GENERALIZATION OF SEMIGROUPS OF TRANSFORMATIONS PRESERVING EQUIVALENCE RELATIONS

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ABSTRACT

Let $T(X)$ denote the full transformations semigroup on a nonempty set X. For an equivalence relation σ on X, let

$$T(X,\sigma) = \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \sigma\}.$$

Then $T(X,\sigma)$ forms a subsemigroup of $T(X)$. In this thesis, we consider a generalization of the semigroup $T(X,\sigma)$ as follows:

$$T(X,\sigma,\rho) = \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \rho\}$$

where σ and ρ are equivalence relations on X with $\rho \subseteq \sigma$. The purpose of this thesis is to find characterizations of the regularity of elements in $T(X,\sigma,\rho)$. Moreover, we present a necessary and sufficient condition under which the semigroup $T(X,\sigma,\rho)$ is regular, inverse and abundant. Besides, necessary and sufficient conditions for two elements of $T(X,\sigma,\rho)$ are inverse of each other are investigated. Finally, the Green's relations on the semigroup $T(X,\sigma,\rho)$ are described.

LIST OF CONTENTS

LIST OF CONTENTS (CONT.)

CHAPTER I

INTRODUCTION

For an arbitrary nonempty set X, let $T(X)$ denote the semigroup (under composition) of all full transformations on X (that is, all mappings $\alpha : X \to X$). It is well-known that every semigroup is isomorphic to a subsemigroup of some full transformation semigroups. Hence in order to study structures of semigroups, it suffices to consider in subsemigroups of $T(X)$. Therefore, several researchers are interested in characterizations of subsemigroups of the full transformation semigroup.

One of great important topics in semigroup theory is regularity consideration. For an element a of a semigroup S, a is called *regular* if there exists $x \in S$ such that $a = axa$. We call that a semigroup S is *regular* if every element of S is regular. Regular semigroups were introduced by Green [1] in his influential 1951 paper "On the structure of semigroups". The concept of regularity in a semigroup was adapted from an analogous condition for rings, already considered by Neumann [2]. It was Green's study of regular semigroups which led him to define his celebrated relations. According to a footnote in Green 1951, the suggestion that the notion of regularity be applied to semigroups was first made by Rees [3, 4]. This property of regular elements was first observed by Thierrin [5] in 1952.

Another important kind of the regularity was introduced by Clifford [6] in 1941, who studied elements a of a semigroup S having the property that there exists $x \in S$ such that $a = axa$ and $ax = xa$, which we call now *completely regular elements*, and semigroups whose any element is completely regular, called *completely regular semigroups*. The complete regularity was also investigated by Croisot [7] in 1953, who also studied elements a of a semigroup S for which $a \in Sa^2$ (resp. $a \in a^2S$), called *left regular* (resp. *right regular*) *elements*, and semigroups whose every element is left regular (resp. right regular), called *left regular* (resp. *right regular*) *semigroups*. Notice that for any elements of a semigroup S which is both left and right regular is also regular in S but the converse is not true in general. In particular, the regularity, the left regularity, the right regularity and the completely regularity are coincide in commutative semigroups.

If a is a regular element of a semigroup S, then the element x, whose existence was postulated by the definition of the regularity, can be chosen such that $a = axa$ and $x = xax$, and any element x satisfying this condition, which is not necessary unique, is called an *inverse* of a. A regular semigroup whose any element has a unique inverse is called an *inverse semigroup.* Inverse semigroups were first defined and investigated by Vagner [8, 9] in 1952 and 1953, who called them *generalized groups,* and independently by Preston [10, 11, 12] in 1954.

An element a of a semigroup S is called *idempotent* if $a^2 = a$. If there exists x in S such that ax is idempotent of S, then we call an element a of S that *E-inversive.* A semigroup S is called an *E-inversive semigroup* if every element of S is E-inversive. Clearly, regular semigroups and finite semigroups are E-inversives. The concept of E-inversive semigroup was introduced by Thierrin [13] in 1955 and the basic properties of E-inversive semigroups were given by Catino and Miccoli [14], Mitsch [15] and Mitsch and Petrich [16, 17].

Another one topic is Green's relations. These were relations on semigroups introduced in 1951 by Green [1]. Let S be a semigroup and $a, b \in S$. If a and b generate the same left principal ideal, that is, $S^1a = S^1b$, then we say that *a and b are $\mathcal{L}$ relate* and write $(a, b) \in \mathcal{L}$. If a and b generate the same right principal ideal, that is, $aS^1 = bS^1$, then we say that *a and b are $\mathcal{R}$ relate* and write $(a, b) \in \mathcal{R}$. If a and b generate the same principal ideal, that is, $S^1aS^1 = S^1bS^1$, then we say that *a and b are $\mathcal{J}$ relate* and write $(a, b) \in \mathcal{J}$. Let $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. It is well-known that these five relations are equivalence relations and called *Green's relations* on S.

On a semigroup S, the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on S are generalizations of the familiar Green's relations $\mathcal{R}$ and $\mathcal{L}$ which first introduced by Pastijn in [18] and adopted by Fountain in [19, 20]. The relation $\mathcal{L}^*$ is defined by the rule that $(a, b) \in \mathcal{L}^*$ if and only if a and b are related by the Green's relation $\mathcal{L}$ in a semigroup T such that S is a subsemigroup of T. The relation $\mathcal{R}^*$ can be defined dually. In a regular semigroup $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$. For instance, Fountain [20] pointed out that a semigroup S has the property that for every $a \in S$ the left ideal S^1a is projective if and only if every $\mathcal{L}^*$-class of S contains idempotent. We call such semigroups *left abundant* (they are also called *rpp semigroups* in the literature). *Right abundant* semigroups are defined dually.

A semigroup is *abundant* if it is both left and right abundant. It is known that a regular semigroup is abundant but the converse is not true. For example, Umar [21] showed that the semigroup of order-decreasing finite full transformations is abundant but not regular. The property of being abundant can be considered as a wide generalization of regularity.

A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$, and we say a subsemigroup B of S is a *bi-ideal* of S if $BSB \subseteq B$. Then quasi-ideals are a generalization of one-sided ideals and bi-ideals are a generalization of quasi-ideals. The notion of quasi-ideals for semigroups was first introduced by Steinfeld [22] in 1956 while the notion of bi-ideals for semigroups was introduced earlier by Good and Hughes [23] in 1952. Kapp [24] used **BQ** to denote the class of semigroups whose sets of bi-ideals and quasi-ideals coincide in 1969. A semigroup S is called a *BQ-semigroup* if $S \in \mathbf{BQ}$. In 1961, Lajos [25] showed that every regular semigroup is BQ-semigroup. Also, in 1972, Mielke [26] described the structure of Green's relations on BQ-semigroups. Later in 2001, Kemprasit and Baupradist [27] showed that for any positive integer n, the multiplicative semigroup $\mathbb{Z}_n$ has the property that the set of bi-ideals and the set of quasi-ideals coincide if and only if either $n = 4$ or n is square-free (there is no $a \in \mathbb{Z}$ such that $a > 1$ and $a^2|n$) and [28] considered BQ-semigroup on the multiplicative interval semigroup and the additive interval semigroup on $\mathbb{R}$ in 2004.

Many years passed, subsemigroups of $T(X)$ have been studied by a number of semigroup theorists. Particularly, characterization of the regularity, the left regularity, the right regularity, the completely regularity, Green's relations, inverse semigroups and abundant semigroups on subsemigroups of $T(X)$ have been investigated, see [21], [29-42].

Let σ be an equivalence relation on a nonempty set X. In 2005, Pei [29] has studied Green's relations and the regularity on a subsemigroup of $T(X)$ defined by

$$T(X,\sigma) = \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \sigma\}$$

and call it the *semigroup of transformations that preserve an equivalence on* X. In 2013, Nammak and Laysilikul [30] gave necessary and sufficient conditions when elements of $T(X,\sigma)$ to be left regular, right regular and completely regular. Pei and Zhou [31] proved that $T(X,\sigma)$ is abundant but not regular if the equivalence relation σ is simple (there is exactly one σ-class A ($\neq X$) containing more than one point and the other

σ-classes are singletons) or 2-bounded (the cardinality of every σ-class is not more than 2). Later, Deng, Zeng and Xu [32] introduced a subsemigroup of $T(X)$ defined by

$$T(X,\sigma^*) := \{\alpha \in T(X) : \forall x,y \in X, (x,y) \in \sigma \text{ if and only if } (x\alpha, y\alpha) \in \sigma\},$$

the so-called *semigroups of transformations that preserve double direction equivalence on* X. They investigated the regularity and Green's relations on $T(X,\sigma^*)$. Later, Laysilikul and Namnak [33] investigated a necessary and sufficient condition for the left regularity, the right regularity and the completely regularity of elements in $T(X,\sigma^*)$. Deng [34] discussed the Green's *-relations, certain *-ideal and certain Rees quotient semigroup for the semigroup $T(X,\sigma^*)$ and proved that regular and abundant in the semigroup $T(X,\sigma^*)$ coincided. Mendes-Gonçalves and Sullivan [35] introduced a subsemigroup of $T(X)$ defined by

$$E(X,\sigma) = \{\alpha \in T(X) : \forall x,y \in X, (x,y) \in \sigma \text{ implies } (x,y) \in \ker\alpha\},$$

where $\ker\alpha = \{(x,y) \in X \times X : x\alpha = y\alpha\}$ and call it the *semigroup of transformations restricted by an equivalence* σ. The authors characterized Green's relations on the largest regular subsemigroup of $E(X,\sigma)$. They also showed that if $|X| \geq 2$ and $\sigma \neq I_X = \{(x,x) : x \in X\}$, then $E(X,\sigma)$ is not isomorphic to $T(Z)$ for any set Z. Sun and Wang [36] proved that $E(X,\sigma)$ is right abundant but not left abundant whenever the equivalence σ on the set X ($|X| \geq 3$) is non-trivial.

The semigroups $T(X,\sigma)$ and $E(X,\sigma)$ motivate us to define $T(X,\sigma,\rho)$ as follows:

$$T(X,\sigma,\rho) = \{\alpha \in T(X) : \forall x,y \in X, (x,y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \rho\}$$

where σ and ρ are equivalence relations on a nonempty set X with $\rho \subseteq \sigma$. It is easy to see that $T(X,\sigma,\rho)$ is a subsemigroup of $T(X)$. Notice that the identity transformation on X, namely i_X, contained in $T(X,\sigma)$ but need not belong to $T(X,\sigma,\rho)$. If $\sigma = I_X$ or $\rho = X \times X$, then $i_X \in T(X,\sigma,\rho)$. In particular, if $\sigma = I_X$ or $\rho = X \times X$, then $T(X,\sigma,\rho) = T(X)$, if $\rho = \sigma$, then $T(X,\sigma,\rho) = T(X,\sigma)$ and if $\rho = I_X$, then $T(X,\sigma,\rho) = E(X,\sigma)$. Hence $T(X,\sigma,\rho)$ is a generalization of $T(X,\sigma)$ and $E(X,\sigma)$.

The purpose of this thesis is to find characterizations of the regularity, the left regularity, the right regularity and the completely regularity of elements in $T(X,\sigma,\rho)$ and to find a necessary and sufficient condition for two elements of $T(X,\sigma,\rho)$ are inverse of each other. Moreover, we aim to study the Green's relations on the semigroup $T(X,\sigma,\rho)$ and to present a necessary and sufficient condition under which the semigroup $T(X,\sigma,\rho)$ is regular, left regular, right regular, completely regular, inverse and abundant.

Let Y be a fixed nonempty subset of X. In 1975, Symons [37] considered the subsemigroup of $T(X)$ defined by

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}.$$

The author described all the automorphisms of this semigroup. Moreover, he determined when the two semigroups of this type are isomorphic (see [37]). In 2005, Nenthein, Youngkhong and Kemprasit [38] characterized regular elements of $T(X,Y)$ and determined the numbers of regular elements in $T(X,Y)$ for a finite set X. Moreover, Nenthein and Kemprasit [39] proved that $T(X,Y)$ is a BQ-semigroup. In 2008, Sanwong and Sommanee [40] described $T(X,Y)$ to be regular and determined the Green's relations on $T(X,Y)$. Also, a class of maximal inverse subsemigroups of $T(X,Y)$ is obtained.

For equivalence relations σ and ρ on a nonempty set X with $\rho \subseteq \sigma$. In this thesis, we show that $T(X,\sigma,\rho)$ can be embeddable in $T(Y,Z)$ for some sets Y, Z with $Z \subseteq Y$ but not isomorphic to $T(Y,Z)$.

Furthermore, we also consider the subsemigroup of $T(X)$ which introduced by Araújo and Konieczny [41], namely,

$$T(X,\sigma,R) = \{\alpha \in T(X) : R\alpha \subseteq R \text{ and } \forall x,y \in X, (x,y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \sigma\},$$

where R is a cross-section of the partition X/σ induced by σ. It is called the *semigroups of transformations preserving an equivalence relation and a cross-section*. Clearly, $T(X,\sigma,R) \subseteq T(X,\sigma)$. They have been proved that the semigroups $T(X,\sigma,R)$ are precisely the centralizers of idempotents of $T(X)$. After year, they discussed the structure of $T(X,\sigma,R)$ in terms of Green's relations, described the regular elements of $T(X,\sigma,R)$ and determined the following classes of the semigroups $T(X,\sigma,R)$: regular, abundant, inverse and completely regular in [42].

In this thesis, we define a new subsemigroup of $T(X,\sigma)$ as follows:

$$T_R(X,\sigma) = \{\alpha \in T(X) : R\alpha = R \text{ and } \forall x,y \in X, (x,y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \sigma\}$$

where R is a cross-section of the partition X/σ induced by an equivalence σ and call it the *semigroups of transformations preserving an equivalence relation and fix a cross-section.* However, the regularity, the left regularity, the right regularity, the completely regularity, Green's relations, inverse and abundant are described on the semigroup $T_R(X, \sigma)$.

The remainder of this thesis consists of six chapters. Chapter II contains precise definitions, notations and some useful results which will be used in the later chapters. In addition, we give some relationships between some subsemigroups of $T(X)$ in the last section. In chapter III, we present characterizations for the regularity, the left regularity, the right regularity and the completely regularity of elements and semigroups in some subsemigroups of $T(X)$. In chapter IV, Green's relations on some subsemigroups of $T(X)$ are described. We investigate some algebraic structures on some subsemigroups of $T(X)$ in chapter V. Chapter VI, we conclude the results of the thesis.

CHAPTER II

PRELIMINARIES

In this chapter, we give some precise definitions, notations and basic results which will be used in our study. Moreover, we will show some necessary propositions that we usually refer to and we show some relationships among subsemigroups of $T(X)$ in the rest of this chapter.

2.1 Elementary concepts

A *binary operation* $\cdot$ on a nonempty set S is a function from $S \times S$ into S. A *semigroup* $(S, \cdot)$ is a nonempty set S with a binary operation denoted by $\cdot$, called *multiplication*, satisfying the associativity law:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

for all $a, b, c \in S$. Generally, the symbol for multiplication will be omitted, so, the *product of elements* a and b from S will be denoted by juxtaposition ab. But, when several operations are present, we may write $a \circ b$ and $a * b$.

A nonempty subset T of a semigroup S is a *subsemigroup* of S if T forms a semigroup under the same binary relation on S.

Theorem 2.1.1. [43] *Let T be a nonempty subset of a semigroup S. Then T is a subsemigroup of S if and only if for every $a, b \in T$, $ab \in T$.*

An element a of a semigroup S is *regular* if $a = axa$ for some $x \in S$. An element a of S is *left* (*right*) *regular* if $a = xa^2$ ($a = a^2x$) for some $x \in S$. An element a of S is *completely regular* if $a = axa$ and $ax = xa$ for some $x \in S$. A semigroup S is a (*left, right, completely*) *regular semigroup* if all its elements are (left, right, completely) regular.

Theorem 2.1.2. [44] *Let S be a semigroup and $a \in S$. Then a is completely regular if and only if a is both a left and a right regular element of S.*

An element x of a semigroup S is an *inverse* of an element a in S if $a = axa$ and $x = xax$. The relationship between regularity and inverses is given by following simple result.

Lemma 2.1.3. [45] *An element a of a semigroup S has an inverse if and only if a is regular.*

A semigroup S in which every element has a unique inverse is an *inverse semigroup*.

An element a of a semigroup S is *idempotent* if $a = a^2$.

Theorem 2.1.4. [45] *A semigroup S is an inverse semigroup if and only if it is regular and its idempotents commute.*

An element a of a semigroup S is *E-inversive* if there exists x in S such that ax is idempotent of S. A semigroup S is called an *E-inversive semigroup* if every element of S is E-inversive.

Theorem 2.1.5. [16] *Every regular element of a semigroup S is E-inversive.*

An element a of a semigroup S is *left zero* (*right zero*) if $ax = a$ ($xa = a$) for all $x \in S$. An element a is *zero* if it is both a left and a right zero element. A semigroup S is a *left zero* (right zero, zero) *semigroup* if every element of S is a left zero (right zero, zero) element.

The next lemma is easy to verify.

Lemma 2.1.6. *The following statements hold.*

(1) *Every left (right) zero of a semigroup S is regular.*

(2) *Every left (right) zero of a semigroup S is left regular.*

(3) *Every left (right) zero of a semigroup S is right regular.*

(4) *Every left (right) zero of a semigroup S is completely regular.*

(5) *Every left (right) zero of a semigroup S is E-inversive.*

Let A and B be subsets of a semigroup S. Denote

$$AB = \{ab : a \in A \text{ and } b \in B\}.$$

If $A = \{a\}$, then we also write aB instead of $\{a\}B$ and similarly $A\{b\} = Ab$ if $B = \{b\}$.

Let S be a semigroup. A nonempty subset A of S is a *left* (*right*) *ideal* of S if $SA \subseteq A$ ($AS \subseteq A$). A left or a right ideal of S is often called a *one-sided ideal* of a semigroup S. Further, A is an (*two-sided*) *ideal* of S if A is both a left and a right ideal of S, i.e. if $SA \cup AS \subseteq A$. A subset A of S is a *quasi-ideal* of S if $SA \cap AS \subseteq A$. A subsemigroup A of S is a *bi-ideal* of S if $ASA \subseteq A$. It is easy to prove that a subset A of S is a quasi-ideal if and only if A is an intersection of left and right ideals of S. Clearly, every quasi-ideal of S is a bi-ideal of S, every left (right) ideal of S is a quasi-ideal of S and every ideal of S is a left and a right ideal of S.

The intersection of ideals of a semigroup S is again an ideal of S if it is a nonempty set. We now introduce the following notions.

Let a be an element of a semigroup S. The *principal left ideal of S generated by an element a* is the intersection of all left ideals which contain a, and it is denoted by $(a)_l$. The *principal right ideal, principal* (*two-sided*) *ideal, principal quasi-ideal, principal bi-ideal* of S are analogously defined and are denoted by $(a)_r$, (a), $(a)_q$ and $(a)_b$, respectively.

Lemma 2.1.7. [43] *Let a be an element of a semigroup S. The following statements hold.*

(1) $(a) = \{a\} \cup aS \cup Sa \cup SaS$.

(2) $(a)_l = \{a\} \cup Sa$.

(3) $(a)_r = \{a\} \cup aS$.

(4) $(a)_q = \{a\} \cup (aS \cap Sa)$.

(5) $(a)_b = \{a, a^2\} \cup aSa$.

A *relation* σ on a set X is a subset of $X \times X$. To simplify notation, we write $x\sigma y$ instead of $(x, y) \in \sigma$, for any elements x and y in X. A special relation that is worth mentioning here is the *identity relation* on X,

$$I_X = \{(x, x) : x \in X\}$$

that is, two elements are related if and only if they are equal. Also we have the *universal relation* $X \times X$, in which everything is related to everything.

Let X be a nonempty set and σ a relation on X. Then

(1) σ is *reflexive* if $(x, x) \in \sigma$ for all $x \in X$.

(2) σ is *symmetric* if $(x, y) \in \sigma$ implies $(y, x) \in \sigma$ for all $x, y \in X$.

(3) σ is *anti-symmetric* if $(x, y), (y, x) \in \sigma$ imply $x = y$ for all $x, y \in X$.

(4) σ is *transitive* if $(x, y), (y, z) \in \sigma$ imply $(x, z) \in \sigma$ for all $x, y, z \in X$.

If σ satisfies (1), (3) and (4), then σ is a *partially order* on X. And we call σ an *equivalence relation* on X if σ satisfies (1), (2) and (4).

Let X be a nonempty set and σ an equivalence relation on X, we denote

$$X/\sigma = \{\{x \in X : (x, a) \in \sigma\} : a \in X\}$$

and call $A \in X/\sigma$ an *equivalence class* or *σ-class*. Denote by $x\sigma$ the σ-class containing x for all $x \in X$. A *cross-section* R of the partition X/σ induced by σ is a set consisting exactly one point in each σ-class.

An equivalence relation σ on X is said to be *T-relation* if there is at most one σ-class containing two or more elements. For every $n \in \mathbb{N}$, an equivalence relation σ on X is said to be *n-bounded* if the cardinality of every σ-class is not more than n.

Let π be a collection of nonempty subsets of X. We say that π is a *partition* of X if π satisfies the following conditions.

