GENERALIZED EXTRAGRADIENT ITERATIVE METHODS FOR THE SPLIT FEASIBILITY PROBLEMS AND THE FIXED POINT PROBLEMS

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ABSTRACT

This thesis is separated into two parts. The first part is to present iterative methods to solve the fixed point problems in Banach spaces and the another part is to propose iterative methods to solve the split type problems which are split feasibility problems and split equilibrium problems in Hilbert spaces. We prove the convergence properties of our proposed iterative methods in many directions. Nevertheless, we give the numerical examples to demonstrate the effectiveness of our theoretical results.

CHAPTER I

INTRODUCTION

The fixed point theory is one of the rapidly growing areas of research. This theory plays a fundamental role in many branches of pure, applied, and computational mathematics, such as nonlinear analysis, optimization, economics. In 1922, the Polish mathematician, Stefan Banach [3] established the important result on fixed point theory known as the Banach Contraction Principle. Many researchers have studied to generalize, improve and extend the fixed point theory. The concept of a multi-valued contraction mapping was initiated by Nadler [50] and Markin [48]. The authors [48, 50] concentrated the ideas of Banach's contraction mappings and multi-valued mappings which use the Haudorff metric and proved some fixed point theorems. Since then there exist extensive literatures on a multi-valued fixed point theory which have applications in several areas, such as convex optimization, control theory, differential inclusion, and economics (see [31] and references cited therein). Approximating a fixed point of nonexpansive and multi-valued nonexpansive mappings has been happened to the difference of the iterative methods. Sokhuma and Kaewkhao [63] proposed the iterative method of a hybrid pair of a single-valued nonexpansive mapping and a multi-valued nonexpansive mapping in 2011. The strong convergence theorem for their iterative method was proven in Banach spaces. Many researchers have generalized, improved and extended their iterative methods for solving various problems.

The convex feasibility problem (CFP), is formulated as the problem of finding a point in the nonempty intersection of finite family of closed convex sets in one space. It has attracted the concentration owing to its extensive applications in several applied disciplines diverse as approximation theorem, image recovery, signal processing, control theory, biomedical engineering, communication and geophysics (see [15, 23, 42, 61, 83]). A special case of convex feasibility problem (CFP) is the split feasibility problem (SFP), where some of the closed convex sets are related to constraints in the range of a nonlinear operator. The split feasibility problem (SFP) induced by Censor and Elfving [14] in 1994 is formulated as the problem of finding a point of a closed convex subset in one space such that the image of the point under a bounded linear operator belongs to a closed convex subset in another spaces. Many researchers have studied and interested this problem due to its application. The well-known iterative methods have been proposed for solving the split feasibility problem (SFP) such as CQ algorithm of Byne [7], the applied Mann iterative process of Xu [76], the subgradient extragradient of Vinh and Hoai [71]. On the other side, Korpelevich [41] presented the extragradient iterative method where two metric projections onto feasibility sets must be found at each iterative step. It improves the usual gradient projection iterative method (see [5, 62]) by performing an additional metric projection step at each iteration. Many researchers have modified and relaxed this iterative method in order to ensure the existence of the solutions of various problems.

To be continue, we are interested in the split equilibrium problem (SEP) which generalizes than the split feasibility problem (SFP) introduced by He [19] in 2010. It is a generalization of various important problems involving split variational inequality problem (SVIP), split minimization problem (SMP), split common fixed point problem (SCFP). To solving the split equilibrium problem is to find a solution of equilibrium problem in one space such that under a bounded linear operator, its image can to find a solution of equilibrium problem in another space. Many researchers have studied and constructed iterative methods for solving the equilibrium problem. The most interested iterative methods are the proximal point iterative method and the extragradient iterative method for the equilibrium problem. To solving the split equilibrium problem has used it for finding solutions (see more [34, 25, 26, 28]).

Motivated and inspired by the work mentioned above, we propose iterative methods for solving all mentioned problems and to investigate its convergence theorems under suitable assumptions. Moreover we give numerical examples to demonstrate our proposed iterative methods.

In the following we give a description of the contents of this thesis.

Chapter II. We include some well-known definitions and some useful results that will be used in our main results of this thesis.

Chapter III. This chapter focuses on the proposed iterative method for the fixed point problems of a hybrid pair of a generalized nonexpansive single-valued mapping and a finite family of multi-valued nonexpansive mappings. The weak and strong convergence theorems of the proposed iterative method are proven in Banach spaces.

Chapter IV. We solve a problem of gradient projection iterative method involving the minimization of the considered function which is ill-posed by using Tikhonov's regularization [76] in the part of the extragradient iterative method and combine with a generalized Ishikawa iterative method for solving the split feasibility and the fixed point problems of pseudo-contractive mappings with Lipschitz assumption on a closed convex subset in Hilbert spaces. On the other hand, we aviod Lipschitzian condition by substituting a generalized Ishikawa iterative method to be a generalized Mann iterative method in the proposed iterative method for solving the split feasibility and the fixed point problems. The weak convergence theorems of our iterative methods in Hilbert spaces are proven. Moreover, we give numerical results and compare its behavior with an Ishikawa-type extragradient iterative method and a Mann-type extragradient iterative method of Ceng et al. [21].

Chapter V. In this chapter, we are interested in constructing iterative method for solving the equilibrium problem. Each iterative method constructed by many researchers for solving this problem has advantage and disadvantage differently, so we focus on the extragradient iterative method for solving the equilibrium problem such that its disadvantage is a condition of a equilibrium bifunction which need Lipschitz-type continuity on a closed convex subset in Hilbert spaces. This condition is very strong and so difficult to approximate. Another interested iterative method is the proximal point iterative method such that its disadvantage is when a equilibrium bifunction is a generalized monotone bifunction such as a pseudomonotone bifunction, a regularized equilibrium problem can not be strongly monotone so the existence and the uniqueness of the solution can not be confirmed. Next in 2015, Khatibzadeh et al. [37] solved this risen problem by using a pseudomonotone bifunction in the proximal point method for finding the solution of the equilibrium problem under different assumptions. Solving the multiple set split equilibrium problem is separated into two parts. In section 5.2, we will propose iterative methods by combining the extragradient iterative method with the proximal point iterative method of a pseudomonotone bifunction under conditions of Khatibzadeh et al. [37] and prove the weak and strong convergence of the proposed iterative methods. In section 5.3 we will propose similar iterative methods in the section 5.2 but in the part of the extragradient iterative method, we avoid Lipschitz-type continuity on a closed convex subset in Hilbert spaces by using linesearch procedures of Tran D.Q. et al. [69] and prove the weak and strong convergence of the proposed iterative methods. Finally, we close this chapter by demonstrating numerical examples which apply the Nash cournot oligopolistic equilibrium problem.

Chapter VI. The conclusion of this thesis is presented.

CHAPTER II

PRELIMINARIES

In this chapter, we present several definitions, notations, and some useful results that will be used in the later chapter.

Throughout this thesis, we denote $\mathbb{R}$ stands for the set of all real numbers and $\mathbb{N}$ the set of all natural numbers.

2.1 Basic results

Definition 2.1.1. [44] A *linear space* or *vector space* X over $\mathbb{R}$ is a set X with the binary operation *addition* defined for elements in X and *scalar multiplication* defined for numbers in $\mathbb{R}$ with elements in X satisfying the following properties: for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$,

(V1) $x + y = y + x$;

(V2) $(x + y) + z = x + (y + z)$;

(V3) there exists an element $0 \in X$ called the zero vector of X such that $x + 0 = x$ for all $x \in X$;

(V4) for every element $x \in X$, there exists an element $-x \in X$ called the additive inverse or the negative of x such that $x + (-x) = 0$;

(V5) $\alpha(x + y) = \alpha x + \alpha y$;

(V6) $(\alpha + \beta)x = \alpha x + \beta x$;

(V7) $(\alpha\beta)x = \alpha(\beta x)$;

(V8) $1 \cdot x = x$.

The elements of a vector space X are called *vectors*, and the elements of $\mathbb{R}$ called *scalars*.

Definition 2.1.2. [44] A norm on a linear space X is a function $\|\cdot\| : X \to \mathbb{R}$ with the following properties:

(N1) $\|x\| \geq 0$ for all $x \in X$;

(N2) $\|x\| = 0$ if and only if $x = 0$;

(N3) $\|\alpha x\| = |\alpha|\|x\|$ for all scalars $\alpha \in \mathbb{R}$ and each $x \in X$;

(N4) $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in X$.

A norm linear space $(X, \|\cdot\|)$ is a linear space X equipped with a norm $\|\cdot\|$.

Definition 2.1.3. [44] An inner product space is a vector space X along with a real-valued function $\langle\cdot,\cdot\rangle$ called an inner product which associates each pair of elements $x, y \in X$. An inner product space satisfies the following properties:

(I1) $\langle x, x\rangle \geq 0$ for all $x \in X$;

(I2) $\langle x, x\rangle = 0$ if and only if $x = 0$;

(I3) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for all scalar $\alpha \in \mathbb{R}$ and each $x, y \in X$;

(I4) $\langle x, y\rangle = \langle y, x\rangle$ for each $x, y \in X$;

(I5) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for each $x, y, z \in X$.

Theorem 2.1.4. [44] (Schwarz inequality) Let X be an inner product space. For each $x, y \in X$, we have

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Definition 2.1.5. [44] A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a normed space X is said to converge (strongly) to an element $x \in X$ if $\lim_{n\to\infty} \|x_n - x\| = 0$. We usually write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$ and call the element x the limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$.

Definition 2.1.6. [44] A sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq N$.

Definition 2.1.7. [65] A normed space is said to be *complete* if every Cauchy sequence is convergent.

Proposition 2.1.8. [44] Let $X \times Y$ be an inner product space. It is complete if and only if both X and Y are complete.

Definition 2.1.9. [66] A Banach space is a complete normed space.

Example 2.1.10. l_p and $L_p[0,1]$, $1 \leq p \leq \infty$ are Banach spaces.

Definition 2.1.11. [66] A Hilbert space is a complete inner product space.

Example 2.1.12. l_2 is a Hilbert space.

Theorem 2.1.13. [44] A strong convergent sequence in a Hilbert space is weak convergent with the same limit. In particular, a weak convergent sequence of a finite dimensional Hilbert space is a strong convergence with the same limit.

Definition 2.1.14. [66] A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a normed space X is said to be bounded if there exists a positive number M such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Theorem 2.1.15. [66] Every bounded sequence in a Hilbert space possesses a weakly convergent subsequence.

Definition 2.1.16. Let A be a nonempty subset of H. The sequence $\{x_n\}$ is said to be *Fejermonotone* if and only if

$$\|x_{n+1} - x\| \leq \|x_n - x\|, \text{ for every } n \in \mathbb{N} \text{ and } x \in A.$$

Let X be a normed space, we denote the set $B(x; r) := \{z \in X : \|x-z\| < r\}$ a ball with center $x \in X$ and radius $r > 0$. Next, we recall some useful sets in a normed space.

Definition 2.1.17. [44] A subset A of a normed space X is said to be open if for each $x \in A$, there exists $r > 0$ such that $B(x;r) \subset A$. A subset B of X is said to be closed if its complement $X \backslash B$ is open.

Definition 2.1.18. [44] Let A be a subset of a normed space X and $x \in H$. Then, x is said to be an interior point of A if there exists $r > 0$ such that $B(x;r) \subset A$. The interior of A is the set of all interior points of A and denoted by $int(A)$.

Definition 2.1.19. [44] Let A be a subset of a normed space X. The closure of A is the smallest closed set containing A and it is denoted by $cl(A)$.

Let us recall useful facts related to convergence and closedness which will be needed later.

Theorem 2.1.20. [44] Let A be a subset of a normed space X. Then,

(1) $x \in cl(A)$ if and only if there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ such that $x_n \to x$ as $n \to +\infty$.

(2) A is closed if and only if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset A$ with $x_n \to x \in X$ as $n \to +\infty$, we have $x \in A$.

2.2 Convexities

Throughout this subsection, we let H be a Hilbert space.

Definition 2.2.1. [66] A subset C of H is said to be *convex* if $\alpha x + (1-\alpha)y \in C$ for every $x, y \in C$ and for every $\alpha \in (0,1)$.

We will mention that the arbitrary intersection operation yields convexity.

Theorem 2.2.2. [4] Let $\{C_j : j \in \mathcal{J}\}$ be an arbitrary collection of convex sets in H. Then, their intersection $\bigcap_{j\in\mathcal{J}} C_j$ is also convex.

Definition 2.2.3. [27] A subset C of H is a *cone* if $\alpha x \in C$ whenever $x \in C$ and $\alpha \in (0, +\infty)$

Definition 2.2.4. [27] A convex subset C of H and $\bar{x} \in C$. A vector $d \in H$ is *normal* to C at $\bar{x}$ if

$$\langle d, x - \bar{x} \rangle \leq 0, \text{ for every } x \in C.$$

Observe that if d is normal, then so is λd for $\lambda \geq 0$. The collection of all normal forms the cone called *normal cone* and is denoted by $N_C\bar{x}$.

A Banach space X is *uniformly convex* if for any two sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$\|x_n\| = \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds.

Proposition 2.2.5. [73] Let X be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing continuous convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|),$$

for all $x, y \in B_r = \{z \in X : \|z\| \leq r\}$ and $\lambda \in [0, 1]$.

In the following definition, we recall the convexity of a real-valued function which goes together with the convexity of a set as we have recalled above.

Definition 2.2.6. [66] A function $f : H \to \mathbb{R}$ is said to be *convex* if for any $x, y \in H$ and for any $\alpha \in (0, 1)$, there holds

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

Next, we recall some semicontinuities of a function on a Hilbert space.

Definition 2.2.7. [66] A function $f : H \to \mathbb{R}$ is said to be *upper semicontinuous* on H if $\{x \in H : f(x) < \lambda\}$ is an open set for all $\lambda \in \mathbb{R}$.

Definition 2.2.8. [66] A function $f : H \to \mathbb{R}$ is said to be *lower semicontinuous* on H if $\{x \in H : f(x) \leq \lambda\}$ is a closed set for all $\lambda \in \mathbb{R}$.

In order to present very useful properties of semicontinuities, we denote the extended real number $[-\infty, +\infty] := \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 2.2.9. [4] Let D be a subset of $[-\infty, +\infty]$. A number $a \in [-\infty, +\infty]$ is the (necessarily unique) *infimum* (or the greatest lower bound) of D if it is a lower bound of D and if, for every lower bound $\bar{a}$ of D, we have $\bar{a} \leq a$. This number is denoted by $\inf(D)$. The *supremum* (or least upper bound) of D is $\sup(D) := -\inf\{-b : b \in D\}$.

Remark 2.2.10. Note that if D is bounded from above in $\mathbb{R}$, we know from the completeness of $\mathbb{R}$ that the supremum of D exists. If D is not bounded from above in $\mathbb{R}$, in this situation, we have $\sup(D) = +\infty$. Similarly, if D is not bounded from below in $\mathbb{R}$, we have the infimum $\inf(D) = -\infty$. In this viewpoint, the set D always admits an infimum and a supremum in $[-\infty, +\infty]$.

Definition 2.2.11. [65] Let $f : H \to \mathbb{R}$ be a function. For a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq H$ the *limit inferior* of $\{f(x_n)\}_{n\in\mathbb{N}}$ in $[-\infty, +\infty]$ is

$$\liminf_{n\to+\infty} f(x_n) := \sup_{n\geq 1} \inf_{n\leq N} f(x_N)$$

and its *limit superior* in $[-\infty, +\infty]$ is

$$\limsup_{n\to+\infty} f(x_n) := \inf_{n\geq 1} \sup_{n\leq N} f(x_N).$$

The following theorem gives the characterization of lower semicontinuity in the term of limit inferior.

Theorem 2.2.12. [65] Let $f : H \to \mathbb{R}$ be a function. Then, f is *lower semicontinuous* at $x \in H$ if and only if, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in H,

$$x_n \to x \quad \text{as} \quad k \to +\infty \;\; \text{implies} \;\; f(x) \leq \liminf_{n\to\infty} f(x_n).$$

It is alike to the lower semicontinuity, we also have the characterization of upper semicontinuity in the term of limit superior.

Theorem 2.2.13. [65] Let $f : H \to \mathbb{R}$ be a function. Then, f is *upper semicontinuous* at $x \in H$ if and only if, for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in H,

$$x_n \to x \text{ as } k \to +\infty \text{ implies } \limsup_{n\to\infty} f(x_n) \leq f(x).$$

Definition 2.2.14. [65] A function $f : H \to \mathbb{R}$ is said to be continuous at $x \in H$ if, it is lower and upper semicontinuous at x.

The following theorem concerning a suficient condition for continuity of a convex function.

Theorem 2.2.15. [70] Assume that H is finite dimensional. Then a convex function $f : H \to \mathbb{R}$ is continuous.

The following definition involving differentiabilities of a function in Hilbert spaces.

Definition 2.2.16. [4] Let $f : H \to \mathbb{R}$ be a function and $x, s \in H$ be given. The *directional derivative* of f at x in the direction s is

$$f'(x,s) = \lim_{t\to 0} \frac{f(x+ts) - f(x)}{t},$$

whenever this limit exists. The function f is said to be Gâteaux differentiable at x if its has directional derivatives $f'(x,s)$ for all $s \in H$ and

$$f'(x,s) = \langle g, s \rangle$$

holds for some $g \in H$. The element g is called Gâteaux derivative or Gâteaux gradient of f at x and is denoted by $\nabla f(x)$.

Definition 2.2.17. [4] Let $f : H \to \mathbb{R}$ be a function and $x \in H$ be given. The function f is said to be Fréchet differentiable or, shortly, differentiable at x if there exists an element $y \in H$ such that

$$\lim_{\|h\| \to 0} \frac{f(x+h) - f(x) - \langle y, h \rangle}{\|h\|}.$$

The element y is called Fréchet derivative or gradient of f at x and is denoted by $Df(x)$.

The following theorem concerning the relationship between these two differentiabilities.

Theorem 2.2.18. [4] Let $f : H \to \mathbb{R}$ be a function and $x \in H$. If f is Fréchet differentiable at x, then it is Gâteaux differentiable at x and $Df(x) = \nabla f(x)$.

Definition 2.2.19. [4] Let $f : H \to \mathbb{R}$ be a function and $x \in H$. An element $g \in H$ is a subgradient of f at x if

$$f(y) \geq f(x) + \langle g, y - x \rangle \text{ for every } y \in H.$$

The set of all subgradients of f at x is called subdifferential of f at x and may be denoted by $\partial f(x)$, i.e., $\partial f(x) = \{g \in H : f(y) \geq f(x) + \langle g, y - x \rangle, \forall y \in H\}$. If $\partial f(x) \neq \emptyset$, we say that f is subdifferentiable at x.

In order to guarantee subdifferentiability of a function, we need the continuity as the following theorem.

Theorem 2.2.20. [85] Let $f : H \to \mathbb{R}$ be a convex function. If f is continuous at some element $x_0 \in H$, then it is subdifferentiable. Furthermore, if f is lower semicontinuous, then it is also subdifferentiable.

We close this subsection by providing the relationship between differentiability and subdifferentiability.

Theorem 2.2.21. [85] Let $f : H \to \mathbb{R}$ be a convex function and $x \in H$. If f is Gâteaux differentiable at x, then $\partial f(x) = \{\nabla f(x)\}$.

Next, we recall some properties of bifunctions on $C \times C$ where C is a closed convex subset in a Hilbert space H.

Definition 2.2.22. [6] A bifunction $f : C \times C \longrightarrow \mathbb{R}$ is said to be

(i) monotone on C if

$$f(x,y) + f(y,x) \leq 0, \text{ for all } x, y \in C;$$

(ii) pseudomonotone on C if

$$f(x,y) \geq 0 \Longrightarrow f(y,x) \leq 0, \text{ for all } x, y \in C;$$

(iii) pseudomonotone on C with respect to $S \subset C$ if

$$f(x,y) \geq 0 \Longrightarrow f(y,x) \leq 0, \text{ for all } x \in S, y \in C;$$

(iv) Lipschitz-type continuous on C if there exist two positive constants L_1, L_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - L_1\|x-y\|^2 - L_2\|y-z\|^2, \text{ for all } x, y, z \in C.$$

2.3 Operators

Throughout this section we also let C be a closed convex subset of a Hilbert space H. We denote the set of fixed points of an operator $T : H \to H$ by

$$Fix(T) := \{x \in H : Tx = x\}.$$

In the following we will recall some useful operators and its properties.

Definition 2.3.1. Let $T : H \to H$ be an operator which is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in H,$$

and T is called to be *firmly nonexpansive* if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2, \text{ for all } x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive.

Next, we recall the definition of the metric projection and its properties.

Definition 2.3.2. Let C be a nonempty subset of H and $x \in H$. If there exists an element $y \in C$ such that

$$\|x - y\| \leq \|x - c\| \text{ for all } c \in C,$$

then the element y is called a *metric projection* of x onto C and is denoted by $P_C x$. Further, if $P_C x$ exists and uniquely determined for all $x \in H$, then the operator $P_C : H \to C$ is called the metric projection onto C.

We can guarantee the existence and uniqueness of the metric projection by the following theorem.

Theorem 2.3.3. [10] Let C be a nonempty closed and convex subset of H. Then for any $x \in H$ there exists a unique metric projection $P_C x$.

The following are some properties of the metric projection which will be used in our main results.

Proposition 2.3.4. Let $x \in H$ and $z \in C$. Then

(1) $z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \leq 0$ for all $y \in C$;

(2) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$ for all $y \in C$;

(3) $\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2$ for all $y \in C$.

Moreover, it is well-known that metric projection $P_C : H \to C$ is firmly nonexpansive, that is,

$$\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2$$
$$\Leftrightarrow \|P_Cx-P_Cy\|^2 \leq \|x-y\|^2 - \|(I-P_C)x-(I-P_C)y\|^2, \ \forall x,y \in H. \tag{2.3.1}$$

Let H be a Hilbert space, the inequalities are hold: for all $x,y,z \in H$, then

$$\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2, \tag{2.3.2}$$

$$\|\alpha x+(1-\alpha)y\|^2 = \alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2 \text{ where } \alpha \in [0,1], \tag{2.3.3}$$

and

$$\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2, \tag{2.3.4}$$

where $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma=1$.

The following definitions are some nonlinear operators which are important basic operators that we will discuss in later chapters.

Definition 2.3.5. $T : H \to H$ is called a *contraction operator* if there exists a positive real number $\rho \in (0,1)$ such that

$$\|Tx-Ty\| \leq \rho\|x-y\| \text{ for all } x,y \in H.$$

Definition 2.3.6. $T : H \to H$ is called a *Lipschitz operator* if there exists a positive real number L such that

$$\|Tx-Ty\| \leq L\|x-y\| \text{ for all } x,y \in H.$$

Definition 2.3.7. Let C be a closed convex subset of H and $T : C \to C$ is called a *uniformly L-Lipschitz operator* if there exists a positive real number L such that

$$\|T^nx-T^ny\| \leq L\|x-y\| \text{ for all } x,y \in C \text{ and } n \in \mathbb{N}.$$

Definition 2.3.8. Let C be a closed convex subset of H and $T : C \to C$ is called a *pseudo-contraction* if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 \text{ for all } x, y \in C,$$

if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \text{ for all } x, y \in C.$$

Proposition 2.3.9. [21] Let T be a pseudo-contractive mapping with the nonempty fixed point set $Fix(T)$, then the following conclusion holds:

$$\langle Ty - y, Ty - x^*\rangle \leq \|Ty - y\|^2, \text{ for all } y \in C, \; x^* \in Fix(T).$$

Next, to overcome the L-Lipschitzian property, we suppose that pseudo-contraction mapping T satisfies the following condition:

$$\langle Ty - y, Ty - x^*\rangle \leq 0 \;\; \forall y \in C, \forall x \in C. \tag{2.3.5}$$

Next, we recall the definition of operators which are used in our main results.

Definition 2.3.10. Let $T : H \to H$ be an operator. Then

(1) T is said to be *monotone* if

$$\langle x - y, Tx - Ty\rangle \geq 0, \text{ for all } x, y \in H;$$

(2) T is said to be *β-stongly monotone* with $\beta > 0$, if

$$\langle x - y, Tx - Ty\rangle \geq \beta\|x - y\|^2, \text{ for all } x, y \in H;$$

(3) T is said to be *ν-inverse strongly monotone* (ν-ism), with $\nu > 0$, if

$$\langle x - y, Tx - Ty\rangle \geq \nu\|Tx - Ty\|^2 \text{ for all } x, y \in H;$$

(4) T is said to be *pseudomonotone*, if

$$\langle Ty, x - y\rangle \geq 0 \Rightarrow \langle Tx, x - y\rangle \geq 0, \text{ for all } x, y \in H.$$

It is easy to check that a β-strongly monotone mapping is monotone and a monotone mapping is pseudomonotone.

Example 2.3.11. Let C be a nonempty closed convex subset of $\mathbb{R}$ and $T : C \to \mathbb{R}$ be a mapping.

(1) If we take $Tx = 1-x$ and $C = [0,1]$, then it is easy to check that the mapping T is a pseudomonotone mapping, neither a monotone mapping nor a strongly monotone mapping.

(2) If a mapping T is defined by $Tx = c$, where c is a constant and $C = \mathbb{R}$. We observe that the mapping T is monotone, but not strongly monotone mapping.

Proposition 2.3.12. [8] Let $T : H \to H$ be a given mapping. Then

(1) T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism;

(2) T is ν-ism, then γT is $\frac{\nu}{\gamma}$-ism, for $\gamma > 0$;

(3) T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.

The following definition is very important called the demiclosed principle.

Definition 2.3.13. Let $T : H \to H$ be a mapping. Then $(I - T)$ is said to be demiclosed at zero if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ we have $x = Tx$.

The following theorem is due to Opial [52] involves the demiclosed principle of a nonexpansive operator.

Theorem 2.3.14. [52] If an operator $T : H \to H$ is nonexpansive with $Fix(T) \neq \emptyset$, then it satisfies the demiclosed principle.

The following definition of operators will play an important role in this thesis. Let H_1 and H_2 be two Hilbert spaces with the inner products $\langle \cdot, \cdot \rangle$ and the associate norm $\|\cdot\|$.

Definition 2.3.15. [44] Let $A : H_1 \to H_2$ be an operator. Then

(1) A is said to be *linear* if

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay \quad \text{for all } x, y \in H_1 \text{ and for all } \alpha, \beta \in \mathbb{R};$$

(2) A is said to be *bounded* if there exists a real number $M > 0$ such that $\|Ax\|_{H_2} \leq M\|x\|_{H_1}$ for all $x \in H_1$;

(3) A is said to be *continuous* at an element $x \in H_1$ if for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset H_1$ such that $x_n \to x \in H_1$ as $n \to \infty$, we have the sequence $\{Ax_n\}_{n\in\mathbb{N}} \subset H_2$ satisfies $Ax_n \to Ax \in H_2$ as $n \to \infty$. And, A is said to be continuous if it is continuous at every element of H_1.

Definition 2.3.16. [44] Let $A : H_1 \to H_2$ be a bounded linear operator. The number

$$\|A\| = \sup_{0 \neq x \in H_1} \left\{ \frac{\|Ax\|_{H_2}}{\|x\|_{H_1}} \right\}$$

is called a *norm* of the operator A.

The following theorem gives some useful properties of a linear operator.

Theorem 2.3.17. [44] Let $A : H_1 \to H_2$ be a linear operator. Then the following statements are true:

(1) If A is bounded, then

$$\|Ax\|_{H_2} \leq \|A\|\|x\|_{H_1} \quad \text{for every } x \in H_1.$$

(2) A is bounded if and only if A is continuous.

Definition 2.3.18. [44] Let $A : H_1 \to H_2$ be a bounded linear operator. An operator $A^* : H_2 \to H_1$ is called an *adjoint operator* of A if

$$\langle Ax, y\rangle_{H_2} = \langle x, A^*y\rangle_{H_1} \text{ for all } x \in H_1 \text{ and for all } y \in H_2.$$

Of course, we can guarantee the well-defined of the adjoint operator of a bounded linear operator by the following theorem.

Theorem 2.3.19. [44] Let $A : H_1 \to H_2$ be a bounded linear operator. Then there exists a unique adjoint operator $A^* : H_2 \to H_1$ of A. Furthermore, the adjoint operator A^* is bounded linear operator with norm

$$\|A^*\| = \|A\|.$$

The following theorem provides a general property of the adjoint operator which is used frequently.

Theorem 2.3.20. [44] Let $A : H_1 \to H_2$ be a bounded linear operator. Then it holds that

$$\|A^*A\| = \|AA^*\| = \|A\|^2.$$

2.4 Further Convergence Tools

Lemma 2.4.1. [74] Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers satisfying

$$a_{n+1} \leq (1 + c_n)a_n + b_n, \text{ for all } n \in \mathbb{N},$$

where $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then:

(i) $\lim_{n\to\infty} a_n$ exists;

(ii) if $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Definition 2.4.2. [59] Let F be a nonempty subset of a Banach space X and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is of monotone type (I) with respect to F if there exist sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty}\delta_n < \infty, \sum_{n=1}^{\infty}\varepsilon_n < \infty$, and $\|x_{n+1} - p\| \leq (1+\delta_n)\|x_n - p\| + \varepsilon_n$ for all $n \in \mathbb{N}$ and $p \in F$.

Recall that $\text{dist}(x, F) = \inf\{\|x-y\| : y \in F\}$ is the distance from a point x to a subset F in X.

Proposition 2.4.3. [59] Let F be a nonempty subset of a Banach space X and let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is of monotone type (I) with respect to F and $\liminf_{n\to\infty} \text{dist}(x_n, F) = 0$, then $\lim_{n\to\infty} x_n = p$ for some $p \in X$ satisfying $\text{dist}(p, F) = 0$. In particular, if F is closed, then $p \in F$.

Lemma 2.4.4. [60] Let X be a uniformly convex Banach space, $\{\lambda_n\}$ be a sequence of real numbers such that $0 < a \leq \lambda_n \leq b < 1$, for all $n \in \mathbb{N}$, $\{x_n\}$ and $\{y_n\}$ be sequences of X satisfying, for some $r \geq 0$,

(i) $\limsup_{n\to\infty} \|x_n\| \leq r$;

(ii) $\limsup_{n\to\infty} \|y_n\| \leq r$;

(iii) $\lim_{n\to\infty} \|\lambda_n x_n + (1-\lambda_n)y_n\| = r$.

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.4.5. [64] Let X be a Banach space which satisfies the Opial property and $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.

Lemma 2.4.6. [87] Let H be a real Hilbert space, C be a closed convex subset of H. Let $T : C \to C$ be a continuous pseudo-contractive mapping. Then

(1) $Fix(T)$ is a closed convex subset of C.

(2) $(I - T)$ is demiclosed at zero.

Moreover, we utilize a weak-cluster point of the sequence $\{x_n\}$, denoted by

$$\omega_W(x_n) = \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Lemma 2.4.7. [40] Let H be a real Hilbert space and $\{x_n\}$ be a bounded sequence in H such that there exists a nonempty closed convex set C of H satisfying:

(1) for every $w \in C$, $\lim_{n\to\infty} \|x_n - w\|$ exists;

(2) each weak-cluster point of the sequence $\{x_n\}$ is in C.

Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.4.8. [52] (Opial condition) For any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for each $y \in H$ with $y \neq x$.

Lemma 2.4.9. [77] Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. If any weak limit point of $\{x_n\}$ belongs to C and

$$\|x_n - u\| \leq \|u - P_C u\|, \text{ for all } n \in \mathbb{N}.$$

Then $x_n \longrightarrow P_C u$ as $n \longrightarrow \infty$.

CHAPTER III

ITERATIVE METHODS FOR SOLVING THE FIXED POINT PROBLEMS

In this chapter, we focus on iterative method which is a hybrid pair of a generalized nonexpansive single-valued mapping and a nonexpansive multi-valued mapping for solving the fixed point problems. Finally we give a numerical example for supporting our main results.

Throughout of this chapter, we denote that D is a nonempty subset of a Banach space X. Let $CB(D)$ and $KC(D)$ be the families of nonempty closed bounded subsets and nonempty compact convex subsets of D, respectively.

3.1 Introduction and preliminaries

The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A,B) = \max\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(y,A)\} \text{ for } A, B \in CB(D),$$

where $\operatorname{dist}(x,B) = \inf\{\|x-y\| : y \in B\}$ is the distance from a point x to a subset B.

Let $t : D \to D$ be a single-valued mapping and $T : D \to CB(D)$ be a multi-valued mapping. Denote that $Fix(t) = \{x \in D : x = tx\}$ is the set of fixed points of t and $Fix(T) = \{x \in D : x \in Tx\}$ is the set of fixed points of T. A point x is called a *common fixed point* of t and T if $x = tx \in Tx$.

Definition 3.1.1. [68] Let $t : D \to D$ and $I : D \to D$ be single-valued mappings. We say that t is generalized I-asymptotically nonexpansive if there exist sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$ such that

$$\|t^n x - t^n y\| \le k_n \|Ix - Iy\| + s_n,$$

for all $x, y \in D$ and $n \in \mathbb{N}$.

If I is an identity mapping, then a single-valued mapping t reduces to a generalized asymptotically nonexpansive mapping. If $s_n = 0$, for all $n \in \mathbb{N}$, and I is an identity mapping, then a single-valued mapping t is called an asymptotically nonexpansive mapping. In particular, if $k_n = 1$, $s_n = 0$, for all $n \in \mathbb{N}$, and I is an identity mapping, a single-valued mapping t reduces to a nonexpansive mapping. The fixed point theorems for generalized I-asymptotically nonexpansive single-valued mappings in uniformly convex Banach spaces can be found in [68].

The following example shows that t is generalized I-asymptotically nonexpansive.

