BACKGROUND EVOLUTION FROM MODIFIED ACTIONS OF GRAVITY FROM GENERAL DISFORMAL TRANSFORMATION

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ABSTRACT

We study theory of modified gravity, namely disformal gravity, which is constructed from disformal metric. We derive the action for disformal gravity from general purely disformal transformation. Then we find the equations of motion for the background universe and find that the disformal gravity does not provide the kinetic driven for cosmic acceleration as usually expected from Galilean-like theories.

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CHAPTER I

INTRODUCTION

1.1 Introduction

In this thesis we study the modified gravity theory and their late-time cosmological consequences. We concentrate on the disformal gravity theory. We also investigate and review some of the related theories.

The modified gravity theories are still alive. It is alternative descriptions of gravity to the standard Einstein's General Relativity (GR). The latter is very beautiful theory that is based on minimal assumptions and fits perfectly to the experimental data [1]. There are still some good reasons in studying the modified gravity such as the dark sector of universe e.g. dark matters and dark energy, the cosmological constant problems , etc. Furthermore, modified gravity models with non-minimal coupling between scalar matter and gravitygains recent observational support [2, 3].

1.2 Cosmology from General Relativity

In General Relativity(GR) the dynamics of the gravitational field $g_{\mu\nu}(x)$ in the presence of matter-energy contents are governed by the **Einstein's field equation**(EFE)(See appendix C)

$$R_{\mu\nu} - (1/2)R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}. \tag{1.1}$$

This equation can be derived from applying Hamilton principle to the **Einstein-Hilbert action** plus the matter action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - 2\Lambda) + S_m. \tag{1.2}$$

The **cosmological constant**, Λ, is typically omitted from equations before the discovery of the cosmic speed-up in 1998. The cosmological considerations based on GR can be started by **Friedmann-Lemaître-Robertson-Walker** (FLRW) **metric**

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + dr^2 d\Omega\right] \quad , \tag{1.3}$$

where $k = +1, -1, 0$ for closed, open, and flat universe, respectively, $d\Omega \equiv d\theta^2 + \sin^2 d\varphi^2$, t is the cosmic time. The **scale factor**, $a(t)$ is an only dynamical variable in this metric. By direct calculations we then obtain the **Einstein tensor**, $G_{\mu\nu}$,(C.14)

$$G_{00} = 3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) \quad , \tag{1.4}$$

$$G_{ij} = -g_{ij}\left(\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) \quad . \tag{1.5}$$

By using EFE (1.1) with the matter described by the perfect fluid e.g. the energy-momentum tensor

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 \\ 0 & g_{ij}p \end{bmatrix}, \tag{1.6}$$

where ρ and p are the energy density and pressure of the matter. The energy-momentum tensor obeys the covariant conservation law

$$\nabla_\mu T^{\mu\nu} = 0 \,. \tag{1.7}$$

The equations (1.4),(1.5) and (1.7) respectively give us the three fundamental equations of cosmology

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \quad , \tag{1.8}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \quad , \tag{1.9}$$

$$\dot{\rho} + 3H(\rho + p) = 0 \quad , \tag{1.10}$$

They are the Friedmann equation, Raychaudhuri equation and conservation equation, respectively. We will always set $k = 0$. This is according to observations and inflationary model, and $H := \frac{\dot{a}}{a} \equiv \frac{da/dt}{a}$. There are only two independent equations from the three. In order to solved the three unknowns $a(t), \rho(t)$, and $p(t)$ we need another independent equation namely the **equation of state**

$$p = w\rho \quad , \tag{1.11}$$

where w is called the **equation of state parameter**. It has different value for different type of matter content. For dust or non-relativistic matter their pressure is very small compared to their energy density (ρ) so we have $w = 0$. For relativistic particles, it move with velocity (v) closed to the speed of light ($v \sim 1$) or radiation we can think of them as electromagnetic field so describe by the energy-momentum tensor

$$T^{\rm EM}_{\mu\nu} = g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \tag{1.12}$$

which is traceless, $g^{\mu\nu}T^{\rm EM}_{\mu\nu} = 0$ (we use the unit $\varepsilon_0 = 1$). From the traceless property of energy-momentum tensor for radiations if we cast it as the perfect fluid this condition reads $-\rho + 3p = 0$, so for radiations $w = 1/3$.

1.3 Dark Energy

The present standard model for the universe is ΛCDM model. In this model the universe contains a cosmological constant (associated with dark energy), and cold dark matter. To describe the acceleration expansion the present epoch of the universe must be dominated by the cosmological constant [4, 5], by equation (1.9) the universe will expand with acceleration at late time as required by observational data . The another view is that Λ is an effective quantity for describing accelerated universe. The underlying quantity describes this phenomenon is another form of matter content called the **dark energy**. The dynamics of the universe that describes by equation(1.9) in this setting becomes

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_{de}(1+3w) > 0. \tag{1.13}$$

Therefore, the dark energy can be described by perfect fluid with $w < \frac{-1}{3}$ (not need to be a constant). The special case when $w \equiv -1$ is corresponds for Λ, because from equations(1.8)and (1.9) we can deduce that $\rho_\Lambda = \frac{\Lambda}{8\pi G} = -p_\Lambda$, We can also see another fact that dark energy has negative pressure contrary to the ordinary (and dark) matter and radiation(ρ is always > 0).

Describing cosmic acceleration with Λ has some issues mainly the observed and theoretical calculated value is very different in this scenario ($\sim$ 120 order of magnitude)[6]. This is the **cosmological constant problem**. Another issue with the constant Λ is the **coincidence problem**, why we live in a special time which the energy density of matter and dark energy are in the same order of magnitude ($\sim \mathcal{O}(1)$). This is because the very small value of Λ. Since ρ_Λ is constant and time-independent so at the very early time of the universe this is unnaturally very small compare to the density of the other type of matter.

To this end the dynamical dark energy models (for example quintessence, three-form etc) has been proposed to try to solve these problems. The dark energy is described by the scalar field , ϕ , adding to the model by modifying the action (1.2)

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R - (1/2)\nabla_\mu\phi\nabla^\mu\phi - V(\phi)) + S_m. \tag{1.14}$$

with equation of state

$$\begin{aligned} w = p_\phi/\rho_\phi &= \frac{X - V(\phi)}{X + V(\phi)} \\ &= \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \end{aligned} \tag{1.15}$$

where $X \equiv -\partial_\alpha\phi\partial^\alpha\phi/2$. The observed quantities predicted by this model are very sensitive with a potentials and many forms of the potential have been studied. Many of them are good for describing dynamics of the universe in inflationary phase and late time acceleration. But they still have internal problems such as fine-tuning and coincidence problem. So far we do not have completely consistent theory of this kind.

Instead of using the canonical form of the Lagrangian for the scalar field, one may consider the general form

$$S = \frac{m_{Pl}^2}{2}\int d^4x\sqrt{-g}\,R + \int d^4x\sqrt{-g}(P(\phi, X) + \mathcal{L}_m(g_{\alpha\beta}, \psi))\,, \tag{1.16}$$

where m_{Pl} is a reduced Planck mass (A.4). This model is called the **k-essence** and may give the solution to the coincidence problem [7].

1.4 Modified Gravity & Scalar-Tensor Theories

From the previous section the another way to describe the cosmic acceleration is modifying Einstein's general relativity. The main paradigm is that GR may be the only an approximately correct limit of the more fundamental theory.

Studying modified gravity theories may be leads to more natural explanations for the early universe, the cosmic acceleration, dark sector of the universe, and may be suit for setting up to quantum theory of gravity.

In GR the action of the theory is the Einstein-Hilbert with cosmological constant plus the matter action

$$S_{GR} = S_{\text{E-H}} + S_m = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(R - 2\Lambda) + S_m. \tag{1.17}$$

By varying an action with respect to the metric field($\delta g^{\mu\nu}$) and by using Hamilton principle($\delta S_{GR} = 0$) lead us to the Einstein field equation(EFE)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.18}$$

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= 8\pi G(T_{\mu\nu} + \frac{\Lambda g_{\mu\nu}}{8\pi G}), \\ &=: 8\pi G(T_{\mu\nu} + T^{\Lambda}_{\mu\nu}), \end{aligned} \tag{1.19}$$

where $T_{\mu\nu} := \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$. The Einstein-Hilbert action is not the only one that can describe the geometric and kinematic part according to the available experimental data. It is the simplest one. For more general **$f(R)$-gravity** the action can be expressed as

$$S_f = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}f(R) + S_m. \tag{1.20}$$

The variation of this action yields the equation of motion

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\nabla_\sigma f'(R) - \nabla_\mu\nabla_\nu f'(R) = 8\pi G T_{\mu\nu}, \tag{1.21}$$

which equivalent to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\left(\frac{T_{\mu\nu}}{f'(R)} - \frac{1}{f'(R)}\left[\frac{1}{2}g_{\mu\nu}\left(Rf'(R) - f(R)\right) + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f'(R)\right]\right)$$
$$=: 8\pi G(\tilde{T}_{\mu\nu} + T^{\text{curv}}_{\mu\nu}). \tag{1.22}$$

The equation (1.22) gives the correct limit to EFE(1.18)(with $\Lambda = 0$) when $f(R) \longrightarrow R$, and we call $T^{\text{curv}}_{\mu\nu}$ the **curvature fluid energy-momentum tensor**. Hence, by generalizing the Einstein-Hilbert action we obtain the field's equation that can be recasts as Einstein-Hilbert action plus dark-energy-like fluid. This type of modified gravity is called $f(R)$-**gravity** .

Instead of generalized the form of an action, we can modified Einstein's gravity by adding special degrees of freedom . The simplest one is a **scalar field** and the simplest model of this kind is the **Brans-Dicke theory**

$$\begin{aligned} S_{BD} &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\nabla^\mu\phi\nabla_\mu\phi - V(\phi)\right] + S_m \\ &= \int d^4x\sqrt{-g}\left[\mathcal{L}_{BD} + \mathcal{L}_m\right], \end{aligned} \tag{1.23}$$

where ω is the free parameter of theory. We can drop out G and adopt $1/\phi$ as a varying gravitational constant. This action is in Jordan frame (the frame which the energy-momentum tensor covariantly conserved, $\nabla_\mu T^{\mu\nu} = 0$, so, the particles follow the geodesics. In this frame the Ricci scalar in the Lagrangian density can be multiplied by some function of the degrees of freedom. In contrary the Einstein frame is the frame which the action is linear in Ricci scalar). In Jordan frame we can view a scalar field as a field coupling with gravity not the matter fields. Transforming this action to Einstein frame can be done by a suitable conformal transformation(precisely, Weyl transformation, for this case

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \phi g_{\mu\nu}. \tag{1.24}$$

The formula for conformal transformations between the metric can transform the action in eq. (1.23) to Einstein's frame

$$S_{BD} = \int d^4x\sqrt{-\tilde{g}}\left(\frac{\tilde{R}}{16\pi G} - \frac{1}{2}\tilde{\nabla}_\mu\varphi\tilde{\nabla}^\mu\varphi - U(\varphi) + \tilde{\mathcal{L}}_m\right) \quad , \tag{1.25}$$

where $\varphi = \sqrt{\frac{2\omega+3}{16\pi G}}\ln\phi$. We can interpret this result that Brans-Dicke theory with un-coupled matter is equivalent to GR with a scalar field coupling to the matter (a coupling term is in $\tilde{\mathcal{L}}_m$) modulo some suitable conformal mapping of the metric.

Furthermore, It can be shown that $f(R)$-gravity conformally equivalent to

Brans-Dicke theory , and so GR coupling minimally with a scalar field by suitable conformal mapping and redefinition of a scalar field.

These are the prototype of modified gravity theories. Many forms of the functions f have been studied. Nevertheless non of them is physically or mathematically complete, mainly, because the field's equation(1.21) is fourth order in the metric, this leading to the **Ostrogradsky instability**. This instability occurs in the model which contains the terms with more than second order in time derivatives of the degrees of freedom (in this case $g_{\mu\nu}$). Indeed, by **Lovelock theorem** in 4-dimensional space-time the Einstein-Hilbert lagrangian of GR ($f(R) = R$) is the only a non-degenerated [1] one that can gives equation of motion with the order less than or equal to 2. To avoid the Ostrogradsky instability the Lagrangian of $f(R)$ gravity must be degenerated one.

As we have shown by examples above, the conformal transformation is importants in scalar-tensor theory. Jacob Bekenstein[13] suggests that the most general mapping between the metric involving one scalar field and preserve diffeomorphisms is the **disformal transformations**

$$g_{\mu\nu} \longmapsto C(\phi, X)g_{\mu\nu} + D(\phi, X)\phi_\mu\phi_\nu \,, \tag{1.26}$$

where $\phi_\mu \equiv \phi_{,\mu} = \frac{\partial\phi}{\partial x^\mu}$, $X \equiv -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$. The functions $C(\phi, X)$ and$D(\phi, X)$ are the arbitrary functions of ϕ and X. Study disformal related theories might gives us insights into gravitational theory. We will discuss this topic in Chapter 3

1.5 Outline & Motivations of this Thesis

The most general scalar-tensor theory which provides equations of motion up to second order and hence free from Ostrogradsky instability is Horndeski theory. In the modern approach, the Horndeski theory can be viewed as the generalized Galileon. By generalizing the Galileon theory to the curved space-time we obtain the equivalent theory of Horndeski theory which is the most general scalar-tensor theory with one scalar field in four dimensions which provides the second-order equations of motion.

The Galileon theory[15, 16] is *the most general theory of one scalar field* that provides at most second order in the equations of motion[14, 12]. The generalization of such model to the curved spacetime leads to the generalized galileon or the **Horndeski theory** [10]. Recently, It has been found that there are a class of theories of extend Horndeski which is a larger class of scalar-tensor theory which the equations of motion possess higher-order derivatives but still free from Ostro-

[1]The non-degenerated (or non-singular) Lagrangian means the Hessian matrix, W_{ab} (For example, in a case of $\mathcal{L} \equiv \mathcal{L}[\phi]$ we have $W_{ab} \equiv \frac{\partial^2\mathcal{L}}{\partial(\partial_0\phi^a)\partial(\partial_0\phi^b)} = \frac{\partial\pi_b}{\partial(\partial_0\phi^a)}$) is invertible, or $\det W_{ab} \neq 0$. For more details see [8, 9]

gradski instability.This follows from Hamiltonian analysis and counting degrees of freedom (dof)[52, 34, 35]. This so-called the **beyond Horndeski theory** or **GLPV theory**[11, 23, 24] or the doubly generalized Galileon(G^3) has the same number of dof of the original Horndeski theory (3 dof). The even further generalization called XG^3 theory and spatially covariant theory of gravity are also exist [19].

These generalized class of the scalar-tensor theories up to XG^3 are related by the generalized version of the conformal transformation called the **disformal transformation**[13]. The class of Horndeski theories is closed under $g_{\mu\nu} \to C(\phi)g_{\mu\nu} + D(\phi)\phi_\mu\phi_\nu$. The class of GLPV theories is closed under $g_{\mu\nu} \to C(\phi)g_{\mu\nu}+D(\phi, X)\phi_\mu\phi_\nu$. The class of XG^3 theories also closed under such transformation in the unitary gauge[19]. For the class of spatially covariant theory of gravity it is still unknown.

However, the study in [22] has shown that the Generalization of Horndeski theory by the transformations $g_{\mu\nu} \to C(\phi, X)g_{\mu\nu}$ or $g_{\mu\nu} \longmapsto C(\phi, X)g_{\mu\nu} + D(\phi, X)\phi_\mu\phi_\nu$ can also provide the second order equations of motion in the proper way. So the result theories are free from Ostrogradski instability. But it is still not clear that these transformations will lead us to more general theory than those we have discussed above or how to relate them together. The meaning of these ghost-freeness in general context are also not clear and the further investigations is still needed.

In our work we discuss the disformal transformation in the form

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + D(\phi, X)\phi_\mu\phi_\nu. \tag{1.27}$$

By apply this transformation to the gravity part of the action (1.16) we obtain (3.32)

$$\begin{aligned}\mathcal{L}_{\text{disf}} &= G_2(\phi, X) + G_4(\phi, X)R + G_{4,X}((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}) \\ &\quad + \gamma D_{,X}X((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}) + 2X\nabla^\mu(\gamma\nabla_\mu D) + \phi_\mu\phi^\nu\nabla^\mu(\gamma\nabla_\nu D) \quad .\end{aligned} \tag{1.28}$$

Since the GLPV theory is closed under this transformation then the above action still belongs to the GLPV class. This ensures us that our result action not propagates any ghost degrees of freedom . But we still need to check it by explicitly calculation and match the extra terms in the second line of the above action to the beyond Horndeski terms in the GLPV action. In the worse case, if these terms is beyond GLPV we need to check that they are still belong to the XG^3 class. We will analyze these extra terms as far as possible to understand them exactly.

Next, we will study the evolution of background universe to see whether or not the purely kinetic part of the disformal scalar field can driven the cosmic acceleration as one might expected from the Galileon-like theories. This investigation will be done in chapter 4

CHAPTER II

HORNDESKI, AND GLPV

2.1 Introduction

The accelerating expansion of the universe in the early time and today stimulates the study of modified gravity with contains more degrees of freedom additional to the standard general relativity. By investigated such theories lead us to further general models, which have their own interesting. In theoretical side, these studies can be done in their own right as the investigations on the structure of the scalar-tensor and the related theories.

2.2 Galileon

Galileon field theory inspired by the decoupling limit of DGP theory can provide cosmic acceleration. This is because in such limit the theory has the galileon symmetry. The galileon field theory is the most general theory of a scalar field in flat spacetime that contains such symmetry [15, 16].

The Lagrangian is generally given by

$$\mathcal{L}_{(n)} = \mathcal{T}^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_n} \phi^{\nu_1}_{\mu_1} \cdots \phi^{\nu_n}_{\mu_n} \quad , \tag{2.1}$$

where $\mathcal{T} \equiv \mathcal{T}(\phi, \partial\phi)$ and $\mathcal{T}^{\mu_1 \dots \mu_n \nu_1 \dots \nu_n} = \mathcal{T}^{([\mu_1 \dots \mu_n],[\nu_1 \dots \nu_n])}$ (completely anti-symmetric in $\{\mu_1 \dots \mu_n\}$ and $\{\nu_1 \dots \nu_n\}$ and symmetric under $\mu_i \Longleftrightarrow \nu_i$) . The action is invariant under the **Galileon symmetry**

$$\phi_\mu \to \phi_\mu + b_\mu, \; \phi \to \phi + c \tag{2.2}$$

in the curved space-time which is the Generalization from the Galileon symmetry in flat Minkowskian space-time

$$\phi \longrightarrow \phi + b_\mu x^\mu + c \quad . \tag{2.3}$$

The first example of the Galileon theory in flat space-time is given by the Lagrangian[17]

$$\mathcal{L}^{\mathrm{Gal},1}_N = \frac{1}{(D-n-1)!} \epsilon^{\mu_1 \dots \mu_{n+1} \sigma_1 \dots \sigma_{D-n-1}} \epsilon_{\nu_1 \dots \nu_{n+1} \sigma_1 \dots \sigma_{D-n-1}} \phi^{\nu_{n+1}} \phi_{\mu_{n+1}} \phi^{\nu_1}_{\mu_1} \dots \phi^{\nu_n}_{\mu_n}, \tag{2.4}$$

where N is a number of the scalar fields in the action. In any particular dimension, the maximum possible values of n is restricted by $n_{\max} + 1 = D =$ the number of the indices of the **Levi-Civita tensor**, ϵ and $N = n + 2$. For example, in four-dimensional space-time the possible value of n are $0, 1, 2, 3$ ($N = 2, 3, 4, 5$) and the possible Lagrangians can be written as

$$\mathcal{L}^{\mathrm{Gal},1}_2 = \frac{1}{3!} \epsilon^{\mu_1 \delta_1 \delta_2 \delta_3} \epsilon_{\nu_1 \delta_1 \delta_2 \delta_3} \phi^{\nu_1} \phi_{\mu_1} \quad , \tag{2.5}$$

$$\mathcal{L}_3^{\mathrm{Gal},1} = \frac{1}{2!}\epsilon^{\mu_1\mu_2\delta_1\delta_2}\epsilon_{\nu_1\nu_2\delta_1\delta_2}\phi^{\nu_2}\phi_{\mu_2}\phi^{\nu_1}_{\mu_1} \quad , \tag{2.6}$$

$$\mathcal{L}_4^{\mathrm{Gal},1} = \frac{1}{1!}\epsilon^{\mu_1\mu_2\mu_3\delta}\epsilon_{\nu_1\nu_2\nu_3\delta}\phi^{\nu_3}\phi_{\mu_3}\phi^{\nu_1}_{\mu_1}\phi^{\nu_2}_{\mu_2} \quad , \tag{2.7}$$

$$\mathcal{L}_5^{\mathrm{Gal},1} = \frac{1}{0!}\epsilon^{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\nu_1\nu_2\nu_3\nu_4}\phi^{\nu_4}\phi_{\mu_4}\phi^{\nu_1}_{\mu_1}\phi^{\nu_2}_{\mu_2}\phi^{\nu_3}_{\mu_3} \quad . \tag{2.8}$$

We can obtain the equations of motion from the Lagrangians $\mathcal{L} \equiv \mathcal{L}(\phi, \partial\phi, \partial\partial\phi)$ by Euler-Lagrange equations

$$\mathcal{E}_N \equiv \frac{\partial \mathcal{L}_N}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_N}{\partial \phi_\mu} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}_N}{\partial \phi_{\mu\nu}} = 0 \quad . \tag{2.9}$$

This gives us the equations of motion

$$\mathcal{E}_N = Nn!\phi^{[\mu_1}_{\mu_1} \dots \phi^{\mu_{N-1}]}_{\mu_{N-1}} = 0 \quad , \tag{2.10}$$

explicitly,

$$\mathcal{E}_2 = 2\Box\phi, \tag{2.11}$$

$$\mathcal{E}_3 = 3(\Box\phi^2 - \phi^2_{\mu\nu}), \tag{2.12}$$

$$\mathcal{E}_4 = 4(\Box\phi^3 - 3\Box\phi\phi^2_{\mu\nu} + 2\phi^3_{\mu\nu}), \tag{2.13}$$

$$\mathcal{E}_5 = 5(\Box\phi^4 - 6\Box\phi^2\phi^2_{\mu\nu} + 3\phi^2_{\mu\nu}\phi^2_{\alpha\beta} + 8\Box\phi\phi^3_{\mu\nu} - 6\phi^4_{\mu\nu}). \tag{2.14}$$

Thus we obtain the second order equations of motion. To generalize the flat space-time galileon to the curved space-time we will use the Generalization $\eta_{\mu\nu} \to g_{\mu\nu}$ and $\partial_\mu \to \nabla_\mu$ to the Lagrangians and the Euler-Lagrange equations. From the Lagrangian $\mathcal{L}^{\mathrm{Gal},1}$ in four dimensional curved space-time we can calculate the equations of motion by using the Euler-Lagrange equations. For example, starting with $\mathcal{L}_4^{\mathrm{Gal},1}$, we can calculate

$$\frac{\partial \mathcal{L}_4}{\partial \phi} = 0 \quad , \frac{\partial \mathcal{L}_4}{\partial \phi_\gamma} = -2\delta^{\mu_1\dots\mu_3}_{\mu_1\dots\mu_3}\delta^\gamma_{\mu_3}\phi^{\nu_3}_\gamma\phi^{\nu_1}_{\mu_1}\phi^{\nu_2}_{\mu_2}, \tag{2.15}$$

$$\Rightarrow \nabla_\gamma(\frac{\partial \mathcal{L}_4}{\partial \phi_\gamma}) = -2\delta^{\mu_1\dots\mu_3}_{\mu_1\dots\mu_3}\phi^{\nu_3}_{\mu_3}\phi^{\nu_1}_{\mu_1}\phi^{\nu_2}_{\mu_2} - 4\delta^{\mu_1\dots\mu_3}_{\mu_1\dots\mu_3}\phi^{\nu_3}\phi_{\mu_1}{}^{\nu_1}{}_{\mu_3}\phi^{\nu_2}_{\mu_2}, \tag{2.16}$$

$$\frac{\partial \mathcal{L}_4}{\partial \phi^\lambda_\gamma} = -2\delta^{\mu_1\dots\mu_3}_{\mu_1\dots\mu_3}\phi^{\nu_3}\phi_{\mu_3}\delta^{\nu_1}_\lambda\delta^\gamma_{\mu_1}\phi^{\nu_2}_{\mu_2}, \tag{2.17}$$

$$\begin{aligned}\Rightarrow \nabla_\gamma\nabla^\lambda(\frac{\partial \mathcal{L}_4}{\partial \phi^\lambda_\gamma}) = & -2\delta^{\mu_1\dots\mu_3}_{\mu_1\dots\mu_3}[2\phi^{\nu_3\lambda}{}_\gamma\phi_{\mu_3}\delta^{\nu_1}_{(\lambda}\delta^{\gamma)}_{\mu_1}\phi^{\nu_1}_{\mu_1} + 2\phi^{\nu_3\lambda}\phi_{\mu_3\gamma}\delta^{\nu_1}_{(\lambda}\delta^{\gamma)}_{\mu_1}\phi^{\nu_1}_{\mu_1} \\ & +2\phi^{\nu_3\lambda}\phi_{\mu_3}\delta^{\nu_1}_{(\lambda}\delta^{\gamma)}_{\mu_1}\phi^{\nu_1}_{\mu_1\gamma} + \phi^{\nu_3}{}_\gamma\phi_{\mu_3}\delta^{\nu_1}_{(\lambda}\delta^{\gamma)}_{\mu_1}\phi^{\nu_1\lambda}_{\mu_1} \\ & +\phi^{\nu_3}\phi_{\mu_3\gamma}\delta^{\nu_1}_{(\lambda}\delta^{\gamma)}_{\mu_1}\phi^{\nu_1\lambda}_{\mu_1} + \phi^{\nu_3}\phi_{\mu_3}\delta^{\nu_1}_{(\lambda}\delta^{\gamma)}_{\mu_1}\phi^{\nu_1\lambda}_{\mu_1}{}_\gamma] \quad .\end{aligned} \tag{2.18}$$

Hence

$$\mathcal{E}_4 \equiv \frac{\partial \mathcal{L}_4}{\partial \phi} - \nabla_\gamma(\frac{\partial \mathcal{L}_4}{\partial \phi_\gamma}) + \nabla_\gamma\nabla^\lambda(\frac{\partial \mathcal{L}_4}{\partial \phi^\lambda_\gamma}), \tag{2.19}$$

$$= 4\phi^{[\mu_1}_{\mu_1}\phi^{\mu_2}_{\mu_2}\phi^{\mu_3]}_{\mu_3} + 10\phi^{[\mu_3}\phi_{\mu_1}{}^{\mu_1}{}_{\mu_3}\phi^{\mu_2]}_{\mu_2} - 2\phi^{[\mu_3}\phi_{\mu_3}\phi^{\mu_2}{}_{\mu_2}{}^{\mu_1]}{}_{\mu_1}, \tag{2.20}$$

$$= 4((\Box\phi^3 - 3\Box\phi\phi^2_{\mu\nu} + 2\phi^3_{\mu\nu}) - \frac{5}{2}R_{\mu\nu}\phi^\mu\phi^\nu\Box\phi + 2R_{\mu\nu\alpha\beta}\phi^\nu\phi^\beta\phi^{\mu\alpha}$$
$$+2R_{\mu\nu}\phi^\mu\phi^{\nu\beta}\phi_\beta + \frac{1}{2}R_{\mu\nu}\phi^{\mu\nu}\phi^\beta\phi_\beta + \frac{1}{4}\nabla^\mu R\phi_\mu\phi_\nu\phi^\nu$$
$$-\frac{1}{2}\nabla_\alpha R_{\mu\nu}\phi^\alpha\phi^\mu\phi^\nu) \quad = 0, \tag{2.21}$$

where $\lceil ... \rfloor \equiv n![...]$. In this case the equations of motion is third order in the metric field due to the covariant differentiation of the Ricci tensor and scalar (the last two terms). To remove these third order terms one may try to integrating by part these third order terms

$$\mathcal{E}_4^{III} \equiv \nabla^\mu R\phi_\mu\phi_\nu\phi^\nu - 2\nabla_\alpha R_{\mu\nu}\phi^\alpha\phi^\mu\phi^\nu,$$
$$= \overbrace{-2\nabla_\nu(G_{\alpha\beta}\phi^\alpha\phi^\beta\phi^\mu)}^{\text{the boundary term}}\overbrace{-R(\Box\phi\phi_\nu\phi^\nu + 2\phi_\mu\phi^{\mu\nu}\phi_\nu) + 2R_{\mu\nu}(2\phi^\mu_\alpha\phi^\nu\phi^\alpha + \Box\phi\phi^\mu\phi^\nu)}^{\text{the second order part}}. \tag{2.22}$$

If we do not want to ignore the boundary term we must add the extra term to the original action to cancele this boundary part of the equations of motion (the same strategy of adding the Gibbons–Hawking–York boundary term to Einstein-Hilbert action). After calculating the equations of motion this new extra term in the action must yields the quantity

$$+2\nabla_\nu(G_{\alpha\beta}\phi^\alpha\phi^\beta\phi^\mu). \tag{2.23}$$

So we can see that

$$-\nabla_\mu(\frac{\partial\mathcal{L}_4^{\text{extra}}}{\partial\phi_\mu}) = +2\nabla_\mu(G_{\alpha\beta}\phi^\alpha\phi^\beta\phi^\mu). \tag{2.24}$$

This implies

$$\mathcal{L}_4^{\text{extra}} = -G_{\alpha\beta}\phi^\alpha\phi^\beta\phi^\mu\phi_\mu \quad . \tag{2.25}$$

Hence the covariantization version of $\mathcal{L}_4^{\text{Gal},1}$ is given by

$$\tilde{S}_4^{\text{Gal},1} = \int d^4x\sqrt{-g}\tilde{\mathcal{L}}_4^{\text{Gal},1} = \int d^4x\sqrt{-g}\left(\epsilon^{\mu_1\mu_2\mu_3\delta}\epsilon_{\nu_1\nu_2\nu_3\delta}\phi^{\nu_3}\phi_{\mu_3}\phi^{\nu_1}_{\mu_1}\phi^{\nu_2}_{\mu_2} - G_{\alpha\beta}\phi^\alpha\phi^\beta\phi^\mu\phi_\mu\right), \tag{2.26}$$

and the final second order equations of motion of $\tilde{\mathcal{L}}_4^{\text{Gal},1}$ then reads (2.21),(2.22)

$$\mathcal{E}_4' = 4\{(\Box\phi^3 - 3\Box\phi\phi^2_{\mu\nu} + 2\phi^3_{\mu\nu}) - 2R_{\mu\nu}\phi^\mu\phi^\nu\Box\phi + 3R_{\mu\nu\alpha\beta}\phi^\nu\phi^\beta\phi^{\mu\alpha}$$
$$+2R_{\mu\nu}\phi^\mu\phi^{\nu\beta}\phi_\beta + \frac{1}{2}R_{\mu\nu}\phi^{\mu\nu}\phi^\beta\phi_\beta - \frac{1}{4}R(\Box\phi\phi_\nu\phi^\nu + 2\phi_\mu\phi^{\mu\nu}\phi_\nu)\}$$
$$= 0 \quad . \tag{2.27}$$

The method of adding the suitable counter term to the action can be considered in more general setting[11, 12, 14]. Such method leads us to the conclusion that the covariant Galilean is equivalent to the Horndeski theory.

2.3 Horndeski

According to covariantization of Galileon theory considered in the previous section, we then obtain the Horndeski action then consists of the following action

$$S^H = \int d^4x\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i\,, \tag{2.28}$$

where

$$\mathcal{L}_2^H = G_2(\phi, Y), \tag{2.29}$$
$$\mathcal{L}_3^H = G_3(\phi, Y)\Box\phi, \tag{2.30}$$
$$\mathcal{L}_4^H = G_4(\phi, Y)R - 2G_{4Y}(\phi, Y)(\Box\phi^2 - \phi_{\mu\nu}^2), \tag{2.31}$$
$$\mathcal{L}_5^H = G_5(\phi, Y)G_{\mu\nu}\phi^{\mu\nu} + (1/3)G_{5Y}(\phi, Y)(\Box\phi^3 - 3\Box\phi\phi_{\mu\nu}^2 + 2(\phi_{\mu\nu})^3) \tag{2.32}$$

Note that theses Lagrangians are in the form of the coefficient functions of (ϕ, Y) multiply with the second derivative of the scalar field, $(\nabla\nabla\phi)$, and/or multiply with the general covariant quantities $G_{\mu\nu}, R$. In anti-symmetric fashion, they are

$$\mathcal{L}_2^H = G_2(\phi, Y), \tag{2.33}$$
$$\mathcal{L}_3^H = G_3(\phi, Y)\delta_\nu^\mu\phi_\mu^\nu, \tag{2.34}$$
$$\mathcal{L}_4^H = G_4(\phi, Y)(1/2)\delta_{\alpha\beta}^{\mu\nu}R_{\mu\nu}^{\alpha\beta} - 2G_{4Y}(\phi, Y)\delta_{\alpha\beta}^{\mu\nu}\phi_\mu^\alpha\phi_\nu^\beta, \tag{2.35}$$
$$\mathcal{L}_5^H = G_5(\phi, Y)(-1/4)\delta_{\alpha\beta\rho}^{\mu\nu\sigma}R_{\mu\nu}^{\alpha\beta}\phi_\sigma^\rho + (1/3)G_{5Y}(\phi, Y)\delta_{\alpha\beta\rho}^{\mu\nu\sigma}\phi_\mu^\alpha\phi_\nu^\beta\phi_\sigma^\rho, \tag{2.36}$$

where $\delta_{\nu_1\ldots\nu_n}^{\mu_1\ldots\mu_n}$ is a generalized Kronecker delta (A.62). This form is more compact than the previous form since we use only $\epsilon^{\mu\nu\rho\sigma}, \epsilon_{\mu\nu\rho\sigma}, R_{\mu\nu}^{\alpha\beta}, \phi_\nu^\mu$, in the construction. This action is equivalents to many gravitational theories [39] depends on the functions $G_2(\phi, Y), G_3(\phi, Y), G_4(\phi, Y)$, and $G_5(\phi, Y)$.