(1) $\cup\pi = X$ and

(2) for every $A, B \in \pi$, $A \cap B \neq \emptyset$ implies $A = B$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of a set X. we write $\mathcal{P} \preceq \mathcal{Q}$ if for every $P \in \mathcal{P}$, there exists $Q \in \mathcal{Q}$ such that $P \subseteq Q$. It is obvious that $\preceq$ is a partial order on the set of all partitions of X.

Theorem 2.1.8. [50] *Let σ be arbitrary equivalence relation on a nonempty set X. Then X/σ is a partition of X.*

Theorem 2.1.9. [50] *Let π be a partition of a nonempty set X. Then $\sigma = \bigcup_{A\in\pi}(A \times A)$ is an equivalence relation on X and $X/\sigma = \pi$.*

By using principal ideals of certain elements of a semigroup S, we can define various very important relations on S. Let $a, b \in S$. Relations $\mathcal{J}, \mathcal{L}, \mathcal{R}, \mathcal{D}$ and $\mathcal{H}$ defined on S by

$$\begin{aligned}
(a,b) \in \mathcal{J} &\Leftrightarrow (a) = (b),\\
(a,b) \in \mathcal{L} &\Leftrightarrow (a)_l = (b)_l,\\
(a,b) \in \mathcal{R} &\Leftrightarrow (a)_r = (b)_r,\\
\mathcal{D} &= \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L},\\
\mathcal{H} &= \mathcal{L} \cap \mathcal{R},
\end{aligned}$$

are *Green's relations* or *Green's equivalences.* For any element a of a semigroup S and $\mathcal{T} \in \{\mathcal{J}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{H}\}$, the $\mathcal{T}$-class of S containing a is denoted by $\mathcal{T}_a$.

Theorem 2.1.10. [45] *Let a be a regular element of a semigroup S. Then for every $x \in \mathcal{D}_a$, x is regular.*

Let S be a semigroup, which has no an identity element. Find a symbol not in S, call it 1. We now extend the definition of $*$ on S to $S \cup \{1\}$ by

$$1 * 1 = 1, 1 * s = s * 1 = s \text{ for all } s \in S \text{ and } a * b = ab \text{ for all } a, b \in S.$$

Then $*$ is associative. Thus we have managed to extend multiplication in S to $S \cup \{1\}$. For an arbitrary semigroup S, the semigroup with identity S^1 is defined by

$$S^1 = \begin{cases} S & \text{if } S \text{ is a semigroup with identity,}\\ S \cup \{1\} & \text{if } S \text{ has no identity.}\end{cases}$$

Green's relations for any semigroup S are well known [45].

Theorem 2.1.11. [45] *Let a and b be elements of a semigroup S. Then the following statements hold.*

(1) $(a,b) \in \mathcal{L}$ *if and only if there exist* $x, y \in S^1$ *such that* $a = xb$ *and* $b = ya$.

(2) $(a,b) \in \mathcal{R}$ *if and only if there exist* $x, y \in S^1$ *such that* $a = bx$ *and* $b = ay$.

(3) $(a,b) \in \mathcal{J}$ *if and only if there exist* $w, x, y, z \in S^1$ *such that* $a = wbx$ *and* $b = yaz$.

Let S be a semigroup. The relation $\mathcal{L}^*$ is defined by the rule that $(a,b) \in \mathcal{L}^*$ on S if and only if a and b are related by the Green's relation $\mathcal{L}$ in a semigroup T such that S is a subsemigroup of T. The relation $\mathcal{R}^*$ can be defined dually.

The following and its dual give a characterization of $\mathcal{L}^*$ and $\mathcal{R}^*$.

Lemma 2.1.12. [20] *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(1) $(a,b) \in \mathcal{L}^*$ *on* S.

(2) *For every* $x, y \in S^1$, $ax = ay$ *if and only if* $bx = by$.

Lemma 2.1.13. [20] *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(1) $(a,b) \in \mathcal{R}^*$ *on* S.

(2) *For every* $x, y \in S^1$, $xa = ya$ *if and only if* $xb = yb$.

A semigroup S is *left* (*right*) *abundant* if every $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) of S contains idempotent. A semigroup S is *abundant* if it is both left and right abundant.

Theorem 2.1.14. [20] *Every regular semigroup is abundant.*

For a nonempty subset A of a semigroup S, $(A)_q$ and $(A)_b$ denote respectively the quasi-ideal and the bi-ideal of S generated by A, that is, $(A)_q$ is the intersection of all quasi-ideals of S containing A and $(A)_b$ is the intersection of all bi-ideals of S containing A. We have the following proposition.

Proposition 2.1.15. [45] *For a nonempty subset A of a semigroup S,*

$$(A)_q = A \cup (SA \cap AS) \text{ and } (A)_b = A \cup A^2 \cup ASA.$$

A semigroup S is a BQ-semigroup if every bi-ideal of S is a quasi-ideal.

Calais [46] gave a characterization of the BQ-semigroups as follows.

Proposition 2.1.16. [46] *A semigroup S is a BQ-semigroup if and only if $(\{x, y\})_b = (\{x, y\})_q$ for all x, $y \in S$.*

Lemma 2.1.17. [39] *Every bi-ideal of a regular semigroup is a BQ-semigroup.*

A *map* $\varphi : X \to Y$ is a subset of $X \times Y$ such that for every $x \in X$, there exists exactly one element $y \in B$ such that $(x, y) \in \varphi$. The *domain* of φ is

$$\text{dom } \varphi = \{x \in X : (x, y) \in \varphi \text{ for some } y \in Y\}$$

and the *image* of the map is

$$X\varphi = \{y \in Y : (x, y) \in \varphi \text{ for some } x \in X\}.$$

If $x \in \text{dom } \varphi$, then $x\varphi$ is called the *image of x under φ*. We denote the *kernel* of φ by $\ker \varphi$ and define it by

$$\ker \varphi = \{(x, y) \in X \times X : x\varphi = y\varphi\}.$$

A special mapping that is worth mentioning here is the *identity mapping* on X, $i_X = \{(x, x) : x \in X\}$ that is, two elements are related if and only if they are equal.

A map $\alpha : X \to X$ is a *constant mapping* if there exists $a \in X$ such that $x\alpha = a$ for all $x \in X$.

Throughout this thesis, maps are written on the right and composed left to right. The composition of two maps is the usual composition, namely, let $\varphi : X \to Y$ and $\psi : Y \to Z$ be two maps, then we define a new map $\varphi \circ \psi : X \to Z$ by

$$x(\varphi \circ \psi) = (x\varphi)\psi.$$

This new map is called the *composition* of φ and ψ.

Let $\varphi : X \to Y$ be a map and $A \subseteq X$ then $\varphi|_A : A \to Y$ is also a map which is defined by

$$x\varphi|_A = x\varphi \text{ for all } x \in A,$$

and called the *restriction* of φ to A.

A map $\varphi : X \to Y$ is *injective* (or *one-to-one*, or is an *injection*) if $x_1\varphi \neq x_2\varphi$ for any two different elements x_1 and x_2 of X. The map φ is *surjective* (or ***onto***, or is a *surjection*) if $X\varphi = Y$. The map φ is *bijective* (or is a *bijection* or a *one-to-one correspondence*) if it is both injective and surjective.

Theorem 2.1.18. [48] *Let X a finite set and let $\varphi : X \to X$ be a map. Then the following statements are equivalent.*

(1) *φ is injective.*

(2) *φ is surjective.*

(3) *φ is bijective.*

The maps $\varphi : X \to Y$ and $\psi : Y \to X$ are *mutually inverse bijections* if $\varphi\psi = i_X$ and $\psi\varphi = i_Y$.

A map $\varphi : S \to T$ where $(S, \cdot)$ and $(T, *)$ are semigroups, is a *morphism* (or *homomorphism*) if, for all $x, y \in S$

$$(x \cdot y)\varphi = (x\varphi) * (y\varphi).$$

A morphism $\varphi : S \to T$ is an *isomorphism* if it is a bijection. We say that S *and T are isomorphic as semigroup.*

Theorem 2.1.19. *Let $\varphi : S \to T$ be an isomorphism of semigroups. If S is a BQ-semigroup, then T is also a BQ-semigroup.*

2.2 Subsemigroups of full transformation semigroups

For a nonempty set X, the map α from X to itself is called a *transformation* on X. The set of all maps $\alpha : X \to X$ forms a semigroup under the composition of maps. This semigroup is called the *full transformation semigroup* on X and is denoted by $T(X)$. Subsemigroups of $T(X)$ are called *transformation semigroups.*

If X is a finite set $\{x_1, x_2, \ldots, x_n\}$, and if $y_1, y_2, \ldots, y_n$ are elements of X, not necessarily distinct, then we shall use the classical notation

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$

to mean that α is the transformation of X defined by

$$x_i\alpha = y_i \text{ for all } i = 1, 2, \ldots, n.$$

Let σ and ρ be equivalence relations on X with $\rho \subseteq \sigma$ and R a cross-section of the partition X/σ induced by σ. The following subsemigroups of $T(X)$ are considered as in [29, 32, 35, 41], respectively, which defined by

$$\begin{aligned} T(X,\sigma) &= \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \sigma\}, \\ E(X,\sigma) &= \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in \sigma \text{ implies } x\alpha = y\alpha\}, \\ T(X,\sigma^*) &= \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in \sigma \text{ if and only if } (x\alpha, y\alpha) \in \sigma\} \text{ and} \\ T(X,\sigma,R) &= \{\alpha \in T(X,\sigma) : R\alpha \subseteq R\}. \end{aligned}$$

Hence $T(X,\sigma)$, $T(X,\sigma^*)$ and $T(X,\sigma,R)$ clearly contain i_X. Now, we define new subsets of $T(X)$ by

$$T(X,\sigma,\rho) = \{\alpha \in T(X) : \forall x, y \in X, (x,y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \rho\}$$

and

$$T_R(X,\sigma) = \{\alpha \in T(X,\sigma) : R\alpha = R\}.$$

Theorem 2.2.1. *Let σ and ρ be equivalence relations on a nonempty set X with $\rho \subseteq \sigma$. Then the following statements hold.*

(1) *$T(X,\sigma,\rho)$ is a semigroup.*

(2) *$T_R(X,\sigma)$ is a semigroup.*

Proof. (1) Since $T(X,\sigma,\rho)$ contains a constant map, we deduce that $T(X,\sigma,\rho) \neq \emptyset$. Let $\alpha,\beta \in T(X,\sigma,\rho)$ and let $x,y \in X$ be such that $(x,y) \in \sigma$. Then $(x\alpha, y\alpha) \in \rho$. Since $\rho \subseteq \sigma$, $(x\alpha, y\alpha) \in \sigma$. It follows that $(x\alpha\beta, y\alpha\beta) \in \rho$. Therefore $\alpha\beta \in T(X,\sigma,\rho)$. Hence $T(X,\sigma,\rho)$ is a subsemigroup of $T(X)$.

(2) Clearly, $i_X \in T_R(X,\sigma)$ and so $T_R(X,\sigma) \neq \emptyset$. Let $\alpha,\beta \in T_R(X,\sigma)$ and let $x,y \in X$ be such that $(x,y) \in \sigma$. Then $(x\alpha, y\alpha) \in \sigma$. Therefore $(x\alpha\beta, y\alpha\beta) \in \sigma$. Hence $\alpha\beta \in T(X,\sigma)$. Since $\alpha,\beta \in T_R(X,\sigma)$, $R\alpha = R$ and $R\beta = R$. This implies that $R\alpha\beta = R\beta = R$. Hence $\alpha\beta \in T_R(X,\sigma)$. We conclude that $T_R(X,\sigma)$ is a subsemigroup of $T(X)$. □

Next, we will briefly recall some characterizations for above semigroups and introduce some notations that will be used in the sequel.

Proposition 2.2.2. *Let σ and ρ be equivalence relations on a nonempty set X with $\rho \subseteq \sigma$. Then the following statements hold.*

(1) *$i_X \in E(X,\sigma)$ if and only if $\sigma = I_X$.*

(2) *$i_X \in T(X,\sigma,\rho)$ if and only if $\sigma = \rho$.*

Proof. (1) Suppose that $i_X \in E(X,\sigma)$. Let $x,y \in X$ be such that $(x,y) \in \sigma$. Then $x = xi_X = yi_X = y$. Hence $\sigma = I_X$.

Conversely, assume that $\sigma = I_X$. Let $x,y \in X$ be such that $(x,y) \in \sigma$. Then $xi_X = x = y = yi_X$. Hence $i_X \in E(X,\sigma)$.

(2) Assume that $i_X \in T(X,\sigma,\rho)$. Let $x,y \in X$ be such that $(x,y) \in \sigma$. Then $(x,y) = (xi_X, yi_X) \in \rho$ and hence $\sigma = \rho$.

Conversely, if $\sigma = \rho$, then $i_X \in T(X,\sigma) = T(X,\sigma,\rho)$. □

The next proposition is easy to verify.

Proposition 2.2.3. *Let σ and ρ be equivalence relations on a nonempty set X with $\rho \subseteq \sigma$ and R a cross-section of the partition X/σ induced by σ. Then the following statements hold.*

(1) $E(X,\sigma) \subseteq T(X,\sigma,\rho) \subseteq T(X,\sigma)$.

(2) $T_R(X,\sigma) \subseteq T(X,\sigma,R) \subseteq T(X,\sigma)$.

For a nonempty set X and $\alpha \in T(X)$, $\pi(\alpha)$ denotes the decomposition of X induced by α, namely

$$\pi(\alpha) = \{y\alpha^{-1} : y \in X\alpha\}$$

and define $\alpha_* : \pi(\alpha) \to X\alpha$ by

$$P\alpha_* = x\alpha \text{ for each } P \in \pi(\alpha) \text{ and } x \in P.$$

Then $\pi(\alpha)$ is a partition of X with $\pi(\alpha) = X/\ker\alpha$ and α_* is a bijection.

For a nonempty subset A of X and $\alpha \in T(X)$, we write

$$\pi_A(\alpha) = \{P \in \pi(\alpha) : P \cap A \neq \emptyset\}.$$

For arbitrary equivalence relation σ and $\alpha \in T(X)$, we denote

$$\pi_\sigma(\alpha) = \{A\alpha^{-1} : A \in X/\sigma \text{ and } A\alpha^{-1} \neq \emptyset\}.$$

Then $\pi_\sigma(\alpha)$ is a partition of X.

For an equivalence σ on a set X and $\varphi : A \to B$ where $A, B \subseteq X$, we say that φ is *σ^*-preserving* if $(x,y) \in \sigma$ if and only if $(x\varphi, y\varphi) \in \sigma$ for all $x, y \in A$.

Let $\alpha, \beta \in T(X,\sigma,\rho)$ and φ a map from $\pi(\alpha)$ into $\pi(\beta)$. If for each $A \in X/\sigma$, there exists $B \in X/\rho$ such that

$$(\pi_A(\alpha))\varphi \subseteq \pi_B(\beta),$$

then φ is said to be *$\sigma\rho$-admissible.* Note that, if $\sigma = \rho$, then φ is said to be *σ-admissible.* If φ is a bijection and both φ and φ^{-1} are $\sigma\rho$-admissible, then φ is said to be *$(\sigma\rho)^*$-admissible* and if $\sigma = \rho$, we say that φ is said to be *σ^*-admissible.*

Proposition 2.2.4. *Let σ and ρ be equivalence relations on X with $\rho \subseteq \sigma$ and let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $\varphi : \pi(\alpha) \to \pi(\beta)$ is $\sigma\rho$-admissible if and only if for every $A \in X/\sigma$, there exists $B \in X/\rho$ such that $B \cap P\varphi \neq \emptyset$ for all $P \in \pi_A(\alpha)$.*

Proof. Suppose that $\varphi : \pi(\alpha) \to \pi(\beta)$ is $\sigma\rho$-admissible. Let $A \in X/\sigma$. Then there exists $B \in X/\rho$ such that

$$(\pi_A(\alpha))\varphi \subseteq \pi_B(\beta).$$

Let $P \in \pi_A(\alpha)$. Then $P\varphi \in \pi_B(\beta)$. Hence $B \cap P\varphi \neq \emptyset$.

Conversely, suppose that for each $A \in X/\sigma$, there exists $B \in X/\rho$ such that $B \cap P\varphi \neq \emptyset$ for all $P \in \pi_A(\alpha)$. Let $A \in X/\sigma$. Then there exists $B \in X/\rho$ such that $B \cap P\varphi \neq \emptyset$ for all $P \in \pi_A(\alpha)$. Let $P \in \pi_A(\alpha)$. Then $P\varphi \in \pi(\beta)$ and $B \cap P\varphi \neq \emptyset$. Thus $P\varphi \in \pi_B(\beta)$. Hence $(\pi_A(\alpha))\varphi \subseteq \pi_B(\beta)$. □

Lemma 2.2.5. *Let σ and ρ be equivalence relations on X with $\rho \subseteq \sigma$. Then $\alpha \in T(X, \sigma, \rho)$ if and only if for every $B \in X/\sigma$, there exists $B' \in X/\rho$ such that $B\alpha \subseteq B'$. Consequently, for each $A \in X/\sigma$, the set $A\alpha^{-1}$ is either $\emptyset$ or a union of some σ-classes.*

Proof. Suppose that $\alpha \in T(X, \sigma, \rho)$. Let $a \in X$. Then there exists $b \in X$ such that $a\alpha = b$. Let $y \in (a\sigma)\alpha$. Then $y = x\alpha$ for some $x \in a\sigma$. Since $(a, x) \in \sigma$ and $\alpha \in T(X, \sigma, \rho)$, we have $(b, y) = (a\alpha, x\alpha) \in \rho$. This means that $y \in b\rho$. Hence $(a\sigma)\alpha \subseteq b\rho$.

Conversely, suppose that for each $A \in X/\sigma$, there exists $B \in X/\rho$ such that $A\alpha \subseteq B$. Let $x, y \in X$ be such that $(x, y) \in \sigma$. By assumption, there exists $b \in X$ such that $x\alpha, y\alpha \in (x\sigma)\alpha \subseteq b\rho$. It follows that $(x\alpha, y\alpha) \in \rho$. Hence $\alpha \in T(X, \sigma, \rho)$, as required. □

Corollary 2.2.6. *Let σ be an equivalence relation on X and $\alpha \in T(X)$. Then the following statements hold.*

(1) *$\alpha \in T(X, \sigma)$ if and only if for every $B \in X/\sigma$, there exists $B' \in X/\sigma$ such that $B\alpha \subseteq B'$.*

(2) *$\alpha \in E(X, \sigma)$ if and only if for every $B \in X/\sigma$, there exists $x \in X$ such that $B\alpha = \{x\}$.*

Proposition 2.2.7. *Let σ and ρ be equivalence relations on X with $\rho \subseteq \sigma$. Then α is a right zero element of $T(X, \sigma, \rho)$ if and only if α is constant.*

Proof. Suppose that α is nonconstant. Then there exist distinct elements $a, b \in X\alpha$. Thus $a = a'\alpha$ and $b = b'\alpha$ for some $a', b' \in X$. Define $\beta \in T(X)$ by

$$x\beta = \begin{cases} a' & \text{if } (x, b') \in \sigma, \\ b' & \text{otherwise.} \end{cases}$$

Then $(b'\sigma)\beta = \{a'\} \subseteq a'\rho$ and $A\beta = \{b'\} \subseteq b'\rho$ for all $A \in X/\sigma \setminus \{b'\sigma\}$. By Lemma 2.2.5, we deduce that $\beta \in T(X, \sigma, \rho)$. Since $b'\beta\alpha = a'\alpha = a \neq b = b'\alpha$, we conclude that $\beta\alpha \neq \alpha$. This proves that α is not a right zero element of $T(X, \sigma, \rho)$. $\square$

Corollary 2.2.8. *Let σ be an equivalence relation on X. Then the following statements hold.*

(1) *α is a right zero element of $T(X, \sigma)$ if and only if α is constant.*

(2) *α is a right zero element of $E(X, \sigma)$ if and only if α is constant.*

As a consequence of Proposition 2.2.7, a necessary and sufficient condition for $T(X, \sigma, \rho)$ to be a right zero semigroup can be given as follows.

Corollary 2.2.9. *Let σ and ρ be equivalence relations on X with $\rho \subseteq \sigma$. Then $T(X, \sigma, \rho)$ is a right zero semigroup if and only if $\sigma = X \times X$ and $\rho = I_X$.*

Proof. Assume that $T(X, \sigma, \rho)$ is a right zero semigroup and $\sigma \neq X \times X$. Then there exist $a, b \in X$ such that $a\sigma \neq b\sigma$. Thus $a \neq b$. Define $\alpha \in T(X)$ by

$$x\alpha = \begin{cases} a & \text{if } (x, a) \in \sigma, \\ b & \text{otherwise.} \end{cases}$$

By Lemma 2.2.5, $\alpha \in T(X, \sigma, \rho)$. Since $a\alpha = a$ and $b\alpha = b$, $a, b \in X\alpha$ and hence α is nonconstant. By Proposition 2.2.7, we obtain that α is not a right zero element of $T(X, \sigma, \rho)$. This is a contradiction. Hence $\sigma = X \times X$. Next, we will show that $\rho = I_X$. Let $c, d \in X$ be such that $(c, d) \in \rho$. Define $\alpha \in T(X)$ by

$$x\alpha = \begin{cases} c & \text{if } x = c, \\ d & \text{otherwise.} \end{cases}$$

As a result $(c, d) \in \rho$, we get that $\alpha \in T(X, \sigma, \rho)$. It then follows by assumption and Proposition 2.2.7 that α is constant and hence $c = d$. Therefore $\rho = I_X$.

