GREEN'S RELATIONS ON SOME SEMIGROUPS CONTAINING MARGOLIS-MEAKIN EXPANSION

KITSANACHAI SRIPON

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by Mr. Kitsanachai Sripon

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Oral Defense Committee

A. Iampan ....Chair

(Assistant Professor Aiyared Iampan, Ph.D.)

E. Laysirikul ....Advisor

(Assistant Professor Ekkachai Laysirikul, Ph.D.)

Chaiwat Namnak ....Internal Examiner

(Associate Professor Chaiwat Namnak, Ph.D.)

Sureeporn Chaopraknoi ....Internal Examiner

(Assistant Professor Sureeporn Chaopraknoi, Ph.D.)

Approved

....

(Associate Professor Paisarn Muneesawang, Ph.D.)

Dean of the Graduate School

24 JUL 2018

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Kitsanachai Sripon

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Author	Kitsanachai Sripon
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ABSTRACT

Let X be a non-empty set, G a group with the identity 1 and let $f : X \to G$ be a mapping. Denote the Cayley graph of the group G with respect to f by Γ. In this thesis, we let

$$\mathrm{IFin}(\Gamma) \rtimes G = \{(\Gamma', g) : \Gamma' \text{ is a finite subgraph of } \Gamma \text{ with } 1 \in V(\Gamma') \text{ and } g \in G\}$$

$$\text{and } \mathrm{GFin}(\Gamma) \rtimes G = \{(\Gamma', g) : \Gamma' \text{ is a finite subgraph of } \Gamma \text{ with } g \in V(\Gamma')\}.$$

Then $\mathrm{IFin}(\Gamma) \rtimes G$ and $\mathrm{GFin}(\Gamma) \rtimes G$ are semigroups under the following multiplication

$$(\Gamma', g)(\Gamma'', h) = (\Gamma' \cup g\Gamma'', gh) \quad \text{where } g\Gamma'' \text{ is a subgraph with}$$

$V(g\Gamma'') = \{gk : k \in V(\Gamma'')\}$ and $E(g\Gamma'') = \{(gk, x) : (k, x) \in E(\Gamma'')\}$. Regularity and Green's relations for $\mathrm{IFin}(\Gamma) \rtimes G$ and $\mathrm{GFin}(\Gamma) \rtimes G$ are investigated. Moreover, we characterize the natural partial order on $\mathrm{IFin}(\Gamma) \rtimes G$ and $\mathrm{GFin}(\Gamma) \rtimes G$.

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CHAPTER I

INTRODUCTION

For any semigroup S, the notation S^1 means S itself if S contains the identity element 1 and we let $S^1 = S \cup \{1\}$ if S has no identity. Then we define the binary operation $\cdot$ on S^1 by

$$1 \cdot 1 = 1,\ 1 \cdot a = a \cdot 1 = a \text{ and } a \cdot b = ab \text{ for all } a, b \in S.$$

Hence S^1 becomes a semigroup with the identity element 1.

In 1951, Green [3] defined the relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ on an arbitrary semigroup S as follows: for each $a, b \in S$,

$$a\mathcal{L}b \text{ if and only if } a = xb, b = ya \text{ for some } x, y \in S^1,$$

$$a\mathcal{R}b \text{ if and only if } a = bx, b = ay \text{ for some } x, y \in S^1 \text{ and}$$

$$a\mathcal{J}b \text{ if and only if } a = xby, b = uav \text{ for some } x, y, u, v \in S^1.$$

Then he also defined the relations $\mathcal{H}$ and $\mathcal{D}$ by

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \text{ and } \mathcal{D} = \mathcal{L} \circ \mathcal{R}.$$

These five relations are equivalence relations which are called Green's relations on a semigroup S.

An element a of a semigroup S is called regular if $a \in aSa$, that is, $a = axa$ for some $x \in S$. A semigroup S is called a regular semigroup if every element of S is regular. And any regular semigroup S is an inverse semigroup if $E(S)$ is a commutative subsemigroup of S where the set $\{x \in S : x^2 = x\}$ is denoted by $E(S)$.

In 1952, Vagner [13] introduced a natural partial order $\leq$ on an inverse semigroup S as follows:

$$a \leq b \text{ if and only if } a = eb \text{ for some } e \in E(S). \tag{1}$$

Later, Mitsch [10] defined a partial order $\leq$ on an inverse semigroup S by

$$a \leq b \text{ if and only if } ab^{-1} = aa^{-1} \tag{2}$$

where a^{-1}, b^{-1} denote the unique inverses of a and b, respectively and showed that the partial orders (1) and (2) are identical.

In 1980, Nambooripad [11] defined $\leq$ on a regular semigroup S by

$$a \leq b \text{ if and only if } a = eb = bf \text{ for some } e, f \in E(S).$$

Then $(S, \leq)$ is a partially ordered set. This order equivalent to (1) if S is an inverse semigroup.

Later in 1986, Kowol and Mitsch [7] extended the above partial order to any semigroup S by defining $\leq$ on S as follows:

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1.$$

Let X be a non-empty set, let G be a group with identity 1 and let $f : X \to G$ be a function. By the Cayley graph Γ of G with respect to f, we mean the directed graph whose vertex set $V(\Gamma)$ is G and whose edge set $E(\Gamma)$ is $G \times X$, where each $g \in G, x \in X, (g, x)$ denotes an edge with initial vertex g and terminal vertex $g(xf)$. In 1989, Margolis and Meakin [8] let G be X-generated as a group and defined a semigroup

$$M(X; f) = \{(\Gamma', g) : \Gamma' \text{ is a finite connected subgraph of } \Gamma \text{ with } 1, g \in V(\Gamma')\}$$

under the multiplication

$$(\Gamma', g)(\Gamma'', h) = (\Gamma' \cup g\Gamma'', gh)$$

for all $(\Gamma', g), (\Gamma'', h) \in M(X; f)$ and $g\Gamma''$ is a subgraph of Γ with $V(g\Gamma'') = \{gk : k \in V(\Gamma'')\}$ and $E(g\Gamma'') = \{(gk, x) : (k, x) \in E(\Gamma'')\}$. We call $M(X; f)$ the Margolis-Meakin expansion of G with respect to f. Green's relations and some characterizations on $M(X;f)$ were studied in [8].

Recently, [1] introduced a new semigroup defined as follows: let Γ be the Cayley graph of the group G with respect to $f : X \to G$. Let $\mathrm{Fin}(\Gamma)$ be the semigroup of all finite subgraphs of Γ without isolated vertices with $\emptyset$ adjoined under union operation. Let

$$\mathrm{Fin}(\Gamma) \rtimes G = \{(\Gamma', g) : \Gamma' \in \mathrm{Fin}(\Gamma) \text{ and } g \in G\}.$$

Then $\mathrm{Fin}(\Gamma) \rtimes G$ is a semigroup under the multiplication as follows:

$$(\Gamma', g)(\Gamma'', h) = (\Gamma' \cup g\Gamma'', gh) \text{ for all } (\Gamma', g), (\Gamma'', h) \in \mathrm{Fin}(\Gamma) \rtimes G.$$

Clearly, Margolis-Meakin expansion of G with respect to f is a subsemigroup of $\mathrm{Fin}(\Gamma) \rtimes G$ for X-generated group G respect to f. [1] generalized the results of [8] for $\mathrm{Fin}(\Gamma) \rtimes G$.

In our work, we define new semigroups which contain Margolis-Meakin expansion. Let

$$\mathrm{IFin}(\Gamma) \rtimes G = \{(\Gamma', g) : \Gamma' \text{ is a finite subgraph of } \Gamma \text{ with } 1 \in V(\Gamma') \text{ and } g \in G\}$$

$$\text{and } \mathrm{GFin}(\Gamma) \rtimes G = \{(\Gamma', g) : \Gamma' \text{ is a finite subgraph of } \Gamma \text{ with } g \in V(\Gamma')\}.$$

Then it is easy to verify that these two sets are semigroups under the above multiplication.

The purpose of this thesis is to increase understanding about some algebraic structures of $\mathrm{IFin}(\Gamma) \rtimes G, \mathrm{GFin}(\Gamma) \rtimes G$ and some of their subsemigroups. Furthermore, we endow $\mathrm{IFin}(\Gamma) \rtimes G$ and $\mathrm{GFin}(\Gamma) \rtimes G$ with the natural partial order and determine when two elements of these semigroups are related under this order. Also, their maximal and minimal elements of each semigroup are described.

This thesis is divided into five chapters. Chapter I is an introduction to research problems. Chapter II contains definitions, notations and some useful results which are often used in this thesis. In chapter III, we characterize regularity and Green's relations on $\mathrm{IFin}(\Gamma) \rtimes G, \mathrm{GFin}(\Gamma) \rtimes G$ and their subsemigroups. Chapter IV contains characterization of the natural partial order on $\mathrm{IFin}(\Gamma) \rtimes G$ and $\mathrm{GFin}(\Gamma) \rtimes G$. Chapter V, we conclude the results of this thesis.

CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions, notations, and some useful results that will be used in the later chapters.

2.1 Elementary Concepts

Definition 2.1.1. Let S be a non-empty set. By a **binary operation** on S, we mean a function $\cdot$ from $S \times S$ into S and we called $(S, \cdot)$ is **closed**. The image of an element $(a, b) \in S \times S$ under $\cdot$ is denoted by $a \cdot b$. Frequently, we can write ab for $a \cdot b$.

Definition 2.1.2. A binary operation on a set S is said to be **associative** if $a(bc) = (ab)c$ is satisfied for all $a, b, c \in S$. A set together with an associative binary operation is called a **semigroup**.

Definition 2.1.3. A non-empty subset A of a semigroup S is called a **subsemigroup** of S if A is closed under the operation, that is, $ab \in A$ for every $a, b \in A$.

Definition 2.1.4. An element e of a semigroup S is called an **idempotent element** if $e^2 = e$. Denote the set of all idempotent elements in S by $E(S)$.

Definition 2.1.5. An element a of a semigroup S is called a **regular element** if there is $x \in S$ such that $a = axa$. Denote $Reg(S) = \{x \in S : x \text{ is a regular element}\}$.

Definition 2.1.6. A semigroup S is called a **regular semigroup** if every element of S is regular.

Definition 2.1.7. Let x and y be elements of a semigroup S. We say that y is an **inverse** of x if $x = xyx$ and $y = yxy$. We shall denote the set of all inverses of an element x by $V(x)$.

Definition 2.1.8. A semigroup S is said to be an **inverse semigroup** if $|V(x)| = 1$ for all $x \in S$. For any element x in an inverse semigroup S, we denote the inverse element of x by x^{-1}.

Theorem 2.1.9. [4] *A regular semigroup S is an inverse semigroup if and only if $E(S)$ is a commutative subsemigroup of S.*

Definition 2.1.10. A relation $\leq$ on a non-empty set P is called a **partial order** on P if it is reflexive, anti-symmetric and transitive, i.e., for every $a, b, c \in P$,

(1) $a \leq a$ (reflexive);

(2) if $a \leq b$ and $b \leq a$, then $a = b$ (anti-symmetric);

(3) if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitive).

A set with a partial order is called a **partially ordered set**.

Definition 2.1.11. Let $\leq$ be a partial order on a set P and A be a non-empty subset of P. An element $a \in A$ is called the **minimum element** of A if $a \leq x$ for all $x \in A$. Also, an element $a \in A$ is called the **maximum element** of A if $x \leq a$ for all $x \in A$.

Definition 2.1.12. Let $\leq$ be a partial order on a set P and A be a non-empty subset of P. An element $a \in A$ is called a **minimal element** of A if there is no element $x \in A \setminus \{a\}$ such that $x \leq a$. Also, an element $a \in A$ is called a **maximal element** of A if there is no element $x \in A \setminus \{a\}$ such that $a \leq x$.

Definition 2.1.13. Let S be a semigroup. A relation R on S is said to be **left compatible** if

$$(s, t) \in R \text{ implies } (as, at) \in R \text{ for all } s, t, a \in S$$

and **right compatible** if

$$(s, t) \in R \text{ implies } (sa, ta) \in R \text{ for all } s, t, a \in S.$$

It is called **compatible** if it is both left and right compatible.

For any semigroup S, S^1 is either a semigroup S if S has the identity element 1 or a semigroup $S \cup \{1\}$ under the multiplication $\cdot$ defined by

$$1 \cdot 1 = 1,\ 1 \cdot a = a \cdot 1 = a \text{ and } a \cdot b = ab \text{ for all } a, b \in S.$$

Definition 2.1.14. Let a, b be elements of a semigroup S. Then we define

$$a\mathcal{L}b \text{ if and only if } a = xb, b = ya \text{ for some } x, y \in S^1,$$
$$a\mathcal{R}b \text{ if and only if } a = bx, b = ay \text{ for some } x, y \in S^1 \text{ and}$$
$$a\mathcal{J}b \text{ if and only if } a = xby, b = uav \text{ for some } x, y, u, v \in S^1.$$

From [3], we know that $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ are equivalence relations on S. We also have that $\mathcal{D}$ and $\mathcal{H}$ are equivalence relations on S where $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. These five equivalence relations are usually called **Green's relations** on a semigroup S.

Mitsch [6] defined the **natural partial order** $\leq$ on any semigroup S as follows: for $a, b \in S$,

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1.$$

This order coincides with the natural partial order $\leq$ for an inverse semigroup S which defined as follows: for $a, b \in S$,

$$a \leq b \text{ if and only if } a = eb \text{ for some } e \in E(S).$$

2.2 Directed graph and Cayley graph

Definition 2.2.1. A **directed graph** or **digraph** D is a non-empty set of objects called **vertices** together with (possibly empty) set of ordered pairs of vertices of D called **edges**. The vertex set of D is denoted by $V(D)$ and the edge set is denoted by $E(D)$. For $(a, b) \in E(D)$, we say that a is an **initial vertex** of edge (a, b) and b is a **terminal vertex** of (a, b).

Example 2.2.2. *A digraph* Γ *with* $V(\Gamma) = \{a, b, c\}$ *and* $E(\Gamma) = \{(a, c), (c, a), (a, b)\}$ *is illustracted in the following figure.*

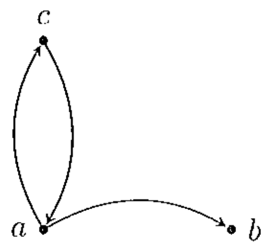

Figure 1 : Digraph Γ

Definition 2.2.3. Let D be a directed graph and $a \in V(D)$. a is called an **isolated vertex** if $(a, b), (c, a) \notin E(D)$ for each $b, c \in V(D)$.

Definition 2.2.4. For directed graphs Γ_1 and Γ_2, we say that Γ_1 is a **subgraph** of Γ_2 and we write $\Gamma_1 \subseteq \Gamma_2$ if $V(\Gamma_1) \subseteq V(\Gamma_2)$ and $E(\Gamma_1) \subseteq E(\Gamma_2)$. We define a directed graph $\Gamma_1 \cup \Gamma_2$ by letting $V(\Gamma_1 \cup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma_1 \cup \Gamma_2) = E(\Gamma_1) \cup V(\Gamma_2)$.

Definition 2.2.5. For a group G and a non-empty subset S of G, the **Cayley graph of** G **relative to** S is a directed graph D with vertex set G and

$$(x, y) \in E(D) \text{ if and only if } y = sx \text{ for some } s \in S.$$

Definition 2.2.6. Let X be a non-empty set, G be a group with the identity 1 and $f : X \to G$ be a function, the **Cayley graph of** G **with respect to** f is denoted by $\Gamma(X; f)$ is a directed graph with vertex set G and edge set is $G \times X$, where

$$(g, x) \text{ denotes an edge with initial vertex } g \text{ and terminal vertex } g(xf)$$

for all $g \in G$ and $x \in X$.