2.3.1 Horndeski theory in ADM formalism

The Lagrangian density of Horndeski theory in ADM variables and in the *unitary gauge* fixing condition can be constructed by setting

$$\phi = \phi(t), \quad \text{and choosing } n_\mu = -\gamma\nabla_\mu\phi, \text{ where } \gamma = \frac{1}{\sqrt{-Y}} \tag{2.37}$$

In ADM formalism, we foliate the spacetime continuum$(\mathcal{M})$ to the *equal-time space-like hypersurfaces* (Σ_t) that change with time parameter. To do so, we write the line element as

$$ds^2 = (-N^2 + N^a N_a)dt^2 + q_{ab}dx^a dx^b, \tag{2.38}$$
$$= -N^2dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt). \tag{2.39}$$

where N is a lapse function and N^a is a shift vector. The spacetime metric and its inverse can be represents as

$$g_{\mu\nu} = \begin{bmatrix} -N^2 + N^a N_a & N_b \\ N_a & q_{ab} \end{bmatrix}, \quad g^{\mu\nu} = \begin{bmatrix} \frac{-1}{N^2} & \frac{N^b}{N^2} \\ \frac{N^a}{N^2} & q^{ab} - \frac{N^a N^b}{N^2} \end{bmatrix} \tag{2.40}$$

For the free particles free-falling in the spacetime (following the geodesic) their

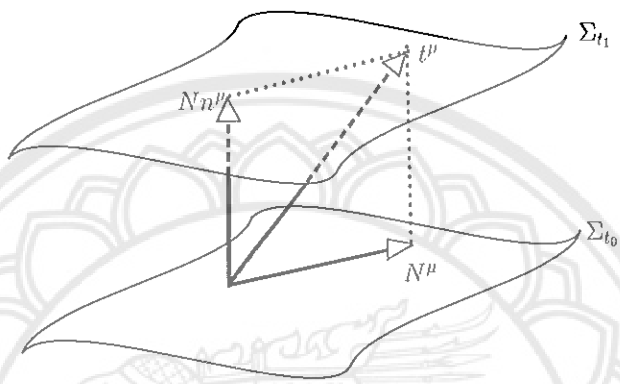

Figure 1 Space-time foliation.

world-lines define the time flows. We call a vector tangents to this flow at each point a *time-flow vector*, t^μ. In their free-falling frames at particular point in the space-time the metric field for describing the particle motion is the flat Minkowsian metric adapted from metric for curved metric at that point space-time metric . The observer in such frame may expect that a particle will follow the geodesic defined by the flat Minkowsian metric since she don't feel gravitation, but, indeed the shift in position occurs since a particle really follows the geodesic defined by general space-time metric. *Such shifts occur when the gravity is not uniform distribution*, equivalently we can say such free falling frame *has inertia* . Imagine that we are live inside the falling elevator in uniform gravitational field and place the ball at some height above the floor of this elevator. The deviation of path of such ball will not be detected, but if so, the non-uniformity of gravitational field was detected so the shifts in position of this ball occurred. Such shifts are describe by the shift vector $N^\mu(x)$. At each infinitesimal region the shift vector is a projection of the time vector as shown in figure 1. Therefore we can associate the vector perpendicular to the hypersurface with the time-flow and the shift vectors

$$Nn^\mu := t^\mu - N^\mu \; , \tag{2.41}$$

where n^μ is a unit vector orthogonal to the hypersurface(Σ_t)

$$g_{\mu\nu} n^\mu n^\nu = -1, \tag{2.42}$$

$$g_{\mu\nu} n^\mu N^\nu = 0, \tag{2.43}$$

and we call $N(x)$ the lapse function. It is dynamical and captures the deviation of speed of the clock from the local Minkowsian sense cause by non-uniformity of the gravitational field. The vector n^μ can be interpreted as the gradient of some scalar function that constant on each hypersurface. Such time function, $T(x)$, is usually defined via

$$n_\alpha = -N\nabla_\alpha T. \tag{2.44}$$

2.3.1.1 The Extrinsic Curvature. Any vector field in space-time can be decomposes in to the spatial and temporal part by the help of the vector field, n^μ

$$v = \underbrace{-g(v,n)n}_{\perp} + \underbrace{(v + g(v,n)n)}_{\parallel}. \tag{2.45}$$

(This equation using coordinate-free nonation. See appendix ??). We call the vector with $g(v,n) = 0$ *spatial.* For the covariant derivative this space-time decomposition reads

$$\nabla_u v = \underbrace{-g(\nabla_u v,n)n}_{\equiv K(u,v)n} + \underbrace{(\nabla_u v + g(\nabla_u v,n)n)}_{\equiv D_u v}. \tag{2.46}$$

We call $K(\cdot,\cdot)$ (or $K_{\mu\nu}$) the *extrinsic curvature* and D_u (or D_μ) the *spatial covariant derivative*

$$K(u,v) \;:=\; -g(\nabla_u v,n), \tag{2.47}$$
$$D_u v \;:=\; \nabla_u v + g(\nabla_u v,n)n, \tag{2.48}$$
$$\therefore \nabla_u v \;=\; D_u v + K(u,v)n. \tag{2.49}$$

Note that the projection of any vector to the hypersurface is given by

$$v_\parallel \;=\; v + g(v,n)n, \tag{2.50}$$
$$\Rightarrow (v_\parallel)^\mu \;=\; (\delta^\mu_\nu + n_\nu n^\mu)v^\nu, \tag{2.51}$$
$$:= \; q^\mu_\nu v^\nu, \tag{2.52}$$

where

$$q^\mu_\nu = (\delta^\mu_\nu + n_\nu n^\mu). \tag{2.53}$$

We call q^μ_ν the *projection operator.* By lowering the contravariant index we obtain

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu\,. \tag{2.54}$$

This is the induced metric on the hypersurface. It is a spatial object which we need only three-dimensional coordinate for describe it. By appropriate coordinate transformation, one can use only the spatial indices to describe it and we can write it as q_{ab} which will be turned out to be equals to g_{ab}. Note also that from (2.42)

$$g(n,n) \equiv n_\mu n^\mu = -1, \tag{2.55}$$

we obtain the useful identity for ADM analysis reads

$$\nabla_u g(n,n) = 0 \quad \Rightarrow \quad g(\nabla_u n, n) = 0, \tag{2.56}$$

$$n^\alpha \nabla_\beta n_\alpha = 0. \tag{2.57}$$

by applying the above identity (2.56) to the relation (2.48), one has

$$\nabla_u n = D_u n. \tag{2.58}$$

This means $\nabla_u n (= (u^\alpha \nabla_\alpha n_\beta) dx^\beta)$ is a spatial object. In the coordinate basis, the equation (2.58) means

$$u^\mu \nabla_\mu n_\nu = q^\alpha_\nu u^\mu \nabla_\mu n_\alpha, \tag{2.59}$$

$$= (\delta^\alpha_\nu + n^\alpha n_\nu) u^\mu \nabla_\mu n_\alpha, \tag{2.60}$$

$$= u^\mu \nabla_\mu n_\nu + 0, \tag{2.61}$$

the projection of this object is equals to itself so it is spatial and from (2.49) we also have

$$K(u,n) = K(n,u) = 0, \quad \text{or } K_{\mu\nu} n^\nu = 0,\ K_{\mu\nu} n^\mu = 0. \tag{2.62}$$

This is because in the case of *torsion-free*, $K_{\mu\nu}$ is symmetric. Therefore, both slots of $K(\cdot,\cdot)$ deal only with the spatial part of the vectors. Hence,

$$K(u,v) = K(u_\parallel, v_\parallel) = -g(\nabla_{u_\parallel} v_\parallel, n), \tag{2.63}$$

$$= -\nabla_{u_\parallel} \underbrace{g(v_\parallel, n)}_{0} + g(v_\parallel, \nabla_{u_\parallel} n), \tag{2.64}$$

$$= g\big(v + (v\cdot n)n, \nabla_{(u+(u\cdot n)n)} n\big), \tag{2.65}$$

$$= g\big(v, \nabla_{u+(u\cdot n)n} n\big) + (v\cdot n) g\big(n, \nabla_{u+(u\cdot n)n} n\big), \tag{2.66}$$

$$= g\big(v, \nabla_u n\big) + (u\cdot n) g\big(v, \nabla_n n\big) + (v\cdot n) \underbrace{g\big(n, \nabla_u n\big)}_{0}$$

$$+ (v\cdot n)(u\cdot n) \underbrace{g\big(n, \nabla_n n\big)}_{0}, \tag{2.67}$$

$$K(u,v) = g\big(v, \nabla_u n\big) + (u\cdot n) g\big(v, \nabla_n n\big). \tag{2.68}$$

In the coordinate basis we have

$$K_{\mu\nu} \equiv K(\partial_\mu, \partial_\nu) = g(\partial_\nu, \nabla_\mu n^\beta\, \partial_\beta) + n_\mu g(\partial_\nu, n^\lambda \nabla_\lambda n^\beta\, \partial_\beta), \tag{2.69}$$

$$= \nabla_\mu n^\beta g_{\nu\beta} + n_\mu n^\lambda \nabla_\lambda n^\beta g_{\nu\beta}, \tag{2.70}$$

$$= \nabla_\mu n_\nu + n_\mu n^\lambda \nabla_\lambda n_\nu, \tag{2.71}$$

$$K_{\mu\nu} = \nabla_\mu n_\nu + n_\mu a_\nu, \tag{2.72}$$

where $a_\mu (\equiv n^\nu \nabla_\nu a_\mu)$ is a acceleration vector[2] which is related to the lapse function,

[2]Note that the acceleration vector is a spatial object. It is in the form of $\nabla_u n(=$

$N(x)$, by the help of (2.44)

$$\begin{aligned}
a_\alpha &= n^\mu \nabla_\mu n_\alpha && (2.73)\\
&= -n^\mu \nabla_\mu (N \nabla_\alpha T) && (2.74)\\
&= -n^\mu \nabla_\mu N \nabla_\alpha T - N n^\mu \underbrace{\nabla_\mu (\nabla_\alpha T)}_{\nabla_\alpha \nabla_\mu T} && (2.75)\\
&= \frac{1}{N} n_\alpha n^\mu \nabla_\mu N - N n^\mu \nabla_\alpha (-\frac{1}{N} n_\mu) && (2.76)\\
&= \frac{1}{N} n_\alpha n^\mu \nabla_\mu N - \underbrace{n^\mu \nabla_\alpha n_\mu}_{0} + N n^\mu n_\mu \nabla_\alpha (\frac{1}{N}) && (2.77)\\
&= \frac{1}{N} n_\alpha n^\mu \nabla_\mu N - \frac{1}{N} \underbrace{n^\mu n_\mu}_{-1} \nabla_\alpha N && (2.78)\\
&= \frac{1}{N} (\nabla_\alpha N + n_\alpha n^\mu \nabla_\mu N) = \frac{1}{N} D_\alpha N && (2.79)\\
a_\alpha &= D_\alpha \ln N. && (2.80)
\end{aligned}$$

The temporal and spatial part of $\nabla_\mu n_\nu$ in (2.72) can be inspected by the first index μ

$$\nabla_\mu n_\nu = K_{\mu\nu} - n_\mu a_\nu = (\nabla_\mu n_\nu)_\parallel + (\nabla_\mu n_\nu)_\perp \tag{2.81}$$

The equation (2.72) obviously equivalents to

$$K_{\mu\nu} = q_\mu^\alpha \nabla_\alpha n_\nu. \tag{2.82}$$

Since the above quantity is spatial object then alternatively we can write it with the additional projective operator

$$K_{\mu\nu} = q_\mu^\alpha q_\nu^\beta \nabla_\alpha n_\beta. \tag{2.83}$$

From this expression the geometric interpretation can be readily understood. The extrinsic curvature tells us that how n^μ or how the orientations of the global hypersurface change as they are embedded in the space-time. Computing the Lie derivative of the induced metric

$$\pounds_n q_{\mu\nu} = n_\mu a_\nu + n_\nu a_\mu + \nabla_\mu n_\nu + \nabla_\nu n_\mu = 2K_{\mu\nu}\,, \tag{2.84}$$

leads us to the another form of the extrinsic curvature [40]

$$\begin{aligned}
K_{\mu\nu} &= \frac{1}{2} \pounds_n q_{\mu\nu}\,, && (2.85)\\
&= \frac{1}{2N} (\pounds_t q - \pounds_N q)_{\mu\nu}\,. && (2.86)
\end{aligned}$$

$(u^\alpha \nabla_\alpha n_\beta) dx^\beta)$ (2.58).

In ADM coordinate in which the metric is given by equation(2.40)

$$K_{ab} = \frac{1}{2N}(\dot{q} - \pounds_N q)_{ab}, \quad (2.87)$$

$$= \frac{1}{2N}(\dot{q}_{ab} - D_a N_b - D_b N_a), \quad (2.88)$$

where D_a is covariant derivative on the hypersurface, indices $a, b, \cdots = 1, 2, 3$ are for the spatial coordinate on the hypersurface .

2.3.1.2 ADM formalism. In this formalism we parametrize the spacetime to the space and time such that

$$x^\mu \equiv \Big\{x^0, x^1, x^2, x^3\Big\} \longmapsto \Big\{\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3\Big\} \equiv \Big\{t, x^a\Big\}, \quad (2.89)$$

or simply

$$x^\mu = \Big\{t, x^a\Big\}, \quad (2.90)$$

which "= " actually means "*equals to ... in ADM coordinate*". It is clearly that

$$\frac{dx^\mu}{dt} = \Big\{1, 0, 0, 0\Big\} \equiv t^\mu, \quad N^\mu = \Big\{0, N^a\Big\}. \quad (2.91)$$

Consequently, by (2.41) we have

$$n^\mu = \frac{1}{N}\Big\{1, -N^a\Big\}. \quad (2.92)$$

We list here some useful identities,

$$g_{\mu\nu}t^\mu t^\nu = g_{\mu\nu}\frac{\partial x^\mu}{\partial \tilde{x}^0}\frac{\partial x^\nu}{\partial \tilde{x}^0} = g_{00}\,, \quad (2.93)$$

$$g_{\mu\nu}n^\mu n^\nu = g_{\mu\nu}e^\mu_0 e^\nu_0 = \eta_{00} = -1\,. \quad (2.94)$$

Where e^I_μ is a frame field (A.31), the last line $e^\mu_0 = n^\mu$ because in Minkowskian sense

$$n^I := \{1, 0, 0, 0\}\,, \quad (2.95)$$

where the index I runs for $\{0, 1, 2, 3\}$ is the Minkowskian index. Therefore in general space-time view, we write

$$n^\mu = e^\mu_I n^I\,, \quad (2.96)$$

$$= e^\mu_I \begin{bmatrix} 1 \\ \vec{0} \end{bmatrix} = e^\mu_0\,. \quad (2.97)$$

From (2.93)

$$g_{00} = g_{\mu\nu}(Nn^\mu + N^\mu)(Nn^\nu + N^\nu)\,, \quad (2.98)$$

$$= -N^2 + N_\mu N^\mu = N^2 + N_a N^a, \quad (2.99)$$

$$g_{\mu\nu}t^\mu N^\nu = g_{\mu\nu}(Nn^\mu + N^\mu)N^\nu, \quad (2.100)$$

$$g_{\mu b}\frac{\partial x^{\mu}}{\partial \tilde{x}^{0}}N^{b} = 0 + N_a N^a, \tag{2.101}$$

$$\therefore g_{0b} = N_b. \tag{2.102}$$

These give us the form of the metric in ADM coordinate represents as (2.40)

$$g_{\mu\nu} = \begin{bmatrix} -N^2 + N^a N_a & N_b \\ N_a & q_{ab} \end{bmatrix}.$$

In order to find the inverse of this metric, we represent the relevant vectors by the column vector

$$t^{\mu} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad n^{\mu} = \frac{1}{N}\begin{bmatrix} 1 \\ -N^a \end{bmatrix}, \quad N^{\mu} = \begin{bmatrix} 0 \\ N^a \end{bmatrix}. \tag{2.103}$$

The equation (2.40),(2.103) allow us to write

$$n^{\mu} \otimes n^{\nu} = \frac{1}{N^2}\begin{bmatrix} 1 \\ -N^a \end{bmatrix}\begin{bmatrix} 1 & -N^b \end{bmatrix}, \tag{2.104}$$

$$= \frac{1}{N^2}\begin{bmatrix} 1 & -N^b \\ -N^a & N^a N^b \end{bmatrix}, \tag{2.105}$$

$$g^{\mu\nu} = q^{\mu\nu} - n^{\mu} \otimes n^{\nu}, \tag{2.106}$$

$$= q^{\mu\nu} + \frac{1}{N^2}\begin{bmatrix} -1 & N^b \\ N^a & -N_a N^b \end{bmatrix}, \tag{2.107}$$

$$t_{\mu} = \begin{bmatrix} -N^2 + N^a N_a & N_b \\ N_a & q_{ab} \end{bmatrix}\begin{bmatrix} 1 \\ \vec{0} \end{bmatrix} = \begin{bmatrix} -N^2 + N_a N^a \\ N_a \end{bmatrix}, \tag{2.108}$$

$$n_{\mu} = \begin{bmatrix} -N^2 + N^a N_a & N_b \\ N_a & q_{ab} \end{bmatrix}\frac{1}{N}\begin{bmatrix} 1 \\ -N^b \end{bmatrix} = \begin{bmatrix} -N \\ 0 \end{bmatrix}, \tag{2.109}$$

$$N_{\mu} = t_{\mu} - N n^{\mu} = \begin{bmatrix} N_a N^a \\ N_a \end{bmatrix}, \tag{2.110}$$

$$n_{\mu} \otimes n_{\nu} = \begin{bmatrix} N^2 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.111}$$

$$q_{\mu\nu} = g_{\mu\nu} + n_{\mu}n_{\nu}, \tag{2.112}$$

$$= \begin{bmatrix} N_a N^a & N_b \\ N_a & q_{ab} \end{bmatrix}. \tag{2.113}$$

From (2.107) we need to know $q^{\mu\nu}$ for computing the inverse metric. This is can be done from the requirement that

$$q^{\mu\nu}q_{\nu\alpha} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\left(= \delta^{\nu}_{\alpha} - \delta^{\nu}_{0} \otimes \delta^{0}_{\alpha}\right), \tag{2.114}$$

and by observation that

$$\underbrace{\begin{bmatrix} N_a N^a & N^b \\ N^a & q_{ab} \end{bmatrix}}_{q_{\mu\nu}} \begin{bmatrix} 0 & 0 \\ 0 & q^{bc} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \delta^c_a \end{bmatrix}, \quad \text{where } q_{ab}q^{bc} = \delta^c_a, \tag{2.115}$$

one can realize that

$$q^{\mu\nu} = \begin{bmatrix} 0 & 0 \\ 0 & q^{ab} \end{bmatrix}. \tag{2.116}$$

Applying this result to (2.107), one obtains

$$g^{\mu\nu} = \begin{bmatrix} 0 & 0 \\ 0 & q^{ab} \end{bmatrix} + \frac{1}{N^2} \begin{bmatrix} -1 & N^b \\ N^a & -N_a N^b \end{bmatrix}, \tag{2.117}$$

then the inverse metric in ADM coordinate reads

$$g^{\mu\nu} = \begin{bmatrix} \frac{-1}{N^2} & \frac{N^b}{N^2} \\ \frac{N^a}{N^2} & q^{ab} - \frac{N^a N^b}{N^2} \end{bmatrix}.$$

The important part of ADM analysis is the *Gauß-Codazzi equation* which is related the four-dimensional curvature to the three-dimensional one. The Riemann (intrinsic) curvature tensor on the hypersurface is defined by

$$-{}^3R^{\alpha}{}_{\gamma\beta\sigma}A_\alpha \;=\; (D_\beta D_\sigma - D_\sigma D_\beta)A_\gamma\,, \tag{2.118}$$

where A_μ is a covariant vector field. This directly implies

$$\begin{aligned} D_\nu D_\beta A_\alpha &= q^\sigma_\alpha n^\rho K_{\nu\beta}\nabla_\rho A_\sigma - K_{\nu\alpha}K^\lambda_\beta A_\lambda + q^\gamma_\nu q^\rho_\beta q^\sigma_\alpha \nabla_\gamma\nabla_\rho A_\sigma, && (2.119)\\ \Rightarrow (D_\nu D_\beta - D_\beta D_\nu)A_\alpha &= (-K_{\nu\alpha}K^\lambda_\beta + K_{\beta\alpha}K^\lambda_\nu)A_\lambda + q^\gamma_\nu q^\rho_\beta q^\sigma_\alpha(\nabla_\gamma\nabla_\rho - \nabla_\rho\nabla_\gamma)A_\sigma, \\ \Rightarrow -{}^3R^\lambda{}_{\alpha\nu\beta}A_\lambda &= (-K_{\nu\alpha}K^\lambda_\beta + K_{\beta\alpha}K^\lambda_\nu)A_\lambda - q^\gamma_\nu q^\rho_\beta q^\sigma_\alpha R^\lambda{}_{\sigma\gamma\rho}A_\lambda, && (2.120)\\ \Rightarrow {}^3R \equiv {}^3R^\lambda{}_{\alpha\nu\beta}q^\nu_\lambda q^{\alpha\beta} &= K_{\mu\nu}K^{\mu\nu} - K^2 + q^\gamma_\lambda q^{\rho\sigma}R^\lambda{}_{\sigma\gamma\rho}, && (2.121) \end{aligned}$$

but

$$R \equiv g^{\rho\sigma}\delta^\gamma_\lambda R^\lambda{}_{\sigma\gamma\rho} = (q^{\rho\sigma}q^\gamma_\lambda - 2n^\sigma n^\rho q^\gamma_\lambda)R^\lambda{}_{\sigma\gamma\rho}, \tag{2.122}$$

so

$$\begin{aligned} {}^3R &= K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2n^\sigma n^\rho q^\gamma_\lambda R^\lambda{}_{\sigma\gamma\rho}, && (2.123)\\ &= K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2n^\rho q^\gamma_\lambda[\nabla_\gamma, \nabla_\rho]n^\lambda, && (2.124)\\ &= K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2(\nabla_\mu(n^\nu\nabla_\nu n^\mu) - K_{\mu\nu}K^{\mu\nu} - \nabla_\mu(n^\mu K) + K^2), && (2.125)\\ \Rightarrow \quad & R = {}^3R + K_{\mu\nu}K^{\mu\nu} - K^2 - 2\nabla_\mu(a^\mu - n^\mu K) \quad . && (2.126) \end{aligned}$$

This equation is useful for expressing the Einstein-Hilbert action to the ADM variables. Considering equation (2.123), it can be recast as

$$
\begin{aligned}
{}^3R &= K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2n^\sigma n^\rho(\delta^\gamma_\lambda + n^\gamma n_\lambda)R^\lambda{}_{\sigma\gamma\rho},\\
&= K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2n^\sigma n^\rho R_{\rho\sigma} + \underbrace{2n^\sigma n^\rho n^\gamma n_\lambda R^\lambda{}_{\sigma\gamma\rho}}_{0},
\end{aligned}
$$

$$
R_{\mu\nu}n^\mu n^\nu = \frac{1}{2}\left({}^3R - K_{\mu\nu}K^{\mu\nu} + K^2 - R\right), \tag{2.127}
$$

where we have used the symmetric property for the last term in second line. Then

$$
\begin{aligned}
G_{\mu\nu}n^\mu n^\nu &\equiv R_{\mu\nu}n^\mu n^\nu - \frac{1}{2}g_{\mu\nu}n^\mu n^\nu R,\\
&= \frac{1}{2}\left({}^3R - K_{\mu\nu}K^{\mu\nu} + K^2 - R\right) + \frac{1}{2}R,\\
G_{\mu\nu}n^\mu n^\nu &= \frac{1}{2}\left({}^3R - K_{\mu\nu}K^{\mu\nu} + K^2\right). \qquad (2.128)
\end{aligned}
$$

This identity will be used later. The equation (2.120) also gives us the relation for Riemann curvature tensor

$$
-{}^3R^\lambda{}_{\alpha\nu\beta} = -K_{\nu\alpha}K^\lambda_\beta + K_{\beta\alpha}K^\lambda_\nu - q^\gamma_\nu q^\rho_\beta q^\sigma_\alpha R^\lambda{}_{\sigma\gamma\rho}, \tag{2.129}
$$

$$
{}^3R_{\lambda\alpha\nu\beta} = K_{\nu\alpha}K_{\lambda\beta} - K_{\alpha\beta}K_{\lambda\nu} + q^\delta_\lambda q^\gamma_\nu q^\rho_\beta q^\sigma_{\alpha\beta} R_{\delta\sigma\gamma\rho}. \tag{2.130}
$$

The equation (2.130) is called the *Gauß-Codazzi equation*. Starting from this equation we can also derive the relation for Ricci tensor

$$
-{}^3R_{\alpha\beta} \equiv -q^\nu_\lambda\, {}^3R^\lambda{}_{\alpha\nu\beta}, \tag{2.131}
$$

$$
= -K_{\lambda\alpha}K^\lambda_\beta + K_{\beta\alpha}K^\lambda_\lambda - (\delta^\nu_\lambda + n^\nu n_\lambda)q^\gamma_\nu q^\rho_\beta q^\sigma_\alpha R^\lambda{}_{\sigma\gamma\rho}. \tag{2.132}
$$

The second term in the parentheses then vanishes due to $n^\nu q^\gamma_\nu = 0$, which yields

$$
{}^3R_{\alpha\beta} = K_{\alpha\lambda}K^\lambda_\beta - KK_{\alpha\beta} + q^\gamma_\nu q^\rho_\beta q^\sigma_\alpha R^\nu{}_{\sigma\gamma\rho}, \tag{2.133}
$$

$$
= K_{\alpha\lambda}K^\lambda_\beta - KK_{\alpha\beta} + (\delta^\gamma_\nu + n^\gamma n_\nu)q^\rho_\beta q^\sigma_\alpha R^\nu{}_{\sigma\gamma\rho}, \tag{2.134}
$$

$$
= K_{\alpha\lambda}K^\lambda_\beta - KK_{\alpha\beta} + q^\rho_\beta q^\sigma_\alpha R_{\sigma\rho} + q^\rho_\beta q^\sigma_\alpha n^\gamma n_\nu R^\nu{}_{\sigma\gamma\rho}, \tag{2.135}
$$

$$
{}^3R_{\alpha\beta} = K_{\alpha\lambda}K^\lambda_\beta - KK_{\alpha\beta} + (R_{\alpha\beta})_\parallel + (n^\gamma n_\nu R^\nu{}_{\alpha\gamma\beta})_\parallel. \tag{2.136}
$$

2.3.1.3 In the unitary gauge. In this gauge

$$
\phi(t, x^a) = \phi(t), \tag{2.137}
$$

$$
n_\mu = -\gamma\phi_\mu, \quad \gamma = \frac{1}{\sqrt{-Y}}. \tag{2.138}
$$

The minus sign in front of Y appears because ϕ_μ is time-like. Therefore $\phi_\mu\phi^\mu < 0$. Let us find the spatially geometric description corresponding for the second derivatives of the scalar field

$$\nabla_\mu n_\nu = -\nabla_\mu \gamma \phi_\nu - \gamma \phi_{\mu\nu}. \tag{2.139}$$

$$\nabla_\mu \gamma = \frac{\gamma^3}{2} Y_\mu \tag{2.140}$$

$$\therefore \nabla_\mu n_\nu = -\frac{\gamma^3}{2} Y_\mu \phi_\nu - \gamma \phi_{\mu\nu}. \tag{2.141}$$

From (2.72)

$$\nabla_\mu n_\nu = K_{\mu\nu} - n_\mu a_\nu, \tag{2.142}$$

$$-\frac{\gamma^3}{2} Y_\mu \phi_\nu - \gamma \phi_{\mu\nu} = K_{\mu\nu} - n_\mu a_\nu, \tag{2.143}$$

$$\boxed{\phi_{\mu\nu} = -\frac{1}{\gamma}(K_{\mu\nu} - n_\mu a_\nu) - \frac{\gamma^2}{2} Y_\mu \phi_\nu} \tag{2.144}$$

From (2.141) :

$$n^\mu \nabla_\mu n_\nu \equiv a_\nu = -\frac{\gamma^3}{2} n^\mu Y_\mu \phi_\nu + \gamma^2 \phi^\mu \phi_{\mu\nu}. \tag{2.145}$$

But

$$Y_\mu = 2\phi^\beta \phi_{\beta\mu}. \tag{2.146}$$

therefore we have

$$a_\mu = -\frac{\gamma^3}{2} n^\lambda Y_\lambda \phi_\mu + \frac{\gamma^2}{2} Y_\mu, \tag{2.147}$$

$$n_\nu a_\mu = -\frac{\gamma^3}{2} n^\lambda Y_\lambda \phi_\mu n_\nu - \frac{\gamma^3}{2} Y_\mu \phi_\nu. \tag{2.148}$$

Finally, we have

$$-\frac{\gamma^2}{2} Y_\mu \phi_\nu = \frac{1}{\gamma} n_\nu a_\mu + \frac{\gamma^2}{2} \phi^\lambda Y_\lambda n_\mu n_\nu. \tag{2.149}$$

Substituting of this result to (2.144) gives

$$\boxed{\phi_{\mu\nu} = -\frac{1}{\gamma}(K_{\mu\nu} - n_\mu a_\nu - a_\mu n_\nu) + \frac{\gamma^2}{2} \phi^\lambda Y_\lambda n_\mu n_\nu} \tag{2.150}$$

We also obtain

$$\Box \phi = -\frac{1}{\gamma} K - \frac{\gamma^2}{2} \phi^\lambda Y_\lambda, \tag{2.151}$$

and by contracting both side of (2.150) we obtain

$$\begin{aligned} -\gamma \phi^\nu \phi_{\mu\nu} &= -\frac{a_\mu}{\gamma} + \frac{\gamma}{2} n^\lambda n_\mu Y_\lambda, \\ a_\mu &= \frac{\gamma^2}{2}(Y_\mu + n^\lambda n_\mu Y_\lambda), \end{aligned}$$

$$a_\mu = \frac{\gamma^2}{2} q_\mu^\lambda Y_\lambda, \tag{2.152}$$

which means that $\frac{2}{\gamma^2} a_\mu$ is a spatial part of Y_μ. Indeed, from (2.147) we can notice that Y_μ can be decomposed in to (3+1)-style as

$$Y_\mu = \frac{2}{\gamma^2} a_\mu + \gamma \phi^\lambda Y_\lambda n_\mu, \tag{2.153}$$

where the first term is spatial part and the second term is temporal part.

The goal of the next derivation is to rewrite the Horndeski action in terms of the functions of $(\phi, Y(\equiv \phi_\mu \phi^\mu))$ and the spatial quantities such as ${}^3R_{abcd}, {}^3R_{ab}, {}^3R, K_{ab}, q_{ab}$. It is useful to see how the basic building blocks of ADM formalism look like in this gauge

$$\begin{aligned} \phi = \phi(t) \Rightarrow \phi_\mu &= \left\{\dot{\phi}, 0, 0, 0\right\}, \\ \frac{-\phi_\mu}{\sqrt{-Y}} &= \left\{\frac{-\dot{\phi}}{\sqrt{-Y}}, 0, 0, 0\right\}, \\ n_\mu = \left\{\frac{-\dot{\phi}}{\sqrt{-Y}}, 0, 0, 0\right\} &= -\delta_\mu^0 \frac{\dot{\phi}}{\sqrt{-Y}}. \end{aligned} \tag{2.154}$$

Comparing equation (2.154) with (2.109) we obtain

$$\frac{\dot{\phi}}{\sqrt{-Y} N} = 1, \quad \text{or} \quad \frac{1}{\gamma} = \frac{\dot{\phi}}{N} \quad . \tag{2.155}$$

Rearranging it to the another form

$$N\, dt = \frac{1}{\sqrt{-Y}} d\phi, \tag{2.156}$$

which means that at each point in configuration space (ϕ, Y) we can find corresponding point in (N, t)

$$(\phi, Y) \mapsto (N, t). \tag{2.157}$$

We can write coefficient functions of Horndeski Lagrangians as a functions of (N, t). Note also that

$$\begin{aligned} n^\mu &= -\phi^\mu / \sqrt{-Y} = -g^{\mu\nu} \phi_\nu / \sqrt{-Y}, &(2.158)\\ &= g^{\mu 0} \dot{\phi} / \sqrt{-Y}, &(2.159)\\ &= \left\{\frac{\dot{\phi}}{\sqrt{-Y} N^2}, \frac{-N^a \dot{\phi}}{N^2 \sqrt{-Y}}\right\}. &(2.160) \end{aligned}$$

Back to the Horndeski Lagrangian, we classify it to $\mathcal{L}_2^H, \mathcal{L}_3^H, \mathcal{L}_4^H, \mathcal{L}_5^H$, by the power of $\nabla\nabla\phi$ in each Lagrangian. Transforming these Lagrangian in to the spatial geometric term may be mixed the power of $\nabla\nabla\phi$. In such case we call the Lagrangians are in the *mixed-form*. If we rearrange the term with the same number of power together in the same Lagrangian, we call the Lagrangians are in the *pure-form*. The number of power of $\nabla\nabla\phi$ is dimensionally equivalent to the main spatial geometric quantities as the following

$$K, K_{ab} \sim \nabla\nabla\phi \Rightarrow \#\nabla\nabla\phi = 1, \quad (2.161)$$
$${}^3R_{abcd}, {}^3R_{ab}, {}^3R \sim \nabla\nabla\phi\,\nabla\nabla\phi \Rightarrow \#\nabla\nabla\phi = 2, \quad (2.162)$$

the first line concludes from (2.150), while the second line concludes from (2.130). Therefore, our requirements is the new form of Lagrangians will be consist of the quantities with dimensionally equivalent to $(\nabla\nabla\phi)^n$, and does not consist of $n_\mu \sim \nabla\phi$ or $a_\mu \sim \nabla\phi\nabla\nabla\phi$ or $Y_\lambda \sim \nabla\phi\nabla\nabla\phi$.

Let us consider the Horndeski Lagrangians. Since (2.29)

$$\mathcal{L}_2^H = G_2(\phi, Y) = G_2(\phi(t), \frac{-\dot{\phi}^2}{N^2}) = \bar{A}_2(N,t) =: G_2(N,t),$$

is a function of (N,t) by default, so nothing to do with it. The next piece is (2.30)

$$\mathcal{L}_3^H = G_3(N,t)\Box\phi,$$

by using (2.151) we obtain

$$\mathcal{L}_3^H = -\frac{1}{\gamma}G_3K - \frac{\gamma^2}{2}G_3\phi^\lambda Y_\lambda, \quad (2.163)$$

which the first term fits into the requirements but not for the second term because the terms in the dimension of $\nabla\phi$ appeare. Replacing the term $\phi^\lambda Y_\lambda$ by using (2.151) cannot fixed this situation. We can try to integrate by parts

$$\begin{aligned}
\mathcal{L}_3^H &= \nabla_\mu\big(G_3\phi^\mu\big) - \nabla_\mu G_3\,\phi^\mu,\\
&= \nabla_\mu\big(G_3\phi^\mu\big) - \Big(G_{3\phi}\phi_\mu\phi^\mu + G_{3Y}Y_\mu\phi^\mu\Big),\\
&= \nabla_\mu\big(G_3\phi^\mu\big) - \Big(G_{3\phi}Y + G_{3Y}Y_\mu\phi^\mu\Big),\\
&= \nabla_\mu\big(G_3\phi^\mu\big) - \Big(G_{3\phi}Y + G_{3Y}\big[-\frac{2\Box\phi}{\gamma^2} - \frac{2K}{\gamma^3}\big]\Big),\\
&= \nabla_\mu\big(G_3\phi^\mu\big) - \Big(G_{3\phi}Y + G_{3Y}2Y\Box\phi + \frac{2Y}{\gamma}G_{3Y}K\Big).
\end{aligned}$$

At this stage we have

$$\mathcal{L}_3^H = -G_{3\phi}Y - \frac{2Y}{\gamma}G_{3Y}K - G_{3Y}2Y\Box\phi + \nabla_\mu\mathcal{B}_1^\mu, \quad (2.164)$$

where the boundary term

$$\nabla_\mu\mathcal{B}_{1a}^\mu \equiv \nabla_\mu\big(G_3\phi^\mu\big). \quad (2.165)$$

Therefore, up to the boundary terms

$$\mathcal{L}_3^H = -G_{3\phi}Y - \frac{2Y}{\gamma}G_{3Y}K - G_{3Y}2Y\Box\phi. \tag{2.166}$$

The first two terms are in the required form but the last term is not ($\Box\phi = -\frac{1}{\gamma}K - \frac{\gamma^2}{2}\phi^\lambda Y_\lambda$). The strategy for getting rid of the last term is redefine the function to be the another function plus the extra term

$$\begin{aligned} G_3 &= F_3 + \text{the extra term}, \\ &= F_3 + \mathcal{A}_3. \end{aligned} \tag{2.167}$$

We add the extra term because we hope that it may help in canceling the unwanted piece. The Lagrangian then reads

$$\mathcal{L}_3^H = -F_{3\phi}Y - \frac{2Y}{\gamma}F_{3Y}K - \mathcal{A}_{3\phi}Y - \frac{2Y}{\gamma}\mathcal{A}_{3Y}K - (F_3 + \mathcal{A}_3)_Y 2Y\Box\phi. \tag{2.168}$$

We expect that these bad terms will be canceled out (or, at least, become another good terms). If we guess that

$$\mathcal{A}_{3\phi}Y + \frac{2Y}{\gamma}\mathcal{A}_{3Y}K + (F_3 + \mathcal{A}_3)_Y 2Y\Box\phi = 0, \tag{2.169}$$

then we have

$$\begin{aligned} 0 &= \mathcal{A}_{3\phi}Y + \frac{2Y}{\gamma}\mathcal{A}_{3Y}K + (F_3 + \mathcal{A}_3)_Y 2Y\Big(-\frac{K}{\gamma} - \frac{\gamma^2}{2}\phi^\lambda Y_\lambda\Big), \\ &= \mathcal{A}_{3\phi}Y + \frac{2Y}{\gamma}\mathcal{A}_{3Y}K - \frac{2YK}{\gamma}F_{3Y} - \gamma^2 Y F_{3Y}\phi^\lambda Y_\lambda - \frac{2Y}{\gamma}\mathcal{A}_{3Y}K - \gamma^2 Y\mathcal{A}_{3Y}\phi^\lambda Y_\lambda, \\ &= \mathcal{A}_{3\phi}Y - \frac{2YK}{\gamma}F_{3Y} + F_{3Y}\phi^\lambda Y_\lambda + \mathcal{A}_{3Y}\phi^\lambda Y_\lambda, \\ \Big(\nabla_\mu \mathcal{A}_3\Big)\phi^\mu &= \frac{2Y}{\gamma}F_{3Y}K - F_{3Y}\phi^\lambda Y_\lambda, \\ &= -2YF_{3Y}\Big(-\frac{K}{\gamma} + \frac{1}{2Y}\phi^\lambda Y_\lambda\Big), \\ &= -2YF_{3Y}\Big(-\frac{K}{\gamma} - \frac{\gamma^2}{2}\phi^\lambda Y_\lambda\Big), \\ &= -2YF_{3Y}\,\Box\phi, \\ &= \nabla_\mu\Big(2YF_{3Y}\Big)\,\phi^\mu + \nabla_\mu\mathcal{B}^\mu_{1b}, \end{aligned}$$

where the boundary term given by

$$\nabla_\mu\mathcal{B}^\mu_{1b} \equiv \nabla_\mu\Big(-2Y\; F_{3Y}\phi^\mu\Big). \tag{2.170}$$

Finally, we obtain

$$\mathcal{A}_3 = 2YF_{3Y}. \tag{2.171}$$

If we redefine,

$$\boxed{G_3 = F_3 + 2YF_{3Y}} \tag{2.172}$$

then the Lagrangian, $\mathcal{L}_3^H$, (2.168) becomes

$$\mathcal{L}_3^H = -F_{3\phi}Y - \frac{2Y}{\gamma}F_{3Y}K, \tag{2.173}$$

up to the boundary terms (2.165),(2.170)

$$\nabla_\mu \mathcal{B}_1^\mu = \nabla_\mu\Big(F_3\phi^\mu\Big). \tag{2.174}$$

We are quite lucky because, indeed, there is no proper method for solving these kind of problems, but integrating by parts and using (the trial) auxiliary functions are the effective tools. We may observe that *the redefinition of the coefficient function in the term that contains Y_λ by $\sim A(\phi, Y) + 2YA_Y(\phi, Y)$, where $A(\phi, Y)$ is some auxiliary function tends to simplify the problems.* This is because

$$(A + 2YA_Y)Y^\mu\phi_\mu = \nabla^\mu(\phi_\mu A) + \frac{K}{\gamma}A - A_\phi Y. \tag{2.175}$$

The Lagrangian now follows our requirements. But, the dimensions of $\nabla\nabla\phi$ of both terms are not the same such that the first term is zero, $(\#\nabla\nabla\phi = 0)$, while the second is one, $(\#\nabla\nabla\phi = 1)$, so this Lagrangian is in the *mixed-form*. Next, consider

$$\mathcal{L}_4^H = G_4 R - 2G_{4Y}(\Box\phi^2 - \phi_{\mu\nu}^2). \tag{2.176}$$

By using (2.126), (2.151), (2.150)

$$\begin{aligned}
\mathcal{L}_4^H &= G_4\Big({}^3R + K_{\mu\nu}K^{\mu\nu} - K^2 - 2\nabla_\mu(a^\mu - n^\mu K)\Big) - 2G_{4Y}\Big\{\Big(\frac{K^2}{\gamma^2} + \gamma K\phi^\lambda Y_\lambda \\
&\quad + \frac{\gamma^4}{4}\phi^\lambda Y_\lambda\phi^\beta Y_\beta\Big) - \Big(\frac{1}{\gamma^2}K_{\mu\nu}K^{\mu\nu} - \frac{2}{\gamma^2}a_\mu a^\mu + \frac{\gamma^4}{4}\phi^\lambda Y_\lambda\phi^\beta Y_\beta\Big)\Big\}, \\
&= G_4\Big({}^3R + K_{\mu\nu}K^{\mu\nu} - K^2 - 2\nabla_\mu(a^\mu - n^\mu K)\Big) \\
&\quad -2G_{4Y}\Big(\frac{K^2}{\gamma^2} + \gamma K\phi^\lambda Y_\lambda - \frac{1}{\gamma^2}K_{\mu\nu}K^{\mu\nu} + \frac{2}{\gamma^2}a_\mu a^\mu\Big), \\
&= G_4\Big({}^3R + K_{\mu\nu}K^{\mu\nu} - K^2 - 2\nabla_\mu(a^\mu - n^\mu K)\Big) \\
&\quad +2YG_{4Y}\Big(K^2 - K_{\mu\nu}K^{\mu\nu}\Big) - 2G_{4Y}\Big(\gamma K\phi^\lambda Y_\lambda + \frac{2}{\gamma^2}a_\mu a^\mu\Big), \\
&= G_4\Big({}^3R - 2\nabla_\mu(a^\mu - n^\mu K)\Big) \\
&\quad +(2YG_{4Y} - G_4)\Big(K^2 - K_{\mu\nu}K^{\mu\nu}\Big) - 2G_{4Y}\Big(- Kn^\lambda Y_\lambda + \frac{2}{\gamma^2}(\frac{\gamma^2}{2}q_\mu^\lambda Y_\lambda)a^\mu\Big), \\
&= G_4\,{}^3R + (2YG_{4Y} - G_4)\Big(K^2 - K_{\mu\nu}K^{\mu\nu}\Big) - 2G_4\nabla_\mu(a^\mu - n^\mu K)
\end{aligned}$$

$$
\begin{aligned}
& -2G_{4Y}Y_\lambda\Big(-Kn^\lambda + q^\lambda_\mu a^\mu\Big), \\
= \;& G_4\,{}^3R + \big(2YG_{4Y} - G_4\big)\Big(K^2 - K_{\mu\nu}K^{\mu\nu}\Big) - \nabla_\mu\big(2G_4(a^\mu - n^\mu K)\big) \\
& -2G_{4Y}Y_\lambda\Big(-Kn^\lambda + q^\lambda_\mu a^\mu - a^\lambda + n^\lambda K\Big) + 2G_{4\phi}\phi_\mu\Big(a^\mu - n^\mu K\Big),
\end{aligned}
$$

where we have used (2.152) for the term $a_\mu a^\mu$. The form of this Lagrangian now becomes

$$
\mathcal{L}^H_4 = G_4\,{}^3R + \big(2YG_{4Y} - G_4\big)\Big(K^2 - K_{\mu\nu}K^{\mu\nu}\Big) - 2\sqrt{-Y}G_{4\phi}K + \nabla_\mu\mathcal{B}^\mu_2, \quad (2.177)
$$

where the boundary term

$$
\nabla_\mu\mathcal{B}^\mu_2 = -\nabla_\mu\big(2G_4(a^\mu - n^\mu K)\big). \quad (2.178)
$$

Next, consider the last piece, $\mathcal{L}^H_5$, and will follow closely Ref [41]

$$
\mathcal{L}^H_5 = G_5(\phi, Y)G_{\mu\nu}\phi^{\mu\nu} + (1/3)G_{5Y}(\phi, Y)(\Box\phi^3 - 3\Box\phi\phi^2_{\mu\nu} + 2(\phi_{\mu\nu})^3).
$$