Example 3.1.2. Define

$$tx = \sin\frac{1}{x}, Ix = \frac{1}{x}, \text{ where } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \text{ and } x \neq 0.$$

This implies that

$$\begin{aligned}
|\sin^n(\frac{1}{x}) - \sin^n(\frac{1}{y})| &\leq |\sin(\frac{1}{x}) + \sin(\frac{1}{y})| + \frac{1}{2n} \\
&\leq |2\sin(\frac{1}{2x} - \frac{1}{2y})| + \frac{1}{2n} \\
&\leq 2|\arcsin(\sin(\frac{1}{2x} - \frac{1}{2y}))| + \frac{1}{2n} \\
&= 2|\frac{1}{2x} - \frac{1}{2y}| + \frac{1}{2n} \\
&= |\frac{1}{x} - \frac{1}{y}| + \frac{1}{2n} \\
&= |Ix - Iy| + \frac{1}{2n}.
\end{aligned}$$

Moreover, we know that t is not nonexpansive. Set $x = \frac{2}{\pi}$ and $y = -\frac{2}{\pi}$

$$|\sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2})| = |\sin(\frac{\pi}{2}) + \sin(\frac{\pi}{2})|$$

$$|1 + 1| = |2|.$$

$$|x - y| = |\frac{2}{\pi} - (-\frac{2}{\pi})| = \frac{4}{\pi} = |1.273|.$$

Therefore $|Tx - Ty| \geq |x - y|$.

The following example shows that the fixed point set of a generalized I-asymptotically nonexpansive mapping is not necessarily closed.

Example 3.1.3. [54] Define a single-valued mapping $t : [-\frac{2}{3}, \frac{2}{3}] \to [-\frac{2}{3}, \frac{2}{3}]$ by

$$tx \begin{cases} x, & \text{if } x \in [-\frac{2}{3}, 0) \\ 0, & \text{if } x = 0 \\ x^4, & \text{if } x \in (0, \frac{2}{3}], \end{cases} \tag{3.1.1}$$

and

$$Ix = x \text{ for all } x \in [-\frac{2}{3}, \frac{2}{3}].$$

Then t is generalized I-asymptotically nonexpansive and $Fix(t) = [-\frac{2}{3}, 0)$ which is not closed.

In 2011, Sokhuma and Kaewkhao [63] proposed the iterative method of a pair of a nonexpansive single-valued mapping t and a nonexpansive multi-valued mapping T as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n t y_n, \ n \in \mathbb{N}, \end{cases} \tag{3.1.2}$$

where $x_1 \in D, z_n \in Tx_n$ and $0 < a \leq \alpha_n, \beta_n \leq b < 1$. They assured the existence of a strong convergence theorem for the iterative method (3.1.2) in uniformly convex Banach spaces.

In 2011, Eslamian and Abkar [29] introduced the iterative method for a pair of a finite family of asymptotically nonexpansive single-valued mappings $\{t_i\}_{i=1}^N$ and a finite family of quasi-nonexpansive multi-valued mappings $\{T_i\}_{i=1}^N$ as follows:

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \ n \in \mathbb{N}, \end{cases} \tag{3.1.3}$$

where $x_1 \in D$, $z_n^{(i)} \in T_i x_n$, and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, ..., N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = \sum_{i=0}^N \beta_n^{(i)} = 1$.

In 2015, Suantai and Phuengrattana [55] extended the results of [29, 30, 63] in uniformly convex Banach spaces. They introduced the following iterative method for a pair of a finite family of generalized asymptotically nonexpansive single-valued mappings $\{t_i\}_{i=1}^N$ and a finite family of quasi-nonexpansive multi-valued mappings $\{T_i\}_{i=1}^N$:

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \ n \in \mathbb{N}, \end{cases} \tag{3.1.4}$$

where $x_1 \in D$, $z_n^{(i)} \in T_i x_n$, and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, ..., N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = \sum_{i=0}^N \beta_n^{(i)} = 1$. They [55] proved the weak and strong convergence theorems of the iterative method defined in (3.1.4) in Banach spaces.

Inspired by the above convincing, in this chapter, an iterative method for a hybrid pair of a finite family of generalized I-asymptotically nonexpansive single-valued mappings and a finite family of generalized nonexpansive multi-valued mappings is established. Moreover, the weak and strong convergence theorems of the proposed iterative method in Banach spaces are proven. The obtained results can be viewed as an improvement and extension of the several results in [29, 30, 33, 55, 63, 82].

3.2 Convergence theorems

Let D be a nonempty closed convex subset of a Banach space X. Suppose that $\{I_i\}_{i=1}^N$ is a finite family of asymptotically nonexpansive self-mappings on D with a sequence of real numbers $\{\nu_n^{(i)}\} \subset [1, \infty)$ with $\lim_{n\to\infty} \nu_n^{(i)} = 1$. Therefore

$$\|I_i^n x - I_i^n y\| \le \nu_n^{(i)} \|x - y\|,$$

for all $x, y \in D$, for all $i = 1, 2, ..., N$ and for all $n \in \mathbb{N}$.

Assume that $\{t_i\}_{i=1}^N$ is a finite family of generalized I_i-asymptotically nonexpansive self-mappings on D with the sequences of real numbers $\{k_n^{(i)}\} \subset [1, \infty)$ and $\{s_n^{(i)}\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n^{(i)} = 1$ and $\lim_{n\to\infty} s_n^{(i)} = 0$. Therefore

$$\|t_i^n x - t_i^n y\| \le k_n^{(i)} \|I_i^n x - I_i^n y\| + s_n^{(i)},$$

for all $x, y \in D$, for all $i = 1, 2, ..., N$ and for all $n \in \mathbb{N}$.

Letting $k_n = \max_{1 \leq i \leq N}\{k_n^{(i)}\}$ and $s_n = \max_{1 \leq i \leq N}\{s_n^{(i)}\}$. It follows that $\lim_{n\to\infty} k_n = 1$, and $\lim_{n\to\infty} s_n = 0$ and

$$\|t_i^n x - t_i^n y\| \leq k_n \|I_i^n x - I_i^n y\| + s_n,$$

for all $x, y \in D$, for all $i = 1, 2, ..., N$ and for all $n \in \mathbb{N}$.

Let $\nu_n = \max_{1 \leq i \leq N}\{\nu_n^{(i)}\}$. It is clear that $\lim_{n\to\infty} \nu_n = 1$ and

$$\|I_i^n x - I_i^n y\| \leq \nu_n \|x - y\|,$$

for all $x, y \in D$, for all $i = 1, 2, ..., N$ and for all $n \in \mathbb{N}$.

Put $r_n = \max\{k_n, \nu_n\}$. Thus we have $\lim_{n\to\infty} r_n = 1$, $\|I_i^n x - I_i^n y\| \leq r_n \|x - y\|$ and

$$\|t_i^n x - t_i^n y\| \leq k_n \|I_i^n x - I_i^n y\| + s_n \leq r_n^2 \|x - y\| + s_n,$$

for all $x, y \in D$, for all $i = 1, 2, ..., N$ and for all $n \in \mathbb{N}$.

We now prove the following lemma for helping to prove the theorem in this section.

Lemma 3.2.1. Let D be a nonempty closed convex subset of a Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of generalized I_i-asymptotically nonexpansive single-valued mappings on D with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(r_n^3 - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of asymptotically nonexpansive single-valued mappings on D with a sequence $\{\nu_n\} \subset [1, \infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N Fix(t_i) \cap \bigcap_{i=1}^N Fix(I_i) \cap \bigcap_{i=1}^N Fix(T_i)$ is nonempty closed and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, ..., N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \ \ z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \ \ n \in \mathbb{N}, \end{cases} \tag{3.2.1}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, ..., N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$.

Proof. Assume that $p \in \mathcal{F}$. Therefore

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n^{(0)} x_n + \sum_{i=1}^{N} \alpha_n^{(i)} t_i^n y_n - p\| \\
&= \|\alpha_n^{(0)} x_n + \sum_{i=1}^{N} \alpha_n^{(i)} t_i^n y_n - \sum_{i=0}^{N} \alpha_n^{(i)} p\| \\
&= \|\alpha_n^{(0)} x_n + \sum_{i=1}^{N} \alpha_n^{(i)} t_i^n y_n - \alpha_n^0 p - \sum_{i=1}^{N} \alpha_n^{(i)} p\| \\
&\leq \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \alpha_n^{(i)} \|t_i^n y_n - p\| \\
&\leq \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \alpha_n^{(i)} \Big(r_n^2 \|y_n - p\| + s_n \Big),
\end{aligned}$$

for all $i = 1, 2, ..., N$.

Since

$$\begin{aligned}
\|y_n - p\| &= \|\beta_n^{(0)} x_n + \sum_{i=1}^{N} \beta_n^{(i)} I_i^n z_n^{(i)} - p\| \\
&= \|\beta_n^{(0)} x_n + \sum_{i=1}^{N} \beta_n^{(i)} I_i^n z_n^{(i)} - \sum_{i=0}^{N} \beta_n^{(i)} p\| \\
&\leq \beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} \|I_i^n z_n^{(i)} - p\| \\
&\leq \beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} r_n \|z_n^{(i)} - p\| \\
&= \beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} r_n \operatorname{dist}(z_n^{(i)}, T_i p) \\
&\leq \beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} r_n H(T_i x_n, T_i p) \\
&\leq \beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} r_n \|x_n - p\|,
\end{aligned}$$

we obtain that

$$\begin{aligned}
\|x_{n+1} - p\| \leq& \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \alpha_n^{(i)} \Big(r_n^2 \|y_n - p\| + s_n \Big) \\
\leq& \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \alpha_n^{(i)} \Big(r_n^2 (\beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} r_n \|x_n - p\|) + s_n \Big)
\end{aligned}$$

$$
\begin{aligned}
&=(\alpha_n^{(0)} + r_n^2 \sum_{i=1}^{N} \alpha_n^{(i)} \beta_n^{(0)})\|x_n - p\| + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \sum_{i=1}^{N} \beta_n^{(i)} \|x_n - p\| + \sum_{i=1}^{N} \alpha_n^{(i)} s_n \\
&\leq(\alpha_n^{(0)} + r_n^2 \sum_{i=1}^{N} \alpha_n^{(i)} \beta_n^{(0)})\|x_n - p\| + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \sum_{i=1}^{N} \beta_n^{(i)} \|x_n - p\| + s_n \\
&\leq\alpha_n^{(0)}\|x_n - p\| + r_n(r_n^2 \sum_{i=1}^{N} \alpha_n^{(i)} \beta_n^{(0)})\|x_n - p\| \\
&\quad + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \sum_{i=1}^{N} \beta_n^{(i)} \|x_n - p\| + s_n \\
&=\alpha_n^{(0)}\|x_n - p\| + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \sum_{i=0}^{N} \beta_n^{(i)} \|x_n - p\| + s_n \\
&=\alpha_n^{(0)}\|x_n - p\| + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \|x_n - p\| + s_n \\
&\leq r_n^3 \alpha_n^{(0)}\|x_n - p\| + r_n^3 \sum_{i=1}^{N} \alpha_n^{(i)} \|x_n - p\| + s_n \\
&=r_n^3 \sum_{i=0}^{N} \alpha_n^{(i)} \|x_n - p\| + s_n \\
&=r_n^3\|x_n - p\| + s_n \\
&=(1 + (r_n^3 - 1))\|x_n - p\| + s_n.
\end{aligned}
$$

It now follows from Lemma 2.4.1, we have $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in \mathcal{F}$. This completes the proof. □

Theorem 3.2.2. Let D be a nonempty closed convex subset of a Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of generalized I_i-asymptotically nonexpansive single-valued mappings on D with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(r_n^3 - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of asymptotically nonexpansive single-valued mappings on D with a sequence $\{\nu_n\} \subset [1,\infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N Fix(t_i) \cap \bigcap_{i=1}^N Fix(I_i) \cap \bigcap_{i=1}^N Fix(T_i)$ is nonempty closed and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, ..., N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.2.2}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, ..., N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$ if and only if $\liminf_{n\to\infty} \mathrm{dist}(x_n, \mathcal{F}) = 0$.

Proof. The necessity is obvious. For proving the converse, suppose that $\liminf_{n\to\infty} \mathrm{dist}(x_n, \mathcal{F}) = 0$. It follows from the proof of Lemma 3.2.1, we can conclude that the sequence $\{x_n\}$ is of monotone type (I) with respect to $\mathcal{F}$. By Proposition 2.4.3, we obtain that the sequence $\{x_n\}$ converges to a point in $\mathcal{F}$. □

The following lemma is a main tool for proving our results.

Lemma 3.2.3. Let D be a nonempty closed convex subset of a uniformly convex Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized I_i-asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^\infty (r_n^3 - 1) < \infty$ and $\sum_{n=1}^\infty s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of uniformly Γ-Lipschitzian and asymptotically nonexpansive single-valued mappings of D into itself with a sequence $\{\nu_n\} \subset [1,\infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N Fix(t_i) \cap \bigcap_{i=1}^N Fix(I_i) \cap \bigcap_{i=1}^N Fix(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, ..., N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.2.3}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ such that $0 < a \le \alpha_n^{(i)}, \beta_n^{(i)} \le b < 1$ for all $i = 1, 2, ..., N$, $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then we have the followings:

(i) $\lim_{n\to\infty} \|x_n - I_i^n z_n^{(i)}\| = 0$ for all $i = 1, 2, ..., N$;

(ii) $\lim_{n\to\infty} \|x_n - t_i x_n\| = 0$ for all $i = 1, 2, ..., N$;

(iii) if $\lim_{n\to\infty} \|z_n^{(i)} - I_i^n z_n^{(i)}\| = 0$, then $\lim_{n\to\infty} \|x_n - I_i x_n\| = 0$ for all $i = 1, 2, ..., N$.

Proof. (i) Let $p \in \mathcal{F}$. We conclude from Lemma 3.2.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists. Assume that $\lim_{n\to\infty} \|x_n - p\| = c$. This implies that, for each $i = 1, 2, ..., N$,

$$\begin{aligned}
\|t_i^n y_n - p\| &\le r_n^2 \|y_n - p\| + s_n \\
&= r_n^2 \|\beta_n^{(0)} x_n + \sum_{i=1}^{N} \beta_n^{(i)} I_i^n z_n^{(i)} - p\| + s_n \\
&\le r_n^2 \beta_n^{(0)} \|x_n - p\| + r_n^2 \sum_{i=1}^{N} \beta_n^{(i)} \|I_i^n z_n^{(i)} - p\| + s_n \\
&\le r_n^2 \beta_n^{(0)} \|x_n - p\| + r_n^2 \sum_{i=1}^{N} \beta_n^{(i)} \nu_n \|z_n^{(i)} - p\| + s_n \\
&\le r_n^2 \beta_n^{(0)} \|x_n - p\| + r_n^3 \sum_{i=1}^{N} \beta_n^{(i)} \|z_n^{(i)} - p\| + s_n \\
&= r_n^2 \beta_n^{(0)} \|x_n - p\| + r_n^3 \sum_{i=1}^{N} \beta_n^{(i)} \operatorname{dist}(z_n^{(i)}, T_i p) + s_n \\
&\le r_n^2 \beta_n^{(0)} \|x_n - p\| + r_n^3 \sum_{i=1}^{N} \beta_n^{(i)} H(T_i x_n, T_i p) + s_n \\
&\le r_n^3 \beta_n^{(0)} \|x_n - p\| + r_n^3 \sum_{i=1}^{N} \beta_n^{(i)} \|x_n - p\| + s_n \\
&= r_n^3 \|x_n - p\| + s_n.
\end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \|t_i^n y_n - p\| \le \limsup_{n\to\infty} (r_n^2 \|y_n - p\| + s_n) \le \limsup_{n\to\infty} (r_n^3 \|x_n - p\| + s_n).$$

Since $\lim_{n\to\infty} r_n = 1$ and $\lim_{n\to\infty} s_n = 0$, we have

$$\limsup_{n\to\infty} \|t_i^n y_n - p\| \le \limsup_{n\to\infty} \|y_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = c. \tag{3.2.4}$$

Because of $\lim_{n\to\infty}\|x_{n+1}-p\| = \lim_{n\to\infty}\|\alpha_n^{(0)}(x_n-p)+\sum_{i=1}^N \alpha_n^{(i)}(t_i^n y_n - p)\|$ and by Lemma 2.4.4, we obtain that

$$\lim_{n\to\infty}\|x_n - t_i^n y_n\| = 0 \text{ for all } i = 1,2,...,N. \tag{3.2.5}$$

Since

$$\begin{aligned}
\|x_{n+1}-p\| &\le \alpha_n^{(0)}\|x_n-p\| + \sum_{i=1}^N \alpha_n^{(i)}\|t_i^n y_n - p\| \\
&= \Big(1-\sum_{i=1}^N \alpha_n^{(i)}\Big)\|x_n-p\| + \sum_{i=1}^N \alpha_n^{(i)}\|t_i^n y_n - p\| \\
&\le \Big(1-\sum_{i=1}^N \alpha_n^{(i)}\Big)\|x_n-p\| + \sum_{i=1}^N \alpha_n^{(i)}(r_n^2\|y_n-p\|) + s_n),
\end{aligned}$$

we have

$$\|x_{n+1}-p\| - \|x_n-p\| \le \sum_{i=1}^N \alpha_n^{(i)}(r_n^2\|y_n-p\| - \|x_n-p\| + s_n).$$

This implies that

$$\begin{aligned}
\frac{\|x_{n+1}-p\| - \|x_n-p\|}{bN} + \|x_n-p\| &\le \frac{\|x_{n+1}-p\| - \|x_n-p\|}{\sum_{i=1}^N \alpha_n^{(i)}} + \|x_n-p\| \\
&\le r_n^2\|y_n-p\| - \|x_n-p\| + s_n + \|x_n-p\| \\
&= r_n^2\|y_n-p\| + s_n.
\end{aligned}$$

By (3.2.4), this yields

$$\begin{aligned}
c &= \liminf_{n\to\infty}\Big(\frac{\|x_{n+1}-p\| - \|x_n-p\|}{bN} + \|x_n-p\|\Big) \\
&\le \liminf_{n\to\infty}(r_n^2\|y_n-p\| + s_n) \\
&= \liminf_{n\to\infty}\|y_n-p\| \\
&\le \limsup_{n\to\infty}\|y_n-p\| \le c.
\end{aligned}$$

Since

$$\|I_i^n z_n^{(i)} - p\| \le \nu_n\|z_n^{(i)} - p\| = \nu_n \mathrm{dist}(z_n^{(i)}, T_i p) \le \nu_n H(T_i x_n, T_i p) \le \nu_n\|x_n - p\|,$$

it follows that

$$\limsup_{n\to\infty} \|I_i^n z_n^{(i)} - p\| \le \limsup_{n\to\infty} \nu_n \|x_n - p\| = c.$$

Therefore

$$c = \lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} \|\beta_n^{(0)}(x_n - p) + \sum_{i=1}^{N} \beta_n^{(i)}(I_i^n z_n^{(i)} - p)\|.$$

By Lemma 2.4.4, we have

$$\lim_{n\to\infty} \|x_n - I_i^n z_n^{(i)}\| = 0 \text{ for all } i = 1, 2, ..., N.$$

(ii) Since t_i is generalized I_i-asymptotically nonexpansive, for all $i = 1, 2, ..., N$, we obtain that

$$\begin{aligned}\|t_i^n x_n - x_n\| &= \|t_i^n x_n - t_i^n y_n + t_i^n y_n - x_n\| \\ &\le \|t_i^n x_n - t_i^n y_n\| + \|t_i^n y_n - x_n\| \\ &\le r_n^2 \|x_n - y_n\| + s_n + \|t_i^n y_n - x_n\|.\end{aligned}$$

Using the definition of $\{x_n\}$, we have $y_n - x_n = \sum_{i=1}^{N} \beta_n^{(i)}(I_i^n z_n^{(i)} - x_n)$. This implies that

$$\begin{aligned}\|t_i^n x_n - x_n\| &\le r_n^2 \sum_{i=1}^{N} \beta_n^{(i)} \|I_i^n z_n^{(i)} - x_n\| + \|t_i^n y_n - x_n\| + s_n \\ &\le r_n^2 \|I_i^n z_n^{(i)} - x_n\| + \|t_i^n y_n - x_n\| + s_n.\end{aligned}$$

By (i) and (3.2.5), we obtain that

$$\lim_{n\to\infty} \|x_n - t_i^n x_n\| = 0 \text{ for all } i = 1, 2, ..., N. \qquad (3.2.6)$$

For each $i = 1, 2, ..., N$, we have

$$\begin{aligned}\|x_n - t_i x_n\| \le & \|x_n - x_{n+1}\| + \|x_{n+1} - t_i^{n+1} x_{n+1}\| + \|t_i^{n+1} x_{n+1} - t_i^{n+1} x_n\| \\ & + \|t_i^{n+1} x_n - t_i x_n\| \\ \le & \|x_n - x_{n+1}\| + \|x_{n+1} - t_i^{n+1} x_{n+1}\| + L\|x_{n+1} - x_n\| + \|t_i^{n+1} x_n - t_i x_n\| \\ \le & (1 + L)\|x_n - x_{n+1}\| + \|x_{n+1} - t_i^{n+1} x_{n+1}\| + L\|t_i^n x_n - x_n\|\end{aligned}$$

$$\leq (1+L)\sum_{i=1}^{N}\alpha_n^{(i)}\|x_n - t_i^n y_n\| + \|x_{n+1} - t_i^{n+1}x_{n+1}\| + L\|t_i^n x_n - x_n\|.$$

By using (3.2.5) and (3.2.6), we can conclude that $\lim_{n\to\infty}\|x_n - t_i x_n\| = 0$ for all $i = 1, 2, ..., N$.

(iii) Since

$$\begin{aligned}
\|I_i^n x_n - x_n\| &\leq \|I_i^n x_n - I_i^n z_n^{(i)}\| + \|I_i^n z_n^{(i)} - x_n\| \\
&\leq \nu_n\|x_n - z_n^{(i)}\| + \|I_i^n z_n^{(i)} - x_n\| \\
&\leq \nu_n(\|x_n - I_i^n z_n^{(i)}\| + \|I_i^n z_n^{(i)} - z_n^{(i)}\|) + \|I_i^n z_n^{(i)} - x_n\|
\end{aligned}$$

and by (i), we obtain that

$$\lim_{n\to\infty}\|I_i^n x_n - x_n\| = 0. \tag{3.2.7}$$

Since

$$\begin{aligned}
\|x_n - I_i x_n\| \leq &\|x_n - x_{n+1}\| + \|x_{n+1} - I_i^{n+1}x_{n+1}\| + \|I_i^{n+1}x_{n+1} - I_i^{n+1}x_n\| \\
&+ \|I_i^{n+1}x_n - I_i x_n\| \\
\leq &\|x_n - x_{n+1}\| + \|x_{n+1} - I_i^{n+1}x_{n+1}\| + \Gamma\|x_{n+1} - x_n\| + \|I_i^{n+1}x_n - I_i x_n\| \\
\leq &(1+\Gamma)\|x_n - x_{n+1}\| + \|x_{n+1} - I_i^{n+1}x_{n+1}\| + \Gamma\|I_i^n x_n - x_n\| \\
\leq &(1+\Gamma)\sum_{i=1}^{N}\alpha_n^{(i)}\|x_n - t_i^n y_n\| + \|x_{n+1} - I_i^{n+1}x_{n+1}\| + \Gamma\|I_i^n x_n - x_n\|,
\end{aligned}$$

and by (3.2.5) and (3.2.7), we can conclude that $\lim_{n\to\infty}\|x_n - I_i x_n\| = 0$ for all $i = 1, 2, ..., N$. □

Next, we prove a strong convergence theorem of the proposed iterative method in a uniformly convex Banach space. Moreover, we add uniformly L-Lipschitzian of mappings $\{t_i\}_{i=1}^N$ and $\{T_i\}_{i=1}^N$ satisfy condition (E) in order to reduce closedness of $\mathcal{F}$.

Definition 3.2.4. ([1]) A multi-valued mapping $T : D \to CB(D)$ is said to satisfy condition (E_μ) where $\mu \geq 0$ if for each $x, y \in D$,

$$\text{dist}(x, Ty) \leq \mu\text{dist}(x, Tx) + \|x - y\|.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Remark 3.2.5. We observe that if T is nonexpansive, then T satisfies the condition (E_1).

Theorem 3.2.6. Let D be a nonempty compact convex subset of a uniformly convex Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized I_i-asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(r_n^3 - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of uniformly Γ-Lipschitzian and asymptotically nonexpansive single-valued mappings of D into itself with a sequence $\{\nu_n\} \subset [1,\infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ satisfying condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^N Fix(t_i) \cap \bigcap_{i=1}^N Fix(I_i) \cap \bigcap_{i=1}^N Fix(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1,2,...,N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.2.8}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1,2,...,N$ such that $0 < a \leq \alpha_n^{(i)}, \beta_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Suppose that $\lim_{n\to\infty} \|z_n^{(i)} - I_i^{(i)} z_n^{(i)}\| = 0$ for all $i = 1,2,...,N$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.

Proof. Using Lemma 3.2.1, we obtain that $\{x_n\}$ is bounded. By the compactness of D, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging strongly to $p \in D$. By condition (E), there exists $\mu \geq 1$ such that

$$\begin{aligned} \text{dist}(p, T_i p) &\leq \|p - x_{n_j}\| + \text{dist}(x_{n_j}, T_i p) \\ &\leq \|p - x_{n_j}\| + \mu \text{dist}(x_{n_j}, T_i x_{n_j}) + \|x_{n_j} - p\| \\ &= 2\|x_{n_j} - p\| + \mu \text{dist}(x_{n_j}, T_i x_{n_j}) \\ &\leq 2\|x_{n_j} - p\| + \mu \|x_{n_j} - z_{n_j}^{(i)}\| \end{aligned}$$

$$\leq 2\|x_{n_j} - p\| + \mu\|x_{n_j} - I_i^{n_j} z_{n_j}^{(i)}\| + \mu\|I_i^{n_j} z_{n_j}^{(i)} - z_{n_j}^{(i)}\|,$$

for all $i = 1, 2, ..., N$. By using Lemma 3.2.3 (i), we obtain that $p \in T_i p$ for all $i = 1, 2, ..., N$. This implies that $p \in \bigcap_{i=1}^{N} Fix(T_i)$. Since t_i is uniformly L-Lipschitzian, we have

$$\begin{aligned} \|t_i p - p\| &\leq \|t_i p - t_i x_{n_j}\| + \|t_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - p\| \\ &\leq L\|x_{n_j} - p\| + \|t_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - p\| \\ &= (L+1)\|x_{n_j} - p\| + \|t_i x_{n_j} - x_{n_j}\|, \end{aligned}$$

for all $i = 1, 2, ..., N$. By Lemma 3.2.3 (ii), we obtain that $t_i p = p$ for all $i = 1, 2, ..., N$. This implies that $p \in \bigcap_{i=1}^{N} Fix(t_i)$. Since I_i is uniformly Γ-Lipschitzian, we have

$$\begin{aligned} \|I_i p - p\| &\leq \|I_i p - I_i x_{n_j}\| + \|I_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - p\| \\ &\leq \Gamma\|x_{n_j} - p\| + \|I_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - p\| \\ &= (\Gamma+1)\|x_{n_j} - p\| + \|I_i x_{n_j} - x_{n_j}\|, \end{aligned}$$

for all $i = 1, 2, ..., N$. By Lemma 3.2.3 (iii), we obtain that $I_i p = p$ for all $i = 1, 2, ..., N$. It follows that $p \in \bigcap_{i=1}^{N} Fix(I_i)$. Thus $p \in \mathcal{F}$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we have $\lim_{n\to\infty} \|x_n - p\| = \lim_{j\to\infty} \|x_{n_j} - p\| = 0$. Hence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. □

We now illustrate the following example for supporting Theorem 3.2.6.

Example 3.2.7. Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $D = [0, 1]$. Define single-valued mappings t_1, t_2, I_1, and I_2 on D as follows:

$$t_1 x = \arctan x,\ t_2 x = x^2,\ I_1 x = x \text{ and } I_2 x = \frac{x}{2}.$$

Define multi-valued mappings T_1 and T_2 on D by

$$T_1 x = [0, \frac{x}{3}] \text{ and } T_2 x = [\frac{x}{4}, \frac{x}{2}].$$

Let $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^{2} \beta_n^{(i)} I_i^n z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^{2} \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.2.9}$$

where $\alpha_n^{(0)} = \frac{1}{12n}$, $\alpha_n^{(1)} = \frac{12n-1}{36n}$, $\alpha_n^{(2)} = \frac{12n-1}{18n}$, $\beta_n^{(0)} = \frac{1}{10n}$, $\beta_n^{(1)} = \frac{10n-1}{30n}$, $\beta_n^{(2)} = \frac{10n-1}{15n}$, for all $n \in \mathbb{N}$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 0, where $\{0\} = \bigcap_{i=1}^{2} Fix(t_i) \cap \bigcap_{i=1}^{2} Fix(I_i) \cap \bigcap_{i=1}^{2} Fix(T_i)$.

Solution We first show that t_1 is a generalized I_1-asymptotically nonexpansive and uniformly L-Lipschitzian single-valued mapping. Let $k_n = 1$ and $s_n = (\frac{2}{3})^n$ for all $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$. Since

$$\frac{|t_1^n x - t_1^n y|}{|x-y|} \leq 1 + \left(\frac{2}{3|x-y|}\right)^n \text{ for all } x, y \in D,$$

we have

$$|t_1^n x - t_1^n y| \leq |x-y| + s_n \text{ for all } n \in \mathbb{N},$$

and we can show that t_1 is a uniformly L-Lipschitzian mapping with $L = 1$. Since t_2 is a single-valued nonexpansive mapping of D, we have t_2 is a uniformly L-Lipschitzian and generalized I_2-asymptotically nonexpansive single-valued mapping of D. Moreover $\bigcap_{i=1}^{2} Fix(t_i) = \{0\} = \bigcap_{i=1}^{2} Fix(I_i)$. Both T_1 and T_2 are quasi-nonexpansive multi-valued mappings satisfying condition (E) and $\bigcap_{i=1}^{2} Fix(T_i) = \{0\}$ (see [55]). Thus $\bigcap_{i=1}^{2} Fix(t_i) \cap \bigcap_{i=1}^{2} Fix(I_i) \cap \bigcap_{i=1}^{2} Fix(T_i) = \{0\}$.

For every $n \in \mathbb{N}$, $\alpha_n^{(0)} = \frac{1}{12n}$, $\alpha_n^{(1)} = \frac{12n-1}{36n}$, $\alpha_n^{(2)} = \frac{12n-1}{18n}$, $\beta_n^{(0)} = \frac{1}{10n}$, $\beta_n^{(1)} = \frac{10n-1}{30n}$, $\beta_n^{(2)} = \frac{10n-1}{15n}$. Therefore the sequences $\{\alpha_n^{(0)}\}$, $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\beta_n^{(0)}\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ satisfy all assumptions in Theorem 3.2.6. By putting $z_n^{(1)} = \frac{x_n}{3}$, and $z_n^{(2)} = \frac{x_n}{2}$ for all $n \in \mathbb{N}$ and by using the algorithm 3.2.9 with the initial point $x_1 = 0.5$. The sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 0, where $\{0\} = \bigcap_{i=1}^{2} Fix(t_i) \cap \bigcap_{i=1}^{2} Fix(I_i) \cap \bigcap_{i=1}^{2} Fix(T_i)$.

Table 1 : The value of the sequences $\{x_n\}$ and $\{y_n\}$ in Example 3.2.7

n	x_n	y_n
1	0.5000000	0.1750000
2	0.1133181	0.0355693
3	0.0168875	0.0050975
4	0.0021379	0.0006325
5	0.0002512	0.0000734
6	0.0000283	0.0000082
7	0.0000031	0.0000009
8	0.0000003	0.0000001
9	0.0000000	0.0000000

Finally, we prove a weak convergence theorem of the proposed iterative method in a uniformly convex Banach space.

Theorem 3.2.8. Let D be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized I_i-asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(r_n^3 - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of uniformly Γ-Lipschitzian and asymptotically nonexpansive single-valued mappings of D into itself with a sequence $\{\nu_n\} \subset [1,\infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $KC(D)$ satisfying condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^N Fix(t_i) \cap \bigcap_{i=1}^N Fix(I_i) \cap \bigcap_{i=1}^N Fix(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, ..., N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \ \ z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \ \ n \in \mathbb{N}, \end{cases} \tag{3.2.10}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, ..., N$ such that $0 < a \leq \alpha_n^{(i)}, \beta_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Suppose that

$\lim_{n\to\infty} \|z_n^{(i)} - I_i^{(i)} z_n^{(i)}\| = 0$ for all $i = 1, 2, ..., N$. Then the sequence $\{x_n\}$ converges weakly to a point in $\mathcal{F}$.