Example 2.2.7. *Let* $G = \{1, g, h, gh\}$ *be the Klein four-group "an abelian group with 4 elements and* $a = a^{-1}$ *for all* $a \in G$*" with identity 1 and let* $X = \{x, y\}$*. Define* $f_1 : X \to G$ *by* $xf_1 = g$ *and* $yf_1 = h$*. We also define* $f_2 : X \to G$ *by* $xf_2 = 1$ *and* $yf_2 = h$*. Then the Cayley graphs* $\Gamma(X; f_1)$ *and* $\Gamma(X; f_2)$ *are shown in the following figures.*

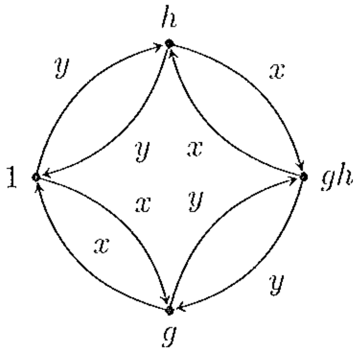

Figure 2 : Digraph $\Gamma(X; f_1)$

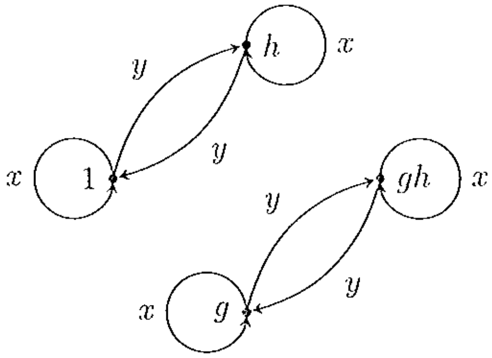

Figure 3 : Digraph $\Gamma(X; f_2)$

2.3 Some semigroups containing Margolis-Meakin Expansion.

In the rest of this thesis, we let Γ be the Cayley graph of group G with respect to mapping $f : X \to G$ where X is a non-empty set. We denote the identity of G by 1. To define new semigroups containing Margolis-Meakin expansion, we need the following proposition and lemma.

Proposition 2.3.1. *Let Γ' be a finite subgraph of Γ and $g \in G$. The following statements hold:*

(1) $|V(\Gamma')| = |V(g\Gamma')|$ *and* $|E(\Gamma')| = |E(g\Gamma')|$.

(2) *If* $g\Gamma' \subseteq \Gamma'$, *then* $g\Gamma' = \Gamma'$.

(3) *If* $\Gamma'' \subseteq \Gamma'$, *then* $g\Gamma'' \subseteq g\Gamma'$.

(4) $g\Gamma'$ *is a finite subgraph of* Γ.

Proof. (1) Define $\varphi : V(\Gamma') \to V(g\Gamma')$ by

$$h\varphi = gh \quad \text{for all } h \in V(\Gamma').$$

Clearly, φ is a bijection. Hence $|V(\Gamma')| = |V(g\Gamma')|$. Similarly, $|E(\Gamma')| = |E(g\Gamma')|$.

(2) Assume that $g\Gamma' \subseteq \Gamma'$. Then $V(g\Gamma') \subseteq V(\Gamma')$ and $E(g\Gamma') \subseteq E(\Gamma')$. From (1) and Γ' is a finite subgraph of Γ, we have $g\Gamma' = \Gamma'$.

(3) is obvious.

(4) From (1), we have $g\Gamma'$ is a finite graph. Since $V(g\Gamma') \subseteq V(\Gamma)$ and $E(g\Gamma') \subseteq E(\Gamma)$, we obtain that $g\Gamma'$ is a finite subgraph of Γ. $\square$

Lemma 2.3.2. *Let Γ_1, Γ_2 and Γ_3 be finite subgraphs of Γ and $g_1, g_2 \in G$. Then*

$$\Gamma_1 \cup g_1(\Gamma_2 \cup g_2\Gamma_3) = (\Gamma_1 \cup g_1\Gamma_2) \cup g_1g_2\Gamma_3.$$

Proof. We consider

$$\begin{aligned} V(\Gamma_1 \cup g_1(\Gamma_2 \cup g_2\Gamma_3)) &= V(\Gamma_1) \cup V(g_1(\Gamma_2 \cup g_2\Gamma_3)) \\ &= V(\Gamma_1) \cup \{g_1h : h \in V(\Gamma_2) \text{ or } h \in V(g_2\Gamma_3)\} \\ &= V(\Gamma_1) \cup (V(g_1\Gamma_2) \cup \{g_1g_2h' : h' \in V(\Gamma_3)\}) \\ &= (V(\Gamma_1) \cup V(g_1\Gamma_2)) \cup V(g_1g_2\Gamma_3) \\ &= V((\Gamma_1 \cup g_1\Gamma_2) \cup g_1g_2\Gamma_3). \end{aligned}$$

Similarly, we have $E(\Gamma_1 \cup g_1(\Gamma_2 \cup g_2\Gamma_3)) = E((\Gamma_1 \cup g_1\Gamma_2) \cup g_1g_2\Gamma_3)$. We conclude that $\Gamma_1 \cup g_1(\Gamma_2 \cup g_2\Gamma_3) = (\Gamma_1 \cup g_1\Gamma_2) \cup g_1g_2\Gamma_3$. $\square$

Now, we consider the following sets:

$$\begin{aligned} \mathrm{IFin}(\Gamma) \rtimes G &= \{(\Gamma', g) : \Gamma' \text{ is a finite subgraph of } \Gamma \text{ with } 1 \in V(\Gamma') \text{ and } g \in G\}, \\ \mathrm{GFin}(\Gamma) \rtimes G &= \{(\Gamma', g) : \Gamma' \text{ is a finite subgraph of } \Gamma \text{ with } g \in V(\Gamma')\}, \\ \mathrm{IFin}^*(\Gamma) \rtimes G &= \{(\Gamma', g) \in \mathrm{IFin}(\Gamma) \rtimes G : \Gamma' \text{ has no isolated vertex}\} \text{ and} \\ \mathrm{GFin}^*(\Gamma) \rtimes G &= \{(\Gamma', g) \in \mathrm{GFin}(\Gamma) \rtimes G : \Gamma' \text{ has no isolated vertex}\}. \end{aligned}$$

Theorem 2.3.3. $\mathrm{IFin}(\Gamma) \rtimes G$ *is a semigroup under the multiplication*

$$(\Gamma', g)(\Gamma'', h) = (\Gamma' \cup g\Gamma'', gh)$$

where $g\Gamma''$ is a subgraph of Γ with $V(g\Gamma'') = \{gk : k \in V(\Gamma'')\}$ and $E(g\Gamma'') = \{(gk, x) : (k, x) \in E(\Gamma'')\}$.

Proof. Let $(\Gamma_1, g_1), (\Gamma_2, g_2), (\Gamma_3, g_3) \in \mathrm{IFin}(\Gamma) \rtimes G$. From Proposition 2.3.1(4), we have $g_1\Gamma_2$ is a finite subgraph of Γ. Since $1 \in V(\Gamma_1)$, we get that $\Gamma_1 \cup g_1\Gamma_2$ is a finite subgraph of Γ and $1 \in V(\Gamma_1 \cup g_1\Gamma_2)$. Hence $(\Gamma_1 \cup g_1\Gamma_2, g_1g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. This means that $(\Gamma_1, g_1)(\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. It follows from Lemma 2.3.2 that

$$\begin{aligned}
(\Gamma_1, g_1)((\Gamma_2, g_2)(\Gamma_3, g_3)) &= (\Gamma_1, g_1)(\Gamma_2 \cup g_2\Gamma_3, g_2g_3) \\
&= (\Gamma_1 \cup g_1(\Gamma_2 \cup g_2\Gamma_3), g_1(g_2g_3)) \\
&= ((\Gamma_1 \cup g_1\Gamma_2) \cup g_1g_2\Gamma_3, (g_1g_2)g_3) \\
&= (\Gamma_1 \cup g_1\Gamma_2, g_1g_2)(\Gamma_3, g_3) \\
&= ((\Gamma_1, g_1)(\Gamma_2, g_2))(\Gamma_3, g_3).
\end{aligned}$$

Hence $\mathrm{IFin}(\Gamma) \rtimes G$ is a semigroup. □

Theorem 2.3.4. *$\mathrm{GFin}(\Gamma) \rtimes G$ is a semigroup under the multiplication*

$$(\Gamma', g)(\Gamma'', h) = (\Gamma' \cup g\Gamma'', gh)$$

where $g\Gamma''$ is a subgraph of Γ with $V(g\Gamma'') = \{gk : k \in V(\Gamma'')\}$ and $E(g\Gamma'') = \{(gk, x) : (k, x) \in E(\Gamma'')\}$.

Proof. Let $(\Gamma_1, g_1), (\Gamma_2, g_2), (\Gamma_3, g_3) \in \mathrm{GFin}(\Gamma) \rtimes G$. From Proposition 2.3.1(4), we have $g_1\Gamma_2$ is a finite subgraph of Γ. Since $g_1g_2 \in V(g_1\Gamma_2)$, we get that $\Gamma_1 \cup g_1\Gamma_2$ is a finite subgraph of Γ and $g_1g_2 \in V(\Gamma_1 \cup g_1\Gamma_2)$. Hence $(\Gamma_1 \cup g_1\Gamma_2, g_1g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. This means that $(\Gamma_1, g_1)(\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. Note from the proof of Theorem 2.3.3 that $(\Gamma_1, g_1)((\Gamma_2, g_2)(\Gamma_3, g_3)) = ((\Gamma_1, g_1)(\Gamma_2, g_2))(\Gamma_3, g_3)$. Hence $\mathrm{GFin}(\Gamma) \rtimes G$ is a semigroup. □

Lemma 2.3.5. *Let Γ' be a finite subgraph of Γ and $g \in G$. If Γ' has no isolated vertex, then $g\Gamma'$ has no isolated vertex.*

Proof. Suppose that Γ' has no isolated vertex. We will show that $g\Gamma'$ has no isolated vertex. Let $k \in V(g\Gamma')$. Then $k = gh$ for some $h \in V(\Gamma')$. Thus there exists

$(h', x) \in E(\Gamma')$ such that $h = h'$ or $h = h'xf$. Therefore $k = gh'$ or $k = gh'xf$. From $(gh', x) \in E(g\Gamma')$, we conclude that $g\Gamma'$ has no isolated vertex. $\square$

It is clear that $\text{IFin}^*(\Gamma) \rtimes G$ and $\text{GFin}^*(\Gamma) \rtimes G$ are subsets of $\text{IFin}(\Gamma) \rtimes G$ and $\text{GFin}(\Gamma) \rtimes G$, respectively. The above lemma verifies that $\text{IFin}^*(\Gamma) \rtimes G$ and $\text{GFin}^*(\Gamma) \rtimes G$ are closed.

Now, we will give an example and some characterizations of these semigroups.

Example 2.3.6. *Let $G = \{1, g, h, gh\}$ be the Klein four-group with identity 1 and $X = \{x, y\}$. Define $f : X \to G$ by $xf = g$ and $yf = h$. Denote the Cayley graph of G with respect to f by Γ. Then we consider directed graphs Γ_1 and Γ_2 defined as follow:*

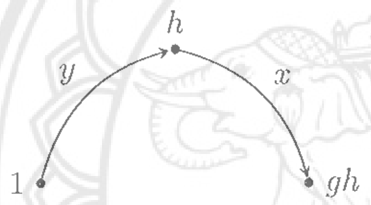

Figure 4: Digraph Γ_1

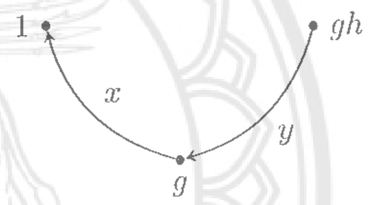

Figure 5: Digraph Γ_2

Then, we have $(\Gamma_1, h), (\Gamma_2, gh) \in (\text{IFin}^*(\Gamma) \rtimes G) \cap (\text{GFin}^*(\Gamma) \rtimes G)$. Note that $(\Gamma_1, h)(\Gamma_2, gh) = (\Gamma_1 \cup h\Gamma_2, g)$ and $(\Gamma_2, gh)(\Gamma_1, h) = (\Gamma_2 \cup gh\Gamma_1, g)$.

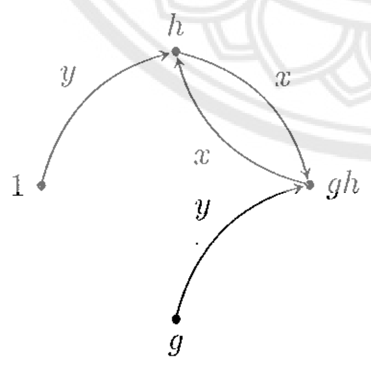

Figure 6: Digraph $\Gamma_1 \cup h\Gamma_2$

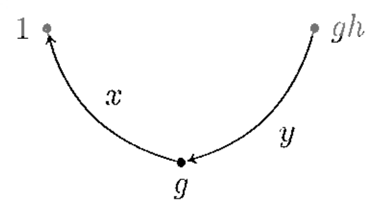

Figure 7: Digraph $\Gamma_2 \cup gh\Gamma_1$

Since $(g, y) \in E(\Gamma_1 \cup h\Gamma_2)$ and $(g, y) \notin E(\Gamma_2 \cup gh\Gamma_1)$, we get that $\text{IFin}^*(\Gamma) \rtimes G$ and $\text{GFin}^*(\Gamma) \rtimes G$ are not commutative.

Proposition 2.3.7. *Let* $(\Gamma', g) \in \mathrm{IFin}(\Gamma) \rtimes G$. *Then* (Γ', g) *is an idempotent element if and only if* $g = 1$. *In this case,* $E(\mathrm{IFin}(\Gamma) \rtimes G)$ *is a commutative subsemigroup of* $\mathrm{IFin}(\Gamma) \rtimes G$.

Proof. Assume that (Γ', g) is an idempotent element. Then $(\Gamma', g) = (\Gamma', g)(\Gamma', g) = (\Gamma' \cup g\Gamma', g^2)$. That is $\Gamma' = \Gamma' \cup g\Gamma'$ and $g = g^2$. Since G has only one idempotent, we obtain $g = 1$.

The converse is clear. □

Proposition 2.3.8. *Let* $(\Gamma', g) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* (Γ', g) *is an idempotent element if and only if* $g = 1$. *In this case,* $E(\mathrm{GFin}(\Gamma) \rtimes G)$ *is a commutative subsemigroup of* $\mathrm{GFin}(\Gamma) \rtimes G$.

Proof. It follows from the proof of Proposition 2.3.7. □

CHAPTER III

REGULARITY AND GREEN'S RELATIONS

In this chapter, we discuss on regularity and Green's relations for $\mathrm{IFin}(\Gamma) \rtimes G$, $\mathrm{GFin}(\Gamma) \rtimes G$ and their subsemigroups.

3.1 $\mathrm{IFin}(\Gamma) \rtimes G$ and it's subsemigroups

Theorem 3.1.1. *Let $(\Gamma', g) \in \mathrm{IFin}(\Gamma) \rtimes G$. Then (Γ', g) is a regular element if and only if $g \in V(\Gamma')$.*

Proof. Assume that (Γ', g) is a regular element. Then there exists $(\Gamma'', h) \in \mathrm{IFin}(\Gamma) \rtimes G$ such that

$$\begin{aligned}(\Gamma', g) &= (\Gamma', g)(\Gamma'', h)(\Gamma', g)\\ &= (\Gamma' \cup g\Gamma'', gh)(\Gamma', g)\\ &= (\Gamma' \cup g\Gamma'' \cup gh\Gamma', ghg).\end{aligned}$$

Thus $\Gamma' = \Gamma' \cup g\Gamma'' \cup gh\Gamma'$ and $g = ghg$. Since G is a group, we conclude $1 = gh$. Therefore $\Gamma' = \Gamma' \cup g\Gamma''$ and so $g\Gamma'' \subseteq \Gamma'$. This implies that $g = g1 \in V(g\Gamma'') \subseteq V(\Gamma')$.