Conversely, assume that $\sigma = X \times X$ and $\rho = I_X$. It's an obviously fact that every element of $T(X,\sigma,\rho)$ is constant. It follows readily from Proposition 2.2.7 that $T(X,\sigma,\rho)$ is a right zero semigroup. □

Corollary 2.2.10. *Let σ be an equivalence relation on X. Then the following statements hold.*

(1) *$T(X,\sigma)$ is a right zero semigroup if and only if $|X| = 1$.*

(2) *$E(X,\sigma)$ is a right zero semigroup if and only if $\sigma = X \times X$.*

Lemma 2.2.11. [50] *Let σ be an equivalence relation on X and $\alpha \in T(X,\sigma^*)$. If $P \in \pi(\alpha)$, then there exists $A \in X/\sigma$ such that $P \subseteq A$.*

Lemma 2.2.12. [32] *Let $\alpha \in T(X,\sigma^*)$. Then $\pi_\sigma(\alpha) = X/\sigma$.*

Lemma 2.2.13. *Let σ an be equivalence relation on X and R a cross-section of the partition X/σ induced by σ. Suppose that $\alpha \in T(X,\sigma,R)$ and $r,s \in R$. If $x \in r\sigma$ with $x\alpha \in s\sigma$, then $(r\sigma)\alpha \subseteq s\sigma$ and $r\alpha = s$.*

Proof. Suppose that $x \in r\sigma$ with $x\alpha \in s\sigma$. Let $y \in r\sigma$. Then $(x,y) \in \sigma$. Since $\alpha \in T(X,\sigma)$, $(x\alpha, y\alpha) \in \sigma$ and thus $y\alpha \in s\sigma$ whence $(r\sigma)\alpha \subseteq s\sigma$. Since $r\alpha \in s\sigma \cap R$, it follows that $r\alpha = s$. □

2.3 Relationships between some subsemigroups of the full transformation semigroups

In this section, we characterize the conditions under which some subsemigroups of $T(X)$ are equal.

Theorem 2.3.1. *Let σ and ρ be equivalence relations on a nonempty set X with $\rho \subseteq \sigma$ and R a cross-section of the partition X/σ induced by σ. Then the following statements hold.*

(1) *$T(X,\sigma,\rho) = E(X,\sigma)$ if and only if $\rho = I_X$.*

(2) *$T(X,\sigma,\rho) = T(X,\sigma)$ if and only if $\sigma = \rho$.*

(3) $T(X,\sigma,\rho) = T(X,\sigma^*)$ *if and only if* $\rho = X \times X$.

(4) $T(X,\sigma,\rho) = T(X,\sigma,R)$ *if and only if* $\sigma = I_X$.

(5) $T(X,\sigma,\rho) = T_R(X,\sigma)$ *if and only if* $|X| = 1$.

(6) $T(X,\sigma,\rho) = T(X)$ *if and only if* $\sigma = I_X$ *or* $\rho = X \times X$.

Proof. (1) It follows from $E(X,\sigma) = T(X,\sigma,I_X)$.

(2) Clearly.

(3) Assume that $T(X,\sigma,\rho) = T(X,\sigma^*)$. Since $i_X \in T(X,\sigma^*) = T(X,\sigma,\rho)$ and by Theorem 2.2.2, we have $\rho = \sigma$. Let $x, y \in X$. Define $\alpha \in T(X,\sigma,\rho)$ by $z\alpha = x$ for all $z \in X$. By assumption, $\alpha \in T(X,\sigma^*)$ and by the definition of α, we get that $x\alpha = y\alpha$, so $(x\alpha, y\alpha) \in \sigma$. From $\alpha \in T(X,\sigma^*)$, it follows that $(x,y) \in \sigma = \rho$. Hence $\rho = X \times X$.

Conversely, assume that $\rho = X \times X$. Then $\sigma = \rho$. Thus $T(X,\sigma^*) \subseteq T(X,\sigma) = T(X,\sigma,\rho)$ by (2). Let $\alpha \in T(X,\sigma,\rho)$. Then by (2), we have $\alpha \in T(X,\sigma)$. Since $\sigma = X \times X$, we deduce that $(x,y) \in \sigma$ for all $x, y \in X$ with $(x\alpha, y\alpha) \in \sigma$. Therefore $\alpha \in T(X,\sigma^*)$. Hence $T(X,\sigma,\rho) \subseteq T(X,\sigma^*)$. We conclude that $T(X,\sigma,\rho) = T(X,\sigma^*)$.

(4) Assume that $T(X,\sigma,\rho) = T(X,\sigma,R)$. Let $x, y \in X$ be such that $(x,y) \in \sigma$. Define $\alpha \in T(X,\sigma,\rho)$ by $z\alpha = x$ for all $z \in X$. By assumption, we have $\alpha \in T(X,\sigma,R)$. Then $\{x\} = R\alpha \subseteq R$. By symmetry, we can show that $y \in R$. Since $(x,y) \in \sigma$ and by the definition of R, it follows that $x = y$. Hence $\sigma = I_X$.

Conversely, assume that $\sigma = I_X$. Then $\sigma = \rho$ and $R = X$. By (2), we have $T(X,\sigma,R) \subseteq T(X,\sigma) = T(X,\sigma,\rho)$. For each $\alpha \in T(X,\sigma,\rho)$, we have that $R\alpha = X\alpha \subseteq X = R$. This implies that $T(X,\sigma,R) = T(X,\sigma,\rho)$.

(5) Assume that $T(X,\sigma,\rho) = T_R(X,\sigma)$. Let $x, y \in X$. Define $\alpha \in T(X,\sigma,\rho)$ by $z\alpha = x$ for all $z \in X$. By assumption, we have $\alpha \in T_R(X,\sigma)$. Therefore $R = R\alpha \subseteq X\alpha = \{x\}$. This implies that $R = \{x\}$. Similarly, we can show that $y \in R$ and hence $x = y$. We conclude that $|X| = 1$.

The converse is clear.

(6) Assume that $T(X,\sigma,\rho) = T(X)$ and $\sigma \neq I_X$. Let $x, y \in X$. Suppose that

$(x, y) \notin \rho$. Since $\sigma \neq I_X$, there exists $A \in X/\sigma$ such that $|A| \geq 2$. Let $a, b \in A$ be distinct. Define $\alpha \in T(X)$ by

$$z\alpha = \begin{cases} x & \text{if } z = a, \\ y & \text{otherwise.} \end{cases}$$

By assumption, we get that $\alpha \in T(X, \sigma, \rho)$. But $(a, b) \in \sigma$ and $(a\alpha, b\alpha) = (x, y) \notin \rho$, which is a contradiction. Hence $(x, y) \in \rho$. We conclude that $\rho = X \times X$.

The converse is clear. □

Corollary 2.3.2. *Let σ be an equivalence relation on a nonempty set X and R a cross-section of the partition X/σ induced by σ. Then the following statements hold.*

(1) $T(X, \sigma) = E(X, \sigma)$ *if and only if* $\sigma = I_X$.

(2) $T(X, \sigma) = T(X, \sigma^*)$ *if and only if* $\sigma = X \times X$.

(3) $T(X, \sigma) = T(X, \sigma, R)$ *if and only if* $\sigma = I_X$.

(4) $T(X, \sigma) = T_R(X, \sigma)$ *if and only if* $|X| = 1$.

(5) $T(X, \sigma) = T(X)$ *if and only if* $\sigma = I_X$ *or* $\sigma = X \times X$.

(6) $E(X, \sigma) = T(X, \sigma^*)$ *if and only if* $|X| = 1$.

(7) $E(X, \sigma) = T(X, \sigma, R)$ *if and only if* $\sigma = I_X$.

(8) $E(X, \sigma) = T_R(X, \sigma)$ *if and only if* $|X| = 1$.

(9) $E(X, \sigma) = T(X)$ *if and only if* $\sigma = I_X$.

Theorem 2.3.3. *Let σ be an equivalence relation on a nonempty set X and R a cross-section of the partition X/σ induced by σ. Then the following statements hold.*

(1) $T(X, \sigma^*) = T(X, \sigma, R)$ *if and only if* $|X| = 1$.

(2) $T(X, \sigma^*) = T_R(X, \sigma)$ *if and only if R is finite and* $\sigma = I_X$.

(3) $T(X, \sigma^*) = T(X)$ *if and only if* $\sigma = X \times X$.

Proof. (1) Assume that $T(X,\sigma^*) = T(X,\sigma,R)$ and $|X| > 1$. Let $r \in R$ and $x \in X \setminus \{r\}$. Define $\alpha \in T(X)$ by $z\alpha = r$ for all $z \in X$. Then $\alpha \in T(X,\sigma)$. Thus $R\alpha = \{r\} \subseteq R$ and we then have $\alpha \in T(X,\sigma,R)$. By assumption, we obtain that $\alpha \in T(X,\sigma^*)$. Claim that $\sigma = X \times X$. For each $a, b \in X$, we have $a\alpha = b\alpha$. It follows from $\alpha \in T(X,\sigma^*)$ that $(a,b) \in \sigma$. This implies that $\sigma = X \times X$. Hence $x \notin R$. Define $\beta \in T(X)$ by $z\beta = x$ for all $z \in X$. As a result $\sigma = X \times X$, $\beta \in T(X,\sigma^*)$ and so $\beta \in T(X,\sigma,R)$. But $r\beta = x \notin R$, which leads to a contradiction with $R\beta \subseteq R$. Hence $|X| = 1$.

The converse is clear.

(2) Assume that $T(X,\sigma^*) = T_R(X,\sigma)$. Suppose that R is an infinite set. Then R has a countable infinite subset $\{r_n : n \in \mathbb{N}\}$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} r_{n+1} & \text{if } x \in r_n\sigma \text{ for all } n \in \mathbb{N}, \\ x & \text{otherwise.} \end{cases}$$

Then $(r_n\sigma)\alpha \subseteq r_{n+1}\sigma$ for all $n \in \mathbb{N}$ and $(x\sigma)\alpha \subseteq x\sigma$ for all $x\sigma \in X/\sigma \setminus \{r_n\sigma : n \in \mathbb{N}\}$. By Theorem 2.2.6, we have $\alpha \in T(X,\sigma)$. On the other hand, let $x, y \in X$ be such that $(x\alpha, y\alpha) \in \sigma$. If $(x\alpha)\sigma \in \{r_n\sigma : n \in \mathbb{N}\}$, then $(x, r_k) \in \sigma$ for some $k \in \mathbb{N} \setminus \{1\}$. By the definition of α, we then have $x, y \in r_{k-1}\sigma$. Therefore $(x,y) \in \sigma$. If $(x\alpha)\sigma \notin \{r_n\sigma : n \in \mathbb{N}\}$, then $(x,y) = (x\alpha, y\alpha) \in \sigma$. Hence $\alpha \in T(X,\sigma^*)$. By assumption, we obtain that $\alpha \in T_R(X,\sigma)$. This implies that $r_1 \in R = R\alpha \subseteq X\alpha$. This is a contradiction. Hence R is finite.

Next, we will show that $\sigma = I_X$. Let a, $b \in X$ be such that $(a,b) \in \sigma$. Then there exists $r \in R$ such that $a, b \in r\sigma$. Define α, $\beta : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in r\sigma, \\ x & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} b & \text{if } x \in r\sigma, \\ x & \text{otherwise.} \end{cases}$$

Then $(x\sigma)\alpha \subseteq x\sigma$ and $(y\sigma)\beta \subseteq y\sigma$ for all $x, y \in X$. This implies that α, $\beta \in T(X,\sigma^*)$. By assumption, we obtain α, $\beta \in T_R(X,\sigma)$. By Lemma 2.2.13, $a = r\alpha = r$ and $b = r\beta = r$. This implies that $a = b$ and hence σ is the identity relation.

Conversely, assume that R is finite and $\sigma = I_X$. Let $\alpha \in T_R(X, \sigma)$. Then $\alpha \in T(X, \sigma)$. On the other hand, let $x, y \in X$ be such that $(x\alpha, y\alpha) \in \sigma$. Then $x\alpha, y\alpha \in r\sigma$ for some $r \in R$. By Lemma 2.2.13, $r_x\alpha = r$ and $r_y\alpha = r$ for some $r_x, r_y \in R$ with $(x, r_x), (y, r_y) \in \sigma$. Since $\alpha|_R : R \to R$ is a surjection on R and R is finite, it follows that $\alpha|_R : R \to R$ is also an injection. This implies that $r_x = r_y$ and thus $(x, y) \in \sigma$. Hence $T_R(X, \sigma) \subseteq T(X, \sigma^*)$. For the reverse inclusion, let $\alpha \in T(X, \sigma^*)$. Then $\alpha \in T(X, \sigma)$. Let $r \in R$. Then $r\alpha \in s\sigma$ for some $s \in R$. Since σ is the identity relation, $r\alpha = s$ and so $R\alpha \subseteq R$. Therefore $\alpha|_R$ is a mapping from R into R. Claim that $\alpha|_R : R \to R$ is an injection. Let $r_1, r_2 \in R$ be such that $r_1\alpha = r_2\alpha$. Then $(r_1\alpha, r_2\alpha) \in \sigma$. Since $\alpha \in T(X, \sigma^*)$, $(r_1, r_2) \in \sigma$. By the definition of R, we have that $r_1 = r_2$. So we have the claim. As a result R is finite, we obtain that $\alpha|_R : R \to R$ is a surjection. Hence $R\alpha = R$. This shows that $\alpha \in T_R(X, \sigma)$ and hence $T_R(X, \sigma) \subseteq T(X, \sigma^*)$. We conclude that $T_R(X, \sigma) = T(X, \sigma^*)$.

(3) Assume that $T(X, \sigma^*) = T(X)$. Let $x \in X$. Define $\alpha \in T(X)$ by $z\alpha = x$ for all $z \in X$. Thus $\alpha \in T(X, \sigma^*)$. For each $a, b \in X$, we have $a\alpha = b\alpha$. This implies that $(a, b) \in \sigma$. Hence $\sigma = X \times X$.

The converse is follows from (2) and (5) of Corollary 2.3.2. □

Theorem 2.3.4. *Let σ be an equivalence relation on a nonempty set X and R a cross-section of the partition X/σ induced by σ. Then the following statements hold.*

(1) *$T(X, \sigma, R) = T_R(X, \sigma)$ if and only if $\sigma = X \times X$.*

(2) *$T(X, \sigma, R) = T(X)$ if and only if $\sigma = I_X$.*

(3) *$T_R(X, \sigma) = T(X)$ if and only if $|X| = 1$.*

Proof. (1) The necessity is clear. To prove the sufficiency, we assume that $T(X, \sigma, R) = T_R(X, \sigma)$. Let $r \in R$. Define $\alpha \in T(X)$ by $x\alpha = r$ for all $x \in X$. Then $\alpha \in T(X, \sigma)$. Therefore $R\alpha = \{r\} \subseteq R$, we then have $\alpha \in T(X, \sigma, R)$. By assumption, we obtain that $\alpha \in T_R(X, \sigma)$. Therefore $\{r\} = R\alpha = R$. It follows that $\sigma = X \times X$.

(2) Follows from (3) and (5) of Corollary 2.3.2.

(3) Follows from (4) and (5) of Corollary 2.3.2. □

CHAPTER III

REGULARITY FOR SOME SUBSEMIGROUPS OF FULL TRANSFORMATION SEMIGROUPS

In this chapter, we characterize the regular, left regular, right regular and completely regular elements of some subsemigroups of $T(X)$. Moreover, we give a necessary and sufficient condition for some subsemigroups of $T(X)$ to be left regular, right regular and completely regular semigroups.

3.1 Regularity for the full transformation semigroups

In this section, we describe the regularity, the left regularity, the right regularity and the completely regularity for $T(X)$. The following results are quoted from [49] Theorem 5 and [50] Theorems 3.1.2, 3.1.3 and 3.1.4, respectively.

Theorem 3.1.1. [49] *Every element of $T(X)$ is regular. Hence $T(X)$ is a regular semigroup.*

Theorem 3.1.2. [50] *Let $\alpha \in T(X)$. Then α is left regular of $T(X)$ if and only if for every $P \in \pi(\alpha)$, $P \cap X\alpha \neq \emptyset$.*

Theorem 3.1.3. [50] *Let $\alpha \in T(X)$. Then α is right regular of $T(X)$ if and only if $\alpha|_{X\alpha}$ is an injection.*

Theorem 3.1.4. [50] *Let $\alpha \in T(X)$. Then α is completely regular of $T(X)$ if and only if for every $P \in \pi(\alpha), |P \cap X\alpha| = 1$.*

By Theorem 3.1.1, $T(X)$ is a regular semigroup. But not necessary for the left regular semigroup, the right regular semigroup and the completely regular semigroup. As we see in the next example.

Example 3.1.5. Define $\alpha : \mathbb{N} \to \mathbb{N}$ by

$$x\alpha = \begin{cases} 1 & \text{if } x \in \{1, 2\}, \\ 2 & \text{if } x = 3, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \in T(\mathbb{N})$. We set $P = 2\alpha^{-1} \in \pi(\alpha)$. Then $P = \{3\}$. Since $X\alpha = \mathbb{N} \setminus \{3\}$, $P \cap X\alpha = \emptyset$. By Theorem 3.1.2, we get that α is not a left regular element of $T(X)$. Hence $T(X)$ is not a left regular semigroup. We note that $1, 2 \in X\alpha$ and $1\alpha = 2\alpha$. It follows from Theorem 3.1.3 that $T(X)$ is not a right semigroup. Also, α is not completely regular by Theorem 2.1.2.

Next, we characterize the semigroup $T(X)$ which is a left regular semigroup, a right regular semigroup and a completely regular semigroup, respectively.

Theorem 3.1.6. *$T(X)$ is a left regular semigroup if and only if $|X| \leq 2$.*

Proof. Assume that $T(X)$ is a left regular semigroup. Suppose that $|X| > 2$. Then there exist distinct elements $a, b, c \in X$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in \{a, b\}, \\ b & \text{otherwise.} \end{cases}$$

Then $\alpha \in T(X)$. Since $c\alpha = b$, $b\alpha^{-1} \in \pi(\alpha)$. Note that $b\alpha^{-1} = X \setminus \{a, b\}$ and $X\alpha = \{a, b\}$, we obtain $b\alpha^{-1} \cap X\alpha = \emptyset$. It follows from Theorem 3.1.2 that α is not a left regular element of $T(X)$. This is a contradiction. Hence $|X| \leq 2$.

Conversely, suppose that $|X| \leq 2$. If $|X| = 1$, then $T(X)$ contains only the identity transformation on X. It is obviously that $T(X)$ is a left regular semigroup. Suppose that $|X| = 2$, say $X = \{a, b\}$. Then

$$T(X) = \left\{ \begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \begin{pmatrix} a & b \\ b & b \end{pmatrix} \right\}.$$

By virtue of Theorem 3.1.2, every element of $T(X)$ is left regular. Hence $T(X)$ is a left regular semigroup. □

Theorem 3.1.7. *$T(X)$ is a right regular semigroup if and only if $|X| \leq 2$.*

Proof. Assumse that $T(X)$ is a right regular semigroup. Suppose that $|X| > 2$. Then there exist distinct elements $a, b, c \in X$. Define $\alpha \in T(X)$ as in the same proof of Theorem 3.1.6. Since $a, b \in X\alpha$ and $a\alpha = b\alpha$, $\alpha|_{X\alpha}$ is not injective. Theorem 3.1.3 tells us that α is not right regular. This is a contradiction. Hence $|X| \leq 2$.

Conversely, suppose that $|X| \leq 2$. It is easy to verify by Theorem 3.1.3 that every element of $T(X)$ is right regular. □

As an immediate consequence of Theorems 2.1.2, 3.1.6 and 3.1.7, we have the following

Corollary 3.1.8. *$T(X)$ is a completely regular semigroup if and only if $|X| \leq 2$.*

3.2 Regularity for the generalization of semigroups of transformations preserving equivalence relations

Throughout this section, let σ and ρ be equivalence relations on a nonempty set X with $\rho \subseteq \sigma$. We recall

$$T(X, \sigma, \rho) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \rho\}.$$

The purpose of this section is to give necessary and sufficient conditions for elements in $T(X, \sigma, \rho)$ to be regular, left regular, right regular and completely regular. Also, the relations σ and ρ for which $T(X, \sigma, \rho)$ is a regular semigroup, a left regular semigroup, a right regular semigroup and a completely regular semigroup are considered.

Firstly, we describe the regularity, the left regularity, the right regularity and the completely regularity for elements of the semigroup $T(X, \sigma, \rho)$, respectively.

Theorem 3.2.1. *Let $\alpha \in T(X, \sigma, \rho)$. Then α is regular of $T(X, \sigma, \rho)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\rho$ such that $A \cap X\alpha \subseteq B\alpha$.*

Proof. Suppose that α is regular of $T(X, \sigma, \rho)$. Then $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X, \sigma, \rho)$. Let $A \in X/\sigma$. By Lemma 2.2.5, $A\beta \subseteq B$ for some $B \in X/\rho$. If $A \cap X\alpha = \emptyset$, then $A \cap X\alpha \subseteq B\alpha$. Assume that $A \cap X\alpha \neq \emptyset$. Let $y \in A \cap X\alpha$. Then $y = x\alpha$ for some $x \in X$

and hence $y\beta \in A\beta \subseteq B$. Therefore $y = x\alpha = x\alpha\beta\alpha = y\beta\alpha \in B\alpha$. Hence $A \cap X\alpha \subseteq B\alpha$.