Proof. By using Lemma 3.2.1, we obtain that $\{x_n\}$ is bounded. Since X is a uniformly convex Banach space, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to $p \in D$. By Lemma 3.2.3, we have $\lim_{j\to\infty} \|x_{n_j} - I_i^{n_j} z_{n_j}^{(i)}\| = 0$, $\lim_{j\to\infty} \|x_{n_j} - t_i x_{n_j}\| = 0$ and $\lim_{n\to\infty} \|I_i x_{n_j} - x_{n_j}\| = 0$ for all $i = 1, 2, ..., N$. We will show that $p \in \mathcal{F}$. Since $T_1 p$ is compact, for all $j \in \mathbb{N}$, we can choose $w_{n_j} \in T_1 p$ such that $\|x_{n_j} - w_{n_j}\| = \mathrm{dist}(x_{n_j}, T_1 p)$ and the sequence $\{w_{n_j}\}$ has a convergent subsequence $\{w_{n_k}\}$ with $\lim_{k\to\infty} w_{n_k} = w \in T_1 p$. By using condition (E), we obtain that

$$\mathrm{dist}(x_{n_k}, T_1 p) \leq \mu \mathrm{dist}(x_{n_k}, T_1 x_{n_k}) + \|x_{n_k} - p\|.$$

This yields

$$\begin{aligned}
\|x_{n_k} - w\| &\leq \|x_{n_k} - w_{n_k}\| + \|w_{n_k} - w\| \\
&= \mathrm{dist}(x_{n_k}, T_1 p) + \|w_{n_k} - w\| \\
&\leq \mu \mathrm{dist}(x_{n_k}, T_1 x_{n_k}) + \|x_{n_k} - p\| + \|w_{n_k} - w\| \\
&\leq \mu \|x_{n_k} - z_{n_k}^{(1)}\| + \|x_{n_k} - p\| + \|w_{n_k} - w\| \\
&\leq \mu \|x_{n_k} - I_1^{n_k} z_{n_k}^{(1)}\| + \mu \|I_1^{n_k} z_{n_k}^{(1)} - z_{n_k}^{(1)}\| + \|x_{n_k} - p\| + \|w_{n_k} - w\|.
\end{aligned}$$

It follows that

$$\limsup_{k\to\infty} \|x_{n_k} - w\| \leq \limsup_{k\to\infty} \|x_{n_k} - p\|.$$

From the Opial property, we can conclude that $p = w \in T_1 p$. Similarly, it can be shown that $p \in T_i p$ for all $i = 1, 2, ..., N$. Therefore $p \in \bigcap_{i=1}^{N} Fix(T_i)$. By mathematical induction, we obtain that

$$\lim_{j\to\infty} \|x_{n_j} - t_i^m x_{n_j}\| = 0 \text{ for each } m \in \mathbb{N}, \tag{3.2.11}$$

for all $i = 1, 2, ..., N$. Indeed, the conclusion it is true for $m = 1$. Suppose the conclusion holds for $m \geq 1$. Since t_i is a uniformly L-Lipschitzian single-valued mapping, we obtain that

$$\begin{aligned}\|x_{n_j} - t_i^{m+1}x_{n_j}\| &\leq \|x_{n_j} - t_i^m x_{n_j}\| + \|t_i^m x_{n_j} - t_i^{m+1}x_{n_j}\| \\ &\leq \|x_{n_j} - t_i^m x_{n_j}\| + L\|x_{n_j} - t_i x_{n_j}\|.\end{aligned}$$

This implies that $\lim_{j\to\infty}\|x_{n_j} - t_i^{m+1}x_{n_j}\| = 0$ for all $i = 1, 2, ..., N$. Therefore (3.2.11) holds. Since t_i is generalized I_i-asymptotically nonexpansive, we obtain that

$$\begin{aligned}\limsup_{j\to\infty}\|t_i^m x_{n_j} - t_i^m p\| &\leq \limsup_{j\to\infty}(k_m\|I_i^m x_{n_j} - p\| + s_m) \\ &\leq \limsup_{j\to\infty}(k_m(\nu_m\|x_{n_j} - p\|) + s_m) \\ &\leq \limsup_{j\to\infty}(r_m^2\|x_{n_j} - p\| + s_m).\end{aligned}$$

It follows that

$$\limsup_{m\to\infty}\left(\limsup_{j\to\infty}\|t_i^m x_{n_j} - t_i^m p\|\right) \leq \limsup_{j\to\infty}\|x_{n_j} - p\|. \tag{3.2.12}$$

By Proposition 2.2.5, there exists a strictly increasing continuous convex function $g : [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned}\left\|x_{n_j} - \frac{p + t_i^m p}{2}\right\|^2 &= \left\|\frac{1}{2}(x_{n_j} - p) + \frac{1}{2}(x_{n_j} - t_i^m p)\right\|^2 \\ &\leq \frac{1}{2}\|x_{n_j} - p\|^2 + \frac{1}{2}\|x_{n_j} - t_i^m p\|^2 - \frac{1}{4}g(\|p - t_i^m p\|).\end{aligned}$$

Therefore

$$\begin{aligned}\limsup_{j\to\infty}\left\|x_{n_j} - \frac{p + t_i^m p}{2}\right\|^2 \leq \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j} - p\|^2 + \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j} - t_i^m p\|^2 \\ - \frac{1}{4}g(\|p - t_i^m p\|).\end{aligned} \tag{3.2.13}$$

Since X satisfies the Opial property and $\{x_{n_j}\}$ converges weakly to p, we obtain that

$$\limsup_{j\to\infty}\|x_{n_j} - p\|^2 \leq \limsup_{j\to\infty}\|x_{n_j} - \frac{p + t_i^m p}{2}\|^2.$$

By using (3.2.13), we have

$$\frac{1}{4}g(\|p - t_i^m p\|) \leq \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j} - p\|^2 + \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j} - t_i^m p\|^2$$

$$-\limsup_{j\to\infty}\left\|x_{n_j}-\frac{p+t_i^mp}{2}\right\|^2$$
$$\leq \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j}-p\|^2+\frac{1}{2}\limsup_{j\to\infty}\|x_{n_j}-t_i^mp\|^2$$
$$-\limsup_{j\to\infty}\|x_{n_j}-p\|^2.$$

Therefore

$$g(\|p-t_i^mp\|)\leq 2\limsup_{j\to\infty}\|x_{n_j}-t_i^mp\|^2-2\limsup_{j\to\infty}\|x_{n_j}-p\|^2. \tag{3.2.14}$$

Using (3.2.11), (3.2.12), and (3.2.14), these yield

$$\limsup_{m\to\infty} g(\|p-t_i^mp\|)\leq 2\limsup_{m\to\infty}(\limsup_{j\to\infty}\|x_{n_j}-t_i^mp\|^2)-2\limsup_{j\to\infty}\|x_{n_j}-p\|^2$$
$$\leq 0.$$

Therefore $\lim_{m\to\infty} g(\|p-t_i^mp\|)=0$ for all $i=1,2,...,N$. By using the properties of g, we have $\lim_{m\to\infty}\|p-t_i^mp\|=0$ for all $i=1,2,...,N$. This implies that

$$\|t_ip-p\|\leq \|t_ip-t_i^{m+1}p\|+\|t_i^{m+1}p-p\|$$
$$\leq L\|p-t_i^mp\|+\|t_i^{m+1}p-p\|.$$

This implies that $t_ip=p$ for all $i=1,2,...,N$. Therefore $p\in\bigcap_{i=1}^N Fix(t_i)$. Similarly, we can prove that $p\in\bigcap_{i=1}^N Fix(I_i)$. Thus we obtain $p\in\mathcal{F}$. We now show that $\{x_n\}$ converges weakly to p. Suppose on the contrary. Then there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\{x_{n_l}\}$ converges weakly to $q\in D$ and $q\neq p$. By the same argument as above, we can show that $q\in\mathcal{F}$. By Lemma 3.2.1, we obtain that $\lim_{n\to\infty}\|x_n-p\|$ and $\lim_{n\to\infty}\|x_n-q\|$ exist. Using Lemma 2.4.5, we obtain that $q=p$. Hence $\{x_n\}$ converges weakly to a point in $\mathcal{F}$. This completes the proof. □

We now present the following example for supporting Theorem 3.2.8.

Example 3.2.9. Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $D=[0,+\infty)$. Define single-valued mappings t_1,t_2,I_1, and I_2 on D as follows:

$$t_1x=\frac{x}{1+2x},\ t_2x=\frac{x}{2(1+x)},\ I_1x=\frac{x}{1+x}\ \text{and}\ I_2x=\frac{x}{1+2x}.$$

Define multi-valued mappings T_1 and T_2 on D by

$$T_1x = [0, \frac{x}{5}] \text{ and } T_2x = [\frac{x}{2}, \frac{x}{4}].$$

Let $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^{2} \beta_n^{(i)} I_i^n z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^{2} \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.2.15}$$

where $\alpha_n^{(0)} = \frac{1}{12n}$, $\alpha_n^{(1)} = \frac{12n-1}{36n}$, $\alpha_n^{(2)} = \frac{12n-1}{18n}$, $\beta_n^{(0)} = \frac{1}{10n}$, $\beta_n^{(1)} = \frac{10n-1}{30n}$, $\beta_n^{(2)} = \frac{10n-1}{15n}$, for all $n \in \mathbb{N}$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 0, where $\{0\} = \bigcap_{i=1}^{2} Fix(t_i) \cap \bigcap_{i=1}^{2} Fix(I_i) \cap \bigcap_{i=1}^{2} Fix(T_i)$.

Solution We first show that t_1 is a generalized I_1-asymptotically nonexpansive and uniformly L-Lipschitzian single-valued mapping. Let $k_n = 1$ and $s_n = (\frac{1}{2})^n$ for all $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$. Since

$$\begin{aligned} |t_1^n x - t_1^n y| \leq |\frac{x}{1+2nx} - \frac{y}{1+2ny}| &\leq |\frac{x-y}{(1+2nx)(1+2ny)}| \\ &\leq |\frac{x-y}{(1+nx)(1+ny)}| \\ &\leq |\frac{x}{1+nx} - \frac{y}{1+ny}| + \frac{1}{2^n} \\ &\leq k_n |I_1^n x - I_1^n y| + \frac{1}{2^n} \end{aligned}$$

for all $x, y \in D$ and $n \in \mathbb{N}$. Similarly, t_2 is generalized I_2-asymptotically nonexpansive with $k_n = 1$ and $s_n = (\frac{1}{2})^n$ for all $n \in \mathbb{N}$. Moreover $\bigcap_{i=1}^{2} Fix(t_i) = \{0\} = \bigcap_{i=1}^{2} Fix(I_i)$. Both T_1 and T_2 are quasi-nonexpansive multi-valued mappings satisfying condition (E) and $\bigcap_{i=1}^{2} Fix(T_i) = \{0\}$ (see [55]). Furthermore, we have $\bigcap_{i=1}^{2} Fix(t_i) \cap \bigcap_{i=1}^{2} Fix(I_i) \cap \bigcap_{i=1}^{2} Fix(T_i) = \{0\}$.

For every $n \in \mathbb{N}$, $\alpha_n^{(0)} = \frac{1}{12n}$, $\alpha_n^{(1)} = \frac{12n-1}{36n}$, $\alpha_n^{(2)} = \frac{12n-1}{18n}$, $\beta_n^{(0)} = \frac{1}{10n}$, $\beta_n^{(1)} = \frac{10n-1}{30n}$, $\beta_n^{(2)} = \frac{10n-1}{15n}$. Therefore the sequences $\{\alpha_n^{(0)}\}$, $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\beta_n^{(0)}\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ satisfy all assumptions in Theorem 3.2.8. By putting $z_n^{(1)} = \frac{x_n}{5}$ and $z_n^{(2)} = \frac{x_n}{2}$ for all $n \in \mathbb{N}$ and by using the algorithm 3.2.15 with the initial point $x_1 = 5$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 0, where $\{0\} = \bigcap_{i=1}^{2} Fix(t_i) \cap \bigcap_{i=1}^{2} Fix(I_i) \cap \bigcap_{i=1}^{2} Fix(T_i)$.

Table 2 : The value of the sequences $\{x_n\}$ and $\{y_n\}$ in Example 3.2.9

n	x_n	y_n
1	5.000000	0.937500
2	0.664154	0.172128
3	0.101141	0.038240
4	0.024007	0.010239
⋮	⋮	⋮
10	0.000018	0.000008
11	0.000006	0.000003
12	0.000002	0.000001
13	0.000001	0.000000
14	0.000000	0.000000

CHAPTER IV

ITERATIVE METHODS FOR THE SPLIT FEASIBILITY AND FIXED POINT PROBLEMS

In this chapter, we construct iterative methods by combining the extragradient with regularization method due to a generalized Ishikawa-type and Mann-type iterative methods for solving split feasibility and fixed point problems.

4.1 Introduction and Prelimilaries

In this section, we denote that C is a nonempty subset of a Hilbert space H. The fixed point problems for the mapping $T : C \to C$ is the following:

$$\text{find} \quad x \in C \quad \text{such that} \quad Tx = x.$$

Denote $Fix(T) = \{x \in C : Tx = x\}$ be the set of solutions of the fixed point problems.

In 1953, Mann [47] introduced the Mann iterative method as follows:

$$x_{n+1} = (1 - \sigma_n)x_n + \sigma_n T x_n, \quad \text{for all } n \in \mathbb{N}, \tag{4.1.1}$$

where $\{\alpha_n\} \subset [0, 1]$.

And then in 1974, Ishikawa [45] introduced the Ishikawa iterative method as follows:

$$\begin{cases} y_n = (1 - \sigma_n)x_n + \sigma_n T x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T y_n, \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{4.1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset [0, 1]$.

Next, Noor [51] introduced three-step iterative method as follows:

$$\begin{cases} y_n = (1 - \sigma_n)x_n + \sigma_n T x_n, \\ z_n = (1 - \beta_n)x_n + \beta_n T y_n, \\ x_{n+1} = (1 - \gamma_n)x_n + \gamma_n T z_n, \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{4.1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}, \{\gamma_n\} \subset [0,1]$. Clearly, Mann and Ishikawa iterative methods are special cases of Noor iteration.

The above iterative methods have been extensively studied by many authors for approximating fixed points of nonlinear mappings and solutions of nonlinear operator equations.

On the other hand, let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. The split feasibility problem (SFP) has the following property:

$$\text{find} \quad x \in C \quad \text{such that} \quad Ax \in Q.$$

where $A : H_1 \to H_2$ is a bounded linear operator with its adjoint of A^*. Denote $\Gamma_0 = \{x \in C : Ax \in Q\}$ the set of solutions of the split feasibility problems (SFP) and $\Gamma = \{x \in Fix(T) \cap C : Ax \in Fix(S) \cap Q\}$ the set of solutions of the split feasibility and fixed point problems where $T : H_1 \to H_1$ and $S : H_2 \to H_2$.

Censor and Elfving [14] introduced the split feasibility problem (SFP) in finite-dimensional Hilbert spaces for modeling inverse problems which arise in phase retrievals and medical image reconstruction [7]. The split feasibility problem (SFP) can also be applied to intensity-modulated radiation therapy (IMRT) [16, 17, 18] and have been used in signal processing and image reconstruction, see [7, 16, 8, 56, 75, 78, 86].

The original iterative method for solving the split feasibility problems (SFP) is given in [14] under assuming the existence of the inverse of A. We know that the finding of the inverse of A is difficult so this iterative method has not become popular. A more popular iterative method for solving the split feasibility problems (SFP) is the CQ iterative method which introduced by Byrne [14] because it is found to be a gradient-projection method (GPM) in convex minimization and a special case of the proximal forward-backward splitting method [24].

Many researchers have studied the CQ iterative method and its variant form, refer to [12, 13, 72, 76, 79, 80, 84]. In 2010, Xu [76] applied a Mann-type iterative method to the split feasibility problems (SFP) and proposed an average CQ iterative method which proposed in the following:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C(I - \gamma A^*(I - P_Q A)x_n \text{ for all } n \in \mathbb{N}, \tag{4.1.4}$$

where $\{\alpha_n\}$ is a real number sequence in $(0,1)$ and γ is a constant in $(0, \frac{1}{\|A\|^2})$. This was proven to be weakly convergent to a solution of the split feasibility problem (SFP).

For solving the split feasibility and fixed point problems, in 2012, Ceng et al. [12] proposed an iterative method by combining the extragradient iterative method which was introduced by Korpelevich [41] with the idea of Nadezhkina and Takahashi [49]. The authors proposed iterative process in the following:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = P_C(I - \lambda_n(I - P_Q A))x_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(SP_C(I - \lambda_n(I - P_Q A)))y_n, \text{ for all } n \in \mathbb{N}, \end{cases} \tag{4.1.5}$$

where $\{\alpha_n\}$ is a real number sequence in $(0,1)$ and $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{\|A\|^2})$. They proved that the sequences generated by their iterative method converge weakly to an element of the solutions of the split feasibility and the fixed point problems of a nonexpansive mapping S on C.

In 2014, Yao et al. [81] studied the split feasibility and fixed point problems. They [81] constructed an iterative method as the following:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ u_n = P_C(\alpha_n u + (1-\alpha_n)(x_n - \delta A^*(I - SP_Q)Ax_n)), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n T u_n), \text{ for all } n \in \mathbb{N}, \end{cases} \tag{4.1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three real number sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$, $S : H_2 \to H_2$ is a nonexpansive mapping and $T : H_1 \to H_1$ is an L-Lipschitzian pseudo-contractive mapping. They [81] proved that the sequences generated by their iterative method converge strongly to solutions of the split feasibility and the fixed point problems.

Very recently, Ceng et al. [21] had motivation and inspiration from the work of Ceng et al. [12] and Yao et al. [81]. They proposed an Ishikawa-type extragradient iterative method for pseudo-contractive mappings with Lipschitz assumption on T. For given $x_0 \in C$ as the following:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ax_n), \\ z_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ay_n), \\ w_n = (1 - \alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} = (1 - \beta_n)z_n + \beta_n T w_n, \text{ for all } n \in \mathbb{N}. \end{cases} \tag{4.1.7}$$

Moreover, they proposed a Mann-type extragradient iterative method for pseudo-contractive mappings without Lipschitz assumption on T as the following:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ax_n), \\ z_n = P_C(x_n - \lambda_n A^*(I - SP_Q)Ay_n), \\ x_{n+1} = (1 - \alpha_n)z_n + \alpha_n T z_n, \text{ for all } n \in \mathbb{N}, \end{cases} \tag{4.1.8}$$

where $S : Q \to Q$ is a nonexpansive mapping and they [21] proved that their sequences generated by their iterative methods converges weakly to a solution of the split feasibility and the fixed point problems.

The mathematical term well-posed problem stems from a definition given by Jacques Hadamard [32]. He believed that mathematical models of physical phenomena should have the properties that:

(1) a solution exists;

(2) the solution is unique;

(3) the solution's behavior changes continuously with the initial conditions.

Problems that are not well-posed in the sense of Hadamard are termed ill-posed. If the problem is well-posed, then it stands a good chance of solution on a computer using a stable algorithm. If it is not well-posed, it needs to be re-formulated for numerical treatment. Typically this involves including additional assumptions, such as smoothness of solution. This process is known as regularization. Tikhonov regularization is one of the most commonly used for regularization of linear ill-posed problems.

Throughout this research, we assume that the solution set of the split feasibility problem is nonempty. Let $f : H_1 \to \mathbb{R}$ be a continuous differentiable function, the minimization problem

$$\min_{x\in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 \tag{4.1.9}$$

is ill-posed. Therefore, Xu [76] considered the following Tikhonov regularized problem:

$$\min_{x\in C} f^{\alpha}(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{4.1.10}$$

where $\alpha > 0$ is the regularization parameter.

We observe that the gradient

$$\nabla f^{\alpha}(x) = \nabla f(x) + \alpha I x = A^*(I - P_Q)Ax + \alpha I x$$

is $(\alpha + \|A\|^2)$- Lipschitz continuous and α-strongly monotone.

Lemma 4.1.1. [21] Let Q be a nonempty closed convex supset of a Hilbert space H and $S : Q \to Q$ be a nonexpansive mapping. Set $\nabla f^S = A^*(I - SP_Q)A$, then

$$\langle x - y, \nabla f^S(x) - \nabla f^S(y)\rangle \geq \frac{1}{2\|A\|^2}\|\nabla f^S(x) - \nabla f^S(y)\|^2. \tag{4.1.11}$$

In 2012, Ceng et al. [11] proposed iterative method by combining the regularization method and extragradient method due to Nadezhkina and Takahashi [49] and they proved that the sequence generated by their iterative method converge weakly to an element of the solution of the split feasibility and fixed point problems.

We can use fixed point algorithms to solve the split feasibility problem on the basis of the following observation.

Let $\lambda > 0$ and assume that $x^* \in \Gamma$. Then $Ax^* \in Q$ which implies that $(I - P_Q)Ax^* = 0$, and thus, $\lambda(I - P_Q)Ax^* = 0$. Hence, we have the fixed point equation $x^* = (I - \lambda(I - P_Q)A)x^*$. Requiring that $x^* \in C$, we consider the fixed point equation

$$x^* = P_C(I - \lambda(I - P_Q)A)x^* = P_C(I - \lambda\nabla f)x^*. \tag{4.1.12}$$

It is proven in [76] that the solutions of the fixed point equation (4.1.12) are exactly the solutions of the split feasibility problems; namely, for given $x^* \in C$, x^* solves the split feasibility problem if and only if x^* solves the fixed point (4.1.12).

According to these motivations, we introduce the iterative methods by using a combination of an extragradient method with regularization due to a generalized Ishikawa iterative method for solving the split feasibility and the fixed point problems of pseudo-contractive mappings with Lipschitz assumption on C and nonexpansive mappings on Q. On the other hand, we aviod Lipschitzian condition by proposing an iterative method which combine an extragradient method with regularization due to a generalized Mann iterative method for solving the split feasibility and the fixed point problems. We establish weak convergence theorems for sequences generated by the proposed iterative processes. Finally we give numerical results and compare its behavior with an Ishikawa-type extragradient iterative method and a Mann-type extragradient iterative method of Ceng et al. [21].

4.2 Convergence theorems

The generalized Ishikawa-type extragradient with regularization iterative method for pseudo-contractive mappings with Lipschitz assumption

In this section, we propose the generalized Ishikawa-type extragradient with regularization iterative method for pseudo-contractive mappings with Lipschitz assumption for solving the split feasibility and fixed point problems.

Theorem 4.2.1. Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint of A^*. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\Big), \\ z_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)y_n\Big), \\ w_n = (1-\sigma_n)z_n + \sigma_n T z_n, \\ s_n = (1-\beta_n)z_n + \beta_n T w_n, \\ x_{n+1} = (1-\gamma_n)z_n + \gamma_n T s_n, \end{cases} \tag{4.2.1}$$

where $\{\lambda_n\} \subset [\kappa, \tau]$ for some $\kappa, \tau \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0,\infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $0 < a < \gamma_n < b < \beta_n < c < \sigma_n < d < \frac{1}{\sqrt{L^2+1}+1+L^2}$. Then the sequence $\{x_n\}$ generated by algorithm (4.2.1) converges weakly to an element of Γ.

Proof. Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in Fix(T) \cap C$ and $Ax^* \in Fix(S) \cap Q$. Set $v_n = P_Q A x_n$, $u_n = x_n - \lambda_n(A^*(I - SP_Q)Ax_n + \alpha_n I)x_n$, $\nabla f^{S\alpha_n} = A^*(I - SP_Q)A + \alpha_n I$ and $\nabla f^S = A^*(I - SP_Q)A$, for all $n \geq 0$. Since P_C is nonexpansive, we have

$$\begin{aligned} \|y_n - x^*\|^2 =& \|P_C u_n - x^*\|^2 \leq \|u_n - x^*\|^2 \\ =& \|x_n - \lambda_n\Big(A^*(I - SP_Q)A + \alpha_n I\Big)x_n - x^*\|^2 \\ =& \|x_n - x^*\|^2 + 2\lambda_n\langle x_n - x^*, A^*(SP_Q - I)Ax_n\rangle \\ &+ \lambda_n^2\|A^*(SP_Q - I)Ax_n\|^2 - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle. \end{aligned} \tag{4.2.2}$$

From A is a linear operator with its adjoint A^*, we obtain that

$$\begin{aligned} \langle x_n - x^*, A^*(Sv_n - Ax_n)\rangle &= \langle Ax_n - Ax^*, Sv_n - Ax_n\rangle \\ &= \langle Ax_n - Ax^* + Sv_n - Ax_n - Sv_n + Ax_n, Sv_n - Ax_n\rangle \\ &= \langle Sv_n - Ax^*, Sv_n - Ax_n\rangle - \|Sv_n - Ax_n\|^2. \end{aligned} \tag{4.2.3}$$

In combination with (2.3.3), we get that

$$\langle Sv_n - Ax^*, Sv_n - Ax_n\rangle = \frac{1}{2}(\|Sv_n - Ax^*\|^2 + \|Sv_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2). \quad (4.2.4)$$

Since S is a nonexpansive mapping and (2.3.1), we have

$$\begin{aligned}\|Sv_n - Ax^*\|^2 &= \|SP_QAx_n - SP_QAx^*\|^2\\ &\leq \|P_QAx_n - P_QAx^*\|^2\\ &\leq \|Ax_n - Ax^*\|^2 - \|v_n - Ax_n\|^2. \end{aligned} \quad (4.2.5)$$

In view of (4.2.3), (4.2.4) and (4.2.5), it follows that

$$\begin{aligned}&\langle x_n - x^*, A^*(Sv_n - Ax_n)\rangle\\ &= \frac{1}{2}(\|Sv_n - Ax^*\|^2 + \|Sv_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2) - \|Sv_n - Ax_n\|^2\\ &\leq \frac{1}{2}(\|Ax_n - Ax^*\|^2 - \|v_n - Ax^*\|^2 + \|Sv_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2)\\ &\quad - \|Sv_n - Ax_n\|^2\\ &= -\frac{1}{2}\|v_n - Ax^*\|^2 - \frac{1}{2}\|Sv_n - Ax_n\|^2. \end{aligned} \quad (4.2.6)$$

Substituting (4.2.6) into (4.2.3) and the assumption of $\{\lambda_n\}$, this implies that

$$\begin{aligned}\|y_n - x^*\|^2 =&\|x_n - x^*\|^2 + \lambda_n^2\|A\|^2\|Sv_n - Ax_n\|^2 + 2\lambda_n\langle x_n - x^*, A^*(Sv_n - Ax_n)\rangle\\ &- \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle\\ \leq&\|x_n - x^*\|^2 + \lambda_n^2\|A\|^2\|Sv_n - Ax_n\|^2\\ &+ 2\lambda_n\big(-\frac{1}{2}\|v_n - Ax_n\|^2 - \frac{1}{2}\|Sv_n - Ax_n\|^2\big)\\ &- \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle\\ =&\|x_n - x^*\|^2 - \lambda_n\|v_n - Ax_n\|^2 - \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2\\ &- \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle\\ \leq&\|x_n - x^*\|^2 - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle. \end{aligned} \quad (4.2.7)$$

Now, we will show that

$$\langle \nabla f^{S\alpha_n}(x) - \nabla f^{S\alpha_n}(y), x - y\rangle \geq \frac{1}{\alpha_n + 2\|A\|^2}\|\nabla f^{S\alpha_n}(x) - \nabla f^{S\alpha_n}(y)\|^2. \quad (4.2.8)$$

By Lemma 4.1.1, we have

$$\langle x-y, \nabla f^S(x) - \nabla f^S(y)\rangle \geq \frac{1}{2\|A\|^2}\|\nabla f^S(x) - \nabla f^S(y)\|^2.$$

Observe that

$$\begin{aligned}
&(\alpha_n + 2\|A\|^2)\langle \nabla f^{S\alpha_n}(x) - \nabla f^{S\alpha_n}(y), x-y\rangle \\
&\quad = (\alpha_n + 2\|A\|^2)\Big(\alpha_n\|x-y\|^2 + \langle \nabla f^S(x) - \nabla f^S(y), x-y\rangle\Big) \\
&\quad = \alpha_n^2\|x-y\|^2 + \alpha_n\langle \nabla f^S(x) - \nabla f^S(y), x-y\rangle + 2\alpha_n\|A\|^2\|x-y\|^2 \\
&\qquad + 2\|A\|^2\langle \nabla f^S(x) - \nabla f^S(y), x-y\rangle \\
&\quad \geq \alpha_n^2\|x-y\|^2 + \alpha_n\langle \nabla f^S(x) - \nabla f^S(y), x-y\rangle + 2\alpha_n\|A\|^2\|x-y\|^2 \\
&\qquad + \|\nabla f^S(x) - \nabla f^S(y)\|^2 \\
&\quad \geq \alpha_n^2\|x-y\|^2 + 2\alpha_n\langle \nabla f^S(x) - \nabla f^S(y), x-y\rangle + \|\nabla f^S(x) - \nabla f^S(y)\|^2 \\
&\quad = \|\alpha_n(x-y) + \nabla f^S(x) - \nabla f^S(y)\|^2 \\
&\quad = \|\nabla f^{S\alpha_n}(x) - \nabla f^{S\alpha_n}(y)\|^2.
\end{aligned}$$

By Proposition 2.3.4(2), we get that

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|x_n - \lambda_n\nabla f^{S\alpha_n}(y_n) - x^*\|^2 - \|x_n - \lambda_n\nabla f^{S\alpha_n}(y_n) - z_n\|^2 \\
&= \|x_n - x^*\|^2 - 2\lambda_n\langle x_n - x^*, \nabla f^{S\alpha_n}(y_n)\rangle + \lambda_n^2\|\nabla f^{S\alpha_n}(y_n)\|^2 \\
&\quad - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, \nabla f^{S\alpha_n}(y_n)\rangle - \lambda_n^2\|\nabla f^{S\alpha_n}(y_n)\|^2 \\
&= \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle \nabla f^{S\alpha_n}(y_n), x^* - z_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - z_n\|^2 - 2\lambda_n\Big(\langle \nabla f^{S\alpha_n}(y_n) - \nabla f^{S\alpha_n}(x^*), y_n - x^*\rangle \\
&\quad + \langle \nabla f^{S\alpha_n}(x^*), x^* - y_n\rangle + \langle \nabla f^{S\alpha_n}(y_n), y_n - z_n\rangle\Big) \\
&\leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle \nabla f^{S\alpha_n}(y_n), y_n - z_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2\langle x_n - \lambda_n\nabla f^{S\alpha_n}(y_n) - y_n, z_n - y_n\rangle.
\end{aligned}$$

Combining (4.2.8) with Proposition 2.3.4(1), we have

$$\langle x_n - \lambda_n\nabla f^{S\alpha_n}(y_n) - y_n, z_n - y_n\rangle$$

$$= \langle x_n - \lambda_n \nabla f^{S\alpha_n}(x_n) - y_n, z_n - y_n\rangle + \lambda_n\langle \nabla f^{S\alpha_n}(x_n) - \nabla f^{S\alpha_n}(y_n), z_n - y_n\rangle$$
$$\leq \lambda_n\langle \nabla f^{S\alpha_n}(x_n) - \nabla f^{S\alpha_n}(y_n), z_n - y_n\rangle$$
$$\leq \lambda_n\|\nabla f^{S\alpha_n}(x_n) - \nabla f^{S\alpha_n}(y_n)\|\|z_n - y_n\|$$
$$\leq \lambda_n(\alpha_n + 2\|A\|^2)\|x_n - y_n\|\|z_n - y_n\|. \tag{4.2.9}$$

The hypothesis of $\{\lambda_n\}$ and (4.2.9), it follows that

$$\begin{aligned}\|z_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\ &\quad + 2\langle x_n - \lambda_n \nabla f^{S\alpha_n}(y_n) - y_n, z_n - y_n\rangle \\ &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\ &\quad + 2\lambda_n(\alpha_n + 2\|A\|^2)\|x_n - y_n\|\|z_n - y_n\| \\ &\leq \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 + \|z_n - y_n\|^2 \\ &\quad + \lambda_n^2(\alpha_n + 2\|A\|^2)^2\|x_n - y_n\|^2 \\ &= \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2\|x_n - y_n\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \tag{4.2.10}$$

Likewise, we get that

$$\|z_n - x^*\|^2 = \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2\|z_n - y_n\|. \tag{4.2.11}$$

Since T is a pseudo-contractive mapping, we obtain that

$$\|Tz_n - x^*\|^2 \leq \|z_n - x^*\|^2 + \|z_n - Tz_n\|^2, \tag{4.2.12}$$

and

$$\begin{aligned}\|Tw_n - x^*\|^2 &= \|T((1-\sigma_n)z_n + \sigma_n Tz_n) - x^*\|^2 \\ &\leq \|(1-\sigma_n)(z_n - x^*) + \sigma_n(Tz_n - x^*)\|^2 \\ &\quad + \|(1-\sigma_n)z_n + \sigma_n Tz_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2. \end{aligned} \tag{4.2.13}$$

Again using (2.3.3) and T is an L-Lipschitzian pseudo-contractive mapping, this implies that

$$\|(1-\sigma_n)z_n + \sigma_n Tz_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2$$

$$= \|(1-\sigma_n)(z_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)) + \sigma_n(Tz_n - T((1-\sigma_n)z_n + \sigma_n Tz_n))\|^2$$
$$= (1-\sigma_n)\|z_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2 + \sigma_n\|Tz_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2$$
$$- \sigma_n(1-\sigma_n)\|z_n - Tz_n\|^2$$
$$\leq (1-\sigma_n)\|z_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2 + \sigma_n L^2\|z_n - Tz_n\|^2$$
$$- \sigma_n(1-\sigma_n)\|z_n - Tz_n\|^2$$
$$= (1-\sigma_n)\|z_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2 - \sigma_n(1-\sigma_n - L^2)\|z_n - Tz_n\|^2. \tag{4.2.14}$$

Combining with (2.3.3) and (4.2.12), we get that

$$\begin{aligned}
&\|(1-\sigma_n)(z_n - x^*) + \sigma_n(Tz_n - x^*)\|^2 \\
&\qquad = (1-\sigma_n)\|z_n - x^*\|^2 + \sigma_n\|Tz_n - x^*\|^2 - \sigma_n(1-\sigma_n)\|z_n - Tz_n\|^2 \\
&\qquad \leq (1-\sigma_n)\|z_n - x^*\|^2 + \sigma_n[\|z_n - x^*\|^2 + \|z_n - Tz_n\|^2] \\
&\qquad - \sigma_n(1-\sigma_n)\|z_n - Tz_n\|^2 \\
&\qquad = \|z_n - x^*\|^2 + \sigma_n^2\|z_n - Tz_n\|^2.
\end{aligned} \tag{4.2.15}$$

By (4.2.14) and (4.2.15), it follows that

$$\begin{aligned}
\|Tw_n - x^*\|^2 &= \|T((1-\sigma_n)z_n + \sigma_n Tz_n) - x^*\|^2 \\
&\leq \|(1-\sigma_n)(z_n - x^*) + \sigma_n(Tz_n - x^*)\|^2 \\
&+ \|(1-\sigma_n)z_n + \sigma_n Tz_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2 \\
&= \|z_n - x^*\|^2 + (1-\sigma_n)\|z_n - T((1-\sigma_n)z_n + \sigma_n Tz_n)\|^2 \\
&- \sigma_n(1 - 2\sigma_n - \sigma_n^2 L^2)\|z_n - Tz_n\|^2.
\end{aligned} \tag{4.2.16}$$