Assume that $g \in V(\Gamma')$. Define a subgraph Γ'' of Γ by $V(\Gamma'') = \{1\}$ and $E(\Gamma'') = \emptyset$. Clearly, $(\Gamma'', g^{-1}) \in \mathrm{IFin}(\Gamma) \rtimes G$. Then we have

$$\begin{aligned}(\Gamma', g)(\Gamma'', g^{-1})(\Gamma', g) &= (\Gamma' \cup g\Gamma'' \cup gg^{-1}\Gamma', gg^{-1}g)\\ &= (\Gamma' \cup g\Gamma'', g).\end{aligned}$$

Note that $g\Gamma''$ is a finite subgraph of Γ with $V(g\Gamma'') = \{g\}$ and $E(g\Gamma'') = \emptyset$. Therefore

$$(\Gamma', g)(\Gamma'', g^{-1})(\Gamma', g) = (\Gamma' \cup g\Gamma'', g) = (\Gamma', g).$$

We conclude that (Γ', g) is a regular element. □

Corollary 3.1.2. *Let G be a group. Then* $\mathrm{IFin}(\Gamma) \rtimes G$ *is not a regular semigroup if and only if* $|G| > 1$.

Proof. Suppose that $|G| > 1$. Choose $g \in G \setminus \{1\}$ and define a subgraph Γ' of Γ by

$$V(\Gamma') = \{1\} \text{ and } E(\Gamma') = \emptyset.$$

Hence $(\Gamma', g) \in \mathrm{IFin}(\Gamma) \rtimes G$ but $g \notin V(\Gamma')$. From Theorem 3.1.1, $\mathrm{IFin}(\Gamma) \rtimes G$ is not a regular semigroup.

Assume that $|G| = 1$. To show that $\mathrm{IFin}(\Gamma) \rtimes G$ is a regular semigroup, let $(\Gamma', g) \in \mathrm{IFin}(\Gamma) \rtimes G$. This means that $1 \in V(\Gamma')$. By assumption, we have $g = 1 \in V(\Gamma')$. From Theorem 3.1.1, we have (Γ', g) is a regular element. Hence $\mathrm{IFin}(\Gamma) \rtimes G$ is a regular semigroup. □

Theorem 3.1.3. *$Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ is the maximum regular subsemigroup of* $\mathrm{IFin}(\Gamma) \rtimes G$.

Proof. It follows from Theorem 3.1.1 that $Reg(\mathrm{IFin}(\Gamma) \rtimes G) = \{(\Gamma', g) \in \mathrm{IFin}(\Gamma) \rtimes G : g \in V(\Gamma')\}$. Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Then

$$(\Gamma_1, g_1)(\Gamma_2, g_2) = (\Gamma_1 \cup g_1\Gamma_2, g_1g_2).$$

Since $g_1g_2 \in V(g_1\Gamma_2)$, we conclude that $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ is closed. Let H be a regular subsemigroup of $\mathrm{IFin}(\Gamma) \rtimes G$. It is clear that $H \subseteq Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Therefore $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ is the maximum regular subsemigroup of $\mathrm{IFin}(\Gamma) \rtimes G$. □

Corollary 3.1.4. *$Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ is an inverse semigroup and* $(\Gamma', g)^{-1} = (g^{-1}\Gamma', g^{-1})$ *for all* $(\Gamma', g) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$.

Proof. It follows from Theorems 3.1.3, 2.1.9 and Proposition 2.3.7 that $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ is an inverse semigroup. Let $(\Gamma', g) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Then $1, g \in V(\Gamma')$. We note that

$$1 = g^{-1}g \in V(g^{-1}\Gamma') \text{ and } g^{-1} = g^{-1}1 \in V(g^{-1}\Gamma').$$

Therefore $(g^{-1}\Gamma', g^{-1}) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ by Theorem 3.1.1 and Proposition 2.3.1(4). Consider

$$\begin{aligned}(\Gamma', g)(g^{-1}\Gamma', g^{-1})(\Gamma', g) &= (\Gamma' \cup gg^{-1}\Gamma', gg^{-1})(\Gamma', g)\\ &= (\Gamma', 1)(\Gamma', g)\\ &= (\Gamma', g)\end{aligned}$$

and

$$\begin{aligned}(g^{-1}\Gamma', g^{-1})(\Gamma', g)(g^{-1}\Gamma', g^{-1}) &= (g^{-1}\Gamma' \cup g^{-1}\Gamma', g^{-1}g)(g^{-1}\Gamma', g^{-1})\\ &= (g^{-1}\Gamma', 1)(g^{-1}\Gamma', g^{-1})\\ &= (g^{-1}\Gamma', g^{-1}).\end{aligned}$$

Hence $(g^{-1}\Gamma', g^{-1})$ is an inverse element of (Γ', g). □

Theorem 3.1.5. *Let $(\Gamma', g) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. Then (Γ', g) is a regular element if and only if $g \in V(\Gamma')$.*

Proof. Assume that (Γ', g) is a regular element in $\mathrm{IFin}^*(\Gamma) \rtimes G$. Since $\mathrm{IFin}^*(\Gamma) \rtimes G$ is a subsemigroup of $\mathrm{IFin}(\Gamma) \rtimes G$, we have (Γ', g) is a regular element in $\mathrm{IFin}(\Gamma) \rtimes G$. By Theorem 3.1.1, we obtain that $g \in V(\Gamma')$.

Assume that $g \in V(\Gamma')$. Note that $1 = g^{-1}g \in V(g^{-1}\Gamma')$. We obtain that $(g^{-1}\Gamma', g^{-1}) \in \mathrm{IFin}(\Gamma) \rtimes G$. By Lemma 2.3.5, $g^{-1}\Gamma'$ has no isolated vertex. Thus $(g^{-1}\Gamma', g^{-1}) \in \mathrm{IFin}^*(\Gamma) \rtimes G$ and then we have

$$\begin{aligned}(\Gamma', g)(g^{-1}\Gamma', g^{-1})(\Gamma', g) &= (\Gamma' \cup gg^{-1}\Gamma', gg^{-1})(\Gamma', g)\\ &= (\Gamma', 1)(\Gamma', g)\\ &= (\Gamma', g).\end{aligned}$$

We conclude that (Γ', g) is a regular element. □

Corollary 3.1.6. *Let $(\Gamma', g) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. Then $\mathrm{IFin}^*(\Gamma) \rtimes G$ is not a regular semigroup if and only if $|G| \geq 3$ or ($|G| = 2$ and $1 \in Xf$).*

Proof. Assume that $|G| \geq 3$ or ($|G| = 2$ and $1 \in Xf$).

Case 1 : $|G| \geq 3$. Let $x \in X$. Then $xf \in G$ and so $G \setminus \{1, xf\} \neq \emptyset$. Let $g \in G \setminus \{1, xf\}$. Define a subgraph Γ' of Γ by $V(\Gamma') = \{1, xf\}$ and $E(\Gamma') = \{(1, x)\}$. Thus $(\Gamma', g) \in \mathrm{IFin}^*(\Gamma) \rtimes G$ and $g \notin V(\Gamma')$. Therefore (Γ', g) is not a regular element by Theorem 3.1.1.

Case 2 : $|G| = 2$ and $1 \in Xf$. Let $x \in X$ be such that $xf = 1$ and choose $g \in G \setminus \{1\}$. Define a subgraph Γ' of Γ by $V(\Gamma') = \{1\}$ and $E(\Gamma') = \{(1, x)\}$. Then $(\Gamma', g) \in \mathrm{IFin}^*(\Gamma) \rtimes G$ and $g \notin V(\Gamma')$. Hence (Γ', g) is not a regular element by Theorem 3.1.1.

Assume that ($|G| = 2$ and $1 \notin Xf$) or $|G| = 1$. Clearly if $|G| = 1$, then $\mathrm{IFin}^*(\Gamma) \rtimes G$ is a regular semigroup. Suppose that $|G| = 2$ and $1 \notin Xf$. Let $(\Gamma', g) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. If $g = 1$, then $g \in V(\Gamma')$. Suppose that $g \neq 1$. Then $G = \{1, g\}$. Since $1 \in V(\Gamma')$ and Γ' has no isolated vertex, there exists $(h, x) \in E(\Gamma')$ such that $1 = h$ or $1 = h(xf)$. From $xf \neq 1$, we get $xf = g$. If $1 = h$, then $g = 1xf = hxf \in V(\Gamma')$. If $1 \neq h$, then $g = h \in V(\Gamma')$. By Theorem 3.1.5, we get (Γ', g) is a regular element. Hence $\mathrm{IFin}^*(\Gamma) \rtimes G$ is a regular semigroup. □

Example 3.1.7. *Let Γ be a Cayley graph of $G = \{1, g\}$ with respect to $f : \{x\} \to G$ where $xf = g$. It follows from Corollary 3.1.2 that $\mathrm{IFin}(\Gamma) \rtimes G$ is not regular. But $\mathrm{IFin}^*(\Gamma) \rtimes G$ is a regular semigroup since Corollary 3.1.6.*

Now, we focus our attention on Green's relations for the semigroup $\mathrm{IFin}(\Gamma) \rtimes G$.

Theorem 3.1.8. *Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. Then $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ if and only if $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$.*

Proof. Assume that $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$. Then we get that $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ or $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)$ and $(\Gamma_2, g_2) = (\Gamma_4, g_4)(\Gamma_1, g_1)$ for some $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{IFin}(\Gamma) \rtimes G$. If $(\Gamma_1, g_1) = (\Gamma_2, g_2)$, then $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$. Suppose that $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)$

and $(\Gamma_2, g_2) = (\Gamma_4, g_4)(\Gamma_1, g_1)$ for some $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{IFin}(\Gamma) \rtimes G$. So $(\Gamma_1, g_1) = (\Gamma_3 \cup g_3\Gamma_2, g_3g_2)$ and $(\Gamma_2, g_2) = (\Gamma_4 \cup g_4\Gamma_1, g_4g_1)$. This implies that

$$\Gamma_1 = \Gamma_3 \cup g_3\Gamma_2 \text{ and } g_1 = g_3g_2.$$

We get $g_3 = g_1g_2^{-1}$. Thus $g_1g_2^{-1}\Gamma_2 = g_3\Gamma_2 \subseteq \Gamma_1$ which means $g_2^{-1}\Gamma_2 \subseteq g_1^{-1}\Gamma_1$ by Proposition 2.3.1(3). Similarly, $\Gamma_2 = \Gamma_4 \cup g_4\Gamma_1$ and $g_2 = g_4g_1$. So $g_4 = g_2g_1^{-1}$ and then $g_2g_1^{-1}\Gamma_1 = g_4\Gamma_1 \subseteq \Gamma_2$. We conclude that $g_1^{-1}\Gamma_1 \subseteq g_2^{-1}\Gamma_2$ by Proposition 2.3.1(3). Therefore $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$.

Assume that $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$. It is clear that $(\Gamma_1, g_1g_2^{-1}), (\Gamma_2, g_2g_1^{-1}) \in \mathrm{IFin}(\Gamma) \rtimes G$. Consider

$$\begin{aligned}(\Gamma_1, g_1g_2^{-1})(\Gamma_2, g_2) &= (\Gamma_1 \cup g_1g_2^{-1}\Gamma_2, g_1g_2^{-1}g_2)\\ &= (\Gamma_1 \cup g_1g_1^{-1}\Gamma_1, g_1) \qquad \text{by assumption}\\ &= (\Gamma_1, g_1)\end{aligned}$$

and

$$\begin{aligned}(\Gamma_2, g_2g_1^{-1})(\Gamma_1, g_1) &= (\Gamma_2 \cup g_2g_1^{-1}\Gamma_1, g_2g_1^{-1}g_1)\\ &= (\Gamma_2 \cup g_2g_2^{-1}\Gamma_2, g_2) \qquad \text{by assumption}\\ &= (\Gamma_2, g_2).\end{aligned}$$

Hence $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$. □

Theorem 3.1.9. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$ *and* ($g_1 = g_2$ *or* $g_1, g_2 \in V(\Gamma_1)$).

Proof. Assume that $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$. Then we get that $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ or $(\Gamma_1, g_1) = (\Gamma_2, g_2)(\Gamma_3, g_3)$ and $(\Gamma_2, g_2) = (\Gamma_1, g_1)(\Gamma_4, g_4)$ for some $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{IFin}(\Gamma) \rtimes G$. If $(\Gamma_1, g_1) = (\Gamma_2, g_2)$, then $\Gamma_1 = \Gamma_2$ and $g_1 = g_2$. Suppose that $(\Gamma_1, g_1) = (\Gamma_2, g_2)(\Gamma_3, g_3)$ and $(\Gamma_2, g_2) = (\Gamma_1, g_1)(\Gamma_4, g_4)$ for some $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{IFin}(\Gamma) \rtimes G$. Thus $\Gamma_1 = \Gamma_2 \cup g_2\Gamma_3$ and $\Gamma_2 = \Gamma_1 \cup g_1\Gamma_4$. It follows that $\Gamma_2 \subseteq \Gamma_1, g_2\Gamma_3 \subseteq \Gamma_1$, $\Gamma_1 \subseteq \Gamma_2$ and

$g_1\Gamma_4 \subseteq \Gamma_2$. Hence $\Gamma_2 = \Gamma_1$. We note that $g_2 = g_2 1 \in V(g_2\Gamma_3) \subseteq V(\Gamma_1)$ and similarly, we get $g_1 \in V(\Gamma_2)$.

If $\Gamma_1 = \Gamma_2$ and $g_1 = g_2$, then $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$. Suppose that $\Gamma_1 = \Gamma_2$ and $g_1, g_2 \in V(\Gamma_1)$. From $g_1, g_2 \in V(\Gamma_1)$, we have $1 = g_2^{-1}g_2 \in V(g_2^{-1}\Gamma_1)$ and $1 = g_1^{-1}g_1 \in V(g_1^{-1}\Gamma_2)$. This means that $(g_2^{-1}\Gamma_1, g_2^{-1}g_1), (g_1^{-1}\Gamma_2, g_1^{-1}g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$ by Proposition 2.3.1(4). Note that

$$(\Gamma_2, g_2)(g_2^{-1}\Gamma_1, g_2^{-1}g_1) = (\Gamma_2 \cup g_2g_2^{-1}\Gamma_1, g_2g_2^{-1}g_1) = (\Gamma_1, g_1)$$

and

$$(\Gamma_1, g_1)(g_1^{-1}\Gamma_2, g_1^{-1}g_2) = (\Gamma_1 \cup g_1g_1^{-1}\Gamma_2, g_1g_1^{-1}g_2) = (\Gamma_2, g_2).$$

Hence $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$. □

As an immediate consequence of the previous theorems, we get the following result.

Theorem 3.1.10. *Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. Then $(\Gamma_1, g_1)\mathcal{H}(\Gamma_2, g_2)$ if and only if $\Gamma_1 = \Gamma_2$, $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_1$ and ($g_1 = g_2$ or $g_1, g_2 \in V(\Gamma_1)$).*

Theorem 3.1.11. *Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. Then $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ if and only if $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$ or ($g_2 \in V(\Gamma_2)$ and $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ for some $g \in V(\Gamma_2)$).*

Proof. Assume that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$. Then there exists $(\Gamma_3, g_3) \in \mathrm{IFin}(\Gamma) \rtimes G$ such that $(\Gamma_1, g_1)\mathcal{L}(\Gamma_3, g_3)$ and $(\Gamma_3, g_3)\mathcal{R}(\Gamma_2, g_2)$. From Theorems 3.1.8 and 3.1.9, we get that

$$g_1^{-1}\Gamma_1 = g_3^{-1}\Gamma_3,\ \Gamma_3 = \Gamma_2 \text{ and } (g_3 = g_2 \text{ or } g_3, g_2 \in V(\Gamma_2).$$

If $g_3 = g_2$, then $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$. If $g_3, g_2 \in V(\Gamma_2)$, then $g_1^{-1}\Gamma_1 = g_3^{-1}\Gamma_3 = g_3^{-1}\Gamma_2$.

If $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$, then $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ by Theorem 3.1.8. Since $\mathcal{L} \subseteq \mathcal{D}$, we have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$. Suppose that $g_2 \in V(\Gamma_2)$ and $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ for some $g \in$

$V(\Gamma_2)$. Then $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g)$ by Theorem 3.1.8. Since $g, g_2 \in V(\Gamma_2)$, we have $(\Gamma_2, g)\mathcal{R}(\Gamma_2, g_2)$ by Theorem 3.1.9. These imply that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$. □

Theorem 3.1.12. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \text{IFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_2, g_2) \in \text{IFin}(\Gamma) \rtimes G(\Gamma_1, g_1)\text{IFin}(\Gamma) \rtimes G$ *if and only if there exists* $g \in V(\Gamma_2)$ *such that* $gg_1 \in V(\Gamma_2)$ *and* $g\Gamma_1 \subseteq \Gamma_2$.