Consider the first term, using the integration by parts gives

$$
\begin{aligned}
G_5\,G_{\mu\nu}\phi^{\mu\nu} &= \nabla^\mu(G_5\,G_{\mu\nu}\,\phi^\nu) - G_{5Y}G_{\mu\nu}Y^\mu\phi^\nu - G_{5\phi}\gamma^{-2}G_{\mu\nu}n^\mu n^\mu, \quad (2.179)\\
&= \nabla^\mu(G_5\,G_{\mu\nu}\,\phi^\nu) - G_{5Y}G_{\mu\nu}Y^\mu\phi^\nu \\
&\quad - \frac{1}{2\gamma^2}G_{5\phi}\Big({}^3R - K_{\mu\nu}K^{\mu\nu} + K^2\Big), \quad (2.180)
\end{aligned}
$$

where in the third line we have used the identity (2.128). Then, let us transform the quantities inside the second term in the Lagrangian. These can be done by the using of (2.150) and (2.151)

$$
\begin{aligned}
\Box\phi^3 &= -\Big(\frac{K}{\gamma} + \frac{\gamma^2}{2}\phi^\lambda Y_\lambda\Big)^3, \\
&= -\frac{K^3}{\gamma^3} - \frac{3}{2}K^2\phi^\lambda Y_\lambda - \frac{3}{4}\gamma^3K(\phi^\lambda Y_\lambda)^2 - \frac{1}{8}\gamma^6(\phi^\lambda Y_\lambda)^3. \quad (2.181)\\
\Box\phi\phi_{\mu\nu}\phi^{\mu\nu} &= -\Big(\frac{K}{\gamma} + \frac{\gamma^2}{2}\phi^\lambda Y_\lambda\Big)\Big\{\frac{1}{\gamma^2}K_{\alpha\beta}K^{\alpha\beta} - \frac{2}{\gamma^2}a_\mu a^\mu + \frac{\gamma^2}{4}(\phi^\beta Y_\beta)^2\Big\}, \\
&= -\Big\{\frac{1}{\gamma^3}K\,K_{\alpha\beta}K^{\alpha\beta} - \frac{2}{\gamma^3}Ka_\mu a^\mu + \frac{\gamma^3}{4}K(\phi^\beta Y_\beta)^2 + \frac{1}{2}K_{\alpha\beta}K^{\alpha\beta}\phi^\lambda Y_\lambda \\
&= -a_\mu a^\mu\phi^\lambda Y_\lambda + \frac{\gamma^6}{8}(\phi^\lambda Y_\lambda)^3\Big\}. \quad (2.182)\\
\phi^\mu_\nu\,\phi^\nu_\rho\,\phi^\rho_\mu &= -\frac{1}{\gamma^3}\Big\{K^\mu_\nu K^\nu_\rho K^\rho_\mu - 3K_{\mu\nu}a^\mu a^\nu\Big\} + \frac{3}{2}\phi^\omega Y_\omega a_\nu a^\nu - \frac{\gamma^6}{8}(\phi^\lambda Y_\lambda)^3. \quad (2.183)
\end{aligned}
$$

From (2.181), (2.182), and (2.183), we obtain

$$\Box\phi^3 - 3\Box\phi\phi_{\mu\nu}^2 + 2(\phi_{\mu\nu})^3 = -\frac{K^3}{\gamma^3} - \frac{3}{2}\phi^\lambda Y_\lambda\Big(K^2 - K_{\alpha\beta}K^{\alpha\beta}\Big) + \frac{3}{\gamma^3}K\,K_{\alpha\beta}K^{\alpha\beta} - \frac{2}{\gamma^3}K^\mu_\nu K^\nu_\rho K^\rho_\mu + \frac{6}{\gamma^3}\Big(K^\mu_\nu a^\nu a_\mu - K\,a_\mu a^\mu\Big)\,. \tag{2.184}$$

Then, at this step we have

$$\begin{aligned}\mathcal{L}_5^H &= -G_{5Y}G_{\mu\nu}Y^\mu\phi^\nu - \frac{1}{2\gamma^2}G_{5\phi}\Big({}^3R - K_{\mu\nu}K^{\mu\nu} + K^2\Big)\\ &\quad +(1/3)G_{5Y}\Big\{-\frac{K^3}{\gamma^3} - \frac{3}{2}\phi^\lambda Y_\lambda\Big(K^2 - K_{\alpha\beta}K^{\alpha\beta}\Big) + \frac{3}{\gamma^3}K\,K_{\alpha\beta}K^{\alpha\beta} - \frac{2}{\gamma^3}K^\mu_\nu K^\nu_\rho K^\rho_\mu\\ &\quad +\frac{6}{\gamma^3}\Big(K^\mu_\nu a^\nu a_\mu - K\,a_\mu a^\mu\Big)\Big\} + \nabla_\mu \mathcal{B}^\mu_3\,,\end{aligned} \tag{2.185}$$

where the boundary term

$$\nabla_\mu \mathcal{B}^\mu_{3a} \equiv \nabla^\mu(G_5\,G_{\mu\nu}\,\phi^\nu). \tag{2.186}$$

Note that, the bad terms are the terms that contain Y_μ, ϕ^μ(or n^μ), and $a^\mu(\equiv \frac{\gamma^2}{2}q^\lambda_\mu Y_\lambda)$. As we have observed in the case of $\mathcal{L}_3^H$, the coefficient function in front of the Y_μ term will be redefined. In this case we try

$$G_{5Y} = \tilde{F}_{5Y} + \frac{\tilde{F}_5}{2Y}. \tag{2.187}$$

Applying this only for the bad terms, the Lagrangian then looks like (up to the boundary term)

$$\begin{aligned}\mathcal{L}_5^H &= -(\tilde{F}_{5Y} + \frac{\tilde{F}_5}{2Y})G_{\mu\nu}Y^\mu\phi^\nu - \frac{1}{2\gamma^2}G_{5\phi}\Big({}^3R - K_{\mu\nu}K^{\mu\nu} + K^2\Big)\\ &\quad +(1/3)(\tilde{F}_{5Y} + \frac{\tilde{F}_5}{2Y})\Big\{-\frac{3}{2}\phi^\lambda Y_\lambda\Big(K^2 - K_{\alpha\beta}K^{\alpha\beta}\Big) + \frac{6}{\gamma^3}\Big(K^\mu_\nu a^\nu a_\mu - K\,a_\mu a^\mu\Big)\Big\}\\ &\quad +(1/3)G_{5Y}\Big\{-\frac{K^3}{\gamma^3} + \frac{3}{\gamma^3}K\,K_{\alpha\beta}K^{\alpha\beta} - \frac{2}{\gamma^3}K^\mu_\nu K^\nu_\rho K^\rho_\mu\Big\}\,.\end{aligned} \tag{2.188}$$

We need to manipulate only the bad term, so we write

$$\mathcal{L}_5^H \equiv {}^{\mathfrak{GD}}\mathcal{L}_5^H + {}^{\mathfrak{BD}}\mathcal{L}_5^H\,, \tag{2.189}$$

$$\begin{aligned}{}^{\mathfrak{GD}}\mathcal{L}_5^H &= -\frac{1}{2\gamma^2}G_{5\phi}\Big({}^3R - K_{\mu\nu}K^{\mu\nu} + K^2\Big)\\ &\quad +(1/3)G_{5Y}\Big\{-\frac{K^3}{\gamma^3} + \frac{3}{\gamma^3}K\,K_{\alpha\beta}K^{\alpha\beta} - \frac{2}{\gamma^3}K^\mu_\nu K^\nu_\rho K^\rho_\mu\Big\},\end{aligned} \tag{2.190}$$

$${}^{\mathfrak{BD}}\mathcal{L}_5^H = -(\tilde{F}_{5Y} + \frac{\tilde{F}_5}{2Y})G_{\mu\nu}Y^\mu\phi^\nu + (1/3)(\tilde{F}_{5Y} + \frac{\tilde{F}_5}{2Y})\Big\{-\frac{3}{2}\phi^\lambda Y_\lambda\Big(K^2 - K_{\alpha\beta}K^{\alpha\beta}\Big)$$

$$
\begin{aligned}
& +\frac{6}{\gamma^3}\Big(K^\mu_\nu a^\nu a_\mu - K\, a_\mu a^\mu\Big)\Big\}, \\
= \;& -(\tilde{F}_{5Y}+\frac{\tilde{F}_5}{2Y})G_{\mu\nu}Y^\mu\phi^\nu + (1/3)(\tilde{F}_{5Y}+\frac{\tilde{F}_5}{2Y})\Big\{-\frac{3}{2}\phi^\lambda Y_\lambda\Big(K^2 - K_{\alpha\beta}K^{\alpha\beta}\Big) \\
& +\frac{6}{\gamma^3}\,\frac{\gamma^2}{2}q^\lambda_\mu Y_\lambda\,\Big(K^\mu_\nu a^\nu - K\, a^\mu\Big)\Big\}\,, \\
= \;& -(\tilde{F}_{5Y}+\frac{\tilde{F}_5}{2Y})G_{\mu\nu}Y^\mu\phi^\nu + (1/3)(\tilde{F}_{5Y}+\frac{\tilde{F}_5}{2Y})\Big\{-\frac{3}{2}\phi^\lambda Y_\lambda\Big(K^2 - K_{\alpha\beta}K^{\alpha\beta}\Big) \\
& +\frac{3}{\gamma}Y_\lambda\,\Big(K^\lambda_\nu a^\nu - K\, a^\lambda\Big)\Big\}, \\
= \;& -(\tilde{F}_{5Y}+\frac{\tilde{F}_5}{2Y})Y^\mu\cdot G_{\mu\nu}\phi^\nu - (\tilde{F}_{5Y}+\frac{\tilde{F}_5}{2Y})Y^\mu\cdot\Big\{\frac{1}{2}\phi_\mu\Big(K^2 - K_{\alpha\beta}K^{\alpha\beta}\Big) \\
& -\frac{1}{\gamma}\,\Big(K_{\mu\nu}a^\nu - K\, a_\mu\Big)\Big\},
\end{aligned}
\tag{2.191}
$$

By using

$$
(A_Y+\frac{A}{2Y})Y^\mu\cdot\square\!\!\!/\, = \nabla^\mu(A\square\!\!\!/) + \frac{A}{2Y}Y^\mu\square\!\!\!/ - A(\square\!\!\!/)^\mu - \square\!\!\!/ A_\phi\phi^\mu, \tag{2.192}
$$

where $\square\!\!\!/ \equiv$ '*something*'. In this case $A = \tilde{F}_5$ and $\square\!\!\!/ = -\tilde{F}_5 G_{\mu\nu}\phi^\nu - \frac{\phi_\mu}{2}\tilde{F}_5(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + \frac{\tilde{F}_5}{\gamma}(K_{\mu\nu}a^\nu - K\,a_\mu)$. The equation (2.191) now becomes

$$
\begin{aligned}
{}^{\mathfrak{BD}}\mathcal{L}^H_5 \;=\;& \nabla^\mu\Big(-\tilde{F}_5 G_{\mu\nu}\phi^\nu - \frac{\phi_\mu}{2}\tilde{F}_5(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + \frac{\tilde{F}_5}{\gamma}(K_{\mu\nu}a^\nu - K\,a_\mu)\Big) \\
& -\frac{\gamma^2}{2}\tilde{F}_5\Big(-G_{\mu\nu}\phi^\nu Y^\mu - \frac{\phi_\mu Y^\mu}{2}(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + \frac{1}{\gamma}Y^\mu(K_{\mu\nu}a^\nu - Ka_\mu)\Big) \\
& -\tilde{F}_5\Big(-G_{\mu\nu}\phi^{\nu\mu} - \frac{\square\phi}{2}(K^2 - K_{\alpha\beta}K^{\alpha\beta}) - \frac{\phi_\mu}{2}\nabla^\mu(K^2 - K_{\alpha\beta}K^{\alpha\beta}) \\
& +\frac{1}{\gamma}\nabla^\mu(K_{\mu\nu}a^\nu - Ka_\mu) - \frac{\gamma Y^\mu}{2}(K_{\mu\nu}a^\nu - Ka_\mu)\Big) \\
& -\Big(-G_{\mu\nu}\phi^\nu\phi^\mu\tilde{F}_{5\phi} - \frac{\phi_\mu}{2}\phi^\mu\tilde{F}_{5\phi}(K^2 - K_{\alpha\beta}K^{\alpha\beta})\Big)\,, \qquad (2.193) \\
\equiv\;& \nabla_\mu\mathcal{B}^\mu_{3b} + \mathcal{A} + \mathcal{B} + \mathcal{C}, \qquad (2.194)
\end{aligned}
$$

where the boundary terms

$$
\nabla_\mu\mathcal{B}^\mu_{3b} = \nabla^\mu\Big(-\tilde{F}_5 G_{\mu\nu}\phi^\nu - \frac{\phi_\mu}{2}\tilde{F}_5(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + \frac{\tilde{F}_5}{\gamma}(K_{\mu\nu}a^\nu - K\,a_\mu)\Big), \tag{2.195}
$$

while $\mathcal{A} = \frac{A}{2Y}Y^\mu\square\!\!\!/, \mathcal{B} = -A(\square\!\!\!/)^\mu$, and $\mathcal{C} = -\square\!\!\!/ A_\phi\phi^\mu$. Consider

$$
\begin{aligned}
\mathcal{C} \;=\;& -\Big(-G_{\mu\nu}\phi^\nu\phi^\mu\tilde{F}_{5\phi} - \frac{\phi_\mu}{2}\phi^\mu\tilde{F}_{5\phi}(K^2 - K_{\alpha\beta}K^{\alpha\beta})\Big), \qquad (2.196) \\
=\;& \frac{1}{\gamma^2}\Big(G_{\mu\nu}n^\mu n^\nu - \frac{1}{2}(K^2 - K_{\alpha\beta}K^{\alpha\beta})\Big)\,\tilde{F}_{5\phi}, \\
=\;& \frac{1}{\gamma^2}\tilde{F}_{5\phi}\Big(\frac{1}{2}({}^3R - K_{\mu\nu}K^{\mu\nu} + K^2) - \frac{1}{2}K^2 + \frac{1}{2}K_{\alpha\beta}K^{\alpha\beta}\Big),
\end{aligned}
$$

$$\mathcal{C} = \frac{1}{2\gamma^2}\tilde{F}_{5\phi}\,{}^3R, \tag{2.197}$$

which is in a good form. Next, we will consider the more complicate piece of the Lagrangian

$$\begin{aligned}\mathcal{A}+\mathcal{B} &= \tilde{F}_5\Big(\frac{\gamma^2}{2}G_{\mu\nu}\phi^\nu Y^\mu + \phi_\mu Y^\mu\frac{\gamma^2}{4}(K^2 - K_{\alpha\beta}K^{\alpha\beta})\underbrace{-\tfrac{\gamma}{2}Y^\mu(K_{\mu\nu}a^\nu - K\,a_\mu)}_{+\varnothing} \\ &\quad +G_{\mu\nu}\phi^{\nu\mu} + \frac{\Box\phi}{2}(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + \frac{\phi_\mu}{2}\nabla^\mu(K^2 - K_{\alpha\beta}K^{\alpha\beta}) \\ &\quad -\frac{1}{\gamma}\nabla^\mu(K_{\mu\nu}a^\nu - K a_\mu)\underbrace{+\tfrac{\gamma Y^\mu}{2}(K_{\mu\nu}a^\nu - K a_\mu)}_{-\varnothing}\Big).\end{aligned} \tag{2.198}$$

In the first term, we will use(2.147), $\frac{\gamma^2}{2}Y^\mu = a^\mu + \frac{\gamma^3}{2}n^\lambda Y_\lambda \phi^\mu$, then

$$\begin{aligned}\mathcal{A}+\mathcal{B} &= \tilde{F}_5\Big(\{a^\mu + \frac{\gamma^3}{2}n^\lambda Y_\lambda\phi^\mu\}G_{\mu\nu}\phi^\nu + \phi_\mu Y^\mu\frac{\gamma^2}{4}(K^2 - K_{\alpha\beta}K^{\alpha\beta}) \\ &\quad +G_{\mu\nu}\phi^{\nu\mu} + \frac{\Box\phi}{2}(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + \frac{\phi_\mu}{2}\nabla^\mu(K^2 - K_{\alpha\beta}K^{\alpha\beta}) \\ &\quad -\frac{1}{\gamma}\nabla^\mu(K_{\mu\nu}a^\nu - K a_\mu)\Big),\end{aligned} \tag{2.199}$$

$$\begin{aligned}&= -\frac{\tilde{F}_5}{\gamma}\Big(-\gamma G_{\mu\nu}\phi^{\nu\mu} + \frac{\gamma^3}{2}\phi^\lambda Y_\lambda G_{\mu\nu}n^\mu n^\nu + G_{\mu\nu}n^\nu a^\mu + n_\mu Y^\mu\frac{\gamma^2}{4}(K^2 - K_{\alpha\beta}K^{\alpha\beta}) \\ &\quad -\gamma\frac{\Box\phi}{2}(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + \frac{n_\mu}{2}\nabla^\mu((K^2 - K_{\alpha\beta}K^{\alpha\beta})) \\ &\quad +\nabla^\mu(K_{\mu\nu}a^\nu - K a_\mu)\Big).\end{aligned} \tag{2.200}$$

The first three terms look similar to $G_{\mu\nu}K^{\mu\nu}$, (2.150). Therefore, we may need some relations around this quantity. From equation (2.136), we have

$${}^3R_{\mu\nu}K^{\mu\nu} = R_{\mu\nu}K^{\mu\nu} + n^\rho n^\sigma R_{\mu\rho\nu\sigma}K^{\mu\nu} - K\,K^2_{\mu\nu} + k^3_{\mu\nu}\,, \tag{2.201}$$

which implies

$$\begin{aligned}G_{\mu\nu}K^{\mu\nu} &= R_{\mu\nu}K^{\mu\nu} - \frac{1}{2}K\,R \\ &= {}^3R_{\mu\nu}K^{\mu\nu} - n^\rho n^\sigma R_{\mu\rho\nu\sigma}K^{\mu\nu} + K\,K^2_{\mu\nu} - k^3_{\mu\nu} - \frac{1}{2}K\,R.\end{aligned}$$

Replacing the extrinsic curvature on the left-hand side of the above equation by (2.150)

$$K_{\mu\nu} = -\gamma\phi^{\mu\nu} + \frac{\gamma^3}{2}\phi^\lambda Y_\lambda n^\mu n^\nu + n^\mu a^\nu + n^\nu a^\mu. \tag{2.202}$$

Finally, we obtain

$$\begin{aligned}&-\gamma G_{\mu\nu}\phi^{\mu\nu}+\frac{\gamma^3}{2}\phi^\lambda Y_\lambda G_{\mu\nu}n^\mu n^\nu+G_{\mu\nu}n^\mu a^\nu+G_{\mu\nu}a^\mu n^\nu\\&={}^3R_{\mu\nu}K^{\mu\nu}-R_{\mu\rho\nu\sigma}n^\rho n^\sigma K^{\mu\nu}+KK^2_{\mu\nu}-K^3_{\mu\nu}-\frac{1}{2}KR.\end{aligned}\tag{2.203}$$

Use this identity for the first three terms and use (2.151)

$$-\gamma\Box\phi=K+\frac{\gamma^3}{2}\phi^\lambda Y_\lambda,\tag{2.204}$$

for the $\Box\phi$ term of the equation (2.200). It now becomes

$$\begin{aligned}\mathcal{A}+\mathcal{B}\;=\;&-\frac{\tilde{F}_5}{\gamma}\Big(\underbrace{-G_{\mu\nu}n^\mu a^\nu}_{-R_{\mu\nu}n^\mu a^\nu}+{}^3R_{\mu\nu}K^{\mu\nu}-R_{\mu\rho\nu\sigma}n^\rho n^\sigma K^{\mu\nu}+KK^2_{\mu\nu}-K^3_{\mu\nu}-\frac{1}{2}KR\\&\underbrace{+n_\mu Y^\mu\tfrac{\gamma^2}{4}(K^2-K_{\alpha\beta}K^{\alpha\beta})}_{\varnothing}+(\frac{1}{2}K^3-\frac{1}{2}KK^2_{\alpha\beta})\underbrace{+\tfrac{\gamma^3}{4}\phi^\lambda Y_\lambda(K^2-K_{\alpha\beta}K^{\alpha\beta})}_{-\varnothing}\\&+\frac{n_\mu}{2}\nabla^\mu(K^2-K_{\alpha\beta}K^{\alpha\beta})+\nabla^\mu(K_{\mu\nu}a^\nu-Ka_\mu)\Big),\end{aligned}\tag{2.205}$$

$$\begin{aligned}=\;&-\frac{\tilde{F}_5}{\gamma}\Big(-R_{\mu\nu}n^\mu a^\nu+{}^3R_{\mu\nu}K^{\mu\nu}-R_{\mu\rho\nu\sigma}n^\rho n^\sigma K^{\mu\nu}+\frac{1}{2}KK^2_{\mu\nu}-K^3_{\mu\nu}\\&+\frac{1}{2}K^3-\frac{1}{2}K({}^3R+K_{\mu\nu}K^{\mu\nu}-K^2-2\nabla_\mu(a^\mu-n^\mu K))\\&+\frac{n_\mu}{2}\nabla^\mu(K^2-K_{\alpha\beta}K^{\alpha\beta})+\nabla^\mu(K_{\mu\nu}a^\nu-Ka_\mu)\Big),\end{aligned}\tag{2.206}$$

$$\begin{aligned}=\;&-\frac{\tilde{F}_5}{\gamma}\Big(-R_{\mu\nu}n^\mu a^\nu-R_{\mu\rho\nu\sigma}n^\rho n^\sigma K^{\mu\nu}+{}^3G_{\mu\nu}K^{\mu\nu}-K^3_{\mu\nu}+K^3+K\nabla_\mu(a^\mu-n^\mu K)\\&+\frac{n_\mu}{2}\nabla^\mu(K^2-K_{\alpha\beta}K^{\alpha\beta})+\nabla^\mu(K_{\mu\nu}a^\nu-Ka_\mu)\Big)\end{aligned}\tag{2.207}$$

where ${}^3G_{\mu\nu}\equiv{}^3R_{\mu\nu}-\frac{1}{2}q_{\mu\nu}{}^3R$. Next, we will take care about the terms with covariant derivative

$$\begin{aligned}\mathcal{A}+\mathcal{B}\;=\;&-\frac{\tilde{F}_5}{\gamma}\Big({}^3G_{\mu\nu}K^{\mu\nu}-R_{\mu\nu}n^\mu a^\nu-R_{\mu\rho\nu\sigma}n^\rho n^\sigma K^{\mu\nu}-K^3_{\mu\nu}+K^3\\&+K\nabla_\mu a^\mu-Kn^\mu\nabla_\mu K-\underbrace{K^2\nabla_\mu n^\mu}_{K^3}+\frac{n_\mu}{2}\cdot 2\{K\nabla^\mu K-K^{\alpha\beta}\nabla^\mu K_{\alpha\beta}\}\\&+a^\nu\nabla^\mu K_{\mu\nu}+K_{\mu\nu}\nabla^\mu a^\nu-K\nabla^\mu a_\mu-a_\mu\nabla^\mu K\Big),\end{aligned}\tag{2.208}$$

$$\begin{aligned}=\;&-\frac{\tilde{F}_5}{\gamma}\Big({}^3G_{\mu\nu}K^{\mu\nu}-R_{\mu\nu}n^\mu a^\nu-R_{\mu\rho\nu\sigma}n^\rho n^\sigma K^{\mu\nu}-K^3_{\mu\nu}\\&-n_\mu K^{\alpha\beta}\nabla^\mu K_{\alpha\beta}+a^\nu\nabla^\mu K_{\mu\nu}+K_{\mu\nu}\nabla^\mu a^\nu-a_\mu\nabla^\mu K\Big).\end{aligned}\tag{2.209}$$

We will try to modify the second and the third terms in more simple form. For the second term, we consider

$$R_{\mu\nu}n^{\mu} = R^{\alpha}{}_{\mu\alpha\nu}n^{\mu} \;=\; \nabla_\alpha\nabla_\nu n^\alpha - \nabla_\nu\nabla_\alpha n^\alpha,$$
$$= \nabla_\alpha(K^\alpha_\nu - n_\nu a^\alpha) - \nabla_\nu K,$$
$$= \nabla_\alpha K^\alpha_\nu - n_\nu\nabla_\alpha a^\alpha - a^\alpha\nabla_\alpha n_\nu - \nabla_\nu K. \qquad (2.210)$$

Therefore,

$$R_{\mu\nu}n^\mu a^\nu \;=\; a^\nu\nabla_\alpha K^\alpha_\nu - a^\alpha a^\nu\nabla_\alpha n_\nu - a^\nu\nabla_\nu K,$$
$$= a^\nu\nabla_\alpha K^\alpha_\nu - K_{\alpha\nu}a_\nu a^\alpha - a^\nu\nabla_\nu K. \qquad (2.211)$$

For the third term, we have

$$R_{\mu\rho\nu\sigma}n^\rho \;=\; \nabla_\nu\nabla_\sigma n_\mu - \nabla_\sigma\nabla_\nu n_\mu,$$
$$= \nabla_\nu(K_{\sigma\mu} - n_\sigma a_\mu) - \nabla_\sigma(K_{\nu\mu} - n_\nu a_\mu),$$
$$= \nabla_\nu K_{\sigma\mu} - a_\mu\nabla_\nu n_\sigma - n_\sigma\nabla_\nu a_\mu - \nabla_\sigma K_{\mu\nu} + n_\nu\nabla_\sigma a_\mu + a_\mu\nabla_\sigma n_\nu. \qquad (2.212)$$

Then

$$R_{\mu\rho\nu\sigma}n^\rho n^\sigma = n^\sigma\nabla_\nu K_{\sigma\mu} - a_\mu\underbrace{n^\sigma\nabla_\nu n_\sigma}_{0} + \nabla_\nu a_\mu - n^\sigma\nabla_\sigma K_{\mu\nu} + n^\sigma n_\nu\nabla_\sigma a_\mu + n^\sigma a_\mu\nabla_\sigma n_\nu. \qquad (2.213)$$

Hence, the third term reads

$$R_{\mu\rho\nu\sigma}n^\rho n^\sigma K^{\mu\nu} \;=\; n^\sigma K^{\mu\nu}\nabla_\nu K_{\sigma\mu} + K^{\mu\nu}\nabla_\nu a_\mu - n^\sigma K^{\mu\nu}\nabla_\sigma K_{\mu\nu} + \underbrace{n^\sigma K^{\mu\nu}a_\mu\nabla_\sigma n_\nu}_{K^{\mu\nu}a_\mu a_\nu},$$
$$= \nabla_\nu(K^{\mu\nu}\underbrace{n^\sigma K_{\sigma\mu}}_{0}) - \nabla_\nu K^{\mu\nu}\underbrace{n^\sigma K_{\sigma\mu}}_{0}\underbrace{-\nabla_\nu n^\sigma K^{\mu\nu}K_{\mu\sigma}}_{-K^3_{\mu\nu}}$$
$$+K^{\mu\nu}\nabla_\nu a_\mu - n^\sigma K^{\mu\nu}\nabla_\sigma K_{\mu\nu} + K^{\mu\nu}a_\mu a_\nu,$$
$$R_{\mu\rho\nu\sigma}n^\rho n^\sigma K^{\mu\nu} \;=\; -K^3_{\mu\nu} + K^{\mu\nu}\nabla_\nu a_\mu - n^\sigma K^{\mu\nu}\nabla_\sigma K_{\mu\nu} + K^{\mu\nu}a_\mu a_\nu. \qquad (2.214)$$

Inserting (2.211) and (2.214) into (2.209) yields

$$\mathcal{A}+\mathcal{B} \;=\; -\frac{\tilde{F}_5}{\gamma}\Big({}^3G_{\mu\nu}K^{\mu\nu} + \big[-a^\nu\nabla_\alpha K^\alpha_\nu + K_{\alpha\nu}a_\nu a^\alpha + a^\nu\nabla_\nu K\big]$$
$$+\big[K^3_{\mu\nu} - K^{\mu\nu}\nabla_\nu a_\mu + n^\sigma K^{\mu\nu}\nabla_\sigma K_{\mu\nu} - K^{\mu\nu}a_\mu a_\nu\big]$$
$$-K^3_{\mu\nu} - n_\mu K^{\alpha\beta}\nabla^\mu K_{\alpha\beta} + a^\nu\nabla^\mu K_{\mu\nu} + K_{\mu\nu}\nabla^\mu a^\nu - a_\mu\nabla^\mu K\Big) \qquad (2.215)$$

$$\mathcal{A}+\mathcal{B} \;=\; -\frac{\tilde{F}_5}{\gamma}\,{}^3G_{\mu\nu}K^{\mu\nu}. \qquad (2.216)$$

From the equations (2.186), (2.189), (2.190), (2.194), (2.195), (2.197) and (2.216), we can conclude that

$$\mathcal{L}_5^H = -\sqrt{-Y}\tilde{F}_5\, K^{\mu\nu}\Big({}^3R_{\mu\nu} - \frac{1}{2}q_{\mu\nu}\,{}^3R\Big) - \frac{1}{3}(-Y)^{3/2}G_{5Y}\mathcal{K} + \frac{1}{2}Y\,(G_{5\phi} - \tilde{F}_{5\phi})\,{}^3R$$
$$+\frac{1}{2}YG_{5\phi}(\,K^2 - K_{\mu\nu}K^{\mu\nu}) + \nabla_\mu\mathcal{B}_3^\mu, \tag{2.217}$$

where

$$\mathcal{K} \equiv [K^3] \equiv \delta^{\alpha\beta\gamma}_{\mu\nu\rho}K^\mu_\alpha K^\nu_\beta K^\rho_\gamma = K^3 - 3KK^2_{\mu\nu} + 2K^3_{\mu\nu}, \tag{2.218}$$

and the boundary terms

$$\nabla_\mu\mathcal{B}_3^\mu \equiv \nabla_\mu\mathcal{B}^\mu_{3a} + \nabla_\mu\mathcal{B}^\mu_{3b} = \nabla^\mu\Big((G_5 - \tilde{F}_5)G_{\mu\nu}\phi^\nu - \frac{\phi_\mu}{2}\tilde{F}_5(K^2 - K_{\alpha\beta}K^{\alpha\beta}) + \frac{\tilde{F}_5}{\gamma}(K_{\mu\nu}a^\nu - K\,a_\mu)\Big). \tag{2.219}$$

In ADM coordinate and up to the boundary terms

$$\mathcal{L}_5^H = -\sqrt{-Y}\tilde{F}_5\, K^{ab}\Big({}^3R_{ab} - \frac{1}{2}q_{ab}\,{}^3R\Big) - \frac{1}{3}(-Y)^{3/2}G_{5Y}\mathcal{K} + \frac{1}{2}Y\,(G_{5\phi} - \tilde{F}_{5\phi})\,{}^3R$$
$$+\frac{1}{2}YG_{5\phi}(\,K^2 - K_{ab}K^{ab}). \tag{2.220}$$

We will denote Horndeski Lagrangians in the final spatially covariant form different from their original form to remark that they are in ADM variables and in unitary gauge. We list them below in the mixed-form

$$\mathcal{L}_2^{H^{3+1}_{mixed}} = G_2, \tag{2.221}$$
$$\mathcal{L}_3^{H^{3+1}_{mixed}} = -F_{3\phi}Y - \frac{2Y}{\gamma}F_{3Y}K + \nabla_\mu\mathcal{B}_1^\mu, \tag{2.222}$$
$$\mathcal{L}_4^{H^{3+1}_{mixed}} = G_4\,{}^3R + (2YG_{4Y} - G_4)\Big(K^2 - K_{\mu\nu}K^{\mu\nu}\Big) - 2\sqrt{-Y}G_{4\phi}K + \nabla_\mu\mathcal{B}_2^\mu, \tag{2.223}$$
$$\mathcal{L}_5^{H^{3+1}_{mixed}} = -\sqrt{-Y}\tilde{F}_5\, K^{\mu\nu}\Big({}^3R_{\mu\nu} - \frac{1}{2}q_{\mu\nu}\,{}^3R\Big) - \frac{1}{3}(-Y)^{3/2}G_{5Y}\mathcal{K} + \frac{1}{2}Y\,(G_{5\phi} - \tilde{F}_{5\phi})\,{}^3R$$
$$+\frac{1}{2}YG_{5\phi}(\,K^2 - K_{\mu\nu}K^{\mu\nu}) + \nabla_\mu\mathcal{B}_3^\mu, \tag{2.224}$$

where $G_2, G_3, G_4,$ and G_5 are the coefficient functions of the general covariant Horndeski action, F_3 and $\tilde{F}_5$ are given by

$$G_3 \equiv F_3 + 2YF_{3Y},$$
$$G_{5Y} \equiv \tilde{F}_{5Y} + \frac{\tilde{F}_5}{2Y}\quad.$$

The total boundary terms now becomes

$$\begin{aligned}\nabla_\mu \mathcal{B}^\mu &\equiv \nabla_\mu\Big(\mathcal{B}_1^\mu + \mathcal{B}_2^\mu + \mathcal{B}_3^\mu\Big)\,,\\ &= \nabla_\mu\Big(-\frac{1}{\gamma}F_3 n^\mu - 2G_4(a^\mu - n^\mu K) - \frac{1}{\gamma}(G_5 - \tilde{F}_5)G_{\mu\nu}n^\nu + \frac{n_\mu}{2\gamma}\tilde{F}_5(K^2 - K_{\alpha\beta}K^{\alpha\beta})\\ &\quad + \frac{\tilde{F}_5}{\gamma}(K_{\mu\nu}a^\nu - K\, a_\mu)\Big).\end{aligned} \tag{2.225}$$

Then the Horndeski Lagrangians in the pure-form(up to the boundary terms) read

$$\mathcal{L}_2^{H,3+1} \equiv A_2(t,N), \tag{2.226}$$
$$\mathcal{L}_3^{H,3+1} \equiv A_3(t,N)K, \tag{2.227}$$
$$\mathcal{L}_4^{H,3+1} \equiv A_4(t,N)\Big(K^2 - K_{\mu\nu}K^{\mu\nu}\Big) + B_4(t,N)\,{}^3R, \tag{2.228}$$
$$\begin{aligned}\mathcal{L}_5^{H,3+1} \equiv\ & A_5(t,N)\Big(K^3 - 2KK_{\mu\nu}K^{\mu\nu} + 3K_{\mu\nu}K^{\nu\rho}K^\mu_\rho\Big)\\ & + B_5(t,N)K^{\mu\nu}\Big({}^3R_{\mu\nu} - (1/2)q_{\mu\nu}\,{}^3R\Big),\end{aligned} \tag{2.229}$$

which are equivalents to the form written in ADM coordinate

$$\mathcal{L}_2^{H,3+1} \equiv A_2(t,N), \tag{2.230}$$
$$\mathcal{L}_3^{H,3+1} \equiv A_3(t,N)K, \tag{2.231}$$
$$\mathcal{L}_4^{H,3+1} \equiv A_4(t,N)\Big(K^2 - K_{ab}K^{ab}\Big) + B_4(t,N)\,{}^3R, \tag{2.232}$$
$$\begin{aligned}\mathcal{L}_5^{H,3+1} \equiv\ & A_5(t,N)\Big(K^3 - 2KK_{ab}K^{ab} + 3K_{ab}K^{bc}K^a_c\Big)\\ & + B_5(t,N)K^{ab}\Big({}^3R_{ab} - (1/2)q_{ab}\,{}^3R\Big),\end{aligned} \tag{2.233}$$

where the coefficient functions are given by

$$A_2 = G_2 - Y\,F_{3\phi}\,, \tag{2.234}$$
$$A_3 = 2(-Y)^{3/2}F_{3Y} - 2(-Y)^{1/2}G_{4\phi}\,, \tag{2.235}$$
$$A_4 = 2YG_{4Y} - G_4 + \frac{1}{2}YG_{5\phi}\,, \tag{2.236}$$
$$B_4 = G_4 + \frac{1}{2}Y\Big(G_{5\phi} - \tilde{F}_{5\phi}\Big)\,, \tag{2.237}$$
$$A_5 = -\frac{1}{3}(-Y)^{3/2}G_{5Y} \tag{2.238}$$
$$B_5 = -(-Y)^{1/2}\tilde{F}_5\,. \tag{2.239}$$

Please note that these six functions are not linearly independent. They are depend on four independent functions $G_2, G_3, G_4,$ andG_5.

2.4 Gleyzes-Langlois-Piazza-Vernizzi Theories (GLPV)

Relaxing of the conditions (2.234), (2.235), (2.236), (2.237), (2.238), and (2.239), and respect $A2, A3, A_4, A_5, B_4$, and B_5 as the independent functions gives us a larger class of the healthy scalar-tensor theories called *GLPV theory*. The Lagrangians then read

$$\mathcal{L}_2^{GLPV,3+1} \equiv A_2(t,N), \tag{2.240}$$
$$\mathcal{L}_3^{GLPV,3+1} \equiv A_3(t,N)K, \tag{2.241}$$
$$\mathcal{L}_4^{GLPV,3+1} \equiv A_4(t,N)\Big(K^2 - K_{ab}K^{ab}\Big) + B_4(t,N)\,{}^3R, \tag{2.242}$$
$$\mathcal{L}_5^{GLPV,3+1} \equiv A_5(t,N)\Big(K^3 - 2KK_{ab}K^{ab} + 3K_{ab}K^{bc}K^a_c\Big) + B_5(t,N)K^{ab}\Big({}^3R_{ab} - (1/2)q_{ab}\,{}^3R\Big), \tag{2.243}$$

Therefore, this theory emerges from the generalisation of Horndeski theory in ADM variables with unitary gauge fixing. The general covariance version of the theories in this class can be recovered by using the so called the *Stueckelberg trick* which gives us the covariant version of GLPV.

The counting of degrees of freedom for GLPV theory has been done in [34]. In contrary to the standard GR, The Hamiltonian analysis for GLPV theory gives us fourteen second class constraints. Therefore the number of phase space degrees of freedom becomes $20 - 14 = 6$ implies the number of physical dof is 3. two of them describe gravity and the another one describes a scalar field. This scalar dof can appears explicitly in the formulation by means of the Stueckelberg trick as we will be shown below.

Since unitary gauge fixing condition breaks general covariance by introducing time using the scalar field. To recover the full spacetime general covariance back, one must do the reverse, introducing the scalar field (the Stueckelberg field) by the helps of time parameter. Then, the Stueckelberg trick in our case is induced by the mapping

$$t \longmapsto \phi\,. \tag{2.244}$$

Therefore

$$\dot{\phi} \longmapsto \frac{\partial\phi}{\partial\phi} = 1\,, \tag{2.245}$$

and from (2.155), one has

$$N \longmapsto \gamma \quad . \tag{2.246}$$

Furthermore, for the tensorial quantities we deselected the preferred ADM coordinate back to the general spacetime coordinate and the scalar quantities then threat as the quantities associated to the scalar field induced by the mapping (2.244).