Likewise, since T is a pseudo-contractive mapping, we get that

$$\|Ts_n - x^*\|^2 \leq \|s_n - x^*\|^2 + \|s_n - Ts_n\|^2. \tag{4.2.17}$$

Consider

$$\begin{aligned}
\|Ts_n - x^*\|^2 &= \|T((1-\beta_n)z_n + \beta_n Tw_n) - x^*\|^2 \\
&\leq \|(1-\beta_n)(z_n - x^*) + \beta_n(Tw_n - x^*)\|^2
\end{aligned}$$

$$+\|(1-\beta_n)z_n+\beta_nTw_n-T((1-\beta_n)z_n+\beta_nTw_n)\|^2, \qquad (4.2.18)$$

and by combining with (2.3.3) and (4.2.16), we get that

$$\begin{aligned}
&\|(1-\beta_n)(z_n-x^*)+\beta_n(Tw_n-x^*)\|^2\\
&=(1-\beta_n)\|z_n-x^*\|^2+\beta_n\|Tw_n-x^*\|^2-\beta_n(1-\beta_n)\|z_n-Tw_n\|^2\\
&\le(1-\beta_n)\|z_n-x^*\|^2+\beta_n\Big(\|z_n-x^*\|^2+(1-\sigma_n)\|z_n-Tw_n\|^2\\
&\quad-\sigma_n(1-2\sigma_n-\sigma_n^2L^2)\|z_n-Tz_n\|^2\Big)-\beta_n(1-\beta_n)\|z_n-Tw_n\|^2\\
&\le\|z_n-x^*\|^2-\beta_n(\sigma_n-\beta_n)\|z_n-Tw_n\|^2-\beta_n\sigma_n(1-2\sigma_n-\sigma_n^2L^2)\|z_n-Tz_n\|^2.
\end{aligned} \qquad (4.2.19)$$

Again using (2.3.3) and T is an L-Lipschitzian pseudo-contractive mapping , we get

$$\begin{aligned}
&\|(1-\beta_n)z_n+\beta_nTw_n-T((1-\beta_n)z_n+\beta_nTw_n)\|^2\\
&=\|(1-\beta_n)(z_n-T((1-\beta_n)z_n+\beta_nTw_n))+\beta_n(Tw_n-T((1-\beta_n)z_n+\beta_nTw_n))\|^2\\
&=(1-\beta_n)\|z_n-T((1-\beta_n)z_n+\beta_nTw_n)\|^2+\beta_n\|Tw_n-T((1-\beta_n)z_n+\beta_nTw_n)\|^2\\
&\quad-\beta_n(1-\beta_n)\|z_n-Tw_n\|^2\\
&=(1-\beta_n)\|z_n-T((1-\beta_n)z_n+\beta_nTw_n)\|^2+\beta_nL^2\|w_n-((1-\beta_n)z_n+\beta_nTw_n)\|^2\\
&\quad-\beta_n(1-\beta_n)\|z_n-Tw_n\|^2.
\end{aligned} \qquad (4.2.20)$$

Since $w_n=(1-\sigma_n)z_n+\sigma_nTz_n$ and $\sigma_n<d<\frac{1}{\sqrt{L^2+1}+1+L^2}$, we have

$$\begin{aligned}
\|w_n-(1-\beta_n)z_n-\beta_nTw_n\|^2 &=\|(1-\sigma_n)z_n+\sigma_nTz_n-(1-\beta_n)z_n-\beta_nTw_n\|^2\\
&=\beta_n^2\|z_n-Tw_n\|^2+\sigma_n^2\|z_n-Tz_n\|^2\\
&\quad-2\beta_n\sigma_n\langle z_n-Tw_n,z_n-Tz_n\rangle\\
&=\beta_n^2\|z_n-Tw_n\|^2+\sigma_n^2\|z_n-Tz_n\|^2\\
&\quad-2\beta_n\sigma_n\langle z_n-Tw_n+Tz_n-Tz_n,z_n-Tz_n\rangle\\
&=\beta_n^2\|z_n-Tw_n\|^2+\sigma_n^2\|z_n-Tz_n\|^2\\
&\quad-2\beta_n\sigma_n\|z_n-Tz_n\|^2-2\beta_n\sigma_n\langle Tz_n-Tw_n,z_n-Tz_n\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2 \\
&\quad - 2\beta_n\sigma_n\|z_n - Tz_n\|^2 + 2\beta_n\sigma_n\|Tz_n - Tw_n\|\|Tz_n - z_n\| \\
&\leq \beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2 \\
&\quad - 2\beta_n\sigma_n\|z_n - Tz_n\|^2 + 2\beta_n\sigma_n^2 L\|z_n - Tz_n\|\|Tz_n - z_n\| \\
&= \beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2 \\
&\quad - 2\beta_n\sigma_n(1 - \sigma_n L)\|z_n - Tz_n\|^2 \\
&\leq \beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2. \qquad (4.2.21)
\end{aligned}
$$

Combining (4.2.20) with (4.2.21), we obtain that

$$
\begin{aligned}
&\|(1-\beta_n)z_n + \beta_n Tw_n - T((1-\beta_n)z_n + \beta_n Tw_n)\|^2 \\
&\quad = (1-\beta_n)\|z_n - Ts_n\|^2 + \beta_n L^2\|w_n - ((1-\beta_n)z_n + \beta_n Tw_n)\|^2 \\
&\quad\quad - \beta_n(1-\beta_n)\|z_n - Tw_n\|^2 \\
&\quad \leq (1-\beta_n)\|z_n - Ts_n\|^2 + \beta_n L^2(\beta_n^2\|z_n - Tw_n\|^2 + \sigma_n^2\|z_n - Tz_n\|^2) \\
&\quad\quad - \beta_n(1-\beta_n)\|z_n - Tw_n\|^2 \\
&\quad = (1-\beta_n)\|z_n - Ts_n\|^2 + \beta_n\sigma_n^2 L^2\|z_n - Tz_n\|^2 \\
&\quad\quad - \beta_n(1-\beta_n-\beta_n^2 L^2)\|z_n - Tw_n\|^2. \qquad (4.2.22)
\end{aligned}
$$

From (4.2.18), (4.2.19) and (4.2.22), this implies that

$$
\begin{aligned}
\|Ts_n - x^*\|^2 &= \|T((1-\beta_n)z_n + \beta_n Tw_n) - x^*\|^2 \\
&\leq \|(1-\beta_n)(z_n - x^*) + \beta_n(Tw_n - x^*)\|^2 \\
&\quad + \|(1-\beta_n)z_n + \beta_n Tw_n - T((1-\beta_n)z_n + \beta_n Tw_n)\|^2 \\
&\leq \|z_n - x^*\|^2 - \beta_n(\sigma_n - \beta_n)\|z_n - Tw_n\|^2 \\
&\quad - \beta_n\sigma_n(1 - 2\sigma_n - \sigma_n^2 L^2)\|z_n - Tz_n\|^2 \\
&\quad + (1-\beta_n)\|z_n - Ts_n\|^2 + \beta_n\sigma_n^2 L^2\|z_n - Tz_n\|^2 \\
&\quad - \beta_n(1-\beta_n-\beta^2 L^2)\|z_n - Tw_n\|^2 \\
&= \|z_n - x^*\|^2 + (1-\beta_n)\|z_n - Ts_n\|^2 \\
&\quad - \beta_n\Big((\sigma_n - \beta_n) + (1-\beta_n-\beta_n^2 L^2)\Big)\|Tw_n - z_n\|^2
\end{aligned}
$$

$$- \beta_n\sigma_n(1-\sigma_n(2+L^2)-\sigma_n^2L^2)\|z_n - Tz_n\|^2. \tag{4.2.23}$$

Since $\beta_n < c < \sigma_n < d < \frac{1}{\sqrt{L^2+1}+1+L^2}$, it obtains that

$$1-\beta_n-\beta_n^2L^2 > 0 \text{ and } 1-\sigma_n(2+L^2)-\sigma_n^2L^2 > 0.$$

Therefore

$$\|Ts_n - x^*\|^2 \leq \|z_n - x^*\|^2 + (1-\beta_n)\|z_n - Ts_n\|^2. \tag{4.2.24}$$

From (2.3.3), (4.2.1) and (4.2.24), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 =& \|(1-\gamma_n)z_n + \gamma_n Ts_n - x^*\|^2 \\
=&(1-\gamma_n)\|z_n - x^*\|^2 + \gamma_n\|Ts_n - x^*\|^2 - \gamma_n(1-\gamma_n)\|z_n - Ts_n\|^2 \\
\leq&(1-\gamma_n)\|z_n - x^*\|^2 + \gamma_n(\|z_n - x^*\|^2 + (1-\beta_n)\|z_n - Ts_n\|^2) \\
&- \gamma_n(1-\gamma_n)\|z_n - Ts_n\|^2 \\
=&\|z_n - x^*\|^2 - \gamma_n(\beta_n - \gamma_n)\|z_n - Ts_n\|^2 \\
\leq&\|z_n - x^*\|^2.
\end{aligned} \tag{4.2.25}$$

This together with (4.2.11) implies that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|,$$

for every $x^* \in \Gamma$ and for all $n \geq 0$. Therefore $\{x_n\}$ generated by algorithm (4.2.1) is the Féjermonotone with respect to Γ. Thus we obtain $\lim_{n\to\infty}\|x_n - x^*\|$ exists immediately, it follows that $\{x_n\}$ is bounded and the sequence $\{\|x_n - x^*\|\}$ is monotonically decreasing. Moreover, $\{y_n\}$ and $\{z_n\}$ are also bounded sequences from (4.2.7) and (4.2.10) immediately. Combining (4.2.9) and (4.2.25), this implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - (1-\lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2.
\end{aligned}$$

It follows that

$$(1-\lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

and so

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4.2.26}$$

Likewise, we get

$$\lim_{n\to\infty} \|z_n - y_n\| = 0.$$

In combination (4.2.26), (4.2.7) and $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\begin{aligned}
&\lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 + \lambda_n\|v_n - Ax^*\|^2 \\
&\quad\le \|x_n - x^*\|^2 - \|y_n - x^*\|^2 - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \\
&\quad\le (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\| - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|v_n - Ax_n\| = 0 \text{ and } \lim_{n\to\infty} \|Sv_n - Ax_n\| = 0.$$

So $\lim_{n\to\infty} \|v_n - Sv_n\| = 0$. From (4.2.25), we get that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \le& \|z_n - x^*\|^2 - \gamma_n(\beta_n - \gamma_n)\|z_n - Ts_n\|^2 \\
\le& \|x_n - x^*\|^2 - \gamma_n(\beta_n - \gamma_n)\|z_n - Ts_n\|^2.
\end{aligned}$$

It follows that

$$\gamma_n(\beta_n - \gamma_n)\|z_n - Ts_n\|^2 \le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

and so

$$\lim_{n\to\infty} \|z_n - Ts_n\| = 0.$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|z_n - Tz_n\| \le& \|z_n - Ts_n\| + \|Ts_n - Tz_n\| \\
\le& \|z_n - Ts_n\| + L\|s_n - z_n\|.
\end{aligned}$$

Since $s_n = (1 - \beta_n)z_n + \beta_n Tw_n$, we have

$$\|z_n - Tz_n\| \le \|z_n - Ts_n\| + L\|s_n - z_n\|$$

$$=\|z_n - Ts_n\| + \beta_n L\|z_n - Tw_n\|. \tag{4.2.27}$$

Since $w_n = (1-\sigma_n)z_n + \sigma_n T z_n$, we get that

$$\begin{aligned}\|z_n - Tw_n\| \leq &\|z_n - Ts_n\| + \|Ts_n - Tw_n\| \\ \leq &\|z_n - Ts_n\| + L\|s_n - w_n\| \\ = &\|z_n - Ts_n\| + \sigma_n L\|z_n - Tz_n\| + \beta_n L\|z_n - Tw_n\|. \end{aligned} \tag{4.2.28}$$

So,

$$(1-\beta_n L)\|Tw_n - z_n\| \leq \|z_n - Ts_n\| + \sigma_n L\|z_n - Tz_n\|. \tag{4.2.29}$$

By (4.2.27) and the previous inequations, we get

$$\begin{aligned}\|z_n - Tz_n\| \leq &\|z_n - Ts_n\| + \beta_n L\|z_n - Tw_n\| \\ = &\|z_n - Ts_n\| + \beta_n L\left(\frac{1}{1-\beta_n L}\|z_n - Ts_n\| + \frac{\sigma_n L}{1-\beta_n L}\|z_n - Tz_n\|\right) \\ = &\left(1 + \frac{\beta_n L}{1-\beta_n L}\right)\|z_n - Ts_n\| + \frac{\sigma_n \beta_n L^2}{1-\beta_n L}\|z_n - Tz_n\|.\end{aligned}$$

This implies that

$$\|z_n - Tz_n\| \leq \left(\frac{1}{1-\beta_n L - \sigma_n \beta_n L^2}\right)\|z_n - Ts_n\|.$$

Therefore

$$\lim_{n\to\infty}\|z_n - Tz_n\| = 0.$$

By (4.2.29), we have

$$\lim_{n\to\infty}\|z_n - Tw_n\| = 0.$$

Using the firm nonexpansiveness of P_C, (2.3.1) and (4.2.7), we have

$$\begin{aligned}\|y_n - x^*\|^2 = \|P_C u_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|P_C u_n - u_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \|y_n - u_n\|^2.\end{aligned}$$

It follows that

$$\|y_n - u_n\|^2 \leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2$$

$$\leq (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\|.$$

From (4.2.26), we have

$$\lim_{n\to\infty} \|y_n - u_n\| = 0.$$

Since the sequence $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{x}$.

Therefore, from the above conclusions, we can obtain that

$$\begin{cases} x_{n_i} \rightharpoonup \hat{x}, \\ y_{n_i} \rightharpoonup \hat{x}, \\ u_{n_i} \rightharpoonup \hat{x}, \end{cases} \quad \text{and} \quad \begin{cases} z_{n_i} \rightharpoonup \hat{x}, \\ Ax_{n_i} \rightharpoonup A\hat{x}, \\ v_{n_i} \rightharpoonup A\hat{x}, \end{cases} \tag{4.2.30}$$

it is applied by Lemma 2.4.6, we have

$$\hat{x} \in Fix(T) \text{ and } A\hat{x} \in Fix(S).$$

From $y_{n_i} = P_C u_{n_i} \in C$ and $v_{n_i} = P_Q Ax_{n_i}$ and combine with (4.2.30), we get that

$$\hat{x} \in C \text{ and } A\hat{x} \in Q.$$

Therefore

$$\hat{x} \in C \cap Fix(T) \text{ and } A\hat{x} \in Q \cap Fix(S).$$

We can conclude that $\hat{x} \in \Gamma$ and this shows that $\omega_W(x_n) \subset \Gamma$. Since the $\lim_{n\to\infty} \|x_n - x^*\|$ exists for every $x^* \in \Gamma$ and every subsequence of $\{x_n\}$ converges weakly to $x^* \in \Gamma$, it is immediate from Lemma 2.4.7 that $\{x_n\}$ converges weakly to $x^* \in \Gamma$. This completes the proof. □

Next, utilizing Theorem 4.2.1, we give the following corollary when defining iterative method becomes combining an extragradient method with regularization due to the Ishikawa iterative method.

Corollary 4.2.2. Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\Big), \\ z_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)y_n\Big), \\ w_n = (1-\sigma_n)z_n + \sigma_n T z_n, \\ x_{n+1} = (1-\beta_n)z_n + \beta_n T w_n, \end{cases} \tag{4.2.31}$$

where $\{\lambda_n\} \subset [\kappa, \tau]$ for some $\kappa, \tau \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $0 < a < \beta_n < b < \sigma_n < c < \frac{1}{\sqrt{L^2+1}+1+L^2}$ Then the sequence $\{x_n\}$ generated by algorithm (4.2.31) converges weakly to an element of Γ.

Proof. Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in Fix(T) \cap C$ and $Ax^* \in Fix(S) \cap Q$. Set $v_n = P_Q A x_n$, $u_n = x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n$, $\nabla f^{S\alpha_n} = A^*(I - SP_Q)A + \alpha_n I$ and $\nabla f^S = A^*(I - SP_Q)A$, for all $n \geq 0$. As in Theorem 4.2.1, we have

$$\begin{aligned} \|y_n - x^*\|^2 \leq & \|x_n - x^*\|^2 - \lambda_n\|v_n - Ax_n\|^2 - \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 \\ & - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle, \end{aligned} \tag{4.2.32}$$

$$\begin{aligned} \|z_n - x^*\|^2 & \leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2\|x_n - y_n\|^2 \\ & \leq \|x_n - x^*\|^2, \end{aligned} \tag{4.2.33}$$

$$\|z_n - x^*\|^2 = \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2\|z_n - y_n\|, \tag{4.2.34}$$

and

$$\begin{aligned} \|Tw_n - x^*\|^2 \leq \|z_n - x^*\|^2 + (1-\sigma_n)\|z_n - T((1-\sigma_n)z_n + \sigma_n T z_n)\|^2 \\ - \sigma_n(1 - 2\sigma_n - \sigma_n^2 L^2)\|z_n - Tz_n\|^2. \end{aligned}$$

Since $b < \sigma_n < c < \frac{1}{\sqrt{L^2+1}+1+L^2}$, we obtain that

$$\|Tw_n - x^*\|^2 \leq \|z_n - x^*\|^2 + (1-\sigma_n)\|z_n - T((1-\sigma_n)z_n + \sigma_n T z_n)\|^2. \tag{4.2.35}$$

From (4.2.32) and (4.2.35), this implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 =&\|(1-\beta_n)z_n+\beta_nTw_n-x^*\|^2\\ =&(1-\beta_n)\|z_n-x^*\|^2+\beta_n\|Tw_n-x^*\|^2-\beta_n(1-\beta_n)\|z_n-Tw_n\|^2\\ \le&(1-\beta_n)\|z_n-x^*\|^2+\beta_n(\|z_n-x^*\|^2+(1-\sigma_n)\|z_n-Tw_n\|^2)\\ &-\beta_n(1-\beta_n)\|z_n-Tw_n\|^2\\ =&\|z_n-x^*\|^2-\beta_n(\sigma_n-\beta_n)\|z_n-Tw_n\|^2\\ \le&\|z_n-x^*\|^2\le\|x_n-x^*\|^2. \end{aligned} \tag{4.2.36}$$

This implies that $\{x_n\}$ is a bounded sequence and the sequence $\{\|x_n-x^*\|\}$ is monotonically decreasing. Thus we obtain $\lim_{n\to\infty}\|x_n-x^*\|$ exists immediately, Moreover, $\{y_n\}$ and $\{z_n\}$ are also bounded sequences. In the same process of a proof in Theorem 4.2.1, we get that

$$\lim_{n\to\infty}\|z_n-y_n\|=\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|y_n-u_n\|=0,$$

and by (4.2.32), we obtain that

$$\lim_{n\to\infty}\|v_n-Ax_n\|=\lim_{n\to\infty}\|Sv_n-Ax_n\|=0.$$

From (4.2.36), we observe that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le&\|z_n-x^*\|^2-\beta_n(\sigma_n-\beta_n)\|z_n-Tw_n\|^2\\ \le&\|x_n-x^*\|^2-\beta_n(\sigma_n-\beta_n)\|z_n-Tw_n\|^2. \end{aligned} \tag{4.2.37}$$

Thus

$$\beta_n(\sigma_n-\beta_n)\|z_n-Tw_n\|^2 \le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2, \tag{4.2.38}$$

taking the limit of $n\to\infty$,

$$\lim_{n\to\infty}\|z_n-Tw_n\|=0.$$

For all $n\in\mathbb{N}$, we have

$$\|z_n-Tz_n\|\le\|z_n-Tw_n\|+\|Tw_n-Tz_n\|$$

$$\leq \|z_n - Tw_n\| + L\|(1-\sigma_n)z_n + \sigma_n T z_n - z_n\|$$
$$\leq \|z_n - Tw_n\| + \sigma_n L\|z_n - Tz_n\|.$$

It follows that

$$(1-\sigma_n L)\|z_n - Tz_n\| \leq \|z_n - Tw_n\|.$$

Therefore

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0.$$

Consequently, all of conditions in Theorem 4.2.1 are satisfied and we can conclude that Corollary 4.2.2 can be obtained from Theorem 4.2.1 immediately. □

Next, utilizing Theorem 4.2.1, we give the following corollary when omit $\{z_n\}$ in the iterative method of Theorem 4.2.1.

Corollary 4.2.3. Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\Big), \\ w_n = (1-\sigma_n)y_n + \sigma_n T y_n, \\ s_n = (1-\beta_n)y_n + \beta_n T w_n, \\ x_{n+1} = (1-\gamma_n)y_n + \gamma_n T s_n, \end{cases} \tag{4.2.39}$$

where $\{\lambda_n\} \subset [\kappa, \tau]$ for some $\kappa, \tau \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0,\infty), \Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $0 < a < \gamma_n < b < \beta_n < c < \sigma_n < d < \frac{1}{\sqrt{L^2+1}+1+L^2}$. Then the sequence $\{x_n\}$ generated by algorithm (4.2.39) converges weakly to an element of Γ.

Proof. Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in Fix(T) \cap C$ and $Ax^* \cap Fix(S) \in Q$. Set $v_n = P_Q A x_n$, $u_n = x_n - \lambda_n(A^*(I -$

$SP_Q)A + \alpha_n I)x_n$, $\nabla f^{S\alpha_n} = A^*(I - SP_Q)A + \alpha_n I$ and $\nabla f^S = A^*(I - SP_Q)A$, for all $n \geq 0$. As in Theorem 4.2.1, we have

$$\begin{aligned} \|y_n - x^*\|^2 \leq & \|x_n - x^*\|^2 - \lambda_n\|v_n - Ax_n\|^2 - \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 \\ & - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \end{aligned} \tag{4.2.40}$$

and

$$\|x_{n+1} - x^*\|^2 \leq \|y_n - x^*\|^2. \tag{4.2.41}$$

This implies that $\{x_n\}$ is a bounded sequence. In view of (4.2.40) and (4.2.41), we obtain that

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|Sv_n - Ax_n\| = 0. \tag{4.2.42}$$

Therefore

$$\lim_{n\to\infty} \|v_n - Sv_n\| = 0.$$

Since P_C is firmly nonexpansive, we have

$$\begin{aligned} \|y_n - x^*\|^2 = \|P_C u_n - x^*\|^2 &\leq \|u_n - x^*\|^2 - \|P_C u_n - u_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \|y_n - u_n\|^2. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{4.2.43}$$

By $u_n = x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n$ and (4.2.42), it follows that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0.$$

Combining with the previous equation and (4.2.43) implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

In process in the proof of Theorem 4.2.1, we have $\lim_{n\to\infty} \|y_n - Ts_n\| = 0$. It leads to prove that $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$. Consequently, all of conditions in Theorem 4.2.1 are satisfied and we can conclude that Corollary 4.2.3 can be obtained from Theorem 4.2.1 immediately. □

Next, utilizing Theorem 4.2.1, we give the following corollary by setting $S : H_2 \to H_2$ to be identity mapping in Theorem 4.2.1.

Corollary 4.2.4. Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = P_C\Big(x_n - \lambda_n(A^*(I-P_Q)A + \alpha_n I)x_n\Big), \\ z_n = P_C\Big(x_n - \lambda_n(A^*(I-P_Q)A + \alpha_n I)y_n\Big), \\ w_n = (1-\sigma_n)z_n + \sigma_n T z_n, \\ s_n = (1-\beta_n)z_n + \beta_n T w_n, \\ x_{n+1} = (1-\gamma_n)z_n + \gamma_n T s_n, \end{cases} \tag{4.2.44}$$

where $\{\lambda_n\} \subset [\kappa, \tau]$ for some $\kappa, \tau \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $0 \leq \gamma_n < a < \beta_n < b < \sigma_n < c < \frac{1}{\sqrt{L^2+1}+1+L^2}$. Then the sequence $\{x_n\}$ generated by algorithm (4.2.44) converges weakly to an element of Γ.

Example 4.2.5. [22] Let H be the real Hilbert space $\mathbb{R}^2$ under the usual Euclidean inner product.

If $x = (a, b) \in H$, define $x^\perp \in H$ to be $(b, -a)$. Let $K := \{x \in H : \|x\| \leq 1\}$ and set

$$K_1 := \{x \in H : \|x\| \leq \frac{1}{2}\} \quad \text{and} \quad K_2 := \{x \in H : \frac{1}{2} \leq \|x\| \leq 1\}.$$

Define $T : K \to K$ as follows:

$$Tx = \begin{cases} x + x^\perp & \text{if } x \in K_1, \\ \frac{x}{\|x\|} - x + x^\perp & \text{if } x \in K_2. \end{cases} \tag{4.2.45}$$

Then T is Lipschitz $L = 5$ and pseudocontractive and $F(T) = \{0\}$.

We next show that Example 4.2.5 satisfies all asumptions in Theorem 4.2.1 in order to show that the convergence of the iterative method defined in Theorem

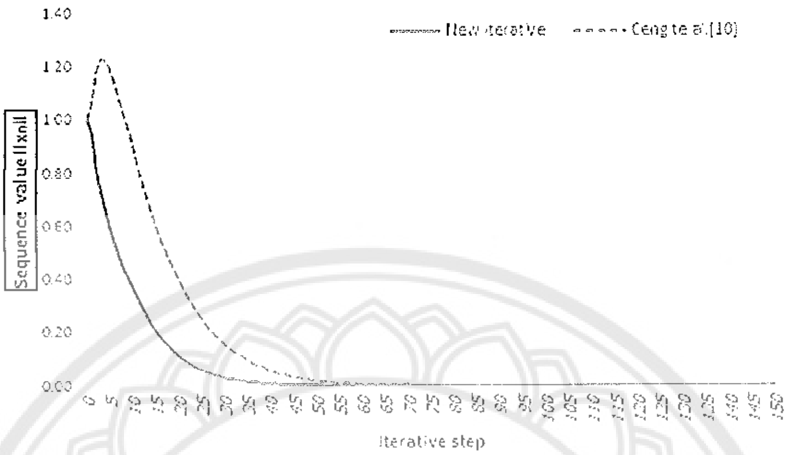
4.2.1 and compare its behavior with Ishikawa-type extragradient iterative method of Ceng et al. [21].

Example 4.2.6. Let $H_1 = H_2 = \mathbb{R}^2$ under the usual Euclidean inner product. Let $C = \{x \in H : \|x\| \leq 1\}$. and T as in Example 4.2.5 for all $x \in C$. Let $Q = \mathbb{R}^2$ and $Sx = \frac{1}{3}x$ for all $x \in \mathbb{R}^2$. Set $Ax = \frac{1}{2}x$ for all $x \in \mathbb{R}^2$. Let $\lambda_n = \frac{n+1}{n+5}$, $\alpha_n = \frac{1}{(n+1)^2}$, $\sigma_n = 0.03$, $\beta_n = 0.025$, $\gamma_n = 0.01$ for all $n \in \mathbb{N}$. It is easy to see that $\Gamma = \{0\}$. Let $x_0 = (0.8, 0.6)$, then the sequence $\{x_n\}$ generated iteratively by (4.2.1) converges to 0.

Table 1 for Example 4.2.6

No.of iteration	New iterative.	Iterative of Ceng et al. [21]
0	(0.800000, 0.600000)	(0.800000, 0.600000)
10	(0.253946, 0.249351)	(0.633949, 0.605189)
20	(0.080841, 0.079395)	(0.284389, 0.271512)
⋮	⋮	⋮
80	(0.000035, 0.000034)	(0.000514, 0.000491)
90	(0.000009, 0.000009)	(0.000164, 0.000157)
91	(0.000008, 0.000008)	(0.000146, 0.000140)
⋮	⋮	⋮
110	(0.000001, 0.000001)	(0.000016, 0.000015)
111	(0.000001, 0.000001)	(0.000014, 0.000014)
112	(0.000000, 0.000000)	(0.000013, 0.000012)

Figure 1. The convergence of $\{x_n\}$ of Theorem 4.2.1 and Theorem 3.1 [21]



The generalized Mann-type extragradient with regularization iterative method for pseudo-contractive mappings without Lipschitz assumption

We propose the generalized Mann-type extragradient with regularization iterative method for pseudo-contractive mappings without Lipschitz assumption for solving the split feasibility and fixed point problems.

Theorem 4.2.7. Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be a continuous pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\Big), \\ z_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)y_n\Big), \\ x_{n+1} = \sigma_n z_n + \beta_n T z_n + \gamma_n x_n, \ n \geq 0, \end{cases} \tag{4.2.46}$$

where $\{\lambda_n\} \subset [\kappa, \tau]$ for some $\kappa, \tau \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty), \Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $\{\gamma_n\}, \{\beta_n\}, \{\sigma_n\} \subset (a, b) \subset (0, 1)$ such that $\gamma_n + \beta_n + \sigma_n = 1$. Then the sequence $\{x_n\}$ generated by algorithm (4.2.46) converges weakly to an element of Γ.

Proof. Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in Fix(T) \cap C$ and $Ax^* \in Fix(S) \cap Q$. Set $v_n = P_Q Ax_n$, $u_n = x_n - \lambda_n(A^*(I -$

$SP_Q)Ax_n + \alpha_n I)x_n$, $\nabla f^{S\alpha_n} = A^*(I - SP_Q)A + \alpha_n I$ and $\nabla f^S = A^*(I - SP_Q)A$, for all $n \geq 0$. In the same way is proven in Theorem 4.2.1

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \lambda_n\|v_n - Ax_n\|^2 - \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \tag{4.2.47}$$

and

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2. \tag{4.2.48}$$

Likewise, we obtain that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|z_n - y_n\|^2.$$

In view of (2.3.3), (2.3.4), (4.2.47), and (4.2.48), this implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 =& \|\gamma_n x_n + \sigma_n z_n + \beta_n T z_n - x^*\|^2 \\
=& \gamma_n\|x_n - x^*\|^2 + \sigma_n\|z_n - x^*\|^2 + \beta_n\|Tz_n - x^*\|^2 \\
& - \gamma_n\sigma_n\|x_n - z_n\|^2 - \gamma_n\beta_n\|x_n - Tz_n\|^2 - \sigma_n\beta_n\|z_n - Tz_n\|^2 \\
=& \gamma_n\|x_n - x^*\|^2 + \sigma_n\|z_n - x^*\|^2 + \beta_n\big(\langle Tz_n - z_n, Tz_n - x^*\rangle \\
& + \langle z_n - x^*, Tz_n - x^*\rangle\big) - \gamma_n\sigma_n\|x_n - z_n\|^2 - \gamma_n\beta_n\|x_n - Tz_n\|^2 \\
& - \sigma_n\beta_n\|z_n - Tz_n\|^2 \\
\leq& \gamma_n\|x_n - x^*\|^2 + (\sigma_n + \beta_n)\|z_n - x^*\|^2 - \gamma_n\sigma_n\|x_n - z_n\|^2 \\
& - \gamma_n\beta_n\|x_n - Tz_n\|^2 - \sigma_n\beta_n\|z_n - Tz_n\|^2 \\
\leq& \gamma_n\|x_n - x^*\|^2 + (\sigma_n + \beta_n)\big(\|x_n - x^*\|^2 \\
& - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2\big) \\
& - \gamma_n\sigma_n\|x_n - z_n\|^2 - \gamma_n\beta_n\|x_n - Tz_n\|^2 - \sigma_n\beta_n\|z_n - Tz_n\|^2 \\
\leq& \|x_n - x^*\|^2 - (\sigma_n + \beta_n)(1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2 \\
& - \gamma_n\sigma_n\|x_n - z_n\|^2 - \gamma_n\beta_n\|x_n - Tz_n\|^2 - \sigma_n\beta_n\|z_n - Tz_n\|^2.
\end{aligned} \tag{4.2.49}$$

By the hypothesis of $\{\lambda_n\}$, we have

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|.$$

This implies that $\{\|x_n - x^*\|\}$ is a nonincreasing sequence and obtain that the limit of the sequence $\{\|x_n - x^*\|\}$ exists, we get that $\{x_n\}$ is a bounded sequence. From (4.2.49), we have

$$\begin{aligned}(\sigma_n + \beta_n)(1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2 + \gamma_n\sigma_n\|x_n - z_n\|^2 \\ + \gamma_n\beta_n\|x_n - Tz_n\|^2 + \sigma_n\beta_n\|z_n - Tz_n\|^2 \\ \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.\end{aligned}$$

By the hypothesis of the parameter σ_n, β_n and γ_n , we obtain that

$$\lim_{n\to\infty} \|x_n - z_n\| = \lim_{n\to\infty} \|x_n - Tz_n\| = \lim_{n\to\infty} \|z_n - Tz_n\| = 0,$$

and

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4.2.50}$$

Likewise, we have

$$\lim_{n\to\infty} \|z_n - y_n\| = 0.$$

Combining with (4.2.47), this implies that

$$\begin{aligned}&\lambda_n\|v_n - Ax_n\|^2 + \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \\ &\leq (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\| - \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle.\end{aligned}$$

By the hypothesis of $\{\alpha_n\}$, $\{\lambda_n\}$ and (4.2.50), it follows that

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|Sv_n - Ax_n\| = 0.$$

Therefore,

$$\lim_{n\to\infty} \|v_n - Sv_n\| = 0.$$

In the proof of Theorem 4.2.1, we get that

$$\lim_{n\to\infty} \|y_n - u_n\| = \lim_{n\to\infty} \|u_n - x_n\| = 0.$$

Consequently, all of conditions in Theorem 4.2.1 are satisfied and we can conclude that Theorem 4.2.7 can be obtained from Theorem 4.2.1 immediately. □

Similarly as before subsection, utilizing Theorem 4.2.7, we give the following Corollary when changing the generalized Mann-type iterative method is the Mann-type iterative method.

Corollary 4.2.8. Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be a continuous pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\Big), \\ z_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)y_n\Big), \\ x_{n+1} = (1 - \beta_n)z_n + \beta_n T z_n, \ n \geq 0, \end{cases} \tag{4.2.51}$$

where $\{\lambda_n\} \subset [\kappa, \tau]$ for some $\kappa, \tau \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty), \Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $\{\beta_n\} \subset (0, 1)$ such that $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. Then the sequence $\{x_n\}$ generated by algorithm (4.2.51) converges weakly to an element of Γ.