Proof. Assume that $(\Gamma_2, g_2) \in \text{IFin}(\Gamma) \rtimes G(\Gamma_1, g_1)\text{IFin}(\Gamma) \rtimes G$. Then $(\Gamma_2, g_2) = (\Gamma_3, g_3)(\Gamma_1, g_1)(\Gamma_4, g_4)$ for some $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \text{IFin}(\Gamma) \rtimes G$. This implies that $(\Gamma_2, g_2) = (\Gamma_3 \cup g_3\Gamma_1 \cup g_3g_1\Gamma_4, g_3g_1g_4)$. Hence $\Gamma_2 = \Gamma_3 \cup g_3\Gamma_1 \cup g_3g_1\Gamma_4$ and $g_2 = g_3g_1g_4$. Then $g_3\Gamma_1 \subseteq \Gamma_2$. We note that $g_3 \in V(g_3\Gamma_1) \subseteq V(\Gamma_2)$ and $g_3g_1 \in V(g_3g_1\Gamma_4) \subseteq V(\Gamma_2)$.

Assume that there exists $g \in V(\Gamma_2)$ such that $gg_1 \in V(\Gamma_2)$ and $g\Gamma_1 \subseteq \Gamma_2$. Note that $1 = g_1^{-1}g^{-1}(gg_1) \in V(g_1^{-1}g^{-1}\Gamma_2)$. Then $(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) \in \text{IFin}(\Gamma) \rtimes G$ by Proposition 2.3.1(4). Consider

$$\begin{aligned}(\Gamma_2, g)(\Gamma_1, g_1)(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) &= (\Gamma_2, g)(\Gamma_1 \cup g^{-1}\Gamma_2, g^{-1}g_2)\\ &= (\Gamma_2 \cup g\Gamma_1 \cup \Gamma_2, g_2)\\ &= (\Gamma_2, g_2) \text{ since } g\Gamma_1 \subseteq \Gamma_2.\end{aligned}$$

Thus $(\Gamma_2, g_2) \in \text{IFin}(\Gamma) \rtimes G(\Gamma_1, g_1)\text{IFin}(\Gamma) \rtimes G$. □

It is well-known that for a finite semigroup, we have $\mathcal{D} = \mathcal{J}$ and we only have $\mathcal{D} \subseteq \mathcal{J}$ in the general case. The following theorem verifies that $\mathcal{D}$ and $\mathcal{J}$ are identical on $\text{IFin}(\Gamma) \rtimes G$ although the semigroup is infinite.

Theorem 3.1.13. $\mathcal{D}$ *and* $\mathcal{J}$ *on* $\text{IFin}(\Gamma) \rtimes G$ *are equal.*

Proof. Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \text{IFin}(\Gamma) \rtimes G$ be such that $(\Gamma_1, g_1)\mathcal{J}(\Gamma_2, g_2)$. Then there exist $(\Gamma_3, g_3), (\Gamma_4, g_4), (\Gamma_5, g_5), (\Gamma_6, g_6) \in (\text{IFin}(\Gamma) \rtimes G)^1$ such that $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$. There are 7 cases to consider.

Case 1 : $(\Gamma_3, g_3), (\Gamma_4, g_4), (\Gamma_5, g_5)$ and (Γ_6, g_6) are not the identity. Then $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$. By Theorem 3.1.12, there exists $h_1 \in V(\Gamma_1)$ such that $h_1 g_2 \in V(\Gamma_1)$, $h_1\Gamma_2 \subseteq \Gamma_1$ and there exists $h_2 \in V(\Gamma_2)$ such that $h_2 g_1 \in V(\Gamma_2)$, $h_2\Gamma_1 \subseteq \Gamma_2$. Then $h_2 h_1 \Gamma_2 \subseteq h_2\Gamma_1 \subseteq \Gamma_2$. From Proposition 2.3.1(2), we obtian that

$$h_2 h_1 \Gamma_2 = h_2 \Gamma_1 = \Gamma_2.$$

Similarly, we get $h_1 h_2 \Gamma_1 = \Gamma_1 = h_1 \Gamma_2$. Since $h_1 g_2 \in V(\Gamma_1) = V(h_1 h_2 \Gamma_1)$, we have $h_1 g_2 = h_1 h_2 k$ for some $k \in V(\Gamma_1)$. Thus $g_2 = h_2 k \in V(h_2\Gamma_1) = V(\Gamma_2)$. Similarly, we have $g_1 \in V(\Gamma_1)$. This implies that

$$g_2^{-1}\Gamma_2 = (h_2 k)^{-1}\Gamma_2 = k^{-1}h_2^{-1}h_2\Gamma_1 = k^{-1}\Gamma_1.$$

By Theorem 3.1.11, we then have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$.

Case 2 : $(\Gamma_3, g_3) = (\Gamma_4, g_4) = (\Gamma_5, g_5) = (\Gamma_6, g_6)$ is the identity. Then $(\Gamma_1, g_1) = (\Gamma_2, g_2)$. Hence $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$.

Case 3 : $(\Gamma_4, g_4) = (\Gamma_6, g_6)$ is the identity. Then $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)$. Hence we obtain that $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$. Since $\mathcal{L} \subseteq \mathcal{D}$, we then have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$.

Case 4 : $(\Gamma_3, g_3) = (\Gamma_5, g_5)$ is the identity. Then $(\Gamma_1, g_1) = (\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_1, g_1)(\Gamma_6, g_6)$. Hence we obtain that $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$. Since $\mathcal{R} \subseteq \mathcal{D}$, we then have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$.

Case 5 : $(\Gamma_3, g_3) = (\Gamma_6, g_6)$ is the identity. Then $(\Gamma_1, g_1) = (\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)$. Thus $(\Gamma_1, g_1) = (\Gamma_2 \cup g_2\Gamma_4, g_2 g_4)$ implies $\Gamma_1 = \Gamma_2 \cup g_2\Gamma_4$ and $g_1 = g_2 g_4$. So $\Gamma_2 \subseteq \Gamma_1$. Similarly, we have $(\Gamma_2, g_2) = (\Gamma_5 \cup g_5\Gamma_1, g_5 g_1)$ implies $\Gamma_2 = \Gamma_5 \cup g_5\Gamma_1$ and $g_2 = g_5 g_1$. Then $g_5 = g_2 g_1^{-1}$. We obtain that $g_2 g_1^{-1}\Gamma_1 \subseteq \Gamma_2$ and so $g_2 g_1^{-1}\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_1$. By Proposition 2.3.1(2), we have

$$g_2 g_1^{-1}\Gamma_1 = \Gamma_1 = \Gamma_2.$$

Hence $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$ and then $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ from Theorem 3.1.11.

Case 6 : (Γ_3, g_3) is the identity. Then $(\Gamma_1, g_1) = (\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$. Thus $(\Gamma_1, g_1) = (\Gamma_1, 1)(\Gamma_2, g_2)(\Gamma_4, g_4)$. It follows from Case 1 that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$.

Case 7 : (Γ_4, g_4) is the identity. Then $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$. Then $(\Gamma_1, g_1) = (\Gamma_3 \cup g_3\Gamma_2, g_3g_2)$ implies $\Gamma_1 = \Gamma_3 \cup g_3\Gamma_2$ and $g_1 = g_3g_2$. Therefore $g_3 = g_1g_2^{-1}$ and $g_3\Gamma_2 \subseteq \Gamma_1$. Thus $g_1g_2^{-1}\Gamma_2 = g_3\Gamma_2 \subseteq \Gamma_1$. This means that $g_2^{-1}\Gamma_2 \subseteq g_1^{-1}\Gamma_1$ by Proposition 2.3.1(3). Since $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$ and Theorem 3.1.12, there exists $g \in V(\Gamma_2)$ such that $gg_1 \in V(\Gamma_2)$ and $g\Gamma_1 \subseteq \Gamma_2$. This implies that $g_2^{-1}g\Gamma_1 \subseteq g_2^{-1}\Gamma_2 \subseteq g_1^{-1}\Gamma_1$. From Proposition 2.3.1(2), we have $g_2^{-1}g\Gamma_1 = g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$. Thus $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$. By Thorem 3.1.11, we deduce $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$. □

Example 3.1.14. *Recall Γ_1 and Γ_2 from Example 2.3.6. We note that $gh \in V(\Gamma_2)$ and $h^{-1}\Gamma_1 = g^{-1}\Gamma_2$ where $g \in V(\Gamma_2)$. By Theorem 3.1.11, we conclude $(\Gamma_1, h)\mathcal{D}(\Gamma_2, gh)$. In fact, we can verify that $(\Gamma_1, h)\mathcal{L}(\Gamma_2, g)\mathcal{R}(\Gamma_2, gh)$. Note that $((\Gamma_1, h), (\Gamma_2, gh)) \notin \mathcal{L}$ and $((\Gamma_1, h), (\Gamma_2, gh)) \notin \mathcal{R}$ via Theorem 3.1.8 and Theorem 3.1.9, respectively.*

From Theorem 3.1.3, we have $Reg(\mathrm{IFin}(\Gamma) \rtimes G) = \{(\Gamma', g) \in \mathrm{IFin}(\Gamma) \rtimes G : g \in V(\Gamma')\}$ is a subsemigroup of $\mathrm{IFin}(\Gamma) \rtimes G$. Then, we now characterize Green's relations on $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$.

Theorem 3.1.15. *Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Then $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ if and only if $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$.*

Proof. Assume that $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ on $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Since $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ is a subsemigroup of $\mathrm{IFin}(\Gamma) \rtimes G$, we have $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ on $\mathrm{IFin}(\Gamma) \rtimes G$. By Theorem 3.1.8, we get that $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$.

Assume that $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$. Then $\Gamma_1 = g_1g_2^{-1}\Gamma_2$ and $g_2g_1^{-1}\Gamma_1 = \Gamma_2$. Since

$(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$, we have $1 \in V(\Gamma_1)$ and $1 \in V(\Gamma_2)$. Thus $g_1g_2^{-1} = g_1g_2^{-1} \in V(g_1g_2^{-1}\Gamma_2) = V(\Gamma_1)$ and $g_2g_1^{-1} = g_2g_1^{-1} \in V(g_2g_1^{-1}\Gamma_1) = V(\Gamma_2)$. Therefore $(\Gamma_1, g_1g_2^{-1}), (\Gamma_2, g_2g_1^{-1}) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. We conclude that

$$(\Gamma_1, g_1g_2^{-1})(\Gamma_2, g_2) = (\Gamma_1, g_1) \text{ and } (\Gamma_2, g_2g_1^{-1})(\Gamma_1, g_1) = (\Gamma_2, g_2).$$

Hence $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$. □

Theorem 3.1.16. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. *Then* $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$.

Proof. Assume that $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ on $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Since $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ is a subsemigroup of $\mathrm{IFin}(\Gamma) \rtimes G$, we have $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ on $\mathrm{IFin}(\Gamma) \rtimes G$. By Theorem 3.1.9, we get that $\Gamma_2 = \Gamma_1$.

Suppose that $\Gamma_1 = \Gamma_2$. Claim that $(g_2^{-1}\Gamma_1, g_2^{-1}g_1), (g_1^{-1}\Gamma_2, g_1^{-1}g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Since $g_1 \in V(\Gamma_1)$ and by assumption, we then have $g_1 \in V(\Gamma_2)$. Similarly, $g_2 \in V(\Gamma_1)$. Then $1 = g_2^{-1}g_2 \in V(g_2^{-1}\Gamma_1)$ and $1 = g_1^{-1}g_1 \in V(g_1^{-1}\Gamma_2)$. Clearly, $g_2^{-1}g_1 \in V(g_2^{-1}\Gamma_1)$ and $g_1^{-1}g_2 \in V(g_1^{-1}\Gamma_2)$. This implies that $(g_2^{-1}\Gamma_1, g_2^{-1}g_1), (g_1^{-1}\Gamma_2, g_1^{-1}g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ by Proposition 2.3.1(4). Note that

$$(\Gamma_2, g_2)(g_2^{-1}\Gamma_1, g_2^{-1}g_1) = (\Gamma_1, g_1) \text{ and } (\Gamma_1, g_1)(g_1^{-1}\Gamma_2, g_1^{-1}g_2) = (\Gamma_2, g_2).$$

Hence $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$. □

As an immediate consequence of the previous theorems, we get the following result.

Theorem 3.1.17. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. *Then* $(\Gamma_1, g_1)\mathcal{H}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$ *and* $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_1$.

Theorem 3.1.18. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. *Then* $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ *if and only if* $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ *for some* $g \in V(\Gamma_2)$.

Proof. Assume that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Since $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ is a subsemigroup of $\mathrm{IFin}(\Gamma) \rtimes G$, we have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\mathrm{IFin}(\Gamma) \rtimes G$. By Theorem 3.1.11 and $g_2 \in V(\Gamma_2)$, we get that $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ for some $g \in V(\Gamma_2)$.

Suppose that $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ for some $g \in V(\Gamma_2)$. By Theorem 3.1.15, $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g)$. Since $g \in V(\Gamma_2)$, we have $(\Gamma_2, g)\mathcal{R}(\Gamma_2, g_2)$ from Theorems 3.1.1 and 3.1.16. These imply that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$. □

Theorem 3.1.19. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. *Then* $(\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)(\Gamma_1, g_1)Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ *if and only if there exists* $g \in V(\Gamma_2)$ *such that* $g\Gamma_1 \subseteq \Gamma_2$.

Proof. Assume that $(\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)(\Gamma_1, g_1)Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Since $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ is a subset of $\mathrm{IFin}(\Gamma) \rtimes G$, we obtain that $(\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{IFin}(\Gamma) \rtimes G$. By Theorem 3.1.12, we get that $g\Gamma_1 \subseteq \Gamma_2$ for some $g \in V(\Gamma_2)$.

Assume that there exists $g \in V(\Gamma_2)$ such that $g\Gamma_1 \subseteq \Gamma_2$. Thus $gg_1 \in V(g\Gamma_1) \subseteq V(\Gamma_2)$. Note that $1 = g_1^{-1}g^{-1}gg_1 \in V(g_1^{-1}g^{-1}\Gamma_2)$ and $g_1^{-1}g^{-1}g_2 \in V(g_1^{-1}g^{-1}\Gamma_2)$. Then we have $(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ by Proposition 2.3.1(4) and Theorem 3.1.1. We note that

$$(\Gamma_2, g)(\Gamma_1, g_1)(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) = (\Gamma_2, g_2).$$

Thus $(\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)(\Gamma_1, g_1)Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. □

Corollary 3.1.20. $\mathcal{D}$ *and* $\mathcal{J}$ *on* $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ *are equal.*

Proof. It is obvious that $\mathcal{D} \subseteq \mathcal{J}$. Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ be such that $(\Gamma_1, g_1)\mathcal{J}(\Gamma_2, g_2)$ on $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Since $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ is a subsemigroup of $\mathrm{IFin}(\Gamma) \rtimes G$, we have $(\Gamma_1, g_1)\mathcal{J}(\Gamma_2, g_2)$ on $\mathrm{IFin}(\Gamma) \rtimes G$. By Theorem 3.1.13, we then have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\mathrm{IFin}(\Gamma) \rtimes G$. We conclude that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ via Theorems 3.1.11 and 3.1.18. □

Next, we characterize Green's relations for the semigroup $\mathrm{IFin}^*(\Gamma) \rtimes G$.

Theorem 3.1.21. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ *if and only if* $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$.

Proof. Assume that $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ on $\mathrm{IFin}^*(\Gamma) \rtimes G$. Since $\mathrm{IFin}^*(\Gamma) \rtimes G$ is a subsemigroup of $\mathrm{IFin}(\Gamma) \rtimes G$, we have $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ on $\mathrm{IFin}(\Gamma) \rtimes G$. By Theorem 3.1.8, we get $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$.