For example,

$$
\begin{aligned}
n^a \mapsto n^\mu &= -\gamma\phi^\mu \, . && (2.247)\\
q_{ab} \longmapsto q_{\mu\nu} &= g_{\mu\nu} + \gamma^2\phi_\mu\phi_\nu \, , && (2.248)\\
&= g_{\mu\nu} - \frac{1}{Y}\phi_\mu\phi_\nu \, , && (2.249)\\
D_a N \mapsto D_\mu N &= q^\lambda_\mu \nabla_\mu N \, , && (2.250)\\
&= (\delta^\lambda_\mu + \gamma^2\phi^\lambda\phi_\mu)\nabla_\mu(\gamma) \, , && (2.251)\\
&= -\frac{1}{2\sqrt{-Y}}[(\ln Y)_\mu - \frac{\phi^\lambda\phi_\mu}{Y}(\ln Y)_\lambda] \, , && (2.252)\\
K_{ab} \mapsto K_{\mu\nu} &= q^\lambda_\mu \nabla_\lambda n_\nu = (\delta^\lambda_\mu + \gamma^2\phi^\lambda\phi_\mu)\nabla_\lambda(-\gamma\phi_\nu) \, , && (2.253)\\
&= \frac{-1}{\sqrt{-Y}}\left\{\phi_{\mu\nu} - \phi_{(\mu}\nabla_{\nu)}\ln Y + \frac{1}{2Y}\phi^\lambda\phi_\mu\phi_\nu\nabla_\lambda \ln Y\right\} && (2.254)\\
K = q_{ab}K^{ab} \mapsto q^{\mu\nu}K_{\mu\nu} &= g^{\mu\nu}K_{\mu\nu} \, , && (2.255)\\
&= g^{\mu\nu}[\frac{-1}{\sqrt{-Y}}\left\{\phi_{\mu\nu} - \phi_{(\mu}\nabla_{\nu)}\ln Y + \frac{1}{2Y}\phi^\lambda\phi_\mu\phi_\nu \ln Y\right\}] \, , && (2.256)\\
&= -\gamma\Box\phi - \frac{\gamma^3}{2}\phi^\lambda Y_\lambda \, , && (2.257)\\
&= \frac{-1}{\sqrt{-Y}}\Big(\Box\phi - \frac{1}{2}\phi^\lambda(\ln Y)_\lambda\Big) \, , && (2.258)\\
a_a \mapsto a_\mu &= n^\nu\nabla_\nu n_\mu \, , && (2.259)\\
&= \frac{1}{2Y}\phi^\nu\phi_\mu\nabla_\nu \ln Y - \frac{1}{2}\nabla_\mu \ln Y \, , && (2.260)
\end{aligned}
$$

where the more complicate quantities in the GLPV Lagrangian ${}^3R$ and ${}^3R_{\alpha\beta}$ can be computed by using these quantities. By (2.244) and (2.246). Now, we can proceeding in calculate the covariant form of the GLPV Lagrangians

$$\mathcal{L}_2^{GLPV,3+1} \equiv A_2(t,N) \longmapsto A_2(\phi,Y) \, . \qquad (2.261)$$

By using (2.254), one has

$$\mathcal{L}_3^{GLPV,3+1} = -A_3\gamma\Box\phi - A_3\frac{\gamma^3}{2}\phi^\lambda Y_\lambda \, . \qquad (2.262)$$

Observing that

$$C_Y Y_\mu \phi^\mu = \nabla_\mu(\phi^\mu C) - C\,\Box\phi - C_\phi Y \, , \qquad (2.263)$$

we can redefine the second term in (2.262) as $A_3\frac{\gamma^3}{2} \equiv \tilde{C}_{3Y}$. Then we can write

$$
\begin{aligned}
\mathcal{L}_3^{GLPV,3+1} &= -A_3\gamma\Box\phi - \tilde{C}_{3Y}\phi^\lambda Y_\lambda \, , && (2.264)\\
&= -A_3\gamma\Box\phi - \nabla_\mu(\phi^\mu\tilde{C}_3) + \tilde{C}_3\,\Box\phi + \tilde{C}_{3\phi}Y \, , && (2.265)\\
&= (-A_3\gamma + \tilde{C}_3)\Box\phi + \tilde{C}_{3\phi}Y - \nabla_\mu(\phi^\mu\tilde{C}_3) \, . && (2.266)
\end{aligned}
$$

Ignoring the boundary term and using

$$A_3 \frac{\gamma^3}{2} \equiv \tilde{C}_{3Y} \Rightarrow -A\,\gamma = 2Y\,C_Y, \tag{2.267}$$

we obtain

$$\mathcal{L}_3^{GLPV,3+1} = (\tilde{C}_3 + 2Y\,\tilde{C}_{3Y})\Box\phi + \tilde{C}_{3\phi}Y. \tag{2.268}$$

For L_4, by using (2.127)

$$\begin{aligned} \mathcal{L}_4^{GLPV,3+1} &= A_4\Big(K^2 - K_{\mu\nu}K^{\mu\nu}\Big) + B_4\Big(K_{\mu\nu}K^{\mu\nu} - K^2 + R + 2n^\sigma n^\rho R_{\rho\sigma}\Big), \\ &= B_4\,R + (A_4 - B_4)\Big(K^2 - K_{\mu\nu}K^{\mu\nu}\Big) + 2B_4 n^\mu n^\nu R_{\mu\nu}\,. \end{aligned} \tag{2.269}$$

We will calculate the necessary quantities as the following

$$\begin{aligned} K^2 &= \Big(-\gamma\Box\phi - \frac{\gamma^3}{2}\phi^\lambda Y_\lambda\Big)^2, \\ &= \gamma^2\Box\phi^2 + \gamma^4\Box\phi\,\phi^\lambda Y_\lambda + \frac{\gamma^6}{4}\phi^\lambda Y_\lambda \phi^\omega Y_\omega && (2.270) \\ &= \gamma^2\Big\{\Box\phi^2 + 2\,\gamma^2\Box\phi\,\triangle\cdot\phi + \gamma^4(\triangle\cdot\phi)^2\Big\}\,. && (2.271) \end{aligned}$$

$$\begin{aligned} K^{\mu\nu}K_{\mu\nu} &= \gamma^2\Big\{\overset{(a)}{\phi_{\mu\nu}} + \overset{(b)}{2\gamma^2\phi_{(\mu}\phi_{\nu)\beta}\phi^\beta} + \overbrace{\gamma^4\phi^\lambda\phi_\mu\phi_\nu\phi_{\lambda\beta}\phi^\beta}^{c}\Big\} && (2.272) \\ &\quad \times\Big\{\overbrace{\phi^{\mu\nu}}^{d} + \overbrace{2\gamma^2\phi^{(\mu}\phi^{\nu)\omega}\phi^\omega}^{e} + \overbrace{\gamma^4\phi^\eta\phi^\mu\phi^\nu\phi_{\eta\omega}\phi^\omega}^{f}\Big\}, && (2.273) \\ &= \gamma^2\Big\{\overbrace{\phi^{\mu\nu}\phi_{\mu\nu}}^{ad} + \overbrace{2\gamma^2\triangle\cdot\triangle}^{ae} + \overbrace{\gamma^4(\triangle\cdot\phi)^2}^{af} + \overbrace{2\gamma^2\triangle\cdot\triangle}^{bd} && (2.274) \\ &\quad + \overbrace{2\gamma^4\big(Y\triangle\cdot\triangle + (\triangle\cdot\phi)^2\big)}^{be} - \overbrace{2\gamma^4(\triangle\cdot\phi)^2}^{bf} + \overbrace{\gamma^4(\triangle\cdot\phi)^2}^{cd} && (2.275) \\ &\quad - \overbrace{2\,\gamma^4(\triangle\cdot\phi)^2}^{ce} + \overbrace{\gamma^8\,Y^2(\triangle\cdot\phi)^2}^{cf}\Big\}, && (2.276) \\ &= \gamma^2\Big\{\phi^2_{\mu\nu} + 2\,\gamma^2\triangle\cdot\triangle + \gamma^4\,(\triangle\cdot\phi)^2\Big\}, && (2.277) \\ & && (2.278) \end{aligned}$$

where we have introduced a shorthand notation for using between the calculations such that

$$\begin{aligned} \triangle_\mu &:= \phi_{\mu\nu}\,\phi^\nu\,, && (2.279) \\ \bigcirc_\mu^\nu &:= \phi_{\mu\beta}\phi^{\beta\nu}\,. && (2.280) \\ & && (2.281) \end{aligned}$$

Note also that, $2\,\triangle\cdot\phi = Y_\mu\phi^\mu$ and $4\,\triangle\cdot\triangle = Y_\mu Y^\mu$. Therefore

$$\begin{aligned}\mathcal{L}_4^{GLPV} &= B_4\,R + \gamma^2\,(A_4 - B_4)\Big\{(\Box\phi^2 + 2\,\gamma^2\Box\phi\,\triangle\cdot\phi + \gamma^4(\triangle\cdot\phi)^2) \quad (2.282)\\ &\quad -(\phi_{\mu\nu}^2 + 2\,\gamma^2\triangle\cdot\triangle + \gamma^4\,(\triangle\cdot\phi)^2)\Big\} + 2\gamma^2\,B_4\phi^\mu\phi^\nu R_{\mu\nu}\,,\\ &= B_4\,R + \gamma^2\,(A_4 - B_4)\Big\{\Box\phi^2 + 2\,\gamma^2\Box\phi\,\triangle\cdot\phi \quad (2.283)\\ &\quad -\phi_{\mu\nu}^2 - 2\,\gamma^2\triangle\cdot\triangle\Big\} + 2\gamma^2\,B_4\phi^\mu\phi^\nu R_{\mu\nu}\,.\end{aligned}$$

For the last term

$$\begin{aligned}\gamma^2\,B_4\phi^\mu\phi^\nu R_{\mu\nu} &= \gamma^2\,B_4\phi^\nu[\nabla_\mu,\nabla_\nu]\phi^\mu\,, \quad (2.284)\\ &= \gamma^2\,B_4\phi^\nu\nabla_\mu\nabla_\nu\phi^\mu - \gamma^2\,B_4\phi^\nu\nabla_\nu\nabla_\mu\phi^\mu\,, \quad (2.285)\\ &\equiv A - B\,. \quad (2.286)\end{aligned}$$

$$\begin{aligned}A &= \gamma^2\,B_4\phi^\nu\nabla_\mu\nabla_\nu\phi^\mu = \nabla_\mu\big[\gamma^2\,B_4\phi^\nu\nabla_\nu\phi^\mu\big] - \nabla_\mu\big(\gamma^2\,B_4\phi^\nu\big)\nabla_\nu\phi^\mu\,, \quad (2.287)\\ &= \nabla_\mu\big[\gamma^2\,B_4\phi^\nu\nabla_\nu\phi^\mu\big] - (\nabla_\mu\gamma^2)B_4\,\phi^\nu\phi^\mu_\nu - \gamma^2(\nabla_\mu B_4)\phi^\nu\phi^\mu_\nu - \gamma^2 B_4\phi^\mu_\nu\phi^\nu_\mu\,,\\ &= \nabla\cdot\big[\gamma^2\,B_4\triangle\big] - B_4\gamma^4 Y_\mu\phi^\nu\phi^\mu_\nu - \gamma^2\left(B_{4Y}Y_\mu + B_{4\phi}\,\phi_\mu\right)\phi^\nu\phi^\mu_\nu - \gamma^2\,B_4\phi_{\mu\nu}\phi^{\mu\nu}\,,\\ &= \nabla\cdot\big[\gamma^2\,B_4\triangle\big] - B_4\gamma^4(Y\cdot\triangle) - \gamma^2 B_{4Y}(Y\cdot\triangle) - \gamma^2 B_{4\phi}(\triangle\cdot\phi) - \gamma^2 B_4\,\phi^2_{\mu\nu}\,,\\ &= \nabla\cdot\big[\gamma^2\,B_4\triangle\big] - 2B_4\gamma^4(\triangle\cdot\triangle) - 2\gamma^2 B_{4Y}(\triangle\cdot\triangle) - \gamma^2 B_{4\phi}(\triangle\cdot\phi) - \gamma^2 B_4\,\phi^2_{\mu\nu}\,.\end{aligned}$$
(2.288)

Similarly, for the term B

$$\begin{aligned}B &= \gamma^2 B_4\,\phi^\nu\nabla_\nu(\Box\phi)\,, \quad (2.289)\\ &= \ldots \quad (2.290)\\ &= \nabla\cdot\big[\gamma^2 B_4\Box\phi\,\phi\big] - 2\gamma^4 B_4(\triangle\cdot\phi)\Box\phi - 2\gamma^2 B_{4Y}(\triangle\cdot\phi)\Box\phi - \gamma^2 B_{4\phi}\,Y\Box\phi - \gamma^2 B_4\Box\phi^2\,.\end{aligned}$$
(2.291)

Then the equation (2.269) becomes

$$\begin{aligned}\mathcal{L}_4^{GLPV} &= B_4\,R + \gamma^2\,(A_4 - B_4)\Big\{\Box\phi^2 + 2\gamma^2\,\Box\phi\,\triangle\cdot\phi - \phi^2_{\mu\nu} - 2\gamma^2\,\triangle\cdot\triangle\Big\}\\ &\quad +2\gamma^2\Big\{-2B_4\gamma^2\triangle\cdot\triangle - 2B_{4Y}(\triangle\cdot\triangle) - B_{4\phi}(\triangle\cdot\phi) - B_4\phi^2_{\mu\nu} + 2\gamma^2 B_4(\triangle\cdot\phi)\Box\phi\\ &\quad +2B_{4Y}(\triangle\cdot\phi)\Box\phi + B_{4\phi}Y\Box\phi + B_4\Box\phi^2\Big\}\,.\end{aligned}$$
(2.292)

By comparing with the Galileon Lagrangian (2.8)

$$\begin{aligned}\mathcal{L}_4^{\mathrm{Gal},1} &= \frac{1}{1!}\epsilon^{\mu_1\mu_2\mu_3\delta}\epsilon_{\nu_1\nu_2\nu_3\delta}\phi^{\nu_3}\phi_{\mu_3}\phi^{\nu_1}_{\mu_1}\phi^{\nu_2}_{\mu_2}\\ &= -Y(\Box\phi^2 - \phi^2_{\mu\nu}) - 2\,\phi_\mu\phi^{\mu\nu}\phi_{\nu\alpha}\phi^\alpha + 2\Box\phi\,\phi_{\mu\nu}\phi^\mu\phi^\nu\,,\\ &= -Y(\Box\phi^2 - \phi^2_{\mu\nu}) - 2\,(\triangle\cdot\triangle) + 2\Box\phi\,(\triangle\cdot\phi)\,. \quad (2.293)\end{aligned}$$

We con observe that the first bracket of (2.292) proportional to the the $\mathcal{L}_4^{\mathrm{Gal},1}$, we may use this clue to simplify our equation into the terms of galileon. The equation

(2.292) becomes so far

$$\begin{aligned}
\mathcal{L}_4^{GLPV} &= B_4\,R + \gamma^4\,(A_4 - B_4)\mathcal{L}_4^{gal,1} + 2\,\gamma^4\Big\{B_4\,\mathcal{L}_4^{gal,1} + 2YB_{4Y}(\Delta\cdot\Delta) - 2YB_{4Y}(\Delta\cdot\phi)\Box\phi \\
&\quad +YB_{4\phi}(\Delta\cdot\phi) - B_{4\phi}Y^2\Box\phi\Big\}\,, \qquad (2.294)\\
&= B_4\,R + \gamma^4\,(A_4 - B_4)\mathcal{L}_4^{gal,1} + 2\,\gamma^4\Big\{B_4\,\mathcal{L}_4^{gal,1} - YB_{4Y}\mathcal{L}_4^{gal,1} - (\Box\phi^2 - \phi_{\mu\nu}^2)Y^2B_{4Y} \\
&\quad +YB_{4\phi}(\Delta\cdot\phi) - B_{4\phi}Y^2\Box\phi\Big\}\,, \qquad (2.295)\\
&= B_4\,R + \frac{(A_4 + B_4 - 2YB_{4Y})}{Y^2}\mathcal{L}_4^{gal,1} - 2B_{4Y}\left(\Box\phi^2 - \phi_{\mu\nu}^2\right) - 2\gamma^2B_{4\phi}(\Delta\cdot\phi) \\
&\quad -2B_{4\phi}\Box\phi\,. \qquad (2.296)
\end{aligned}$$

The last two terms then read

$$\begin{aligned}
-2\gamma^2B_{4\phi}(\Delta\cdot\phi) - 2B_{4\phi}\Box\phi &= -2\gamma^2B_{4\phi}\phi_{\mu\nu}\phi^\mu\phi^\nu - 2B_{4\phi}\Box\phi\,, \qquad (2.297)\\
&= -\gamma^2B_{4\phi}Y_\nu\phi^\nu - 2B_{4\phi}\Box\phi\,. \qquad (2.298)
\end{aligned}$$

For the first term we will use the trick (2.263). Therefore, we must define

$$C_{4Y} := \gamma^2\,B_{4\phi},\ \text{or}\ C_4 = \int \gamma^2\,B_{4\phi}\,dY, \qquad (2.299)$$

which implies (up to the boundary term)

$$\begin{aligned}
-2\gamma^2B_{4\phi}(\Delta\cdot\phi) - 2B_{4\phi}\Box\phi &= -C_{4Y}\,Y_\mu\phi^\mu - 2B_{4\phi}\,\Box\phi\,,\\
&= -\{-C_4\,\Box\phi - C_{4\phi}\,Y\} - 2\gamma^{-2}C_{4Y}\Box\phi\\
&= C_{4\phi}Y + (C_4 + 2YC_{4Y})\Box\phi\,. \qquad (2.300)
\end{aligned}$$

Substitute this last piece into the equation (2.296), we have

$$\begin{aligned}
\mathcal{L}_4^{GLPV} &= B_4\,R + \frac{(A_4 + B_4 - 2YB_{4Y})}{Y^2}\mathcal{L}_4^{gal,1} - 2B_{4Y}\left(\Box\phi^2 - \phi_{\mu\nu}^2\right)\\
&\quad +C_{4\phi}Y + (C_4 + 2YC_{4Y})\Box\phi + \nabla_\mu\mathcal{B}^\mu_{bH4}\,. \qquad (2.301)
\end{aligned}$$

This is the covariant form of the Lagrangian $\mathcal{L}_4^{GLPV}$. Next, we will consider $\mathcal{L}_5^{GLPV}$ in the covariant form, start with the spatially covariant form (2.243) in the general coordinate

$$\begin{aligned}
\mathcal{L}_5^{GLPV,3+1} &= A_5\left(K^3 - 2KK_{\mu\nu}K^{\mu\nu} + 3K_{\mu\nu}K^{\nu\rho}K^\mu_\rho\right)\\
&\quad + B_5\,K^{\mu\nu}\left({}^3R_{\mu\nu} - (1/2)q_{\mu\nu}\,{}^3R\right)\,.
\end{aligned}$$

We will calculate the covariant form of the first three terms as the following

$$\begin{aligned}
K^3 &= -\gamma\Big(\Box\phi + \gamma^2(\Delta\cdot\phi)\Big)\,\gamma^2\Big(\Box\phi^2 + 2\gamma^2\Box\phi(\Delta\cdot\phi) + \gamma^4(\Delta\cdot\phi)^2\Big)\,,\\
&= -\gamma^3\Big(\Box\phi^3 + 3\gamma^2\Box\phi^2(\Delta\cdot\phi) + 3\gamma^4\Box\phi(\Delta\cdot\phi)^2 + \gamma^6(\Delta\cdot\phi)^3\Big)\,. \qquad (2.302)
\end{aligned}$$

$$KK_{ab}K^{ab} \;\longmapsto\; K\,K^{\mu\nu}K_{\mu\nu}\,,$$

$$
\begin{aligned}
&= -\gamma\Big(\Box\phi + \gamma^2(\triangle\cdot\phi)\Big)\,\gamma^2\Big(\phi_{\mu\nu}^2 + 2\gamma^2(\triangle\cdot\triangle) + \gamma^4(\triangle\cdot\phi)^2\Big)\,,\\
&= -\gamma^3\Big(\Box\phi\,\phi_{\mu\nu}^2 + 2\gamma^2\Box\phi(\triangle\cdot\triangle) + \gamma^4\Box\phi(\triangle\cdot\phi)^2 + \gamma^2\,\phi_{\mu\nu}^2(\triangle\cdot\phi)\\
&\quad +2\gamma^4(\triangle\cdot\triangle)(\triangle\cdot\phi) + \gamma^6(\triangle\cdot\phi)^3\Big)\,. \qquad (2.303)
\end{aligned}
$$

$$
\begin{aligned}
K_{\mu\nu}K^{\nu\alpha} &= \gamma^2\Big(\phi_{\mu\nu}\phi^{\nu\alpha} + \gamma^2\triangle_\mu\triangle^\alpha + \gamma^2\,\phi^\alpha(\phi\cdot\bigcirc)_\mu + \gamma^4\triangle_\mu\,\phi^\alpha(\triangle\cdot\phi)\\
&\quad \gamma^2\,\phi_\mu(\phi\cdot\bigcirc)^\alpha + \gamma^4\phi_\mu\,\phi^\alpha(\triangle\cdot\triangle) + \gamma^6\phi_\mu\,\phi^\alpha(\triangle\cdot\phi)^2 + \gamma^4\,\phi_\mu\,\triangle^\alpha(\triangle\cdot\phi)\Big)\,, \qquad (2.304)
\end{aligned}
$$

$$
\begin{aligned}
\therefore K_{\mu\nu}K^{\nu\alpha}K^\mu_\alpha &= -\gamma^3\Big\{\phi^\mu_\alpha + \gamma^2\phi_\alpha\triangle^\mu + \ldots\Big\}\,,\\
&= -\gamma^3\Big(\phi_{\mu\nu}\,\phi^{\nu\alpha}\,\phi^\mu_\alpha + 3\gamma^3(\phi\cdot\bigcirc\cdot\triangle) + 3\gamma^4(\triangle\cdot\triangle)(\triangle\cdot\phi) + \gamma^6(\triangle\cdot\phi)^3\Big)\,. \qquad (2.305)
\end{aligned}
$$

Then the combination of the first three terms in term of the scalar field reads

$$
\begin{aligned}
&K^3 - 2KK_{\mu\nu}K^{\mu\nu} + 3K_{\mu\nu}K^{\nu\rho}K^\mu_\rho \equiv [K]^3 - 2[K]\,[K^2] + 3\,[K^3]\,,\\
&= -\gamma^3\Big(\Box\phi^3 + 3\gamma^2\,\Box\phi^2(\triangle\cdot\phi) + 3\gamma^4\,\Box\phi(\triangle\cdot\phi)^2 + \gamma^6(\triangle\cdot\phi)^3 - 3\,\Box\phi\,\phi_{\mu\nu}^2\\
&\quad -6\gamma^2\,\Box\phi(\triangle\cdot\triangle) - 3\gamma^4\Box\phi(\triangle\cdot\phi)^2 - 3\gamma^2\,\phi_{\mu\nu}^2(\triangle\cdot\phi) - 6\gamma^4(\triangle\cdot\triangle)(\triangle\cdot\phi)\\
&\quad -3\gamma^6(\triangle\cdot\phi)^3 + 2[\phi^3] + 6\gamma^2(\phi\cdot\bigcirc\cdot\triangle) + 6\gamma^4(\triangle\cdot\triangle)(\triangle\cdot\phi) + 2\gamma^6(\triangle\cdot\phi)^3\Big)\,, \qquad (2.306)
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma^5\Big(Y\Box\phi^3 - 3Y\Box\phi\,\phi_{\mu\nu}^2 + 2Y\phi_{\mu\nu}^3 - 3(\triangle\cdot\phi)\Box\phi^2\\
&\quad +3(\triangle\cdot\phi)\phi_{\alpha\beta}^2 - 6(\phi\cdot\bigcirc\cdot\triangle) + 6\Box\phi\,(\triangle\cdot\triangle)\Big)\,, \qquad (2.307)\\
&= -\gamma^5\mathcal{L}_5^{gal,1} \equiv -\gamma^5\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\phi^\alpha\phi_\mu\phi^\beta_\nu\phi^\gamma_\rho\phi^\delta_\sigma\,. \qquad (2.308)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_5 &= A_5\left(K^3 - 3K_{\mu\nu}K^{\mu\nu}K + 2K^\mu_\nu K^\nu_\alpha K^\alpha_\mu\right)\\
&\quad +B_5\Big(K^{\mu\nu\,3}R_{\mu\nu} - \frac{1}{2}\,K^{\,3}R\Big)\,, \qquad (2.309)\\
&= A_5\left(K^3 - 3K_{\mu\nu}K^{\mu\nu}K + 2K^\mu_\nu K^\nu_\alpha K^\alpha_\mu\right)\\
&\quad +B_5\Big(K^{\mu\nu}K^\alpha_\mu K_{\alpha\nu} - K\,K^{\mu\nu}K_{\mu\nu} + K^{\mu\nu}\,R_{\mu\nu} + K_{\mu\nu}n^\alpha\,n_\beta R^\beta{}_{\mu\alpha\nu} - \frac{1}{2}K\,K_{\mu\nu}K^{\mu\nu}\\
&\quad +\frac{1}{2}K^3 - \frac{1}{2}R\,K - K\,R_{\mu\nu}n^\mu n^\nu\Big)\,, \qquad (2.310)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_5 &= A_5\left(K^3 - 3K_{\mu\nu}K^{\mu\nu}K + 2K^\mu_\nu K^\nu_\alpha K^\alpha_\mu\right) \qquad (\equiv \mathscr{M})\\
&\quad +B_5\Big(K^{\mu\nu}K_{\nu\alpha}K^\alpha_\mu - \frac{3}{2}K\,K_{\mu\nu}K^{\mu\nu} + \frac{1}{2}K^3\Big) \qquad (\equiv \mathscr{N})\\
&\quad +B_5\Big(K_{\mu\nu}n^\alpha n_\beta R^\beta{}_{\mu\alpha\nu} - K\,R_{\mu\nu}n^\mu n^\nu\Big) \quad (\equiv \mathscr{P})
\end{aligned}
$$

$$+B_5\, K^{\mu\nu}G_{\mu\nu}\,. \qquad (\equiv \mathscr{Q}) \tag{2.311}$$

$$\begin{aligned}\mathscr{P} \;=\; & B_5\Big(\big\{-K^3_{\mu\nu} + K^{\mu\nu}\nabla_\nu a_\mu - n^\sigma K^{\mu\nu}\nabla_\sigma K_{\mu\nu} + K^{\mu\nu}a_\mu a_\nu\big\} \\ & -\big\{K\,n^\sigma\nabla^\mu K_{\sigma\mu} + K\nabla^\mu a_\mu - K\,n^\sigma\nabla_\sigma K + K\,n^\sigma n^\mu\nabla_\sigma a_\mu \\ & +K\,n^\sigma a_\mu\nabla_\sigma n^\mu\big\}\Big)\,,\end{aligned} \tag{2.312}$$

$$\mathscr{Q} \;=\; -B_5\gamma\phi_{\mu\nu}G^{\mu\nu} + B_5 n_\mu a_\nu G^{\mu\nu} + B_5 n_\nu a_\mu G^{\mu\nu} + B_5\frac{\gamma^3}{2}\phi^\lambda Y_\lambda n_\mu n_\nu G^{\mu\nu}\,, \tag{2.313}$$

$$= -B_5\gamma\phi_{\mu\nu}G^{\mu\nu} + 2B_5R_{\mu\nu}n^\mu a^\nu + B_5\frac{\gamma^3}{2}\phi^\lambda Y_\lambda\big\{R_{\mu\nu}n^\mu n^\nu + \frac{1}{2}R\big\}\,, \tag{2.314}$$

$$\begin{aligned}= & -B_5\gamma\phi_{\mu\nu}G^{\mu\nu} + 2B_5R_{\mu\nu}n^\mu\big(\frac{\gamma^2}{2}Y^\nu - \frac{\gamma^3}{2}n^\lambda Y_\lambda\phi^\nu\big) \\ & +B_5\frac{\gamma^3}{2}\phi^\lambda Y_\lambda\big\{R_{\mu\nu}n^\mu n^\nu + \frac{1}{2}R\big\}\,,\end{aligned} \tag{2.315}$$

$$\begin{aligned}= & \big(B_{5\phi}\phi_\mu + B_{5Y}Y_\mu\big)\gamma\phi_\nu G^{\mu\nu} + \frac{1}{2}B_5\gamma^3Y_\mu\phi_\nu G^{\mu\nu} + \gamma^2B_5R_{\mu\nu}n^\mu Y^\nu \\ & -B_5\frac{\gamma^3}{2}\phi^\lambda Y_\lambda R_{\mu\nu}n^\mu n^\nu + \frac{\gamma^3}{4}B_5\phi^\lambda Y_\lambda R\,,\end{aligned} \tag{2.316}$$

$$\begin{aligned}\mathscr{Q} \;=\; & B_{5Y}Y_\mu\gamma\phi_\nu G^{\mu\nu} + \gamma B_{5\phi}\phi_\mu\phi_\nu G^{\mu\nu} + B_5\frac{\gamma^3}{2}\big(R_{\mu\nu}Y^\mu\phi^\nu - \frac{1}{2}RY^\lambda\phi_\lambda\big) \\ & -\gamma^3B_5R_{\mu\nu}\phi^\mu Y^\nu - B_5\frac{\gamma^3}{2}\phi^\lambda Y_\lambda R_{\mu\nu}n^\mu n^\nu + \frac{B_5\gamma^3}{4}\phi^\lambda Y_\lambda R\,,\end{aligned} \tag{2.317}$$

$$\begin{aligned}= & B_{5Y}Y_\mu\gamma\phi_\nu G^{\mu\nu} + \gamma B_{5\phi}\phi_\mu\phi_\nu G^{\mu\nu} - B_5\frac{\gamma^3}{2}R_{\mu\nu}Y^\mu\phi^\nu - B_5\frac{\gamma^3}{2}\phi^\lambda Y_\lambda R_{\mu\nu}n^\mu n^\nu\,, \\ = & B_{5Y}Y_\mu\gamma\phi_\nu G^{\mu\nu} + \gamma B_{5\phi}\phi_\mu\phi_\nu G^{\mu\nu} - B_5\gamma R_{\mu\nu}a^\mu\phi^\nu\,,\end{aligned} \tag{2.318}$$

where we have used

$$a_\mu = \frac{\gamma^2}{2}Y_\mu - \frac{\gamma^3}{2}n^\lambda Y_\lambda\phi_\mu\,. \tag{2.319}$$

Then, we will define

$$\tilde{G}_{5Y} = -B_{5Y}\gamma\,. \tag{2.320}$$

Hence, we have so far

$$\begin{aligned}\mathscr{Q} \;=\; & -\tilde{G}_{5Y}Y_\mu\phi_\nu G^{\mu\nu} - \tilde{G}_{5\phi}Y_\mu\phi_\nu G^{\mu\nu} + \tilde{G}_{5\phi}\phi_\mu\phi_\nu G^{\mu\nu} \\ & +\gamma B_{5\phi}\phi^\mu\phi^\nu G_{\mu\nu} + B_5R_{\mu\nu}a^\mu n^\nu\,,\end{aligned} \tag{2.321}$$

$$\begin{aligned}= & -\nabla_\mu(\tilde{G}_5)\phi_\nu G^{\mu\nu} + \tilde{G}_{5\phi}G_{\mu\nu}\phi^\mu\phi^\nu + \gamma B_{5\phi}G_{\mu\nu}\phi^\mu\phi^\nu + B_5R_{\mu\nu}a^\mu n^\nu\,, \\ \overset{u.t.b}{=} & \tilde{G}_5\phi^{\mu\nu}G_{\mu\nu} + (\tilde{G}_{5\phi} + \gamma\,B_{5\phi})\,G_{\mu\nu}\phi^\mu\phi^\nu + B_5(a^\nu\nabla_\alpha K^\alpha_\nu - K_{\alpha\nu}a^\nu a^\alpha - a^\nu\nabla_\nu K)\,,\end{aligned} \tag{2.322}$$

$$\begin{aligned}\mathscr{P}+\mathscr{Q} &= B_5\Big(-K^3_{\mu\nu}+K^{\mu\nu}\nabla_\nu a_\mu - n^\sigma K^{\mu\nu}\nabla_\sigma K_{\mu\nu}+K_{\mu\nu}a^\mu a^\nu \\ &\quad -Kn^\sigma\nabla^\mu K_{\sigma\mu}-K\nabla^\mu a_\mu + Kn^\sigma\nabla_\sigma K - Kn^\sigma n^\mu\nabla_\sigma a_\mu - Kn^\sigma a_\mu\nabla_\sigma n^\mu \\ &\quad +a^\nu\nabla_\alpha K^\alpha_\nu - K_{\nu\alpha}a^\nu a^\alpha - a^\nu\nabla_\nu K\Big)+\tilde{G}_{5\phi}G_{\mu\nu}\phi^{\mu\nu} \\ &\quad +(\tilde{G}_{5\phi}+\gamma B_{5\phi})G_{\mu\nu}\phi^\mu\phi^\nu\ , \qquad (2.323)\\ &= B_5\Big(-K^3_{\mu\nu}+\nabla_\nu(K^{\mu\nu}a_\mu)\overbrace{-n^\sigma K^{\mu\nu}\nabla_\sigma K_{\mu\nu}}^{-\frac{1}{2}n^\sigma\nabla_\sigma(K^{\mu\nu}K_{\mu\nu})}\overbrace{-Kn^\sigma\nabla^\mu K_{\sigma\mu}}^{+KK_{\sigma\mu}\nabla^\mu n^\sigma} \\ &\quad -n^\mu(Ka_\mu)+\underbrace{Kn^\sigma\nabla_\sigma K}_{\frac{1}{2}n^\sigma\nabla_\sigma K^2}\underbrace{-Kn^\sigma n^\mu\nabla_\sigma a_\mu - Kn^\sigma a_\mu\nabla_\sigma n^\mu}_{0}\Big) \\ &\quad +\tilde{G}_{5\phi}G_{\mu\nu}\phi^{\mu\nu}+(\tilde{G}_{5\phi}+\gamma B_{5\phi})G_{\mu\nu}\phi^\mu\phi^\nu\ , \qquad (2.324)\end{aligned}$$

$$\begin{aligned}\mathscr{P}+\mathscr{Q} &\overset{u.t.b}{=} \boxed{B_5(-K^3_{\mu\nu})}-\nabla_\nu B_5\,K^{\mu\nu}a_\mu+\frac{1}{2}\nabla_\sigma B_5\,n^\sigma K^{\mu\nu}K_{\mu\nu}+\boxed{\frac{1}{2}B_5KK_{\mu\nu}K^{\mu\nu}} \\ &\quad \boxed{+B_5KK_{\sigma\mu}K^{\mu\sigma}}+\nabla^\mu B_5\,Ka_\mu-\frac{1}{2}\nabla_\sigma B_5 n^\sigma K^2\boxed{-\frac{1}{2}B_5K^3} \\ &\quad +\tilde{G}_{5\phi}G_{\mu\nu}\phi^{\mu\nu}+(\tilde{G}_{5\phi}+\gamma B_{5\phi})G_{\mu\nu}\phi^\mu\phi^\nu\ , \qquad (2.325)\\ &= -\mathscr{N}-\nabla_\nu B_5\,K^{\mu\nu}a_\mu+\frac{1}{2}\nabla_\sigma B_5\,n^\sigma K^{\mu\nu}K_{\mu\nu}+\nabla^\mu B_5\,Ka_\mu-\frac{1}{2}\nabla_\sigma B_5 n^\sigma K^2 \\ &\quad +\tilde{G}_{5\phi}G_{\mu\nu}\phi^{\mu\nu}+(\tilde{G}_{5\phi}+\gamma B_{5\phi})G_{\mu\nu}\phi^\mu\phi^\nu\ , \qquad (2.326)\end{aligned}$$

$$\begin{aligned}\mathscr{N}+\mathscr{P}+\mathscr{Q} &= -\nabla_\nu B_5\,K^{\mu\nu}a_\mu-\frac{1}{2}\nabla_\sigma B_5 n^\sigma K^{\mu\nu}K_{\mu\nu}+\nabla^\mu B_5 Ka_\mu-\frac{1}{2}\nabla_\sigma B_5 n^\sigma K^2 \\ &\quad +\tilde{G}_{5\phi}G_{\mu\nu}\phi^{\mu\nu}+(\tilde{G}_{5\phi}+\gamma B_{5\phi})G_{\mu\nu}\phi^\mu\phi^\nu\ , \qquad (2.327)\\ &= -B_{5Y}Y_\nu K^{\mu\nu}a_\mu - B_{5\phi}\overbrace{\phi_\nu K^{\mu\nu}}^{0}a_\mu+\frac{1}{2}B_{5Y}Y_\sigma n^\sigma K^{\mu\nu}K_{\mu\nu} \\ &\quad +\frac{1}{2}B_{5\phi}\phi_\sigma n^\sigma K^{\mu\nu}K_{\mu\nu}+B_{5Y}Y^\mu a_\mu K+B_{5\phi}\overbrace{\phi^\mu a_\mu}^{0}K-\frac{1}{2}B_{5Y}Y_\sigma n^\sigma K^2 \\ &\quad -\frac{1}{2}B_{5\phi}\phi_\sigma n^\sigma K^2+\tilde{G}_{5\phi}G_{\mu\nu}\phi^{\mu\nu}+(\tilde{G}_{5\phi}+\gamma B_{5\phi})G_{\mu\nu}\phi^\mu\phi^\nu\ , \qquad (2.328)\\ &= B_{5Y}\Big(\underbrace{-K_{\mu\nu}a_\mu Y_\nu+\tfrac{1}{2}Y_\sigma n^\sigma K^{\mu\nu}K_{\mu\nu}+Y^\mu a_\mu K-\tfrac{1}{2}Y_\sigma n^\sigma K^2}_{\equiv -S}\Big) \\ &\quad +B_{5\phi}\Big(\frac{1}{2}\phi_\sigma n^\sigma K^{\mu\nu}K_{\mu\nu}-\frac{1}{2}\phi_\sigma n^\sigma K^2\Big) \\ &\quad +\tilde{G}_{5\phi}G_{\mu\nu}\phi^{\mu\nu}+(\tilde{G}_{5\phi}+\gamma B_{5\phi})G_{\mu\nu}\phi^\mu\phi^\nu\ , \qquad (2.329)\end{aligned}$$

using the Stueckelberg's trick for this part

$$\begin{aligned} S &= K_{\mu\nu}a_\mu Y_\nu - \frac{1}{2}Y_\sigma n^\sigma K^{\mu\nu}K_{\mu\nu} - Y^\mu a_\mu K + \frac{1}{2}Y_\sigma n^\sigma K^2 \,, & (2.330)\\ &= -\gamma\big(\phi^{\mu\nu} + \gamma^4(\triangle\cdot\phi)\phi^\mu\phi^\nu + \gamma^2\phi^\mu\triangle^\nu + \gamma^2\phi^\nu\triangle^\mu\big)\big(\gamma^2\triangle_\mu + \gamma^4(\triangle\cdot\phi)\phi_\mu\big)\,2\triangle_\nu \\ &\quad +\frac{\gamma}{2}\,(2\triangle_\sigma)\,\phi^\sigma\,\gamma^2\big(\phi^2_{\mu\nu} + 2\gamma^2(\triangle\cdot\triangle) + \gamma^4(\triangle\cdot\phi)^2\big) \\ &\quad -2\triangle^\mu\,\big(\gamma^2\triangle_\mu + \gamma^4(\triangle\cdot\phi)\,\phi_\mu\big)\big(-\gamma\Box\phi - \gamma^3(\triangle\cdot\phi)\big) \\ &\quad -\frac{\gamma}{2}\,(2\triangle_\sigma)\phi^\sigma\big(\gamma\,\Box\phi + \gamma^3(\triangle\cdot\phi)\big)^2 \,, & (2.331)\\ &= -2\gamma^3\Big((\phi\cdot\bigcirc\cdot\triangle) + \gamma^2(\triangle\cdot\phi)(\triangle\cdot\triangle) + \gamma^4(\triangle\cdot\phi)(\triangle\cdot\phi)^2 + \gamma^6(\triangle\cdot\phi)^2Y(\triangle\cdot\phi) \\ &\quad +\gamma^2(\triangle\cdot\phi)(\triangle\cdot\triangle) + \gamma^4(\triangle\cdot\phi)Y(\triangle\cdot\triangle) + \gamma^2(\triangle\cdot\phi)(\triangle\cdot\triangle) + \gamma^4(\triangle\cdot\phi)(\triangle\cdot\phi)^2\Big) \\ &\quad +\gamma^3(\triangle\cdot\phi)\Big(\phi^2_{\mu\nu} + 2\gamma^2(\triangle\cdot\triangle) + \gamma^4(\triangle\cdot\phi)^2\Big) \\ &\quad +2\gamma^3\Big((\triangle\cdot\triangle) + \gamma^2(\triangle\cdot\phi)^2\Big)\Big(\Box\phi + \gamma^2(\triangle\cdot\phi)\Big) \\ &\quad -\gamma^3(\triangle\cdot\phi)\Big(\Box\phi^2 + 2\gamma^2\,\Box\phi(\triangle\cdot\phi) + \gamma^4(\triangle\cdot\phi)^2\Big) \,, & (2.332)\\ &= -2\gamma^3\Bigg[\Big((\phi\cdot\bigcirc\cdot\triangle) + 2\gamma^2(\triangle\cdot\phi)(\triangle\cdot\triangle) + 2\gamma^4(\triangle\cdot\phi)^3 - \gamma^4(\triangle\cdot\phi)^3\Big) \\ &\quad +\Big(-\frac{1}{2}(\triangle\cdot\phi)\phi^2_{\mu\nu} - \gamma^2(\triangle\cdot\triangle)(\triangle\cdot\phi) - \frac{1}{2}\gamma^4(\triangle\cdot\phi)^3\Big) \\ &\quad +\Big(-(\triangle\cdot\triangle)\Box\phi - \gamma^2(\triangle\cdot\triangle)(\triangle\cdot\phi) - \gamma^2\Box\phi(\triangle\cdot\phi)^2 - \gamma^4(\triangle\cdot\phi)^3\Big) \\ &\quad +\Big(\frac{1}{2}(\triangle\cdot\phi)\Box\phi^2 + \gamma^2(\triangle\cdot\phi)^2\Box\phi + \frac{1}{2}\gamma^4\Box\phi^3\Big)\Bigg] \,, & (2.333)\\ &= -2\gamma^3\Big((\phi\cdot\bigcirc\cdot\triangle) - \frac{1}{2}(\triangle\cdot\phi)\phi^2_{\mu\nu} - (\triangle\cdot\triangle)\,\Box\phi + \frac{1}{2}(\triangle\cdot\phi)\,\Box\phi^2\Big) \,, & (2.334) \end{aligned}$$