Proof. Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in Fix(T) \cap C$ and $Ax^* \in Fix(S) \cap Q$. Set $v_n = P_Q Ax_n$, $u_n = x_n - \lambda_n(A^*(I - SP_Q)Ax_n + \alpha_n I)x_n$, $\nabla f^{S\alpha_n} = A^*(I - SP_Q)A + \alpha_n I$ and $\nabla f^S = A^*(I - SP_Q)A$, for all $n \geq 0$. In the same way is proven in Theorem 4.2.1

$$\begin{aligned} \|y_n - x^*\|^2 \leq &\|x_n - x^*\|^2 - \lambda_n\|v_n - Ax_n\|^2 - \lambda_n(1 - \lambda_n\|A\|^2)\|Sv_n - Ax_n\|^2 \\ &- \lambda_n\alpha_n\langle 2(u_n - x^*) + \lambda_n\alpha_n x_n, x_n\rangle \end{aligned} \tag{4.2.52}$$

and

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|x_n - y_n\|^2. \tag{4.2.53}$$

Likewise, we obtain that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \lambda_n^2(\alpha_n + 2\|A\|^2)^2)\|z_n - y_n\|^2.$$

By (2.3.5) and (4.2.53), it follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 =&\|(1-\beta_n)z_n-\beta_nTz_n-x^*\|^2\\ =&(1-\beta_n)\|z_n-x^*\|^2+\beta_n\|Tz_n-x^*\|^2-\beta_n(1-\beta_n)\|z_n-Tz_n\|^2\\ =&(1-\beta_n)\|z_n-x^*\|^2+\beta_n\langle Tz_n-z_n,Tz_n-x^*\rangle\\ &+\beta_n\langle z_n-x^*,Tz_n-x^*\rangle-\beta_n(1-\beta_n)\|z_n-Tz_n\|^2\\ \le&\|z_n-x^*\|^2-\beta_n(1-\beta_n)\|z_n-Tz_n\|^2\\ \le&\|x_n-x^*\|^2-(1-\lambda_n^2(\alpha_n+2\|A\|^2)^2)\|x_n-y_n\|^2\\ &-\beta_n(1-\beta_n)\|z_n-Tz_n\|^2. \end{aligned} \tag{4.2.54}$$

Therefore

$$\|x_{n+1}-x^*\|\le\|x_n-x^*\|.$$

Similarly, by the process in Theorem 4.2.7, we have $\{x_n\}$ is a bounded sequence. From (4.2.54), we have

$$\begin{aligned}(1-\lambda_n^2(\alpha_n+2\|A\|^2)^2)\|x_n-y_n\|^2+\beta_n(1-\beta_n)\|z_n-Tz_n\|^2\\ \le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2.\end{aligned} \tag{4.2.55}$$

It follows that

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|z_n-Tz_n\|=0.$$

Similarly, we obtain that

$$\lim_{n\to\infty}\|z_n-y_n\|=0.$$

As the same argument of Therem 4.2.7, we get that

$$\lim_{n\to\infty}\|v_n-Ax_n\|=\lim_{n\to\infty}\|Sv_n-Ax_n\|=\lim_{n\to\infty}\|v_n-Sv_n\|=0.$$

and

$$\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|u_n-y_n\|=0.$$

Consequently, all of conditions in Theorem 4.2.7 are satisfied and we can conclude that Corollary 4.2.8 can be obtained from Theorem 4.2.7 immediately. □

Next, utilizing Theorem 4.2.7, we give the following corollary when omit $\{z_n\}$ in the iterative method of Theorem 4.2.7.

Corollary 4.2.9. Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear with its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be a continuous pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\Big), \\ x_{n+1} = \sigma_n y_n + \beta_n T y_n + \gamma_n x_n, \ n \geq 0, \end{cases} \tag{4.2.56}$$

where $\{\lambda_n\} \subset [\kappa, \tau]$ for some $\kappa, \tau \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $\{\gamma_n\}, \{\beta_n\}, \{\sigma_n\} \subset (a, b) \subset (0, 1)$ such that $\gamma_n + \beta_n + \sigma_n = 1$. Then the sequence $\{x_n\}$ generated by algorithm (4.2.56) converges weakly to an element of Γ.

Proof. Firstly, we will show that the sequence $\{x_n\}$ is bounded. Let $x^* \in \Gamma$. Then $x^* \in C \cap Fix(T)$ and $Ax^* \in Q \cap Fix(S)$. Set $v_n = P_Q Ax_n$, $u_n = x_n - \lambda_n(A^*(I - SP_Q)Ax_n + \lambda_n I)\alpha_n x_n$ for all $n \geq 0$. In the same way is proven in Theorem 4.2.7

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0$$

and

$$\lim_{n\to\infty} \|v_n - Ax_n\| = \lim_{n\to\infty} \|Sv_n - Ax_n\| = \lim_{n\to\infty} \|v_n - Sv_n\| = 0.$$

Similarly to Corollary 4.2.3,

$$\lim_{n\to\infty} \|u_n - y_n\| = \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Consequently, all of conditions in Theorem 4.2.7 are satisfied and we can conclude that Corollary 4.2.9 can be obtained from Theorem 4.2.7 immediately. □

Next, utilizing Theorem 4.2.7, we give the following corollary when define $S : H_2 \to H_2$ to be identity mapping in Theorem 4.2.7.

Corollary 4.2.10. Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let $T : C \to C$ be a continuous pseudo-contractive mapping such that $\Gamma \cap Fix(T) \neq \emptyset$. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = P_C\Big(x_n - \lambda_n(A^*(I - P_Q)A + \alpha_n I)x_n\Big), \\ z_n = P_C\Big(x_n - \lambda_n(A^*(I - P_Q)A + \alpha_n I)y_n\Big), \\ x_{n+1} = \alpha_n z_n + \beta_n T z_n + \gamma_n x_n, \ n \geq 0, \end{cases} \tag{4.2.57}$$

where $\{\lambda_n\} \subset [\kappa, \tau]$ for some $\kappa, \tau \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty} \alpha_n < \infty$ and $\{\gamma_n\}, \{\beta_n\}, \{\sigma_n\} \subset (a, b) \subset (0, 1)$ such that $\gamma_n + \beta_n + \sigma_n = 1$. Then the sequence $\{x_n\}$ generated by algorithm (4.2.57) converges weakly to an element of Γ.

Next, we give numerical example which satisfy with Theorem 4.2.7 in order to show that the convergence of the iterative process defined in Theorem 4.2.7 and compare its behavior with Mann-type extragradient iterative method of Ceng et al. [21].

Example 4.2.11. Let $H_1 = H_2 = \mathbb{R}$. Let $C = \mathbb{R}/\{-1\}$ and $Tx = -\frac{x}{(1+x)}$ for all $x \in C$. Since $\|Tx - Ty\|^2 \leq \|\frac{x-y}{(1+x)(1+y)}\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2$, then T is a continuous pseudocontractive mapping. Let $Q = \mathbb{R}$ and $Sx = \frac{1}{3}x$ for all $x \in \mathbb{R}$. Set $Ax = \frac{1}{2}x$ for all $x \in \mathbb{R}$. Let $\lambda_n = \frac{n+1}{n+5}$, $\alpha_n = \frac{1}{(n+1)^2}$, $\sigma_n = 0.6$ $\beta_n = 0.3$, $\gamma_n = 0.1$ for all $n \in \mathbb{N}$. It is easy to see that $\Gamma = \{0\}$. Let the sequence $\{x_n\}$ be generated iteratively by (4.2.46), then the sequence $\{x_n\}$ converges to 0.

Table 2 for Example 4.2.11

No.of iteration	New iterative.	Ceng et al.[21]
0	2.000000	2.000000
1	1.007543	1.369996
5	0.037257	0.164183
10	0.000267	0.005400
⋮	⋮	⋮
14	0.000005	0.000304
15	0.000002	0.000147
16	0.000001	0.000071
17	0.000000	0.000034

Figure 2. The convergence of $\{x_n\}$ of Theorem 4.2.7 and Theorem 4.1 [21]

CHAPTER V

ITERATIVE METHODS FOR SOLVING THE SPLIT EQUILIBRIUM PROBLEMS

In this chapter, we propose the parallel extragradient-proximal point methods and the parallel extragradient-proximal point methods with linesearch for solving multiple set split equilibrium problem when both equilibrium bifunctions are pseudomonotone to obtain weak and strong convergence theorems of the iterates generated by the proposed iterative methods are obtained under certain assumptions for equilibrium bifunctions and parameters. In addition, we also present a numerical example to satisfy the convergence of the proposed iterative methods.

Throughout this chapter, let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively.

5.1 Introduction and preliminaries

Given a bounded linear operator $A : H_1 \to H_2$, the split equilibrium problem (SEP) introduced by He [34] in 2010 is the following:

$$\begin{cases} \text{Find } \ x^* \in C \ \text{ such that } \ f(x^*,y) \geq 0, \ \text{ for all } \ y \in C, \\ \text{and } \ u^* = Ax^* \in Q \ \text{ such that } \ g(u^*,v) \geq 0, \ \text{ for all } \ v \in Q, \end{cases} \tag{5.1.1}$$

where $f : H_1 \times H_1 \to \mathbb{R}$ and $g : H_2 \times H_2 \to \mathbb{R}$ are bifunctions with $f(x,x) = g(u,u) = 0$ for all $x \in C$ and for all $u \in Q$, respectively.

Obviously, in problem (5.1.1), if $g = 0$ and $Q = H_2$, then the split equilibrium problem (SEP) becomes the following equilibrium problem (EP):

$$\text{Find } \ x^* \in C \ \text{ such that } \ f(x^*,y) \geq 0 \ \text{ for all } \ y \in C.$$

The most interested method for solving solution of the equilibrium problem (EP) based on the proximal point method which consists of solving a regularized equilibrium problem (REP); at the current iteration, given x_n, the next iterate x_{n+1} solves the following problem;

$$\text{Find } \ x \in C \ \text{ such that } \ f(x,y)+\frac{1}{r_n}\langle y-x, x-x_n\rangle \geq 0, \quad \text{for all } \ y \in C, \tag{5.1.2}$$

where f is a bifunction and $r_n > 0$ for all $n \in \mathbb{N}$. Observe that problems (5.1.2) is strongly monotone, if f is monotone. Hence the solution of a regularized equilibrium problem (REP) exists and is unique but when f is a generalized monotone bifunction such as a pseudomonotone bifunction, problem (5.1.2) can not be strongly monotone. So the proximal point method can not be applied to this case.

In 2015, Khatibzadeh et al. [37] solved this risen problem by using pseudomonotone bifunctions in the proximal point method for finding the solution of the equilibrium problem under different assumption and proved the weak convergence of the generated sequences to the proximal point method in Hilbert spaces.

Another method for solving the solution of the equilibrium problem (EP), Tran et al. [69] proposed the extragradient method based on the auxiliary problem principle. Given $x_0 \in C$, the sequences $\{x_n\}$ and $\{y_n\}$ generated by

$$\begin{cases} y_n = \arg\min\{\lambda f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\}, \\ x_{n+1} = \arg\min\{\lambda f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\}, \end{cases} \tag{5.1.3}$$

where λ is a suitable parameter.

The split equilibrium problem is to find a solution of the equilibrium problem such that its image under a given bounded liner operator is a solution of another equilibrium problem.

In 2012, He [34] used the proximal method for obtaining a solution of the split equilibrium problem (SEP) and introduced the following method;

$$\begin{cases} f(u_n^i, y) + \frac{1}{r_n}\langle y - u_n^i, u_n^i - x_n\rangle \geq 0, \quad \text{for all } \ y \in C, i = 1,2,...,N, \\ \tau_n = \frac{u_n^1+...+u_n^N}{N}, \\ F(w_n, z) + \frac{1}{r_n}\langle z - w_n, w_n - \tau_n\rangle \geq 0, \quad \text{for all } \ z \in Q, \\ x_{n+1} = P_C(\tau_n + \mu A^*(w_n - A\tau_n)), \end{cases} \tag{5.1.4}$$

where f and F are monotone bifunctions on C and Q, respectively. Under suitable conditions on bifunctions and parameters, He [34] proved that the sequences $\{x_n\}$ and $\{u_n^i\}$ converge weakly to a solution of the split equilibrium problem (SEP) for all $i = 1, 2, ..., N$.

Over the last several years, many researchers have related problems when f is monotone or pseudomonotone and g is monotone and constructed methods for solving the solution of split equilibrium problem (SEP), see [25, 26, 28].

Recently, Kim and Dinh [39] introduced the multiple set split equilibrium problem (MSSEP) which is stated as follows;

$$\begin{cases} \text{Find } \ x^* \in C := \bigcap_{i=1}^{N} C_i \ \text{ such that } \ f_i(x^*, y) \geq 0, \ \text{ for all } \ y \in C_i, \\ \text{and for all } \ i = 1, 2, ..., N, \ \text{ and such that} \\ \text{the point } \ u^* = Ax^* := \bigcap_{j=1}^{M} Q_j \ \text{ solves } \ g_j(u^*, v) \geq 0, \ \text{ for all } \ v \in Q_j, \\ \text{and for all } \ j = 1, 2, ..., M, \end{cases} \tag{5.1.5}$$

where $C_i \subset H_1$ for all $i = 1, 2, ..., N$, $Q_j \subset H_2$ for all $j = 1, 2, ..., M$ and $f_i : H_1 \times H_1 \to \mathbb{R}$, $g_j : H_2 \times H_2 \to \mathbb{R}$ such that $f_i(x, x) = g_j(u, u) = 0$, for all $x \in C_i$, $i = 1, 2, ..., N$ and $u \in Q_j$, $j = 1, 2, ..., M$. They proposed the iterative method by using extragradient methods for the multiple set split equilibrium problem (MSSEP) and proved the weak and strong convergence theorems in their results.

The multiple set split equilibrium problem (MSSEP) is a generalization of many important problems in applied mathematics including the multiple set split variational inequality problem (MSSVIP) introduced by Censor et al. [20], the split common fixed point problem, see [19, 43] and the split common fixed null problem, see [9, 67].

In this section, we denote the solution set of the equilibrium problem $EP(C_i, f_i)$ by

$$Sol(C_i, f_i) = \{x^* \in C_i \ : \ f_i(x^*, y) \geq 0 \text{ for all } \ y \in C_i\},$$

for all $i = 1, 2, ..., N$, and

$$Sol(Q_j, g_j) = \{u^* \in Q_j \ : \ g_j(u^*, v) \geq 0 \text{ for all } v \in Q_j\},$$

for all $j = 1, 2, ..., M$, is the solution set of the equilibrium problem $EP(Q_j, g_j)$.

Motivated and inspired by the work mentioned above, we propose iterative methods for multiple set split equilibrium problem and obtain the weak and strong convergence theorems of the proposed iterative methods. In section 5.2, we utilize the extragradient method and the proximal method for two pseudomonotone mappings in H_1 and H_2 to obtain the weak convergence theorem. Moreover, in section 5.3, we propose the iterative method by using a combination of the extragradient method with Armijo linesearch type rule for avioding Lipschitz-type continuity of bifunction in H_1 with the proximal method in H_2. In the addition, we combine this method with the shrinking projection method to obtain the strong convergence theorem. Finally we give numerical examples to demonstrate the proposed iterative methods.

5.2 Parallel extragradient-proximal point iterative methods for multiple set split equilibrium problems

In order to solving the multiple set split equilibrium problem (MSSEP), we assume that $f : H_1 \times H_1 \to \mathbb{R}$ with $f(x, x) = 0$ for all $x \in C$ satisfies the following conditions:

Assumption A

(A1) f is pseudomonotone on C with respect to $Sol(C, f)$;

(A2) $f(x, \cdot)$ is convex, lower semicontinuous and subdifferentiable on C for all $x \in C$;

(A3) f is weakly continuous on $C \times C$: that is, if $x, y \in C$ and $\{x_n\}, \{y_n\} \subset C$ converge weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$.

(A4) f is Lipschitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$.

Moreover, we assume that $g : H_2 \times H_2 \to \mathbb{R}$ with $g(u,u) = 0$ for all $u \in Q$ satisfies the following conditions:

Assumption B

(B1) g is pseudomonotone on Q;

(B2) $g(u,\cdot)$ is convex and lower semicontinuous for all $u \in Q$;

(B3) $g(\cdot,v)$ is upper semicontinuous for all $v \in Q$;

(B4) There exists $\theta \geq 0$ such that $g(u,v) + g(v,u) \leq \theta\|u - v\|^2$ for all $u, v \in Q$ (g is called undermonotone and θ is the undermonotonicity constant of g).

Lemma 5.2.1. [46] If g satisfies (B2), (B3) and (B4), then the sequence $\{x_n\}$ generated by the proximal point method is well-defined.

Lemma 5.2.2. [46] If equilibrium bifunction g satisfies (B_2), (B_3), (B_4) and assume that $\lambda > \theta$, then $EP(Q,g)$ has a unique solution.

Lemma 5.2.3. [36] If g satisfies (B1), (B2) and (B3), then the solution set of the equilibrium problem (EP) and the solution set of the convex feasibility problem (CFP) have the same solution set.

Remark that the convex feasibility problem (CFP) is a dual of the equilibrium problem (EP) i.e., finding $x^* \in Q$ such that $f(x,x^*) \leq 0$, for all $x \in Q$.

Lemma 5.2.4. [34] Let H be a real Hilbert space. Then for each $x_1, x_2, ..., x_n \in H$ and $a_1, a_2, ..., a_n \in [0,1]$ with $\sum_{i=1}^{N} a_n^i$, $n \in \mathbb{N}$ we have

$$\|\sum_{i=1}^{N} a_i x_i\|^2 = \sum_{i=1}^{N} a_i\|x_i\|^2 - \sum_{i=1}^{N-1}\sum_{l=i+1}^{N} a_i a_l\|x_i - x_l\|^2. \tag{5.2.1}$$

Lemma 5.2.5. [7] Suppose that $f_i, i = 1, 2, ..., N$, satisfies assumptions (A1), (A2), (A4) such that $\bigcap_{i=1}^{N} Sol(C_i, f_i) \neq \emptyset$. Then, for all $i = 1, 2, ..., N$, we obtain that:

(i) $\rho_n^i[f_i(x_n, y) - f_n(x_n, y_n^i)] \geq \langle y_n^i - x_n, y_n^i - y\rangle$, for all $y \in C_i$;

(ii) $\|z_n^i - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - 2\rho_n^i L_1)\|x_n - y_n^i\|^2 - (1 - 2\rho_n^i L_2)\|y_n^i - z_n^i\|^2$ for all $x^* \in \bigcap_{i=1}^N Sol(C_i, f_i)$ and $n \in \mathbb{N}$.

Lemma 5.2.6. [28] Let the equilibrium bifunction f satisfy assumptions (A1) on $Sol(C, f)$ and (A2) on C, and $x_n \subset C$, $0 < \underline{\rho} \leq \overline{\rho}$, $\{\rho_n\} \subset [\underline{\rho}, \overline{\rho}]$. Let

$$y_n = \arg\min\{f(x_n, y) + \frac{1}{2\rho_n}\|y - x_n\|^2 : y \in C\}, \text{ for all } n \in \mathbb{N}.$$

If $\{x_n\} \subset C$ is bounded, then $\{y_n\}$ is also bounded.

Next, we propose iterative method for solving multiple set split equilibrium problems such that we have motivation and inspiration from the work of Tran et al. [69] and He [34].

Algorithm 1 Parallel extragradient-proximal point methods for multiple set split equilibrium problem.

Initialization. Let $x_0 \in C = \bigcap_{i=1}^N C_i$, choose constants $0 < \underline{\rho} \leq \overline{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}, 0 < \underline{\alpha} \leq \overline{\alpha} \leq 1$. For each $i = 1, 2, ..., N$, choose parameters $\{\rho_n^i\} \subset [\underline{\rho}, \bar{\rho}]$, $\{\alpha_n^i\} \subset [\underline{\alpha}, \bar{\alpha}]$, $\Sigma_{i=1}^N \alpha_n^i = 1$ and $\mu \in (0, \frac{2}{\|A\|^2})$.

Step 1. Solve $2N$ strongly convex optimization programs in parallel

$$\begin{cases} y_n^i = \arg\min\{f_i(x_n, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\}, \\ z_n^i = \arg\min\{f_i(y_n^i, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\}, \end{cases}$$

for $i = 1, 2, ..., N$.

Step 2. Compute $\bar{z}_n = \sum_{i=1}^N \alpha_n^i z_n^i$. and $w_n^j = A\bar{z}_n$.

Step 3. Solve M regularized multiple set equilibrium programs in parallel

$$g_j(w_n^j, v) + \lambda_n^j \langle w_n^j - A\bar{z}_n, v - w_n^j\rangle \geq 0, \quad \text{for all } \ v \in Q_j, \ j = 1, 2, ..., M.$$

Step 4. Set $\bar{w}_n = \arg\max\{\|w_n^j - A\bar{z}_n\| : j = 1, 2, ..., M\}$.

Step 5. Compute $x_{n+1} = P_C(\bar{z}_n + \mu A^*(\bar{w}_n - A\bar{z}_n))$.

Set $n = n + 1$ and go back **Step 1.**

Theorem 5.2.7. Let C_i and Q_j be two closed and convex subsets of real Hilbert spaces H_1 and H_2 for all $i = 1, 2, ..., N, j = 1, 2, ..., M$, respectively. Let f_i be a bifunction satisfying assumption A on C_i for each $i = 1, 2, ..., N$ and g_j be a bifunction satisfying assumption B on Q_j for all $j = 1, 2, ..., M$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n^j\} \subset (\theta, \bar{\gamma}]$, for some $\bar{\gamma} > \theta$ for all $j = 1, 2, ..., M$. In addition the solution set

$$\Omega = \left\{ x^* \in \bigcap_{i=1}^{N} Sol(C_i, f_i) : Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j) \right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{y_n^i\}, \{z_n^i\}$, $i = 1, 2, ..., N$ converge weakly to an element $x^* \in \Omega$ and $\{w_n^j\}$, $j = 1, 2, ..., M$ converges weakly to an element $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$.

Proof. Let $x^* \in \Omega$. Then

$$x^* \in \bigcap_{i=1}^{N} Sol(C_i, f_i) \text{ and } Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j).$$

From Lemma 5.2.4 and Lemma 5.2.5 for all $i = 1, 2, ..., N$, we obtain that

$$\begin{aligned} \|\bar{z}_n - x^*\|^2 &= \|\sum_{i=1}^{N} \alpha_n^i z_n^i - x^*\|^2 = \|\sum_{i=1}^{N} \alpha_n^i (z_n^i - x^*)\|^2 \\ &= \sum_{i=1}^{N} \alpha_n^i \|z_n^i - x^*\|^2 - \sum_{i=1}^{N-1} \sum_{l=i+1}^{N} \alpha_n^i \alpha_n^l \|z_n^i - z_n^l\|^2 \\ &\leq \|x_n - x^*\|^2 - \sum_{i=1}^{N} \alpha_n^i (1 - 2\rho_n^i L_1) \|x_n - y_n^i\|^2 \\ &- \sum_{i=1}^{N} \alpha_n^i (1 - 2\rho_n^i L_2) \|y_n^i - z_n^i\|^2. \end{aligned} \tag{5.2.2}$$

Suppose that $j_n \in \{1, 2, ..., M\}$ such that $\bar{w}_n = w_n^{j_n}$. Since $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$, we have

$$g_j(w_n^{j_n}, Ax^*) + \lambda_n^{j_n} \langle w_n^{j_n} - A\bar{z}_n, Ax^* - w_n^{j_n} \rangle \geq 0.$$

By Lemma 5.2.3, every element of $\bigcap_{j=1}^{M} Sol(Q_j, g_j)$ can solve $CFP(Q_j, g_j)$ for all $j = 1, 2, ..., M$, hence $g_j(w_n^{j_n}, Ax^*) \leq 0$ for all $n \in \mathbb{N}$ and $j_n = 1, 2, ..., M$. It follows that

$$\langle w_n^{j_n} - A\bar{z}_n, Ax^* - w_n^{j_n} \rangle \geq 0.$$

Since

$$\langle w_n^{j_n} - A\bar{z}_n, Ax^* - w_n^{j_n} \rangle \leq \frac{1}{2}\{-\|w_n^{j_n} - Ax^*\|^2 + \|A\bar{z} - Ax^*\|^2 - \|w_n^{j_n} - A\bar{z}\|^2\},$$

we obtain that

$$\|Ax^* - A\bar{z}_n\|^2 - \|w_n^{j_n} - A\bar{z}_n\|^2 - \|Ax^* - w_n^{j_n}\|^2 \geq 0.$$

This implies that

$$\|Ax^* - w_n^{j_n}\|^2 \leq \|Ax^* - A\bar{z}_n\|^2 - \|w_n^{j_n} - A\bar{z}_n\|^2. \tag{5.2.3}$$

Likewise, we get that

$$\langle A(\bar{z}_n - x^*), w_n^{j_n} - A\bar{z}_n \rangle = \frac{1}{2}\{\|w_n^{j_n} - Ax^*\|^2 - \|A\bar{z}_n - Ax^*\|^2 - \|w_n^{j_n} - A\bar{z}_n\|^2\}.$$

Combining with (5.2.3) and previous inequation, we obtain that

$$\langle A(\bar{z}_n - x^*), w_n^{j_n} - A\bar{z}_n \rangle \leq -\|w_n^{j_n} - A\bar{z}_n\|^2.$$

It follows that

$$\langle A(\bar{z}_n - x^*), \bar{w}_n - A\bar{z}_n \rangle \leq -\|\bar{w}_n - A\bar{z}_n\|^2. \tag{5.2.4}$$

Because of the relation (5.2.4) and the definition of x_{n+1}, this implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 =& \|P_C(\bar{z}_n + \mu A^*(\bar{w}_n - A\bar{z}_n)) - P_C x^*\|^2 \\
\leq& \|\bar{z}_n - x^* + \mu A^*(\bar{w}_n - A\bar{z}_n)\|^2 \\
=& \|\bar{z}_n - x^*\|^2 + \mu^2\|A^*(\bar{w}_n - A\bar{z}_n)\|^2 + 2\mu\langle \bar{z}_n - x^*, A^*(\bar{w}_n - A\bar{z}_n)\rangle \\
=& \|\bar{z}_n - x^*\|^2 + \mu^2\|A\|^2\|\bar{w}_n - A\bar{z}_n\|^2 + 2\mu\langle A(\bar{z}_n - x^*), \bar{w}_n - A\bar{z}_n\rangle \\
\leq& \|\bar{z}_n - x^*\|^2 + \mu^2\|A\|^2\|\bar{w}_n - A\bar{z}_n\|^2 - 2\mu\|\bar{w}_n - A\bar{z}_n\|^2
\end{aligned}$$

$$=\|\bar{z}_n - x^*\|^2 - \mu(2-\mu\|A\|^2)\|\bar{w}_n - A\bar{z}_n\|^2. \tag{5.2.5}$$

In combination with (5.2.2), we get that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 =& \|\bar{z}_n - x^*\|^2 - \mu(2-\mu\|A\|^2)\|\bar{w}_n - A\bar{z}_n\|^2\\ \leq& \|x_n - x^*\|^2 - \sum_{n=1}^{N} \alpha_n^i(1-2\rho_n^i L_1)\|x_n - y_n^i\|^2\\ &- \sum_{i=1}^{N} \alpha_n^i(1-2\rho_n^i L_2)\|y_n^i - z_n^i\|^2\\ &- \mu(2-\mu\|A\|^2)\|\bar{w}_n - A\bar{z}_n\|^2.\end{aligned} \tag{5.2.6}$$

By the hypothesis of $\mu \in (0, \frac{2}{\|A\|^2})$, $\{\rho_n^i\} \subset [\underline{\rho}, \bar{\rho}]$, it follows that

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2.$$

This implies that $\{\|x_n - x^*\|\}$ is a nonincreasing sequence, we obtain that the limit of the sequence $\{\|x_n - x^*\|\}$ exists. From (5.2.6), we get that

$$\begin{aligned}&\sum_{n=1}^{N} \alpha_n^i(1-2\rho_n^i L_1)\|x_n - y_n^i\|^2\\ &+ \sum_{i=1}^{N} \alpha_n^i(1-2\rho_n^i L_2)\|y_n^i - z_n^i\|^2 + \mu(2-\mu\|A\|^2)\|\bar{w}_n - A\bar{z}_n\|^2\\ \leq& \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.\end{aligned} \tag{5.2.7}$$

Therefore

$$\lim_{n\to\infty} \|x_n - y_n^i\| = \lim_{n\to\infty} \|y_n^i - z_n^i\| = 0, \text{ for all } i = 1, 2, ..., N, \tag{5.2.8}$$

and

$$\lim_{n\to\infty} \|\bar{w}_n - A\bar{z}_n\| = 0, \tag{5.2.9}$$

by the hypothesis of $\bar{w}_n$, we get

$$\lim_{n\to\infty} \|w_n^{j_n} - A\bar{z}_n\| = 0, \text{ for all } j = 1, 2, ..., M. \tag{5.2.10}$$

Since the limit of $\{\|x_n - x^*\|\}$ exists, $\{x_n\}$ is a bounded sequence. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to x^* as

$k \to \infty$. From Lemma 5.2.6, $\{y_n^i\}$ is a bounded sequence and $\{z_n^i\}$ is also a bounded sequence. It obtain that $\{y_{n_k}^i\}$ and $\{z_{n_k}^i\}$ converge weakly to x^* for all $i = 1, 2, ..., N$. Hence from (5.2.2), $\{\bar{z}_{n_k}\}$ converges weakly to x^*. Consequently $\{A\bar{z}_{n_k}\}$ converges weakly to Ax^*. By (5.2.10), we have $\{w_{n_k}^{j_{n_k}}\}$ converges weakly to Ax^* as $k \to \infty$ for all $j = 1, 2, ..., M$.

Note that $x_n \in C = \bigcap_{i=1}^N C_i$, since C_i is a closed and convex set for all $i = 1, 2, ..., N$, we have C is also a closed and convex set. So $x^* \in C$ that is $x^* \in C_i$ for all $i = 1, 2, ..., N$. Similarly, Q_j is a closed and convex set for all $j = 1, 2, ..., M$. So $Ax^* \in Q$ that is $Ax^* \in Q_j$ for all $j = 1, 2, ..., M$. From Lemma 5.2.5, for all $i = 1, 2, ..., N$, this implies that

$$\rho_{n_k}^i [f_i(x_{n_k}, y) - f_i(x_{n_k}, y_{n_k}^i)] \geq \langle y_{n_k}^i - x_{n_k}, y_{n_k}^i - y \rangle,$$

for all $y \in C_i, i = 1, 2, ..., N$. Since

$$\langle y_{n_k}^i - x_{n_k}, y_{n_k}^i - y \rangle \geq -\|y_{n_k}^i - x_{n_k}\| \|y_{n_k}^i - y\|,$$

we obtain that

$$\rho_{n_k}^i [f_i(x_{n_k}, y) - f_i(x_{n_k}, y_{n_k}^i)] \geq -\|y_{n_k}^i - x_{n_k}\| \|y_{n_k}^i - y\|.$$

Hence

$$f_i(x_{n_k}, y) - f_i(x_{n_k}, y_{n_k}^i) \geq -\frac{1}{\rho_{n_k}^i} \|y_{n_k}^i - x_{n_k}\| \|y_{n_k}^i - y\|.$$

Taking the limit $k \longrightarrow \infty$, we get that

$$f_i(x^*, y) - f_i(x^*, x^*) \geq 0 \text{ for all } y \in C_i, i = 1, 2, ..., N.$$

Therefore $x^* \in \bigcap_{i=1}^N Sol(C_i, f_i)$. In addition, for each $j = 1, 2, ..., M$, by using Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned} 0 &\leq g_j(w_n^j, v) + \lambda_n^j \langle w_n^j - A\bar{z}_n, v - w_n^j \rangle \\ &\leq g_j(w_n^j, v) + \lambda_n^j \|w_n^j - A\bar{z}_n\| \|v - w_n^j\|. \end{aligned}$$

From the hypothesis of $\{\lambda_n^j\}$ and $\{\|v - w_n^j\|\}$ are bounded sequences for all $j = 1, 2, ..., M$, we have

$$0 \leq \liminf_{n \longrightarrow \infty} g_j(w_n^j, v), \ \text{ for all } \ v \in Q_j, j = 1, 2, ..., M. \tag{5.2.11}$$

Under upper semicontinuity of $g(\cdot, v)$, $w_{n_k}^{j_{n_k}} \rightharpoonup Ax^*$ for all $j = 1, 2, ..., M$ and (5.2.11), we obtain that

$$g_j(Ax^*, v) \geq \limsup_{n \to \infty} g_j(w_{n_k}^{j_{n_k}}, v) \geq 0,$$

for all $v \in Q_j$, $j = 1, 2, ..., M$. Therefore $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$. Finally, we prove that $\{x_n\}$ converges weakly to x^*. Suppose that there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \rightharpoonup \bar{x}$ with $\bar{x} \neq x^*$. By Opial's condition, we obtain that

$$\begin{aligned}
\liminf_{m \to \infty} \|x_{n_m} - \bar{x}\| &< \liminf_{m \to \infty} \|x_{n_m} - x^*\| \\
&= \liminf_{k \to \infty} \|x_{n_k} - x^*\| \\
&< \liminf_{k \to \infty} \|x_{n_k} - \bar{x}\| \\
&= \liminf_{m \to \infty} \|x_{n_m} - \bar{x}\|.
\end{aligned}$$

This is a contradiction and so $\{x_n\}$ converges weakly to x^*.

From (5.2.8), we also have $y_n^i \rightharpoonup x^*$, $z_n^i \rightharpoonup x^*$, for all $i = 1, 2, ..., N$. Therefore $\bar{z}_n \rightharpoonup x^*$ and $A\bar{z}_n \rightharpoonup Ax^*$. Consequently by (5.2.10), $w_n^{j_n} \rightharpoonup Ax^*$ for all $j = 1, 2, ..., M$. □

The following result is an immediate consequence of Theorem 5.2.7 when $N = M = 1$. Then $C_1 = C$ and $Q_1 = Q$, we get the following corollary.