Assume that $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$. It is clear that $(\Gamma_1, g_1g_2^{-1}), (\Gamma_2, g_2g_1^{-1}) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. We can verify that

$$(\Gamma_1, g_1g_2^{-1})(\Gamma_2, g_2) = (\Gamma_1, g_1) \text{ and } (\Gamma_2, g_2g_1^{-1})(\Gamma_1, g_1) = (\Gamma_2, g_2).$$

Hence $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$. □

Theorem 3.1.22. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$ *and* ($g_1 = g_2$ *or* $g_1, g_2 \in V(\Gamma_1)$).

Proof. Assume that $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ on $\mathrm{IFin}^*(\Gamma) \rtimes G$. Since $\mathrm{IFin}^*(\Gamma) \rtimes G$ is a subsemigroup of $\mathrm{IFin}(\Gamma) \rtimes G$, we have $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ on $\mathrm{IFin}(\Gamma) \rtimes G$. By Theorem 3.1.9, $\Gamma_1 = \Gamma_2$ and ($g_1 = g_2$ or $g_1, g_2 \in V(\Gamma_1)$).

If $\Gamma_1 = \Gamma_2$ and $g_1 = g_2$, then $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$. Suppose that $\Gamma_1 = \Gamma_2$ and $g_1, g_2 \in V(\Gamma_1)$. Note that $(g_2^{-1}\Gamma_1, g_2^{-1}g_1), (g_1^{-1}\Gamma_2, g_1^{-1}g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. By Lemma 2.3.5, we have $g_2^{-1}\Gamma_1$ and $g_1^{-1}\Gamma_2$ have no isolated vertex. This means that $(g_2^{-1}\Gamma_1, g_2^{-1}g_1), (g_1^{-1}\Gamma_2, g_1^{-1}g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. We see that

$$(\Gamma_2, g_2)(g_2^{-1}\Gamma_1, g_2^{-1}g_1) = (\Gamma_1, g_1) \text{ and } (\Gamma_1, g_1)(g_1^{-1}\Gamma_2, g_1^{-1}g_2) = (\Gamma_2, g_2).$$

Hence $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$. □

As an immediate consequence of the previous theorems, we obtain the following result.

Theorem 3.1.23. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{H}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$, $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_1$ *and* ($g_1 = g_2$ *or* $g_1, g_2 \in V(\Gamma_1)$).

Theorem 3.1.24. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ *if and only if* $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$ *or* ($g_2 \in V(\Gamma_2)$ *and* $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ *for some* $g \in V(\Gamma_2)$).

Proof. Assume that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\mathrm{IFin}^*(\Gamma) \rtimes G$. Since $\mathrm{IFin}^*(\Gamma) \rtimes G$ is a subsemigroup of $\mathrm{IFin}(\Gamma) \rtimes G$, we have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\mathrm{IFin}(\Gamma) \rtimes G$. By Theorem 3.1.11, $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$ or ($g_2 \in V(\Gamma_2)$ and $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ for some $g \in V(\Gamma_2)$).

If $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$, then $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ on $\mathrm{IFin}^*(\Gamma) \rtimes G$ by Theorem 3.1.21. Since $\mathcal{L} \subseteq \mathcal{D}$, we have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\mathrm{IFin}^*(\Gamma) \rtimes G$. Suppose that $g_2 \in V(\Gamma_2)$ and $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ for some $g \in V(\Gamma_2)$. Then $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g)$ on $\mathrm{IFin}^*(\Gamma) \rtimes G$ from Theorem 3.1.21. Since $g, g_2 \in V(\Gamma_2)$, we have $(\Gamma_2, g)\mathcal{R}(\Gamma_2, g_2)$ from Theorem 3.1.22. These imply that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$. □

Theorem 3.1.25. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{IFin}^*(\Gamma) \rtimes G$ *if and only if there exists* $g \in V(\Gamma_2)$ *such that* $gg_1 \in V(\Gamma_2)$ *and* $g\Gamma_1 \subseteq \Gamma_2$.

Proof. Assume that $(\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{IFin}^*(\Gamma) \rtimes G$. Since $\mathrm{IFin}^*(\Gamma) \rtimes G$ is a subset of $\mathrm{IFin}(\Gamma) \rtimes G$, we get that $(\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{IFin}(\Gamma) \rtimes G$. By Theorem 3.1.12, $gg_1 \in V(\Gamma_2)$ and $g\Gamma_1 \subseteq \Gamma_2$ for some $g \in V(\Gamma_2)$.

Assume that there exists $g \in V(\Gamma_2)$ such that $gg_1 \in V(\Gamma_2)$ and $g\Gamma_1 \subseteq \Gamma_2$. Note that $(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. By Lemma 2.3.5, we have $g_1^{-1}g^{-1}\Gamma_2$ has no isolated vertex. Thus $(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) \in \mathrm{IFin}^*(\Gamma) \times G$. Note that

$$(\Gamma_2, g)(\Gamma_1, g_1)(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) = (\Gamma_2, g_2).$$

Hence $(\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{IFin}^*(\Gamma) \rtimes G$. □

Theorem 3.1.26. $\mathcal{D}$ *and* $\mathcal{J}$ *on* $\mathrm{IFin}^*(\Gamma) \rtimes G$ *are equal.*

Proof. It is obvious that $\mathcal{D} \subseteq \mathcal{J}$. Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \text{IFin}^*(\Gamma) \rtimes G$ and suppose that $(\Gamma_1, g_1)\mathcal{J}(\Gamma_2, g_2)$ on $\text{IFin}^*(\Gamma) \rtimes G$. Since $\text{IFin}^*(\Gamma) \rtimes G$ is a subsemigroup of $\text{IFin}(\Gamma) \rtimes G$, we obtain that $(\Gamma_1, g_1)\mathcal{J}(\Gamma_2, g_2)$ on $\text{IFin}(\Gamma) \rtimes G$. By Theorem 3.1.13, we then have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\text{IFin}(\Gamma) \rtimes G$. We conclude that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\text{IFin}^*(\Gamma) \rtimes G$ via Theorems 3.1.11 and 3.1.24. □

3.2 $\text{GFin}(\Gamma) \rtimes G$ and it's subsemigroups

Theorem 3.2.1. *Let $(\Gamma', g) \in \text{GFin}(\Gamma) \rtimes G$. Then (Γ', g) is a regular element if and only if $1 \in V(\Gamma')$.*

Proof. Assume that (Γ', g) is a regular element. Then there exists $(\Gamma'', h) \in \text{GFin}(\Gamma) \rtimes G$ such that

$$\begin{aligned}(\Gamma', g) &= (\Gamma', g)(\Gamma'', h)(\Gamma', g)\\ &= (\Gamma' \cup g\Gamma'', gh)(\Gamma', g)\\ &= (\Gamma' \cup g\Gamma'' \cup gh\Gamma', ghg).\end{aligned}$$

Thus $\Gamma' = \Gamma' \cup g\Gamma'' \cup gh\Gamma'$ and $g = ghg$. Since G is a group, we conclude $1 = gh$. Therefore $\Gamma' = \Gamma' \cup g\Gamma''$ and so $g\Gamma'' \subseteq \Gamma'$. This implies that $1 = gh \in V(g\Gamma'') \subseteq V(\Gamma')$.

Assume that $1 \in V(\Gamma')$. Note that $g^{-1} = g^{-1}1 \in V(g^{-1}\Gamma')$. Thus $(g^{-1}\Gamma', g^{-1}) \in \text{GFin}(\Gamma) \rtimes G$ by Proposition 2.3.1(4). We see that

$$\begin{aligned}(\Gamma', g)(g^{-1}\Gamma', g^{-1})(\Gamma', g) &= (\Gamma' \cup gg^{-1}\Gamma', gg^{-1})(\Gamma', g)\\ &= (\Gamma', 1)(\Gamma', g)\\ &= (\Gamma', g).\end{aligned}$$

Hence we conclude that (Γ', g) is a regular element. □

From above theorem, we easily obtain that $Reg(\text{GFin}(\Gamma) \rtimes G) = Reg(\text{IFin}(\Gamma) \rtimes G)$ and whence the following corollary is true.

Corollary 3.2.2. *$Reg(\mathrm{GFin}(\Gamma) \rtimes G) = Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ is the maximum regular subsemigroup of* $\mathrm{GFin}(\Gamma) \rtimes G$.

Theorem 3.2.3. *Let $(\Gamma', g) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. Then (Γ', g) is a regular element if and only if $1 \in V(\Gamma')$.*

Proof. Assume that (Γ', g) is a regular element in $\mathrm{GFin}^*(\Gamma) \rtimes G$. Since $\mathrm{GFin}^*(\Gamma) \rtimes G$ is a subsemigroup of $\mathrm{GFin}(\Gamma) \rtimes G$, we have (Γ', g) is a regular element in $\mathrm{GFin}(\Gamma) \rtimes G$. By Theorem 3.2.1, $1 \in V(\Gamma')$.

Assume that $1 \in V(\Gamma')$. Note that $(g^{-1}\Gamma', g^{-1}) \in \mathrm{GFin}(\Gamma) \rtimes G$. By Lemma 2.3.5, $g^{-1}\Gamma'$ has no isolated vertex. Thus $(g^{-1}\Gamma', g^{-1}) \in \mathrm{GFin}^*(\Gamma) \rtimes G$ and

$$(\Gamma', g)(g^{-1}\Gamma', g^{-1})(\Gamma', g) = (\Gamma', g).$$

Hence (Γ', g) is a regular element. □

Next, we can determine Green's relations for elements $\mathrm{GFin}(\Gamma) \rtimes G$.

Theorem 3.2.4. *Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. Then $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ if and only if $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ or ($g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$, $g_1g_2^{-1} \in V(\Gamma_1)$ and $g_2g_1^{-1} \in V(\Gamma_2)$).*

Proof. Assume that $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ and $(\Gamma_1, g_1) \neq (\Gamma_2, g_2)$. We get that $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)$ and $(\Gamma_2, g_2) = (\Gamma_4, g_4)(\Gamma_1, g_1)$ for some $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{GFin}(\Gamma) \rtimes G$. This implies that $\Gamma_1 = \Gamma_3 \cup g_3\Gamma_2$ and $g_1 = g_3g_2$. Hence $g_1g_2^{-1} = g_3 \in V(\Gamma_3) \subseteq V(\Gamma_1)$ and $g_1g_2^{-1}\Gamma_2 = g_3\Gamma_2 \subseteq \Gamma_1$ which means $g_2^{-1}\Gamma_2 \subseteq g_1^{-1}\Gamma_1$ by Proposition 2.3.1(3). Similarly, $\Gamma_2 = \Gamma_4 \cup g_4\Gamma_1$ and $g_2 = g_4g_1$. So $g_2g_1^{-1} = g_4 \in V(\Gamma_4) \subseteq V(\Gamma_2)$ and $g_2g_1^{-1}\Gamma_1 = g_4\Gamma_1 \subseteq \Gamma_2$ which means $g_1^{-1}\Gamma_1 \subseteq g_2^{-1}\Gamma_2$ by Proposition 2.3.1(3). Therefore $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$.

If $(\Gamma_1, g_1) = (\Gamma_2, g_2)$, then $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$. Assume that $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$ and $g_1g_2^{-1} \in V(\Gamma_1)$, $g_2g_1^{-1} \in V(\Gamma_2)$. It is clear that $(\Gamma_1, g_1g_2^{-1}), (\Gamma_2, g_2g_1^{-1}) \in \mathrm{GFin}(\Gamma) \rtimes G$.

Consider

$$\begin{aligned}(\Gamma_1, g_1g_2^{-1})(\Gamma_2, g_2) &= (\Gamma_1 \cup g_1g_2^{-1}\Gamma_2, g_1g_2^{-1}g_2)\\ &= (\Gamma_1 \cup g_1g_1^{-1}\Gamma_1, g_1) \quad \text{by assumption}\\ &= (\Gamma_1, g_1)\end{aligned}$$

and

$$\begin{aligned}(\Gamma_2, g_2g_1^{-1})(\Gamma_1, g_1) &= (\Gamma_2 \cup g_2g_1^{-1}\Gamma_1, g_2g_1^{-1}g_1)\\ &= (\Gamma_2 \cup g_2g_2^{-1}\Gamma_2, g_2) \quad \text{by assumption}\\ &= (\Gamma_2, g_2).\end{aligned}$$

Hence $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$. □

Theorem 3.2.5. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$.

Proof. Assume that $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ and $(\Gamma_1, g_1) \neq (\Gamma_2, g_2)$. Then there exist (Γ_3, g_3), $(\Gamma_4, g_4) \in \mathrm{GFin}(\Gamma) \rtimes G$ such that $(\Gamma_1, g_1) = (\Gamma_2, g_2)(\Gamma_3, g_3)$ and $(\Gamma_2, g_2) = (\Gamma_1, g_1)(\Gamma_4, g_4)$. Thus $\Gamma_1 = \Gamma_2 \cup g_2\Gamma_3$ and $\Gamma_2 = \Gamma_1 \cup g_1\Gamma_4$. It follows that $\Gamma_2 \subseteq \Gamma_1$ and $\Gamma_1 \subseteq \Gamma_2$. Hence $\Gamma_2 = \Gamma_1$.

Suppose that $\Gamma_1 = \Gamma_2$. Note that $g_2^{-1}g_1 \in V(g_2^{-1}\Gamma_1)$ and $g_1^{-1}g_2 \in V(g_1^{-1}\Gamma_2)$. This means that $(g_2^{-1}\Gamma_1, g_2^{-1}g_1), (g_1^{-1}\Gamma_2, g_1^{-1}g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$ via Proposition 2.3.1(4). We see that

$$(\Gamma_2, g_2)(g_2^{-1}\Gamma_1, g_2^{-1}g_1) = (\Gamma_1, g_1) \text{ and } (\Gamma_1, g_1)(g_1^{-1}\Gamma_2, g_1^{-1}g_2) = (\Gamma_2, g_2).$$

Hence $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$. □

As an immediate consequence of the previous theorems, we get the following result.

Theorem 3.2.6. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{H}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$, $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_1$ *and* ($g_1 = g_2$ *or* $g_1g_2^{-1}, g_2g_1^{-1} \in V(\Gamma_1)$).

Theorem 3.2.7. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$ *or* (*there exists* $g \in V(\Gamma_2)$ *such that* $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$, $g_1g^{-1} \in V(\Gamma_1)$ *and* $gg_1^{-1} \in V(\Gamma_2)$).

Proof. Assume that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$. Then there exists $(\Gamma_3, g_3) \in \mathrm{GFin}(\Gamma) \rtimes G$ such that $(\Gamma_1, g_1)\mathcal{L}(\Gamma_3, g_3)$ and $(\Gamma_3, g_3)\mathcal{R}(\Gamma_2, g_2)$. From Theorems 3.2.4 and 3.2.5, we get that $[(\Gamma_1, g_1) = (\Gamma_3, g_3)$ or $(g_1^{-1}\Gamma_1 = g_3^{-1}\Gamma_3, g_1g_3^{-1} \in V(\Gamma_1)$ and $g_3g_1^{-1} \in V(\Gamma_3))]$ and $\Gamma_3 = \Gamma_2$. If $(\Gamma_1, g_1) = (\Gamma_3, g_3)$, then $\Gamma_1 = \Gamma_3 = \Gamma_2$. Suppose that $g_1^{-1}\Gamma_1 = g_3^{-1}\Gamma_3$, $g_1g_3^{-1} \in V(\Gamma_1)$ and $g_3g_1^{-1} \in V(\Gamma_3)$. Therefore $g_1^{-1}\Gamma_1 = g_3^{-1}\Gamma_2$ where $g_3 \in V(\Gamma_2)$.

If $\Gamma_1 = \Gamma_2$, then $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ by Theorem 3.2.5. Since $\mathcal{R} \subseteq \mathcal{D}$, we have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$. Suppose that there exists $g \in V(\Gamma_2)$ such that $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$, $g_1g^{-1} \in V(\Gamma_1)$ and $gg_1^{-1} \in V(\Gamma_2)$. Then $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g)$ by Theorem 3.2.4. From Theorem 3.2.5, we have $(\Gamma_2, g)\mathcal{R}(\Gamma_2, g_2)$. Hence $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$. □

Theorem 3.2.8. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{GFin}(\Gamma) \rtimes G$ *if and only if there exists* $g \in V(\Gamma_2)$ *such that* $g\Gamma_1 \subseteq \Gamma_2$.