$$\begin{aligned} \mathcal{L}_5 &= \mathscr{M} + \mathscr{N} + \mathscr{P} + \mathscr{Q} \,, & (2.335)\\ &= -\gamma^5\,A_5\,\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\phi^\alpha\phi_\mu\phi^\beta_\nu\phi^\gamma_\rho\phi^\delta_\sigma \\ &\quad +2\gamma^3\,B_{5Y}\Big((\phi\cdot\bigcirc\cdot\triangle) - \frac{1}{2}(\triangle\cdot\phi)\phi^2_{\mu\nu} - (\triangle\cdot\triangle)\,\Box\phi + \frac{1}{2}(\triangle\cdot\phi)\,\Box\phi^2\Big) \\ &\quad +\frac{1}{2\gamma}\,B_{5\phi}\Big(K^{\mu\nu}K_{\mu\nu} - K^2\Big) \\ &\quad +\tilde{G}_{5\phi}G_{\mu\nu}\phi^{\mu\nu} + (\tilde{G}_{5\phi} + \gamma B_{5\phi})G_{\mu\nu}\phi^\mu\phi^\nu \,, & (2.336)\\ &= -A_5\,\gamma^5\,\mathcal{L}_5^{gal,1} - B_{5Y}\,\frac{\gamma^3}{3}\Big\{-\mathcal{L}_5^{gal,1} - Y\,\Box\phi^3 + 3Y\,\Box\phi\,\phi^2_{\mu\nu} - 2Y\,\phi^3_{\mu\nu}\Big\} \\ &\quad +\frac{B_{5\phi}}{2\gamma}\Big(K^{\mu\nu}K_{\mu\nu} - K^2\Big) + \tilde{G}_{5\phi}G_{\mu\nu}\phi^{\mu\nu} + (\tilde{G}_{5\phi} + \gamma B_{5\phi})G_{\mu\nu}\phi^\mu\phi^\nu \,, & (2.337)\\ &= \Big(-A_5\gamma^5 + \frac{1}{3}\gamma^3 B_{5Y}\Big)\,\mathcal{L}_5^{gal,1}\underbrace{-\frac{B_{5Y}}{3}\,\gamma}_{\frac{1}{3}\tilde{G}_{5Y}}\Big(\Box\phi^3 - 3\Box\phi\,\phi^2_{\mu\nu} + 2\,\phi^3_{\mu\nu}\Big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{B_{5\phi}}{2\gamma}\Big(K_{\mu\nu}K^{\mu\nu}-K^2\Big)+\Big(-\int B_{5Y\phi}\,\gamma\,dY+\int\frac{d}{dY}\big(\gamma B_{5\phi}\big)dY\Big)G_{\mu\nu}\phi^{\mu}\phi^{\nu}\\
&+\tilde{G}_{5\phi}G_{\mu\nu}\phi^{\mu\nu}\,, \qquad (2.338)\\
=\;&\Big(\frac{-3A_5+(-Y)B_{5Y}}{3(-Y)^{5/2}}\Big)\mathcal{L}_5^{gal,1}+\frac{1}{3}\tilde{G}_{5Y}\Big(\Box\phi^3-3\Box\phi\,\phi_{\mu\nu}^2+2\,\phi_{\mu\nu}^3\Big)\\
&\underbrace{+\tfrac{B_{5\phi}}{2\gamma}\Big(K_{\mu\nu}K^{\mu\nu}-K^2\Big)+\int(\tfrac{1}{2}\gamma^3B_{5\phi})dY\,G_{\mu\nu}\phi^{\mu}\phi^{\nu}}_{\equiv J}\\
&+\tilde{G}_5G_{\mu\nu}\phi^{\mu\nu}\,. \qquad (2.339)
\end{aligned}
$$

$$
\begin{aligned}
J &= \frac{B_{5\phi}}{2\gamma}\Big(K_{\mu\nu}K^{\mu\nu}-K^2\Big)+\frac{1}{2\gamma^2}\int(\gamma^3B_{5\phi})dY\,G_{\mu\nu}n^{\mu}n^{\nu}\,, \qquad (2.340)\\
&= \frac{B_{5\phi}}{2\gamma}\Big(K_{\mu\nu}K^{\mu\nu}-K^2\Big)+\Big(\frac{1}{4\gamma^2}\int(\gamma^3B_{5\phi})dY\Big)\left\{R+2R_{\mu\nu}n^{\mu}n^{\nu}\right\} \qquad (2.341)
\end{aligned}
$$

$$
C_5 \equiv \frac{1}{4\gamma^2}\int B_{5\phi}\gamma^3dY\,, \qquad (2.342)
$$

$$
J = C_5\,R+\frac{B_{5\phi}}{2\gamma}\Big(K_{\mu\nu}K^{\mu\nu}-K^2\Big)+2C_5\,R_{\mu\nu}n^{\mu}n^{\nu}\,, \qquad (2.343)
$$

$$
\begin{aligned}
\frac{d}{dY}4\gamma^2C_5 &= \gamma^3B_{5\phi}\,,\\
4\{\gamma^4C_5+\gamma^2C_{5Y}\} &= \gamma^3B_{5\phi}\,,\\
2C_5+\frac{2}{\gamma^2}C_{5Y} &= \frac{B_{5\phi}}{2\gamma}\,,\\
\frac{B_{5\phi}}{2\gamma} &= 2C_5-2Y\,C_{5Y}\,,
\end{aligned}
$$

$$
J = C_5\,R+(2C_5-2Y\,C_{5Y})\Big(K_{\mu\nu}K^{\mu\nu}-K^2\Big)+2C_5\,R_{\mu\nu}n^{\mu}n^{\nu}\,, \qquad (2.344)
$$

$$
\begin{aligned}
\mathcal{L}_4^{GLPV,3+1} &= B_4\,R+(B_4-A_4)\Big(K_{\mu\nu}K^{\mu\nu}-K^2\Big)+2B_4n^{\mu}n^{\nu}R_{\mu\nu}\\
\longmapsto \mathcal{L}_4^{GLPV} &= B_4\,R-2B_{4Y}(\Box\phi^2-\phi_{\mu\nu}^2)+(C_4+2YC_{4Y})\Box\phi\\
&\quad +Y\,C_{4\phi}+\frac{B_4+A_4-2YB_{4Y}}{Y^2}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\beta\gamma\delta}\phi_{\nu}\phi^{\beta}\phi_{\rho}^{\gamma}\phi_{\sigma}^{\delta}\,, \qquad (2.345)\\
& \qquad (2.346)
\end{aligned}
$$

where $C_4 \equiv \int B_{4\phi}\gamma^2dY$. Therefore

$$
\begin{aligned}
B_4 &\sim C_5 \,, \\
B_4 - A_4 &\sim 2C_5 - 2Y C_{5Y} \,, \\
\therefore A_4 &\sim -C_5 + 2Y C_{5Y} \,, \\
A_4 + B_4 - 2Y\, B_{4Y} &\sim (-C_5 + 2Y C_{5Y}) + C_5 - 2Y C_{5Y} \,, \\
&= 0 \,,
\end{aligned}
$$

$$
J \;\longmapsto\; J^{cov} = C_5\, R - 2C_{5Y}(\Box\phi^2 - \phi_{\mu\nu}^2) + (D_5 + 2Y\, D_{5Y})\Box\phi + Y\, D_{5\phi} \tag{2.347}
$$

where

$$
D_5 \;\equiv\; \int C_{5\phi}\gamma^2 dY \,. \tag{2.348}
$$

$$
\begin{aligned}
\mathcal{L}_{5,M}^{GLPV} &= \tilde{G}_5\, G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}\tilde{G}_{5Y}\Big(\Box\phi^3 - 3\Box\phi\, \phi_{\mu\nu}^2 + 2\,\phi_{\mu\nu}^3\Big) + C_5\, R \\
&\quad -2C_{5Y}(\Box\phi^2 - \phi_{\mu\nu}^2) + (D_5 + 2Y\, D_{5Y})\Box\phi \\
&\quad +Y\, D_{5\phi} - \frac{Y\, B_{5Y} + 3A_5}{3(-Y)^{5/2}}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\phi^\alpha\phi_\mu\phi_\nu^\beta\phi_\rho^\gamma\phi_\sigma^\delta \,,
\end{aligned} \tag{2.349}
$$

We have demonstrate the examples of derivations for the covariant form of $\mathcal{L}_2^{GLPV}, \mathcal{L}_3^{GLPV}$, and $\mathcal{L}_4^{GLPV}$ using the Stueckelberg trick. The covariant form of $\mathcal{L}_5^{GLPV}$ can be found in [24], it reads

$$
\mathcal{L}_{2,M}^{GLPV} = A_2(\phi, Y) \,, \tag{2.350}
$$

$$
\mathcal{L}_{3,M}^{GLPV} = (\tilde{C}_3 + 2Y\, \tilde{C}_{3Y})\Box\phi + Y\, \tilde{C}_{3\phi} \,, \tag{2.351}
$$

$$
\begin{aligned}
\mathcal{L}_{4,M}^{GLPV} &= B_4\, R - 2B_{4Y}(\Box\phi^2 - \phi_{\mu\nu}^2) + (C_4 + 2Y C_{4Y})\Box\phi \\
&\quad +Y\, C_{4\phi} + \frac{B_4 + A_4 - 2Y\, B_{4Y}}{Y^2}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\beta\gamma\delta}\phi_\nu\phi^\beta\phi_\rho^\gamma\phi_\sigma^\delta \,,
\end{aligned} \tag{2.352}
$$

$$
\begin{aligned}
&= B_4\, R - 2B_{4Y}(\Box\phi^2 - \phi_{\mu\nu}^2) + (C_4 + 2Y C_{4Y})\Box\phi \\
&\quad +Y\, C_{4\phi} + F_4\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\beta\gamma\delta}\phi_\nu\phi^\beta\phi_\rho^\gamma\phi_\sigma^\delta \,,
\end{aligned} \tag{2.353}
$$

$$
\begin{aligned}
\mathcal{L}_{5,M}^{GLPV} &= \tilde{G}_5\, G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}\tilde{G}_{5Y}\Big(\Box\phi^3 - 3\Box\phi\, \phi_{\mu\nu}^2 + 2\,\phi_{\mu\nu}^3\Big) + C_5\, R - 2C_{5Y}(\Box\phi^2 - \phi_{\mu\nu}^2) \\
&\quad +(D_5 + 2Y\, D_{5Y})\Box\phi \\
&\quad +Y\, D_{5\phi} - \frac{Y\, B_{5Y} + 3A_5}{3(-Y)^{5/2}}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\phi^\alpha\phi_\mu\phi_\nu^\beta\phi_\rho^\gamma\phi_\sigma^\delta \,,
\end{aligned} \tag{2.354}
$$

$$
\begin{aligned}
&= \tilde{G}_5\, G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}\tilde{G}_{5Y}\Big(\Box\phi^3 - 3\Box\phi\, \phi_{\mu\nu}^2 + 2\,\phi_{\mu\nu}^3\Big) + C_5\, R - 2C_{5Y}(\Box\phi^2 - \phi_{\mu\nu}^2) \\
&\quad +(D_5 + 2Y\, D_{5Y})\Box\phi \\
&\quad +Y\, D_{5\phi} + F_5\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\phi^\alpha\phi_\mu\phi_\nu^\beta\phi_\rho^\gamma\phi_\sigma^\delta \,,
\end{aligned} \tag{2.355}
$$

$$\tilde{C}_3 \equiv \frac{1}{2}\int \gamma^3 A_3 dY \ , \tag{2.356}$$

$$C_4 \equiv \int B_{4\phi}\gamma^2 dY \ , \tag{2.357}$$

$$C_5 \equiv \frac{1}{4\gamma^2}\int B_{5\phi}\gamma^3 dY \ , \tag{2.358}$$

$$D_5 \equiv \int C_{5\phi}\gamma^2 dY \ , \tag{2.359}$$

$$\tilde{G}_5 \equiv -\int B_{5Y}\gamma dY \ , \tag{2.360}$$

$$F_4 \equiv \frac{B_4 + A_4 - 2Y B_{4Y}}{Y^2} \ , \tag{2.361}$$

$$F_5 \equiv -\frac{Y B_{5Y} + 3A_5}{3(-Y)^{5/2}} \ , \tag{2.362}$$

and

$$\begin{aligned}
-\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\beta\gamma\delta}\phi_\nu\phi^\beta\phi^\gamma_\rho\phi^\delta_\sigma &= \delta^{\nu\rho\sigma}_{\beta\gamma\delta}\phi_\nu\phi^\beta\phi^\gamma_\rho\phi^\delta_\sigma \ ,\\
&= Y(\Box\phi^2 - \phi^2_{\mu\nu}) + 2\phi_\mu\phi^{\mu\nu}\phi_{\nu\alpha}\phi^\alpha - 2\Box\phi\,\phi_{\mu\nu}\phi^\mu\phi^\nu \ ,\\
&= Y(\Box\phi^2 - \phi^2_{\mu\nu}) + 2\,(\triangle\cdot\triangle) - 2\Box\phi\,(\triangle\cdot\phi) \ .\\
-\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\phi_\mu\phi^\alpha\phi^\beta_\nu\phi^\gamma_\rho\phi^\delta_\sigma &= \delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}\phi_\mu\phi^\alpha\phi^\beta_\nu\phi^\gamma_\rho\phi^\delta_\sigma \ ,\\
&= Y\Box\phi^3 - 3Y\Box\phi\,\phi^2_{\mu\nu} + 2Y\phi^3_{\mu\nu} - 3\phi_{\mu\nu}\phi^\mu\phi^\nu\Box\phi^2\\
&\quad +3\phi_{\mu\nu}\phi^\mu\phi^\nu\phi^2_{\alpha\beta} - 6\phi^\mu\phi_{\mu\nu}\phi^{\nu\alpha}\phi_{\alpha\beta}\phi^\beta + 6\Box\phi\,\phi^\mu\phi_{\mu\nu}\phi^{\nu\alpha}\phi_\alpha \ .\\
&= +Y\Box\phi^3 - 3Y\Box\phi\,\phi^2_{\mu\nu} + 2Y\phi^3_{\mu\nu} - 3(\triangle\cdot\phi)\Box\phi^2\\
&\quad +3(\triangle\cdot\phi)\phi^2_{\alpha\beta} - 6(\phi\cdot\bigcirc\cdot\triangle) + 6\Box\phi\,(\triangle\cdot\triangle)
\end{aligned}$$

2.5 The Overall Constructions up to GLPV

In this section we will sum up the construction of the Scalar-Tensor theories up to GLPV. We start with Horndeski theory which contains 4 arbitrary functions G_2, G_3, G_4 and G_5 then we perform ADM analysis. We found that the set of relations f ={(2.234),(2.235),(2.236),(2.237),(2.238),(2.239)} maps the Horndeski coefficients to the coefficients A_2, A_3, A_4, A_5, B_4 and B_5 of ADM-Horndeski action (represented by the green arrows in Figure 2). This set of relations will no longer be important at the later stage as we have relaxed it keep only the form of the resulting action and it is now the GLPV action which is not fully in a covariant form. We called it *GLPV as the relaxed ADM-Horndeski*, technically, the coefficients of this action are differ from that of ADM-Horndeski although we still manage to use the same names for them but keep in mind that they are not related to G_2, G_3, G_4 and G_5 anymore since we have thrown away f. We show this structure in Figure 2.

Since GLPV action obtained from decomposition of the Horndeski action into (3+1)-style with the unitary gauge they contain solely the non-covariant terms. By the Stueckelberg's trick we can obtain the *covariantized GLPV*. Now, the set of

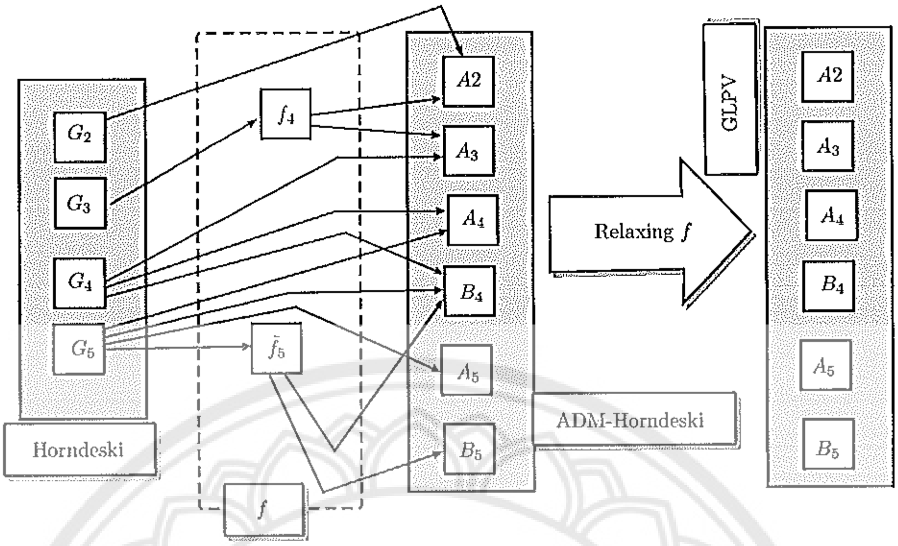

Figure 2 **The mapping of coefficients between Horndeski and ADM-Horndeski Lagrangian.**

relations $g_1 = \{(2.356),(2.357),(2.358),(2.359),(2.360),(4.26),(2.362)\}$ (represented by orange arrows in Figure 3) is the transformations between the coefficients of the both relevant forms of GLPV.

Consider equations (2.351),(2.351),(2.353) and (2.355) , by interchanging their terms we then obtained the new form

$$\mathcal{L}^{GLPV}_{2,M} = A_2(\phi, Y) + Y\left(\tilde{C}_3 + C_4 + D_5\right)_{\phi} , \quad (2.363)$$

$$\mathcal{L}^{GLPV}_{3,M} = \left(\left[\tilde{C}_3 + C_4 + D_5\right] + 2Y\left[\tilde{C}_3 + C_4 + D_5\right]_Y\right)\Box\phi , \quad (2.364)$$

$$\mathcal{L}^{GLPV}_{4,M} = \left(B_4 + C_5\right) R - 2\left(B_4 + C_5\right)_Y\left(\Box\phi^2 - \phi^2_{\mu\nu}\right) + F_4\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\beta\gamma\delta}\phi_\nu\phi^\beta\phi^\gamma_\rho\phi^\delta_\sigma , \quad (2.365)$$

$$\mathcal{L}^{GLPV}_{5,M} = \tilde{G}_5\, G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}\tilde{G}_{5Y}\left(\Box\phi^3 - 3\Box\phi\,\phi^2_{\mu\nu} + 2\,\phi^3_{\mu\nu}\right) + F_5\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\phi^\alpha\phi_\mu\phi^\beta_\nu\phi^\gamma_\rho\phi^\delta_\sigma . \quad (2.366)$$

It is more natural to rewrite

$$C_3 \equiv \tilde{C}_3 + C_4 + D_5 . \quad (2.367)$$

especially when we start from the GLPV action in the forn of Horndeski action plus the extra terms. Then the mappings from the original covariant form to the form of Horndeski are

$$\tilde{G}_2 = A_2 + Y\,C_{3\phi} , \quad (2.368)$$

$$\tilde{G}_3 = C_3 + 2Y\,C_{3Y} , \quad (2.369)$$

$$\tilde{G}_4 = C_5 + B_4 , \quad (2.370)$$

$$\tilde{G}_5 = \tilde{G}_5 , \quad (2.371)$$

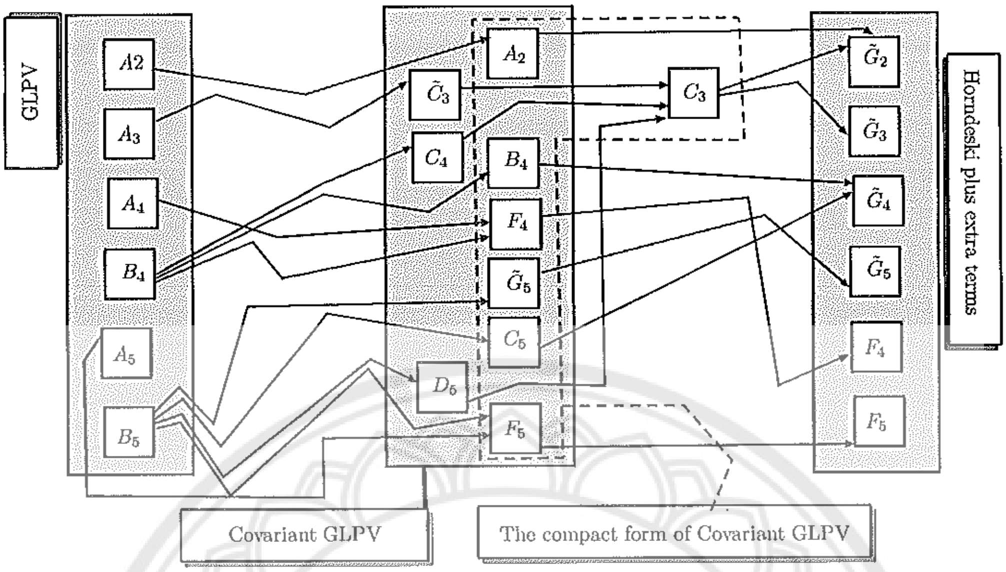

Figure 3 The mappings between a certain forms of GLPV theories.

$$F_4 = F_4 \, , \tag{2.372}$$

$$F_5 = F_5 \, . \tag{2.373}$$

Therefore rearranging the terms of this covariant form we will obtain another covariant forms. We can rearrange the terms to obtain GLPV action in the form of Horndeski action plus another extra covariant terms. It can be done by the transformation $g_3 \circ g_2$ (mapping by the yellow arrows follow by the blue arrows in Figure 3) where $g_2 = \{$ (2.367) $\}$ and $g_3 = \{$ (2.368), (2.369), (2.370), (2.371), (2.372), (2.373) $\}$

In order to transform back from GLPV as the Horndeski plus extra terms to the covariantized GLPV, it is more natural to see the coefficients $\tilde{C}_3 + C_4 + D_5$ as the single function C_3. This process is equivalents to performing g_3^{-1}, and the resulting action is the covariantized GLPV in the compact form.

CHAPTER III

DISFORMAL GRAVITY

3.1 Introduction

A certain class of scalar-tensor theories can be obtained by conformally transform the metric in Einstein-Hilbert action. Therefore we can start from GR and go to such scalar-tensor theories by conformal transformations. If there is no ghost in such theories we may have some viable ST theories. Using the ghost-free condition as a guiding principle Horndeski have shown that the most general scalar-tensor action provides up to second-order equations of motion can be obtained, the Horndeski theory. It turned out that this theory cannot be obtained from Einstein-Hilbert action by means of conformal transformation but the disformal one. The second-order equations of motion of theory guarantee the ghost-free property but not vise-versa, the counter examples exist. Therefore in this chapter we will transform the Einstein-Hilbert action by the disformal transformation. We set the conformal factor equals to unity for simplicity. Then the disformal metric reads

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + D(\phi, X)\phi_\mu\phi_\nu, \tag{3.1}$$

3.2 Basic Quantities in Disformal Gravity

The disformal gravity is a theory of two metrics. In this section we will compute some of the basic quantities in the form of the disformal metric. Firstly, we will compute the volume element of the disformal metric

$$\begin{aligned}
\tilde{g} \equiv & \det \tilde{g}_{\mu\nu} := \frac{1}{4!}\varepsilon^{\mu\nu\gamma\delta}\varepsilon^{\alpha\beta\rho\sigma}\tilde{g}_{\mu\alpha}\tilde{g}_{\nu\beta}\tilde{g}_{\gamma\rho}\tilde{g}_{\delta\sigma}, \qquad (3.2)\\
= & \frac{1}{4!}\varepsilon^{\mu\nu\gamma\delta}\varepsilon^{\alpha\beta\rho\sigma}(Ag_{\mu\alpha} + B\phi_\mu\phi_\alpha)(Ag_{\nu\beta} + B\phi_\nu\phi_\beta)(Ag_{\gamma\rho} + B\phi_\gamma\phi_\rho)(Ag_{\delta\sigma} + B\phi_\delta\phi_\sigma),\\
= & \frac{1}{4!}\varepsilon^{\mu\nu\gamma\delta}\varepsilon^{\alpha\beta\rho\sigma}\left[A^4 g_{\mu\alpha}g_{\nu\beta}g_{\gamma\rho}g_{\delta\sigma} + 4A^3B\phi_\mu\phi_\alpha g_{\nu\beta}g_{\gamma\rho}g_{\delta\sigma}\right],\\
= & A^4 g + 4A^3B\frac{1}{4!}\varepsilon^{\mu\nu\gamma\delta}\varepsilon^{\alpha\beta\rho\sigma}\phi_\mu\phi_\alpha g_{\nu\beta}g_{\gamma\rho}g_{\delta\sigma},\\
= & A^4 g + 4A^3B\frac{1}{4!}(-\sqrt{-g}\epsilon^{\mu\nu\gamma\delta})(-\sqrt{-g}\epsilon^{\alpha\beta\rho\sigma})\phi_\mu\phi_\alpha g_{\nu\beta}g_{\gamma\rho}g_{\delta\sigma},\\
= & A^4 g + 4A^3B(-g)\frac{1}{4!}\epsilon^{\mu\nu\gamma\delta}\epsilon^{\alpha}{}_{\nu\gamma\delta}\phi_\mu\phi_\alpha,\\
= & A^4 g - 4A^3Bg\frac{1}{4!}(-3!g^{\mu\alpha})\phi_\mu\phi_\alpha,\\
= & A^4 g + A^3Bg\phi^\mu\phi_\mu \quad . \qquad (3.3)
\end{aligned}$$

In a more compact form of the volume element

$$\sqrt{-\tilde{g}}d^4x = A^2\sqrt{-g}\sqrt{1-2\frac{B}{A}X}d^4x. \tag{3.4}$$

We note that $d^4x = d^4\tilde{x}$ because x^μ is only a frame which contains no physical data.

The next quantity we want is the inverse of the disformal metric, $\tilde{g}^{\mu\nu}$. By varying the determinant of the disformal metric we obtain the identity

$$\begin{aligned} \delta\tilde{g} &= \tilde{g}\tilde{g}^{\mu\nu}\delta\tilde{g}_{\mu\nu}, && (3.5)\\ &= -\tilde{g}\tilde{g}_{\mu\nu}\delta\tilde{g}^{\mu\nu}. && (3.6)\end{aligned}$$

These identities also hold for the untilde metric. The the inverse metric can be computed by

$$\begin{aligned} \tilde{g}^{\alpha\beta} &= \frac{1}{\tilde{g}}\frac{\delta\tilde{g}}{\delta\tilde{g}_{\alpha\beta}} = \frac{1}{\tilde{g}}\frac{\delta\tilde{g}}{\delta g^{\mu\nu}}\frac{\delta g^{\mu\nu}}{\delta\tilde{g}_{\alpha\beta}} && (3.7)\\ &= \frac{1}{\tilde{g}}\frac{\delta(A^4g + A^3Bgg^{\rho\sigma}\phi_\rho\phi_\sigma)}{\delta g^{\mu\nu}}\frac{1}{A}\frac{\delta g^{\mu\nu}}{\delta g_{\alpha\beta}},\\ &= \frac{1}{A\tilde{g}}\left[A^4\frac{\delta g}{\delta g_{\alpha\beta}} + A^3B\left(\frac{\delta g}{\delta g_{\alpha\beta}}g^{\rho\sigma} + g\frac{\delta g^{\rho\sigma}}{\delta g_{\alpha\beta}}\right)\phi_\rho\phi_\sigma\right],\\ &= \frac{1}{A\tilde{g}}\left[A^4gg^{\alpha\beta} + A^3B\left(gg^{\alpha\beta}g^{\rho\sigma} + g\frac{-g^{\alpha\rho}g^{\beta\sigma}\delta g_{\alpha\beta}}{\delta g_{\alpha\beta}}\right)\phi_\rho\phi_\sigma\right],\\ &= \frac{1}{A\tilde{g}}\left[A^4gg^{\alpha\beta} + A^3B\left(-2gg^{\alpha\beta}X - g\phi^\alpha\phi^\beta\right)\right],\\ &= \frac{1}{A}\frac{A^4gg^{\alpha\beta} + A^3B(-2gg^{\alpha\beta}X - g\phi^\alpha\phi^\beta)}{A^4g - 2A^3BXg},\\ &= \frac{1}{A}\left[g^{\alpha\beta} - \frac{B}{A-2BX}\phi^\alpha\phi^\beta\right]. && (3.8)\end{aligned}$$

Riemann tensor can straightforwardly be computed. Starting by compute the connection(defined by $\tilde{\nabla}_\gamma\tilde{g}_{\mu\nu} = 0$)

$$\begin{aligned} \tilde{\Gamma}^\alpha_{\mu\nu} &= \frac{1}{2}\tilde{g}^{\alpha\beta}\left(\partial_\mu\tilde{g}_{\nu\beta} + \partial_\nu\tilde{g}_{\mu\beta} - \partial_\beta\tilde{g}_{\mu\nu}\right),\\ &= \frac{1}{2}\tilde{g}^{\alpha\beta}((\nabla_\mu\tilde{g}_{\nu\beta} + \Gamma^\lambda_{\mu\nu}\tilde{g}_{\lambda\beta} + \Gamma^\rho_{\mu\beta}\tilde{g}_{\nu\rho}) + (\nabla_\nu\tilde{g}_{\mu\beta} + \Gamma^\lambda_{\mu\nu}\tilde{g}_{\lambda\beta} + \Gamma^\rho_{\nu\beta}\tilde{g}_{\mu\rho})\\ &\quad -(\nabla_\beta\tilde{g}_{\mu\nu} + \Gamma^\lambda_{\beta\mu}\tilde{g}_{\lambda\nu} + \Gamma^\rho_{\beta\nu}\tilde{g}_{\rho\nu})),\\ &= \Gamma^\alpha_{\mu\nu} + \frac{1}{2}\tilde{g}^{\alpha\beta}(\nabla_\mu\tilde{g}_{\nu\beta} + \nabla_\nu\tilde{g}_{\mu\beta} - \nabla_\beta\tilde{g}_{\mu\nu}) \quad . && (3.9)\end{aligned}$$

For the sake of computations, we define

$$\tilde{\Gamma}^\alpha_{\mu\nu} =: \Gamma^\alpha_{\mu\nu} + K^\alpha_{\mu\nu}. \tag{3.10}$$

Then by definition

$$\tilde{R}^\mu{}_{\nu\alpha\beta} \equiv \partial_\alpha\tilde{\Gamma}^\mu_{\beta\nu} - \partial_\beta\tilde{\Gamma}^\mu_{\alpha\nu} + \tilde{\Gamma}^\nu_{\alpha\lambda}\tilde{\Gamma}^\lambda_{\beta\mu} - \tilde{\Gamma}^\nu_{\beta\lambda}\tilde{\Gamma}^\lambda_{\alpha\mu}, \tag{3.11}$$

$$= R^{\mu}{}_{\nu\alpha\beta} + 2\nabla_{[\alpha}K^{\mu}_{\beta]\nu} + 2K^{\mu}_{\lambda[\alpha}K^{\lambda}_{\beta]\nu}. \quad (3.12)$$

The Ricci tensor

$$\tilde{R}_{\beta\nu} = R_{\beta\nu} + 2\nabla_{[\alpha}K^{\alpha}_{\beta]\nu} + 2K^{\alpha}_{\lambda[\alpha}K^{\lambda}_{\beta]\nu}, \quad (3.13)$$

and the Ricci scalar

$$\begin{aligned} \tilde{R} &= \tilde{g}^{\beta\nu}\tilde{R}_{\beta\nu}, & (3.14)\\ &= \frac{1}{A}\left[g^{\beta\nu} - \frac{B}{A-2BX}\phi^{\nu}\phi^{\beta}\right]\left[R_{\beta\nu} + 2\nabla_{[\alpha}K^{\alpha}_{\beta]\nu} + 2K^{\alpha}_{\lambda[\alpha}K^{\lambda}_{\beta]\nu}\right] \ . & (3.15)\end{aligned}$$

3.3 Derivation of the Einstein-Hilbert Lagrangian from Purely Disformal Metric Transformation

If we begin with the metric in the form of the purely disformal metric (3.1) , the inverse metric reads

$$\bar{g}^{\mu\nu} = g^{\mu\nu} - \gamma^2 D\phi^{\mu}\phi^{\nu} \quad \text{, with} \quad \gamma^2 = \frac{1}{1-2DX}, \quad (3.16)$$

and we have introduced (3.10)

$$K^{\alpha}_{\mu\nu} := \bar{\Gamma}^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\mu\nu} = \bar{g}^{\alpha\lambda}(\nabla_{(\mu}\bar{g}_{\nu)\lambda} - \frac{1}{2}\nabla_{\lambda}\bar{g}_{\mu\nu}) \quad . \quad (3.17)$$

Then, we can write the Riemannian tensor as (3.12)

$$\begin{aligned} \bar{R}^{\alpha}{}_{\beta\mu\nu} &= 2\partial_{[\mu}\bar{\Gamma}^{\alpha}_{\nu]\beta} + 2\bar{\Gamma}^{\alpha}_{\gamma[\mu}\bar{\Gamma}^{\gamma}_{\nu]\beta} \quad ,\\ &= R^{\alpha}{}_{\beta\mu\nu} + 2\nabla_{[\mu}K^{\alpha}_{\nu]\beta} + 2K^{\alpha}_{\gamma[\mu}K^{\gamma}_{\nu]\beta} \quad , & (3.18)\end{aligned}$$

and the Ricci scalar as

$$\begin{aligned} \bar{R} &= (g^{\beta\nu} - \gamma^2 D\phi^{\beta}\phi^{\nu})(R_{\beta\nu} + 2\nabla_{[\mu}K^{\mu}_{\nu]\beta} + 2K^{\mu}_{\gamma[\mu}K^{\gamma}_{\nu]\beta}),\\ \bar{R} &= R - \gamma^2 D\phi^{\beta}\phi^{\nu}R_{\beta\nu} + 2g^{\beta\nu}\nabla_{[\mu}K^{\mu}_{\nu]\beta} + 2g^{\beta\nu}K^{\mu}_{\gamma[\mu}K^{\gamma}_{\nu]\beta}\\ &\quad -2\gamma^2 D\phi^{\beta}\phi^{\nu}\nabla_{[\mu}K^{\mu}_{\nu]\beta} - 2\gamma^2 D\phi^{\beta}\phi^{\nu}K^{\mu}_{\gamma[\mu}K^{\gamma}_{\nu]\beta} \quad . & (3.19)\end{aligned}$$

By substituting $\bar{g}_{\mu\nu}$ from (3.16) into $K^{\alpha}_{\mu\nu}$ in (3.17) and after straigthforward calculations we can obtain that

$$
\begin{aligned}
K^{\alpha}_{\nu\beta} \quad &= -\frac{1}{2}(\nabla^{\alpha}D)\phi_{,\beta}\phi_{,\nu} + \gamma^2(\nabla_{(\beta}D)\phi_{,\nu)}\phi^{,\alpha} \\
&+\frac{\gamma^2 D}{2}(\nabla_{\lambda}D)\phi^{,\alpha}\phi^{,\lambda}\phi_{,\beta}\phi_{,\nu} + \gamma^2 D\phi^{,\alpha}\phi_{;\beta\nu} \quad .
\end{aligned}
\tag{3.20}
$$

From appendix D, we list here again the 3rd - 6th terms of (3.19) which are the results from (D.5)(D.15)(D.9) and(D.17), respectively. We will subtract the possible terms at this step before adding to the remaining terms of $\bar{R}$

$$
\begin{aligned}
2g^{\beta\nu}\nabla_{[\mu}K^{\mu}_{\nu]\beta} \;=\; & (1+\gamma^2)(\Box D)X + (\nabla_{\mu}D)(\nabla^{\mu}\gamma^2)X + \gamma^2 D(\Box\phi)^2 - \gamma^2 D\phi^{\mu\nu}\phi_{\mu\nu} \\
&+\frac{1}{2}\phi_{\mu}\phi^{\nu}\{(1+\gamma^2)\nabla^{\mu}\nabla_{\nu}D + (\nabla_{\nu}D)(\nabla^{\mu}\gamma^2)\} \\
&+(\phi_{\mu}\Box\phi - \phi_{\mu\nu}\phi^{\mu\nu})\{\frac{1}{2}(\nabla^{\mu}D)\Box\phi(1+3\gamma^2) + D(\nabla^{\mu}\gamma^2)\Box\phi\} \\
&-\gamma^2 DR_{\mu\nu}\phi^{\mu}\phi^{\nu} \quad .
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
2g^{\nu\beta}K^{\mu}_{\gamma[\mu}K^{\gamma}_{\nu]\beta} \;=\; & \overbrace{\gamma^2(1-\gamma^2)\tfrac{X}{2}D^{\mu}D_{\nu}\phi_{\mu}\phi^{\nu}} + \overbrace{\gamma^2(1-\gamma^2)X^2 D_{\mu}D^{\mu}} \\
&+\phi_{\mu}(\Box\phi)D^{\mu}\{\frac{\gamma^2}{2}(1-\gamma^2)\} + \phi_{\mu\nu}\phi^{\nu}D^{\mu}\{\frac{1}{2}(1-3\gamma^2+2\gamma^4)\} \\
&+\overbrace{\gamma^4 D^2\phi^{\gamma}\phi^{\mu}\phi_{\gamma\mu}(\Box\phi)} - \overbrace{\gamma^4 D^2\phi^{\gamma}\phi^{\mu}\phi_{\nu\gamma}\phi_{\mu}{}^{\nu}} \\
&+\overbrace{\gamma^4 D^2 X D_{\lambda}\phi^{\lambda}\phi^{\mu}\phi^{\nu}\phi_{\mu\nu}} \quad .
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
-2\gamma^2 D\phi^{\beta}\phi^{\nu}\nabla_{[\mu}K^{\mu}_{\nu]\beta} = \quad & -\gamma^2 D\{-2(\Box D)X^2 - (\nabla^{\mu}\nabla_{\nu}D)\phi^{\nu}\phi_{\mu}X \\
&+\phi_{\beta\mu}\phi^{\beta}\Big[(1+\gamma^2)(\nabla^{\mu}D)X\Big] \\
&+(\Box\phi)\phi^{\beta}\Big[2\gamma^2(\nabla_{\beta}D)X(DX-1)\Big] \\
&+\overbrace{\gamma^2 D\Box\phi\phi_{\nu\beta}\phi^{\nu}\phi^{\beta}} - \overbrace{\gamma^2 D\phi^{\beta}\phi_{\beta\mu}\phi^{\mu\nu}\phi_{\nu}}\} \quad .
\end{aligned}
\tag{3.23}
$$

$$-2\gamma^2 D\phi^\beta\phi^\nu K^\mu_{\gamma[\mu}K^\gamma_{\nu]\beta} = \overbrace{\gamma^4 DX^2 D^\mu D_\nu\phi_\mu\phi^\nu} + \overbrace{2\gamma^4 DX^3 D^\mu D_\mu}$$
$$-\overbrace{\gamma^4 D^2 X D^\mu \phi_\mu\phi_{\nu\beta}\phi^\nu\phi^\beta} - 2\gamma^4 D^2 X^2 D^\mu \phi_{\mu\beta}\phi^\beta \quad (3.24)$$