Corollary 5.2.8. Let C and Q be two closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let f be a bifunction satisfying assumption A on C and g be a bifunction satisfying assumption B on Q. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n\} \subset (\theta, \bar{\gamma}]$, for some $\bar{\gamma} > \theta$.

Take $x_0 \in C$, $\{\rho_n\} \subset [\underline{\rho}, \bar{\rho}]$ such that $0 < \underline{\rho} \le \overline{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_n\} \subset [\underline{\alpha}, \bar{\alpha}]$ such that $0 < \underline{\alpha} \le \overline{\alpha} \le 1$ and $\mu \in (0, \frac{2}{\|A\|^2})$. Consider the sequences

$$\begin{cases} y_n = \arg\min\{\rho_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\} \\ z_n = \arg\min\{\rho_n f(y_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\} \\ g(w_n, v) + \lambda_n\langle w_n - Az_n, v - w_n\rangle \ge 0, \quad \text{for all} \quad v \in Q \\ x_{n+1} = P_C(z_n + \mu A^*(w_n - Az_n)). \end{cases}$$

If the solution set

$$\Omega = \left\{ x^* \in Sol(C, f) : Ax^* \in Sol(Q, g) \right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ converge weakly to an element $x^* \in \Omega$ and $\{w_n\}$ converges weakly to an element $Ax^* \in Sol(Q, g)$.

We now present the following example for supporting Theorem 5.2.7.

Example 5.2.9. We apply a Nash-Cournot oligopolistic equilibrium problem result arising in electricity markets and oil markets with **Algorithm 1** to solve the solution of the multiple set of split equilibrium promblem. These problem has been investigated as in [46, 35, 57, 38, 39].

Consider $H_1 = \mathbb{R}^N$, and $H_2 = \mathbb{R}^M$ by $2N = M$. We define

$$C_i = \{x \in \mathbb{R}^N : \ 0 \le x_m \le 80, \ m = 1, 2, ..., N\},$$

and

$$Q_j = \{u \in \mathbb{R}^M : \ 0 \le u_t \le 50, \ t = 1, 2, ..., M\}.$$

Let $A : \mathbb{R}^N \to \mathbb{R}^M$ be given by

$$A(x_1, x_2, x_3, ..., x_{N-1}, x_N) = (\frac{x_1}{2}, x_1 - \frac{x_1}{2}, \frac{x_2}{2}, ..., \frac{x_N}{2}, x_N - \frac{x_N}{2}).$$

The bifunction $f_i, i = 1, 2, ..., N^C$ is given as follows

$$f_i(x, y) = (B_i x + D_i y + b_i)^T(y - x) + c_i(y) - c_i(x),$$

where $B_i, D_i \in \mathbb{R}^{N\times N}$ are symmetric positive semidefinite matrices such that $B_i - D_i$ is also a positive semidefinite matrix, $c_i \in \mathbb{R}^N$ for all $i = 1, 2, ..., N^C$. The bifunction f_i has the form of the one arising from a Nash-Cournot oligopolistic electricity markets equilibrium problem model, see [39, 57] which is equivalent to the profit function f of electricity companies i, that is

$$f_i(x) = p(x)\Sigma_{m\in I_i} x_m - \Sigma_{m\in I_i} c_m(x_m),$$

where $p(x) = 378.4 - 2\Sigma_{m\in I_i} x_m$ is price function of company i, and $c_m(x_m)$ is the cost for generating x_m, we define cost function of generating unit x_m by

$$c_m(x_m) = \max\{c_m^0(x_m), c_m^1(x_m)\}$$

where

$$c_m^0(x_m) = \frac{\alpha_m^0}{2}x_m^2 + \beta_m^0 x_m + \gamma_m^0, \;\; c_m^1(x_m) = \alpha_m^1 x_m + \frac{\beta_m^1}{\beta_m^1 + 1}\gamma_m^{-1/\beta_m^1}(x_m)^{(\beta_m^1+1)/\beta_m^1}.$$

Denote that I_i is the index set of generating unit m and x_m is power of generating unit m.

In this case, the bifunction f_i satisfies the condition of **Assumption A** with Lipschitz-type continuity with constant $L_1^i = L_2^i = \frac{1}{2}\|A_i - B_i\|$ see([35], Lemma 6.2). We choose $L_1 = \max\{L_1^i, i = 1, 2, ...N^C\}$ and $L_2 = \max\{L_2^i, i = 1, 2, ...N^C\}$ and $\rho_n^i = \rho = \frac{1}{4L}$, with $L = \max\{L_1, L_2\}$. By setting $q_i^m = (1, 1, ..., 1)^T \in \mathbb{R}^N$ and define

$$U_i = 2\Sigma_{m=1}^N (1 - q_i^m)(q_i^m)^T, \;\; V_i = \Sigma_{j=1}^N q_i^m (q_i^m)^T,$$

$$b_i = -387.4\Sigma_{m=1}^N q_i^m, \;\text{ and }\; c_i(x) = \Sigma_{m=1}^N c_m(x_m).$$

Then we set $B_i = (U_i + (3/2)V_i)$ and $D_i = (1/2)V_i$, for to show that f is satisfies **Assumption A**, for more detail see [43].

The bifunction g_j has the form of the one arising from a Nash-Cournot oligopolistic oil markets equilibrium problem model, see [38] such that it is the profit function of oil companies j, it is given as follows

$$g_j(u, v) = (E_j u + F_j v + d_j)^T (v - u) + d_j(v) - d_j(u),$$

where $E_j, F_j \in \mathbb{R}^{M\times M}$ are symmetric positive semidefinite matrices such that $E_j - F_j$ is also a positive semidefinite matrix, $d_j \in \mathbb{R}^M$ for all $j = 1, 2, ..., M^Q$. By setting $q_j^t = (1, 1, ..., 1)^T \in \mathbb{R}^{1\times M}$ and define

$$H_j = 2\Sigma_{t=1}^{M}(1 - q_j^t)(q_j^t)^T, \;\; T_j = \Sigma_{t=1}^{M} q_j^t (q_j^t)^T,$$

$$d_j = -8750\Sigma_{t=1}^{M} q_j^t, \;\text{ and } c_j(u) = \Sigma_{t=1}^{M} c_t(u_t).$$

Then we set $E_j = (H_j + T_j)$, $F_j = (1/2)T_j$ for to satisfies **Assumption B.**

Finally, we use the two bifunctions f_i and g_j for our main result in Theorem 5.2.7. Set $N^C = 3$, $M^Q = 1$, $N = 6$, $M = 12$, $\rho_n^i = 2$ for all $i = 1, 2, 3$. Since $\theta = 0$, we can set

$$\lambda_n^j = \begin{cases} |g_j(w_n^j, v) + g_j(v, w_n^j)|/\|w_n^j - v\|^2 \text{ if } w_n^j \neq v, \\ 0.5 \text{ if } w_n^j = v, \end{cases}$$

for all $j = 1$, $\alpha_n^1 = 0.4$, $\alpha_n^2, \alpha_n^3 = 0.3$, $x_0 = (30, 20, 10, 15, 10, 10) \in C = \cap_{i=1}^{3} C_i$ and set $u_t^{1\max} = 50$ for $t = 1, 2, ..., 12$.

Table1. The power generation of the generating unit of each companies

Com.	Gen	x_{min}	x_{max}	Com	Gen	x_{min}	x_{max}	Com	Gen	x_{min}	x_{max}
1	1	0	80	2	1	0	50	3	1	0	60
	2	0	80		2	0	80		2	0	70
	3	0	50		3	0	50		3	0	45
	4	0	55		4	0	45		4	0	55
	5	0	30		5	0	50		5	0	50
	6	0	40		6	0	55		6	0	55

Table2.

The parameters of the cost functions of the generating unit of each companies.

Com.	Gen	α_j^0	β_j^0	γ_j^0	α_j^1	β_j^1	γ_j^0
1	1	0.0400	2.0000	0	2.0000	1	25.0000
	2	0.0350	1.7500	0	1.7500	1	28.5714
	3	0.1250	1.0000	0	1.0000	1	8.0000
	4	0.0166	3.2500	0	3.2500	1	86.2069
	5	0.0500	3.0000	0	3.0000	1	20.0000
	6	0.0500	3.0000	0	3.0000	1	20.0000
2	1	0.0300	3.0000	0	2.0000	1	35.0000
	2	0.0550	1.5500	0	1.7500	1	28.5714
	3	0.0250	1.0000	0	1.0000	1	7.0000
	4	0.2166	3.1500	0	3.2500	1	76.2069
	5	0.0500	2.0000	0	3.0000	1	15.0000
	6	0.0400	3.0000	0	3.0000	1	20.0000
3	1	0.0500	1.0000	0	2.0000	1	29.0000
	2	0.1350	1.9500	0	1.7500	1	27.6714
	3	0.0250	0.8900	0	1.0000	1	5.0000
	4	0.1166	2.2500	0	3.2500	1	90.0069
	5	0.0400	4.0000	0	3.0000	1	15.0000
	6	0.0300	3.0000	0	3.0000	1	25.0000

We implement **Algorithm1** in Matlab R2015b running on a Desktop with Intel(R) Core(TM) i5-4200U CPU with 1.60GHz 2.30GHz, 4 GB Ram. We use the stopping criteria $\frac{\|x_{n+1}-x_n\|}{\max\{1,\|x_n\|\}} \leq \varepsilon$ for terimanation of the algorithm. The Table 3. is the computation results with $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-5}$.

Table3. the break point of each company

iter.	error	μ	Com	x_1	x_2	x_3	x_4	x_5	x_6	Cpu(s)
14	10^{-3}	0.25	1	19.0942	19.0935	19.0909	19.0902	19.0900	19.0903	23.6719
			2	19.0935	19.0933	19.0906	19.0866	19.0897	19.0904	
			3	19.0944	19.0915	19.0895	19.0894	19.0892	19.0906	
			all	20.0197	20.0191	20.0179	20.0172	20.0176	20.0180	
23	10^{-5}	0.25	1	19.0966	19.0968	19.0951	19.0940	19.0942	19.0946	53.0156
			2	19.0959	19.0966	19.0948	19.0903	19.0940	19.0946	
			3	19.0968	19.0948	19.0938	19.0931	19.0935	19.0948	
			all	20.0215	20.0213	20.0206	20.0196	20.0203	20.0206	

We apply Nash-Cournot oligopolistic equilibrium problem result arising in electricity markets and oil markets to solve the split equilibrium problem, which means that we are finding break even point of the trading between electricity companies and oil company. For our main results, we can find break even point of each companies such that the previous research, see [57, 28, 38] can find break even point of total companies but our main results must set the amount of electricity units/oil wells of each companies which are all equal.

Next, we prove a strong convergence theorem of hybrid parallel extragradient-proximal point methods by using previous iterative method with shrink projection method.

Algorithm 2 Hybrid parallel extragradient-proximal point methods for multiple set split equilibrium problem.

Initialization. Let $x_0 \in C = \bigcap_{i=1}^{N} C_i$, choose constants $0 < \underline{\rho} \leq \overline{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $0 < \underline{\alpha} \leq \overline{\alpha} \leq 1$. For each $i = 1, 2, ..., N$, choose parameters $\{\rho_n^i\} \subset [\underline{\rho}, \overline{\rho}]$, $\{\alpha_n^i\} \subset [\underline{\alpha}, \bar{\alpha}]$, $\Sigma_{i=1}^{N} \alpha_n^i = 1$ and $\mu \in (0, \frac{2}{\|A\|^2})$.

Step 1. Solve $2N$ strongly convex optimization programs in parallel

$$\begin{cases} y_n^i = \arg\min\{f_i(x_n, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\}, \\ z_n^i = \arg\min\{f_i(y_n^i, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\}, \end{cases}$$

for $i = 1, 2, ..., N$.

Step 2. Compute $\bar{z}_n = \sum_{i=1}^{N} \alpha_n^i z_n^i$.

Step 3. Solve M regularized multiple set equilibrium programs in parallel

$$g_j(w_n^j, y) + \lambda_n^j \langle w_n^j - A\bar{z}_n, y - w_n^j \rangle \geq 0, \text{ for all } y \in Q_j, \; j = 1, 2, ..., M.$$

Step 4. Set $\bar{w}_n = \arg\max\{\|w_n^j - A\bar{z}_n\| : j = 1, 2, ..., M\}$.

Step 5. Compute $t_n = P_C(\bar{z}_n + \mu A^*(\bar{w}_n - A\bar{z}_n))$.

Step 6. Take $x_{n+1} = P_{C_{n+1}}(x_0)$, where

$$C_{n+1} = \{v \in H : \|t_n - v\| \leq \|\bar{z}_n - v\| \leq \|x_n - v\|\}.$$

Set $n = n + 1$ and go back **Step 1.**

Theorem 5.2.10. Let C_i and Q_j be two closed and convex subsets of real Hilbert spaces H_1 and H_2 for all $i = 1, 2, ..., N, j = 1, 2, ..., M$, respectively. Let f_i be a bifunction satisfying assumption A on C_i for all $i = 1, 2, ..., N$ and g_j be a bifunction satisfying assumption B on Q_j for all $j = 1, 2, ..., M$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n^j\} \subset (\theta, \bar{\gamma}]$, for some $\bar{\gamma} > \theta$ for all $j = 1, 2, ..., M$. In addition the solution set

$$\Omega = \left\{ x^* \in \bigcap_{i=1}^{N} Sol(C_i, f_i) : Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j) \right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{y_n^i\}, \{z_n^i\}$, $i = 1, 2, ..., N$ converge strongly to an element $x^* \in \Omega$ and $\{w_n^j\}$, $j = 1, 2, ..., M$ converges strongly to an element $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$.

Proof. Firstly, we observe that C_{n+1} is a nonempty closed convex set for all $n \in \mathbb{N}$. Let $x^* \in \Omega$. Then $x^* \in \bigcap_{i=1}^{N} Sol(C_i, f_i)$ and $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$. For each $i = 1, 2, ..., N$, by the proof of Theorem 5.2.7, we have

$$\|\bar{z}_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \sum_{i=1}^{N} \alpha_n^i (1 - 2\rho_n^i L_1)\|x_n - y_n^i\|^2 - \sum_{i=1}^{N} \alpha_n^i (1 - 2\rho_n^i L_2)\|y_n^i - z_n^i\|^2. \tag{5.2.12}$$

Suppose that $j_n \in \{1, 2, ..., M\}$ such that $\overline{w}_n = w_n^{j_n}$. Since $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$, we have

$$g_j(w_n^{j_n}, Ax^*) + \lambda_n^{j_n} \langle w_n^{j_n} - A\bar{z}_n, Ax^* - w_n^{j_n} \rangle \geq 0.$$

By the same process of a proof in Theorem 5.2.7, we get that

$$\|Ax^* - w_n^{j_n}\|^2 \leq \|Ax^* - A\bar{z}_n\|^2 - \|w_n^{j_n} - A\bar{z}_n\|^2, \tag{5.2.13}$$

and

$$\langle A(\bar{z}_n - x^*), \bar{w}_n - A\bar{z}_n \rangle \leq -\|\bar{w}_n - A\bar{z}_n\|^2. \tag{5.2.14}$$

By the definition of t_n and (5.2.14), this implies that

$$\begin{aligned}
\|t_n - x^*\|^2 &= \|P_C(\bar{z}_n + \mu A^*(\bar{w}_n - A\bar{z}_n)) - P_C x^*\|^2 \\
&\leq \|\bar{z}_n - x^* + \mu A^*(\bar{w}_n - A\bar{z}_n)\|^2 \\
&= \|\bar{z}_n - x^*\|^2 + \mu^2 \|A^*(\bar{w}_n - A\bar{z}_n)\|^2 + 2\mu \langle \bar{z}_n - x^*, A^*(\bar{w}_n - A\bar{z}_n) \rangle \\
&= \|\bar{z}_n - x^*\|^2 + \mu^2 \|A\|^2 \|\bar{w}_n - A\bar{z}_n\|^2 + 2\mu \langle A(\bar{z}_n - x^*), \bar{w}_n - A\bar{z}_n \rangle \\
&\leq \|\bar{z}_n - x^*\|^2 + \mu^2 \|A\|^2 \|\bar{w}_n - A\bar{z}_n\|^2 - 2\mu \|\bar{w}_n - A\bar{z}_n\|^2 \\
&= \|\bar{z}_n - x^*\|^2 - \mu(2 - \mu\|A\|^2)\|\bar{w}_n - A\bar{z}_n\|^2.
\end{aligned} \tag{5.2.15}$$

By hypothesis of μ, it follows that

$$\|t_n - x^*\|^2 \leq \|\bar{z}_n - x^*\|^2. \tag{5.2.16}$$

Combining (5.2.12), we get that

$$\|t_n - x^*\|^2 \leq \|\bar{z}_n - x^*\|^2 \leq \|x_n - x^*\|^2. \tag{5.2.17}$$

Therefore $\Omega \subset C_{n+1}$. From the definition of C_n, it implies that $x_n = P_{C_n}(x_0)$. By Proposition 2.3.4 and $x_{n+1} \in C_{n+1}$, we have

$$\|x_{n+1} - x_n\|^2 + \|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2, \tag{5.2.18}$$

it follows that

$$\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2.$$

Similarly to $x_{n+1} \in P_{C_{n+1}}(x_0)$ and $x^* \in C_{n+1}$, we get

$$\|x_{n+1} - x_0\|^2 \leq \|x^* - x_0\|^2.$$

This implies that

$$\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2 \leq \|x^* - x_0\|^2. \tag{5.2.19}$$

Consequently, the sequence $\{\|x_n - x_0\|\}$ is a nondecreasing and bounded sequence. Then the limit of the sequence $\{\|x_n - x_0\|\}$ exists. It follows that $\{x_n\}$ is bounded and by Lemma 5.2.6 obtain that $\{y_n^i\}$ is also bounded for all $i = 1, 2, ..., N$. From (5.2.18) and the limit of the sequence $\{\|x_n - x_0\|\}$ exists, we get that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Next, we will show that $x^* \in \Omega$. Since $x_{n+1} \in C_{n+1}$, we obtain that

$$\|x_{n+1} - t_n\| \leq \|x_{n+1} - x_n\|.$$

Since

$$\begin{aligned} \|t_n - x_n\| &\leq \|t_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq 2\|x_{n+1} - x_n\|, \end{aligned} \tag{5.2.20}$$

it follows that

$$\lim_{n\to\infty} \|x_n - t_n\| = 0. \tag{5.2.21}$$

In view of (5.2.12) and (5.2.15), we get that

$$\begin{aligned}\|t_n - x^*\|^2 &= \|\bar{z}_n - x^*\|^2 - \mu(2-\mu\|A\|^2)\|\bar{w}_n - A\bar{z}_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \sum_{n=1}^{N}\alpha_n^i(1-2\rho_n^i L_1)\|x_n - y_n^i\|^2 \\ &- \sum_{i=1}^{N}\alpha_n^i(1-2\rho_n^i L_2)\|y_n^i - z_n^i\|^2 - \mu(2-\mu\|A\|^2)\|\bar{w}_n - A\bar{z}_n\|^2.\end{aligned} \tag{5.2.22}$$

This implies that

$$\begin{aligned}&\sum_{n=1}^{N}\alpha_n^i(1-2\rho_n^i L_1)\|x_n - y_n^i\|^2 \\ &\quad + \sum_{i=1}^{N}\alpha_n^i(1-2\rho_n^i L_2)\|y_n^i - z_n^i\|^2 + \mu(2-\mu\|A\|^2)\|\bar{w}_n - A\bar{z}_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \|t_n - x^*\|^2 \\ &\leq (\|x_n - x^*\| + \|t_n - x^*\|)\|x_n - t_n\|.\end{aligned} \tag{5.2.23}$$

Taking the limit $n \to \infty$ and combining (5.2.21), we have

$$\lim_{n\to\infty}\|x_n - y_n^i\| = \lim_{n\to\infty}\|y_n^i - z_n^i\| = 0, \text{ for all } i = 1,2,...,N, \tag{5.2.24}$$

and

$$\lim_{n\to\infty}\|\bar{w}_n - A\bar{z}_n\| = 0. \tag{5.2.25}$$

By hypothesis of $\bar{w}_n$, we get

$$\lim_{n\to\infty}\|w_n^{j_n} - A\bar{z}_n\| = 0, \text{ for all } j = 1,2,...,M. \tag{5.2.26}$$

Next, we will show that any weak accumulation of $\{x_n\}$ belongs to Ω. Suppose that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to x^* that is $x_{n_k} \rightharpoonup x^*$. Since $\{x_n\}$ is a bounded sequence, by Lemma 5.2.6, $\{y_n^i\}$ is a bounded sequence, it follows that $\{z_n^i\}$ is a bounded sequence. We get $y_{n_k}^i \rightharpoonup x^*$, $z_{n_k}^i \rightharpoonup x^*$ for all $i = 1,2,...,N$, and by (5.2.13), (5.2.25), we obtain that $w_{n_k}^{j_{n_k}} \rightharpoonup Ax^*$ for all $j = 1, 2, ..., M$. From Lemma 5.2.5, for each $i = 1, 2, ..., N$, this implies that

$$\rho_{n_k}^i [f_i(x_{n_k}, y) - f_i(x_{n_k}, y_{n_k}^i)] \geq \langle y_{n_k}^i - x_{n_k}, y_{n_k}^i - y \rangle, \text{ for all } y \in C_i, i = 1, 2, ..., N.$$

Since

$$\langle y_{n_k}^i - x_{n_k}, y_{n_k}^i - y \rangle \geq -\|y_{n_k}^i - x_{n_k}\| \|y_{n_k}^i - y\|,$$

we obtain that

$$\rho_{n_k}^i [f_i(x_{n_k}, y) - f_i(x_{n_k}, y_{n_k}^i)] \geq -\|y_{n_k}^i - x_{n_k}\| \|y_{n_k}^i - y\|.$$

Hence

$$f_i(x_{n_k}, y) - f_i(x_{n_k}, y_{n_k}^i) \geq -\frac{1}{\rho_{n_k}^i} \|y_{n_k}^i - x_{n_k}\| \|y_{n_k}^i - y\|.$$

Taking the limit $k \to \infty$, we get from

$$f_i(x^*, y) - f_i(x^*, x^*) \geq 0 \text{ for all } y \in C_i, i = 1, 2, ..., N.$$

Therefore $x^* \in \bigcap_{i=1}^N Sol(C_i, f_i)$. In addition, for each $j = 1, 2, ..., M$, by using Cauchy-Schwartz inequality, we get that

$$\begin{aligned} 0 &\leq g_j(w_n^j, v) + \lambda_n^j \langle w_n^j - A\bar{z}_n, v - w_n^j \rangle \\ &\leq g_j(w_n^j, v) + \lambda_n^j \|w_n^j - A\bar{z}_n\| \|v - w_n^j\|. \end{aligned} \tag{5.2.27}$$

From hypothesis of $\{\lambda_n^j\}$ and $\{\|v - w_n^j\|\}$ are bounded sequences for all $j = 1, 2, ..., M$, we see that

$$0 \leq \liminf_{n \to \infty} g_j(w_n^j, v), \text{ for all } v \in Q_j, j = 1, 2, ..., M. \tag{5.2.28}$$

Under upper semicontinuity of $g(\cdot, v)$, $w_{n_k}^{j_{n_k}} \rightharpoonup Ax^*$ for all $j = 1, 2, ..., M$ and (5.2.28), we obtain that

$$g_j(Ax^*, v) \geq \limsup_{n \to \infty} g_j(w_{n_k}^{j_{n_k}}, v) \geq 0,$$

for all $v \in Q_j$, $j = 1, 2, ..., M$. Therefore $Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j)$ and we can conclude that $x^* \in \Omega$.

Finally, in view of (5.2.19), we have $\|x_n - x_0\| \leq \|x^* - x_0\|$ where $x^* = P_\Omega(x_0)$. It is immediate from Lemma 2.4.9 that $\{x_n\}$ converges strongly to x^*. From (5.2.24), we also have $\{y_n^i\}$ and $\{z_n^i\}$ converge strongly to an element $x^* \in \Omega$ for all $i = 1, 2, ..., N$ and $\{w_n^j\}$ converges strongly to an element $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$ for all $j = 1, 2, ...M$ by (5.2.26). □

Likewise Theorem 5.2.7, when $N = M = 1$, then $C_1 = C$ and $Q_1 = Q$, we get the following corollary immediately.

Corollary 5.2.11. Let C and Q be two closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let f be a bifunction satisfying assumption A on C and g be a bifunction satisfying assumption B on Q. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n\} \subset (\theta, \bar{\gamma}]$, for some $\bar{\gamma} > \theta$. Take $x_0 \in C$,$\{\rho_n\} \subset [\underline{\rho}, \bar{\rho}]$ such that $0 < \underline{\rho} \leq \overline{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_n\} \subset [\underline{\alpha}, \bar{\alpha}]$ such that $0 < \underline{\alpha} \leq \overline{\alpha} \leq 1$ and $\mu \in (0, \frac{2}{\|A\|^2})$. Consider the sequences

$$\begin{cases} y_n = \arg\min\{\rho_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\} \\ z_n = \arg\min\{\rho_n f(y_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\} \\ g(w_n, v) + \lambda_n \langle w_n - Az_n, v - w_n\rangle \geq 0, \quad \text{for all} \quad v \in Q \\ x_{n+1} = P_C(z_n + \mu A^*(w_n - Az_n)). \end{cases}$$

If the solution set

$$\Omega = \left\{ x^* \in Sol(C, f) : Ax^* \in Sol(Q, g) \right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to an element $x^* \in \Omega$ and $\{w_n\}$ converges strongly to an element $Ax^* \in Sol(Q, g)$.

5.3 Parallel extragradient-proximal iterative methods with linesearch for multiple set split equilibrium problems

In previous subsection, we focus a Lipschitzian-type continuous on C property of f i.e., if there exist two positive constants c_1, c_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2, \text{ for all } x,y,z \in C \text{ and } Q.$$

This condition is very strong and so difficult to approximate. To aviod this requirement, Tran et al.[69] proposed linesearch procedures to obtain extragradient method for solving equilibrium problem.

Consequently, in this subsection, we reduce this condition **Assumption A** of a bifuncton f and use linesearch procedures of Tran et al. [69]. Next, we recall lemma which use in the part of our main result.

Let f be an equilibrium bifunction defined on $C \times C$. For $x, y \in C$, we denoted by $\partial_2 f(x,y)$ the subgradient of the convex function $\partial_2 f(x,\cdot)$ at y, that is

$$\partial_2 f(x,y) = \{\xi \in H : f(x,z) \geq f(x,y) + \langle \xi, z-y\rangle, \forall z \in C\}.$$

In particular

$$\partial_2 f(x,x) = \{\xi \in H : f(x,z) \geq \langle \xi, z-y\rangle, \forall z \in C\}.$$

Let Δ be an open convex set containing C. The next lemma can be considered as infinite-dimensional version of Theorem 24.5 in [58].

Lemma 5.3.1. Let C be a nonempty closed convex subset of a real Hilbert space H and $f : C \to \mathbb{R}$ be a convex, subdifferentiable, lower semicontinouns function on C. Then x^* is a solution to the following convex optimization problem $\min\{g(x) : x \in C\}$ if and only if $0 \in \partial f(x^*) + N_C(x^*)$ where $\partial f(\cdot)$ the subgradient of f and $N_C(x^*)$ is the normal cone of C at x^*.

Lemma 5.3.2. [28] Suppose that $x^* \in Sol(C,f), f(x,\cdot)$ is convex and subdifferentiable on C for all $x \in C$ and that f is pseudomonotone on C. Then, we have

(i) The Armijo linesearch rule (6.4.4) is well defined;

(ii) $f(z_n, x_n) > 0$;

(iii) $0 \in \partial_2 f(z_n, x_n)$;

(iv) $\|u_n - x^*\| \leq \|x_n - x^*\| - \gamma_n(2-\gamma_n)(\sigma_n\|\varepsilon_n\|)^2$.

Nevertheless, we propose the iterative method by using a combination of the extragradient method with Armijo linesearch type rule for avioding Lipschitz-type continuity of bifunction f in H_1 to obtain the weak convergence theorem for solving multiple set split equilibrium problems (MSSEP).

In order to solving the multiple set split equilibrium problems (MSSEP), we assume that $f : H_1 \times H_1 \longrightarrow \mathbb{R}$ with $f(x,x) = 0$ for all $x \in C$ satisfies the following conditions:

Assumption A

(A1) f is pseudomonotone on C with respect to $Sol(C, f)$;

(A2) $f(x, \cdot)$ is convex, lower semicontinouns and subdifferentiable on C for all $x \in C$;

(A3) f is weakly continuous on $C \times C$: that is, if $x, y \in C$ and $\{x_n\}, \{y_n\} \subset C$ converge weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$.

Moreover, we assume that $g : H_2 \times H_2 \longrightarrow \mathbb{R}$ with $g(u,u) = 0$ for all $u \in Q$ satisfies the following conditions:

Assumption B

(B1) g is pseudomonotone on Q;

(B2) $g(u, \cdot)$ is convex, lower semicontinuous for all $u \in Q$;

(B3) $g(\cdot, v)$ is upper semicontinuous for all $v \in Q$;

(B4) There exists $\kappa \geq 0$ such that $g(u,v) + g(v,u) \leq \kappa\|u-v\|^2$ for all $u, v \in Q$ (g is called undermonotone and κ is the undermonotonicity constant of g).

Algorithm 3 Parallel Extragradient-Proximal Methods with linesearch **Initialization.** Let $x_0 \in C = \bigcap_{i=1}^N C_i$, choose constants $\eta, \theta \in (0,1), 0 < \underline{\rho} \leq \overline{\rho}, 0 < \underline{\alpha} \leq \overline{\alpha} \leq 1$ and $0 < \underline{\gamma} < \overline{\gamma} < 2$. For each $i = 1, 2, ..., N, j = 1, 2, ..., M$, choose parameters $\{\rho_n^i\} \subset [\underline{\rho}, \overline{\rho}], \gamma_n \subset [\underline{\rho}, \overline{\rho}], \{\alpha_n^i\} \subset [\underline{\alpha}, \overline{\alpha}], \Sigma_{i=1}^N \alpha_n^i = 1$ and $\mu \in (0, \frac{2}{\|A\|^2})$.

Step 1. Solve N strongly convex optimization programs in parallel

$$y_n^i = \arg\min\{f_i(x_n, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\} \tag{5.3.1}$$

for $i = 1, 2, ..., N$. If $y_n^i = x_n$ then set $x_n = u_n^i$ and go to back step 4. Otherwise go to step 2

Step 2. Armijo linesearch rule. Find m_n as the smallest positive integer number m such that

$$\begin{cases} z_{n,m}^i = (1-\eta^m)x_n + \eta^m y_n^i, \\ f_i(z_{n,m}^i, x_n) - f_i(z_{n,m}^i, y_n^i) \geq \frac{\theta}{2\rho_n^i}\|x_n - y_n^i\|^2. \end{cases} \tag{5.3.2}$$

Set $\eta_n = \eta^{m_n}, z_n^i = z_{n,m_n}^i$.

Step 3. Select $\varepsilon_n^i \in \partial_2 f_i(z_n^i, x_n)$ and compute $\sigma_n^i = \frac{f_i(z_n^i, x_n)}{\|\varepsilon_n^i\|^2}$ and $u_n^i = P_{C_i}(x_n - \gamma_n \sigma_n^i \varepsilon_n^i)$.

Step 4. Compute $\bar{u}_n = \sum_{i=1}^N \alpha_n^i u_n^i$.

Step 5. Solve M regularized multiple set equilibrium programs in parallel

$$g_j(w_n^j, y) + \lambda_n^j \langle w_n^j - A\bar{u}_n, y - w_n^j \rangle \geq 0 \quad y \in Q_j, \; j = 1, 2, ..., M. \tag{5.3.3}$$

Step 6. Set $\bar{w}_n = \arg\max\{\|w_n^j - A\bar{u}_n\| : j = 1, 2, ..., M\}$.

Step 7. Compute $x_{n+1} = P_C(\bar{u}_n + \mu A^*(\bar{w}_n - A\bar{u}_n))$.

Set $n = n + 1$ and go back **Step 1.**

The following lemma presents that if Algorithm 1 stop at step 1 then a solution of the multiple set equilibrium problem in C go to step 5.

Lemma 5.3.3. If Algorithm 1 terminates at step 1, then x_n is a solution to the multiple set problem in C.

Proof. If the algorithm terminates at step 1, then $x_n = y_n^i$. Since $y_n^i = x_n$. Since y_n^i is the solution to the convex optimization problem (5.3.1), we get that

$$f_i(x_n, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 \geq f_i(x_n, y_n^i) + \frac{1}{2\rho_n^i}\|y_n^i - x_n\|^2 \ \forall y \in C_i.$$

By Lemma 2.2 [69], we can conclude that x_n is a solution to the equilibrium problem. □

If Algorithm 1 does not stop, it go to step 2. The next step Lemma 5.3.2 show that there always a positive integer m such that condition (6.4.4) in step 2 is satisfied. Next go to step 3 and step 4 and In step 5 we will find solution of the equilibrium problem in $Q_j \subset H_2$ such that condition (5.3.3) is well-defined by Lemma 5.2.1. Next we will find the most difference between solution of the equilibrium problem in $Q_j \subset H_2$ with sum of the equilibrium problem in $C_i \subset H_1$ for all $i = 1, 2, ..., N$ which it is forwarded to H_2 and finally compute to step 7.