Proof. Assume that $(\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{GFin}(\Gamma) \rtimes G$. Then $(\Gamma_2, g_2) = (\Gamma_3, g_3)(\Gamma_1, g_1)(\Gamma_4, g_4)$ for some $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{GFin}(\Gamma) \rtimes G$. This implies that $(\Gamma_2, g_2) = (\Gamma_3 \cup g_3\Gamma_1 \cup g_3g_1\Gamma_4, g_3g_1g_4)$. Hence $\Gamma_2 = \Gamma_3 \cup g_3\Gamma_1 \cup g_3g_1\Gamma_4$ and $g_2 = g_3g_1g_4$. Therefore $g_3\Gamma_1 \subseteq \Gamma_2$ and $g_3 \in V(\Gamma_3) \subseteq V(\Gamma_2)$.

Assume that there exists $g \in V(\Gamma_2)$ such that $g\Gamma_1 \subseteq \Gamma_2$. Note that $g_1^{-1}g^{-1}g_2 \in V(g_1^{-1}g^{-1}\Gamma_2)$. Then $(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$ since Proposition 2.3.1(4). Consider

$$
\begin{aligned}
(\Gamma_2, g)(\Gamma_1, g_1)(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) &= (\Gamma_2, g)(\Gamma_1 \cup g^{-1}\Gamma_2, g^{-1}g_2) \\
&= (\Gamma_2 \cup g\Gamma_1 \cup \Gamma_2, g_2)
\end{aligned}
$$

$$= (\Gamma_2, g_2) \text{ since } g\Gamma_1 \subseteq \Gamma_2.$$

Thus $(\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{GFin}(\Gamma) \rtimes G$. □

Theorem 3.2.9. *$\mathcal{D}$ and $\mathcal{J}$ on* $\mathrm{GFin}(\Gamma) \rtimes G$ *are equal.*

Proof. It is obvious that $\mathcal{D} \subseteq \mathcal{J}$. Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$ be such that $(\Gamma_1, g_1)\mathcal{J}(\Gamma_2, g_2)$. There exist $(\Gamma_3, g_3), (\Gamma_4, g_4), (\Gamma_5, g_5), (\Gamma_6, g_6) \in (\mathrm{GFin}(\Gamma) \rtimes G)^1$ such that $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$. There are 7 cases to consider.

Case 1 : $(\Gamma_3, g_3), (\Gamma_4, g_4), (\Gamma_5, g_5)$ and (Γ_6, g_6) are not the identity. Then $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$. By Theorem 3.2.8, there exists $h_1 \in V(\Gamma_1)$ such that $h_1\Gamma_2 \subseteq \Gamma_1$ and there exists $h_2 \in V(\Gamma_2)$ such that $h_2\Gamma_1 \subseteq \Gamma_2$. Then $h_2h_1\Gamma_2 \subseteq h_2\Gamma_1 \subseteq \Gamma_2$. From Proposition 2.3.1(2), we obtian that

$$h_2h_1\Gamma_2 = h_2\Gamma_1 = \Gamma_2.$$

Similarly, we get $h_1h_2\Gamma_1 = \Gamma_1 = h_1\Gamma_2$. Since $g_1 \in V(\Gamma_1) = V(h_1\Gamma_2)$, we have $g_1 = h_1k$ for some $k \in V(\Gamma_2)$. This implies that

$$g_1^{-1}\Gamma_1 = (h_1k)^{-1}\Gamma_1 = k^{-1}h_1^{-1}\Gamma_1 = k^{-1}h_1^{-1}h_1\Gamma_2 = k^{-1}\Gamma_2.$$

Note that $g_1k^{-1} = h_1kk^{-1} = h_1 \in V(\Gamma_1)$ and $kg_1^{-1} = k(h_1k)^{-1} = kk^{-1}h_1^{-1} = h_1^{-1}$. Since $\Gamma_2 = h_2\Gamma_1$, we have $h_2^{-1}\Gamma_2 = h_2^{-1}h_2\Gamma_1 = \Gamma_1$. Thus $1 = h_2^{-1}h_2 \in V(h_2^{-1}\Gamma_2) = V(\Gamma_1)$. From $\Gamma_1 = h_1\Gamma_2$, we obtain $h_1^{-1}\Gamma_1 = \Gamma_2$. This means that $h_1^{-1} = h_1^{-1}1 \in V(h_1^{-1}\Gamma_1) = V(\Gamma_2)$. By Theorem 3.2.7, we then have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$.

Case 2 : $(\Gamma_3, g_3) = (\Gamma_4, g_4) = (\Gamma_5, g_5) = (\Gamma_6, g_6)$ is the identity. Then $(\Gamma_1, g_1) = (\Gamma_2, g_2)$. Hence $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$.

Case 3 : $(\Gamma_4, g_4) = (\Gamma_6, g_6)$ is the identity. Then $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)$. We get that $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$. Since $\mathcal{L} \subseteq \mathcal{D}$, we obtain that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$.

Case 4 : $(\Gamma_3, g_3) = (\Gamma_5, g_5)$ is the identity. Then $(\Gamma_1, g_1) = (\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_1, g_1)(\Gamma_6, g_6)$. We get that $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$. Since $\mathcal{R} \subseteq \mathcal{D}$, we obtain that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$.

Case 5 : $(\Gamma_3, g_3) = (\Gamma_6, g_6)$ is the identity. Then $(\Gamma_1, g_1) = (\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)$. Then $\Gamma_1 = \Gamma_2 \cup g_2\Gamma_4$, and $\Gamma_2 = \Gamma_5 \cup g_5\Gamma_1$. Thus $\Gamma_2 \subseteq \Gamma_1$, $g_5\Gamma_1 \subseteq \Gamma_2$. So $g_5\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_1$. From Proposition 2.3.1(2), we have $g_5\Gamma_1 = \Gamma_1$ and then $\Gamma_1 = \Gamma_2$. By Theorem 3.2.7, we get $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$.

Case 6 : (Γ_4, g_4) is the identity. Then $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)$ and $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$. Then $(\Gamma_1, g_1) = (\Gamma_3 \cup g_3\Gamma_2, g_3g_2)$ so $\Gamma_1 = \Gamma_3 \cup g_3\Gamma_2$ and $g_1 = g_3g_2$. Thus $g_3 = g_1g_2^{-1}$ and $g_1g_2^{-1}\Gamma_2 \subseteq \Gamma_1$. This means that $g_2^{-1}\Gamma_2 \subseteq g_1^{-1}\Gamma_1$ by Proposition 2.3.1(3). Since $(\Gamma_2, g_2) = (\Gamma_5, g_5)(\Gamma_1, g_1)(\Gamma_6, g_6)$ and Theorem 3.2.8, there exists $g \in V(\Gamma_2)$ such that $g\Gamma_1 \subseteq \Gamma_2$. This implies that $g_2^{-1}g\Gamma_1 \subseteq g_2^{-1}\Gamma_2 \subseteq g_1^{-1}\Gamma_1$. From Proposition 2.3.1(2), we have $g_2^{-1}g\Gamma_1 = g_1^{-1}\Gamma_1$. Then $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$. Note that $g_1g_2^{-1} = g_3 \in V(\Gamma_3) \subseteq V(\Gamma_1)$. Since $g_2^{-1}g\Gamma_1 = g_2^{-1}\Gamma_2$, we have $g\Gamma_1 = \Gamma_2$. Then $g \in V(\Gamma_2) = V(g\Gamma_1)$ which implies $1 \in V(\Gamma_1)$. From $g_2g_1^{-1}\Gamma_1 = \Gamma_2$ and $1 \in V(\Gamma_1)$, we get that $g_2g_1^{-1} \in V(\Gamma_2)$. Hence $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ by Theorem 3.2.7.

Case 7 : (Γ_5, g_5) is the identity. Then $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_2, g_2) = (\Gamma_1, g_1)(\Gamma_6, g_6)$. Then $\Gamma_2 = \Gamma_1 \cup g_1\Gamma_6$ and $g_2 = g_1g_6$. Since $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2)(\Gamma_4, g_4)$ and Theorem 3.2.8, there exists $g \in V(\Gamma_1)$ such that $g\Gamma_2 \subseteq \Gamma_1$. We obtain that $g\Gamma_2 \subseteq \Gamma_1 \subseteq \Gamma_2$. From Proposition 2.3.1(2), we have $g\Gamma_2 = \Gamma_2$. Therefore $\Gamma_1 = \Gamma_2$. We conclude that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ by Theorem 3.2.7. □

Example 3.2.10. *Recall the Cayley graph Γ from Example 2.2.7. We let Γ_1 and Γ_2 be subgraphs of Γ with $V(\Gamma_1) = \{1, h, gh, g\}$, $V(\Gamma_2) = \{1, h, gh, g\}$, $E(\Gamma_1) = \{(1, x)\}$ and $E(\Gamma_2) = \{(gh, x)\}$. Then $(\Gamma_1, g), (\Gamma_2, h) \in \mathrm{GFin}(\Gamma) \rtimes G$. Therefore $g^{-1}\Gamma_1 = h^{-1}\Gamma_2$ and but $gh \notin V(\Gamma_1)$. Hence $((\Gamma_1, g), (\Gamma_2, h)) \notin \mathcal{L}$ via Theorem 3.2.4.*

Next, we characterize Green's relations for the semigroup $\mathrm{GFin}^*(\Gamma) \rtimes G$.

Theorem 3.2.11. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ *if and only if* $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ *or* $(g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2,\ g_1g_2^{-1} \in V(\Gamma_1)$ *and* $g_2g_1^{-1} \in V(\Gamma_2))$.

Proof. Assume that $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ on $\mathrm{GFin}^*(\Gamma) \rtimes G$. Since $\mathrm{GFin}^*(\Gamma) \rtimes G$ is a subsemigroup of $\mathrm{GFin}(\Gamma) \rtimes G$, we have $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ on $\mathrm{GFin}(\Gamma) \rtimes G$. From Theorem 3.2.4, we get $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ or $(g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2,\ g_1g_2^{-1} \in V(\Gamma_1)$ and $g_2g_1^{-1} \in V(\Gamma_2))$.

If $(\Gamma_1, g_1) = (\Gamma_2, g_2)$, then $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$. Suppose that $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$, $g_1g_2^{-1} \in V(\Gamma_1)$ and $g_2g_1^{-1} \in V(\Gamma_2)$. Clearly, $(\Gamma_1, g_1g_2^{-1}), (\Gamma_2, g_2g_1^{-1}) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. Note that

$$(\Gamma_1, g_1g_2^{-1})(\Gamma_2, g_2) = (\Gamma_1, g_1) \text{ and } (\Gamma_2, g_2g_1^{-1})(\Gamma_1, g_1) = (\Gamma_2, g_2).$$

Hence $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$. □

Theorem 3.2.12. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$.

Proof. Assume that $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ on $\mathrm{GFin}^*(\Gamma) \rtimes G$. Since $\mathrm{GFin}^*(\Gamma) \rtimes G$ is a subsemigroup of $\mathrm{GFin}(\Gamma) \rtimes G$, we have $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ on $\mathrm{GFin}(\Gamma) \rtimes G$. By Theorem 3.2.5, we have $\Gamma_1 = \Gamma_2$.

Suppose that $\Gamma_1 = \Gamma_2$. We note that $(g_2^{-1}\Gamma_1, g_2^{-1}g_1), (g_1^{-1}\Gamma_2, g_1^{-1}g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. By Lemma 2.3.5, we have $g_2^{-1}\Gamma_1$ and $g_1^{-1}\Gamma_2$ have no isolated vertex. Thus $(g_2^{-1}\Gamma_1, g_2^{-1}g_1), (g_1^{-1}\Gamma_2, g_1^{-1}g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. We see that

$$(\Gamma_2, g_2)(g_2^{-1}\Gamma_1, g_2^{-1}g_1) = (\Gamma_1, g_1) \text{ and } (\Gamma_1, g_1)(g_1^{-1}\Gamma_2, g_1^{-1}g_2) = (\Gamma_2, g_2).$$

Hence $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$. □

As an immediate consequence of the previous theorems, we get the following result.

Theorem 3.2.13. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{H}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$, $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_1$ *and* ($g_1 = g_2$ *or* $g_1g_2^{-1}, g_2g_1^{-1} \in V(\Gamma_1)$).

Theorem 3.2.14. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$ *or* (*there exists* $g \in V(\Gamma_2)$ *such that* $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$, $g_1g^{-1} \in V(\Gamma_1)$ *and* $gg_1^{-1} \in V(\Gamma_2)$).

Proof. Assume that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\mathrm{GFin}^*(\Gamma) \rtimes G$. Since $\mathrm{GFin}^*(\Gamma) \rtimes G$ is a subsemigroup of $\mathrm{GFin}(\Gamma) \rtimes G$, we have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\mathrm{GFin}(\Gamma) \rtimes G$. From Theorem 3.2.7, there exists $g \in V(\Gamma_2)$ such that $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$, $g_1g^{-1} \in V(\Gamma_1)$, $gg_1^{-1} \in V(\Gamma_2)$ and $g \in V(\Gamma_2)$.

If $\Gamma_1 = \Gamma_2$, then $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ on $\mathrm{GFin}^*(\Gamma) \rtimes G$ from Theorem 3.2.12. Since $\mathcal{R} \subseteq \mathcal{D}$, we have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\mathrm{GFin}^*(\Gamma) \rtimes G$. Suppose that there exists $g \in V(\Gamma_2)$ such that $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$, $g_1g^{-1} \in V(\Gamma_1)$ and $gg_1^{-1} \in V(\Gamma_2)$. Then $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g)$ on $\mathrm{GFin}^*(\Gamma) \rtimes G$ by Theorem 3.2.11. From Theorem 3.2.12, we have $(\Gamma_2, g)\mathcal{R}(\Gamma_2, g_2)$ on $\mathrm{GFin}^*(\Gamma) \rtimes G$. These imply that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\mathrm{GFin}^*(\Gamma) \rtimes G$. □

Theorem 3.2.15. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{GFin}^*(\Gamma) \rtimes G$ *if and only if there exists* $g \in V(\Gamma_2)$ *such that* $g\Gamma_1 \subseteq \Gamma_2$.

Proof. Assume that $(\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{GFin}^*(\Gamma) \rtimes G$. Since $\mathrm{GFin}^*(\Gamma) \rtimes G$ is a subset of $\mathrm{GFin}(\Gamma) \rtimes G$, we get that $(\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{GFin}(\Gamma) \rtimes G$. By Theorem 3.2.8, $g\Gamma_1 \subseteq \Gamma_2$ for some $g \in V(\Gamma_2)$.

Assume that there exists $g \in V(\Gamma_2)$ such that $g\Gamma_1 \subseteq \Gamma_2$. We note that $(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. By Lemma 2.3.5, we have $g_1^{-1}g^{-1}\Gamma_2$ has no isolated vertex. Then $(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. Note that

$$(\Gamma_2, g)(\Gamma_1, g_1)(g_1^{-1}g^{-1}\Gamma_2, g_1^{-1}g^{-1}g_2) = (\Gamma_2, g_2).$$

Thus $(\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{GFin}^*(\Gamma) \rtimes G$. □

Theorem 3.2.16. *$\mathcal{D}$ and $\mathcal{J}$ on $\mathrm{GFin}^*(\Gamma) \rtimes G$ are equal.*

Proof. Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$ be such that $(\Gamma_1, g_1)\mathcal{J}(\Gamma_2, g_2)$ on $\mathrm{GFin}^*(\Gamma) \rtimes G$. Since $\mathrm{GFin}^*(\Gamma) \rtimes G$ is a subsemigroup of $\mathrm{GFin}(\Gamma) \rtimes G$, we have $(\Gamma_1, g_1)\mathcal{J}(\Gamma_2, g_2)$ on $\mathrm{GFin}(\Gamma) \rtimes G$. By Theorem 3.2.9, we then have $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\mathrm{GFin}(\Gamma) \rtimes G$. We conclude that $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ on $\mathrm{GFin}^*(\Gamma) \rtimes G$ via Theorems 3.2.7 and 3.2.14. $\square$

CHAPTER IV

THE NATURAL PARTIAL ORDER

In this chapter, we characterize the natural partial order on $\text{IFin}(\Gamma) \rtimes G$, $\text{GFin}(\Gamma) \rtimes G$ and $Reg(\text{IFin}(\Gamma) \rtimes G)$.