Conversely, assume that for each $A \in X/\sigma$, there exists $B \in X/\rho$ such that $A \cap X\alpha \subseteq B\alpha$. Let $A \in X/\sigma$ be such that $A \cap X\alpha \neq \emptyset$. By assumption, we choose and fix $A' \in X/\rho$ with $A \cap X\alpha \subseteq A'\alpha$. For each $y \in A \cap X\alpha$, we choose and fix an element $y' \in A'$ such that $y = y'\alpha$. Let $x_A \in A'$. Define $\beta_A : A \to A'$ by

$$x\beta_A = \begin{cases} x' & \text{if } x \in X\alpha, \\ x_A & \text{otherwise.} \end{cases}$$

Let $\beta : X \to X$ be defined by

$$\beta|_A = \begin{cases} \beta_A & \text{if } A \cap X\alpha \neq \emptyset, \\ C_A & \text{otherwise} \end{cases}$$

for all $A \in X/\sigma$ and C_A is a constant map from A into X. Since X/σ is a partition of X, β is well-defined. By the definition of β, we get that $A\beta \subseteq A'$ for all $A \in X/\sigma$ with $A \cap X\alpha \neq \emptyset$ and $A\beta \subseteq \{x\}$ for all $A \in X/\sigma$ with $A \cap X\alpha = \emptyset$ and $x \in X$. By Lemma 2.2.5, we obtain that $\beta \in T(X, \sigma, \rho)$. And $x\alpha\beta\alpha = (x\alpha)'\alpha = x\alpha$ for all $x \in X$. This shows that α is a regular element of $T(X, \sigma, \rho)$, as desired. □

Corollary 3.2.2. *Let α be a regular element of $T(X, \sigma, \rho)$. Then the following statements hold.*

(1) *For every $A \in X/\rho$, there exists $B \in X/\rho$ such that $A \cap X\alpha \subseteq B\alpha$.*

(2) *For every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $A \cap X\alpha \subseteq B\alpha$.*

Corollary 3.2.3. [29] *Let $\alpha \in T(X, \sigma)$. Then α is regular of $T(X, \sigma)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $A \cap X\alpha \subseteq B\alpha$.*

Corollary 3.2.4. *Let $\alpha \in E(X, \sigma)$. Then α is regular of $E(X, \sigma)$ if and only if for every $A \in X/\sigma$, $|A \cap X\alpha| \leq 1$.*

Proof. Suppose that α is regular of $E(X, \sigma)$. By Theorem 3.2.1, there exists $B \in X/I_X$ such that $A \cap X\alpha \subseteq B\alpha$. As a result $|B| = 1$, we then have $|A \cap X\alpha| \leq |B\alpha| = 1$.

Conversely, assume that $|A \cap X\alpha| \leq 1$ for all $A \in X/\sigma$. Let $A \in X/\sigma$. If $|A \cap X\alpha| = 0$, then $A \cap X\alpha = \emptyset$. Thus $A \cap X\alpha \subseteq B\alpha$ for some $B \in X/I_X$. If $|A \cap X\alpha| = 1$, then there exists a unique $a \in A \cap X\alpha$. Thus $a = b\alpha$ for some $b \in X$. This implies that $\{b\} \in X/I_X$. Since $\alpha \in E(X,\sigma)$, we deduce that $\{b\}\alpha = \{a\} = A \cap X\alpha$. Hence $A \cap X\alpha \subseteq B\alpha$ for some $B \in X/I_X$. It follows that Theorem 3.2.1 that α is a regular element of $E(X,\sigma)$. □

Theorem 3.2.5. *Let $\alpha \in T(X,\sigma,\rho)$. Then α is left regular of $T(X,\sigma,\rho)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\rho$ such that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$.*

Proof. Assume that α is left regular of $T(X,\sigma,\rho)$. Then $\alpha = \beta\alpha^2$ for some $\beta \in T(X,\sigma,\rho)$. Let $A \in X/\sigma$. By Lemma 2.2.5, there is $B \in X/\rho$ such that $A\beta \subseteq B$. Suppose that $P \in \pi_A(\alpha)$ and let $x \in P \cap A$. Since $A\beta \subseteq B$, we have that $x\beta \in B$. Hence $x\alpha = x\beta\alpha^2 = (x\beta\alpha)\alpha$ which implies that $x\beta\alpha \in P$ as we wish to show.

Conversely, for each $A \in X/\sigma$, we choose $A' \in X/\rho$ such that for every $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in A'$. Let $x \in X$. Since X/σ and $\pi(\alpha)$ are partitions of X, there exist $A \in X/\sigma$ and $P \in \pi(\alpha)$ such that $x \in A$ and $x \in P$. Hence $P \in \pi_A(\alpha)$. By assumption, we choose $x' \in A'$ such that $x'\alpha \in P$ and $A' \in X/\rho$. We also have that $x'\alpha\alpha = x\alpha$. Define $\beta : X \to X$ by $x\beta = x'$ for all $x \in X$. Then β is well-defined. Let $x, y \in X$ be such that $(x,y) \in \sigma$. Then there exists $A \in X/\sigma$ such that $x, y \in A$. By the definition of β, $x\beta, y\beta \in A'$ where $A' \in X/\rho$. Hence $\beta \in T(X,\sigma,\rho)$. If $x \in X$, then $x\beta\alpha^2 = x'\alpha\alpha = x\alpha$ which gives $\alpha = \beta\alpha^2$. Therefore α is left regular, as required. □

Corollary 3.2.6. [50] *Let $\alpha \in T(X,\sigma)$. Then α is left regular of $T(X,\sigma)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$.*

Corollary 3.2.7. *Let $\alpha \in E(X,\sigma)$. Then α is left regular of $E(X,\sigma)$ if and only if for every $P \in \pi(\alpha)$, $P \cap X\alpha \neq \emptyset$.*

Proof. The necessity is clear from Theorem 3.1.2. To prove the sufficiency, we suppose that $P \cap X\alpha \neq \emptyset$ for all $P \in \pi(\alpha)$. Let $A \in X/\sigma$ and $P \in \pi_A(\alpha)$. Then $P \in \pi(\alpha)$. Since $\alpha \in E(X,\sigma)$, $A\alpha = P\alpha$. This implies that $A \subseteq (P\alpha)\alpha^{-1} = P$. Hence $\pi_A(\alpha) = \{P\}$. By assumption, $P \cap X\alpha \neq \emptyset$ and so $x\alpha \in P$ for some $x \in X$. Let $B = \{x\} \in X/I_X$. For each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$. By Theorem 3.2.5, we conclude that α is left regular of $E(X,\sigma)$. □

Theorem 3.2.8. *Let $\alpha \in T(X,\sigma,\rho)$. Then α is right regular of $T(X,\sigma,\rho)$ if and only if the following statements hold.*

(1) $\alpha|_{X\alpha}$ *is an injection.*

(2) *For every $x, y \in X\alpha$, $(x\alpha, y\alpha) \in \sigma$ implies $(x,y) \in \rho$.*

Proof. Assume that α is right regular of $T(X,\sigma,\rho)$. Then $\alpha = \alpha^2\beta$ for some $\beta \in T(X,\sigma,\rho)$. It follows from Theorem 3.1.3 that $\alpha|_{X\alpha}$ is an injection. Let $x,y \in X\alpha$ be such that $(x\alpha, y\alpha) \in \sigma$. Thus $x = x'\alpha$ and $y = y'\alpha$ for some $x', y' \in X$. Since $\beta \in T(X,\sigma,\rho)$, we have $(x\alpha\beta, y\alpha\beta) \in \rho$. Hence

$$(x,y) = (x'\alpha, y'\alpha) = (x'\alpha^2\beta, y'\alpha^2\beta) = (x\alpha\beta, y\alpha\beta) \in \rho$$

which means that (2) holds.

Conversely, assume that the conditions (1) and (2) hold. Let $A \in X/\sigma$ be such that $A \cap X\alpha^2 \neq \emptyset$. We choose and fix an element $x_A \in A \cap X\alpha^2$. For each $x \in A \cap X\alpha^2$, there exists a unique $x' \in X\alpha$ such that $x = x'\alpha$ by $\alpha|_{X\alpha}$ is injective. We observe that $(x'\alpha, x'_A\alpha) = (x, x_A) \in \sigma$. It follows from (2) that $(x', x'_A) \in \rho$. Define $\beta_A : A \to X$ by

$$x\beta_A = \begin{cases} x' & \text{if } x \in X\alpha^2, \\ x'_A & \text{otherwise.} \end{cases}$$

Then we define the map $\beta : X \to X$ by

$$\beta|_A = \begin{cases} \beta_A & \text{if } A \cap X\alpha^2 \neq \emptyset, \\ C_A & \text{otherwise,} \end{cases}$$

for all $A \in X/\sigma$ and C_A is a constant mapping from A into X. Since X/σ is a partition of X, β is well-defined. Let $x,y \in X$ be such that $(x,y) \in \sigma$. Then $x,y \in A$ for some $A \in X/\sigma$. By the definition of β, we have $(x\beta, y\beta) = (x\beta|_A, y\beta|_A)$. If $A \cap X\alpha^2 = \emptyset$, then $x\beta = y\beta$ and so $(x\beta, y\beta) \in \rho$ by reflexivity of ρ. If $A \cap X\alpha^2 \neq \emptyset$, by the definition of β_A we then have $(x\beta_A, x'_A), (y\beta_A, x'_A) \in \rho$. By transitivity of ρ, $(x\beta, y\beta) \in \rho$, hence $\beta \in T(X,\sigma,\rho)$.

Finally, to show that $\alpha = \alpha^2\beta$, let $x \in X$, so $x\alpha^2 \in X\alpha^2$. Then there exists $A \in X/\sigma$ such that $x\alpha^2 \in A$. By the definition of β_A, $x\alpha^2\beta_A = (x\alpha^2)'$ where $(x\alpha^2)'\alpha = x\alpha^2 = (x\alpha)\alpha$. Since $(x\alpha^2)'$ is unique, $(x\alpha^2)' = x\alpha$. Thus $x\alpha^2\beta = x\alpha^2\beta_A = x\alpha$. Therefore α is right regular, as asserted. □

Corollary 3.2.9. [50] *Let $\alpha \in T(X,\sigma)$. Then α is right regular of $T(X,\sigma)$ if and only if the following statements hold.*

(1) *$\alpha|_{X\alpha}$ is an injection.*

(2) *For every $x,y \in X\alpha$, $(x\alpha, y\alpha) \in \sigma$ implies $(x,y) \in \sigma$.*

Corollary 3.2.10. *Let $\alpha \in E(X,\sigma)$. Then the following statements are equivalent.*

(1) *α is right regular of $E(X,\sigma)$.*

(2) *$\alpha|_{X\alpha}$ is an injection.*

(3) *For every $x,y \in X\alpha$, $(x\alpha, y\alpha) \in \sigma$ implies $x = y$.*

Proof. (1) $\Rightarrow$ (3) It follows from Theorem 3.2.8.

(3) $\Rightarrow$ (2) Assume that (3) holds. Let $x,y \in X\alpha$ be such that $x\alpha = y\alpha$. Then $(x\alpha, y\alpha) \in \sigma$. By assumption, we deduce that $x = y$. Hence $\alpha|_{X\alpha}$ is an injection.

(2) $\Rightarrow$ (1) Suppose that (2) holds. Let $x,y \in X\alpha$ be such that $(x\alpha, y\alpha) \in \sigma$. Then $x\alpha\alpha = y\alpha\alpha$. By assumption, we have that $x\alpha = y\alpha$ and $x = y$. It follows from Theorem 3.2.8 that α is a right regular element of $E(X,\sigma)$. □

Theorem 3.2.11. *Let $\alpha \in T(X,\sigma,\rho)$. Then α is completely regular of $T(X,\sigma,\rho)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\rho$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A(\alpha)$.*

Proof. Suppose that α is completely regular of $T(X,\sigma,\rho)$. Then by Theorem 2.1.2, we have α is both a left and a right regular element of $T(X,\sigma,\rho)$. Let $A \in X/\sigma$. By Theorem 3.2.5, there exists $B \in X/\rho$ such that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$. Thus $\emptyset \neq P \cap B\alpha \subseteq P \cap X\alpha \neq \emptyset$ for all $P \in \pi_A(\alpha)$. Let $P \in \pi_A(\alpha)$ and let $x,y \in P \cap X\alpha$. Then $x\alpha = y\alpha$. By Theorem 3.2.8(1), we get that $x = y$ and hence $|P \cap X\alpha| = 1$. It follows from $B \cap X\alpha \subseteq P \cap X\alpha$ that $|P \cap B\alpha| = |P \cap X\alpha| = 1$.

Conversely, suppose that for every $A \in X/\sigma$, there exists $B \in X/\rho$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A(\alpha)$. We will show that α is completely regular by using Theorem 2.1.2. Firstly, let $A \in X/\sigma$. Then there exists $B \in X/\rho$ such that $|P \cap B\alpha| = 1$ for all $P \in \pi_A(\alpha)$. Thus for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$.

Applying Theorem 3.2.5, we have α is left regular of $T(X,\sigma,\rho)$.

Finally, we will show that α is right regular by using Theorem 3.2.8. Let $x,y \in X\alpha$ be such that $x\alpha = y\alpha$. Thus $x,y \in P \cap X\alpha$ where $P = (x\alpha)\alpha^{-1} \in \pi_A(\alpha)$ and $A \in X/\sigma$ with $x \in A$. By assumption, we get $|P \cap X\alpha| = 1$ and so $x = y$ hence $\alpha|_{X\alpha}$ is an injection. Let $x,y \in X\alpha$ be such that $(x\alpha, y\alpha) \in \sigma$. Then $x\alpha, y\alpha \in A'$ for some $A' \in X/\sigma$. By the hypothesis, there exists $B' \in X/\rho$ such that

$$|P' \cap B'\alpha| = |P' \cap X\alpha| = 1 \text{ for all } P' \in \pi_{A'}(\alpha).$$

Let $P_1' = (x\alpha)\alpha^{-1} \in \pi_{A'}(\alpha)$ and $P_2' = (y\alpha)\alpha^{-1} \in \pi_{A'}(\alpha)$. Thus $x \in P_1' \cap X\alpha$ and $y \in P_2' \cap X\alpha$. By assumption, we get that $|P_1' \cap B'\alpha| = |P_1' \cap X\alpha| = 1$ and $|P_2' \cap B'\alpha| = |P_2' \cap X\alpha| = 1$. Since $x \in P_1' \cap X\alpha$ and $P_1' \cap B\alpha \subseteq P_1' \cap X\alpha$, we deduce that $x = b_1'\alpha$ for some $b_1' \in B'$. Similarly, $y = b_2'\alpha$ for some $b_2' \in B'$. Since $(b_1', b_2') \in \sigma$ and $\alpha \in T(X,\sigma,\rho)$, it follows that $(x,y) = (b_1'\alpha, b_2'\alpha) \in \rho$. □

Corollary 3.2.12. [50] *Let $\alpha \in T(X,\sigma)$. Then α is completely regular of $T(X,\sigma)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A(\alpha)$.*

Corollary 3.2.13. *Let $\alpha \in E(X,\sigma)$. Then α is completely regular of $E(X,\sigma)$ if and only if for every $P \in \pi(\alpha)$, $|P \cap X\alpha| = 1$.*

Secondly, we show that the semigroup $T(X,\sigma,\rho)$ does not necessary for the left regular semigroup, the right regular semigroup and the completely regular semigroup. As we see in the next example.

Example 3.2.14. Let $X = \{1,2,3,4,5,6,7,8\}$, $X/\sigma = \{\{1,2,3\},\{4,5\},\{6,7,8\}\}$ and $X/\rho = \{\{1,2\},\{3\},\{4,5\},\{6,8\},\{7\}\}$. Let $\alpha \in T(X,\sigma,\rho)$ be defined by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 6 & 3 & 3 & 2 & 1 & 2 \end{pmatrix}.$$

Since $\{6,7,8\}\alpha \nsubseteq B\alpha$ for all $B \in X/\rho$ and by Theorem 3.2.1, we deduce that α is not a regular element of $T(X,\sigma,\rho)$. Note that $6,8 \in X\alpha$ and $6\alpha = 8\alpha$. By Theorem 3.2.8, we have that α is not a right regular element of $T(X,\sigma,\rho)$. Next, we will show that α is not a left regular element of $T(X, \sigma, \rho)$ by using Theorem 3.2.5. Let $A = \{4, 5\}$. Then $\pi_A(\alpha) = \{A\}$. Since $A \cap X\alpha = \emptyset$, $x\alpha \notin A$ for all $x \in X$. This implies that α does not satisfy Theorem 3.2.5. Hence α is not a left regular element of $T(X, \sigma, \rho)$. Also, α is not a completely regular element of $T(X, \sigma, \rho)$.

Finally, we give necessary and sufficient conditions for the semigroup $T(X, \sigma, \rho)$ to be regular, left regular, right regular or completely regular.

Theorem 3.2.15. *$T(X, \sigma, \rho)$ is a regular semigroup if and only if one of the following statements holds.*

(1) $\sigma = I_X$.

(2) $\rho = X \times X$.

(3) $\sigma = X \times X$ *and* $\rho = I_X$.

Proof. Assume that $T(X, \sigma, \rho)$ is a regular semigroup, $\sigma \neq I_X$ and $\rho \neq X \times X$. Firstly, we will show that $\sigma = X \times X$. Suppose that $\sigma \neq X \times X$. Since $\sigma \neq I_X$, there exist distinct elements $a, b \in X$ such that $(a, b) \in \sigma$. Then $a, b \in A$ for some $A \in X/\sigma$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A, \\ b & \text{otherwise.} \end{cases}$$

Then $A\alpha \subseteq a\rho$ and $A'\alpha \subseteq b\rho$ for all $A' \in X/\sigma \setminus \{A\}$. By Lemma 2.2.5, we have that $\alpha \in T(X, \sigma, \rho)$. It follows from assumption that α is regular. By Theorem 3.2.1, there exists $B \in X/\rho$ such that $A \cap X\alpha \subseteq B\alpha$. Since $\sigma \neq X \times X$ and $a, b \in A$, it follows that $A \cap X\alpha = \{a, b\}$. Thus $a = x\alpha$ and $b = y\alpha$ for some $x, y \in B$. By the definition of α, we get that $x \in A$ and $y \in X \setminus A$. These imply that $B \cap A \neq \emptyset$ and $B \cap (X \setminus A) \neq \emptyset$, a contradiction. Thereby $\sigma = X \times X$.

Finally, to show that $\rho = I_X$, suppose not. Then there exist distinct elements $c, d \in X$ such that $(c, d) \in \rho$. Then $c, d \in A$ for some $A \in X/\rho$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} c & \text{if } x \in A, \\ d & \text{otherwise.} \end{cases}$$

Since $(c,d) \in \rho$, $\alpha \in T(X,\sigma,\rho)$. By assumption, we get that α is regular and by Corollary 3.2.2(1), there exists $B \in X/\rho$ such that $A \cap X\alpha \subseteq B\alpha$. Since $\rho \neq X \times X$ and $c, d \in A$, we get that $A \cap X\alpha = \{c,d\}$. Thus $c = x\alpha$ and $d = y\alpha$ for some $x, y \in B$. Therefore $x \in A$ and $y \in X \setminus A$, which implies that $B \cap A \neq \emptyset$ and $B \cap (X \setminus A) \neq \emptyset$. This is a contradiction. Hence $\rho = I_X$.

Conversely, assume that one of the converse conditions holds. If $\sigma = I_X$ or $\rho = X \times X$, then by Theorem 2.3.1(6), we have $T(X,\sigma,\rho) = T(X)$. Thus $T(X,\sigma,\rho)$ is a regular semigroup by Theorem 3.1.1. If $\sigma = X \times X$ and $\rho = I_X$, then $T(X,\sigma,\rho)$ is a right zero semigroup by Corollary 2.2.9. We conclude that $T(X,\sigma,\rho)$ is a regular semigroup by Lemma 2.1.6(1). □

Theorem 3.2.15 can be summarized as follows:

Corollary 3.2.16. *$T(X,\sigma)$ is a regular semigroup if and only if $\sigma = I_X$ or $\sigma = X \times X$.*

Corollary 3.2.17. *$E(X,\sigma)$ is a regular semigroup if and only if $\sigma = I_X$ or $\sigma = X \times X$.*

Theorem 3.2.18. *If $|X| \leq 2$, then $T(X,\sigma,\rho)$ is a left regular semigroup.*

Proof. Suppose that $|X| \leq 2$. Then $\sigma, \rho \in \{I_X, X \times X\}$. If $\sigma = \rho$, then by Theorem 2.3.1(6) we have $T(X,\sigma,\rho) = T(X)$. It follows from Theorem 3.1.6 that $T(X,\sigma,\rho)$ is a left regular semigroup. If $\sigma \neq \rho$, then $\sigma = X \times X$ and $\rho = I_X$. By Corollary 2.2.9, we obtain that $T(X,\sigma,\rho)$ is a right zero semigroup. Hence by Lemma 2.1.6(2), we have $T(X,\sigma,\rho)$ is a left regular semigroup. □

Theorem 3.2.19. *Let $|X| \geq 3$. Then $T(X,\sigma,\rho)$ is a left regular semigroup if and only if $\sigma = X \times X$ and $\rho = I_X$.*

Proof. Suppose that $\sigma \neq X \times X$ or $\rho \neq I_X$. We consider two cases as follows:

Case 1. $\sigma \neq X \times X$. Then there exist $A, B \in X/\sigma$ with $A \neq B$. Let $a \in A$ and $b \in B$. Since $|X| \geq 3$, there is an element $c \in X \setminus \{a,b\}$.