Theorem 5.3.4. Let C_i and Q_j be two closed and convex subsets of real Hilbert spaces H_1 and H_2 for all $i = 1, 2, ..., N, j = 1, 2, ..., M$, respectively. Let f_i be bifunction consistent with assumptions A on C_i for each $i = 1, 2, ..., N$ and g_j be a bifunction consistent with assumptions B on Q_j for all $j = 1, 2, ..., M$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n^j\}$ be a bounded sequence such that $\{\lambda_n^j\} \subset (\kappa_{n-1}, +\infty)$ for all $n \in \mathbb{N}$. In addition the solution set

$$\Omega = \left\{ x^* \in \bigcap_{i=1}^{N} Sol(C_i, f_i) : Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j) \right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{u_n^i\}, i = 1, 2, ..., N$ converges weakly to an element $x^* \in \Omega$ and $\{w_n^j\}$, $j = 1, 2, ..., M$ converges weakly to an element $Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j)$.

Proof. Let $x^* \in \Omega$. Then

$$x^* \in \bigcap_{i=1}^N Sol(C_i, f_i) \text{ and } Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j).$$

For each $i = 1, 2, ..., N$, from Lemma 5.3.2(vi), we have

$$\begin{aligned} \|u_n^i - x^*\|^2 =& \|P_{C_i}(x_n - \gamma_n \sigma_n^i \varepsilon_n^i) - x^*\| \\ \leq& \|x_n - x^*\|^2 - \gamma_n(2 - \gamma_n)(\sigma_n^i \|\varepsilon_n^i\|)^2 \\ \leq& \|x_n - x^*\|^2. \end{aligned} \tag{5.3.4}$$

For each $i = 1, 2, ..., N$, from Lemma 5.2.4 and (5.3.4), we obtain that

$$\begin{aligned} \|\bar{u}_n - x^*\|^2 =& \|\sum_{i=1}^N \alpha_n^i u_n^i - x^*\|^2 \\ =& \sum_{i=1}^N \alpha_n^i \|u_n^i - x^*\|^2 - \sum_{i=1}^{N-1} \sum_{l=i+1}^N \alpha_i \alpha_l \|u_n^i - u_n^l\|^2 \\ \leq& \sum_{i=1}^N \alpha_n^i \|u_n^i - x^*\|^2 \\ \leq& \|x_n - x^*\|^2. \end{aligned} \tag{5.3.5}$$

Suppose that $j_n \in \{1, 2, ..., M\}$ such that $\bar{w}_n = w_n^{j_n}$. Since $Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j)$, we have

$$g_j(w_n^{j_n}, Ax^*) + \lambda_n^{j_n} \langle w_n^{j_n} - A\bar{u}_n, Ax^* - w_n^{j_n} \rangle \geq 0.$$

By Lemma 5.2.3, every element of $\bigcap_{j=1}^M Sol(Q_j, g_j)$ can solve $CFP(Q_j, g_j)$ for all $j = 1, 2, ..., M$, hence $g_j(w_n^{j_n}, Ax^*) \leq 0$ for all $n \in \mathbb{N}$ and $j_n = 1, 2, ..., M$. It follows that

$$\langle w_n^{j_n} - A\bar{u}_n, Ax^* - w_n^{j_n} \rangle \geq 0.$$

Since

$$\|Ax^* - A\bar{u}_n\|^2 - \|w_n^{j_n} - A\bar{u}_n\|^2 - \|Ax^* - w_n^{j_n}\|^2 \geq 0,$$

which implies that

$$\|Ax^* - w_n^{j_n}\|^2 \leq \|Ax^* - A\bar{u}_n\|^2 - \|w_n^{j_n} - A\bar{u}_n\|^2. \tag{5.3.6}$$

From the following fact, we have

$$\langle A(\bar{u}_n - x^*), w_n^{j_n} - A\bar{u}_n\rangle = \frac{1}{2}\{\|w_n^{j_n} - Ax^*\|^2 - \|A\bar{u}_n - Ax^*\|^2 - \|w_n^{j_n} - A\bar{u}_n\|^2\}.$$

It follows that

$$\langle A(\bar{u}_n - x^*), w_n^{j_n} - A\bar{u}_n\rangle \leq -\|w_n^{j_n} - A\bar{u}_n\|^2.$$

Hence

$$\langle A(\bar{u}_n - x^*), \bar{w}_n - A\bar{u}_n\rangle \leq -\|\bar{w}_n - A\bar{u}_n\|^2.$$

By the definition of x_{n+1} and previous inequation, we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|P_C(\bar{u}_n + \mu A^*(\bar{w}_n - A\bar{u}_n)) - P_C x^*\|^2 \\
&\leq \|\bar{u}_n - x^* + \mu A^*(\bar{w}_n - A\bar{u}_n)\|^2 \\
&= \|\bar{u}_n - x^*\|^2 + \mu^2\|A^*(\bar{w}_n - A\bar{u}_n)\|^2 + 2\mu\langle \bar{u}_n - x^*, A^*(\bar{w}_n - A\bar{z}_n)\rangle \\
&= \|\bar{u}_n - x^*\|^2 + \mu^2\|A\|^2\|\bar{w}_n - A\bar{u}_n\|^2 + 2\mu\langle A(\bar{u}_n - x^*), \bar{w}_n - A\bar{u}_n\rangle \\
&\leq \|\bar{u}_n - x^*\|^2 + \mu^2\|A\|^2\|\bar{w}_n - A\bar{u}_n\|^2 - 2\mu\|\bar{w}_n - A\bar{u}_n\|^2 \\
&= \|\bar{u}_n - x^*\|^2 - \mu(2 - \mu\|A\|^2)\|\bar{w}_n - A\bar{u}_n\|^2. \qquad (5.3.7)
\end{aligned}$$

In combination with (5.3.5), we get that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\bar{u}_n - x^*\|^2 - \mu(2 - \mu\|A\|^2)\|\bar{w}_n - A\bar{u}_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \mu(2 - \mu\|A\|^2)\|\bar{w}_n - A\bar{u}_n\|^2.
\end{aligned}$$

Since $\mu \in (0, \frac{2}{\|A\|^2})$, we have

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2.$$

This implies that $\{\|x_n - x^*\|\}$ is a nonincreasing sequence. Thus the limit of the sequence $\{\|x_n - x^*\|\}$ exists.

In view of (5.3.5), (5.3.8), we get

$$\|x_{n+1} - x^*\|^2 \leq \|\bar{u}_n - x^*\|^2 \leq \Sigma_{i=1}^N \alpha_n^i \|u_n^i - x^*\|^2 \leq \|x_n - x^*\|^2. \tag{5.3.8}$$

From (5.3.8), we conclude that

$$\mu(2 - \mu\|A\|^2)\|\bar{w}_n - A\bar{u}_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \tag{5.3.9}$$

Since the limit of the sequence $\{\|x_n - x^*\|\}$ exists and $\mu \in (0, \frac{2}{\|A\|^2})$, we have

$$\lim_{n\to\infty} \|\bar{u}_n - x^*\| = \lim_{n\to\infty} \Sigma_{i=1}^N \alpha_n^i \|u_n^i - x^*\|^2 = \lim_{n\to\infty} \|x_n - x^*\| \tag{5.3.10}$$

and

$$\lim_{n\to\infty} \|\bar{w}_n - A\bar{u}_n\| = 0. \tag{5.3.11}$$

This implies that

$$\lim_{n\to\infty} \|w_n^{j_n} - A\bar{u}_n\| = 0, \text{ for all } j = 1, 2, ..., M. \tag{5.3.12}$$

By (5.3.4), we have

$$\|u_n^i - x^*\|^2 \leq \|x_n - x^*\|^2 - \gamma_n(2 - \gamma_n)(\sigma_n^i \|\varepsilon_n^i\|)^2,$$

this implies that

$$\gamma_n(2 - \gamma_n)(\sigma_n^i \|\varepsilon_n^i\|)^2 \leq \|x_n - x^*\|^2 - \|u_n^i - x^*\|^2.$$

So

$$\gamma_n(2 - \gamma_n)\Sigma_{i=1}^N \alpha_n^i (\sigma_n^i \|\varepsilon_n^i\|)^2 \leq \|x_n - x^*\|^2 - \Sigma_{i=1}^N \alpha_n^i \|u_n^i - x^*\|^2.$$

Taking the limit $n \to \infty$ and the definition of γ_n, α_n^i, we obtain that

$$\lim_{n\to\infty} \sigma_n^i \|\varepsilon_n^i\| = 0 \text{ for each } i = 1, 2, ..., N. \tag{5.3.13}$$

Since the limit of $\{\|x_n - x^*\|\}$ exists, $\{x_n\}$ is a bounded sequence, From Lemma 5.2.6, we obtain that $\{y_n^i\}$ is also bounded. From step 3. and (5.3.13) yield

$$\lim_{n\to\infty} f_i(z_n^i, x_n) = \lim_{n\to\infty} [\sigma_n^i \|\varepsilon_n^i\|] = 0 \text{ for each } i = 1, 2, ..., N. \tag{5.3.14}$$

By the Algorithm 3. and f is a equilibrium bifunction, we have

$$0 = f_i(z_n^i, z_n^i) = f(z_n^i, (1-\eta^n)x_n + \eta^n y_n^i)$$
$$\leq (1-\eta^n) f_i(z_n^i, x_n) + \eta_n f_i(z_n^i, y_n^i).$$

So

$$f_i(z_n^i, x_n) \geq \eta_n [f_i(z_n^i, x_n) - f_i(z_n^i, y_n^i)]$$
$$\geq \frac{\theta}{2\rho_n^i} \eta_n \|x_n - y_n^i\|^2.$$

From (5.3.14), this implies that

$$\lim_{n\to\infty} \eta^n \|x_n - y_n^i\|^2 = 0 \text{ for each } i = 1, 2, ..., N. \tag{5.3.15}$$

We now consider two distinct cases

Case 1. $\limsup_{n\to\infty} \eta^n > 0.$

Then there exist $\bar{\eta} > 0$ and a subsequence $\{\eta^{n_k}\} \subset \{\eta^n\}$ such that $\eta^{n_k} > \bar{\eta}$ for each k. From (5.3.15), we get

$$\lim_{k\to\infty} \|x_{n_k} - y_{n_k}^i\| = 0 \text{ for each } i = 1, 2, ..., N. \tag{5.3.16}$$

Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to x^* as $k \to \infty$. From $\lim_{k\to\infty} x_{n_k} = x^*$ and (5.3.16), we have $\lim_{k\to\infty} y_{n_k}^i = x^*$ for each $i = 1, 2, ..., N$. For each $y \in C_i$ for each $i = 1, 2, ..., N$, by the definition of y_n^i,

$$y_{n_k}^i = \arg\min\{f_i(x_{n_k}, y) + \frac{1}{2\rho_n^i}\|y - x_{n_k}\|^2 : y \in C_i\},$$

we have

$$0 \in \partial_2 f_i(x_{n_k}^i, y_{n_k}^i) + \frac{1}{\rho_{n_k}^i}(y_{n_k}^i - x_{n_k}) + N_{C_i}(y_{n_k}^i).$$

So there exists $\hat{\varepsilon}_{n_k}^i \in \partial_2 f_i(x_{n_k}, y_{n_k}^i)$ such that

$$\langle \hat{\varepsilon}_{n_k}^i, y - y_{n_k}^i \rangle + \frac{1}{\rho_{n_k}^i}\langle y_{n_k}^i - x_{n_k}, y - y_{n_k}^i \rangle \geq 0 \text{ for each } y \in C_i, i = 1, 2, ..., N.$$

Combining this with

$$f_i(x_{n_k}, y) - f_i(x_{n_k}, y^i_{n_k}) \geq \langle \hat{\varepsilon}^i_{n_k}, y - y^i_{n_k} \rangle \quad \text{for each} \quad y \in C_i, i = 1, 2, ..., N,$$

yields

$$f_i(x_{n_k}, y) - f_i(x_{n_k}, y^i_{n_k}) + \frac{1}{\rho^i_{n_k}} \langle y^i_{n_k} - x_{n_k}, y - y^i_{n_k} \rangle \geq 0 \quad \text{for each} \quad y \in C_i, i = 1, 2, ..., N. \tag{5.3.17}$$

Since

$$\langle y^i_{n_k} - x_{n_k}, y - y^i_{n_k} \rangle \leq \|y^i_{n_k} - x_{n_k}\| \|y - y^i_{n_k}\|,$$

by (5.3.17), we get that

$$f_i(x_{n_k}, y) - f_i(x_{n_k}, y^i_{n_k}) + \frac{1}{\rho^i_{n_k}} \|y^i_{n_k} - x_{n_k}\| \|y - y^i_{n_k}\| \geq 0 \quad \text{for each} \quad y \in C_i, i = 1, 2, ..., N. \tag{5.3.18}$$

Taking the limit of $k \to \infty$ and by the continuity of f_i, we obtain that

$$f_i(x^*, y) - f_i(x^*, x^*) \geq 0 \quad \text{for each} \quad y \in C_i, i = 1, 2, ..., N.$$

Hence $f_i(x^*, y) \geq 0$ for each $y \in C_i, i = 1, 2, ..., N$. So $x^* \in \bigcap_{i=1}^N Sol(C_i, f_i)$.

Case 2. $\limsup_{n\to\infty} \eta^n = 0$.

From the boundedness of $\{y^i_n\}$, then there exists a subsequence $\{y^i_{n_k}\} \subset \{y^i_n\}$ such that $\{y^i_{n_k}\} \rightharpoonup \bar{y}$ as $k \to \infty$. Replacing y by $x^i_{n_k}$ in (5.3.17), we have

$$f_i(x_{n_k}, y^i_{n_k}) + \frac{1}{\rho^i_{n_k}} \|y^i_{n_k} - x_{n_k}\|^2 \leq 0 \quad \text{for each} \quad i = 1, 2, ..., N. \tag{5.3.19}$$

In the other hand, by the Armijo linesearch rule, for m_{n_k-1}, there exists $z^i_{n_k, m_{n_k-1}}$ such that

$$f_i(z^i_{n_k,m_{n_k-1}}, x_{n_k}) - f_i(z^i_{n_k,m_{n_k-1}}, y^i_{n_k}) < \frac{\theta}{2\rho^i_{n_k}} \|y^i_{n_k} - x_{n_k}\|^2.$$

By (5.3.19), we obtain that

$$f_i(x_{n_k}, y^i_{n_k}) \leq -\frac{1}{\rho^i_{n_k}} \|y^i_{n_k} - x_{n_k}\| \leq \frac{2}{\theta} [f_i(z^i_{n_k,m_{n_k-1}}, y^i_{n_k}) - f_i(z^i_{n_k,m_{n_k-1}}, x_{n_k})]. \tag{5.3.20}$$

According to the algorithm, we have

$$z^i_{n_k,m_{n_k-1}} = (1-\eta^{m_{n_k}-1})x_{n_k} + \eta^{m_{n_k}-1}y^i_{n_k}.$$

Since $\eta^{m_{n_k}-1} \to 0, x_{n_k}$ converges weakly to x^* and y_{n_k} converges weakly to $\bar{y}$, we have $z^i_{n_k,m_{n_k-1}} \rightharpoonup x^*$ as $k \to \infty$. Moreover, $\{\frac{1}{\rho^i_{n_k}}\|y^i_{n_k} - x_{n_k}\|^2\}$ is bounded, without loss of generality, we may assume that $\lim_{k\to\infty}\|y^i_{n_k} - x_{n_k}\|^2$ exists.
Taking $k \to \infty$ in (5.3.20), we obtain that

$$f_i(x^*,\bar{y}) \leq -\lim_{k\to+\infty}\frac{1}{\rho^i_{n_k}}\|y^i_{n_k} - x_{n_k}\|^2 \leq \frac{2}{\theta}f_i(x^*,\bar{y}).$$

Therefore, $f_i(x^*,\bar{y}) = 0$ and $\lim_{k\to+\infty}\|y^i_{n_k} - x_{n_k}\|^2 = 0$. By **Case 1.**, it is immediate that $x^* \in \bigcap_{i=1}^N Sol(C_i, f_i)$. In addition, for each $j = 1, 2, ..., M$, by using Cauchy-Schwartz inequality gets that

$$\begin{aligned} 0 &\leq g_j(w^j_n, v) + \lambda^j_n\langle w^j_n - A\bar{u}_n, v - w^j_n\rangle \\ &\leq g_j(w^j_n, v) + \lambda^j_n\|w^j_n - A\bar{u}_n\|\|v - w^j_n\|. \end{aligned}$$

Since $\{\lambda^j_n\}$ and $\{\|v - w^j_n\|\}$ are bounded sequences for all $j = 1, 2, ..., M$, we have

$$0 \leq \liminf_{n\longrightarrow\infty} g_j(w^j_n, v), \text{ for all } v \in Q_j, j = 1, 2, ..., M. \tag{5.3.21}$$

Under upper semicontinuity of $g(\cdot, v)$, $w^{j_{n_k}}_{n_k} \rightharpoonup Ax^*$ for all $j = 1, 2, ..., M$ and (5.3.21), we obtain that

$$g_j(Ax^*, v) \geq \limsup_{n\to\infty} g_j(w^{j_{n_k}}_{n_k}, v) \geq 0,$$

for all $v \in Q_j$, $j = 1, 2, ..., M$. Therefore $Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j)$ and we can conclude that $x^* \in \Omega$.

Finally, we prove that $\{x_n\}$ converges weakly to x^*. Suppose that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ with $x^* \neq \bar{x}$.
By Opial's condition, we obtain that

$$\liminf_{m\to\infty}\|x_{n_m} - \bar{x}\| < \liminf_{m\to\infty}\|x_{n_m} - x^*\|$$

$$
\begin{aligned}
&= \liminf_{k\to\infty} \|x_{n_k} - x^*\| \\
&< \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| \\
&= \liminf_{m\to\infty} \|x_{n_m} - \bar{x}\|.
\end{aligned}
$$

This is a contradiction and so $\{x_n\}$ converges weakly to x^*.

It is immediate from (5.3.10), we also have $u_n^i \rightharpoonup x^*$, for all $i = 1, 2, ..., N$. Therefore $\bar{u}_n \rightharpoonup x^*$ and $A\bar{u}_n \rightharpoonup Ax^*$. Consequently by (5.3.12), $w_n^{j_n} \rightharpoonup Ax^*$ for all $j = 1, 2, ..., M$. □

A particular case of the multiple set split equilibrium problem is the split equilibrium problem, that is $N = M = 1$, then $C_i = C$ and $Q_j = Q$, we get the following corollary immediately.

Algorithm 2 Parallel Extragradient-Proximal Methods with linesearch

Initialization. Let $x_0 \in C$, choose constants $\eta, \theta \in (0,1), 0 < \underline{\rho} \leq \overline{\rho}$ and $0 < \underline{\gamma} < \overline{\gamma} < 2$. Choose parameters $\{\rho_n\} \subset [\underline{\rho}, \overline{\rho}], \gamma_n \subset [\underline{\rho}, \overline{\rho}], \{\alpha_n\} \subset [\underline{\alpha}, \overline{\alpha}]$ and $\mu \in (0, \frac{2}{\|A\|^2})$.

Step 1. Solve N strongly convex optimization programs in parallel

$$y_n = \arg\min\{f(x_n, y) + \frac{1}{2\rho_n}\|y - x_n\|^2 : y \in C\}. \tag{5.3.22}$$

If $y_n = x_n$ then set $x_n = u_n$ and go to back step 4. Otherwise go to step 2

Step 2. Armijo linesearch rule. Find m_n as the smallest positive integer number m such that

$$\begin{cases} z_{n,m} = (1 - \eta^m)x_n + \eta^m y_n, \\ f(z_{n,m}, x_n) - f(z_{n,m}, y_n) \geq \frac{\theta}{2\rho_n}\|x_n - y_n\|^2. \end{cases} \tag{5.3.23}$$

Set $\eta_n \in \eta^{m_n}, z_n = z_{n,m_n}$.

Step 3. Select $\varepsilon_n \in \partial_2 f(z_n, x_n)$ and compute $\sigma_n = \frac{f(z_n, x_n)}{\|\varepsilon_n\|^2}$ and $u_n = P_C(x_n - \gamma_n \sigma_n \varepsilon_n)$.

Step 4. Solve M regularized multiple set equilibrium programs in parallel

$$g(w_n, y) + \lambda_n \langle w_n - Au_n, y - w_n \rangle \geq 0 \quad y \in Q.$$

Step 5. Compute $x_{n+1} = P_C(\bar{u}_n + \mu A^*(w_n - A\bar{u}_n))$.

Set $n = n + 1$ and go back **Step 1.**

Corollary 5.3.5. Let C and Q be two closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let f be a bifunction consistent with assumptions A on C and g be a bifunction consistent with assumptions B on Q. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n\}$ be a bounded sequence such that $\{\lambda_n\} \subset (\kappa_{n-1}, +\infty)$ for all $n \in \mathbb{N}$. In addition the solution set

$$\Omega = \left\{ x^* \in Sol(C, f) : Ax^* \in Sol(Q, g) \right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{u_n\}$ converge weakly to an element $x^* \in \Omega$ and $\{w_n\}$ converges weakly to an element $Ax^* \in Sol(Q, g)$.

We now present the following example for supporting Theorem 5.3.4

Example 5.3.6. we define the bifunction $f_i : C_i \times C_i \to \mathbb{R}, i = 1, 2, 3$, which apply in the Nash-Cournot equilibrium model in [69]. We define by

$$f_i(x, y) = \langle (P_i + \frac{3}{2}Q_i)x + Q_i y + q_i, y - x \rangle + \langle d_i, arctan(x - y) \rangle$$

where P_i and Q_i are symmetric positive semidefinite. Define $q_1 = (1, 1, 0, 0, 2, 0)^T$, $q_2 = (1, 2, 0, 1, 2, 0)^T, q_3 = (1, 0, 2, 0, 2, 1)^T$ and $d_i = rand(1, 6)$ for all $i = 1, 2, 3$. We set $arctan(x-y) = (arctan(x_1-y_1), ..., arctan(x_6-y_6))^T$. Since $f_i(x, y)+f_i(y, x) = -(y - x)P_i(y - x) - (y - x)\frac{1}{2}Q_i(y - x)$, and P_i and Q_i are symmetric positive semidefinite, we get that the bifunction f_i satisfies both condition A1, A2 and A3, We choose $\rho_n^i = 0.5$.

We define $A : \mathbb{R}^6 \to \mathbb{R}^3$ by

$$A(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 - x_2, x_3 - x_4, x_5 - x_6),$$

A is bounded linear operator such that $\|Ax\| \leq \|x\|$.

Next we consider the bifunction $g_j : Q_j \times Q_j \to \mathbb{R}$ which define by

$$g_j(x, y) = \langle (A_j + B_j)x + B_j y + t_j, y - x \rangle$$

where A_j and B_j are symmetric positive semidefinite. Define $t_1 = (1, 2, 1)^T$. Since $g_1(x, y) + g_1(y, x) = -(y - x)A_1(y - x)$ and A_1 is symmetric positive semidefinite, we get that the bifunction g_j satisfies condition B1, and we easy to check B2, B3 and we get that 0 is the undermonotone constant of g_j.

We choose $\lambda_n = \|B_j - A_j\|$ if $B_j \neq A_j$ and $\lambda_n = 0.5$ if $B_j = A_j$. Define $x_0 = (0.5, 1.2, 1.7, 1.5, 0.5, 1.8)$, $w_0 = (2, 2, 2.5)$ and

$$C = \bigcap_{i=1}^{3} C_i = \{x \in \mathbb{R}^{+6} : 0 \leq x_k \leq 2; k = 1, 2, 3, 4, 5, 6\},$$

$$Q = Q_1 = \{x \in \mathbb{R}^{+3} : 0 \leq x_k \leq 3; k = 1, 2, 3\}.$$

We implement Algorithm1 in Matlab R2015b running on a Desktop with Intel(R) Core(TM) i5-4200U CPU with 1.60GHz 2.30GHz, 4 GB Ram. We use the stopping criteria $\frac{\|x_{n+1} - x_n\|}{max\{1, \|x_n\|\}} \leq \varepsilon$ for terimanation of Algorithm 1 and set $\varepsilon = 10^{-5}$.

The results are reported in the table below

error	μ	γ	iterative	Cpu(s)
10^{-4}	0.5	0.25	10	11.8438
	0.25	0.25	41	72.8125
	0.25	0.5	806	5.6563
	0.25	1.5	278	19.2656

Algorithm 3 Hybrid Parallel Extragradient-Proximal Methods with line-search

Initialization. Let $x_0 \in C = \bigcap_{i=1}^{N} C_i$, choose constants $\eta, \theta \in (0, 1), 0 < \underline{\rho} \leq \overline{\rho}, 0 < \underline{\alpha} \leq \overline{\alpha} \leq 1$ and $0 < \underline{\gamma} < \overline{\gamma} < 2$. For each $i = 1, 2, ..., N, j = 1, 2, ..., M$, choose parameters $\{\rho_n^i\} \subset [\underline{\rho}, \overline{\rho}], \gamma_n \subset [\underline{\rho}, \overline{\rho}], \{\alpha_n^i\} \subset [\underline{\alpha}, \overline{\alpha}], \Sigma_{i=1}^{N} \alpha_n^i = 1$ and $\mu \in (0, \frac{2}{\|A\|^2})$.

Step 1. Solve N strongly convex optimization programs in parallel

$$y_n^i = \arg\min\{f_i(x_n, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\}, \tag{5.3.24}$$

for $i = 1, 2, ..., N$. If $y_n^i = x_n$ then set $x_n = u_n^i$ and go to back step 4. Otherwise go to step 2.

Step 2. Armijo linesearch rule.

Find m_n as the smallest positive integer number m such that

$$\begin{cases} z_{n,m}^i = (1 - \eta^m)x_n + \eta^m y_n^i, \\ f_i(z_{n,m}^i, x_n) - f_i(z_{n,m}^i, y_n^i) \geq \frac{\theta}{2\rho_n^i}\|x_n - y_n^i\|^2. \end{cases} \tag{5.3.25}$$

Set $\eta_n = \eta^{m_n}, z_n^i = z_{n,m_n}^i$.

Step 3. Select $\varepsilon_n^i \in \partial_2 f_i(z_n^i, x_n)$ and compute $\sigma_n^i = \frac{f_i(z_n^i, x_n)}{\|\varepsilon_n^i\|^2}$ and $u_n^i = P_{C_i}(x_n - \gamma_n \sigma_n^i \varepsilon_n^i)$.

Step 4. Compute $\bar{u}_n = \sum_{i=1}^N \alpha_n^i u_n^i$.

Step 5. Solve M regularized multiple set equilibrium programs in parallel

$$g_j(w_n^j, y) + \lambda_n^j \langle w_n^j - A\bar{u}_n, y - w_n^j\rangle \geq 0 \ \ y \in Q_j, \ j = 1, 2, ..., M.$$

Step 6. Set $\bar{w}_n = \arg\max\{\|w_n^j - A\bar{u}_n\| : j = 1, 2, ..., M\}$.

Step 7. Compute $t_n = P_C(\bar{u}_n + \mu A^*(\bar{w}_n - A\bar{u}_n))$.

Step 8. Take $x_{n+1} = P_{C_{n+1}}(x_0)$, where

$$C_{n+1} = \{v \in C_n : \|t_n - v\| \leq \|\bar{u}_n - v\| \leq \|x_n - v\|\}.$$

Set $n = n + 1$ and go back **Step 1.**

Theorem 5.3.7. Let C_i and Q_j be two closed and convex subsets of real Hilbert spaces H_1 and H_2 for all $i = 1, 2, ..., N, j = 1, 2, ..., M$, respectively. Let f_i be a bifunction consistent with assumptions A on C_i for all $i = 1, 2, ..., N$ and g_j

be a bifunction consistent with assumptions B on Q_j for all $j = 1, 2, ..., M$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n^j\}$ be a bounded sequence such that $\{\lambda_n^j\} \subset (\kappa_{n-1}, +\infty)$ for all $n \in \mathbb{N}$. In addition the solution set

$$\Omega = \left\{ x^* \in \bigcap_{i=1}^{N} Sol(C_i, f_i) : Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j) \right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{u_n^i\}$, $i = 1, 2, ..., N$ converge strongly to an element $x^* \in \Omega$ and $\{w_n^j\}$, $j = 1, 2, ..., M$ converges strongly to an element $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$.

Proof. Firstly, we observe that C_{n+1} is a nonempty closed convex set for all $n \in \mathbb{N}$. Let $x^* \in \Omega$. Then

$$x^* \in \bigcap_{i=1}^{N} Sol(C_i, f_i) \text{ and } Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j).$$

For each $i = 1, 2, ..., N$, by the proof of Theorem 5.3.4, we have

$$\|u_n^i - x^*\|^2 \leq \|x_n - x^*\|^2 - \gamma_n(2 - \gamma_n)(\sigma_n^i \|\varepsilon_n^i\|)^2, \tag{5.3.26}$$

and

$$\|\bar{u}_n - x^*\|^2 \leq \Sigma_{i=1}^{N} \alpha_n^i \|u_n^i - x^*\|^2 \leq \|x_n - x^*\|^2, \tag{5.3.27}$$

Suppose that $j_n \in \{1, 2, ..., M\}$ such that $\overline{w}_n = w_n^{j_n}$. Since $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$, by the same process of a proof in Theorem 5.3.4, we get that

$$\|Ax^* - w_n^{j_n}\|^2 \leq \|Ax^* - A\bar{u}_n\|^2 - \|w_n^{j_n} - A\bar{u}_n\|^2, \tag{5.3.28}$$

and

$$\langle A(\bar{u}_n - x^*), \overline{w}_n - A\bar{u}_n \rangle \leq -\|\overline{w}_n - A\bar{u}_n\|^2. \tag{5.3.29}$$

By the definition of t_n and (5.3.29), this implies that

$$\|t_n - x^*\|^2 = \|P_C(\bar{u}_n + \mu A^*(\overline{w}_n - A\bar{u}_n)) - P_C x^*\|^2$$

$$\begin{aligned}
&\leq \|\bar{u}_n - x^* + \mu A^*(\bar{w}_n - A\bar{u}_n)\|^2 \\
&= \|\bar{u}_n - x^*\|^2 + \mu^2\|A^*(\bar{w}_n - A\bar{u}_n)\|^2 + 2\mu\langle \bar{u}_n - x^*, A^*(\bar{w}_n - A\bar{u}_n)\rangle \\
&= \|\bar{u}_n - x^*\|^2 + \mu^2\|A\|^2\|\bar{w}_n - A\bar{u}_n\|^2 + 2\mu\langle A(\bar{u}_n - x^*), \bar{w}_n - A\bar{u}_n\rangle \\
&\leq \|\bar{u}_n - x^*\|^2 + \mu^2\|A\|^2\|\bar{w}_n - A\bar{u}_n\|^2 - 2\mu\|\bar{w}_n - A\bar{u}_n\|^2 \\
&= \|\bar{u}_n - x^*\|^2 - \mu(2 - \mu\|A\|^2)\|\bar{w}_n - A\bar{u}_n\|^2. \qquad (5.3.30)
\end{aligned}$$

By hypothesis of μ, it follows that

$$\|t_n - x^*\|^2 \leq \|\bar{u}_n - x^*\|^2, \qquad (5.3.31)$$

and combine with (5.3.27), we get from

$$\|t_n - x^*\|^2 \leq \|\bar{u}_n - x^*\|^2 \leq \|x_n - x^*\|^2. \qquad (5.3.32)$$

Therefore $\Omega \in C_{n+1}$. From the definition of C_n, it implies that $x_n = P_{C_n}(x_0)$. By Proposition 2.3.4 and $x_{n+1} \in C_{n+1}$, we have

$$\|x_{n+1} - x_n\|^2 + \|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2, \qquad (5.3.33)$$

it follows that

$$\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2.$$

Similarly to $x_{n+1} \in P_{C_{n+1}}(x_0)$ and $x^* \in C_{n+1}$, we get

$$\|x_{n+1} - x_0\|^2 \leq \|x^* - x_0\|^2.$$

This implies that

$$\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2 \leq \|x^* - x_0\|^2. \qquad (5.3.34)$$

Consequently, the sequence $\{\|x_n - x_0\|\}$ is a nondecreasing and bounded sequence. Then the limit of the sequence $\{\|x_n - x_0\|\}$ exists and $\{x_n\}$ is a bounded sequence. By Lemma 5.2.6 obtain that $\{y_n^i\}$ is also bounded for all $i = 1, 2, ..., N$. From (5.3.33) and the limit of the sequence $\{\|x_n - x_0\|\}$ exists, we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Next, we will show that $x^* \in \Omega$. Since $x_{n+1} \in C_{n+1}$, we obtain that

$$\|x_{n+1} - t_n\| \leq \|x_{n+1} - \bar{u}\| \leq \|x_{n+1} - x_n\|.$$

Since

$$\begin{aligned} \|t_n - x_n\| &\leq \|t_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq 2\|x_{n+1} - x_n\|, \end{aligned} \tag{5.3.35}$$

$$\begin{aligned} \|\bar{u}_n - x_n\| &\leq \|\bar{u} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq 2\|x_{n+1} - x_n\|, \end{aligned} \tag{5.3.36}$$

it follows that

$$\lim_{n\to\infty} \|t_n - x_n\| = \lim_{n\to\infty} \|\bar{u}_n - x_n\| = 0. \tag{5.3.37}$$

In view of (5.3.27) and (5.3.30), we get that

$$\begin{aligned} \|t_n - x^*\|^2 =& \|\bar{u}_n - x^*\|^2 - \mu(2 - \mu\|A\|^2)\|\bar{w}_n - A\bar{u}_n\|^2 \\ \leq& \|x_n - x^*\|^2 - \mu(2 - \mu\|A\|^2)\|\bar{w}_n - A\bar{u}_n\|^2. \end{aligned} \tag{5.3.38}$$

This implies that

$$\begin{aligned} \mu(2 - \mu\|A\|^2)\|\bar{w}_n - A\bar{u}_n\|^2 &\leq \|x_n - x^*\|^2 - \|t_n - x^*\|^2 \\ &\leq (\|x_n - x^*\| + \|t_n - x^*\|)\|x_n - t_n\|. \end{aligned}$$

Taking the limit $n \to \infty$, we have

$$\lim_{n\to\infty} \|\bar{w}_n - A\bar{u}_n\| = 0, \tag{5.3.39}$$

by hypothesis of $\bar{w}_n$, we get

$$\lim_{n\to\infty} \|w_n^{j_n} - A\bar{u}_n\| = 0, \ \text{ for all } \ j = 1, 2, ..., M. \tag{5.3.40}$$

By the same way in the proof of Theorem 5.3.4, we obtain that

$$\lim_{n\to\infty} \sigma_n^i \|\varepsilon_n^i\| = 0 \ \text{ for each } \ i = 1, 2, ..., N. \tag{5.3.41}$$

From step 3. and (5.3.41) yield

$$\lim_{n\to\infty} f_i(z_n^i, x_n) = \lim_{n\to\infty} [\sigma_n^i \|\varepsilon_n^i\|] = 0 \text{ for each } i = 1, 2, ..., N. \tag{5.3.42}$$

By the algorithm 4. and f is a equilibrium bifunction, we have

$$\begin{aligned} 0 = f_i(z_n^i, z_n^i) &= f(z_n^i, (1-\eta^n)x_n + \eta^n y_n^i) \\ &\leq (1-\eta^n) f_i(z_n^i, x_n) + \eta_n f_i(z_n^i, y_n^i). \end{aligned}$$

So

$$\begin{aligned} f_i(z_n^i, x_n) &\geq \eta^n [f_i(z_n^i, x_n) - f_i(z_n^i, y_n^i)] \\ &\geq \frac{\theta}{2\rho_n^i} \eta^n \|x_n - y_n^i\|^2. \end{aligned}$$

From (5.3.42), this implies that

$$\lim_{n\to\infty} \eta^n \|x_n - y_n^i\|^2 = 0 \text{ for each } i = 1, 2, ..., N. \tag{5.3.43}$$

Next, we will show that any weak accumulation of $\{x_n\}$ belongs to Ω. Suppose that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to x^* that is $x_{n_k} \rightharpoonup x^*$. From (5.3.37) and (5.3.40) we have $t_n \rightharpoonup x^*$, $\bar{u}_n \rightharpoonup x^*$ and $w_n^{j_n} \rightharpoonup Ax^*$.
By (5.3.43), we get that

$$\lim_{k\to\infty} \eta^{n_k} \|x_{n_k} - y_{n_k}^i\|^2 = 0 \text{ for each } i = 1, 2, ..., N. \tag{5.3.44}$$

We now consider two distinct cases

Case 1. $\limsup_{n\to\infty} \eta^n > 0$.