We begin with the natural partial order on $\text{IFin}(\Gamma) \rtimes G$.

Theorem 4.1.1. *Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \text{IFin}(\Gamma) \rtimes G$. Then $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ if and only if $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ or ($\Gamma_2 \subseteq \Gamma_1$ and $g_2 = g_1 \in V(\Gamma_1)$).*

Proof. Assume that $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ and $(\Gamma_1, g_1) \neq (\Gamma_2, g_2)$. Then there exist $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \text{IFin}(\Gamma) \rtimes G$ such that $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2) = (\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_1, g_1) = (\Gamma_1, g_1)(\Gamma_4, g_4)$. Thus $(\Gamma_1, g_1) = (\Gamma_3 \cup g_3\Gamma_2, g_3g_2) = (\Gamma_2 \cup g_2\Gamma_4, g_2g_4)$ and $(\Gamma_1, g_1) = (\Gamma_1 \cup g_1\Gamma_4, g_1g_4)$. These mean that $\Gamma_1 = \Gamma_2 \cup g_2\Gamma_4$ and $g_1 = g_2g_4 = g_1g_4$. Thus $\Gamma_2 \subseteq \Gamma_1$, $g_2\Gamma_4 \subseteq \Gamma_1$ and $g_2 = g_1$. Since $1 \in V(\Gamma_4)$, we have $g_2 = g_21 \in V(g_2\Gamma_4) \subseteq V(\Gamma_1)$.

If $(\Gamma_1, g_1) = (\Gamma_2, g_2)$, then $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$. Suppose that $\Gamma_2 \subseteq \Gamma_1$ and $g_2 = g_1 \in V(\Gamma_1)$. Claim that $(g_1^{-1}\Gamma_1, 1) \in \text{IFin}(\Gamma) \rtimes G$. Since $g_1 \in V(\Gamma_1)$, we have $1 = g_1^{-1}g_1 \in V(g_1^{-1}\Gamma_1)$. Thus $(g_1^{-1}\Gamma_1, 1) \in \text{IFin}(\Gamma) \rtimes G$ by Proposition 2.3.1(4). Consider

$$(\Gamma_1, 1)(\Gamma_2, g_2) = (\Gamma_1 \cup \Gamma_2, g_2) = (\Gamma_1, g_1),$$

$$(\Gamma_2, g_2)(g_1^{-1}\Gamma_1, 1) = (\Gamma_2 \cup g_1g_1^{-1}\Gamma_1, g_1) = (\Gamma_1, g_1) \quad \text{and}$$

$$(\Gamma_1, g_1)(g_1^{-1}\Gamma_1, 1) = (\Gamma_1 \cup g_1g_1^{-1}\Gamma_1, g_1) = (\Gamma_1, g_1).$$

Hence $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$. □

Example 4.1.2. *Let G be a cyclic group of order 3 generated by g and $f : \{x\} \to G$ be a mapping defined by $xf = g$. The following diagram is a Hasse diagram of elements in $\text{IFin}(\Gamma) \rtimes G$ whose second component is g ordered by the natural partial order.*

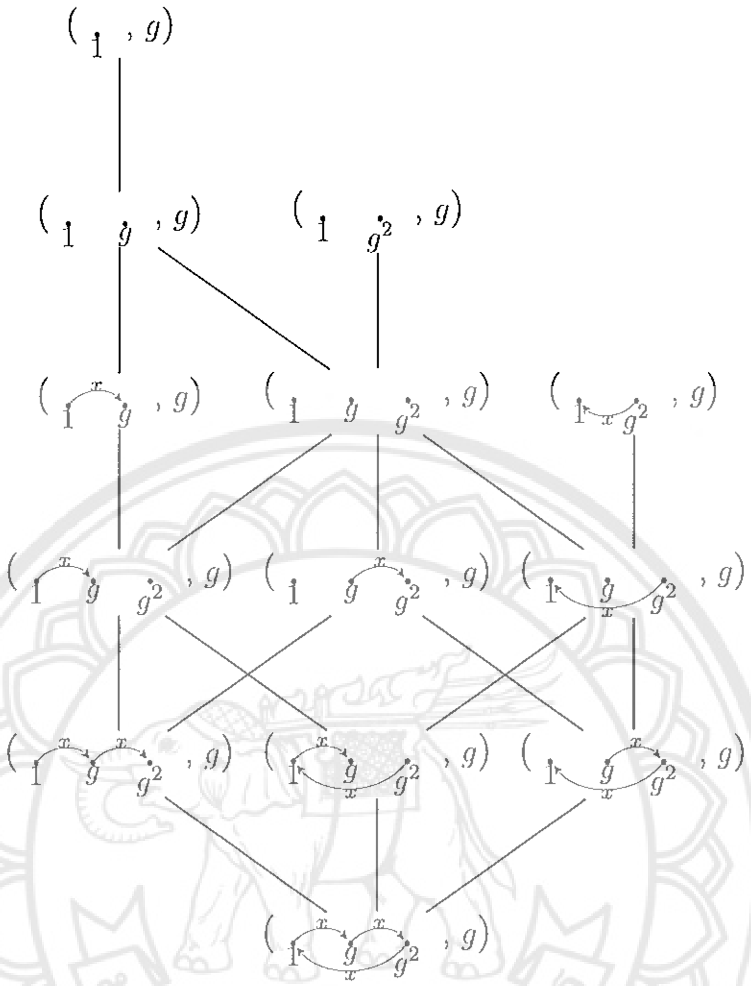

Figure 8 : Hasse diagram 1

Theorem 4.1.3. *The natural partial order on* IFin(Γ) ⋊ *G is left compatible.*

Proof. Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$ be such that $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ and $(\Gamma_1, g_1) \neq (\Gamma_2, g_2)$. By Theorem 4.1.1, we have $\Gamma_2 \subseteq \Gamma_1$ and $g_2 = g_1 \in V(\Gamma_1)$. Let $(\Gamma_3, g_3) \in \mathrm{IFin}(\Gamma) \rtimes G$. We will show that $(\Gamma_3, g_3)(\Gamma_1, g_1) \leq (\Gamma_3, g_3)(\Gamma_2, g_2)$. It suffices to prove that $(\Gamma_3 \cup g_3\Gamma_1, g_3g_1) \leq (\Gamma_3 \cup g_3\Gamma_2, g_3g_2)$. Note that $g_3\Gamma_2 \subseteq g_3\Gamma_1$ by Proposition 2.3.1(3). Thus $\Gamma_3 \cup g_3\Gamma_2 \subseteq \Gamma_3 \cup g_3\Gamma_1$ and $g_3g_1 = g_3g_2$. Since $g_1 \in V(\Gamma_1)$, we have $g_3g_1 \in V(\Gamma_3 \cup g_3\Gamma_1)$. Hence $(\Gamma_3, g_3)(\Gamma_1, g_1) \leq (\Gamma_3, g_3)(\Gamma_2, g_2)$ via Theorem 4.1.1. Therefore $\leq$ is left compatible. □

Theorem 4.1.4. *Let $g \in G$ and $\emptyset_1$ be a digraph with $V(\emptyset_1) = \{1\}$ and $E(\emptyset_1) = \emptyset$. Then the following statements are hold:*

(1) *$(\emptyset_1, g)$ is a maximal element under the natural partial order on $\mathrm{IFin}(\Gamma) \rtimes G$.*

(2) *If Γ is finite, then (Γ, g) is a minimal element under the natural partial order on $\mathrm{IFin}(\Gamma) \rtimes G$.*

(3) *If Γ is infinite, then $\mathrm{IFin}(\Gamma) \rtimes G$ has no minimal element under the natural partial order.*

Proof. (1) and (2) are obvious.

(3) Assume that Γ is infinite and let $(\Gamma', g) \in \mathrm{IFin}(\Gamma) \rtimes G$. Since $\Gamma' \neq \Gamma$, we have $V(\Gamma') \neq V(\Gamma)$ or $E(\Gamma') \neq E(\Gamma)$.

Case 1 : $V(\Gamma') \neq V(\Gamma)$. Choose $h \in V(\Gamma) \setminus V(\Gamma')$. Define Γ'' by $V(\Gamma'') = V(\Gamma') \cup \{g, h\}$ and $E(\Gamma'') = E(\Gamma')$. From $(\Gamma', g) \in \mathrm{IFin}(\Gamma) \rtimes G$, we then have $(\Gamma'', g) \in \mathrm{IFin}(\Gamma) \rtimes G$. It follows from Theorem 4.1.1 that $(\Gamma'', g) \leq (\Gamma', g)$ and clearly that $(\Gamma'', g) \neq (\Gamma', g)$.

Case 2 : $E(\Gamma') \neq E(\Gamma)$. Choose $(h, x) \in E(\Gamma) \setminus E(\Gamma')$. Define Γ'' by $V(\Gamma'') = V(\Gamma') \cup \{g, h, hxf\}$ and $E(\Gamma'') = E(\Gamma') \cup \{(h, x)\}$. From $(\Gamma', g) \in \mathrm{IFin}(\Gamma) \rtimes G$, we then have $(\Gamma'', g) \in \mathrm{IFin}(\Gamma) \rtimes G$. It follows from Theorem 4.1.1 that $(\Gamma'', g) \leq (\Gamma', g)$ and clearly that $(\Gamma'', g) \neq (\Gamma', g)$.

Hence $\mathrm{IFin}(\Gamma) \rtimes G$ has no minimal element under the natural partial order. □

Corollary 4.1.5. *Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Then $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ if and only if $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ or $(\Gamma_2 \subseteq \Gamma_1$ and $g_1 = g_2)$.*

Theorem 4.1.6. *Let $g \in G$ and $\emptyset_g$ be a digraph with $V(\emptyset_g) = \{1, g\}$ and $E(\emptyset_g) = \emptyset$. Then the following statements are hold:*

(1) *$(\emptyset_g, g)$ is a maximal element under the natural partial order on $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$.*

(2) *If* Γ *is finite, then* (Γ, g) *is a minimal element under the natural partial order on* $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$.

(3) *If* Γ *is infinite, then* $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ *has no minimal element under the natural partial order.*

Proof. (1) and (2) are obvious.

(3) Assume that Γ is infinite and let $(\Gamma', g) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. Since $\Gamma' \neq \Gamma$, we have $V(\Gamma') \neq V(\Gamma)$ or $E(\Gamma') \neq E(\Gamma)$.

Case 1 : $V(\Gamma') \neq V(\Gamma)$. Choose $h \in V(\Gamma) \setminus V(\Gamma')$. Define Γ'' by $V(\Gamma'') = V(\Gamma') \cup \{h\}$ and $E(\Gamma'') = E(\Gamma')$. From $(\Gamma', g) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$, we then have $(\Gamma'', g) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. It follows from Corollary 4.1.5 that $(\Gamma'', g) \leq (\Gamma', g)$ and clearly that $(\Gamma'', g) \neq (\Gamma', g)$.

Case 2 : $E(\Gamma') \neq E(\Gamma)$. Choose $(h, x) \in E(\Gamma) \setminus E(\Gamma')$. Define Γ'' by $V(\Gamma'') = V(\Gamma') \cup \{h, hxf\}$ and $E(\Gamma'') = E(\Gamma') \cup \{(h, x)\}$. From $(\Gamma', g) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$, we then have $(\Gamma'', g) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. It follows from Corollary 4.1.5 that $(\Gamma'', g) \leq (\Gamma', g)$ and clearly that $(\Gamma'', g) \neq (\Gamma', g)$.

Hence $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ has no minimal element under the natural partial order. □

Theorem 4.1.7. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ *if and only if* $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ *or* ($\Gamma_2 \subseteq \Gamma_1$, $g_1 = g_2$ *and* $1 \in V(\Gamma_1)$).

Proof. Assume that $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ and $(\Gamma_1, g_1) \neq (\Gamma_2, g_2)$. Then there exist $(\Gamma_3, g_3), (\Gamma_4, g_4) \in \mathrm{GFin}(\Gamma) \rtimes G$ such that $(\Gamma_1, g_1) = (\Gamma_3, g_3)(\Gamma_2, g_2) = (\Gamma_2, g_2)(\Gamma_4, g_4)$ and $(\Gamma_1, g_1) = (\Gamma_1, g_1)(\Gamma_4, g_4)$. Thus $(\Gamma_1, g_1) = (\Gamma_3 \cup g_3\Gamma_2, g_3g_2) = (\Gamma_2 \cup g_2\Gamma_4, g_2g_4)$ and $(\Gamma_1, g_1) = (\Gamma_1 \cup g_1\Gamma_4, g_1g_4)$. This means that $\Gamma_1 = \Gamma_3 \cup g_3\Gamma_2 = \Gamma_2 \cup g_2\Gamma_4$ and $g_1 = g_3g_2 = g_2g_4 = g_1g_4$. So $g_1 = g_2$ which implies $g_3 = 1$. We get that $1 = g_3 \in V(\Gamma_3) \subseteq V(\Gamma_1)$. Clearly, $\Gamma_2 \subseteq \Gamma_1$.

If $(\Gamma_1, g_1) = (\Gamma_2, g_2)$, then $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$. Suppose that $\Gamma_2 \subseteq \Gamma_1$, $g_1 = g_2$ and $1 \in V(\Gamma_1)$. Then $1 = g_2^{-1}g_2 \in V(g_2^{-1}\Gamma_1)$. Thus $(g_2^{-1}\Gamma_1, 1) \in \mathrm{GFin}(\Gamma) \rtimes G$ by Proposition 2.3.1(4). Consider

$$(\Gamma_1, 1)(\Gamma_2, g_2) = (\Gamma_1 \cup \Gamma_2, g_2) = (\Gamma_1, g_1),$$
$$(\Gamma_2, g_2)(g_2^{-1}\Gamma_1, 1) = (\Gamma_2 \cup \Gamma_1, g_1) = (\Gamma_1, g_1) \text{ and}$$
$$(\Gamma_1, g_1)(g_2^{-1}\Gamma_1, 1) = (\Gamma_1 \cup g_2 g_2^{-1}\Gamma_1, g_1) = (\Gamma_1, g_1).$$

Hence $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$. □

Theorem 4.1.8. *The natural partial order on* $\mathrm{GFin}(\Gamma) \rtimes G$ *is right compatible.*

Proof. Let $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$ be such that $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ and $(\Gamma_1, g_1) \neq (\Gamma_2, g_2)$. By Theorem 4.1.7, we have $\Gamma_2 \subseteq \Gamma_1$, $g_1 = g_2$ and $1 \in V(\Gamma_1)$. Let $(\Gamma_3, g_3) \in \mathrm{GFin}(\Gamma) \rtimes G$. We will show that $(\Gamma_1, g_1)(\Gamma_3, g_3) \leq (\Gamma_2, g_2)(\Gamma_3, g_3)$. It suffices to prove that $(\Gamma_1 \cup g_1\Gamma_3, g_1g_3) \leq (\Gamma_2 \cup g_2\Gamma_3, g_2g_3)$. Since $\Gamma_2 \subseteq \Gamma_1$, $g_1 = g_2$ and $1 \in V(\Gamma_1)$, we have $\Gamma_2 \cup g_2\Gamma_3 \subseteq \Gamma_1 \cup g_1\Gamma_3$, $g_1g_3 = g_2g_3$ and $1 \in V(\Gamma_1 \cup g_1\Gamma_3)$. Thus $(\Gamma_1, g_1)(\Gamma_3, g_3) \leq (\Gamma_2, g_2)(\Gamma_3, g_3)$ by Theorem 4.1.7. We conclude that $\leq$ is right compatible. □

Example 4.1.9. *Recall the Cayley graph* Γ *from Example 2.2.7. Consider subgraphs* Γ_1, Γ_2 *and* Γ_3 *of* Γ *defined as follow:* $V(\Gamma_1) = \{1, g, gh\}$, $E(\Gamma_1) = \{(1, x), (g, y)\}$, $V(\Gamma_2) = \{g, gh\}$, $E(\Gamma_2) = \{(g, y)\}$, $V(\Gamma_3) = \{h, gh\}$ *and* $E(\Gamma_3) = \{(gh, x)\}$. *Clearly that* $(\Gamma_1, g) \leq (\Gamma_2, g)$ *via Theorem 4.1.7. Since* $1 \notin V(\Gamma_3 \cup h\Gamma_1)$, *we obtain that* $(\Gamma_3, h)(\Gamma_1, g) \not\leq (\Gamma_3, h)(\Gamma_2, g)$. *Hence the natural partial order on* $\mathrm{GFin}(\Gamma) \rtimes G$ *is not left compatible.*

Theorem 4.1.10. *Let* $g \in G$ *and* $\emptyset_g$ *be a digraph with* $V(\emptyset_g) = \{g\}$ *and* $E(\emptyset_g) = \emptyset$. *Then the following statements are hold:*

(1) $(\emptyset_g, g)$ *is a maximal element under the natural partial order on* $\mathrm{GFin}(\Gamma) \rtimes G$.