Subcase 1.1 Either $c \in A$ or $c \in B$. Without loss of generality, we may assume that $c \in A$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A, \\ c & \text{otherwise.} \end{cases}$$

Then $\alpha \in T(X,\sigma,\rho)$. Suppose that α is a left regular element of $T(X,\sigma,\rho)$. Since $A \neq B$, $X\alpha = \{a,c\}$. Let $P = c\alpha^{-1} \in \pi(\alpha)$. Then $P = X \setminus A$ and $P \in \pi_B(\alpha)$. Since $a, c \in A \cap X\alpha$ and by the definition of α, we have that $P \cap X\alpha = \emptyset$. By Theorem 3.2.5, there is $B' \in X/\rho$ such that $x\alpha \in P$ for some $x \in B'$. This is a contradiction. Hence α is not a left regular element of $T(X,\sigma,\rho)$.

Subcase 1.2 $c \notin A$ and $c \notin B$. Then there is $C \in X/\sigma$ such that $c \in C$. Thus $C \notin \{A,B\}$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} a & \text{if } x \in A \cup B, \\ b & \text{otherwise.} \end{cases}$$

Thus $\beta \in T(X,\sigma,\rho)$. Suppose that β is left regular of $T(X,\sigma,\rho)$. Let $P = b\beta^{-1} \in \pi(\beta)$. Then $P = X \setminus (A \cup B)$ and $P \in \pi_C(\beta)$. By Theorem 3.2.5, there is $C' \in X/\rho$ such that $x\beta \in P$ for some $x \in C'$. This is a contradiction with $P \cap X\beta = (X \setminus (A \cup B)) \cap \{a,b\} = \emptyset$. Hence β is not a left regular element of $T(X,\sigma,\rho)$.

Case 2. $\rho \neq I_X$. Then there are distinct elements $a, b \in X$ such that $(a,b) \in \rho$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} a & \text{if } x \in \{a,b\}, \\ b & \text{otherwise.} \end{cases}$$

Since $(a,b) \in \rho$, we have that $\gamma \in T(X,\sigma,\rho)$. By $|X| \geq 3$, we obtain that $b \in X\gamma$. Let $P = b\gamma^{-1}$. Then $P \in \pi(\gamma)$. Since $P\gamma = \{b\}$, $p \notin \{a,b\}$ for all $p \in P$. This implies that $P \cap X\gamma = \emptyset$. By Theorem 3.1.2, we deduce that γ is not a left regular element of $T(X)$. Consequently, γ is not a left regular element of $T(X,\sigma,\rho)$.

From the two cases, we conclude that $T(X,\sigma,\rho)$ is not a left regular semigroup.

The converse necessarily follows from Corollary 2.2.9 and Lemma 2.1.6. □

Theorems 3.2.18 and 3.2.19 can be summarized as follows:

Corollary 3.2.20. *The following statements hold.*

(1) *$T(X,\sigma)$ is a left regular semigroup if and only if $|X| \leq 2$.*

(2) *$E(X,\sigma)$ is a left regular semigroup if and only if $|X| \leq 2$ or $\sigma = X \times X$.*

The proof of the next result is similar to Theorem 3.2.18.

Theorem 3.2.21. *If $|X| \leq 2$, then $T(X, \sigma, \rho)$ is a right regular semigroup.*

Theorem 3.2.22. *Let $|X| \geq 3$. Then $T(X, \sigma, \rho)$ is a right regular semigroup if and only if $\sigma = X \times X$ and $\rho = I_X$.*

Proof. Suppose that $\sigma \neq X \times X$ or $\rho \neq I_X$. We consider two cases as follows.

Case 1. $\sigma \neq X \times X$. Then there exist $A, B \in X/\sigma$ with $A \neq B$. Let $a \in A$ and $b \in B$. Since $|X| \geq 3$, there is an element $c \in X \setminus \{a, b\}$.

Subcase 1.1 Either $c \in A$ or $c \in B$. Without loss of generality, we may assume that $c \in A$. Define $\alpha \in T(X, \sigma, \rho)$ as in the same proof of Theorem 3.2.19. Since $a, c \in X\alpha$ and $a\alpha = c\alpha$, $\alpha|_{X\alpha}$ is not injective.

Subcase 1.2 $c \notin A$ and $c \notin B$. Define $\beta \in T(X, \sigma, \rho)$ as in the same proof of Theorem 3.2.19. Since $a, b \in X\beta$ and $a\beta = b\beta$, $\beta|_{X\beta}$ is not injective.

Case 2. $\rho \neq I_X$. Then there exist distinct elements $a, b \in X$ such that $(a, b) \in \rho$. Define $\gamma \in T(X, \sigma, \rho)$ as in the same proof of Theorem 3.2.19. Since $|X| \geq 3$, $a, b \in X\gamma$. Note that $a\gamma = b\gamma$, which implies that $\gamma|_{X\gamma}$ is not injective.
From the two cases, they follow from Theorem 3.2.8 that $T(X, \sigma, \rho)$ is not a right regular semigroup.

The converse necessarily follows from Corollary 2.2.9 and Lemma 2.1.6. □

Theorems 3.2.21 and 3.2.22 can be summarized as follows:

Corollary 3.2.23. *The following statements hold.*

(1) *$T(X, \sigma)$ is a right regular semigroup if and only if $|X| \leq 2$.*

(2) *$E(X, \sigma)$ is a right regular semigroup if and only if $|X| \leq 2$ or $\sigma = X \times X$.*

As an immediate consequence of Theorems 2.1.2, 3.2.18, 3.2.19, 3.2.21 and 3.2.22, we have the following

Corollary 3.2.24. *$T(X, \sigma, \rho)$ is a completely regular semigroup if and only if one of the following statements holds.*

(1) $|X| \leq 2$.

(2) *$\sigma = X \times X$ and $\rho = I_X$.*

Corollary 3.2.24 can be summarized as follows:

Corollary 3.2.25. *The following statements hold.*

(1) $T(X, \sigma)$ *is a completely regular semigroup if and only if* $|X| \leq 2$.

(2) $E(X, \sigma)$ *is a completely regular semigroup if and only if* $|X| \leq 2$ *or* $\sigma = X \times X$.

3.3 Regularity for the semigroups of transformations that preserve double direction equivalence

In this section, let σ be an equivalence relation on X. We recall

$$T(X, \sigma^*) := \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in \sigma \text{ if and only if } (x\alpha, y\alpha) \in \sigma\}.$$

Deng et al. [32] have given some characterizations of the regularity for elements on $T(X, \sigma^*)$. Later, Laysirikul and Namnak investigated the left regularity, the right regularity and the completely regularity for elements of $T(X, \sigma^*)$. Hence we give a necessary and sufficient condition for $T(X, \sigma^*)$ to be left regular, right regular and completely regular.

The following results are quoted from [32] Theorem 2.7.2 and [50] Theorems 3.4.4, 3.4.5 and 3.4.6, respectively.

Theorem 3.3.1. [32] *Let* $\alpha \in T(X, \sigma^*)$. *Then* α *is regular of* $T(X, \sigma^*)$ *if and only if for every* $A \in X/\sigma$, $A \cap X\alpha \neq \emptyset$.

Theorem 3.3.2. [50] *Let* $\alpha \in T(X, \sigma^*)$. *Then* α *is left regular of* $T(X, \sigma^*)$ *if and only if for every* $P \in \pi(\alpha)$, $P \cap X\alpha \neq \emptyset$.

Theorem 3.3.3. [50] *Let* $\alpha \in T(X, \sigma^*)$. *Then* α *is right regular of* $T(X, \sigma^*)$ *if and only if the following statements hold.*

(1) $\alpha|_{X\alpha}$ *is an injection.*

(2) *If there exists* $A \in X/\sigma$ *such that* $A \cap X\alpha^2 = \emptyset$, *then there exists an injection* $\varphi : \{A \in X/\sigma : A \cap X\alpha^2 = \emptyset\} \to \{A \in X/\sigma : A \cap X\alpha = \emptyset\}$.

Theorem 3.3.4. [50] *Let $\alpha \in T(X, \sigma^*)$. Then α is completely regular of $T(X, \sigma^*)$ if and only if for every $P \in \pi(\alpha)$, $|P \cap X\alpha| = 1$.*

Now, we show the semigroup $T(X, \sigma^*)$ does not necessary for the left regular semigroup, the right regular semigroup and completely regular semigroup. As we see in the next example.

Example 3.3.5. Let $A_1 = \{1\}, A_2 = \{2,3\}, A_3 = \{4,5,6\}$ and for $n \geq 4$

$$A_n = \left\{ x \in \mathbb{N} : \frac{(n-1)n}{2} < x \leq \frac{n(n+1)}{2} \right\}.$$

Define $\sigma = \bigcup_{i \in \mathbb{N}} (A_i \times A_i)$. Clearly, σ is an equivalence relation on $\mathbb{N}$. Now, we define $\alpha : \mathbb{N} \to \mathbb{N}$ by

$$x\alpha = \min A_{n+1} \text{ for all } x \in A_n \text{ and for each } A_n \in \mathbb{N}/\sigma.$$

Since $\mathbb{N}/\sigma$ is a partition of $\mathbb{N}$, α is well-defined. To show that $\alpha \in T(\mathbb{N}, \sigma^*)$, since $A_n\alpha \subseteq A_{n+1}$ for all $n \in \mathbb{N}$ and by Corollary 2.2.6, $\alpha \in T(\mathbb{N}, \sigma)$. On the other hand, let $x, y \in \mathbb{N}$ be such that $(x\alpha, y\alpha) \in \sigma$. By the definition of α, we then have $x\alpha = y\alpha = \min A_n$ for some $n \in \mathbb{N} \setminus \{1\}$. Therefore $x, y \in A_{n-1}$ and hence $(x, y) \in \sigma$. We deduce that $\alpha \in T(\mathbb{N}, \sigma^*)$ as required. We note that $\alpha|_{\mathbb{N}\alpha}$ is injective but since $\{A \in \mathbb{N}/\sigma : A \cap X\alpha^2 = \emptyset\} = \{A_1, A_2\}$ and $\{A \in \mathbb{N}/\sigma : A \cap X\alpha = \emptyset\} = \{A_1\}$, there is no injection from

$$\{A \in \mathbb{N}/\sigma : A \cap X\alpha^2 = \emptyset\} \text{ to } \{A \in \mathbb{N}/\sigma : A \cap X\alpha = \emptyset\}.$$

Therefore α does not satisfy condition (2) in Theorem 3.3.3. Hence α is not right regular of $T(\mathbb{N}, \sigma^*)$. Also, α is not completely regular of $T(\mathbb{N}, \sigma^*)$. Since $2 \in \mathbb{N}\alpha$, $2\alpha^{-1} \in \pi(\alpha)$ and so $2\alpha^{-1} = \{1\}$. It follows from $1 \notin \mathbb{N}\alpha$ that $2\alpha^{-1} \cap \mathbb{N}\alpha = \emptyset$. By Theorem 3.3.2, we get that α is not left regular of $T(\mathbb{N}, \sigma^*)$.

Theorem 3.3.6. [32] *$T(X, \sigma^*)$ is a regular semigroup if and only if X/σ is finite.*

Theorem 3.3.7. *$T(X, \sigma^*)$ is a left regular semigroup if and only if X/σ is finite and σ is both a T-relation and 2-bounded.*

Proof. Assume that $T(X, \sigma^*)$ is a left regular semigroup and X/σ is an infinite set. Then X/σ has a countable infinite subset $\{A_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, we choose and fix an element $a_n \in A_n$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a_{n+1} & \text{if } x \in A_n \text{ for all } n \in \mathbb{N}, \\ x & \text{otherwise.} \end{cases}$$

Then $A_n\alpha \subseteq A_{n+1}$ for all $n \in \mathbb{N}$ and $A\alpha \subseteq A$ for all $A \in X/\sigma \setminus \{A_n : n \in \mathbb{N}\}$. By Corollary 2.2.6, we deduce that $\alpha \in T(X,\sigma)$. On the other hand, let $(x\alpha, y\alpha) \in \sigma$. Then $x\alpha, y\alpha \in A$ for some $A \in X/\sigma$. If $A \in \{A_n : n \in \mathbb{N}\}$, then $A = A_n$ for some $n \in \mathbb{N}$ with $n > 1$. Therefore $x, y \in A_{n-1}$ and so $(x,y) \in \sigma$. If $A \notin \{A_n : n \in \mathbb{N}\}$, then $(x,y) = (x\alpha, y\alpha) \in \sigma$. Hence $\alpha \in T(X,\sigma^*)$. Note that $A_1 \cap X\alpha = \emptyset$ and $A_1 = a_2\alpha^{-1} \in \pi(\alpha)$. By Theorem 3.3.2, we get that α is not left regular. This is a contradiction. Hence X/σ is finite.

Suppose that σ is not a T-relation. Then there are disjoint $A, B \in X/\sigma$ such that $|A|, |B| \geq 2$. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ be such that $a_1 \neq a_2$ and $b_1 \neq b_2$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} a_1 & \text{if } x \in B, \\ b_1 & \text{if } x = a_1, \\ b_2 & \text{if } x \in A \setminus \{a_1\}, \\ x & \text{otherwise.} \end{cases}$$

Then $A\beta \subseteq B$, $B\beta \subseteq A$ and $C\beta \subseteq C$ for all $C \in X/\sigma \setminus \{A, B\}$. By Corollary 2.2.6, we then have $\alpha \in T(X,\sigma)$. On the other hand, let $(x\beta, y\beta) \in \sigma$. Then $x\beta, y\beta \in C$ for some $C \in X/\sigma$. If $C = A$, then $x, y \in B$ and so $(x,y) \in \sigma$. Similarly, if $C = B$, then $x, y \in A$ and so $(x,y) \in \sigma$. Otherwise, $(x,y) = (x\beta, y\beta) \in \sigma$. Therefore $\beta \in T(X,\sigma^*)$. Since $A \setminus \{a_1\} = b_2\beta^{-1} \in \pi(\beta)$ and $A \setminus \{a_1\} \cap X\beta = \emptyset$ and by Theorem 3.3.2, we have that β is not left regular. This is a contradiction. Hence σ is a T-relation.

Next, we will show that σ is 2-bounded. Suppose that σ is not 2-bounded. Then there exists $C \in X/\sigma$ such that $|C| \geq 3$. Let c_1, c_2, c_3 be all distinct elements of C. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} c_1 & \text{if } x \in \{c_1, c_2\}, \\ c_2 & \text{if } x = c_3, \\ x & \text{otherwise.} \end{cases}$$

Then $D\gamma \subseteq D$ for all $D \in X/\sigma$. We can see that $\gamma \in T(X,\sigma^*)$. Since $\{c_3\} = c_2\gamma^{-1} \in \pi(\gamma)$ and $c_3 \notin X\gamma$ and by Theorem 3.3.2, we get that γ is not left regular. This is a contradiction. Hence σ is 2-bounded, as required.

Conversely, suppose that X/σ is finite and σ is both 2-bounded and a T-relation. Let $\alpha \in T(X, \sigma^*)$ and $P \in \pi(\alpha)$. From Lemma 2.2.11, there exists $A \in X/\sigma$ such that $P \subseteq A$. By Theorems 3.3.6 and 3.3.1, $A \cap X\alpha \neq \emptyset$. Since σ is both 2-bounded and a T-relation, $|A| = 1$ or $|A| = 2$. If $|A| = 1$, then $A = P$ and hence $P \cap X\alpha = A \cap X\alpha \neq \emptyset$. Suppose that $|A| = 2$ and $P \cap X\alpha = \emptyset$. Let $x \in A \cap X\alpha$. Then $x \notin P$. Since $|A| = 2$, we get that P has only one element, say, $P = \{p\}$. This implies that $(x\alpha, p\alpha) \in \sigma$ and $x\alpha \neq p\alpha$ from $x \notin P$. Since σ is a T-relation, A is only one σ-class such that $|A| > 1$. Thus $\{x\alpha, p\alpha\} = A$ and so $P \subseteq \{x\alpha, p\alpha\} \subseteq X\alpha$. This is a contradiction. Hence $P \cap X\alpha \neq \emptyset$. By Theorem 3.3.2, we obtain that α is left regular of $T(X, \sigma^*)$. Consequently, we conclude that $T(X, \sigma^*)$ is a left regular semigroup. □

Theorem 3.3.8. *$T(X, \sigma^*)$ is a right regular semigroup if and only if X/σ is finite and σ is both a T-relation and 2-bounded.*

Proof. Assume that $T(X, \sigma^*)$ is a right regular semigroup and X/σ is an infinite set. Define $\alpha \in T(X, \sigma^*)$ as in the same proof of Theorem 3.3.7. By the definition of α, we get that $\{A \in X/\sigma : A \cap X\alpha = \emptyset\} = \{A_1\}$ and $\{A \in X/\sigma : A \cap X\alpha^2 = \emptyset\} = \{A_1, A_2\}$. Then there is no an injection $\varphi : \{A \in X/\sigma : A \cap X\alpha^2 = \emptyset\} \to \{A \in X/\sigma : A \cap X\alpha = \emptyset\}$. By Theorem 3.3.3, α is not right regular. This is a contradiction. Hence X/σ is finite.

We will show that σ is a T-relation. Suppose not. Then there are disjoint $A, B \in X/\sigma$ such that $|A|, |B| \geq 2$. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ be such that $a_1 \neq a_2$ and $b_1 \neq b_2$. Define $\beta \in T(X, \sigma^*)$ as in the same proof of Theorem 3.3.7. Since $b_1, b_2 \in X\beta$ and $b_1\beta = b_2\beta$, $\beta|_{X\beta}$ is not injective. By Theorem 3.3.3, we have that β is not right regular. This is a contradiction. Hence σ is a T-relation.

Finally, we will show that σ is 2-bounded. Suppose that σ is not 2-bounded. Then there exists $C \in X/\sigma$ such that $|C| \geq 3$. Let c_1, c_2, c_3 be all distinct elements of C. Define $\gamma \in T(X, \sigma^*)$ as in the same proof of Theorem 3.3.7. Since $c_1, c_2 \in X\gamma$ and $c_1\gamma = c_2\gamma$, $\gamma|_{X\gamma}$ is not injective. By Theorem 3.3.3, we have that γ is not right regular. This is a contradiction. Hence σ is 2-bounded.

Conversely, assume that X/σ is a finite set and σ is both 2-bounded and a T-relation. Let $\alpha \in T(X, \sigma^*)$. For each $x, y \in X\alpha$. If $x\alpha = y\alpha$, then $(x\alpha, y\alpha) \in \sigma$. By the definition of α, we have $(x, y) \in \sigma$. Suppose that $x \neq y$. By the definition of

σ, we get that $\{x,y\} \in X/\sigma$ and it is only one element of X/σ with containing more than one point. Since $x,y \in X\alpha$, $x = x'\alpha$ and $y = y'\alpha$ for some $x',y' \in X$. This implies that $(x',y') \in \sigma$ and $x' \neq y'$. Therefore $\{x,y\} = \{x',y'\}$. If $x = x'$, then $y = y'$ and hence $x = x\alpha = y\alpha = y$. If $x \neq x'$, then $x = y'$ and $y = x'$. Thus $x = x'\alpha = y\alpha = x\alpha = y'\alpha = y$. This is a contradiction. Hence $x = y$. Consequently, $\alpha|_{X\alpha}$ is an injection. Since $\alpha^2 \in T(X,\sigma^*)$ and X/σ is finite, by Theorems 3.3.6 and 3.3.1, it follows that $A \cap X\alpha^2 \neq \emptyset$ for all $A \in X/\sigma$. By Theorem 3.3.3, α is right regular. Consequently, $T(X,\sigma^*)$ is a right regular semigroup. □

As a consequence of Theorems 2.1.2, 3.3.7 and 3.3.8, we have the following.

Corollary 3.3.9. *$T(X,\sigma^*)$ is a completely regular semigroup if and only if X/σ is finite and σ is both a T-relation and 2-bounded.*

3.4 Regularity for the semigroups of transformations preserving an equivalence relation and a cross-section

In this section, let σ be an equivalence relation on X and R a cross-section of the partition X/σ induced by σ. We recall that

$$T(X,\sigma,R) = \{\alpha \in T(X) : R\alpha \subseteq R \text{ and } \forall x,y \in X, (x,y) \in \sigma \text{ implies } (x\alpha,y\alpha) \in \sigma\}.$$

Araújo and Konieczny [42] have given some characterizations of the regularity for elements on $T(X,\sigma,R)$. We characterize the left regularity, the right regularity and completely regularity on $T(X,\sigma,R)$.

Firstly, we investigate the condition under which an element in $T(X,\sigma,R)$ is regular, left regular, right regular and completely regular, respectively.

