nauk Gruzin SSR. 1984;114:489–491.
8. Hamel A, Löhne A. Minimal element theorems and Ekelands principle with set relations. J Nonlinear Convex Anal. 2006;7:19–37.
9. Hernández E, Rodríguez-Marín L. Nonconvex scalarization in set optimization with set-valed maps. J Math Anal Appl. 2007;325:1–18.
10. Jahn J, Ha TXD. New order relations in set optimization. J Optim Theory Appl. 2011;148:209–236.
11. Kuroiwa D. On set-valued optimization. Nonlinear Anal. 2001;47:1395–1400.
12. Kuroiwa D. Generalized minimality in set optimization. In: Hamel AH, Heyde F, Löhne A, Rudloff B, Schrage C, editors. Set Optimization and Applications-The State of the Art: From Set Relations to Set-Valued Risk Measures. Springer; 2015. p. 293–311.

13. Long XJ, Peng JW. Generalized B-well-posedness for set optimization problems. J Optim Theory Appl. 2013;157:612–623.

14. Dhingra M, Lalitha CS. Well-setness and scalarization in set optimization. Optim Lett. 2016;10:1657–1667.

15. Neukel N. Order relations of sets and its application in socio-economics. Appl Math Sci, 2013;7:5711–5739.

16. Hamel A, Heyde F. Duality for set-valued measures of risk. SIAM J Financial Math. 2010;1:66–95.

17. Ansari QH, Yao JC. Recent Developments in Vector Optimization. Berlin: Springer; 2011.

18. Hamel AH, Heyde F, Löhne A, Rudloff B, Schrage C. Set Optimization and Applications-The State of the Art: From Set Relations to Set-Valued Risk Measures. Berlin: Springer; 2015.

19. Khan AA, Tammer C, Zălinescu C. Set-valued optimization. Berlin: Springer; 2016.

20. Kuroiwa D. Some duality theorems of set-valued optimization with natural criteria. In: Proceedings of the international conference on nonlinear analysis and convex analysis. New York: World Scientific, River Edge; 1999. p. 221-228.

21. Crespi GP, Schrage C. Set optimization meets variational inequalities. In: Hamel A, Heyde F, Löhne A, Rudloff B, Schrage C, editors. Set Optimization and Applications-The State of the Art. Berlin: Springer; 2015. p. 213–247.

22. Jahn J. Vectorization in set optimization. J Optim Theory Appl. 2015;167:783-795.

23. Jahn J. Vector Optimization: Theory, Applications, and Extensions. Berlin: Springer; 2004.

24. Alonso M, Rodríguez-Marín L. Set-relations and optimality conditions in set-valued maps. Nonlinear Anal. 2005;63:1167–1179.

25. Hernández E, Rodríguez-Marín L. Existence theorems for set optimization problems. Nonlinear Anal. 2007;67:1276–1736.

26. Sach PH. Solution existence in bifunction-set optimization. J Optim Theory Appl. 2018;176:1–16.

27. Chen J, Ansari QH, Yao JC. Characterizations of set order relations and constrained set optimization problems via oriented distance function. Optimization. 2017;66:1741–1754.

28. Ha TXD. A Hausdorff-type distance, a directional derivative of a set-valued map and applications in set optimization. Optimization. 2018;67:1031-1050.

29. Hernández E, Rodríguez-Marín L. Optimality conditions for set-valued maps with set optimization. Nonlinear Anal. 2009;70:3057–3064.

30. Köbis E, Tam LT, Tammer C. A generalized scalarization method in set optimization with respect to variable domination structures. Vietnam J Math. 2018;46:95–125.

31. Karaman E, Soyertem M, Atasever Güvenç I, Tozkan D, Küçük M, Küçük Y. Partial order relations on family of sets and scalarizations for set optimization. Positivity. 2017;22:783–802.

32. Ha TXD. Some variants of the Ekeland variational principle for a set-valued map. J Optim Theory Appl. 2005;124:187–206.

33. Hernández E, Rodríguez-Marín L. Lagrangian duality in set-valued optimization. J Optim Theory Appl. 2007;134:119–134.

34. Gutiérrez C, Miglierina E, Molho E, Novo V. Pointwise well-posedness in set optimization with cone proper sets. Nonlinear Anal. 2012;75:1822–1833.

35. Long XJ, Peng JW, Peng ZY. Scalarization and pointwise well-posedness for set optimization problems. J Global Optim. 2015;62:763–773.

36. Han Y, Huang N. Well-posedness and stability of solutions for set optimization problems. Optimization. 2017;66:17–33.

37. Xu YD, Zhang PP. On the Stability of the Solution Set Mappings to Parametric Set Optimization Problems. J Oper Res Soc China. 2016;4:255–263.

38. Xu YD, Li SJ. Continuity of the solution set mappings to a parametric set optimization problem. Optimization Lett. 2014;8:2315–2327.

39. Gaydu M, Geoffroy MH, Jean-Alexis C, Nedelcheva D. Stability of minimizers of set optimization problems. Positivity. 2017;21:127–141.

40. Levitin ES, Polyak BT. Convergence of minimizing sequences in conditional extremum problem. Soviet Math Dokl. 1966;7:764–767.

41. Dontchev AL, Zolezzi T. Well-Posed Optimization Problems. Lecture Notes in Mathematics. Berlin: Springer-Verlag; 1993.

42. Lucchetti R, Revalski J. Recent Development in Well-posed Variational Problems. Dordrecht: Kluwer Academic Publishers; 1995.

43. Huang XX. Extended and strongly extended well-posedness of set-valued optimization problems. Math Meth Oper Res. 2001;53:101–116.