We comeback to the Ricci scalar with the above results

$$\bar{R} = (g^{\beta\nu} - \gamma^2 D\phi^\beta\phi^\nu)(R_{\beta\nu} + 2\nabla_{[\nu}K^\mu_{\nu]\beta} + 2K^\mu_{\gamma[\mu}K^\gamma_{\nu]\beta}), \quad (3.25)$$

$$\begin{aligned}\bar{R} &= R - \gamma^2 D\phi^\beta\phi^\nu R_{\beta\nu} + 2g^{\beta\nu}\nabla_{[\mu}K^\mu_{\nu]\beta} + 2g^{\beta\nu}K^\mu_{\gamma[\mu}K^\gamma_{\nu]\beta} \\ &\quad -2\gamma^2 D\phi^\beta\phi^\nu\nabla_{[\mu}K^\mu_{\nu]\beta} - 2\gamma^2 D\phi^\beta\phi^\nu K^\mu_{\gamma[\mu}K^\gamma_{\nu]\beta} \quad .\end{aligned} \quad (3.26)$$

$$\begin{aligned}&= R - 2\gamma^2 DR_{\beta\nu}\phi^\beta\phi^\nu + \gamma^2 D((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}) \\ &+(\Box D)X\overbrace{(1+\gamma^2+2DX\gamma^2)}^{2\gamma^2} + X(\nabla_\mu D)(\nabla^\mu\gamma^2) \\ &+\frac{1}{2}\phi_\mu\phi^\nu(\nabla^\mu\nabla_\nu D)\overbrace{(1+\gamma^2+2DX\gamma^2)}^{2\gamma^2} + \frac{1}{2}\phi_\mu\phi^\nu(\nabla^\mu\gamma^2)(\nabla_\nu D) \\ &+X^2 D_\mu D^\mu\overbrace{(\gamma^2-\gamma^4+\gamma^4 2DX)}^{=0} \\ &+\phi_\mu\Box\phi\Big[\underbrace{\frac{1}{2}(1+3\gamma^2+\gamma^2-\gamma^4-4\gamma^4 DX(DX-1))}_{4\gamma^2}D^\mu + D(\nabla^\mu\gamma^2)\Big] \\ &-\phi_{\mu\nu}\phi^\nu\Big[\underbrace{\frac{1}{2}(1+3\gamma^2-1+3\gamma^2-2\gamma^4+4D^2X^2\gamma^4+2DX\gamma^2(1+\gamma^2))}_{4\gamma^2}D^\mu \\ &+D(\nabla^\mu\gamma^2)\Big],\end{aligned} \quad (3.27)$$

$$\begin{aligned}\bar{R} &= R - 2\gamma^2 DR_{\beta\nu}\phi^\beta\phi^\nu + \gamma^2 D((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}) + 2\gamma^2 X(\Box D) \\ &+X(\nabla_\mu D)(\nabla^\mu\gamma^2) + \frac{1}{2}\phi_\mu\phi^\nu\Big[2\gamma^2\nabla^\mu\nabla_\nu D + 2\gamma(\nabla^\mu\gamma)(\nabla_\nu D)\Big] \\ &+\phi_\mu(\Box\phi)\Big[2\gamma^2 D^\mu + 2\gamma D(\nabla^\mu\gamma)\Big]\end{aligned}$$

$$-\phi_{\mu\nu}\phi^{\nu}\Big[2\gamma^2 D^{\mu} + 2\gamma D(\nabla^{\mu}\gamma)\Big] \quad . \tag{3.28}$$

$$\begin{aligned}
\mathcal{L}_{GF} &= \frac{1}{\gamma}\tilde{R} = \frac{1}{\gamma}R - 2\gamma D R_{\beta\nu}\phi^{\beta}\phi^{\nu} + \gamma D((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}) \\
&\quad + 2\gamma X(\Box D) + 2X(\nabla_{\nu}D)(\nabla^{\nu}\gamma) + \frac{1}{2}\phi_{\mu}\phi^{\nu}\Big[2\gamma(\nabla^{\mu}\nabla_{\nu}D) + 2(\nabla^{\mu}\gamma)(\nabla_{\nu}D)\Big] \\
&\quad + \phi_{\mu}(\Box\phi)\overbrace{\Big[2\gamma D^{\mu} + 2D(\nabla^{\mu}\gamma)\Big]}^{2\nabla^{\mu}(\gamma D)} \\
&\quad - \phi_{\mu\nu}\phi^{\nu}\overbrace{\Big[2\gamma D^{\mu} + 2D(\nabla^{\mu}\gamma)\Big]}^{2\nabla^{\mu}(\gamma D)},
\end{aligned} \tag{3.29}$$

where GF denotes the Galileon frame. By using the relation

$$\begin{aligned}
\nabla_{\nu}(2\gamma D(\phi^{\nu}\Box\phi - \phi_{\mu}\phi^{\mu\nu})) &= 2\nabla_{\nu}(\gamma D)(\phi^{\nu}\Box\phi - \phi_{\mu}\phi^{\mu\nu}) + 2\gamma D((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}) \\
&\quad - 2\gamma D R_{\beta\nu}\phi^{\nu}\phi^{\beta},
\end{aligned} \tag{3.30}$$

We obtain the more compact form of the action

$$\begin{aligned}
\mathcal{L}_{GF} &= \frac{1}{\gamma}R - \gamma D((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}) + 2\gamma X(\Box D) + 2X(\nabla_{\mu}D)(\nabla^{\mu}\gamma) \\
&\quad + \phi_{\mu}\phi^{\nu}\Big[\gamma\nabla^{\mu}\nabla_{\nu}D + (\nabla^{\mu}\gamma)(\nabla_{\nu}D)\Big],
\end{aligned} \tag{3.31}$$

or even more compact

$$\begin{aligned}
\mathcal{L}_{GF} &= \frac{1}{\gamma}R - \gamma D((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}) + 2X\nabla^{\mu}(\gamma\nabla_{\mu}D) \\
&\quad + \phi_{\mu}\phi^{\nu}\nabla^{\mu}(\gamma\nabla_{\nu}D).
\end{aligned} \tag{3.32}$$

By using integration by parts with the last two terms one obtains

$$\mathcal{L} = \frac{R}{\gamma} - \gamma D\big(\Box\phi^2 - \phi_{\mu\nu}\phi^{\mu\nu}\big) + \gamma\Big[\nabla_{\mu}D\phi_{\alpha}\phi^{\alpha\mu} - \nabla_{\nu}D\,\Box\phi\,\phi^{\nu}\Big] \,. \tag{3.33}$$

3.4 Disformal Action in Covariant GLPV form We will consider the disformal gravity Lagrangian (3.33) (we set $16\pi G = 1$ in this section)

$$\mathcal{L} = \frac{R}{\gamma} - \gamma D\big(\Box\phi^2 - \phi_{\mu\nu}\phi^{\mu\nu}\big) + \gamma\Big[\nabla_\mu D\phi_\alpha\phi^{\alpha\mu} - \nabla_\nu D\,\Box\phi\,\phi^\nu\Big]\,. \tag{3.34}$$

It is a covariant action so we will compare it with covariant GLPV action (2.363),(2.364), (2.365),(2.366), and (2.367)

$$\mathcal{L}_2 = A_2(\phi, Y) + Y\,C_{3\phi}\,, \tag{3.35}$$

$$\mathcal{L}_3 = (C_3 + 2Y\,C_{3Y})\Box\phi\,, \tag{3.36}$$

$$\mathcal{L}_4 = (B_4 + C_5)\,R - 2(B_4 + C_5)_Y(\Box\phi^2 - \phi_{\mu\nu}^2) + F_4\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\beta\gamma\delta}\phi_\nu\phi^\beta\phi_\rho^\gamma\phi_\sigma^\delta\,, \tag{3.37}$$

$$\mathcal{L}_5 = \tilde{G}_5\,G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}\tilde{G}_{5Y}\Big(\Box\phi^3 - 3\Box\phi\,\phi_{\mu\nu}^2 + 2\,\phi_{\mu\nu}^3\Big) + F_5\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\phi^\alpha\phi_\mu\phi_\nu^\beta\phi_\rho^\gamma\phi_\sigma^\delta\,. \tag{3.38}$$

Consider the case of $\mathcal{L}_5 = C_5 = 0$. In this case

$$\begin{aligned}\mathcal{L}_4 &= B_4\,R - 2B_{4Y}(\Box\phi^2 - \phi_{\mu\nu}^2) \\ &\quad + \left(\frac{B_4 + A_4 - 2Y\,B_{4Y}}{Y^2}\right)\Big(-Y(\Box\phi^2 - \phi_{\mu\nu}^2) - 2\phi_\mu\phi^{\mu\nu}\phi_{\nu\alpha}\phi^\alpha + 2\Box\phi\phi_{\mu\nu}\phi^\mu\phi^\nu\Big)\,, \\ &= B_4\,R - \frac{B_4 + A_4}{Y}\Big(\Box\phi^2 - \phi_{\mu\nu}^2\Big) \\ &\quad + \frac{2(B_4 + A_4 - 2Y\,B_{4Y})}{Y^2}\Big(\Box\phi\phi_{\mu\nu}\phi^\mu\phi^\nu - \phi_\mu\phi^{\mu\nu}\phi_{\nu\alpha}\phi^\alpha\Big)\,. \end{aligned} \tag{3.39}$$

We then expand our disformal action

$$\begin{aligned}\mathcal{L} &= \frac{R}{\gamma} - \gamma D\big(\Box\phi^2 - \phi_{\mu\nu}\phi^{\mu\nu}\big) \\ &\quad + \gamma\Big[D_\phi\phi_\mu\phi_\alpha\phi^{\alpha\mu} + D_Y Y_\mu\phi_\alpha\phi^{\alpha\mu} - D_\phi\phi_\nu\,\Box\phi\,\phi^\nu - D_Y Y_\nu\,\Box\phi\,\phi^\nu\Big]\,, \qquad (3.40) \\ &= \frac{R}{\gamma} - \gamma D\big(\Box\phi^2 - \phi_{\mu\nu}\phi^{\mu\nu}\big) \\ &\quad + \gamma\Big[D_\phi\phi_\mu\phi_\alpha\phi^{\alpha\mu} + 2D_Y\phi_{\beta\mu}\phi^\beta\phi_\alpha\phi^{\alpha\mu} - D_\phi\phi_\nu\,\Box\phi\,\phi^\nu - 2D_Y\phi_{\beta\nu}\phi^\beta\,\Box\phi\,\phi^\nu\Big]\,,\end{aligned}$$

$$
\begin{aligned}
= \; & \frac{R}{\gamma} - \gamma D\big(\Box\phi^2 - \phi_{\mu\nu}\phi^{\mu\nu}\big) \\
& -2\gamma D_Y\left[\phi_{\beta\nu}\phi^{\beta}\,\Box\phi\,\phi^{\nu} - \phi_{\beta\mu}\phi^{\beta}\phi_{\alpha}\phi^{\alpha\mu}\right] \\
& -\gamma D_{\phi} Y\,\Box\phi + \gamma D_{\phi}\phi_{\mu}\phi_{\alpha}\phi^{\alpha\mu}\,. \qquad (3.41)
\end{aligned}
$$

The last term is not seems to match to any terms in covariant GLPV action (3.35), (3.36) and (3.39), but one can see that

$$B_4 = 1/\gamma\,. \qquad (3.42)$$

From (3.39) and (3.41) one may expect that

$$\frac{B_4 + A_4}{Y} = \gamma D\,, \qquad (3.43)$$

which consequently implies

$$A_4 = \gamma D Y - \frac{1}{\gamma}\,. \qquad (3.44)$$

From (3.42) and (3.44) one obtains

$$B_4 + A_4 - 2Y B_{4Y} = -\gamma Y^2 D_Y\,. \qquad (3.45)$$

Therefore the third terms of (3.39) and (3.41) are consistence with each others then the last term of (3.41) should contribute to $\mathcal{L}_2$ and $\mathcal{L}_3$. To deal with it we define an auxiliary function

$$\gamma D_{\phi} \equiv E + Y E_Y\,. \qquad (3.46)$$

Then we have

$$
\begin{aligned}
\gamma D_{\phi} Y_{\mu}\phi^{\mu} &= -D_{\phi}Y^2 - EY\Box\phi\,, \\
&= -D_{\phi}Y^2 - \Box\phi\int\gamma D_{\phi}dY\,. \qquad (3.47)
\end{aligned}
$$

up to the boundary term. equation (3.41) now becomes

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}D_\phi Y^2 - \left(\gamma D_\phi Y + \frac{1}{2}\int \gamma D_\phi dY\right)\Box\phi \\ &+ \frac{R}{\gamma} - \gamma D\left(\Box\phi^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right) \\ &- 2\gamma D_Y\left[\phi_{\beta\nu}\phi^\beta\,\Box\phi\,\phi^\nu - \phi_{\beta\mu}\phi^\beta\phi_\alpha\phi^{\alpha\mu}\right]\end{aligned} \tag{3.48}$$

It is clear that

$$C_3 = -\frac{1}{2}\int \gamma D_\phi dY\ , \quad A_2 = -\frac{1}{2}D_\phi Y^2 - Y C_{3\phi}\ . \tag{3.49}$$

Now one can see that the disformal gravity action is in the class of GLPV theories.

CHAPTER IV

BACKGROUND EVOLUTION

4.1 Introduction

In order to investigate the situations in which the gravity theory in the previous chapter can drive accelerated expansion of the late-time universe, we study the evolution of the FLRW universe for this theory of gravity. The evolution equations can be obtained using the FLRW metric given by

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j \,, \tag{4.1}$$

where δ_{ij} is the Kronecker delta, and we will work in the time gauge, i.e., $\phi = \phi(t)$. We work with the action from the previous chapter (3.35),(3.36),(3.37) including the matter action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\sum_{i=2}^{4}\mathcal{L}_i \quad + S_m \,, \tag{4.2}$$

where

$$\begin{aligned}
\mathcal{L}_2 &= A_2(\phi, Y) + Y\,C_{3\phi} \,, && (4.3)\\
\mathcal{L}_3 &= (C_3 + 2Y\,C_{3Y})\Box\phi \,, && (4.4)\\
\mathcal{L}_4 &= B_4\,R - \frac{B_4 + A_4}{Y}\left(\Box\phi^2 - \phi^2_{\mu\nu}\right)\\
&\quad + \frac{2(B_4 + A_4 - 2Y\,B_{4Y})}{Y^2}\left(\Box\phi\phi_{\mu\nu}\phi^\mu\phi^\nu - \phi_\mu\phi^{\mu\nu}\phi_{\nu\alpha}\phi^\alpha\right) . && (4.5)
\end{aligned}$$

with $B_4 = 1/\gamma$, $C_3 = -\frac{1}{2}\int \gamma D_\phi dY$, $A_2 = -\frac{1}{2}D_\phi Y^2 - YC_{3\phi}$, $A_4 = \gamma DY - \frac{1}{\gamma}$, and $\kappa = 8\pi G$. In the case of GR we have $D = 0, \gamma = 1, C_3 = 0, B_4 = 1, A_2 = 0, A_4 = -1$.

4.2 Equations of Motion

The equations of motion can be derived from [77] (See also [11])

$$\mathcal{E}_{\rm H} + 6H^2Y^2\left(5F_4 + 2YF_{4Y}\right) = -2\kappa\rho_m, \quad (4.6)$$

$$\mathcal{P}_{\rm H} + 2Y\left[-\left(3H^2 + 2\dot{H}\right)YF_4 - 4H\dot{Y}F_4 - 2HY\dot{Y}F_{4Y} - 2HY\dot{\phi}F_{4\phi}\right] = -2\kappa p_m \quad (4.7)$$

ρ_m and p_m are the energy density and pressure of matter respectively, and the equation of motion for the scalar field reads

$$\dot{J} + 3HJ = P_\phi \quad (4.8)$$

with

$$J = J_{\rm H} - 24H^2Y\dot{\phi}F_4 - 12H^2Y^2\dot{\phi}F_{4Y} \quad (4.9)$$

$$P_\phi = P_{{\rm H}\phi} + 6H^2Y^2F_{4\phi} \quad (4.10)$$

$$\mathcal{E}_{\rm H} := \sum_{a=2}^{4}\mathcal{E}_a, \quad \mathcal{P}_{\rm H} := \sum_{a=2}^{4}\mathcal{P}_a, \quad (4.11)$$

$$J_{\rm H} := -\dot{\phi}G_{2Y} - 6HYG_{3Y} - 2\dot{\phi}G_{3\phi} - 3H^2\dot{\phi}(G_{4Y} + 2YG_{4YY}) - 12HYG_{4Y\phi} \quad (4.12)$$

where

$$\mathcal{E}_2 = 2YG_{2Y} - G_2, \quad (4.13)$$

$$\mathcal{E}_3 = -6HY\dot{\phi}G_{3Y} - YG_{3\phi}, \quad (4.14)$$

$$\mathcal{E}_4 = -6H^2G_4 + 24H^2Y(G_{4Y} + YG_{4YY}) - 12HY\dot{\phi}G_{4Y\phi} - 6H\dot{\phi}G_{4\phi}, \quad (4.15)$$

$$\mathcal{P}_2 = G_2, \quad (4.16)$$

$$\mathcal{P}_3 = -Y(G_{3\phi} - 2\ddot{\phi}G_{3Y}), \quad (4.17)$$

$$\begin{aligned}\mathcal{P}_4 = {} & 2(3H^2 + 2\dot{H})G_4 - 4(3H^2Y + H\dot{Y} + 2\dot{H}Y)G_{4Y} - 8HY\dot{Y}G_{4YY} \\ & +4Y(\ddot{\phi} - 2H\dot{\phi})G_{4Y\phi} + 2(\ddot{\phi} + 2H\dot{\phi})G_{4\phi} - 2YG_{4\phi\phi}. \quad (4.18)\end{aligned}$$

$$P_{\mathrm{H}\phi} = G_{2\phi} - Y(G_{3\phi\phi} - 2\ddot{\phi}G_{3Y\phi}) + 6(2H^2 + \dot{H})G_{4\phi} + 6H(\dot{Y} + 2HY)G_{4Y\phi}. \quad (4.19)$$

In this case we have the arbitrary functions A_2, C_3, B_4, A_4. We can recast our action in Horndeski form plus extra term (F_4) by using

$$F_4 = \frac{B_4 + A_4 - 2YB_{4Y}}{Y^2}, \quad (4.20)$$
$$G_2 = A_2 + YC_{3\phi}, \quad (4.21)$$
$$G_3 = C_3 + 2YC_{3Y}, \quad (4.22)$$
$$G_4 = B_4. \quad (4.23)$$

The first-order differentiation for the C_3 term reads

$$C_{3Y} = -\frac{\gamma}{2}D_\phi, \quad (4.24)$$
$$C_{3\phi} = -\frac{1}{Y}(A_2 + \frac{1}{2}D_\phi Y^2). \quad (4.25)$$

We then have

$$F_4 = -\gamma D_Y \quad (4.26)$$
$$G_2 = A_2 - (A_2 + \frac{1}{2}D_\phi Y^2) = -\frac{1}{2}D_\phi Y^2 \quad (4.27)$$
$$G_3 = C_3 - \gamma Y D_\phi \quad (4.28)$$
$$G_4 = \frac{1}{\gamma}. \quad (4.29)$$

Recall that

$$\gamma = \frac{1}{\sqrt{1 + DY}}. \quad (4.30)$$

We then have

$$\gamma_\phi = -\frac{\gamma^3}{2}D_\phi Y \quad (4.31)$$
$$\gamma_Y = -\frac{\gamma^3}{2}(D + YD_Y) \quad (4.32)$$

$$G_{2Y} = -\frac{Y^2}{2} D_{\phi Y} - Y D_\phi \,. \tag{4.33}$$

Therefore from (4.13), (4.27) and (4.33) we have

$$\mathcal{E}_2 = 2YG_{2Y} - G_2, \tag{4.34}$$

$$= -Y^3 D_{\phi Y} - 2Y^2 D_\phi + \frac{1}{2} D_\phi Y^2 \tag{4.35}$$

Next, consider (4.28)

$$G_3 = C_3 - \gamma Y D_\phi$$

$$G_{3Y} = -\frac{3}{2}\gamma D_\phi + \frac{\gamma^3}{2} Y D_\phi (D + Y D_Y) - \gamma Y D_{\phi Y} \,, \tag{4.36}$$

$$G_{3\phi} = -\frac{A_2}{Y} - \frac{1}{2} D_\phi Y - \gamma Y D_{\phi\phi} + \frac{\gamma^3}{2} D_\phi^2 Y^2 \,. \tag{4.37}$$

We then have (4.14) as

$$\mathcal{E}_3 = -6HY\dot{\phi}\big(-\frac{3}{2} D_\phi + \frac{\gamma^3}{2} Y\, D_\phi (D + Y\, D_Y) - \gamma Y D_{\phi Y}\big) + A_2 + \frac{1}{2} D_\phi Y^2 + \gamma Y^2 D_{\phi\phi} - \frac{\gamma^3}{2} D_\phi^2 Y^3 \,. \tag{4.38}$$

Next, consider (4.29)

$$G_4 = \sqrt{1 + DY} \tag{4.39}$$

$$G_{4\phi} = \frac{\gamma}{2} Y D_\phi \tag{4.40}$$

$$G_{4Y} = \frac{\gamma}{2}(Y D_Y + D) \tag{4.41}$$

$$G_{4Y\phi} = -\frac{\gamma^3}{2} D_\phi Y (D + D_Y Y) + \frac{\gamma}{2}(Y D_{Y\phi} + D_\phi) \tag{4.42}$$

$$G_{4YY} = -\frac{\gamma^3}{2}(D + Y D_Y)^2 + \frac{\gamma}{2}(2D_Y + Y D_{YY}) \tag{4.43}$$

We then obtain

$$\mathcal{E}_4 = -\frac{6H^2}{\gamma} + 24H^2 Y \big(\frac{\gamma}{2} Y D_Y + \frac{\gamma}{2} D - \frac{\gamma^3}{4} Y (D + Y D_Y)^2 + \frac{\gamma Y}{2}(2D_Y + D_{YY} Y)\big)$$

$$-12HY\dot{\phi}\big(-\frac{\gamma^3}{4}D_\phi Y(D+YD_Y)+\frac{\gamma}{2}(YD_{Y\phi}+D_\phi)\big)-\frac{6H\gamma}{2}\dot{\phi}YD_\phi\ , \quad (4.44)$$

we also have

$$F_{4Y}=\frac{\gamma^3}{2}D_Y(D+YD_Y)+\gamma D_{YY}\ . \quad (4.45)$$

Therefore, from (4.13),(4.14),(4.15),(4.26) and (4.45) the equations of motion Eq.(4.6) becomes

$$\begin{aligned}&\Big(-Y^3D_{\phi Y}-2Y^2D_\phi+\frac{1}{2}D_\phi Y^2-6HY\dot{\phi}\big(-\frac{3}{2}D_\phi+\frac{\gamma^3}{2}Y\,D_\phi(D+Y\,D_Y)-\gamma YD_{\phi Y}\Big)\\&+A_2+\frac{1}{2}D_\phi Y^2+\gamma Y^2D_{\phi\phi}-\frac{\gamma^3}{2}D_\phi^2Y^3-\frac{6H^2}{\gamma}+24H^2Y\big(\frac{\gamma}{2}YD_Y+\frac{\gamma}{2}D\\&-\frac{\gamma^3}{4}Y(D+YD_Y)^2+\frac{\gamma Y}{2}(2D_Y+D_{YY}Y)\big)-12HY\dot{\phi}\big(-\frac{\gamma^3}{4}D_\phi Y(D+YD_Y)\\&+\frac{\gamma}{2}(YD_{Y\phi}+D_\phi)\big)-\frac{6H\gamma}{2}\dot{\phi}YD_\phi\Big)-30H^2Y^2\gamma D_Y+12H^2Y^3\Big\{\frac{\gamma^3}{2}D_Y(D+YD_Y)\\&-\gamma D_{YY}\Big\}=-2\kappa\rho_m\ . \end{aligned} \quad (4.46)$$

After simplification, it yields a modified Friedmann equation:

$$0 \ = \ \left(A_2-2YA_{2,Y}\right)-2\kappa\rho_m+6H^2\gamma^3\left(1-Y^2D_{,Y}\right)\ , \quad (4.47)$$

it is an equivalent equation of the one which derived from vary an action with respect to the shift N. We can use the same method to obtain other equations of motion, for Eq.(4.7) we obtain

$$0 \ = \ -2H\gamma^3\dot{\phi}\big(D_{,\phi}Y-2(D+YD_{,Y})\ddot{\phi}\big)+2\gamma\big(2\dot{H}+3H^2\big)+A_2+2\kappa p_m \quad (4.48)$$

where a dot denotes a derivative with respect to time, $H=\dot{a}/a$ is the Hubble parameter.

Since the disformal gravity considered in this work is a sub class of the GLPV theory which is the covariantized Galileon theory [78], we first check whether the acceleration of the universe can be driven by the kinetic terms of scalar field as in the

Galileon theory[79, 80]. In the flat FLRW background, we have $\gamma = 1/\sqrt{1 - D\dot{\phi}^2}$, so that $D\dot{\phi}^2$ should lie within the range $(-\infty, 1)$. In addition, it follows from the above equations that γ should be unity ($D \sim 0$) during matter dominated epoch (This condition is needed for the structure formation) and should be larger than unity during the acceleration of the universe. Hence, $0 \leq D\dot{\phi}^2 < 1$ throughout the evolution of the universe. Therefore, our equations of motion now approximately reduce to

$$3H^2 \approx \kappa(\rho_m - \frac{1}{2\kappa}(A_2 - 2YA_{2Y})) \,, \tag{4.49}$$

$$-2\dot{H} - 3H^2 \approx \kappa(p_m + \frac{1}{2\kappa}A_2) \,. \tag{4.50}$$

In this sense we can deduce that

$$\rho_\phi \approx -\frac{1}{2\kappa}(A_2 - 2YA_{2Y}) \,, \tag{4.51}$$

$$p_\phi \approx \frac{1}{2\kappa}A_2 \,, \tag{4.52}$$

$$w_T \approx -\frac{2\dot{H}}{3H^2} - 1 = \frac{p_m + \frac{1}{2\kappa}A_2}{\rho_m - \frac{1}{2\kappa}(A_2 - 2YA_{2Y})} \,, \tag{4.53}$$

$$w_\phi \approx \frac{A_2}{2YA_{2Y} - A_2} \,, \tag{4.54}$$

and acceleration equation for the dust-filled background ($p_m = 0$) then reads

$$4\frac{\ddot{a}}{a} = \frac{2\kappa}{3}\left\{\frac{1}{2\kappa}(A_2 - 2YA_{2Y}) + \frac{3}{2\kappa}A_2 - \rho_m\right\} \,, \tag{4.55}$$

the main contribution in the above equation that can make $\ddot{a} > 0$ is proportional to $-\rho_\phi/3 + p_\phi$. From this rough analysis, we expect that for the disformal gravity considered here, the accelerated expansion of the universe cannot be driven by kinetic terms of the scalar field. We will check this analysis using numerical integration below. An equation of motion from variation of the action with respect to ϕ (4.8) becomes

$$
\begin{aligned}
0 \;=\; & \ddot{\phi}\Big[A_{2,Y} + 2YA_{2,YY} + 3H^2\gamma^5\big[D\big(1 - Y^2D_{,Y} + 2Y^3D_{,YY}\big) - 2YD^2\big]\Big] \\
& + 2Y\big(5D_{,Y} - 3Y^2D_{,Y}^2 + 2YD_{,YY}\big) + 3H\dot{\phi}\Big(A_{2,Y} - 2\gamma^3Y\big(D + YD_{,Y}\big)\big(\frac{3}{2}H^2 + \dot{H}\big)\Big) \\
& + \frac{1}{2}\Big(A_{2,\phi} - 2YA_{2,Y\phi} + 3H^2\gamma^3\big[3\gamma^2Y^2D_{,\phi}(D + YD_{,Y}) - 2Y^2D_{,\phi Y} - YD_{,\phi}\big]\Big)\,.
\end{aligned}
\tag{4.56}
$$

For concreteness, we choose the disformal coupling of the form

$$
D \equiv M^{-4\lambda_2-4}\mathrm{e}^{-\lambda_1\phi}(-Y)^{\lambda_2}\,, \tag{4.57}
$$

and choose A_2 as

$$
A_2 \equiv M_k^{4-4\lambda_3}(-Y)^{\lambda_3} - 2M_v^4\mathrm{e}^{-\lambda_4\phi}\,, \tag{4.58}
$$

Here, M, M_k and M_v are the constant parameters with dimension of mass and in the case of flat FLRW universe they are all equal to zero, while λ_1, λ_2, λ_3 and λ_4 are the dimensionless constant parameters. For the homogeneous and isotropic universe, $Y = -\dot{\phi}^2$, and therefore the field ϕ may be classified as a phantom field when the kinetic term in A_2 is proportional to Y^{λ_3}. We choose the above form of A_2 because this form can be easily reduced to the canonical form, and as discussed above, the potential term of the scalar field is needed to drive an accelerated expansion of the universe. The above form of the disformal coefficient D is chosen because this form is the simplest form that can be used to study the influence of the kinetic-dependent disformal coefficient. For this choice of D and A_2, the equations of motion become

$$
0 \;=\; 6H^2\big(\gamma^3 + \lambda_2(\gamma^3 - \gamma)\big) + M_k^{4-4\lambda_3}(-Y)^{\lambda_3}(1 - 2\lambda_3) - 2M_v^4e^{-\lambda_4\phi} - 2\kappa\rho_m\,, \tag{4.59}
$$

$$
\begin{aligned}
0 \;=\; & 4\gamma\frac{\ddot{a}}{a} - 2\gamma^3H\dot{\phi}\Big(\lambda_1 - 2D\Big(\lambda_2 + 1\Big)\ddot{\phi}\Big) + 2\gamma H\Big(H + \lambda_1\dot{\phi}\Big) + M_k^{4-4\lambda_3}(\dot{\phi})^{2\lambda_3} \\
& - 2M_v^4\mathrm{e}^{-\lambda_4\phi} + 2\kappa p_m\,,
\end{aligned}
\tag{4.60}
$$

$$
\begin{aligned}
0 \;=\; & +\ddot{\phi}\Big(M_k^{4-4\lambda_3}\lambda_3(2\lambda_3 - 1)(\dot{\phi})^{2\lambda_3} - 3\gamma^3DH^2\Big(\lambda_2 + 1\Big)Y\Big(\lambda_2\Big(3\gamma^2DY - 2\Big) + 3\gamma^2DY - 1\Big)\Big) \\
& + 3HM_k^{4-4\lambda_3}\lambda_3(\dot{\phi})^{2\lambda_3}\dot{\phi} + YM_v^4\lambda_4\mathrm{e}^{-\lambda_4\phi} + \frac{Y}{2}\Big[-3\gamma^3DH\Big(H\lambda_1Y\big(\lambda_2(3\gamma^2DY - 2)
\end{aligned}
$$

$$+3\gamma^2 DY-1\big)-2\big(\lambda_2+1\big)\big(3H^2+2\dot{H}\big)\dot{\phi}\Big)\Big]. \tag{4.61}$$

4.3 Evolution of Background Universe

We briefly study the evolution of the universe at late time by solving the above equations numerically. Substituting $M_v^4 \mathrm{e}^{-\lambda_4 \phi}$ from eq. (4.59) into eq. (4.61), we can write eq. (4.61) as

$$\begin{aligned}\phi'' &= \frac{H'}{H}\phi'\Big[2\big(9\gamma\left(\gamma^2-1\right)^2\left(\lambda_2+1\right)^2+3\gamma\left(\gamma^2-1\right)\times\left(2\lambda_2+1\right)\left(\lambda_2+1\right)-2\lambda_3^2\Omega_k+\lambda_3\Omega_k\big)\Big]^{-1}\\ &\times\phi'\Big(3\gamma^3\Big(2\lambda_2\big(2\frac{H'}{H}+2\lambda_1\phi'+\lambda_4\phi'+3\big)+4\frac{H'}{H}+5\lambda_1\phi'+2\lambda_4\phi'+6\Big)\\ &-3\gamma\Big(\lambda_2\big(4\frac{H'}{H}+\lambda_1\phi'+2\lambda_4\phi'+6\big)+4\frac{H'}{H}+2\lambda_1\phi'+6\Big)+\lambda_4\left(\Omega_k-6\Omega_m\right)\phi'\\ &-2\lambda_3\Omega_k\left(\lambda_4\phi'+3\right)-9\gamma^5\lambda_1\left(\lambda_2+1\right)\phi'\Big),\end{aligned} \tag{4.62}$$

where a prime denotes a derivative with respect to $\ln a$

$$\frac{\mathrm{d}}{\mathrm{d}\ln a}=\frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t},$$

$\Omega_k \equiv M_k^{4-4\lambda_3}(H\dot{\phi})^{2\lambda_3}/H^2$, $\Omega_m = \kappa\rho_m/3H^2 = \Omega_m^0 \mathrm{e}^{-3N}/(H^2/H_0^2)$, H_0 and Ω_m^0 are the present value of the Hubble parameter and Ω_m respectively. The function H'/H can be computed by combining eq. (4.60) with eq. (4.59) and setting $p_m = 0$, so that we obtain

$$\begin{aligned}\frac{H'}{H} &= \Big[2\gamma\big(3\gamma^5\left(\lambda_2+1\right)^2-3\gamma^3\left(\lambda_2+1\right)-3\gamma\lambda_2\left(\lambda_2+1\right)+\left(1-2\lambda_3\right)\lambda_3\Omega_k\big)\Big]^{-1}\times\\ &\Big[27\gamma^8\left(\lambda_2+1\right)^3-9\gamma^6\left(\lambda_2+1\right)^2\left(7\lambda_2+6\right)+9\gamma^4\left(\lambda_2+1\right)^2\left(5\lambda_2+3\right)-9\gamma^2\lambda_2\left(\lambda_2+1\right)^2\\ &-9\gamma^5\left(\lambda_2+1\right)^2\left(\lambda_3\Omega_k+3\Omega_m\right)+3\gamma^3\left(\lambda_2+1\right)\times\big(2\left(2\lambda_2-\lambda_3+2\right)\lambda_3\Omega_k+3\left(4\lambda_2+5\right)\Omega_m\big)\\ &-3\gamma\left(\lambda_2+1\right)\big(\left(\lambda_2-2\lambda_3+1\right)\lambda_3\Omega_k+3\left(\lambda_2+2\right)\Omega_m\big)+\lambda_3\left(2\lambda_3-1\right)\Omega_k\left(\lambda_3\Omega_k+3\Omega_m\right)\\ &+\left(\gamma^2-1\right)\gamma\phi'\times\Big(\lambda_1\left(1-2\lambda_3\right)\lambda_3\Omega_k+\left(\lambda_2+1\right)\lambda_4\big[6\gamma^3+6\left(\gamma^2-1\right)\gamma\lambda_2\\ &-2\lambda_3\Omega_k+\Omega_k-6\Omega_m\big]\Big)\Big].\end{aligned} \tag{4.63}$$

Setting $\Omega_m^0 = 0.3$, $w_T = -0.97(1-\Omega_m^0) = -0.68$ at present and $M^2 = M_k^2 =$

$M_v^2 = M_p H_0$, where we have restored M_p in this relation to avoid confusion and $w_T \equiv -2\dot{H}/(3H^2) - 1$, we numerically solve eqs. (4.62) and (4.63) by making an integration from the present to the past of the universe, and plot the evolution of $\Delta\Omega_m \equiv (\Omega_m - \Omega_m^\Lambda)/\Omega_m^\Lambda$ and w_T in figs. (4) and (5). Here, Ω_m^Λ is the density parameter of matter computed from ΛCDM model by setting $\Omega_m^\Lambda = 0.3$ at present.

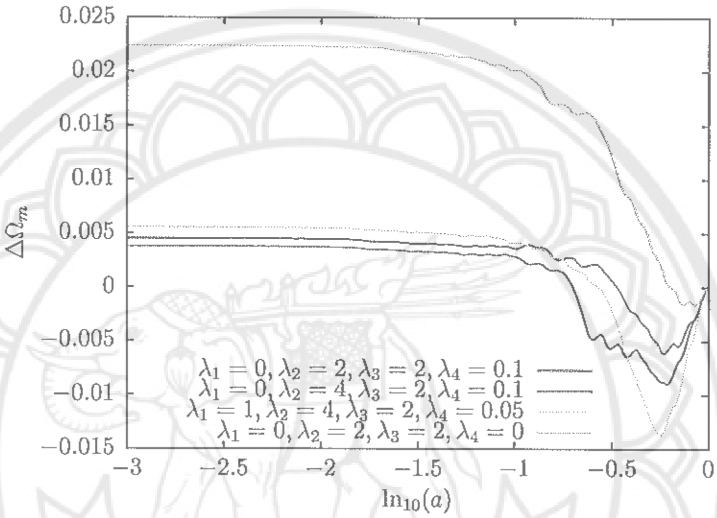

Figure 4 The different density parameter $\Delta\Omega_m$ as a function of $\log_{10} a$ for various values of λ_1, λ_2, λ_3 and λ_4.

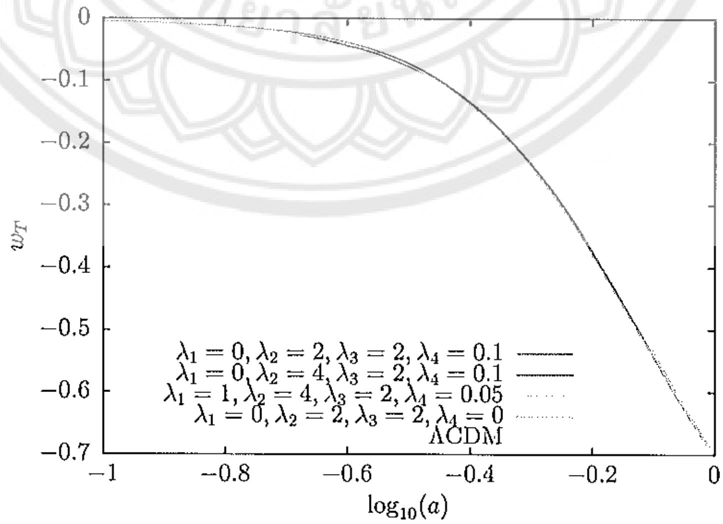

Figure 5 The equation of state parameter w_T as a function of $\log_{10} a$ for various values of λ_1, λ_2, λ_3 and λ_4.

It follows from the plots that the evolution of Ω_m and w_t for the disformal model closely mimics that of evolution for ΛCDM model. In the numerical integration, γ is larger than unity at late time because $-DY = D\dot{\phi}^2$ is close to one, while $\gamma \to 1$ when $-DY$ becomes much smaller than unity during matter dominated epoch. Thus $-DY$ is always smaller than unity throughout the evolution of the universe. Based on the value of the parameters chosen above, $DY < 1$ which implies that $\dot{\phi}/H < 1$, i.e., the field slowly evolves in time compared with the expansion rate of the universe. From our numerical integration, we find that the accelerated expansion of the universe can be achieved if the form of A_2 can satisfy $A_2/(2YA_{2,Y} - A_2) < -1/3$, e.g. $A_2 = (-Y)^{\lambda_3}$ where $\lambda_3 < 1/2$, $A_2 = Y/2 - M_v^4 e^{-\lambda_4 \phi}$ which corresponds to phantom case (a potential driven, not kinetic driven), etc. This is in agreement with the above rough analysis.

CHAPTER V

CONCLUSION

In this thesis, we investigate disformal gravity theory which is in the class of Beyond Horndeski theories. We have reviewed the overall structure of beyond Horndeski up to GLPV in Chapter 2. Then we concentrate our study to the theory generated by purely general disformal transformation $g_{\mu\nu} \longmapsto g_{\mu\nu} + D(\phi, Y)\phi_\mu\phi_\nu$ in Chapter 3. We have obtained the disformal action and have shown that it contains in the class of GLPV theories.