Theorem 3.4.1. [42] *Let $\alpha \in T(X,\sigma,R)$. Then α is regular of $T(X,\sigma,R)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $A \cap X\alpha \subseteq B\alpha$.*

Theorem 3.4.2. *Let $\alpha \in T(X,\sigma,R)$. Then α is left regular of $T(X,\sigma,R)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$.*

Proof. The necessity follows from Corollary 3.2.6. To prove the sufficiency, we suppose that for each $A \in X/\sigma$, we choose $A' \in X/\sigma$ such that for every $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in A'$. Let $x \in X$. Since X/σ and $\pi(\alpha)$ are partitions of X, there exist $A \in X/\sigma$ and $P \in \pi(\alpha)$ such that

$$x \in A \text{ and } x \in P,$$

so $P \in \pi_A(\alpha)$. If $x \notin R$, then by assumption, we choose an element $x' \in A'$ such that $x'\alpha \in P$ and $A' \in X/\sigma$. We also have that $x'\alpha\alpha = x\alpha$. If $x \in R$, then there exists $y \in A'$ such that $y\alpha \in P$ and thus $x\alpha = y\alpha\alpha$. Let $r \in A' \cap R$. Then $(y\alpha\alpha, r\alpha\alpha) \in \sigma$. Since $r\alpha\alpha \in R$ and $x\alpha \in R$, we then have $r\alpha\alpha = x\alpha$. Thus we choose $x' = r$. Define $\beta : X \to X$ by

$$x\beta = x' \text{ for all } x \in X.$$

Clearly, $R\beta \subseteq R$. Let $x, y \in X$ be such that $(x, y) \in \sigma$. Then there exists $A \in X/\sigma$ such that $x, y \in A$. By the definition of β, we obtain that $x\beta, y\beta \in A'$ where $A' \in X/\sigma$. Hence $\beta \in T(X, \sigma, R)$. If $x \in X$, then $x\beta\alpha^2 = x'\alpha\alpha = x\alpha$ which gives $\alpha = \beta\alpha^2$. Therefore α is left regular, as required. □

Theorem 3.4.3. *Let $\alpha \in T(X, \sigma, R)$. Then α is right regular of $T(X, \sigma, R)$ if and only if the following statements hold.*

(1) *$\alpha|_{X\alpha}$ is an injection.*

(2) *For every $x, y \in X\alpha$, $(x\alpha, y\alpha) \in \sigma$ implies $(x, y) \in \sigma$.*

Proof. The necessity follows from Corollary 3.2.9. To prove the sufficiency, we assume that (1) and (2) hold. Let $A \in X/\sigma$ be such that $A \cap X\alpha^2 \neq \emptyset$. Then $A \cap R \cap X\alpha^2 \neq \emptyset$. We choose and fix an element $x_A \in A \cap R \cap X\alpha^2$. For each $x \in A \cap X\alpha^2$, there exists a unique $x' \in X\alpha$ such that $x = x'\alpha$ by $\alpha|_{X\alpha}$ is injective. We observe that $(x'\alpha, x'_A\alpha) = (x, x_A) \in \sigma$. It follows from assumption that $(x', x'_A) \in \sigma$. Define $\beta_A : A \to X$ by

$$x\beta_A = \begin{cases} x' & \text{if } x \in X\alpha^2, \\ x'_A & \text{otherwise.} \end{cases}$$

Then we define the map $\beta : X \to X$ by

$$\beta|_A = \begin{cases} \beta_A & \text{if } A \cap X\alpha^2 \neq \emptyset, \\ i_A & \text{otherwise,} \end{cases}$$

for all $A \in X/\sigma$. Since X/σ is a partition of X, β is well-defined. Let $x, y \in X$ be such that $(x, y) \in \sigma$. Then $x, y \in A$ for some $A \in X/\sigma$. By the definition of β, we have $(x\beta, y\beta) = (x\beta|_A, y\beta|_A)$. If $A \cap X\alpha^2 = \emptyset$, then $(x\beta, y\beta) = (xi_A, yi_A) = (x, y) \in \sigma$. If $A \cap X\alpha^2 \neq \emptyset$, by the definition of β_A we then have $(x\beta_A, x'_A), (y\beta_A, x'_A) \in \sigma$. By transitivity of σ, $(x\beta, y\beta) \in \sigma$, hence $\beta \in T(X, \sigma)$. Since $R\alpha \subseteq R$ and $\alpha|_{X\alpha}$ is injective, it follows that $R\beta \subseteq R$. Consequently, $\beta \in T(X, \sigma, R)$.

Finally, to show that $\alpha = \alpha^2\beta$, let $x \in X$. Then $x\alpha^2 \in X\alpha^2$ and there exists $A \in X/\sigma$ such that $x\alpha^2 \in A$. By the definition of β_A, $x\alpha^2\beta_A = (x\alpha^2)'$ where $(x\alpha^2)'\alpha = x\alpha^2 = (x\alpha)\alpha$. Since $(x\alpha^2)'$ is unique, we get that $(x\alpha^2)' = x\alpha$. Thus $x\alpha^2\beta = x\alpha^2\beta_A = x\alpha$. Hence α is right regular, as asserted. □

As a consequence of Theorems 2.1.2, 3.4.2 and 3.4.3, the following result follows readily.

Corollary 3.4.4. *Let $\alpha \in T(X, \sigma, R)$. Then α is completely regular of $T(X, \sigma, R)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A(\alpha)$.*

Secondly, we show that the semigroup $T(X, \sigma, R)$ does not necessarily for the left regular semigroup, the right regular semigroup and completely regular semigroup. As we see in the next example.

Example 3.4.5. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $X/\sigma = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$ and $R = \{1, 4, 6\}$. Let $\alpha \in T(X, \sigma, R)$ be defined by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 6 & 1 & 3 & 1 & 2 & 1 \end{pmatrix}.$$

Since $6, 8 \in X\alpha$ and $6\alpha = 8\alpha$ by Theorem 3.4.3, we have that α is not a right regular element of $T(X, \sigma, R)$. Next, we will show that α is not a left regular element of $T(X, \sigma, R)$ by using Theorem 3.4.2. Let $A = \{4, 5\}$. Then $\pi_A(\alpha) = \{\{4, 6, 8\}, \{5\}\}$. Since $5 \notin X\alpha$, $5 \neq x\alpha$ for all $x \in X$. This implies that α does not satisfy Theorem 3.4.2. Hence α is not a left regular element of $T(X, \sigma, R)$. Furthermore, α is not a completely regular element of $T(X, \sigma, R)$.

Finally, we investigate the conditions under which the semigroup $T(X,\sigma,R)$ is regular, left regular, right regular and completely regular, respectively.

Theorem 3.4.6. [42] *$T(X,\sigma,R)$ is a regular semigroup if and only if σ is a T-relation or 2-bounded.*

Theorem 3.4.7. *If $|X| \le 2$, then $T(X,\sigma,R)$ is a left regular semigroup.*

Proof. If $|X| = 1$, then done. Suppose that $|X| = 2$, say $X = \{a,b\}$.

Case 1. $|R| = 1$. Assume that $a \in R$. Then

$$T(X,\sigma,R) = \left\{ \begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ a & b \end{pmatrix} \right\}.$$

Then every element of $T(X,\sigma,R)$ is left regular. Hence $T(X,\sigma,R)$ is a left regular semigroup.

Case 2. $|R| = 2$. Then σ is the identity relation. By Theorem 2.3.4(2) and Theorem 3.1.6, we conclude that $T(X,\sigma,R)$ is a left regular semigroup. $\square$

Theorem 3.4.8. *Let $|X| \ge 3$. Then $T(X,\sigma,R)$ is a left regular semigroup if and only if $|X| = 3$ and $|R| = 2$.*

Proof. We will proceed proof by contrapositive of the necessity. We now consider three cases as follows.

Case 1. $|R| = 1$. Then $\sigma = X \times X$. Let $r \in R$. By assumption, there exist distinct elements $a, b \in X \setminus \{r\}$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} r & \text{if } x \neq a, \\ b & \text{otherwise.} \end{cases}$$

Since $\sigma = X \times X$ and $r\alpha = r$, we get that $\alpha \in T(X,\sigma,R)$. Let $P = b\alpha^{-1} \in \pi(\alpha)$. Then $P = \{a\}$. Note that $a \notin X\alpha$, that is, $x\alpha \neq a$ for all $x \in X$. Since $|X/\sigma| = 1$, it follows that α is not satisfy condition of Theorem 3.4.2, which implies that α is not left regular. Hence $T(X,\sigma,R)$ is not a left regular semigroup.

Case 2. $|R| = 2$ and $|X| > 3$. Then there exists $r \in R$ such that $|r\sigma| \ge 2$. Let $a \in r\sigma \setminus \{r\}$ and $s \in R$ with $r \neq s$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} r & \text{if } x \in \{r,s,a\}, \\ a & \text{otherwise.} \end{cases}$$

Since $(a,r) \in \sigma$ and $r\beta = s\beta = r \in R$, we have that $\beta \in T(X,\sigma,R)$. Noting that $|X| > 3$, we obtain $a \in X\beta$. Let $P = a\beta^{-1} \in \pi(\beta)$. Then $P = X \setminus \{r,s,a\}$ and so $P \cap X\beta = \emptyset$. Therefore β is not satisfy condition of Theorem 3.4.2. Consequently, β is not left regular. Hence $T(X,\sigma,R)$ is not a left regular semigroup.

Case 3. $|R| \geq 3$. Let $r,s,t \in R$ be distinct elements. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} r & \text{if } x \in r\sigma \cup s\sigma, \\ s & \text{otherwise.} \end{cases}$$

Clearly, $R\gamma \subseteq R$ and by Corollary 2.2.6, we obtain $\gamma \in T(X,\sigma,R)$. Let $P = s\gamma^{-1} \in \pi(\gamma)$. Then $P = X \setminus (r\sigma \cup s\sigma)$ and thus $P \cap X\gamma = \emptyset$. Therefore γ is not satisfy condition of Theorem 3.4.2 and thus γ is not left regular. Hence $T(X,\sigma,R)$ is not a left regular semigroup.

Conversely, suppose that $|X| = 3$ and $|R| = 2$. Let $X = \{a,b,c\}$. Without loss of generality, we may assume that $X/\sigma = \{\{a\},\{b,c\}\}$ and $R = \{a,b\}$. Therefore $T(X,\sigma,R)$ is the set

$$\left\{ \begin{pmatrix} a & b & c \\ a & a & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & b & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & a & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & b & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & b & c \end{pmatrix} \right\}.$$

By virtue of Theorem 3.4.2, every element of $T(X,\sigma,R)$ is left regular. Hence $T(X,\sigma,R)$ is a left regular semigroup. □

The proof of the next result is similar to Theorem 3.4.7.

Theorem 3.4.9. *If $|X| \leq 2$, then $T(X,\sigma,R)$ is a right regular semigroup.*

Theorem 3.4.10. *Let $|X| \geq 3$. Then $T(X,\sigma,R)$ is a right regular semigroup if and only if $|X| = 3$ and $|R| = 2$.*

Proof. The necessity is proved by contraposition. We distinguish three cases as follows.

Case 1. $|R| = 1$. Let $r \in R$. By assumption, there exist distinct elements $a,b \in X \setminus \{r\}$. Define $\alpha \in T(X,\sigma,R)$ as in the same proof of Theorem 3.4.8. Since $r,b \in X\alpha$ and $r\alpha = b\alpha$, $\alpha|_{X\alpha}$ is not injective.

Case 2. $|R| = 2$ and $|X| > 3$. Then there is $r \in R$ such that $|r\sigma| \geq 2$. Let $a \in r\sigma \setminus \{r\}$ and $s \in R$ with $r \neq s$. Define $\beta \in T(X,\sigma,R)$ as in the same proof of

Theorem 3.4.8. Since $|X| > 3$, we have $s, a \in X\beta$. Note that $s\beta = a\beta$. Thus $\beta|_{X\beta}$ is not an injection.

Case 3. $|R| \geq 3$. Let r, s be distinct elements in R. Define $\gamma \in T(X, \sigma, R)$ as in the same proof of Theorem 3.4.8. By the definition of R, we have $r, s \in X\gamma$ and so $r\gamma = s\gamma$. Therefore $\alpha|_{X\alpha}$ is not injective.

From the three cases, they follow from Theorem 3.4.3 that $T(X, \sigma, R)$ is not a right regular semigroup.

Conversely, suppose that $|X| = 3$ and $|R| = 2$. Let $X = \{a, b, c\}$. Without loss of generality, we may assume that $X/\sigma = \{\{a\}, \{b, c\}\}$ and $R = \{a, b\}$. Therefore $T(X, \sigma, R)$ is the set

$$\left\{ \begin{pmatrix} a & b & c \\ a & a & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & b & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & a & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & b & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & b & c \end{pmatrix} \right\}.$$

It is easy to check that every element of $T(X, \sigma, R)$ is right regular by using Theorem 3.4.3. Hence $T(X, \sigma, R)$ is a right regular semigroup. □

As an immediate consequence of Theorems 2.1.2, 3.4.7, 3.4.8, 3.4.9 and 3.4.10, we have the following.

Corollary 3.4.11. *$T(X, \sigma, R)$ is a completely regular semigroup if and only if one of the following statements holds.*

(1) $|X| \leq 2$.

(2) $|X| = 3$ *and* $|R| = 2$.

3.5 Regularity for the semigroups of transformations preserving an equivalence relation and fix a cross-section

In this section, let σ be an equivalence relation on X and R a cross-section of the partition X/σ induced by σ. We review again that

$$T_R(X, \sigma) = \{\alpha \in T(X) : R\alpha = R \text{ and } \forall x, y \in X, (x, y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \sigma\}.$$

The aim of this section is to present necessary and sufficient conditions for elements of $T_R(X,\sigma)$ and the semigroup $T_R(X,\sigma)$ which is regular, left regular, right regular and completely regular.

Theorem 3.5.1. *Let $\alpha \in T_R(X,\sigma)$. Then α is regular of $T_R(X,\sigma)$ if and only if $\alpha|_R$ is an injection.*

Proof. Suppose that α is regular. Then there exists $\beta \in T_R(X,\sigma)$ such that $\alpha = \alpha\beta\alpha$. Let $r, s \in R$ be such that $r\alpha = s\alpha$. Since $\beta \in T_R(X,\sigma)$, $R\beta = R$ and hence $r = r'\beta$ and $s = s'\beta$ for some $r', s' \in R$. Since $R\alpha = R$, there exist $r'', s'' \in R$ such that $r' = r''\alpha$ and $s' = s''\alpha$. We have that

$$r' = r''\alpha = r''\alpha\beta\alpha = r'\beta\alpha = r\alpha = s\alpha = s'\beta\alpha = s''\alpha\beta\alpha = s''\alpha = s'.$$

This implies that $r = r'\beta = s'\beta = s$. Hence $\alpha|_R$ is an injection, as required.

Conversely, assume that $\alpha|_R$ is an injection. Claim that for every $r \in R$, there exists $r' \in R$ such that $r\sigma \cap X\alpha = (r'\sigma)\alpha$. Let $r \in R$. Since $R\alpha = R$, there is $r' \in R$ such that $r = r'\alpha$. Since $\alpha \in T(X,\sigma)$, it then follows that

$$(r'\sigma)\alpha \subseteq r\sigma \cap X\alpha.$$

For the reverse inclusion, if $y \in r\sigma \cap X\alpha$, then $y = x\alpha$ for some $x \in X$. This implies that $x \in s\sigma$ for some $s \in R$ and so $s\alpha = r$. By assumption and $s\alpha = r'\alpha$, we have $s = r'$. Hence $y \in (r'\sigma)\alpha$. This shows that $r\sigma \cap X\alpha = (r'\sigma)\alpha$. So we have the claim.

For each $r \in R$, we choose $r' \in R$ such that $r\sigma \cap X\alpha = (r'\sigma)\alpha$. Thus $r = r'\alpha$. For each $y \in (r\sigma \cap X\alpha) \setminus \{r\}$, we choose and fix an element $y' \in r'\sigma$ such that $y'\alpha = y$. Define $\beta_r : r\sigma \to r'\sigma$ by

$$x\beta_r = \begin{cases} x' & \text{if } x \in X\alpha, \\ r' & \text{otherwise.} \end{cases}$$

Then β_r is well-defined, $(r\sigma)\beta_r \subseteq r'\sigma$ and $r\beta_r = r' \in R$. Let $\beta : X \to X$ be defined by $\beta|_{r\sigma} = \beta_r$ for all $r \in R$. Since R is a cross-section of the partition X/σ induced by σ, β is well-defined. By Corollary 2.2.6, $\beta \in T(X,\sigma)$. Obviously, $R\beta \subseteq R$. Let $r \in R$. Then $r\alpha = s$ for some $s \in R$. Thus $s\beta_s = s'$ for some $s' \in R$ with $s'\alpha = s$. Therefore $s'\alpha = r\alpha$. By assumption, we have that $s' = r$ and thus $s\beta = s\beta|_{s\sigma} = s\beta_s = s' = r$. It follows that

$R\beta = R$ and therefore $\beta \in T_R(X,\sigma)$. Let $x \in X$. Then $x\alpha \in r\sigma$ for some $r \in R$. Thus

$$x\alpha\beta\alpha = (x\alpha)\beta|_{r\sigma}\alpha = (x\alpha)\beta_r\alpha = (x\alpha)'\alpha = x\alpha$$

and therefore $\alpha = \alpha\beta\alpha$. Hence α is regular. □

Theorem 3.5.2. *Let $\alpha \in T_R(X,\sigma)$ be such that $\alpha|_R$ is an injection. Then α is left regular of $T_R(X,\sigma)$ if and only if for every $P \in \pi(\alpha)$, $P \cap X\alpha \neq \emptyset$.*

Proof. The necessity is clear from Theorem 3.1.2. To prove the sufficiency, we suppose that $P \cap X\alpha \neq \emptyset$ for all $P \in \pi(\alpha)$. We will construct $\beta \in T_R(X,\sigma)$ with $\alpha = \beta\alpha^2$. For each $x \in X$, there exists a unique $P_x \in \pi(\alpha)$ such that $x \in P_x$. By assumption, we have $P_x \cap X\alpha \neq \emptyset$. If $x \notin R$, then we choose and fix an element $x_{P_x} \in P_x \cap X\alpha$ and $x'_{P_x} \in X$ such that $x'_{P_x}\alpha = x_{P_x}$. If $x \in R$, then $x \in P_x \cap X\alpha$. Thus we select $x_{P_x} = x$. Since $R\alpha = R$, we can select $x'_{P_x} \in R$ with $x'_{P_x}\alpha = x_{P_x}$. Define $\beta : X \to X$ by

$$x\beta = x'_{P_x} \text{ for all } x \in X.$$

Then β is well-defined. Since $R\alpha = R$ and $\alpha|_R$ is injective, we deduce that $R\beta = R$. Let $x, y \in X$ be such that $(x,y) \in \sigma$. Then $x, y \in r\sigma$ for some $r \in R$. Since $\pi(\alpha)$ is a partition of X, we note that $x \in P_x$ and $y \in P_y$ for some $P_x, P_y \in \pi(\alpha)$. If $z \in P_x$, then $x\alpha = z\alpha$ and $z \in s\sigma$ for some $s \in R$. It follows from Lemma 2.2.13 that $r\alpha = s\alpha$. By the definition of α, $r = s$ and thus $P_x \subseteq r\sigma$. By symmetry, we also have $P_y \subseteq r\sigma$. Since $x_{P_x} \in P_x$ and $x_{P_y} \in P_y$, we then have $(x_{P_x}, x_{P_y}) \in \sigma$. Claim that $(x'_{P_x}, x'_{P_y}) \in \sigma$. Let $x'_{P_x} \in s\sigma$ and $x'_{P_y} \in t\sigma$ where $s, t \in R$. Since $(x'_{P_x}\alpha, x'_{P_y}\alpha) = (x_{P_x}, x_{P_y}) \in \sigma$, it follows that $s\alpha = t\alpha$. Thus $s = t$ by $\alpha|_R$ is injective. So we have the claim. Hence we conclude that $(x\beta, y\beta) = (x'_{P_x}, x'_{P_y}) \in \sigma$. Consequently, $\beta \in T_R(X,\sigma)$.

Finally, to show that $\alpha = \beta\alpha^2$, let $x \in X$. Then

$$x\beta\alpha^2 = x'_{P_x}\alpha^2 = x_{P_x}\alpha = x\alpha,$$

which implies that $\alpha = \beta\alpha^2$. Therefore α is left regular. □

Example 3.5.3. Let $X = \mathbb{Z}$, $X/\sigma = \{\{0,1\}\} \cup \{\{x\} : x \in \mathbb{Z} \setminus \{0,1\}\}$ and $R = \mathbb{Z} \setminus \{0\}$. Define $\alpha, \beta : X \to X$ by

$$x\alpha = \begin{cases} 1 & \text{if } x \leq 1, \\ -\frac{x}{2} & \text{if } x \in 2\mathbb{N}, \\ \frac{x+1}{2} & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} x & \text{if } x \in \{-1,0,1\}, \\ 2x-1 & \text{if } x \geq 2, \\ -x & \text{if } x \in -2\mathbb{N}, \\ \frac{x-1}{2} & \text{otherwise.} \end{cases}$$

It is easy to check that $\alpha, \beta \in T(X, \sigma)$. Note that $(2\mathbb{N})\alpha = -\mathbb{N}$, $(2\mathbb{N}+1)\alpha = \mathbb{N} \setminus \{1\}$ and $(-\mathbb{N} \cup \{1\})\alpha = \{1\}$, imply that

$$R\alpha = (2\mathbb{N})\alpha \cup (2\mathbb{N}+1)\alpha \cup (-\mathbb{N} \cup \{1\})\alpha = -\mathbb{N} \cup (\mathbb{N} \setminus \{1\}) \cup \{1\} = \mathbb{Z} \setminus \{0\} = R.$$

Thus $\alpha \in T_R(X, \sigma)$. Consider

$\{-1,1\}\beta = \{-1,1\}$, $(\mathbb{N}\setminus\{1\})\beta = \{3,5,7,\ldots\}$, $(-2\mathbb{N})\beta = 2\mathbb{N}$ and $(-2\mathbb{N}-1)\beta = -\mathbb{N}\setminus\{-1\}$.