44. Tikhonov, AN. On the stability of the functional optimization problem. USSR Comput Math Math Phys. 1966;6:28–33.

45. Crespi GP, Guerraggio A, Rocca M. Well posedness in vector optimization problems and vector variational inequalities. J Optim Theory Appl. 2007;132:213–226.

46. Anh LQ, Duy TQ. Tykhonov well-posedness for lexicographic equilibrium problems. Optimization. 2016;65:1929–1948.

47. Wangkeeree R, Anh LQ, Boonman P. Well-posedness for general parametric quasi-variational inclusion problems. Optimization. 2017;66:93–111.

48. Loridan P. Well-posedness in vector optimization. In: Lucchetti R, Revalski J, editors. Recent Developments in Well-Posed Variational Problems. Dordrecht: Kluwer Academic Publishers; 1995. p. 171–192

49. Bednarczuck, EM. Well-posedness of vector optimization problem. In: Jahn J, Krabs W, editors. Recent Advances and Historical Development of Vector

Optimization Problems-Lecture Notes in Economics and Mathematical Systems. Berlin: Springer; 1987. p. 51–61.

50. Miglierina E, Molho E, Rocca M. Well-posedness and scalarization in vector optimization. J Optim Theory Appl. 2005;126:391–409.

51. Bednarczuck EM. An approach to well-posedness in vector optimization: consequences to stability and parametric optimization. Control Cybern. 1994;23:107–122.

52. Anh LQ, Khanh PQ, Van DTM, Yao JC. Well-posedness for vector quasiequilibria. Taiwan J Math. 2009;13:713–737.

53. Fang YP, Hu R, Huang NJ. Extended B-well-posedness and property (H) for set-valued vector optimization with convexity. J Optim Theory Appl. 2007;135:445–458.

54. Zolezzi T. Well-posedness criteria in optimization with application to the calculus of variations. Non Anal TMA. 1995;25:437–453.

55. Zolezzi T. Extended well-posedness of optimization problems. J Optim Theory Appl. 1996;91:257–266.

56. Anh LQ, Khanh PQ. Continuity of solution maps of parametric quasiequilibrium problems. J Global Optim. 2010;46:247–259.

57. Bednarczuk EM, Penot JP. Metrically well-set minimization problems. Appl Math Optim. 1992;26:273–285.

58. Miglierina E, Molho E. Well-posedness and convexity in vector optimization. Math Meth Oper Res. 2003;58:375–385.

59. Huang XX, Yang XQ. Generalized Levitin-Polyak well-posedness in constrained optimization. SIAM J Optim. 2006;17:243–258.

60. Lalitha C, Chatterjee P. Levitin–Polyak well-posedness for constrained quasiconvex vector optimization problems. J Global Optim. 2014;59:191–205.

61. Zhang WY, Li SJ, Teo KL. Well-posedness for set optimization problems. Nonlinear Anal. 2009;71:3769–3778.

62. Gerth C, Weidner P. Nonconvex separation theorems and some applications in vector optimization. J Optim Theory Appl. 1990;67:297-320.

63. Crespi GP, Kuroiwa D, Rocca M. Convexity and global well-posedness in set-optimization. Taiwan J Math. 2014;18:1897–1908.

64. Khoshkhabar-amiranloo S, Khorram E. Scalarization of Levitin-Polyak well-posed set optimization problems. Optimization. 2017;66:113–127.

65. Anh LQ, Khanh PQ. Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces. J Global Optim. 2007;37:449–465.

66. Anh LQ, Khanh PQ. On the Hölder continuity of solutions to parametric multivalued vector equilibrium problems. J Math Anal Appl. 2006;321:308–315.

67. Kuroiwa D. Existence theorems of set optimization with set-valued maps. J Inf Optim Sci. 2003;24:73–84.

68. Araya Y. Four types of nonlinear scalarizations and some applications in set optimization. Nonlinear Anal. 2012;75:3821–3835.

69. Willard S. General Topology. Alberta: Addison-Wesley; 2004.

70. Komornik V. Lectures on Functional Analysis and the Lebesgue Integral. London: Springer; 2016.

71. Kuroiwa D. Natural criteria of set-valued optimization. Manuscript, Shimane University, Japan; 1998.

72. Aubin JP, Frankowska H. Set-Valued Analysis. Boston: Springer; 2009.

73. Göpfert A, Riahi H, Tammer C, Zălinescu C. Variational Methods in Partially Ordered Spaces. New York: Springer; 2003.

74. Ferro F. A minimax theorem for vector valued functions, Part 2. J Optim Theory Appl. 1991;68:35–48.

75. Fan K. A generalization of Tychonoffs fixed point theorem. Math Ann. 1961;142:305-310.

76. Kuratowski K. Topology, Vol. 2. New York: Academic Press; 1968.

77. Milovanovíc-Arandjelovíc MM. Measures of noncompactness on uniform spaces the axiomatic approach. Filomat. 2001;15:221–225.

78. Morgan J, Scalzo V. Pseudocontinuity in optimization and nonzero-sum games. J Optim Theory Appl. 2004;120:181–197.

79. Park S. Some coincidence theorems on acyclic multifunctions and applications to KKM theory. Fixed Point Theory and Applications. 1992;7:248–277

80. Sach PH. On a class of generalized vector quasiequilibrium problems with set-valued maps. J Optim Theory Appl. 2008;139:337–350.

81. Oettli W, Schlager D. Existence of equilibria for monotone multivalued mappings. Math Meth Oper Res. 1998;48:219–228.

82. Crespi GP, Dhingra M, Lalitha CS. Pointwise and global well-posedness in set optimization: a direct approach. Ann Oper Res. 2018;269:1–18.