In Chapter 4 we studied the evolution of background universe in a general purely disformal gravity theory in which the gravity action results from purely disformal transformation on the Einstein-Hilbert action. We wrote the gravity action in the form of the covariant GLPV theory and find their equations of motions.We discussed the cosmic evolution for this model , and found that the accelerated expansion of the universe *cannot be driven by kinetic terms* of the scalar field from the disformal coefficient function $D(\phi, X)$ as in the Galileon theory. The accelerated expansion of the late-time universe can be achieved if the Lagrangian of the scalar field A_2 (the potential terms) satisfies $w_\phi \equiv A_2/(2YA_{2Y} - A_2) < -1/3$ which corresponds to the case of potential driven.

APPENDIX

APPENDIX A TECHNICAL SUPPLEMENTARY

A.1 Notation with details Throughout the thesis we use the notations such that the local Minkowskian metric at every points in the space-time is given by

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta_{\mu\nu} \quad , \tag{A.1}$$

where the lowercase greek indices $\mu, \nu, \cdots = 0, 1, 2, 3$ are the 4-dimensional space-time indices. We use the unit where $c = 1, \epsilon_0 = 1$ so $\mu_0 = 1$ (by $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$). This make the length and time the same unit and reduce the number of fundamental (dimensions of) unit into two : mass(or energy) and length(or time). We use $h = 2\pi$(or $\hbar = 1$). By $E = hc/\lambda$, the fundamental unit is remains only one the energy(or mass) unit. We can indicate the dimensions of any quantities by the exponent of energy and this number is called the *canonical dimensions.* In this unit the Newton constant is write explicitly and sometimes in the form of the *reduced Planck unit.*

In SI unit, the Planck mass

$$M_{Pl} = \sqrt{\frac{\hbar c}{G}}. \tag{A.2}$$

In the natural unit

$$G = \frac{1}{M_{Pl}^2}, \tag{A.3}$$

and the Newton constant in the form of reduced Planck units (we us the subscript Pl for both Planck and reduced Planck unit)

$$8\pi G = \frac{1}{m_{Pl}^2} = l_{Pl}^2 = l_{Pl}^2, \tag{A.4}$$

these are the same thing in the natural unit. We also notice that the canonical dimension of G is -2.

A.2 Coordinate-free notation In section 2.3.1.1 we have used the coordinate-free notation. The vector, v, in this notation can be translated to the coordinate based notation as

$$v = v^{\mu}\partial_{\mu} \,. \tag{A.5}$$

The scalar product, $u \cdot v$ or $g(u, v)$, can be translated to the coordinate based notation as

$$g(u,v) = g(u^{\mu}\partial_{\mu}, v^{\nu}\partial_{\nu}) = u^{\mu}v^{\nu}g(\partial_{\mu}, \partial_{\nu}) = u^{\mu}v^{\nu}g_{\mu\nu} \,. \tag{A.6}$$

The covariant derivative with respect to vector u can be translated to the coordinate based notation as

$$\nabla_{u} = \nabla_{u^{\mu}\partial_{\mu}} = u^{\mu}\nabla_{\partial_{\mu}} \equiv u^{\mu}\nabla_{\mu} \,. \tag{A.7}$$

For example, we can write equation (2.45) as

$$v^{\mu} = \underbrace{-g_{\alpha\beta}v^{\alpha}n^{\beta}\, n^{\mu}}_{\perp} + \underbrace{(v^{\mu} + g_{\alpha\beta}v^{\alpha}n^{\beta}n^{\mu})}_{\parallel}. \tag{A.8}$$

For more details see [81, 82]

A.3 Some geometric objects We us symmetrized and anti-symmetrized bracket defined as

$$A_{(\mu_1 \ldots \mu_l)} : = \frac{1}{l!}\sum_{\sigma} A_{\sigma(\mu_1)\ldots\sigma(\mu_l)}, \tag{A.9}$$

$$A_{[\mu_1 \ldots \mu_l]} : = \frac{1}{l!}\sum_{\sigma} (-1)^{\sigma} A_{\sigma(\mu_1)\ldots\sigma(\mu_l)}, \tag{A.10}$$

where σ is a permutations mapping.

$$(-1)^\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is a even mapping} \\ -1 & \text{if } \sigma \text{ is a odd mapping} \end{cases} \tag{A.11}$$

Note that $(-1)^\sigma = (-1)^D$, if D is the number of exchanges of indices required to transform $\{\sigma(\mu_1), ..., \sigma(\mu_l)\}$ to $\{\mu_1, ..., \mu_l\}$.

The purely numeric completely antisymmetric *Levi-Civita symbols* for $(1, n-1)-$dimensional space(can be straightforwardly generalised to arbitrary signature) is defined by

$$\begin{aligned} \varepsilon_{\mu_1 ... \mu_n} &:= \begin{cases} +1 & \text{if } \mu_1 ... \mu_n \text{ is an even permutation of } 012...(n-1) \\ -1 & \text{if } \mu_1 ... \mu_n \text{ is an odd permutation of } 012...(n-1) \\ 0 & \text{if otherwise} \end{cases} && \text{(A.12)} \\ &= \delta^0_{\mu_1} \delta^1_{\mu 2} \cdots \delta^{n-1}_{\mu_n} - \delta^1_{\mu_1} \delta^0_{\mu 2} \cdots \delta^{n-1}_{\mu_n} - \text{the remaining odd superscript permutation terms} \\ &\quad + \text{the remaining even superscript permutation terms} && \text{(A.13)} \\ &= \begin{vmatrix} \delta^0_{\mu_1} & \delta^0_{\mu_2} & \cdots & \delta^0_{\mu_n} \\ \delta^1_{\mu_1} & \delta^1_{\mu_2} & \cdots & \delta^1_{\mu_n} \\ \vdots & \vdots & \ldots & \vdots \\ \delta^{n-1}_{\mu_1} & \delta^{n-1}_{\mu_2} & \cdots & \delta^{n-1}_{\mu_n} \end{vmatrix} && \text{(A.14)} \\ &= n! \delta^0_{[\mu_1} \delta^1_{\mu_2} \ldots \delta^{n-1}_{\mu_n]} = n! \delta^{[0}_{[\mu_1} \delta^1_{\mu_2} \ldots \delta^{n-1]}_{\mu_n]} = n! \delta^{[0}_{\mu_1} \delta^1_{\mu_2} \ldots \delta^{n-1]}_{\mu_n}. && \text{(A.15)} \end{aligned}$$

Note that

$$\varepsilon_{\mu_1 ... \mu_n} = \varepsilon^{\mu_1 ... \mu_n} \qquad \text{(numerically)}, \tag{A.16}$$

and

$$\begin{aligned} \varepsilon_{\mu_1 ... \mu_n} \varepsilon^{\mu_1 ... \mu_n} &= (n! \delta^{[0}_{\mu_1} \delta^1_{\mu_2} \ldots \delta^{n-1]}_{\mu_n})(n! \delta^{\mu_1}_{[0} \delta^{\mu_2}_1 \ldots \delta^{\mu_n}_{n-1]}), \\ &= (n!)^2 \delta^{[0}_{[0} \delta^1_1 \ldots \delta^{n-1]}_{n-1]}, \\ &= (n!)^2 \delta^{[0}_0 \delta^1_1 \ldots \delta^{n-1]}_{n-1}, \end{aligned}$$

$$
\begin{aligned}
&= (n!)^2 \frac{1}{n!} \delta_0^0 \delta_1^1 \ldots \delta_{n-1}^{n-1}. \\
&= n! \quad . \qquad \text{(A.17)}
\end{aligned}
$$

We can also obtain

$$
\begin{aligned}
\varepsilon_{\mu_1 \ldots \mu_n} \varepsilon^{\nu_1 \ldots \nu_n} &= (n! \delta^0_{[\mu_1} \delta^1_{\mu_2} \ldots \delta^{n-1}_{\mu_n]})(n! \delta_0^{[\nu_1} \delta_1^{\nu_2} \ldots \delta_{n-1]}^{\nu_n]}), \\
&= (n!)^2 \delta^{[\nu_1}_{[\mu_1} \delta^{\nu_2}_{\mu_2} \ldots \delta^{\nu_n]}_{\mu_n]}, \\
&= (n!) \delta^{[\nu_1}_{\mu_1} \delta^{\nu_2}_{\mu_2} \ldots \delta^{\nu_n]}_{\mu_n}, \\
&= \begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_1}_{\mu_2} & \ldots & \delta^{\nu_1}_{\mu_n} \\ \delta^{\nu_2}_{\mu_1} & \delta^{\nu_2}_{\mu_2} & \ldots & \delta^{\nu_2}_{\mu_n} \\ \vdots & \vdots & \ldots & \vdots \\ \delta^{\nu_n}_{\mu_1} & \delta^{\nu_n}_{\mu_2} & \ldots & \delta^{\nu_n}_{\mu_n} \end{vmatrix} . \qquad \text{(A.18)}
\end{aligned}
$$

We can also obtain these identities by means of the equation (A.14)

$$
\begin{aligned}
\varepsilon_{\mu_1 \ldots \mu_n} \varepsilon^{\nu_1 \ldots \nu_n} &= \begin{vmatrix} \delta^0_{\mu_1} & \delta^0_{\mu_2} & \ldots & \delta^0_{\mu_n} \\ \delta^1_{\mu_1} & \delta^1_{\mu_2} & \ldots & \delta^1_{\mu_n} \\ \vdots & \vdots & \ldots & \vdots \\ \delta^{n-1}_{\mu_1} & \delta^{n-1}_{\mu_2} & \ldots & \delta^{n-1}_{\mu_n} \end{vmatrix} \begin{vmatrix} \delta^{\nu_1}_0 & \delta^{\nu_2}_0 & \ldots & \delta^{\nu_n}_0 \\ \delta^{\nu_1}_1 & \delta^{\nu_2}_1 & \ldots & \delta^{\nu_n}_1 \\ \vdots & \vdots & \ldots & \vdots \\ \delta^{\nu_1}_{n-1} & \delta^{\nu_2}_{n-1} & \ldots & \delta^{\nu_n}_{n-1} \end{vmatrix}, \qquad \text{(A.19)} \\
&= \begin{vmatrix} \delta^0_{\mu_1} & \delta^0_{\mu_2} & \ldots & \delta^0_{\mu_n} \\ \delta^1_{\mu_1} & \delta^1_{\mu_2} & \ldots & \delta^1_{\mu_n} \\ \vdots & \vdots & \ldots & \vdots \\ \delta^{n-1}_{\mu_1} & \delta^{n-1}_{\mu_2} & \ldots & \delta^{n-1}_{\mu_n} \end{vmatrix} \begin{vmatrix} \delta^{\nu_1}_0 & \delta^{\nu_1}_1 & \ldots & \delta_{n-1} \\ \delta^{\nu_2}_0 & \delta^{\nu_2}_1 & \ldots & \delta^{\nu_2}_{n-1} \\ \vdots & \vdots & \ldots & \vdots \\ \delta^{\nu_n}_0 & \delta^{\nu_n}_1 & \ldots & \delta^{\nu_n}_{n-1} \end{vmatrix}, \qquad \text{(A.20)} \\
&= \begin{vmatrix} \delta^{\nu_1}_{\mu_1} & \delta^{\nu_1}_{\mu_2} & \ldots & \delta^{\nu_1}_{\mu_n} \\ \delta^{\nu_2}_{\mu_1} & \delta^{\nu_2}_{\mu_2} & \ldots & \delta^{\nu_2}_{\mu_n} \\ \vdots & \vdots & \ldots & \vdots \\ \delta^{\nu_n}_{\mu_1} & \delta^{\nu_n}_{\mu_2} & \ldots & \delta^{\nu_n}_{\mu_n} \end{vmatrix} . \qquad \text{(A.21)}
\end{aligned}
$$

From this determinant we can obtain the further familiar identities such as

$$
\varepsilon_{\mu_1 \ldots \mu_n} \varepsilon^{\mu_1 \ldots \mu_n} = \begin{vmatrix} \delta^{\mu_1}_{\mu_1} & \delta^{\mu_1}_{\mu_2} & \ldots & \delta^{\mu_1}_{\mu_n} \\ \delta^{\mu_2}_{\mu_1} & \delta^{\mu_2}_{\mu_2} & \ldots & \delta^{\mu_2}_{\mu_n} \\ \vdots & \vdots & \ldots & \vdots \\ \delta^{\mu_n}_{\mu_1} & \delta^{\mu_n}_{\mu_2} & \ldots & \delta^{\mu_n}_{\mu_n} \end{vmatrix}, \qquad \text{(A.22)}
$$

$$= \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & n \end{vmatrix}, \tag{A.23}$$

$$= n!. \tag{A.24}$$

By the same method we obtain

$$\varepsilon_{\mu_1 \dots \mu_n} \varepsilon^{\mu_1 \dots \mu_m \nu_{m+1} \nu_n} = m! \begin{vmatrix} \delta^{\nu_{m+1}}_{\mu_{m+1}} & \delta^{\nu_{m+1}}_{\mu_{m+2}} & \dots & \delta^{\nu_{m+1}}_{\mu_n} \\ \delta^{\nu_{m+2}}_{\mu_{m+1}} & \delta^{\nu_{m+2}}_{\mu_{m+2}} & \dots & \delta^{\nu_{m+2}}_{\mu_n} \\ \vdots & \vdots & \dots & \vdots \\ \delta^{\nu_n}_{\mu_{m+1}} & \delta^{\nu_n}_{\mu_{m+2}} & \dots & \delta^{\nu_n}_{\mu_n} \end{vmatrix} . \tag{A.25}$$

Note that $\varepsilon^{\mu_1 \dots \mu_m \nu_{m+1} \nu_n} = (-1)^{m(n-m)} \varepsilon^{\nu_{m+1} \nu_n \mu_1 \dots \mu_m}$. We emphasize that though $\varepsilon_{\mu_1 \dots \mu_n}$ is a tenosr in flat Lorentzian space(such that for Special Relativity) $\varepsilon^{\mu_1 \dots \mu_n}$ is not. But in flat Euclidean space it is. We will see later. The determinant of $(n \times n)$-matrix is given by

$$\det A_{ij} \equiv |A_{ij}| = \varepsilon_{i_1 i_2 \dots i_n} A_{i_1 0} A_{i_2 1} \dots A_{i_n n-1}, \tag{A.26}$$

$$= \frac{1}{n!} \varepsilon_{i_1 i_2 \dots i_n} \varepsilon_{j_1 j_2 \dots j_n} A_{i_1 j_1} A_{i_2 j_2} \dots A_{i_n j_n}. \tag{A.27}$$

In flat space the volume element of that space (V) is $dx^0 \wedge dx^1 \wedge \dots \wedge dx^{(n-1)}$ and

$$\begin{aligned} V &= dx^0 \wedge dx^1 \wedge \dots \wedge dx^{(n-1)} = \frac{1}{n!} \varepsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n}, \\ &= \frac{1}{n!} \varepsilon_{\mu_1 \dots \mu_n} \varepsilon^{\mu_1 \dots \mu_n} dx^n, \\ &= dx^n. \end{aligned} \tag{A.28}$$

Hence, we obtain the useful relation

$$dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n} = \varepsilon^{\mu_1 \dots \mu_n} dx^n, \tag{A.29}$$

where we use the standard definition

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n} = \sum_{\sigma} (-1)^{\sigma} dx^{\sigma(\mu_1)} \otimes dx^{\sigma(\mu_2)} \otimes \cdots \otimes dx^{\sigma(\mu_1)} \qquad \text{(A.30)}$$

In the general curved space the geometry is described by the metric tensor

$$g_{\mu\nu} = (e^I \otimes e^J)(\partial_\mu, \partial_\nu)\eta_{IJ}, \qquad \text{(A.31)}$$

$$\equiv e^I{}_\mu e^J{}_\nu \eta_{IJ}. \qquad \text{(A.32)}$$

Now, we bring the Greek indices for curved space coordinates and the upper case Latin indices for the flat space. The component of *co-frame field* $(e^J{}_\nu)^3$ is a mapping between flat and curved metric in a local coordinate.

Locally, the line element described by both frames is the same

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^I{}_\mu e^J{}_\nu \eta_{IJ} dx^\mu dx^\nu = \eta_{IJ} e^I e^J. \qquad \text{(A.33)}$$

This is followed from the *strong equivalence principle* ([83],[84](chapter2)). To say geometric data is contained in $e^I{}_\mu$ in locally flat coordinate. Physically, we can't measure the gravitational effects in this frame, but the frame itself subject to gravity (the *free falling frame*). From equation (A.32) we obtain that

$$g = -e^2 \qquad \text{(A.34)}$$

We can think of $e^I{}_\mu$ as the square root of the metric $g_{\mu\nu}$ and think of the curved space-time as the generalised of flat space-time following way

$$\eta_{IJ} dx^I dx^J \longmapsto \eta_{IJ} e^I e^J = \eta_{IJ} e^I{}_\mu e^J{}_\mu dx^\mu dx^\nu, \qquad \text{(A.35)}$$

$$dx^I \longmapsto e^I = e^I{}_\mu dx^\mu. \qquad \text{(A.36)}$$

---

[3]The prefix 'co-' is come from the fact that e^I is a co-vector $(e^I = e^I_\mu dx^\mu)$, in oppose to the vector, $v = v^\mu \partial_\mu$.

Then the volume element is generalised

$$V_{flat} = dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{(n-1)} \longmapsto V = e^0 \wedge e^1 \wedge \cdots \wedge e^{(n-1)} \tag{A.37}$$

So, we have

$$\begin{aligned}
V &= e^0 \wedge e^1 \wedge \cdots \wedge e^{(n-1)}, && \text{(A.38)}\\
&= \frac{1}{n!}\varepsilon_{I_1 I_2 \ldots I_n} e^{I_1} \wedge e^{I_2} \wedge \cdots \wedge e^{I_n}, && \text{(A.39)}\\
&= \frac{1}{n!}\varepsilon_{I_1 I_2 \ldots I_n} e^{I_1}{}_{\mu_1} e^{I_2}{}_{\mu_2} \cdots e^{I_n}{}_{\mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}, && \text{(A.40)}\\
&= \frac{1}{n!}\varepsilon_{I_1 I_2 \ldots I_n} e^{I_1}{}_{\mu_1} e^{I_2}{}_{\mu_2} \cdots e^{I_n}{}_{\mu_n} \varepsilon^{\mu_1 \mu_2 \ldots \mu_n} dx^n, && \text{(A.41)}\\
&= e\, dx^n, && \text{(A.42)}\\
&= \sqrt{-g} dx^n, && \text{(A.43)}\\
&= \frac{1}{n!}\sqrt{-g}\varepsilon_{\mu_1 \mu_2 \ldots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n}. && \text{(A.44)}
\end{aligned}$$

The last line tells us that V is indeed a differential n-form (see [85], par III of [86], chapter14 of [87]). We also call it the *volume form.* Write the volume-form into the standard form of n-form

$$V =: \frac{1}{n!}\epsilon_{\mu_1 \mu_2 \ldots \mu_n} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_n} \equiv: \epsilon. \tag{A.45}$$

The the component of the volume form cam be expressed in term of the Levi-Civita symbol

$$\epsilon_{\mu_1 \mu_2 \ldots \mu_n} = \sqrt{-g}\varepsilon_{\mu_1 \mu_2 \ldots \mu_n}. \tag{A.46}$$

The volume form is a tensor(it is the invariant volume $\sqrt{-g}dx^n$). So, the components of ϵ can be raising and lowering by the metric tensor. Then we get the contravariant version

$$\begin{aligned}
\epsilon^{\mu_1 \mu_2 \ldots \mu_n} &= g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \ldots g^{\mu_n \nu_n} \epsilon_{\nu_1 \nu_2 \ldots \nu_n}, && \text{(A.47)}\\
&= \sqrt{-g} g^{\mu_1 \nu_1} g^{\mu_2 \nu_2} \ldots g^{\mu_n \nu_n} \varepsilon_{\nu_1 \nu_2 \ldots \nu_n}. && \text{(A.48)}
\end{aligned}$$

(A.49)

Then by contraction

$$\epsilon^{\mu_1\mu_2...\mu_n}\varepsilon_{\mu_1\mu_2...\mu_n} = \sqrt{-g}g^{\mu_1\nu_1}g^{\mu_2\nu_2}...g^{\mu_n\nu_n}\varepsilon_{\nu_1\nu_2...\nu_n}\varepsilon_{\mu_1\mu_2...\mu_n}, \quad (A.50)$$

$$= n!\sqrt{-g}\; g^{-1} = n!\frac{-1}{\sqrt{-g}}, \quad (A.51)$$

Compare to the equation (A.17) and we will see that

$$\epsilon^{\mu_1\mu_2...\mu_n} = \frac{-1}{\sqrt{-g}}\varepsilon^{\mu_1\mu_2...\mu_n}, \quad (A.52)$$

and

$$\epsilon^{\mu_1\mu_2...\mu_n}\epsilon_{\mu_1\mu_2...\mu_n} = -n!. \quad (A.53)$$

$$\epsilon^{\mu_1\mu_2...\mu_n}\epsilon_{\nu_1\nu_2...\nu_n} = -\varepsilon^{\mu_1\mu_2...\mu_n}\varepsilon_{\nu_1\nu_2...\nu_n}. \quad (A.54)$$

In the Lorentzian flat space(as in SR) our results imply the relations

$$\epsilon_{\mu_1\mu_2...\mu_n} = \sqrt{-\eta}\varepsilon_{\mu_1\mu_2...\mu_n} = \varepsilon_{\mu_1\mu_2...\mu_n}, \quad (A.55)$$

$$\epsilon^{\mu_1\mu_2...\mu_n} = \frac{-1}{\sqrt{-\eta}}\varepsilon^{\mu_1\mu_2...\mu_n} = -\varepsilon^{\mu_1\mu_2...\mu_n}. \quad (A.56)$$

If we define $dx^0 \wedge dx^1 \wedge ... \wedge dx^{(n-1)} = \varepsilon^{01...(n-1)}dx^n$ as the volume element, then

$$\varepsilon^{01...(n-1)} = -\epsilon^{01...(n-1)} = 1. \quad (A.57)$$

Because

$$\varepsilon^{01...(n-1)} = \varepsilon_{01...(n-1)} \quad \text{numerically}, \quad (A.58)$$

so,

$$\epsilon_{01...(n-1)} = 1. \quad (A.59)$$

The last two equations is a matter of definitions but we choose the most natural one. In general Lorentzian space-time, we have

$$\epsilon_{0123} = \sqrt{-g}, \quad \epsilon^{0123} = -1/\sqrt{-g}. \tag{A.60}$$

We write explicitly for the general relations

$$\epsilon^{\nu_1\nu_2\ldots\nu_n}\epsilon_{\nu_1\nu_2\ldots\nu_p\mu_{p+1}\ldots\mu_n} = -p!\delta^{\nu_{p+1}\ldots\nu_n}_{\mu_{p+1}\ldots\mu_n}, \tag{A.61}$$

where $\delta^{\mu_1..\mu_l}_{\nu_1...\nu_l}$ for $l \leq n$ is a generalised Kronecker delta defined by

$$\delta^{\mu_1..\mu_l}_{\nu_1...\nu_l} \;:=\; l!\delta^{\mu_1..\mu_l}_{[\nu_1...\nu_l]} = (-1)^{\sigma_l}\epsilon_{\nu_1...\nu_l}\epsilon^{\mu_1..\mu_l}, \tag{A.62}$$

where $(-1)^{\sigma_l}$ is equals to 1 or -1 depends on the signature of l-dimensional subspace is Euclidean or Lorentzian, respectively. From (A.38) four-dimensional spacetime volume is

$$^4\epsilon = e^0 \wedge e^1 \wedge e^2 \wedge e^3, \tag{A.63}$$

while the volume of the space represents by (in appropriate coordinate adjustment)

$$^3\epsilon = e^1 \wedge e^2 \wedge e^3. \tag{A.64}$$

Therefore

$$\begin{aligned} ^4\epsilon &= e^0 \wedge {}^3\epsilon, & \text{(A.65)} \\ \frac{1}{4!}\varepsilon_{IJKL}e^I \wedge e^J \wedge e^K \wedge e^L &= e^0 \wedge \frac{1}{3!}\varepsilon_{ijk}e^i \wedge e^j \wedge e^k, & \text{(A.66)} \end{aligned}$$

operate on the left with e_0 both side of equation, we obtain

$$\begin{aligned} \frac{1}{4!}\varepsilon_{IJKL}e^J \wedge e^K \wedge e^L \binom{4}{1}\delta^I_0 &= \frac{1}{3!}\varepsilon_{ijk}e^i \wedge e^j \wedge e^k, & \text{(A.67)} \\ \varepsilon_{0ijk}e^i \wedge e^j \wedge e^k &= \varepsilon_{ijk}e^i \wedge e^j \wedge e^k, & \text{(A.68)} \\ \epsilon_{0ijk}e^i \wedge e^j \wedge e^k &= \epsilon_{ijk}e^i \wedge e^j \wedge e^k, & \text{(A.69)} \end{aligned}$$

$$\epsilon_{0abc}dx^a \wedge dx^b \wedge dx^c = \epsilon_{abc}dx^a \wedge dx^b \wedge dx^c, \quad (A.70)$$

$$\varepsilon_{0abc}\sqrt{q}dx^a \wedge dx^b \wedge dx^c = \varepsilon_{abc}\sqrt{q}dx^a \wedge dx^b \wedge dx^c, \quad (A.71)$$

$$(A.72)$$

Hence,

$$\varepsilon_{0abc} = \varepsilon_{abc}, \quad (A.73)$$

where $\sqrt{q}\varepsilon_{0abc} = \epsilon_{0abc}$.

APPENDIX B VARIATION AND DETERMINATION OF THE METRIC

B.1 The Variation of g The derivations in this section is based on the basic knowledge from appendix A.3

$$g := \frac{1}{4!}\varepsilon^{\alpha\beta\gamma\delta}\varepsilon^{\mu\nu\rho\sigma}g_{\alpha\mu}g_{\beta\nu}g_{\gamma\rho}g_{\delta\sigma}. \tag{B.1}$$

$$\therefore \delta g = \frac{1}{3!}\varepsilon^{\alpha\beta\gamma\delta}\varepsilon^{\mu\nu\rho\sigma}g_{\alpha\mu}g_{\beta\nu}g_{\gamma\rho}\delta g_{\delta\sigma}, \tag{B.2}$$

$$= \frac{-g}{3!}\epsilon^{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma}g_{\alpha\mu}g_{\beta\nu}g_{\gamma\rho}\delta g_{\delta\sigma}, \tag{B.3}$$

$$= \frac{-g}{3!}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma}{}^{\sigma}\delta g_{\delta\sigma}, \tag{B.4}$$

$$= g\, g^{\delta\sigma}\delta g_{\delta\sigma}. \tag{B.5}$$

For the variation of inverse metric

$$g^{-1} := \frac{1}{4!}\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\rho\sigma}g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}g^{\delta\sigma}. \tag{B.6}$$

$$\therefore \delta g^{-1} = \frac{1}{3!}\varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\rho\sigma}g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}\delta g^{\delta\sigma}, \tag{B.7}$$

$$= \frac{-g^{-1}}{3!}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\rho\sigma}g^{\alpha\mu}g^{\beta\nu}g^{\gamma\rho}\delta g^{\delta\sigma}, \tag{B.8}$$

$$= \frac{-g^{-1}}{3!}\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha\beta\gamma}{}_{\sigma}\delta g^{\delta\sigma}, \tag{B.9}$$

$$= g^{-1}g_{\delta\sigma}\delta g^{\delta\sigma}. \tag{B.10}$$

$$\therefore -g^{-2}\delta g = g^{-1}g_{\delta\sigma}\delta g^{\delta\sigma}. \tag{B.11}$$

$$\delta g = -g\, g_{\delta\sigma}\delta g^{\delta\sigma}. \tag{B.12}$$

This result can also obtained from a result (B.5) with the relation $\delta(g_{\mu\nu}g^{\nu\mu}) = 0$, and because $\delta g/g = \delta \ln g$, then the equation(B.5) can be expressed as

$$\delta \ln(\det(g)) = Tr(g\delta g^{-1}) \quad . \tag{B.13}$$

Start from $\delta(g_{\mu\nu}g^{\nu\beta}) = 0$ then we have

$$-g_{\mu\nu}\delta g^{\nu\beta} = g^{\nu\beta}\delta g_{\mu\nu}, \tag{B.14}$$

$$\therefore -g_{\alpha\beta}g_{\mu\nu}\delta g^{\nu\beta} = \delta g_{\mu\alpha} \quad . \tag{B.15}$$

B.2 Determinant of $g_{\mu\nu}$ in ADM variables Consider the metric tensor of space-time in ADM variables

$$g_{\mu\nu} = \begin{bmatrix} -N^2 + N^a N_a & N_b \\ N_a & q_{ab} \end{bmatrix}. \tag{B.16}$$

Calculating determinant of this matrix by Gauß definition of determinant yields

$$\begin{aligned}
g &= \varepsilon^{\mu\nu\alpha\beta} g_{\mu 0} g_{\nu 1} g_{\alpha 2} g_{\beta 3}, \\
&= \varepsilon^{0bcd} g_{00} q_{b1} q_{c2} q_{d3} + \varepsilon^{a\nu\alpha\beta} g_{a0} g_{\nu 1} g_{\alpha 2} g_{\beta 3}, \\
&= (-N^2 + N_a N^a)\varepsilon^{bcd} q_{b1} q_{c2} q_{d3} + \varepsilon^{a0cd} g_{a0} g_{01} g_{c2} g_{d3} + \varepsilon^{ab\alpha\beta} g_{a0} g_{b1} g_{\alpha 2} g_{\beta 3}, \\
&= (-N^2 + N_a N^a)q - \varepsilon^{acd} N_a N_1 q_{c2} q_{d3} + \varepsilon^{ab0d} g_{a0} q_{b1} g_{02} q_{d3} - \varepsilon^{abc0} g_{a0} q_{b1} q_{c2} g_{03}, \\
&= (-N^2 + N_a N^a)q - \varepsilon^{acd} N_a N_1 q_{c2} q_{d3} + \varepsilon^{abd} N_a q_{b1} N_2 q_{d3} - \varepsilon^{abc} N_a q_{b1} q_{c2} N_3, \\
&= (-N^2 + N_a N^a)q - \frac{1}{3!}\varepsilon^{acd}\varepsilon^{prs} N_a N_p q_{rc} q_{sd} + \frac{1}{3!}\varepsilon^{acd}\varepsilon^{prs} N_a q_{pc} N_r q_{sd} - \frac{1}{3!}\varepsilon^{acd}\varepsilon^{prs} N_a N_s q_{pc} q_{rd}, \\
&= (-N^2 + N_a N^a)q - \frac{1}{2}\varepsilon^{acd}\varepsilon^{prs} N_a N_p q_{rc} q_{sd}, \\
&= (-N^2 + N_a N^a)q - \frac{1}{2}q\epsilon^{acd}\epsilon^{prs} N_a N_p q_{rc} q_{sd}, \\
&= (-N^2 + N_a N^a)q - \frac{1}{2}q\epsilon^{a}{}_{rs}\epsilon^{prs} N_a N_p, \\
&= (-N^2 + N_a N^a)q - \frac{1}{2}2qq^{ap} N_a N_p,
\end{aligned}$$

finally, we obtain

$$g = -N^2 q, \tag{B.17}$$

where we have use (see Appendix A.3)

$$\varepsilon_{0abc} = \varepsilon_{abc},\ \varepsilon^{0abc} = \varepsilon^{abc},\ \epsilon_{abc} = \sqrt{q}\varepsilon_{abc},\ \epsilon^{abc} = \frac{1}{\sqrt{q}}\varepsilon^{abc}. \tag{B.18}$$

APPENDIX C BASIC EQUATIONS FROM GENERAL RELATIVITY

The dynamical variable of GR is a metric tensor, $g_{\mu\nu}(x)$. At each point the metric describes the geometry of the space-time via **Riemann curvature tensor**, $R^{\alpha}{}_{\mu\beta\nu}$. This tensor is defined by

$$R^{\alpha}{}_{\mu\beta\nu} := \partial_\beta \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\beta} + \Gamma^{\alpha}_{\beta\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\nu\lambda}\Gamma^{\lambda}_{\mu\beta} \ , \tag{C.1}$$

where the **Levi-Civita connection** (by its name means torsion-free and metric compatible)

$$\Gamma^{\alpha}_{\mu\nu} := \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu} \right) . \tag{C.2}$$

It coincides with the Christoffel symbol $\left\{ {\alpha \atop \mu\nu} \right\}$ Note that it is *not* a tensor. **Ricci tensor** and **Ricci scalar** then respectively reads

$$R_{\mu\nu} := R^{\beta}{}_{\mu\beta\nu} \ \ , R := g^{\mu\nu} R_{\mu\nu} \ . \tag{C.3}$$

We list below the symmetries of Riemann tensor

$$R^{\alpha}{}_{\mu\beta\nu} = R_{\beta\nu}{}^{\alpha}{}_{\mu} \ , \tag{C.4}$$

$$R^{\alpha}{}_{\mu\beta\nu} = -R^{\alpha}{}_{\mu\nu\beta} \ , \tag{C.5}$$

$$R^{\alpha}{}_{\mu\beta\nu} = -R_{\mu}{}^{\alpha}{}_{\beta\nu} \ . \tag{C.6}$$

With $R_{\mu\nu\rho\sigma} = g_{\mu\gamma} R^{\gamma}{}_{\nu\rho\sigma}$, the another symmetries can be expressed as

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 \ , \tag{C.7}$$

$$\nabla_\mu R_{\nu\rho\sigma\gamma} + \nabla_\nu R_{\rho\mu\sigma\gamma} + \nabla_\rho R_{\mu\nu\sigma\gamma} = 0 \ , \tag{C.8}$$

They are called the first and second Bianchi 's identity, respectively. From the later contract it with $g^{\nu\beta}$ yields

$$\nabla_\mu R_{\alpha\lambda} + \nabla^\beta R_{\alpha\mu\beta\lambda} - \nabla_\alpha R_{\mu\lambda} = 0 , \tag{C.9}$$

contract again with $g^{\alpha\lambda}$ yields

$$\nabla_\mu R - \nabla^\beta R_{\mu\beta} - \nabla^\lambda R_{\mu\lambda} = 0 , \tag{C.10}$$

and the further simplifying this gives us

$$\frac{1}{2}\nabla_\nu R - \nabla^\mu R_{\mu\nu} = 0 , \tag{C.11}$$

$$\nabla^\mu\left(\frac{1}{2}g_{\mu\nu}R - R_{\mu\nu}\right) = 0 , \tag{C.12}$$

$$\nabla^\mu G_{\mu\nu} = 0 , \tag{C.13}$$

where $G_{\mu\nu}$ is the **Einstein tensor** defined by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R . \tag{C.14}$$

The Einstein-Hilbert action with cosmological constant (in the unit of $[\hbar]$ and c is explicitly shown up) is given by

$$S_{EH}[g^{\mu\nu}] = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}(R - 2\Lambda) . \tag{C.15}$$

The variation of this action functional reads

$$\begin{aligned} \delta S_{EH} &= \frac{c^3}{16\pi G}\int d^4x\Big[\delta\big(\sqrt{-g}\big)(g^{\mu\nu}R_{\mu\nu} - 2\Lambda) + \sqrt{-g}\delta g^{\mu\nu}\, R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}\Big] , \\ &= \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\Big[\big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R + g_{\mu\nu}\Lambda\big) + g^{\mu\nu}\delta R_{\mu\nu}\Big] . \end{aligned} \tag{C.16}$$

In the last line we have use the results from A.3. Consider the Ricci tensor

$$R_{\mu\nu} = \partial_\beta\Gamma^\beta_{\mu\nu} - \partial_\nu\Gamma^\beta_{\beta\mu} + \Gamma^\beta_{\beta\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\beta_{\nu\lambda}\Gamma^\lambda_{\mu\beta} . \tag{C.17}$$

Choosing the frame that locally flat (means $g_{\mu\nu} \overset{!}{=} \eta_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu} \overset{!}{=} 0$) the Ricci tensor reads

$$R_{\mu\nu} \overset{!}{=} \partial_\beta \Gamma^\beta_{\mu\nu} - \partial_\nu \Gamma^\beta_{\beta\mu} , \tag{C.18}$$

$$\overset{!}{=} \nabla_\beta \Gamma^\beta_{\mu\nu} - \nabla_\nu \Gamma^\beta_{\beta\mu} , \tag{C.19}$$

and then its variation reads

$$\delta R_{\mu\nu} = \nabla_\beta \delta\Gamma^\beta_{\mu\nu} - \nabla_\nu \delta\Gamma^\beta_{\beta\mu} . \tag{C.20}$$

This is a tensor equation since the variation of metric connection is a tensor the both terms on RHS are the tensor quantities, they are the total derivatives and when we plug it back into the Einstein-Hilbert action they turned out to be the boundary terms and not contribute to the equations of motion. Note also that this equation is true for every frames since the tensor equations is frame invariant. It is known as the **Palatini identity**.