This means that $R\beta = R$, hence $\beta \in T_R(X, \sigma)$. We will show that $\alpha = \beta\alpha^2$. Let $x \in X$.

If $x \in \{-1,0,1\}$, then $x\beta\alpha\alpha = x\alpha\alpha = 1\alpha = 1 = x\alpha$.

If $x \geq 2$, then $2x - 1 \in \{3,5,7,\ldots\}$ and hence

$$x\beta\alpha\alpha = (2x-1)\alpha\alpha = \left(\frac{(2x-1)+1}{2}\right)\alpha = \left(\frac{2x}{2}\right)\alpha = x\alpha.$$

If $x \in -2\mathbb{N}$, then $x\beta = -x \in 2\mathbb{N}$ so $x\beta\alpha = -(\frac{-x}{2}) \leq 1$. Thus $x\beta\alpha\alpha = 1 = x\alpha$.

If $x \in \{-3,-5,-7,\ldots\}$, then $x\beta = \frac{x-1}{2} \leq 1$, so that $x\beta\alpha\alpha = 1\alpha = 1 = x\alpha$.

From the above discussion, we conclude that α is left regular. But $\alpha|_R$ is not an injection because $1\alpha = -1\alpha$.

Example 3.5.4. Let $X = \mathbb{N}$, $X/\sigma = \{\{1,2\}\} \cup \{\{x\} : x \in X \text{ with } x \geq 3\}$ and $R = \mathbb{N} \setminus \{2\}$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} 1 & \text{if } x \in \{1,2,3\}, \\ x-1 & \text{otherwise.} \end{cases}$$

It is easy to check that $\alpha \in T_R(X,\sigma)$. Note that $X\alpha = \mathbb{N} \setminus \{2\}$ and $\pi(\alpha) = \{\{1,2,3\}\} \cup \{\{x\} : x \geq 4\}$. Thus $P \cap X\alpha \neq \emptyset$ for all $P \in \pi(\alpha)$. Suppose that $\alpha = \beta\alpha^2$ for some $\beta \in T_R(X,\sigma)$. Since

$$x\alpha^2 = \begin{cases} 1 & \text{if } x \in \{1,2,3,4\}, \\ x-2 & \text{otherwise,} \end{cases}$$

we have that $1\beta, 3\beta \in 1(\alpha^2)^{-1} = \{1,2,3,4\}$ and $x\beta = x+1$ for all $x \geq 4$. Since $1\beta, 3\beta \in R$, there exists $r \in \{1,3,4\}$ such that $r \neq x\beta$ for all $x \in R$. Therefore $r \notin R\beta$. This is a contradiction with $R\beta = R$. Hence α is not left regular.

Next, we give a characterization of right regular elements in $T_R(X,\sigma)$.

Theorem 3.5.5. *Let $\alpha \in T_R(X,\sigma)$. Then α is right regular of $T_R(X,\sigma)$ if and only if $\alpha|_{X\alpha}$ is an injection.*

Proof. The necessity is clear from Theorem 3.4.3. To prove the sufficiency, we suppose that $\alpha|_{X\alpha}$ is an injection. Let $A \in X/\sigma$. Then $A \cap R \cap X\alpha^2 \neq \emptyset$. We choose and fix an element $r_A \in A \cap R \cap X\alpha^2$. For each $x \in A \cap X\alpha^2$, there exists a unique $x' \in X\alpha$ such that $x = x'\alpha$ by $\alpha|_{X\alpha}$ is injective. We observe that $r'_A \in R$ and $(x'\alpha, r'_A\alpha) = (x, r_A) \in \sigma$. If $x' \in s\sigma$ for some $s \in R$, then $s\alpha = r_A = r'_A\alpha$, so that $s = r'_A$ by uniqueness of r'_A. Hence $(x', r'_A) \in \sigma$. Define $\beta_A : A \to r'_A\sigma$ by

$$x\beta_A = \begin{cases} x' & \text{if } x \in X\alpha^2, \\ r'_A & \text{otherwise.} \end{cases}$$

Then we define the map $\beta : X \to X$ by $\beta|_A = \beta_A$ for all $A \in X/\sigma$. Since X/σ is a partition of X, β is well-defined. By Corollary 2.2.6, we have $\beta \in T(X,\sigma)$. We now show that $R\beta = R$. Let $r \in R$. Then $r = r'\alpha$ for a unique $r' \in X\alpha$. Since $R\alpha = R$, $r\beta = r' \in R$. On the other hand, let $r \in R$. Then $r\alpha = s$ for some $s \in R$ and so $s = s'\alpha$ for some $s' \in X\alpha$. By uniqueness of s' and $r \in R = R\alpha$, we have that $r = s'$, so that $s\beta = s' = r$. Hence $R\beta = R$. Consequently, $\beta \in T_R(X,\sigma)$.

Finally, we will show that $\alpha = \alpha^2\beta$, let $x \in X$, so $x\alpha^2 \in X\alpha^2$. Then there exists $A \in X/\sigma$ such that $x\alpha^2 \in A$. By the definition of β_A, $x\alpha^2\beta_A = (x\alpha^2)'$, so that $(x\alpha^2)'\alpha = x\alpha^2 = (x\alpha)\alpha$. Since $(x\alpha^2)'$ is unique, $(x\alpha^2)' = x\alpha$. Thus $x\alpha^2\beta = x\alpha^2\beta_A = x\alpha$. Therefore α is right regular, as asserted. □

Now, we give a characterization of completely regular elements in $T_R(X,\sigma)$.

Corollary 3.5.6. *Let $\alpha \in T_R(X,\sigma)$. Then α is completely regular of $T_R(X,\sigma)$ if and only if for every $P \in \pi(\alpha)$, $|P \cap X\alpha| = 1$.*

Proof. The necessity is clear from Corollary 3.4.4. To prove the sufficiency, we suppose that for every $P \in \pi(\alpha)$, $|P \cap X\alpha| = 1$. Let $x, y \in X\alpha$ be such that $x\alpha = y\alpha$. Then $x, y \in P \cap X\alpha$ for some $P \in \pi(\alpha)$. By assumption, we obtain that $x = y$, so that $\alpha|_{X\alpha}$ is an injection. By Theorem 3.5.5, we have α is right regular. Since $\alpha|_{X\alpha}$ is an injection, also is $\alpha|_R$. It follows from assumption and Theorem 3.5.2 that α is left regular. Hence α is completely regular. □

From Example 3.5.3 and by Theorem 3.5.1, we get that $T_R(X,\sigma)$ is not regular. We also have the following theorem for which characterizes when $T_R(X,\sigma)$ is a regular semigroup.

Theorem 3.5.7. *$T_R(X,\sigma)$ is a regular semigroup if and only if R is finite.*

Proof. Assume that $T_R(X,\sigma)$ is a regular semigroup and R is an infinite set. Let $r \in R$. Then $R\backslash\{r\}$ is also infinite and $|R\backslash\{r\}| = |R|$. Thus there exists a bijection $\varphi : R\backslash\{r\} \to R$. Choose and fix $r' \in R \setminus \{r\}$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} r' & \text{if } x \in r\sigma, \\ s\varphi & \text{if } x \in s\sigma. \end{cases}$$

By Corollary 2.2.6, $\alpha \in T(X,\sigma)$. Since $r\alpha = r'$ and $\varphi : R\backslash\{r\} \to R$, we get that $R\alpha \subseteq R$. Let $s \in R$. Since φ is surjective, $s = t\varphi$ for some $t \in R \setminus \{r\}$. Thus $t \neq r$, it follows that $t\alpha = t\varphi = s$. Therefore $R \subseteq R\alpha$. Hence $\alpha \in T_R(X,\sigma)$. Since $r' \in R$, $r' = r''\varphi$ for some $r'' \in R \setminus \{r\}$. This implies that $r'' \neq r$ and $r''\alpha = r''\varphi = r' = r\alpha$. Consequently, $\alpha|_R$ is not injective. By Theorem 3.5.1, we have α is not regular, which is a contradiction. Hence R is finite.

Conversely, suppose that R is finite. Let $\alpha \in T_R(X,\sigma)$. Then $R\alpha = R$ and so $\alpha|_R : R \to R$ is a surjection. By the finiteness of R and Theorem 2.1.18, $\alpha|_R$ is injective. From Theorem 3.5.1, α is regular. We conclude that $T_R(X,\sigma)$ is a regular semigroup. □

The following theorem characterizes the equivalence relation σ on X for which the semigroup $T_R(X,\sigma)$ is left regular.

Theorem 3.5.8. *Let R be a finite set. Then $T_R(X,\sigma)$ is a left regular semigroup if and only if σ is a T-relation and 2-bounded.*

Proof. Assume that $T_R(X,\sigma)$ is a left regular semigroup and σ is not a T-relation. Then there exist distinct elements $r,s \in R$ such that $|r\sigma|,|s\sigma| \geq 2$. Let $a \in r\sigma \setminus \{r\}$ and $b \in s\sigma \setminus \{s\}$. Define $\beta : X \to X$ by

$$x\beta = \begin{cases} r & \text{if } x \in s\sigma \setminus \{b\}, \\ a & \text{if } x = b, \\ s & \text{if } x \in r\sigma, \\ x & \text{otherwise.} \end{cases}$$

Then β is well-defined and $\beta|_R$ is injective. Since $(r\sigma)\beta \subseteq s\sigma$, $(s\sigma)\beta \subseteq r\sigma$ and $(t\sigma)\beta \subseteq t\sigma$ for all $t \in R \setminus \{r,s\}$, we deduce that $\beta \in T(X,\sigma)$ by Corollary 2.2.6. Since $r\beta = s$ and $s\beta = r$ and $t\beta = t$ for all $t \in R\setminus\{r,s\}$, it follows that $\beta \in T_R(X,\sigma)$. Let $P = a\beta^{-1} \in \pi(\beta)$. Then $P = \{b\}$ and $P \cap X\beta = \emptyset$. By Theorem 3.5.2, we have β is not left regular. This is a contradiction. Hence σ is a T-relation.

Next, we will show that σ is 2-bounded, suppose not. Then there is an element $r \in R$ such that $|r\sigma| \geq 3$. Let a, b be distinct elements in $r\sigma \setminus \{r\}$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} r & \text{if } x \in \{r,a\}, \\ a & \text{if } x = b, \\ x & \text{otherwise.} \end{cases}$$

Since $(r,a) \in \sigma$, $\gamma \in T_R(X,\sigma)$. Clearly, $\gamma|_R$ is injective. Set $P = a\gamma^{-1} \in \pi(\gamma)$, so that $P = \{b\}$ and $P \cap X\gamma = \emptyset$. It follows from Theorem 3.5.2 that γ is not left regular. This is a contradiction. We conclude that σ is 2-bounded.

Conversely, assume that σ is a T-relation and 2-bounded. Let $\alpha \in T_R(X,\sigma)$. Then by Theorems 3.5.7 and 3.5.1, we have that $\alpha|_R$ is an injection. Since $R\alpha = R$ and by assumption, we get that $X\alpha = R$ or $X\alpha = X$. If $X\alpha = X$, then $P \cap X\alpha \neq \emptyset$ for all $P \in \pi(\alpha)$. If $X\alpha = R$, then since $R\alpha = R$, we have $P \cap X\alpha \neq \emptyset$ for all $P \in \pi(\alpha)$. Hence by Theorem 3.5.2 we have that α is left regular. Thus $T_R(X,\sigma)$ is a left regular semigroup. □

From Example 3.5.3 and by Theorem 3.5.5, we get that $T_R(X,\sigma)$ is not regular. We also have the following theorem for which characterizes when $T_R(X,\sigma)$ is a right regular semigroup.

Theorem 3.5.9. *$T_R(X,\sigma)$ is a right regular semigroup if and only if R is finite and σ is a T-relation and 2-bounded.*

Proof. Assume that $T_R(X,\sigma)$ is a right regular semigroup. For each $\alpha \in T_R(X,\sigma)$, by assumption and Theorem 3.5.5 we have $\alpha|_{X\alpha}$ is an injection. Since $R = R\alpha \subseteq X\alpha$, $\alpha|_R$ is also injective. Using Theorems 3.5.1 and 3.5.7, we conclude that R is finite.

Next, to show that σ is a T-relation, suppose not. Then there exist distinct elements $r, s \in R$ such that $|r\sigma|, |s\sigma| \geq 2$. Let $a \in r\sigma \setminus \{r\}$ and $b \in s\sigma \setminus \{s\}$. Define $\beta \in T_R(X,\sigma)$ as in the same proof of Theorem 3.5.8. Since $r, a \in X\beta$ and $r\beta = s = a\beta$, $\beta|_{X\beta}$ is not injective. By Theorem 3.5.5 we have that β is not a right regular element of $T_R(X,\sigma)$. This is a contradiction. Hence σ is a T-relation.

Finally, we will show that σ is 2-bounded. Suppose that σ is not 2-bounded. Then there is an element $r \in R$ such that $|r\sigma| \geq 3$. Let a, b be distinct elements in $r\sigma \setminus \{r\}$. Define $\gamma \in T_R(X,\sigma)$ as in the same proof of Theorem 3.5.8. Since $r, a \in X\gamma$ and $r\gamma = r = a\gamma$, $\gamma|_{X\gamma}$ is not injective. It follows from Theorem 3.5.5 that γ is not a right regular element of $T_R(X,\sigma)$. This is a contradiction. Hence σ is 2-bounded.

Conversely, suppose that R is a finite set and σ is a T-relation and 2-bounded. Let $\alpha \in T_R(X,\sigma)$. Since $R\alpha = R$ and by the definition of σ, it follows that $X\alpha = R$ or $X\alpha = X$. If $X\alpha = R$, then since R is finite by Theorems 3.5.7 and 3.5.1, we deduce that $\alpha|_{X\alpha}$ is an injection. If $X\alpha = X$, then α is surjective. By assumption, we obtain that X is finite. Using Theorem 2.1.18, α is injective. Hence α is right regular element of $T_R(X,\sigma)$ by Theorem 3.5.5 and thus $T_R(X,\sigma)$ is right regular. □

Theorems 2.1.2, 3.5.8 and 3.5.9 can be summarized as follows:

Corollary 3.5.10. *$T_R(X,\sigma)$ is a completely regular semigroup if and only if R is finite and σ is a T-relation and 2-bounded.*

CHAPTER IV

GREEN'S RELATIONS

In this chapter, we present Green's relations on $T(X)$, $T(X,\sigma,\rho)$, $T(X,\sigma^*)$, $T(X,\sigma,R)$ and $T_R(X,\sigma)$. We investigate characterizations of left principal ideal, right principal ideal and principal ideal on $T(X,\sigma,\rho)$ and $T_R(X,\sigma)$. And then we determine when elements of $T(X,\sigma,\rho)$ and $T_R(X,\sigma)$ are equivalence respect to Green's relations.

4.1 Green's relations on some subsemigroups of the full transformation semigroups

In this section, we describe Green's relations on the semigroups $T(X)$, $T(X,\sigma^*)$ and $T(X,\sigma,R)$. Throughout this section, let σ be an equivalence relation on X and R a cross-section of the partition X/σ induced by σ.

Firstly, the following results are quoted from [45] Lemma 2.5, 2.6, 2.8 and Theorem 2.9, respectively.

Theorem 4.1.1. [45] *Let $\alpha,\beta\in T(X)$. Then the following statements hold.*

(1) $(\alpha,\beta)\in\mathcal{L}$ *on* $T(X)$ *if and only if* $X\alpha = X\beta$.

(2) $(\alpha,\beta)\in\mathcal{R}$ *on* $T(X)$ *if and only if* $\ker\alpha=\ker\beta$.

(3) $(\alpha,\beta)\in\mathcal{D}$ *on* $T(X)$ *if and only if* $|X\alpha| = |X\beta|$.

(4) $\mathcal{J}=\mathcal{D}$.

Secondly, we focus our attention on Green's relations for the semigroup $T(X,\sigma^*)$. Before starting the first result, we need the following terminology.

For every $\alpha\in T(X,\sigma^*)$. Denote

$$Z(\alpha)=\{A\in X/\sigma : A\cap X\alpha=\emptyset\}.$$

Theorem 4.1.2. [32] *Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathcal{L}$ *on* $T(X, \sigma^*)$.

(2) $X\alpha = X\beta$.

(3) *There exists a σ^*-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.*

Theorem 4.1.3. [32] *Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathcal{R}$ *on* $T(X, \sigma^*)$.

(2) $\ker\alpha = \ker\beta$ *and* $Z(\alpha) = Z(\beta)$.

(3) *There exists $\lambda \in T(X, \sigma^*)$ such that $\lambda|_{X\alpha} : X\alpha \to X\beta$ is a bijection and $\beta = \alpha\lambda$ and there exists $\mu \in T(X, \sigma^*)$ such that $\mu|_{X\beta} : X\beta \to X\alpha$ is a bijection and $\alpha = \beta\mu$.*

Theorem 4.1.4. [32] *Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathcal{D}$ *on* $T(X, \sigma^*)$.

(2) $|Z(\alpha)| = |Z(\beta)|$ *and there exists $\lambda \in T(X, \sigma^*)$ such that $\lambda|_{X\alpha} : X\alpha \to X\beta$ is a bijection.*

Theorem 4.1.5. [32] *Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements are equivalent.*

(1) $(\alpha, \beta) \in \mathcal{J}$ *on* $T(X, \sigma^*)$.

(2) $|X\alpha| = |X\beta|$ *and there exist $\lambda, \mu \in T(X, \sigma^*)$, for every $A \in X/\sigma$, $A\alpha \subseteq B\beta\lambda$ and $A\beta \subseteq C\alpha\mu$ for some $B, C \in X/\sigma$.*

Finally, we describe Green's relations in the semigroup $T(X, \sigma, R)$.

Theorem 4.1.6. [42] *Let $\alpha, \beta \in T(X, \sigma, R)$. Then $(\alpha, \beta) \in \mathcal{L}$ on $T(X, \sigma, R)$ if and only if for every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.*

Theorem 4.1.7. [42] *Let $\alpha, \beta \in T(X, \sigma, R)$. Then $(\alpha, \beta) \in \mathcal{R}$ on $T(X, \sigma, R)$ if and only if* $\ker\alpha = \ker\beta$.

Theorem 4.1.8. [42] *Let $\alpha, \beta \in T(X, \sigma, R)$. Then $(\alpha, \beta) \in \mathcal{D}$ on $T(X, \sigma, R)$ if and only if there exist a bijection $\varphi : X\alpha \to X\beta$ satisfying*

(1) *$(X\alpha \cap R)\varphi \subseteq R$ and*

(2) *for every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $(A\alpha)\varphi \subseteq B\beta$ and $A\beta \subseteq (C\alpha)\varphi$.*

Theorem 4.1.9. [42] *Let $\alpha, \beta \in T(X, \sigma, R)$. Then $(\alpha, \beta) \in \mathcal{J}$ on $T(X, \sigma, R)$ if and only if there exist mappings $\varphi : X\alpha \to X\beta$ and $\psi : X\beta \to X\alpha$ satisfying*

(1) *$(X\alpha \cap R)\varphi \subseteq R$,*

(2) *for every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $A\beta \subseteq (B\alpha)\varphi \subseteq C$,*

(3) *$(X\alpha \cap R)\psi \subseteq R$ and*

(4) *for every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $A\alpha \subseteq (B\beta)\psi \subseteq C$.*

4.2 Green's relations on the generalization of semigroups of transformations preserving equivalence relations

We let X be a nonempty set and let σ and ρ be equivalence relations on X with $\rho \subseteq \sigma$. Green's relations on $T(X, \sigma, \rho)$ are studied in this section.

The first, to describe $\mathcal{L}$-relation on $T(X, \sigma, \rho)$, the following lemma is needed.

Lemma 4.2.1. *Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then the following statements are equivalent.*

(1) *$\alpha = \lambda\beta$ for some $\lambda \in T(X, \sigma, \rho)$.*

(2) *For every $A \in X/\sigma$, there exists $B \in X/\rho$ such that $A\alpha \subseteq B\beta$.*

(3) *There exists $\sigma\rho$-admissible $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.*

Proof. (1) $\Rightarrow$ (2) Assume that $\alpha = \lambda\beta$ for some $\lambda \in T(X, \sigma, \rho)$. Let $A \in X/\sigma$. Then by Lemma 2.2.5, we have $A\lambda \subseteq B$ for some $B \in X/\rho$. By assumption, we obtain that $A\alpha = A\lambda\beta \subseteq B\beta$.