(2) *If* Γ *is finite, then* (Γ, g) *is a minimal element under the natural partial order on* $\mathrm{GFin}(\Gamma) \rtimes G$.

(3) *If* Γ *is infinite, then* $\mathrm{GFin}(\Gamma) \rtimes G$ *has no minimal element under the natural partial order.*

Proof. (1) and (2) are obvious.

(3) Assume that Γ is infinite and let $(\Gamma', g) \in \mathrm{GFin}(\Gamma) \rtimes G$. Since $\Gamma' \neq \Gamma$, we have $V(\Gamma') \neq V(\Gamma)$ or $E(\Gamma') \neq E(\Gamma)$.

Case 1 : $V(\Gamma') \neq V(\Gamma)$. Choose $h \in V(\Gamma) \setminus V(\Gamma')$. Define Γ'' by $V(\Gamma'') = V(\Gamma') \cup \{1, h\}$ and $E(\Gamma'') = E(\Gamma')$. From $(\Gamma', g) \in \mathrm{GFin}(\Gamma) \rtimes G$, we then have $(\Gamma'', g) \in \mathrm{GFin}(\Gamma) \rtimes G$. It follows from Theorem 4.1.7 that $(\Gamma'', g) \leq (\Gamma', g)$ and clearly that $(\Gamma'', g) \neq (\Gamma', g)$.

Case 2 : $E(\Gamma') \neq E(\Gamma)$. Choose $(h, x) \in E(\Gamma) \setminus E(\Gamma')$. Define Γ'' by $V(\Gamma'') = V(\Gamma') \cup \{1, h, hxf\}$ and $E(\Gamma'') = E(\Gamma') \cup \{(h, x)\}$. From $(\Gamma', g) \in \mathrm{GFin}(\Gamma) \rtimes G$, we then have $(\Gamma'', g) \in \mathrm{GFin}(\Gamma) \rtimes G$. It follows from Theorem 4.1.7 that $(\Gamma'', g) \leq (\Gamma', g)$ and clearly that $(\Gamma'', g) \neq (\Gamma', g)$.

Hence $\mathrm{GFin}(\Gamma) \rtimes G$ has no minimal element under the natural partial order. □

Example 4.1.11. *Let* G *be a cyclic group of order* 3 *generated by* g *and* $f : \{x\} \to G$ *be a mapping defined by* $xf = g$. *The following diagram is a Hasse diagram of elements in* $\mathrm{GFin}(\Gamma) \rtimes G$ *whose second component is* g *ordered by the natural partial order.*

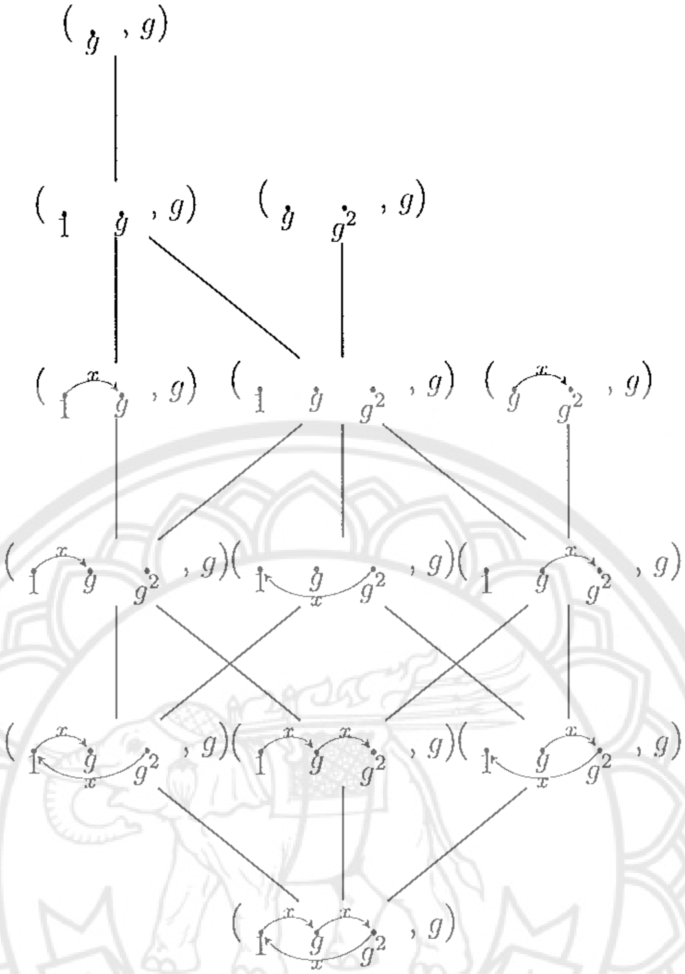

Figure 9 : Hasse diagram 2

CHAPTER V

CONCLUSION

In this thesis, we found that:

Theorem 3.1.1. *Let* $(\Gamma', g) \in \mathrm{IFin}(\Gamma) \rtimes G$. *Then* (Γ', g) *is a regular element if and only if* $g \in V(\Gamma')$.

Corollary 3.1.2. *Let* $g \in G$. *Then* $\mathrm{IFin}(\Gamma) \rtimes G$ *is not a regular semigroup if and only if* $|G| > 1$.

Theorem 3.1.3. $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ *is the maximal regular subsemigroup of* $\mathrm{IFin}(\Gamma) \rtimes G$.

Corollary 3.1.4. $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ *is an inverse semigroup and* $(\Gamma', g)^{-1} = (g^{-1}\Gamma', g^{-1})$ *for all* $(\Gamma', g) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$.

Theorem 3.1.5. *Let* $(\Gamma', g) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* (Γ', g) *is a regular element if and only if* $g \in V(\Gamma')$.

Corollary 3.1.6. *Let* $(\Gamma', g) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* $\mathrm{IFin}^*(\Gamma) \rtimes G$ *is not a regular semigroup if and only if* $|G| \geq 3$ *or* $(|G| = 2$ *and* $1 \in Xf)$.

Theorem 3.1.8. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ *if and only if* $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$.

Theorem 3.1.9. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$ *and* $(g_1 = g_2$ *or* $g_1, g_2 \in V(\Gamma_1))$.

Theorem 3.1.10. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{H}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$, $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_1$ *and* $(g_1 = g_2$ *or* $g_1, g_2 \in V(\Gamma_1))$.

Theorem 3.1.11. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ *if and only if* $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$ *or* $(g_2 \in V(\Gamma_2)$ *and* $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ *for some* $g \in V(\Gamma_2))$.

Theorem 3.1.12. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{IFin}(\Gamma) \rtimes G$ *if and only if there exists* $g \in V(\Gamma_2)$ *such that* $gg_1 \in V(\Gamma_2)$ *and* $g\Gamma_1 \subseteq \Gamma_2$.

Theorem 3.1.13. $\mathcal{D}$ *and* $\mathcal{J}$ *on* $\mathrm{IFin}(\Gamma) \rtimes G$ *are equal.*

Theorem 3.1.15. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. *Then* $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ *if and only if* $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$.

Theorem 3.1.16. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. *Then* $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$.

Theorem 3.1.17. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. *Then* $(\Gamma_1, g_1)\mathcal{H}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$ *and* $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_1$.

Theorem 3.1.18. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. *Then* $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ *if and only if* $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ *for some* $g \in V(\Gamma_2)$.

Theorem 3.1.19. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. *Then* $(\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)(\Gamma_1, g_1)Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ *if and only if there exists* $g \in V(\Gamma_2)$ *such that* $g\Gamma_1 \subseteq \Gamma_2$.

Corollary 3.1.20. $\mathcal{D}$ *and* $\mathcal{J}$ *on* $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ *are equal.*

Theorem 3.1.21. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ *if and only if* $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$.

Theorem 3.1.22. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$ *and* ($g_1 = g_2$ *or* $g_1, g_2 \in V(\Gamma_1)$).

Theorem 3.1.23. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{H}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$, $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_1$ *and* ($g_1 = g_2$ *or* $g_1, g_2 \in V(\Gamma_1)$).

Theorem 3.1.24. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ *if and only if* $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$ *or* ($g_2 \in V(\Gamma_2)$ *and* $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ *for some* $g \in V(\Gamma_2)$).

Theorem 3.1.25. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_2, g_2) \in \mathrm{IFin}^*(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{IFin}^*(\Gamma) \rtimes G$ *if and only if there exists* $g \in V(\Gamma_2)$ *such that* $gg_1 \in V(\Gamma_2)$ *and* $g\Gamma_1 \subseteq \Gamma_2$.

Theorem 3.1.26. $\mathcal{D}$ *and* $\mathcal{J}$ *on* $\mathrm{IFin}^*(\Gamma) \rtimes G$ *are equal.*

Theorem 3.2.1. *Let* $(\Gamma', g) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* (Γ', g) *is a regular element if and only if* $1 \in V(\Gamma')$.

Corollary 3.2.2. $Reg(\mathrm{GFin}(\Gamma) \rtimes G) = Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ *is the maximal regular subsemigroup of* $\mathrm{GFin}(\Gamma) \rtimes G$.

Theorem 3.2.3. *Let* $(\Gamma', g) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. *Then* (Γ', g) *is a regular element if and only if* $1 \in V(\Gamma')$.

Theorem 3.2.4. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ *if and only if* $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ *or* ($g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$, $g_1g_2^{-1} \in V(\Gamma_1)$ *and* $g_2g_1^{-1} \in V(\Gamma_2)$).

Theorem 3.2.5. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$.

Theorem 3.2.6. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{H}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$, $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_1$ *and* ($g_1 = g_2$ *or* $g_1g_2^{-1}, g_2g_1^{-1} \in V(\Gamma_1)$).

Theorem 3.2.7. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$ *or* (*there exists* $g \in V(\Gamma_2)$ *such that* $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$, $g_1g^{-1} \in V(\Gamma_1)$ *and* $gg_1^{-1} \in V(\Gamma_2)$).

Theorem 3.2.8. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{GFin}(\Gamma) \rtimes G$ *if and only if there exists* $g \in V(\Gamma_2)$ *such that* $g\Gamma_1 \subseteq \Gamma_2$.

Theorem 3.2.9. $\mathcal{D}$ *and* $\mathcal{J}$ *on* $\mathrm{GFin}(\Gamma) \rtimes G$ *are equal.*

Theorem 3.2.11. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{L}(\Gamma_2, g_2)$ *if and only if* $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ *or* ($g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_2$, $g_1g_2^{-1} \in V(\Gamma_1)$ *and* $g_2g_1^{-1} \in V(\Gamma_2)$).

Theorem 3.2.12. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{R}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$.

Theorem 3.2.13. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{H}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$, $g_1^{-1}\Gamma_1 = g_2^{-1}\Gamma_1$ *and* $(g_1 = g_2$ *or* $g_1g_2^{-1}, g_2g_1^{-1} \in V(\Gamma_1))$.

Theorem 3.2.14. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1)\mathcal{D}(\Gamma_2, g_2)$ *if and only if* $\Gamma_1 = \Gamma_2$ *or (there exists* $g \in V(\Gamma_2)$ *such that* $g_1^{-1}\Gamma_1 = g^{-1}\Gamma_2$ *and* $g_1g^{-1} \in V(\Gamma_1)$, $gg_1^{-1} \in V(\Gamma_2)$).

Theorem 3.2.15. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G$. *Then* $(\Gamma_2, g_2) \in \mathrm{GFin}^*(\Gamma) \rtimes G(\Gamma_1, g_1)\mathrm{GFin}^*(\Gamma) \rtimes G$ *if and only if there exists* $g \in V(\Gamma_2)$ *such that* $g\Gamma_1 \subseteq \Gamma_2$.

Theorem 3.2.16. $\mathcal{D}$ *and* $\mathcal{J}$ *on* $\mathrm{GFin}^*(\Gamma) \rtimes G$ *are equal.*

Theorem 4.1.1. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{IFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ *if and only if* $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ *or* $(\Gamma_2 \subseteq \Gamma_1$ *and* $g_1 = g_2 \in V(\Gamma_1))$.

Theorem 4.1.3. *The natural partial order on* $\mathrm{IFin}(\Gamma) \rtimes G$ *is left compatible.*

Theorem 4.1.4. *Let* $g \in G$ *and* $\emptyset_1$ *be a digraph with* $V(\emptyset_1) = \{1\}$ *and* $E(\emptyset_1) = \emptyset$. *Then the following statements are hold:*

(1) $(\emptyset_1, g)$ *is a maximal element under the natural partial order on* $\mathrm{IFin}(\Gamma) \rtimes G$.

(2) *If* Γ *is finite, then* (Γ, g) *is a minimal element under the natural partial order on* $\mathrm{IFin}(\Gamma) \rtimes G$.

(3) *If* Γ *is infinite, then* $\mathrm{IFin}(\Gamma) \rtimes G$ *has no minimal element under the natural partial order.*

Corollary 4.1.5. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in Reg(\mathrm{IFin}(\Gamma) \rtimes G)$. *Then* $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ *if and only if* $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ *or* $(\Gamma_2 \subseteq \Gamma_1$ *and* $g_1 = g_2)$.

Theorem 4.1.6. *Let* $g \in G$ *and* $\emptyset_g$ *be a digraph with* $V(\emptyset_g) = \{1, g\}$ *and* $E(\emptyset_g) = \emptyset$. *Then the following statements are hold:*

(1) $(\emptyset_g, g)$ *is a maximal element under the natural partial order on* $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$.

(2) *If* Γ *is finite, then* (Γ, g) *is a minimal element under the natural partial order on* $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$.

(3) *If* Γ *is infinite, then* $Reg(\mathrm{IFin}(\Gamma) \rtimes G)$ *has no minimal element under the natural partial order.*

Theorem 4.1.7. *Let* $(\Gamma_1, g_1), (\Gamma_2, g_2) \in \mathrm{GFin}(\Gamma) \rtimes G$. *Then* $(\Gamma_1, g_1) \leq (\Gamma_2, g_2)$ *if and only if* $(\Gamma_1, g_1) = (\Gamma_2, g_2)$ *or* ($\Gamma_2 \subseteq \Gamma_1$, $g_1 = g_2$ *and* $1 \in V(\Gamma_1)$).

Theorem 4.1.8. *The natural partial order on* $\mathrm{GFin}(\Gamma) \rtimes G$ *is right compatible.*

Theorem 4.1.10. *Let* $g \in G$ *and* $\emptyset_g$ *be a digraph with* $V(\emptyset_g) = \{g\}$ *and* $E(\emptyset_g) = \emptyset$. *Then the following statements are hold:*

(1) $(\emptyset_g, g)$ *is a maximal element under the natural partial order on* $\mathrm{GFin}(\Gamma) \rtimes G$.

(2) *If* Γ *is finite, then* (Γ, g) *is a minimal element under the natural partial order on* $\mathrm{GFin}(\Gamma) \rtimes G$.

(3) *If* Γ *is infinite, then* $\mathrm{GFin}(\Gamma) \rtimes G$ *has no minimal element under the natural partial order.*

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