Then there exist $\bar{\eta} > 0$ and a subsequence $\{\eta^{n_k}\} \subset \{\eta^n\}$ such that $\eta^{n_k} > \bar{\eta}$ for each k. From (5.3.44), we get

$$\lim_{k\to\infty} \|x_{n_k} - y_{n_k}^i\| = 0 \text{ for each } i = 1, 2, ..., N. \tag{5.3.45}$$

Since $x_{n_k} \rightharpoonup x^*$ and (5.3.45), it follows that $y_{n_k}^i \rightharpoonup x^*$ for each $i = 1, 2, ..., N$. For each $y \in C_i$ for each $i = 1, 2, ..., N$, from Theorem 5.3.4, we get that

$$f_i(x_{n_k}, y) - f_i(x_{n_k}, y_{n_k}^i) + \frac{1}{\rho_{n_k}^i} \langle y_{n_k}^i - x_{n_k}, y - y_{n_k}^i \rangle \geq 0 \text{ for each } y \in C_i, i = 1, 2, ..., N,$$

(5.3.46)

and

$$f_i(x_{n_k},y)-f_i(x_{n_k},y^i_{n_k})+\frac{1}{\rho^i_{n_k}}\|y^i_{n_k}-x_{n_k}\|\|y-y^i_{n_k}\| \geq 0 \quad \text{for each } y \in C_i, i=1,2,...,N. \tag{5.3.47}$$

Taking the limit of $k \to \infty$ and by the continuity of f_i, we obtain that

$$f_i(x^*,y)-f_i(x^*,x^*) \geq 0 \quad \text{for each } y \in C_i, i=1,2,...,N.$$

Hence $f_i(x^*,y) \geq 0$ for each $y \in C_i, i=1,2,...,N$. Therefore $x^* \in \bigcap_{i=1}^N Sol(C_i,f_i)$.

Case 2. $\limsup_{n\to\infty} \eta^n = 0$.

From the boundedness of $\{y^i_n\}$, then there exists a subsequence $\{y^i_{n_k}\} \subset \{y^i_n\}$ such that $\{y^i_{n_k}\} \rightharpoonup \bar{y}$. Replacing y by $x^i_{n_k}$ in (5.3.46), we have

$$f_i(x_{n_k},y^i_{n_k})+\frac{1}{\rho^i_{n_k}}\|y^i_{n_k}-x_{n_k}\|^2 \leq 0 \quad \text{for each } i=1,2,...,N. \tag{5.3.48}$$

In the other hand, by the Armijo linesearch rule, for m_{n_k-1}, there exists $z^i_{n_k,m_{n_k-1}}$ such that

$$f_i(z^i_{n_k,m_{n_k-1}},x_{n_k})-f_i(z^i_{n_k,m_{n_k-1}},y^i_{n_k}) < \frac{\theta}{2\rho^i_{n_k}}\|y^i_{n_k}-x_{n_k}\|^2.$$

By (5.3.48), we obtain that

$$f_i(x_{n_k},y^i_{n_k}) \leq -\frac{1}{\rho^i_{n_k}}\|y^i_{n_k}-x_{n_k}\| \leq \frac{2}{\theta}[f_i(z^i_{n_k,m_{n_k-1}},y^i_{n_k})-f_i(z^i_{n_k,m_{n_k-1}},x_{n_k})]. \tag{5.3.49}$$

According to the algorithm, we have

$$z^i_{n_k,m_{n_k-1}} = (1-\eta^{m_{n_k}-1})x_{n_k}+\eta^{m_{n_k}-1}y^i_{n_k}.$$

Since $\eta^{m_{n_k}-1} \to 0, x_{n_k}$ converges weakly to x^* and y_{n_k} converges weakly to $\bar{y}$, we have $z^i_{n_k,m_{n_k-1}} \rightharpoonup x^*$ as $k \to \infty$. Moreover, $\{\frac{1}{\rho^i_{n_k}}\|y^i_{n_k}-x_{n_k}\|^2\}$ is bounded, without loss of generality, we may assume that $\lim_{k\to\infty}\|y^i_{n_k}-x_{n_k}\|^2$ exists.

Taking $k \to \infty$ in (5.3.49), we obtain that

$$f_i(x^*,\bar{y}) \geq -\lim_{k\to+\infty}\frac{1}{\rho^i_{n_k}}\|y^i_{n_k}-x_{n_k}\|^2 \geq \frac{2}{\theta}f_i(x^*,\bar{y}).$$

Therefore, $f_i(x^*, \bar{y}) = 0$ and $\lim_{k\to+\infty} \|y^i_{n_k} - x_{n_k}\|^2 = 0$. By **Case 1.**, it is immediate that $x^* \in \bigcap_{i=1}^N Sol(C_i, f_i)$. In addition, for each $j = 1, 2, ..., M$, by using Cauchy-Schwartz inequality gets that

$$\begin{aligned} 0 &\leq g_j(w^j_n, v) + \lambda^j_n \langle w^j_n - A\bar{u}_n, v - w^j_n\rangle \\ &\leq g_j(w^j_n, v) + \lambda^j_n \|w^j_n - A\bar{u}_n\|\|v - w^j_n\|. \end{aligned}$$

Since $\{\lambda^j_n\}$ and $\{\|v - w^j_n\|\}$ are bounded sequences for all $j = 1, 2, ..., M$, we have

$$0 \leq \liminf_{n\longrightarrow\infty} g_j(w^j_n, v), \quad \text{for all } v \in Q_j, j = 1, 2, ..., M. \tag{5.3.50}$$

Under upper semicontinuity of $g(\cdot, v)$, $\lim_{n\to\infty} w^{j_{n_k}}_{n_k} = Ax^*$ for all $j = 1, 2, ..., M$ and (5.3.50), we obtain that

$$g_j(Ax^*, v) \geq \limsup_{n\to\infty} g_j(w^{j_{n_k}}_{n_k}, v) \geq 0,$$

for all $v \in Q_j$, $j = 1, 2, ..., M$. Therefore $Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j)$. We can conclude that $x^* \in \bigcap_{i=1}^N Sol(C_i, f_i)$ and $Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j)$.

Finally, in view of (5.3.34), we have $\|x_n - x_0\| \leq \|x^* - x_0\|$ where $x^* = P_\Omega(x_0)$. It is immediate from Lemma 2.4.9 that $\{x_n\}$ converges strongly to x^*. From (5.3.43) and (5.3.37), we also have $\{y^i_n\}$ and $\{u^i_n\}$ converge strongly to an element $x^* \in \Omega$ for all $i = 1, 2, ..., N$. and $\{w^j_n\}$ converges strongly to an element $Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j)$ for all $j = 1, 2, ...M$ by (5.3.40). □

The following result is an immediate consequence of Theorem 5.3.7 when $N = M = 1$, then $C_1 = C$ and $Q_1 = Q$, we get the following corollary.

Algorithm 4 Hybrid Parallel Extragradient-Proximal Methods with linesearch

Initialization. Let $x_0 \in C$, choose constants $\eta, \theta \in (0, 1), 0 < \underline{\rho} \leq \overline{\rho}$ and $0 < \underline{\gamma} < \overline{\gamma} < 2$. Choose parameters $\{\rho_n\} \subset [\underline{\rho}, \overline{\rho}], \gamma_n \subset [\underline{\rho}, \overline{\rho}], \{\alpha_n\} \subset [\underline{\alpha}, \overline{\alpha}]$ and $\mu \in (0, \frac{2}{\|A\|^2})$.

Step 1. Solve N strongly convex optimization programs in parallel

$$y_n = \arg\min\{f(x_n, y) + \frac{1}{2\rho_n}\|y - x_n\|^2 : y \in C\}. \tag{5.3.51}$$

If $y_n = x_n$, then set $x_n = u_n$ and go to back step 4.

Step 2. Armijo linesearch rule. Find m_n as the smallest positive integer number m such that

$$\begin{cases} z_{n,m} = (1-\eta^m)x_n + \eta^m y_n, \\ f(z_{n,m}, x_n) - f(z_{n,m}, y_n) \geq \frac{\theta}{2\rho_n}\|x_n - y_n\|^2. \end{cases} \tag{5.3.52}$$

Set $\eta_n = \eta^{m_n}, z_n = z_{n,m_n}$.

Step 3. Select $\varepsilon_n \in \partial_2 f(z_n, x_n)$ and compute $\sigma_n = \frac{f(z_n,x_n)}{\|\varepsilon_n\|^2}$ and $u_n = P_C(x_n - \gamma_n\sigma_n\varepsilon_n)$.

Step 4. Solve M regularized multiple set equilibrium programs in parallel

$$g(w_n, y) + \lambda_n\langle w_n - Au_n, y - w_n\rangle \geq 0 \quad y \in Q.$$

Step 5. Compute $t_n = P_C(\bar{u}_n + \mu A^*(w_n - Au_n))$.

Step 6. Take $x_{n+1} = P_{C_{n+1}}(x_0)$, where

$$C_{n+1} = \{v \in C_n : \|t_n - v\| \leq \|u_n - v\| \leq \|x_n - v\|\}.$$

Set $n = n+1$ and go back **Step 1.**

Corollary 5.3.8. Let C and Q be two closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let f be a bifunction consistent with assumptions A on C and g be a bifunction consistent with assumptions B on Q. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n\}$ be a bounded sequence such that $\{\lambda_n\} \subset (\kappa_{n-1}, +\infty)$ for all $n \in \mathbb{N}$. In addition the solution set

$$\Omega = \left\{x^* \in Sol(C, f) : Ax^* \in Sol(Q, g)\right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{u_n\}$ converge strongly to an element $x^* \in \Omega$ and $\{w_n\}$ converges strongly to an element $Ax^* \in Sol(Q, g)$.

CHAPTER VI

CONCLUSION

In this chapter, we present all results of this thesis including lemmas and theorems.

6.1 Iterative methods for a hybrid pair of generalized I-asymptotically nonexpansive single-valued mappings and generalized nonexpansive multi-valued mappings in Banach spaces

In this section, we present our results about the weak and strong convergence theorems of an iterative method for a hybrid pair of generalized I-asymptotically nonexpansive single-valued mappings and generalized nonexpansive multi-valued mappings in Banach spaces. This results improve and extend the several results in [30, 29, 33, 63, 55, 82].

Theorem 6.1.1. Let D be a nonempty closed convex subset of a Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of generalized I_i-asymptotically nonexpansive single-valued mappings on D with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(r_n^3 - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of asymptotically nonexpansive single-valued mappings on D with a sequence $\{\nu_n\} \subset [1, \infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N Fix(t_i) \cap \bigcap_{i=1}^N Fix(I_i) \cap \bigcap_{i=1}^N Fix(T_i)$ is nonempty closed and $T_ip = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, ..., N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \ \ z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \ \ n \in \mathbb{N}, \end{cases} \tag{6.1.1}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, ..., N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$ if and only if $\liminf_{n\to\infty} \operatorname{dist}(x_n, \mathcal{F}) = 0$.

Next, we prove weak and strong convergence theorems of the proposed iterative method in a uniformly convex Banach space. Moreover, we add uniformly L-Lipschitzian of mappings $\{t_i\}_{i=1}^N$ and $\{T_i\}_{i=1}^N$ satisfy condition (E) in order to reduce closedness of $\mathcal{F}$.

Theorem 6.1.2. Let D be a nonempty compact convex subset of a uniformly convex Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized I_i-asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(r_n^3 - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of uniformly Γ-Lipschitzian and asymptotically nonexpansive single-valued mappings of D into itself with a sequence $\{\nu_n\} \subset [1, \infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ satisfying condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^N Fix(t_i) \cap \bigcap_{i=1}^N Fix(I_i) \cap \bigcap_{i=1}^N Fix(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, ..., N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{cases} \tag{6.1.2}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, ..., N$ such that $0 < a \leq \alpha_n^{(i)}, \beta_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Suppose that $\lim_{n\to\infty} \|z_n^{(i)} - I_i^{(i)} z_n^{(i)}\| = 0$ for all $i = 1, 2, ..., N$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.

Theorem 6.1.3. Let D be a nonempty closed convex subset of a uniformly convex Banach space X with the Opial property. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized I_i-asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^3 - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$ and $\{I_i\}_{i=1}^N$ be a finite family of uniformly Γ-Lipschitzian and asymptotically nonexpansive single-valued mappings of D into itself with a sequence $\{\nu_n\} \subset [1, \infty)$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $KC(D)$ satisfying condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^N Fix(t_i) \cap \bigcap_{i=1}^N Fix(I_i) \cap \bigcap_{i=1}^N Fix(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, ..., N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} I_i^n z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{cases} \tag{6.1.3}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0, 1]$ for all $i = 1, 2, ..., N$ such that $0 < a \leq \alpha_n^{(i)}, \beta_n^{(i)} \leq b < 1$, $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Suppose that $\lim_{n\to\infty} \|z_n^{(i)} - I_i^{(i)} z_n^{(i)}\| = 0$ for all $i = 1, 2, ..., N$. Then the sequence $\{x_n\}$ converges weakly to a point in $\mathcal{F}$.

6.2 Generalized extragradient iterative methods with regularization for solving split feasibility and fixed point problems in Hilbert spaces

In this section, we introduce iterative methods by combining Generalized extragradient iterative methods with regularization due to Ishikawa and Mann iterative methods in solving split feasibility and fixed point problems. Moreover, we prove the weak convergence theorems for proposed iterative methods in Hilbert spaces.

Theorem 6.2.1. Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint of A^*. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be an L-Lipschitzian pseudo-contractive mapping. For

$x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\Big), \\ z_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)y_n\Big), \\ w_n = (1 - \sigma_n)z_n + \sigma_n T z_n, \\ s_n = (1 - \beta_n)z_n + \beta_n T w_n, \\ x_{n+1} = (1 - \gamma_n)z_n + \gamma_n T s_n, \end{cases} \tag{6.2.1}$$

where $\{\lambda_n\} \subset [\kappa, \tau]$ for some $\kappa, \tau \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $0 < a < \gamma_n < b < \beta_n < c < \sigma_n < d < \frac{1}{\sqrt{L^2+1}+1+L^2}$. Then the sequence $\{x_n\}$ generated by algorithm (6.2.1) converges weakly to an element of Γ.

Theorem 6.2.2. Let H_1 and H_2 be two real Hilbert spaces and let C and Q be two nonempty closed convex sets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S : Q \to Q$ be a nonexpansive mapping and let $T : C \to C$ be a continuous pseudo-contractive mapping. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} y_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)x_n\Big), \\ z_n = P_C\Big(x_n - \lambda_n(A^*(I - SP_Q)A + \alpha_n I)y_n\Big), \\ x_{n+1} = \sigma_n z_n + \beta_n T z_n + \gamma_n x_n, \ n \geq 0, \end{cases} \tag{6.2.2}$$

where $\{\lambda_n\} \subset [\kappa, \tau]$ for some $\varpi, \tau \in (0, \frac{1}{\alpha_n + 2\|A\|^2})$, $\{\alpha_n\} \subset (0, \infty)$, $\Sigma_{n=0}^{\infty}\alpha_n < \infty$ and $\{\gamma_n\}, \{\beta_n\}, \{\sigma_n\} \subset (a, b) \subset (0, 1)$ such that $\gamma_n + \beta_n + \sigma_n = 1$. Then the sequence $\{x_n\}$ generated by algorithm (6.2.2) converges weakly to an element of Γ.

6.3 Parallel extragradient-proximal point methods for multiple set split equilibrium problems of pseudomonotone mappings in Hilbert spaces

In this section, we introduce iterative methods by combining the extragradient method with proximal point method for solving multiple set split equilibrium problem when both equilibrium bifunctions are pseudomonotone to obtain weak and strong convergence theorems of the proposed iterative methods in Hilbert spaces.

In order to solving the multiple set split equilibrium problem (MSSEP), we assume that $f : H_1 \times H_1 \to \mathbb{R}$ with $f(x, x) = 0$ for all $x \in C$ satisfies the following conditions:

Assumption A

(A1) f is pseudomonotone on C with respect to $Sol(C, f)$;

(A2) $f(x, \cdot)$ is convex, lower semicontinuous and subdifferentiable on C for all $x \in C$;

(A3) f is weakly continuous on $C \times C$: that is, if $x, y \in C$ and $\{x_n\}, \{y_n\} \subset C$ converge weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$.

(A4) f is Lipschitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$.

Moreover, we assume that $g : H_2 \times H_2 \to \mathbb{R}$ with $g(u, u) = 0$ for all $u \in Q$ satisfies the following conditions:

Assumption B

(B1) g is pseudomonotone on Q;

(B2) $g(u, \cdot)$ is convex and lower semicontinuous for all $u \in Q$;

(B3) $g(\cdot, v)$ is upper semicontinuous for all $v \in Q$;

(B4) There exists $\theta \geq 0$ such that $g(u, v) + g(v, u) \leq \theta \|u - v\|^2$ for all $u, v \in Q$ (g is called undermonotone and θ is the undermonotonicity constant of g).

Algorithm 1 Parallel extragradient-proximal point methods for multiple set split equilibrium problem.

Initialization. Let $x_0 \in C = \bigcap_{i=1}^N C_i$, choose constants $0 < \underline{\rho} \leq \overline{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}, 0 < \underline{\alpha} \leq \overline{\alpha} \leq 1$. For each $i = 1, 2, ..., N$, choose parameters $\{\rho_n^i\} \subset [\underline{\rho}, \bar{\rho}]$, $\{\alpha_n^i\} \subset [\underline{\alpha}, \bar{\alpha}]$, $\Sigma_{i=1}^N \alpha_n^i = 1$ and $\mu \in (0, \frac{2}{\|A\|^2})$.

Step 1. Solve $2N$ strongly convex optimization programs in parallel

$$\begin{cases} y_n^i = \arg\min\{f_i(x_n, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\}, \\ z_n^i = \arg\min\{f_i(y_n^i, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\}, \end{cases}$$

for $i = 1, 2, ..., N$.

Step 2. Compute $\bar{z}_n = \sum_{i=1}^N \alpha_n^i z_n^i$. and $w_n^j = A\bar{z}_n$

Step 3. Solve M regularized multiple set equilibrium programs in parallel

$$g_j(w_n^j, v) + \lambda_n^j \langle w_n^j - A\bar{z}_n, v - w_n^j\rangle \geq 0, \quad \text{for all } \ v \in Q_j, \ j = 1, 2, ..., M.$$

Step 4. Set $\bar{w}_n = \arg\max\{\|w_n^j - A\bar{z}_n\| : j = 1, 2, ..., M\}$.

Step 5. Compute $x_{n+1} = P_C(\bar{z}_n + \mu A^*(\bar{w}_n - A\bar{z}_n))$.

Set $n = n + 1$ and go back **Step 1.**

Theorem 6.3.1. Let C_i and Q_j be two closed and convex subsets of real Hilbert spaces H_1 and H_2 for all $i = 1, 2, ..., N, j = 1, 2, ..., M$, respectively. Let f_i be a bifunction satisfying assumption A on C_i for each $i = 1, 2, ..., N$ and g_j be a bifunction satisfying assumption B on Q_j for all $j = 1, 2, ..., M$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n^j\} \subset (\theta, \bar{\gamma}]$, for some $\bar{\gamma} > \theta$ for all $j = 1, 2, ..., M$. In addition the solution set

$$\Omega = \left\{ x^* \in \bigcap_{i=1}^N Sol(C_i, f_i) : Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j) \right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{y_n^i\}, \{z_n^i\}$, $i = 1, 2, ..., N$ converge weakly to an element $x^* \in \Omega$ and $\{w_n^j\}$, $j = 1, 2, ..., M$ converges weakly to an element $Ax^* \in \bigcap_{j=1}^M Sol(Q_j, g_j)$.

Algorithm 2 Hybrid parallel extragradient-proximal point methods for multiple set split equilibrium problem.

Initialization. Let $x_0 \in C = \bigcap_{i=1}^N C_i$, choose constants $0 < \underline{\rho} \leq \bar{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $0 < \underline{\alpha} \leq \bar{\alpha} \leq 1$. For each $i = 1, 2, ..., N$, choose parameters $\{\rho_n^i\} \subset [\underline{\rho}, \bar{\rho}]$, $\{\alpha_n^i\} \subset [\underline{\alpha}, \bar{\alpha}]$, $\Sigma_{i=1}^N \alpha_n^i = 1$ and $\mu \in (0, \frac{2}{\|A\|^2})$.

Step 1. Solve $2N$ strongly convex optimization programs in parallel

$$\begin{cases} y_n^i = \arg\min\{f_i(x_n, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\}, \\ z_n^i = \arg\min\{f_i(y_n^i, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\}, \end{cases}$$

for $i = 1, 2, ..., N$.

Step 2. Compute $\bar{z}_n = \sum_{i=1}^{N} \alpha_n^i z_n^i$.

Step 3. Solve M regularized multiple set equilibrium programs in parallel

$$g_j(w_n^j, y) + \lambda_n^j \langle w_n^j - A\bar{z}_n, y - w_n^j \rangle \geq 0, \text{ for all } y \in Q_j, \; j = 1, 2, ..., M.$$

Step 4. Set $\bar{w}_n = \arg\max\{\|w_n^j - A\bar{z}_n\| : j = 1, 2, ..., M\}$.

Step 5. Compute $t_n = P_C(\bar{z}_n + \mu A^*(\bar{w}_n - A\bar{z}_n))$.

Step 6. Take $x_{n+1} = P_{C_{n+1}}(x_0)$, where

$$C_{n+1} = \{v \in H : \|t_n - v\| \leq \|\bar{z}_n - v\| \leq \|x_n - v\|\}.$$

Set $n = n + 1$ and go back **Step 1.**

Theorem 6.3.2. Let C_i and Q_j be two closed and convex subsets of real Hilbert spaces H_1 and H_2 for all $i = 1, 2, ..., N, j = 1, 2, ..., M$, respectively. Let f_i be a bifunction satisfying assumption A on C_i for all $i = 1, 2, ..., N$ and g_j be a bifunction satisfying assumption B on Q_j for all $j = 1, 2, ..., M$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n^j\} \subset (\theta, \bar{\gamma}]$, for some $\bar{\gamma} > \theta$ for all $j = 1, 2, ..., M$. In addition the solution set

$$\Omega = \left\{ x^* \in \bigcap_{i=1}^{N} Sol(C_i, f_i) : Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j) \right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{y_n^i\}, \{z_n^i\}$, $i = 1, 2, ..., N$ converge strongly to an element $x^* \in \Omega$ and $\{w_n^j\}$, $j = 1, 2, ..., M$ converges strongly to an element $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$.

6.4 Parallel extragradient-proximal point methods with linesearch for multiple set split equilibrium problems of pseudomonotone mappings in Hilbert spaces

In this section, we introduce iterative methods by combining the extragradient method with linesearch proximal point method for solving multiple set split equilibrium problem when both equilibrium bifunctions are pseudomonotone to obtain weak and strong convergence theorems of the iterates generated by the proposed iterative methods are obtained under certain for equilibrium bifunctions and parameters in Hilbert spaces.

In order to solving the multiple set split equilibrium problems (MSSEP), we assume that $f : H_1 \times H_1 \longrightarrow \mathbb{R}$ with $f(x,x) = 0$ for all $x \in C$ satisfies the following conditions:

Assumption A

(A1) f is pseudomonotone on C with respect to $Sol(C, f)$;

(A2) $f(x, \cdot)$ is convex, lower semicontinouns and subdifferentiable on C for all $x \in C$;

(A3) f is weakly continuous on $C \times C$: that is, if $x, y \in C$ and $\{x_n\}, \{y_n\} \subset C$ converge weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$.

Moreover, we assume that $g : H_2 \times H_2 \longrightarrow \mathbb{R}$ with $g(u, u) = 0$ for all $u \in Q$ satisfies the following conditions:

Assumption B

(B1) g is pseudomonotone on Q;

(B2) $g(u, \cdot)$ is convex, lower semicontinuous for all $u \in Q$;

(B3) $g(\cdot, v)$ is upper semicontinuous for all $v \in Q$;

(B4) There exists $\kappa \geq 0$ such that $g(u,v) + g(v,u) \leq \kappa\|u-v\|^2$ for all $u, v \in Q$ (g is called undermonotone and κ is the undermonotonicity constant of g).

Algorithm 3 Parallel Extragradient-Proximal Methods with linesearch

Initialization. Let $x_0 \in C = \bigcap_{i=1}^N C_i$, choose constants $\eta, \theta \in (0,1), 0 < \underline{\rho} \leq \overline{\rho}, 0 < \underline{\alpha} \leq \overline{\alpha} \leq 1$ and $0 < \underline{\gamma} < \overline{\gamma} < 2$. For each $i = 1, 2, ..., N, j = 1, 2, ..., M$, choose parameters $\{\rho_n^i\} \subset [\underline{\rho}, \overline{\rho}], \gamma_n \subset [\underline{\rho}, \overline{\rho}], \{\alpha_n^i\} \subset [\underline{\alpha}, \overline{\alpha}], \Sigma_{i=1}^N \alpha_n^i = 1$ and $\mu \in (0, \frac{1}{\|A\|^2})$.

Step 1. Solve N strongly convex optimization programs in parallel

$$y_n^i = \arg\min\{f_i(x_n, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\}, \tag{6.4.1}$$

for $i = 1, 2, ..., N$.

Step 2. Armijo linesearch rule. Find m_n as the smallest positive integer number m such that

$$\begin{cases} z_{n,m}^i = (1-\eta^m)x_n + \eta^m y_n^i, \\ f_i(z_{n,m}^i, x_n) - f_i(z_{n,m}^i, y_n^i) \geq \frac{\theta}{2\rho_n^i}\|x_n - y_n^i\|^2. \end{cases} \tag{6.4.2}$$

Set $\eta_n = \eta^{m_n}, z_n^i = z_{n,m_n}^i$.

Step 3. Select $\varepsilon_n^i \in \partial_2 f_i(z_n^i, x_n)$ and compute $\sigma_n^i = \frac{f_i(z_n^i, x_n)}{\|\varepsilon_n^i\|^2}$ and $u_n^i = P_{C_i}(x_n - \gamma_n \sigma_n^i \varepsilon_n^i)$.

Step 4. Compute $\bar{u}_n = \sum_{i=1}^N \alpha_n^i u_n^i$.

Step 5. Solve M regularized multiple set equilibrium programs in parallel

$$g_j(w_n^j, y) + \lambda_n^j\langle w_n^j - A\bar{u}_n, y - w_n^j\rangle \geq 0 \quad y \in Q_j, \; j = 1, 2, ..., M.$$

Step 6. Set $\bar{w}_n = \arg\max\{\|w_n^j - A\bar{u}_n\| : j = 1, 2, ..., M\}$.

Step 7. Compute $x_{n+1} = P_C(\bar{u}_n + \mu A^*(\bar{w}_n - A\bar{u}_n))$.

Set $n = n + 1$ and go back **Step 1.**

Theorem 6.4.1. Let C_i and Q_j be two closed and convex subsets of real Hilbert spaces H_1 and H_2 for all $i = 1, 2, ..., N, j = 1, 2, ..., M$, respectively. Let f_i be a bifunction consistent with assumptions A on C_i for each $i = 1, 2, ..., N$ and g_j be a bifunction consistent with assumptions B on Q_j for all $j = 1, 2, ..., M$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n^j\}$ be a bounded sequence such that $\{\lambda_n^j\} \subset (\kappa_{n-1}, +\infty)$ for all $n \in \mathbb{N}$. In addition the solution set

$$\Omega = \left\{ x^* \in \bigcap_{i=1}^{N} Sol(C_i, f_i) : Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j) \right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{u_n^i\}, i = 1, 2, ..., N$ converge weakly to an element $x^* \in \Omega$ and $\{w_n^j\}, j = 1, 2, ..., M$ converges weakly to an element $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$.

Algorithm 4 Hybrid Parallel Extragradient-Proximal Methods with linesearch

Initialization. Let $x_0 \in C = \bigcap_{i=1}^{N} C_i$, choose constants $\eta, \theta \in (0, 1), 0 < \underline{\rho} \leq \overline{\rho}, 0 < \underline{\alpha} \leq \overline{\alpha} \leq 1$ and $0 < \underline{\gamma} < \overline{\gamma} < 2$. For each $i = 1, 2, ..., N, j = 1, 2, ..., M$, choose parameters $\{\rho_n^i\} \subset [\underline{\rho}, \overline{\rho}], \gamma_n \subset [\underline{\rho}, \overline{\rho}], \{\alpha_n^i\} \subset [\underline{\alpha}, \overline{\alpha}], \Sigma_{i=1}^{N} \alpha_n^i = 1$ and $\mu \in (0, \frac{2}{\|A\|^2})$.

Step 1. Solve N strongly convex optimization programs in parallel

$$y_n^i = \arg\min\{f_i(x_n, y) + \frac{1}{2\rho_n^i}\|y - x_n\|^2 : y \in C_i\}, \tag{6.4.3}$$

for $i = 1, 2, ..., N$.

Step 2. Armijo linesearch rule.

Find m_n as the smallest positive integer number m such that

$$\begin{cases} z_{n,m}^i = (1 - \eta^m)x_n + \eta^m y_n^i, \\ f_i(z_{n,m}^i, x_n) - f_i(z_{n,m}^i, y_n^i) \geq \frac{\theta}{2\rho_n^i}\|x_n - y_n^i\|^2, \end{cases} \tag{6.4.4}$$

Set $\eta_n = \eta^{m_n}, z_n^i = z_{n,m_n}^i$.

Step 3. Select $\varepsilon_n^i \in \partial_2 f_i(z_n^i, x_n)$ and compute $\sigma_n^i = \frac{f_i(z_n^i, x_n)}{\|\varepsilon_n^i\|^2}$ and $u_n^i = P_{C_i}(x_n - \gamma_n \sigma_n^i \varepsilon_n^i)$.

Step 4. Compute $\bar{u}_n = \sum_{i=1}^{N} \alpha_n^i u_n^i$.

Step 5. Solve M regularized multiple set equilibrium programs in parallel

$$g_j(w_n^j, y) + \lambda_n^j \langle w_n^j - A\bar{u}_n, y - w_n^j \rangle \geq 0 \quad y \in Q_j, \; j = 1, 2, ..., M.$$

Step 6. Set $\bar{w}_n = \arg\max\{\|w_n^j - A\bar{u}_n\| : j = 1, 2, ..., M\}$.

Step 7. Compute $t_n = P_C(\bar{u}_n + \mu A^*(\bar{w}_n - A\bar{u}_n))$.

Step 8. Take $x_{n+1} = P_{C_{n+1}}(x_0)$, where

$$C_{n+1} = \{v \in H : \|t_n - v\| \leq \|\bar{u}_n - v\| \leq \|x_n - v\|\}.$$

Set $n = n + 1$ and go back **Step 1.**

Theorem 6.4.2. Let C_i and Q_j be two closed and convex subsets of real Hilbert spaces H_1 and H_2 for all $i = 1, 2, ..., N, j = 1, 2, ..., M$, respectively. Let f_i be a bifunction consistent with assumptions A on C_i for all $i = 1, 2, ..., N$ and g_j be a bifunction consistent with assumptions B on Q_j for all $j = 1, 2, ..., M$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^* and $\{\lambda_n^j\}$ be a bounded sequence such that $\{\lambda_n^j\} \subset (\kappa_{n-1}, +\infty)$ for all $n \in \mathbb{N}$. In addition the solution set

$$\Omega = \left\{ x^* \in \bigcap_{i=1}^{N} Sol(C_i, f_i) : Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j) \right\} \neq \emptyset,$$

then the sequences $\{x_n\}, \{u_n^i\}, i = 1, 2, ..., N$ converge strongly to an element $x^* \in \Omega$ and $\{w_n^j\}, j = 1, 2, ..., M$ converges strongly to an element $Ax^* \in \bigcap_{j=1}^{M} Sol(Q_j, g_j)$.

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