Then we have

$$\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = \frac{c^3\sqrt{-g}}{16\pi G}(G_{\mu\nu} + g_{\mu\nu}\Lambda) . \tag{C.21}$$

By Hamilton principle this gives us the vacuum Einstein equation with cosmological constant, $G_{\mu\nu} + g_{\mu\nu}\Lambda = 0$. To take into the account the matter content we require that

$$\frac{\delta S_{EH}}{\delta g^{\mu\nu}} + \frac{\delta S_M}{\delta g^{\mu\nu}} = \frac{c^3\sqrt{-g}}{16\pi G}\left[(G_{\mu\nu} + g_{\mu\nu}\Lambda) - \frac{8\pi G}{c^4}T_{\mu\nu}\right] . \tag{C.22}$$

Consequently, we have the relation between energy-momentum tensor and the action functional for matter

$$T_{\mu\nu} = \frac{-2c}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} . \tag{C.23}$$

Therefore the above equation can be use to find the energy-momentum tensor for a given action. For example the electtromagnetic action

$$S_{EM}[g^{\mu\nu}, A^{\mu}] = \frac{-1}{4\mu_0}\int d^4x\sqrt{-g}F_{\alpha\beta}F^{\alpha\beta} , \tag{C.24}$$

$$\delta_g S_{EM} = \frac{-1}{4\mu_0}\int d^4x\Big[\delta_g(\sqrt{-g})F_{\alpha\beta}F^{\alpha\beta} + \sqrt{-g}\delta_g(F_{\alpha\beta}F^{\alpha\beta})\Big] , \tag{C.25}$$

$$= \frac{-1}{4\mu_0}\int d^4x\Big[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \sqrt{-g}\delta_g(F_{\alpha\beta}F^{\alpha\beta})\Big] . \tag{C.26}$$

$$\frac{\delta S_{EM}}{\delta g^{\mu\nu}} = \frac{-1}{4\mu_0}\Big[-\frac{1}{2}\sqrt{-g}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \sqrt{-g}\frac{\delta}{\delta g^{\mu\nu}}(F_{\alpha\beta}F^{\alpha\beta})\Big] . \tag{C.27}$$

Consider the last term in the vielbein form

$$F_{\alpha\beta}F^{\alpha\beta} = e^I_\alpha e^J_\beta F_{IJ} e^\alpha_K e^\beta_L F^{KL} . \tag{C.28}$$

We have done here for isolated the flat structure ($g_{\mu\nu}$-independent) from curved structure. Next, we will use the chain rule

$$\frac{\delta}{\delta g^{\mu\nu}} = \frac{\delta}{\delta e^\lambda_P}\frac{\delta e^\lambda_P}{\delta g^{\mu\nu}} . \tag{C.29}$$

From $g^{\mu\nu} = \eta^{MP}e^\mu_M e^\nu_P$ we have

$$\delta g^{\mu\nu} = 2\eta^{MP}e^\mu_M \delta^\nu_\lambda \, \delta e^\lambda_P . \tag{C.30}$$

By using (C.28),(C.29) and (C.30) we can calculating the last term of C.27 as

$$\frac{\delta(F_{\alpha\beta}F^{\alpha\beta})}{\delta g^{\mu\nu}} = \frac{\delta}{\delta e^\lambda_P}(e^I_\alpha e^J_\beta F_{IJ} e^\alpha_K e^\beta_L F^{KL})\,\frac{\delta e^\lambda_P}{\delta g^{\mu\nu}} \tag{C.31}$$

$$= 4e^I_\alpha e^J_\beta e^\alpha_K \frac{\delta e^\beta_L}{\delta e^\lambda_P} F_{IJ}F^{KL}\,\frac{\delta e^\lambda_P}{\delta g^{\mu\nu}} \tag{C.32}$$

$$= (4e^I_\alpha e^J_\beta e^\alpha_K \delta^\beta_\lambda \delta^P_L F_{IJ}F^{KL})(\frac{1}{2}\eta_{MP}e^M_\mu \delta^\lambda_\nu) , \tag{C.33}$$

$$= 2e^I_\alpha e^J_\beta e^\alpha_K \delta^\beta_\lambda \delta^P_L \delta^\lambda_\nu \, e^M_\mu \eta_{MP}\, F_{IJ}F^{KL} , \tag{C.34}$$

$$= 2e^I_\alpha e^J_\nu e^\alpha_K e_{L\mu} F_{IJ}F^{KL} , \tag{C.35}$$

$$= \quad 2F_{\alpha\nu}F^{\alpha}{}_{\mu} = 2g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu}\ . \tag{C.36}$$

Then we obtain

$$\frac{\delta S_{EM}}{\delta g^{\mu\nu}} = \frac{1}{8\mu_0}\sqrt{-g}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{4\mu_0}\sqrt{-g}(2g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu})\ , \tag{C.37}$$

and the energy-momentum tensor of the electtromagnetic field reads

$$T_{\mu\nu} = \frac{-2c}{\sqrt{-g}}\frac{\delta S_{EM}}{\delta g^{\mu\nu}} = \frac{c}{\mu_0}g^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} - \frac{c}{4\mu_0}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\ . \tag{C.38}$$

APPENDIX D DERIVATION OF DISFORMAL ACTION

D.1 Derivation in details

Continue from section 3.3, now we are going to calculate the third and the fifth terms in (3.19) by starting from the second term in (3.18) ,

$$\begin{aligned}
2\nabla_{[\mu}K^{\alpha}_{\nu]\beta} &= \nabla_\mu(-\frac{1}{2}(\nabla^\alpha D)\phi_{,\beta}\phi_{,\nu}) - \nabla_\nu(-\frac{1}{2}(\nabla^\alpha D)\phi_{,\beta}\phi_{,\mu}) + \nabla_\mu(\gamma^2(\nabla_{(\beta}D)\phi_{,\nu)}\phi^{,\alpha}) \\
&\quad -\nabla_\nu(\gamma^2(\nabla_{(\beta}D)\phi_{,\mu)}\phi^{,\alpha}) + \nabla_\mu(\frac{\gamma^2 D}{2}(\nabla_\lambda D)\phi^{,\alpha}\phi^{,\lambda}\phi_{,\beta}\phi_{,\nu}) - \\
&\quad \nabla_\nu(\frac{\gamma^2 D}{2}(\nabla_\lambda D)\phi^{,\alpha}\phi^{,\lambda}\phi_{,\beta}\phi_{,\mu}) + \nabla_\mu(\gamma^2 D\phi^{,\alpha}\phi_{;\beta\nu}) - \nabla_\nu(\gamma^2 D\phi^{,\alpha}\phi_{;\beta\mu}) \quad ,
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}(\nabla_\mu\nabla^\alpha D)\phi_\beta\phi_\nu - \frac{1}{2}(\nabla^\alpha D)\phi_{\beta\mu}\phi_\nu + \frac{1}{2}(\nabla_\nu\nabla^\alpha D)\phi_\beta\phi_\mu + \frac{1}{2}(\nabla^\alpha D)\phi_{\beta\nu}\phi_\mu \\
&\quad +(\nabla_\mu\gamma^2)(\nabla_{(\beta}D)\phi_{\nu)}\phi^\alpha + \gamma^2(\nabla_\mu\nabla_{(\beta}D)\phi_{\nu)}\phi^\alpha + \gamma^2(\nabla_{(\beta}D)\phi_{\nu)\mu}\phi^\alpha + \gamma^2(\nabla_{(\beta}D)\phi_{\nu)}\phi^\alpha{}_\mu \\
&\quad -(\nabla_\nu\gamma^2)(\nabla_{(\beta}D)\phi_{\mu)}\phi^\alpha - \gamma^2(\nabla_\nu\nabla_{(\beta}D)\phi_{\mu)}\phi^\alpha - \gamma^2(\nabla_{(\beta}D)\phi_{\mu)\nu}\phi^\alpha - \gamma^2(\nabla_{(\beta}D)\phi_{\mu)}\phi^\alpha{}_\nu \\
&\quad +(\nabla_{[\mu|}\gamma^2)D(\nabla_\lambda D)\phi^\alpha\phi^\lambda\phi_\beta\phi_{|\nu]} + \gamma^2(\nabla_{[\mu|}D)(\nabla_\lambda D)\phi^\alpha\phi^\lambda\phi_\beta\phi_{|\nu]} \\
&\quad +\gamma^2 D(\nabla_{[\mu|}\nabla_\lambda D)\phi^\alpha\phi^\lambda\phi_\beta\phi_{|\nu]} + \gamma^2 D(\nabla_\lambda D)\phi^\alpha_{[\mu|}\phi^\lambda\phi_\beta\phi_{|\nu]} \\
&\quad +\gamma^2 D(\nabla_\lambda D)\phi^\alpha\phi^\lambda_{[\mu|}\phi_\beta\phi_{|\nu]} + \gamma^2 D(\nabla_\lambda D)\phi^\alpha\phi^\lambda\phi_{\beta[\mu}\phi_{\nu]} \\
&\quad +2(\nabla_{[\mu|}\gamma^2)D\phi^\alpha\phi_{\beta|\nu]} + 2\gamma^2(\nabla_{[\mu|}D)\phi^\alpha\phi_{\beta|\nu]} + 2\gamma^2 D\phi^\alpha_{[\mu|}\phi_{\beta|\nu]} \\
&\quad +2\gamma^2 D\phi^\alpha\phi_{\beta[\nu\mu]} \quad ,
\end{aligned}$$

$$\begin{aligned}
2\nabla_{[\mu}K^{\mu}_{\nu]\beta} &= -(\nabla^\mu\nabla_{[\mu}D)\phi_{\nu]}\phi_\beta - (\nabla^\mu D)\phi_{\beta[\mu}\phi_{\nu]} + (\nabla_\mu\gamma^2)(\nabla_{(\beta}D)\phi_{\nu)}\phi^\mu \\
&\quad -\frac{1}{2}(\nabla_\nu\gamma^2)(\nabla_\mu D)\phi_\beta\phi^\mu + (\nabla_\nu\gamma^2)(\nabla_\beta D)X + \frac{\gamma^2}{2}(\nabla_\beta\nabla_\mu D)\phi_\nu\phi^\mu + \gamma^2(\nabla_\beta\nabla_\nu D)X \\
&\quad -\gamma^2(\nabla_{[\mu}D)\phi_{\nu]\beta}\phi^\mu + \gamma^2(\nabla_{(\beta}D)\phi_{\nu)}\Box\phi - \gamma^2(\nabla_{(\beta}D)\phi_{\mu)}\phi^\mu{}_\nu \\
&\quad +\frac{1}{2}(\nabla_\mu\gamma^2)\phi_\nu D(\nabla_\lambda D)\phi^\mu\phi^\lambda\phi_\beta + D(\nabla_\lambda D)(\nabla_\nu\gamma^2)\phi^\lambda\phi_\beta X \\
&\quad +\frac{\gamma^2}{2}(\nabla_\lambda D)(\nabla_\mu D)\phi_\nu\phi^\mu\phi^\lambda\phi_\beta + \gamma^2(\nabla_\lambda D)(\nabla_\nu D)\phi^\lambda\phi_\beta X
\end{aligned}$$

$$+\frac{\gamma^2}{2}D(\nabla_\lambda\nabla_\mu D)\phi_\nu\phi^\mu\phi^\lambda\phi_\beta+\gamma^2D(\nabla_\lambda\nabla_\nu D)\phi^\lambda\phi_\beta X$$
$$+\frac{\gamma^2 D}{2}(\nabla_\lambda D)\Box\phi\phi_\nu\phi^\lambda\phi_\beta-\frac{\gamma^2 D}{2}(\nabla_\lambda D)\phi^\mu{}_\nu\phi_\mu\phi^\lambda\phi_\beta$$
$$+\frac{\gamma^2 D}{2}(\nabla_\lambda D)\phi^\lambda{}_\mu\phi_\nu\phi^\mu\phi_\beta+\gamma^2D(\nabla_\lambda D)\phi^\lambda{}_\nu\phi_\beta X+\frac{\gamma^2 D}{2}(\nabla_\lambda D)\phi_{\beta\mu}\phi_\nu\phi^\mu\phi^\lambda$$
$$+\gamma^2D(\nabla_\lambda D)\phi_{\beta\nu}\phi^\lambda X+2D(\nabla_{[\mu}\gamma^2)\phi_{\nu]\beta}\phi^\mu+2\gamma^2(\nabla_{[\mu}D)\phi_{\nu]\beta}\phi^\mu$$
$$+\gamma^2D\Box\phi\phi_{\nu\beta}-\gamma^2D\phi^\mu{}_\nu\phi_{\mu\beta}+2\gamma^2D\phi^\mu\phi_{\beta[\nu\mu]}\quad. \tag{D.1}$$

We will use the above quantity in order to calculate the 3rd and the 5th terms in (3.19). The 3rd term is given by

$$2g^{\beta\nu}\nabla_{[\mu}K^\mu_{\nu]\beta}=(\Box D)X$$
$$+\frac{1}{2}(\nabla^\mu\nabla_\nu D)\phi_\mu\phi^\nu-\frac{1}{2}(\nabla^\mu D)\phi_{\mu\nu}\phi^\nu+\frac{1}{2}\nabla^\mu D\Box\phi\phi_\mu+(\nabla_\mu\gamma^2)(\nabla_\nu D)\phi^\mu\phi^\nu$$
$$-\frac{1}{2}(\nabla_\nu\gamma^2)(\nabla_\mu D)\phi^\nu\phi^\mu+(\nabla_\nu\gamma^2)(\nabla^\nu D)X+\frac{\gamma^2}{2}(\nabla^\nu\nabla_\mu D)\phi_\nu\phi^\mu+\gamma^2(\Box D)X$$
$$-\frac{\gamma^2}{2}(\nabla_\mu D)\Box\phi\phi^\mu+\frac{\gamma^2}{2}(\nabla_\nu D)\phi^\nu_\mu\phi^\mu+\gamma^2(\nabla^\nu D)\phi_\nu\Box\phi-\gamma^2(\nabla_{(\nu}D)\phi_{\mu)}\phi^{\mu\nu}$$
$$-(\nabla_\mu\gamma^2)D(\nabla_\lambda D)\phi^\mu\phi^\lambda X+D(\nabla_\lambda D)(\nabla_\nu\gamma^2)\phi^\lambda\phi^\nu X-\overbrace{\gamma^2(\nabla_\lambda D)(\nabla_\mu D)\phi^\mu\phi^\lambda X}^{\oslash}$$
$$+\overbrace{\gamma^2(\nabla_\lambda D)(\nabla_\nu D)\phi^\lambda\phi^\nu X}^{\oslash}-\overbrace{\gamma^2D(\nabla_\lambda\nabla_\mu D)\phi^\mu\phi^\lambda X}^{\oslash\oslash}+\overbrace{\gamma^2D(\nabla_\lambda\nabla_\nu D)\phi^\lambda\phi^\nu X}^{\oslash\oslash}$$
$$-\gamma^2D(\nabla_\lambda D)\Box\phi\phi^\lambda X-\overbrace{\tfrac{\gamma^2 D}{2}(\nabla_\lambda D)\phi^{\mu\nu}\phi_\mu\phi^\lambda\phi_\nu}^{\oslash\oslash\oslash}-\gamma^2D(\nabla_\lambda D)\phi^\lambda_\mu\phi^\mu X$$
$$+\gamma^2D(\nabla_\lambda D)\phi^{\lambda\nu}\phi_\nu X+\overbrace{\tfrac{\gamma^2 D}{2}(\nabla_\lambda D)\phi_{\mu\nu}\phi^\mu\phi^\nu\phi^\lambda}^{\oslash\oslash\oslash}+\gamma^2D(\nabla_\lambda D)\Box\phi\phi^\lambda X$$
$$+D(\nabla_\mu\gamma^2)\Box\phi\phi^\mu-D(\nabla_\nu\gamma^2)\phi^\nu_\mu\phi^\mu+\gamma^2(\nabla_\mu D)\Box\phi\phi^\mu-\gamma^2(\nabla_\nu D)\phi^\nu_\mu\phi^\mu$$
$$+\gamma^2D(\Box\phi)^2-\gamma^2D\phi^{\mu\nu}\phi_{\mu\nu}-\gamma^2DR_{\lambda\mu}\phi^\lambda\phi^\mu \tag{D.2}$$

$$=\ (\Box D)X+\phi_\mu\phi^\nu\{\frac{1}{2}\nabla^\mu\nabla_\nu D+(\nabla^\mu\gamma^2)\nabla_\nu D+\frac{\gamma^2}{2}\nabla^\mu\nabla_\nu D$$
$$-\frac{1}{2}(\nabla^\mu D)(\nabla_\nu\gamma^2)-\overbrace{(\nabla^\mu\gamma^2)(\nabla_\nu D)DX}^{\oslash}+\overbrace{D(\nabla^\mu D)(\nabla_\nu\gamma^2)X}^{\oslash}$$
$$-\overbrace{\gamma^2(\nabla^\mu D)(\nabla_\nu D)X}^{\oslash\oslash}+\overbrace{\gamma^2X(\nabla^\mu D)(\nabla_\nu D)}^{\oslash\oslash}\}$$

$$
\begin{aligned}
&+\phi_{\mu\nu}\phi^{\nu}\{-\frac{1}{2}\nabla^{\mu}D+\frac{\gamma^2}{2}\nabla^{\mu}D-\gamma^2\nabla^{\mu}D-\overbrace{\gamma^2 D(\nabla^{\mu}D)X}^{\oslash\oslash\oslash}\\
&+\overbrace{\gamma^2 D(\nabla^{\mu}D)X}^{\oslash\oslash\oslash}-D\nabla^{\mu}\gamma^2-\gamma^2\nabla^{\mu}D\}\\
&+\phi_{\mu}\{\frac{1}{2}(\nabla^{\mu}D)\Box\phi-\frac{\gamma^2}{2}(\nabla^{\mu}D)\Box\phi+\gamma^2(\nabla^{\mu}D)\Box\phi\\
&-\overbrace{\gamma^2 D(\nabla^{\mu}D)\Box\phi X}^{\oslash\oslash\oslash\oslash}+\overbrace{\gamma^2 D(\nabla^{\mu}D)\Box\phi X}^{\oslash\oslash\oslash\oslash}\\
&+D(\nabla^{\mu}\gamma^2)\Box\phi+\gamma^2(\nabla^{\mu}D)\Box\phi\}\\
&+(\nabla_{\nu}\gamma^2)(\nabla^{\nu}D)X+\gamma^2(\Box D)X+\gamma^2 D(\Box\phi)^2-\gamma^2 D\phi^{\mu\nu}\phi_{\mu\nu}\\
&-\gamma^2 DR_{\mu\nu}\phi^{\mu}\phi^{\nu}\quad , \qquad\qquad (D.3)
\end{aligned}
$$

$$
\begin{aligned}
= \ & (1+\gamma^2)(\Box D)X+(\nabla_{\mu}D)(\nabla^{\mu}\gamma^2)X+\gamma^2 D(\Box\phi)^2-\gamma^2 D\phi^{\mu\nu}\phi_{\mu\nu}\\
&+\frac{1}{2}\phi_{\mu}\phi^{\nu}\{(1+\gamma^2)\nabla^{\mu}\nabla_{\nu}D+(\nabla_{\nu}D)(\nabla^{\mu}\gamma^2)\}\\
&+\phi_{\mu\nu}\phi^{\nu}\{-\frac{1}{2}(\nabla^{\mu}D)(1+3\gamma^2)-D(\nabla^{\mu}\gamma^2)\}\\
&+\phi_{\mu}\{\frac{1}{2}(\nabla^{\mu}D)\Box\phi(1+3\gamma^2)+D(\nabla^{\mu}\gamma^2)\Box\phi\}\\
&-\gamma^2 DR_{\mu\nu}\phi^{\mu}\phi^{\nu}\quad , \qquad\qquad (D.4)
\end{aligned}
$$

$$
\begin{aligned}
2g^{\beta\nu}\nabla_{[\mu}K^{\mu}_{\nu]\beta} = \ & (1+\gamma^2)(\Box D)X+(\nabla_{\mu}D)(\nabla^{\mu}\gamma^2)X+\gamma^2 D(\Box\phi)^2-\gamma^2 D\phi^{\mu\nu}\phi_{\mu\nu}\\
&+\frac{1}{2}\phi_{\mu}\phi^{\nu}\{(1+\gamma^2)\nabla^{\mu}\nabla_{\nu}D+(\nabla_{\nu}D)(\nabla^{\mu}\gamma^2)\}\\
&+(\phi_{\mu}\Box\phi-\phi_{\mu\nu}\phi^{\mu\nu})\{\frac{1}{2}(\nabla^{\mu}D)\Box\phi(1+3\gamma^2)+D(\nabla^{\mu}\gamma^2)\Box\phi\}\\
&-\gamma^2 DR_{\mu\nu}\phi^{\mu}\phi^{\nu}\quad . \qquad\qquad (D.5)
\end{aligned}
$$

Next, calculate the 5th term in (3.19) by using $2\nabla_{[\mu}K^{\mu}_{\nu]\beta}$ from (D.1)

$$
\begin{aligned}
&-2\gamma^2 D\phi^{\beta}\phi^{\nu}\nabla_{[\mu}K^{\mu}_{\nu]\beta}=\\
&-\gamma^2 D\{-2(\Box D)X^2-(\nabla^{\mu}\nabla_{\nu}D)\phi^{\nu}\phi_{\mu}X+(\nabla^{\mu}D)\phi_{\beta\mu}\phi^{\beta}X+\frac{1}{2}(\nabla^{\mu}D)\phi_{\beta\nu}\phi^{\beta}\phi^{\nu}\phi_{\mu}
\end{aligned}
$$

$$
\begin{aligned}
&-2\overbrace{(\nabla_\mu\gamma^2)(\nabla_\beta D)\phi^\mu\phi^\beta X}^{\mathcal{S}}+\overbrace{(\nabla_\nu\gamma^2)(\nabla_\mu D)\phi^\mu\phi^\nu X}^{\mathcal{S}}+\overbrace{(\nabla_\nu\gamma^2)(\nabla_\beta D)\phi^\beta\phi^\nu X}^{\mathcal{S}}\\
&-\overbrace{\gamma^2(\nabla_\beta\nabla_\mu D)\phi^\mu\phi^\beta X}^{\mathcal{SS}}+\overbrace{\gamma^2(\nabla_\beta\nabla_\nu D)\phi^\nu\phi^\beta X}^{\mathcal{SS}}-\overbrace{\tfrac{\gamma^2}{2}(\nabla_\mu D)\phi_{\nu\beta}\phi^\mu\phi^\nu\phi^\beta}^{\mathcal{SSS}}\\
&+\overbrace{\tfrac{\gamma^2}{2}(\nabla_\nu D)\phi_{\mu\beta}\phi^\mu\phi^\nu\phi^\beta}^{\mathcal{SSS}}-2\gamma^2(\nabla_\beta D)\Box\phi\phi^\beta X-\frac{\gamma^2}{2}(\nabla_\beta D)\phi^\mu\phi_{\mu\nu}\phi^\nu\phi^\beta\\
&+\gamma^2(\nabla^\mu D)\phi_{\mu\nu}\phi^\nu X+\overbrace{2(\nabla_\mu\gamma^2)D(\nabla_\lambda D)\phi^\mu\phi^\lambda X^2}^{\mathcal{SSSS}}-\overbrace{2D(\nabla_\lambda D)(\nabla_\nu\gamma^2)\phi^\lambda\phi^\nu X^2}^{\mathcal{SSSS}}\\
&-\overbrace{2\gamma^2(\nabla_\lambda D)(\nabla_\mu D)\phi^\mu\phi^\lambda X^2}^{\oslash}+\overbrace{2\gamma^2(\nabla_\lambda D)(\nabla_\nu D)\phi^\lambda\phi^\nu X^2}^{\oslash}+\overbrace{2\gamma^2 D(\nabla_\lambda\nabla_\mu D)\phi^\mu\phi^\lambda X^2}^{\oslash\oslash}\\
&-\overbrace{2\gamma^2 D(\nabla_\lambda\nabla_\nu D)\phi^\lambda\phi^\nu X^2}^{\oslash\oslash}+2\gamma^2 D(\nabla_\lambda D)\Box\phi\phi^\lambda X^2+\gamma^2 D(\nabla_\lambda D)\phi^\mu{}_\nu\phi_\mu\phi^\nu\phi^\lambda X\\
&+\overbrace{2\gamma^2 D(\nabla_\lambda D)\phi^\lambda{}_\mu\phi^\mu X^2}^{\oslash\oslash\oslash}-\overbrace{2\gamma^2 D(\nabla_\lambda D)\phi^\lambda{}_\nu\phi^\nu X^2}^{\oslash\oslash\oslash}-\overbrace{\gamma^2 D(\nabla_\lambda D)\phi_{\beta\mu}\phi^\beta\phi^\mu\phi^\lambda X}^{\oslash\oslash\oslash\oslash}\\
&+\overbrace{\gamma^2 D(\nabla_\lambda D)\phi_{\beta\nu}\phi^\lambda\phi^\beta\phi^\nu X}^{\oslash\oslash\oslash\oslash}+\overbrace{D(\nabla_\mu\gamma^2)\phi_{\nu\beta}\phi^\mu\phi^\nu\phi^\beta}^{\oslash\oslash\oslash\oslash\oslash}-\overbrace{D(\nabla_\nu\gamma^2)\phi_{\mu\beta}\phi^\mu\phi^\nu\phi^\beta}^{\oslash\oslash\oslash\oslash\oslash}\\
&+0+\gamma^2 D\Box\phi\phi_{\nu\beta}\phi^\nu\phi^\beta-\gamma^2 D\phi^\beta\phi_{\beta\mu}\phi^{\mu\nu}\phi_\nu+0\}\quad, \qquad\qquad (D.6)
\end{aligned}
$$

$$
\begin{aligned}
= \;& -\gamma^2 D\{-2(\Box D)X^2-(\nabla^\mu\nabla_\nu D)\phi^\nu\phi_\mu X+(\nabla^\mu D)\phi_{\beta\mu}\phi^\beta X\\
&+\frac{1}{2}(\nabla^\mu D)\phi_{\beta\nu}\phi^\beta\phi^\nu\phi_\mu-2\gamma^2(\nabla_\beta D)\Box\phi\phi^\beta X-\frac{\gamma^2}{2}(\nabla_\beta D)\phi^\mu\phi_{\mu\nu}\phi^\nu\phi^\beta\\
&+\gamma^2(\nabla^\mu D)\phi_{\mu\nu}\phi^\nu X+2\gamma^2 D(\nabla_\lambda D)\Box\phi\phi^\lambda X^2+\gamma^2 D\Box\phi\phi_{\nu\beta}\phi^\nu\phi^\beta\\
&-\gamma^2 D\phi^\beta\phi_{\beta\mu}\phi^{\mu\nu}\phi_\nu\}\quad, \qquad\qquad (D.7)
\end{aligned}
$$

$$
\begin{aligned}
= \;& -\gamma^2 D\{-2(\Box D)X^2-(\nabla^\mu\nabla_\nu D)\phi^\nu\phi_\mu X\\
&+\phi_{\beta\mu}\phi^\beta\Big[(\nabla^\mu D)X+\gamma^2(\nabla^\mu D)X\Big]\\
&+\phi_{\beta\nu}\phi^\beta\phi^\nu\phi^\mu\overbrace{\Big[\tfrac{1}{2}(\nabla_\mu D)-(\gamma^2/2)(\nabla_\mu D)+\gamma^2 D(\nabla_\mu D)X\Big]}^{(=0)}\\
&+(\Box\phi)\phi^\beta\Big[-2\gamma^2(\nabla_\beta D)X+2\gamma^2 D(\nabla_\beta D)X^2\Big]\\
&+\gamma^2 D\Box\phi\phi_{\nu\beta}\phi^\nu\phi^\beta-\gamma^2 D\phi^\beta\phi_{\beta\mu}\phi^{\mu\nu}\phi_\nu\}\quad, \qquad\qquad (D.8)
\end{aligned}
$$

$$-2\gamma^2 D\phi^\beta\phi^\nu\nabla_{[\mu}K^\mu_{\nu]\beta} = -\gamma^2 D\{-2(\Box D)X^2 - (\nabla^\mu\nabla_\nu D)\phi^\nu\phi_\mu X + \phi_{\beta\mu}\phi^\beta\Big[(1+\gamma^2)(\nabla^\mu D)X\Big] + (\Box\phi)\phi^\beta\Big[2\gamma^2(\nabla_\beta D)X(DX-1)\Big] + \gamma^2 D\Box\phi\phi_{\nu\beta}\phi^\nu\phi^\beta - \gamma^2 D\phi^\beta\phi_{\beta\mu}\phi^{\mu\nu}\phi_\nu\} \quad . \quad (D.9)$$

Next, we will calculate the ramaining 4th and 6th terms in(3.19). For the sake of convenience we will write down everything straigthforwardly

$$\begin{aligned} K^\alpha_{\gamma\mu} &= -\frac{1}{2}(\nabla^\alpha D)\phi_\gamma\phi_\mu + \frac{\gamma^2}{2}(\nabla_\gamma D)\phi_\mu\phi^\alpha + \frac{\gamma^2}{2}(\nabla_\mu D)\phi_\gamma\phi^\alpha \\ &+ \frac{\gamma^2 D}{2}(\nabla_\lambda D)\phi^\alpha\phi^\lambda\phi_\gamma\phi_\mu + \gamma^2 D\phi^\alpha\phi_{\gamma\mu} \quad , \\ K^\gamma_{\nu\beta} &= -\frac{1}{2}(\nabla^\gamma D)\phi_\nu\phi_\beta + \frac{\gamma^2}{2}(\nabla_\nu D)\phi_\beta\phi^\gamma + \frac{\gamma^2}{2}(\nabla_\beta D)\phi_\nu\phi^\gamma \\ &+ \frac{\gamma^2 D}{2}(\nabla_\omega D)\phi^\gamma\phi^\omega\phi_\nu\phi_\beta + \gamma^2 D\phi^\gamma\phi_{\nu\beta} \quad . \end{aligned} \quad (D.10)$$

In the next step all the terms with $\phi_\mu\phi_\nu$ will be vanish,

$$\begin{aligned} K^\alpha_{\gamma[\mu}K^\gamma_{\nu]\beta} &= 0 - \frac{\gamma^2}{4}(\nabla^\alpha D)(\nabla_{[\nu|}D)\phi_\gamma\phi_{|\mu]}\phi_\beta\phi^\gamma - 0 - 0 - \frac{\gamma^2 D}{2}(\nabla^\alpha D)\phi_\gamma\phi_{[\mu|}\phi^\gamma\phi_{|\nu]\beta} \\ &-0 + \frac{\gamma^4}{4}(\nabla_\gamma D)(\nabla_{[\nu}D)\phi_{\mu]}\phi^\alpha\phi_\beta\phi^\gamma + 0 + 0 + \frac{\gamma^4 D}{2}(\nabla_\gamma D)\phi_{[\mu|}\phi^\alpha\phi^\gamma\phi_{|\nu]\beta} \\ &-\frac{\gamma^2}{4}(\nabla_{[\mu|}D)(\nabla^\gamma D)\phi_\gamma\phi^\alpha\phi_{|\nu]}\phi_\beta + 0 + \frac{\gamma^4}{4}(\nabla_{[\mu|}D)(\nabla_\beta D)\phi_\gamma\phi^\alpha\phi_{|\nu]}\phi^\gamma \\ &+\frac{\gamma^4 D}{4}(\nabla_{[\mu|}D)(\nabla_\omega D)\phi_\gamma\phi^\alpha\phi^\gamma\phi^\omega\phi_{|\nu]}\phi_\beta + \frac{\gamma^4 D}{2}(\nabla_{[\mu|}D)\phi_\gamma\phi^\alpha\phi^\gamma\phi_{|\nu]\beta} - 0 \\ &+\frac{\gamma^4 D}{4}(\nabla_\lambda D)(\nabla_{[\nu|}D)\phi^\alpha\phi^\lambda\phi_\gamma\phi_{|\mu]}\phi^\gamma\phi_\beta + 0 + 0 + \frac{\gamma^4 D^2}{2}(\nabla_\lambda D)\phi^\alpha\phi^\lambda\phi_\gamma\phi_{[\mu|}\phi^\gamma\phi_{|\nu]\beta} \\ &-\frac{\gamma^2 D}{2}(\nabla^\gamma D)\phi_{[\nu|}\phi_\beta\phi^\alpha\phi_{\gamma|\mu]} + \frac{\gamma^4 D}{2}(\nabla_{[\nu}D)\phi_\beta\phi^\gamma\phi^\alpha\phi_{\gamma|\mu]} \\ &+\frac{\gamma^4 D}{2}(\nabla_\beta D)\phi_{[\nu|}\phi^\gamma\phi^\alpha\phi_{\gamma|\mu]} + \frac{\gamma^4 D^2}{2}(\nabla_\omega D)\phi^\gamma\phi^\omega\phi_{[\nu|}\phi_\beta\phi^\alpha\phi_{\gamma|\mu]} \\ &+\gamma^4 D^2\phi^\gamma\phi^\alpha\phi_{\gamma[\mu}\phi_{\nu]\beta} \quad , \end{aligned} \quad (D.11)$$

$$\begin{aligned} K^\mu_{\gamma[\mu}K^\gamma_{\nu]\beta} &= \frac{\gamma^2 X}{2}(\nabla^\mu D)(\nabla_{[\nu|}D)\phi_\gamma\phi_\beta + \gamma^2 DX(\nabla^\mu D)\phi_{[\mu|}\phi_{|\nu]\beta} \\ &+\frac{\gamma^4}{4}(\nabla_\gamma D)(\nabla_{[\nu}D)\phi_{\mu]}\phi^\mu\phi_\beta\phi^\gamma + \frac{\gamma^4 D}{2}(\nabla_\gamma D)\phi_{[\mu|}\phi^\mu\phi^\gamma\phi_{|\nu]\beta} \end{aligned}$$

$$
\begin{aligned}
&-\frac{\gamma^2}{4}(\nabla_{[\mu|}D)(\nabla^{\gamma}D)\phi_{\gamma}\phi^{\mu}\phi_{|\nu]}\phi_{\beta} - \frac{\gamma^4 X}{2}(\nabla_{[\mu|}D)(\nabla_{\beta}D)\phi^{\mu}\phi_{|\nu]} \\
&-\frac{\gamma^4 DX}{2}(\nabla_{[\mu|}D)(\nabla_{\omega}D)\phi^{\mu}\phi^{\omega}\phi_{|\nu]}\phi_{\beta} - \gamma^4 DX(\nabla_{[\mu|}D)\phi^{\mu}\phi_{|\nu]\beta} \\
&-\frac{\gamma^4 DX}{2}(\nabla_{\lambda}D)(\nabla_{[\nu|}D)\phi^{\mu}\phi^{\lambda}\phi_{|\mu]}\phi_{\beta} - \gamma^4 D^2 X(\nabla_{\lambda}D)\phi^{\mu}\phi^{\lambda}\phi_{[\mu|}\phi_{|\nu]\beta} \\
&-\frac{\gamma^2 D}{2}(\nabla^{\gamma}D)\phi_{[\nu|}\phi_{\beta}\phi^{\mu}\phi_{\gamma|\mu]} + \frac{\gamma^4 D}{2}(\nabla_{[\nu}D)\phi_{\beta}\phi^{\gamma}\phi^{\mu}\phi_{\gamma|\mu]} \\
&+\frac{\gamma^4 D}{2}(\nabla_{\beta}D)\phi_{[\nu|}\phi^{\gamma}\phi^{\mu}\phi_{\gamma|\mu]} + \frac{\gamma^4 D^2}{2}(\nabla_{\omega}D)\phi^{\gamma}\phi^{\omega}\phi_{[\nu|}\phi_{\beta}\phi^{\mu}\phi_{\gamma|\mu]} \\
&+\gamma^4 D^2\phi^{\gamma}\phi^{\mu}\phi_{\gamma[\mu}\phi_{\nu]\beta} \quad ,
\end{aligned} \tag{D.12}
$$

$$
\begin{aligned}
2g^{\nu\beta}K^{\mu}_{\gamma[\mu}K^{\gamma}_{\nu]\beta} = \;& \gamma^2 X(\nabla^{\mu}D)(\nabla_{[\nu}D)\phi_{\mu]}\phi^{\nu} + 2\gamma^2 DX(\nabla^{\mu}D)\phi_{[\mu}\phi_{\nu]}{}^{\nu} \\
&+\overbrace{\tfrac{\gamma^2}{2}(\nabla_{\gamma}D)(\nabla_{[\mu}D)\phi_{\nu]}\phi^{\mu}\phi^{\nu}\phi^{\gamma}}^{(=0)} + \gamma^4 D(\nabla_{\gamma}D)\phi^{\mu}\phi^{\gamma}\phi_{[\mu}\phi_{\nu]}{}^{\nu} \\
&-\overbrace{\tfrac{\gamma^4}{2}(\nabla^{\gamma}D)(\nabla_{[\mu}D)\phi_{\nu]}\phi^{\mu}\phi^{\nu}\phi_{\gamma}}^{(=0)} - \gamma^4 X(\nabla^{\nu}D)(\nabla_{[\mu}D)\phi_{\nu]}\phi^{\mu} \\
&-\overbrace{\gamma^4 DX(\nabla_{\omega}D)(\nabla_{[\mu}D)\phi_{\nu]}\phi^{\mu}\phi^{\nu}\phi^{\omega}}^{(=0)} - 2\gamma^4 DX(\nabla_{[\mu}D)\phi_{\nu]}{}^{\nu}\phi^{\mu} \\
&-\overbrace{\gamma^4 DX(\nabla_{\lambda}D)(\nabla_{[\nu}D)\phi_{\mu]}\phi^{\mu}\phi^{\nu}\phi^{\lambda}}^{(=0)} - 2\gamma^4 D^2 X(\nabla_{\lambda}D)\phi^{\mu}\phi^{\lambda}\phi_{[\mu}\phi_{\nu]}{}^{\nu} \\
&-\overbrace{\gamma^2 D(\nabla^{\gamma}D)\phi^{\mu}\phi_{\gamma[\mu}\phi_{\nu]}\phi^{\nu}}^{(=0)} + \overbrace{\gamma^4 D(\nabla_{[\nu}D)\phi_{\mu]\gamma}\phi^{\gamma}\phi^{\mu}\phi^{\nu}}^{(=0)} \\
&+\gamma^4 D(\nabla^{\nu}D)\phi^{\gamma}\phi\mu\phi_{[\nu}\phi_{\mu]\gamma} + \overbrace{\gamma^4 D^2(\nabla_{\omega}D)\phi^{\omega}\phi^{\gamma}\phi^{\mu}\phi^{\nu}\phi_{[\nu}\phi\mu]\gamma}^{(=0)} \\
&+2\gamma^4 D^2\phi^{\gamma}\phi\mu\phi_{\gamma[\mu}\phi_{\nu]}{}^{\nu} \quad ,
\end{aligned} \tag{D.13}
$$

$$
\begin{aligned}
= \;& \frac{\gamma^2 X}{2}D^{\mu}D_{\nu}\phi_{\mu}\phi^{\nu} + \gamma^2 X^2 D^{\mu}D_{\mu} + \gamma^2 DX D^{\mu}(\phi_{\mu}(\Box\phi) - \phi_{\mu\nu}\phi^{\nu}) \\
&-\gamma^4 DX D_{\gamma}\phi^{\gamma}\Box\phi - \overbrace{(1/2)\gamma^4 DD_{\gamma}\phi^{\mu}\phi^{\gamma}\phi_{\nu}\phi_{\mu}{}^{\nu}}^{\varnothing} - \frac{\gamma^4 X}{2}D^{\nu}D_{\mu}\phi_{\nu}\phi^{\mu} \\
&-\gamma^4 X^2 D^{\nu}D_{\nu} - \gamma^4 DX D_{\mu}\Box\phi\phi^{\mu} + \gamma^4 DX D_{\nu}\phi_{\mu}{}^{\nu}\phi^{\mu} + 2\gamma^4 D^2 X^2 D_{\lambda}\phi^{\lambda}\Box\phi \\
&+\gamma^4 D^2 X D_{\lambda}\phi^{\mu}\phi^{\lambda}\phi_{\nu}\phi_{\mu}{}^{\nu} + \overbrace{\tfrac{\gamma^4 D}{2}D^{\nu}\phi^{\gamma}\phi^{\mu}\phi_{\nu}\phi_{\mu\gamma}}^{\varnothing} + \gamma^4 DX D^{\nu}\phi^{\gamma}\phi_{\nu\gamma}
\end{aligned}
$$

$$+\gamma^4 D^2\phi^\gamma\phi^\mu\phi_{\gamma\mu}\Box\phi - \gamma^4 D^2\phi^\gamma\phi^\mu\phi_{\gamma\nu}\phi_\mu{}^\nu, \quad \text{(D.14)}$$

Finally, we have got the 4th term of (3.19)

$$\begin{aligned} 2g^{\nu\beta}K^\mu_{\gamma[\mu}K^\gamma_{\nu]\beta} &= \gamma^2(1-\gamma^2)\frac{X}{2}D^\mu D_\nu\phi_\mu\phi^\nu + \gamma^2(1-\gamma^2)X^2 D_\mu D^\mu \\ &+\phi_\mu(\Box\phi)D^\mu\{\frac{\gamma^2}{2}(1-\gamma^2)\} + \phi_{\mu\nu}\phi^\nu D^\mu\{\frac{1}{2}(1-3\gamma^2+2\gamma^4)\} \\ &+\gamma^4 D^2\phi^\gamma\phi^\mu\phi_{\gamma\mu}(\Box\phi) - \gamma^4 D^2\phi^\gamma\phi^\mu\phi_{\nu\gamma}\phi_\mu{}^\nu \\ &+\gamma^4 D^2 X D_\lambda\phi^\lambda\phi^\mu\phi^\nu\phi_{\mu\nu} \quad . \end{aligned} \quad \text{(D.15)}$$

Next, calculate the last term we want

$$\begin{aligned} &K^\mu_{\gamma[\mu}K^\gamma_{\nu]\beta}\phi^\nu\phi^\beta \\ &= \frac{\gamma^2 X}{2}(\nabla^\mu D)(\nabla_{[\nu}D)\phi_{\mu]}\phi_\beta\phi^\beta\phi^\nu + \gamma^2 DX(\nabla^\mu D)\phi_{[\mu}\phi_{\nu]\beta}\phi^\nu\phi^\beta \\ &+0+0+0+0+0+0+0+0+0+0+0+0+0 \quad , \\ &= -\frac{\gamma^2 X^2}{2}(\nabla^\mu D)(\nabla_\nu D)\phi_\mu\phi^\nu - \gamma^2 X^3(\nabla^\nu D)(\nabla_\nu D) \\ &+\frac{\gamma^2 DX}{2}(\nabla^\mu D)\phi_{\nu\beta}\phi_\mu\phi^\nu\phi^\beta + \gamma^2 DX^2(\nabla^\mu D)\phi_{\mu\beta}\phi^\beta. \end{aligned} \quad \text{(D.16)}$$

$$\begin{aligned} -2\gamma^2 D\phi^\beta\phi^\nu K^\mu_{\gamma[\mu}K^\gamma_{\nu]\beta} &= \gamma^4 DX^2 D^\mu D_\nu\phi_\mu\phi^\nu + 2\gamma^4 DX^3 D^\mu D_\mu \\ &-\gamma^4 D^2 X D^\mu\phi_\mu\phi_{\nu\beta}\phi^\nu\phi^\beta - 2\gamma^4 D^2 X^2 D^\mu\phi_{\mu\beta}\phi^\beta \end{aligned} \quad \text{(D.17)}$$

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