(2) $\Rightarrow$ (3) Suppose that (2) holds. To show that $X\alpha \subseteq X\beta$, let $y \in X\alpha$. Then $y = x\alpha$ for some $x \in X$. Thus $x \in A$ for some $A \in X/\sigma$. By (2), there exists $B \in X/\rho$ such that

$$y = x\alpha \in A\alpha \subseteq B\beta \subseteq X\beta.$$

Therefore $X\alpha \subseteq X\beta$. For each $P \in \pi(\alpha)$, we have $P\alpha_* = x\alpha \in X\alpha \subseteq X\beta$ for all $x \in P$. Define $\varphi : \pi(\alpha) \to \pi(\beta)$ by

$$P\varphi = (P\alpha_*)\beta^{-1} \text{ for all } P \in \pi(\alpha).$$

Then φ is well-defined. Let $A \in X/\sigma$ and let $I_A = \{i \in X\alpha : i\alpha^{-1} \cap A \neq \emptyset\}$. For each $i \in I_A$, we let $P_i := i\alpha^{-1}$. Then

$$\pi_A(\alpha) = \{P_i : i \in I_A\} \text{ and } i = P_i\alpha_* \text{ for all } i \in I_A.$$

Let $i \in I_A$. By (2), we have $i \in A\alpha \subseteq B\beta$ for some $B \in X/\rho$. Then $B \cap P_i\varphi = B \cap (P_i\alpha_*)\beta^{-1} = B \cap i\beta^{-1} \neq \emptyset$. Hence φ is $\sigma\rho$-admissible by Proposition 2.2.4. Finally, we will show that $\alpha_* = \varphi\beta_*$. Let $P \in \pi(\alpha)$ and $p \in P$. Then $p\alpha \in X\alpha \subseteq X\beta$ and so $p\alpha = x\beta$ for some $x \in X$. Thus $x \in (p\alpha)\beta^{-1} = (P\alpha_*)\beta^{-1} = P\varphi$. Therefore

$$P\alpha_* = p\alpha = x\beta = P\varphi\beta_*,$$

as required.

(3) $\Rightarrow$ (1) Suppose that $\varphi : \pi(\alpha) \to \pi(\beta)$ is $\sigma\rho$-admissible such that $\alpha_* = \varphi\beta_*$. Let $A \in X/\sigma$. Then $(\pi_A(\alpha))\varphi \subseteq \pi_{A'}(\beta)$ for some $A' \in X/\rho$. For each $x \in A$, we let $P_x = (x\alpha)\alpha^{-1} \in \pi_A(\alpha)$. By assumption and Proposition 2.2.4, we have $P_x\varphi \cap A' \neq \emptyset$. We choose and fix an element $x' \in P_x\varphi \cap A'$. Define $\lambda_A : A \to A'$ by

$$x\lambda_A = x' \text{ for all } x \in A.$$

Let $\lambda \in T(X)$ be such that $\lambda|_A = \lambda_A$ for all $A \in X/\sigma$. Since X/σ is a partition of X, λ is well-defined. Since $A\lambda = A\lambda_A \subseteq A'$ for some $A' \in X/\rho$ and by Lemma 2.2.5, we then have $\lambda \in T(X, \sigma, \rho)$. Let $x \in X$. Then $x \in A$ for some $A \in X/\sigma$. By Proposition 2.2.4, there is $A' \in X/\rho$ such that $x\lambda = x\lambda|_A = x' \in P_x\varphi \cap A'$ where $P_x \in \pi_A(\alpha)$. Since $\alpha_* = \varphi\beta_*$, we obtain that $x\alpha = P_x\alpha_* = P_x\varphi\beta_* = x'\beta = x\lambda\beta$. Hence $\alpha = \lambda\beta$. □

Using Theorem 2.1.11 and Lemma 4.2.1, we can establish the next result.

Theorem 4.2.2. *Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then the following statements are equivalent.*

(1) *$(\alpha, \beta) \in \mathcal{L}$ on $T(X, \sigma, \rho)$.*

(2) *Either $\alpha = \beta$ or for every $A \in X/\sigma$, there exist $B, C \in X/\rho$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.*

(3) *Either $\alpha = \beta$ or there exists a $(\sigma\rho)^*$-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.*

As an immediate consequence of Theorem 4.2.2, we have the following.

Corollary 4.2.3. *Let $\alpha, \beta \in T(X, \sigma, \rho)$ be such that $(\alpha, \beta) \in \mathcal{L}$. Then $X\alpha = X\beta$.*

The following lemma is used for characterizing the $\mathcal{R}$-relation on $T(X, \sigma, \rho)$.

Lemma 4.2.4. *Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $\alpha = \beta\mu$ for some $\mu \in T(X, \sigma, \rho)$ if and only if the following statements hold.*

(1) $\ker \beta \subseteq \ker \alpha$.

(2) *For every $x, y \in X$, $(x\beta, y\beta) \in \sigma$ implies $(x\alpha, y\alpha) \in \rho$.*

Proof. Suppose that $\alpha = \beta\mu$ for some $\mu \in T(X, \sigma, \rho)$. Let $(x, y) \in \ker \beta$. Then $x\beta = y\beta$ and so $x\alpha = x\beta\mu = y\beta\mu = y\alpha$. Thus $(x, y) \in \ker \alpha$ and hence $\ker \beta \subseteq \ker \alpha$. Let $x, y \in X$ be such that $(x\beta, y\beta) \in \sigma$. Since $\alpha = \beta\mu$ and $\mu \in T(X, \sigma, \rho)$, it follows that $(x\alpha, y\alpha) = (x\beta\mu, y\beta\mu) \in \rho$, as required.

Conversely, assume that the conditions (1) and (2) hold. For each $y \in X\beta$, there exists $y' \in X$ such that $y'\beta = y$. Let $A \in X/\sigma$ be such that $A \cap X\beta \neq \emptyset$. We choose and fix an element $x_A \in A \cap X\alpha$. Define $\mu_A : A \to X$ by

$$x\mu_A = \begin{cases} x'\alpha & \text{if } x \in X\beta, \\ x'_A\alpha & \text{otherwise.} \end{cases}$$

Let $x, y \in A$ be such that $x = y$. If $x, y \in X\beta$, then there are $x', y' \in X$ such that $x = x'\beta$ and $y = y'\beta$. Thus $(x', y') \in \ker \beta$ and so $x'\alpha = y'\alpha$ by (1), which implies that

$x\mu_A = y\mu_A$. If $x, y \notin X\beta$, then $x\mu_A = x'_A\alpha = y\mu_A$. Hence μ_A is well-defined. Define $\mu : X \to X$ by

$$\mu|_A = \begin{cases} \mu_A & \text{if } A \cap X\beta \neq \emptyset, \\ C_A & \text{otherwise} \end{cases}$$

for all $A \in X/\sigma$ where C_A is a constant map from A into X. Since X/σ is a partition of X, we have that μ is well-defined. To show that $\mu \in T(X, \sigma, \rho)$, let $A \in X/\sigma$.

Case 1. $A \cap X\beta \neq \emptyset$. Then $x_A \in A \cap X\beta$ with $x_A = x'_A\beta$ for some $x'_A \in X$. For each $x \in A \cap X\alpha$, we have $x'\beta = x$ and $x'_A\beta = x_A$ are elements in A where $x' \in X$. Since $A \in X/\sigma$ and by (2), it follows that $x'\alpha$ and $x'_A\alpha$ are elements in the same ρ-class. We conclude that $A\mu \subseteq B$ for some $B \in X/\rho$.

Case 2. $A \cap X\beta = \emptyset$. Then by reflexivity of the relation ρ, we have $(x\mu, y\mu) = (xC_A, yC_A) \in \rho$. Hence $A\mu \subseteq B$ for some $B \in X/\rho$.

From the two cases and by Lemma 2.2.5, we deduce that $\mu \in T(X, \sigma, \rho)$. Let $x \in X$. Then $x\beta \in X\beta$ and $x\beta \in A$ for some $A \in X/\sigma$ and so $(x\beta)'\beta = x\beta$ for some $(x\beta)' \in X$. Thus $((x\beta)', x) \in \ker\beta$ so that $x\alpha = (x\beta)'\alpha = (x\beta)\mu_A = x\beta\mu$ by (1). This shows that $\alpha = \beta\mu$, as required. □

As an immediate consequence of Theorem 2.1.11 and Lemma 4.2.4, we have the following.

Theorem 4.2.5. *Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $(\alpha, \beta) \in \mathcal{R}$ on $T(X, \sigma, \rho)$ if and only if either $\alpha = \beta$ or the following statements hold.*

(1) $\ker\beta = \ker\alpha$.

(2) *For every $x, y \in X$, $(x\beta, y\beta) \in \sigma$ implies $(x\alpha, y\alpha) \in \rho$.*

(3) *For every $x, y \in X$, $(x\alpha, y\alpha) \in \sigma$ implies $(x\beta, y\beta) \in \rho$.*

However, to describe $\mathcal{R}$-relation again, the following lemma is required.

Lemma 4.2.6. *Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $\alpha = \beta\mu$ for some $\mu \in T(X, \sigma, \rho)$ if and only if there exists a mapping $\varphi : X\beta \to X\alpha$ satisfying*

(1) *$\alpha = \beta\varphi$ and*

(2) *for every* $x, y \in X\beta$, $(x, y) \in \sigma$ *implies* $(x\varphi, y\varphi) \in \rho$.

Proof. Suppose that $\alpha = \beta\mu$ for some $\mu \in T(X, \sigma, \rho)$. For each $y \in X\beta$, there exists $y' \in X$ such that $y = y'\beta$. Define $\varphi : X\beta \to X\alpha$ by

$$x\varphi = x'\alpha \text{ for all } x \in X\beta.$$

Let $x, y \in X\beta$ be such that $x = y$. Then $x'\beta = y'\beta$ for some $x', y' \in X$. Since $\alpha = \beta\mu$ and by Lemma 4.2.4(1), which implies that $(x', y') \in \ker\beta \subseteq \ker\alpha$. Thus $x\varphi = x'\alpha = y'\alpha = y\varphi$. Hence φ is well-defined. Let $x \in X$. Then $x\beta = (x\beta)'\beta$ for some $(x\beta)' \in X$ and so $(x, (x\beta)') \in \ker\beta \subseteq \ker\alpha$. Therefore

$$x\alpha = (x\beta)'\alpha = x\beta\varphi.$$

Hence $\alpha = \beta\varphi$. Let $x, y \in X\beta$ be such that $(x, y) \in \sigma$. Then $x = x'\beta$ and $y = y'\beta$ for some $x', y' \in X$. Thus $(x'\beta, y'\beta) = (x, y) \in \sigma$. By Lemma 4.2.4(2), we deduce that $(x\varphi, y\varphi) = (x'\alpha, y'\alpha) \in \rho$. Hence the necessity follows.

Conversely, suppose that $\varphi : X\beta \to X\alpha$ is a mapping satisfying the conditions (1) and (2). Let $A \in X/\sigma$ be such that $A \cap X\beta \neq \emptyset$. We choose and fix an element $x_A \in A \cap X\beta$. Define $\mu_A : A \to X$ by

$$x\mu_A = \begin{cases} x\varphi & \text{if } x \in X\beta, \\ x_A\varphi & \text{otherwise.} \end{cases}$$

Let $\mu : X \to X$ be defined by

$$\mu|_A = \begin{cases} \mu_A & \text{if } A \cap X\beta \neq \emptyset, \\ C_A & \text{otherwise} \end{cases}$$

for all $A \in X/\sigma$ and C_A is a constant map from A into X. Since X/σ is a partition of X, we have that μ is well-defined. Let $A \in X/\sigma$.

Case 1. $A \cap X\beta \neq \emptyset$. We note that $x_A \in A$. For each $x \in A \cap X\beta$ by (2), it follows that $x\varphi$ and $x_A\varphi$ are elements in the same ρ-class. We conclude that $A\mu \subseteq B$ for some $B \in X/\rho$.

Case 2. $A \cap X\beta = \emptyset$. Then by reflexivity of the relation ρ, we obtain that $(x\mu, y\mu) = (xC_A, yC_A) \in \rho$. Hence $A\mu \subseteq B$ for some $B \in X/\rho$.

From the two cases and by Lemma 2.2.5, we deduce that $\mu \in T(X,\sigma,\rho)$. Finally, we will show that $\alpha = \beta\mu$. Let $x \in X$. Then $x\beta \in A$ for some $A \in X/\rho$. Thus by (1), we obtain

$$x\beta\mu = (x\beta)\mu_A = x\beta\varphi = x\alpha,$$

as required. □

The following theorem is a direct consequence of Theorem 2.1.11 and Lemma 4.2.6.

Theorem 4.2.7. *Let $\alpha, \beta \in T(X,\sigma,\rho)$. Then $(\alpha,\beta) \in \mathcal{R}$ on $T(X,\sigma,\rho)$ if and only if either $\alpha = \beta$ or there exists a bijection $\varphi : X\beta \to X\alpha$ satisfying*

(1) *$\alpha = \beta\varphi$,*

(2) *for every $x, y \in X\beta$, $(x,y) \in \sigma$ implies $(x\varphi, y\varphi) \in \rho$ and*

(3) *for every $x, y \in X\alpha$, $(x,y) \in \sigma$ implies $(x\varphi^{-1}, y\varphi^{-1}) \in \rho$.*

To describe the $\mathcal{J}$-relation on $T(X,\sigma,\rho)$, we first give the following lemma.

Lemma 4.2.8. *Let $\alpha, \beta \in T(X,\sigma,\rho)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in T(X,\sigma,\rho)$ if and only if there exists $\varphi : X\beta \to X$ satisfying*

(1) *for every $x, y \in X\beta$, $(x,y) \in \sigma$ implies $(x\varphi, y\varphi) \in \rho$ and*

(2) *for every $A \in X/\sigma$, there exists $B \in X/\rho$ such that $A\alpha \subseteq (B\beta)\varphi$.*

Proof. Suppose that $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in T(X,\sigma,\rho)$. Let $\varphi = \mu|_{X\beta}$ and let $x, y \in X\beta$ be such that $(x,y) \in \sigma$. Then since $\mu \in T(X,\sigma,\rho)$, we have

$$(x\varphi, y\varphi) = (x\mu|_{X\beta}, y\mu|_{X\beta}) = (x\mu, y\mu) \in \rho.$$

Let $A \in X/\sigma$. By Lemma 2.2.5, there exists $B \in X/\rho$ such that $A\lambda \subseteq B$. Thus $A\alpha = A\lambda\beta\mu \subseteq B\beta\mu = B\beta\mu|_{X\beta} = (B\beta)\varphi$.

Conversely, assume that there exists $\varphi : X\beta \to X$ satisfying the conditions (1) and (2). Let $A \in X/\sigma$ be such that $A \cap X\beta \neq \emptyset$. By (1), $(A \cap X\beta)\varphi \subseteq B$ for some $B \in X/\rho$. Fix some $b_A \in B$ and define $\mu_A : A \to B$ by

$$x\mu_A = \begin{cases} x\varphi & \text{if } x \in X\beta, \\ b_A & \text{otherwise.} \end{cases}$$

Let $\mu : X \to X$ be defined by

$$\mu|_A = \begin{cases} \mu_A & \text{if } A \cap X\beta \neq \emptyset, \\ C_A & \text{otherwise} \end{cases}$$

for all $A \in X/\sigma$ and C_A is a constant map from A into X. Since X/σ is a partition of X, it follows that μ is well-defined. From (1) and Lemma 2.2.5, we have $\mu \in T(X, \sigma, \rho)$.

For each $A \in X/\sigma$, by (2) we choose and fix $B_A \in X/\rho$ such that $A\alpha \subseteq (B_A\beta)\varphi$. Let $x \in A$. Then we choose and fix $b_x \in B_A$ such that $x\alpha = (b_x\beta)\varphi$. Define $\lambda : X \to X$ by $x\lambda = b_x$ for all $x \in X$. Then $\lambda \in T(X, \sigma, \rho)$. Furthermore, for $x \in X$,

$$x\lambda\beta\mu = b_x\beta\mu = (b_x\beta)\varphi = x\alpha,$$

which implies that $\alpha = \lambda\beta\mu$, as desired. □

By Theorem 2.1.11 and Lemma 4.2.8 are useful to obtain this result.

Theorem 4.2.9. *Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $(\alpha, \beta) \in \mathcal{J}$ on $T(X, \sigma, \rho)$ if and only if either $\alpha = \beta$ or there exist mappings $\varphi : X\beta \to X$ and $\psi : X\alpha \to X$ satisfying*

(1) *for every $x, y \in X\beta$, $(x, y) \in \sigma$ implies $(x\varphi, y\varphi) \in \rho$,*

(2) *for every $x, y \in X\alpha$, $(x, y) \in \sigma$ implies $(x\psi, y\psi) \in \rho$ and*

(3) *for every $A \in X/\sigma$, there exist $B, C \in X/\rho$ such that $A\alpha \subseteq (B\beta)\varphi$ and $A\beta \subseteq (C\alpha)\psi$.*

Theorem 4.2.10. *Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $(\alpha, \beta) \in \mathcal{D}$ on $T(X, \sigma, \rho)$ if and only if either $\alpha = \beta$ or there exist a $(\sigma\rho)^*$-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ and a bijection $\psi : X\alpha \to X\beta$ satisfying*

(1) *for every $x, y \in X\alpha$, $(x, y) \in \sigma$ implies $(x\psi, y\psi) \in \rho$,*

(2) *for every $x, y \in X\beta$, $(x, y) \in \sigma$ implies $(x\psi^{-1}, y\psi^{-1}) \in \rho$ and*

(3) *$\alpha_*\psi = \varphi\beta_*$.*

Proof. Suppose that $(\alpha, \beta) \in \mathcal{D}$. Then $(\alpha, \gamma) \in \mathcal{R}$ and $(\gamma, \beta) \in \mathcal{L}$ for some $\gamma \in T(X, \sigma, \rho)$. By Theorem 4.2.5 and Corollary 4.2.3, we have $\pi(\alpha) = \pi(\gamma)$ and $X\beta = X\gamma$, respectively.

Since $(\alpha, \gamma) \in \mathcal{R}$ and by Theorem 4.2.7, there exists a bijection $\psi : X\alpha \to X\beta$ satisfying (1), (2) and

$$\gamma = \alpha\psi.$$

Let $P \in \pi(\gamma) = \pi(\alpha)$ and $x \in P$. Then $P\gamma_* = x\gamma = x\alpha\psi = P\alpha_*\psi$ and hence $\gamma_* = \alpha_*\psi$. Since $(\gamma, \beta) \in \mathcal{L}$ and by Theorem 4.2.2, there exists a $(\sigma\rho)^*$-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that

$$\gamma_* = \varphi\beta_*.$$

Hence $\alpha_*\psi = \varphi\beta_*$ and the assertion follows.

Conversely, assume that $\varphi : \pi(\alpha) \to \pi(\beta)$ is a $(\sigma\rho)^*$-admissible bijection and $\psi : X\alpha \to X\beta$ is a bijection satisfying the conditions (1), (2) and (3). Define $\gamma \in T(X)$ by $x\gamma = (x\alpha)\psi$ for all $x \in X$. Then $\gamma \in T(X, \sigma, \rho)$ by (1) and

$$\gamma = \alpha\psi.$$

Next, we will show that $\pi(\alpha) = \pi(\gamma)$. Let $y \in X\alpha$. Then $\{y\psi\} = (y\alpha^{-1})\alpha\psi = (y\alpha^{-1})\gamma$. Thus $y\alpha^{-1} \subseteq (y\alpha^{-1})\gamma\gamma^{-1} \subseteq (y\psi)\gamma^{-1} \in \pi(\gamma)$. Hence $\pi(\alpha) \preceq \pi(\gamma)$. On the other hand, let $z \in X\gamma$. Then $\{z\psi^{-1}\} = (z\gamma^{-1})\gamma\psi^{-1} = (z\gamma^{-1})\alpha\psi\psi^{-1} = (z\gamma^{-1})\alpha i_{X\alpha} = (z\gamma^{-1})\alpha$. Thus $z\gamma^{-1} \subseteq (z\psi^{-1})\alpha^{-1} \in \pi(\alpha)$ and hence $\pi(\gamma) \preceq \pi(\alpha)$. Consequently, $\pi(\alpha) = \pi(\gamma)$. Let $P \in \pi(\gamma)$ and $x \in P$. Then

$$P\gamma_* = x\gamma = x\alpha\psi = P\alpha_*\psi,$$

it implies that $\gamma_* = \alpha_*\psi$. By (3), we obtain that $\gamma_* = \alpha_*\psi = \varphi\beta_*$. By Theorem 4.2.2, we have that $(\gamma, \beta) \in \mathcal{L}$. It follows from Corollary 4.2.3 that $X\gamma = X\beta$. This implies that $\psi : X\alpha \to X\gamma$ such that $\gamma = \alpha\psi$. From (1) and (2), it follows from Theorem 4.2.7 that $(\alpha, \gamma) \in \mathcal{R}$. Hence $(\alpha, \beta) \in \mathcal{D}$, as required. □

4.3 Green's relations on the semigroups of transformations preserving an equivalence relation and fix a cross-section

In this section, let σ be an equivalence relation on X and R a cross-section of the partition X/σ induced by σ. We focus on Green's relations for regular elements of the semigroup $T_R(X, \sigma)$.

In the case of $\mathcal{R}$-relation, we study for arbitrary elements. First, we need the following lemma.

Lemma 4.3.1. *Let $\alpha, \beta \in T_R(X,\sigma)$. Then $\alpha = \beta\mu$ for some $\mu \in T_R(X,\sigma)$ if and only if* $\ker\beta \subseteq \ker\alpha$.

Proof. Suppose that $\alpha = \beta\mu$ for some $\mu \in T_R(X,\sigma)$. Let $(x,y) \in \ker\beta$. Then $x\beta = y\beta$ and so $x\alpha = x\beta\mu = y\beta\mu = y\alpha$. Thus $(x,y) \in \ker\alpha$ and hence $\ker\beta \subseteq \ker\alpha$.

Conversely, assume that $\ker\beta \subseteq \ker\alpha$. For each $y \in X\beta \setminus R$, there exists $y' \in X$ such that $y = y'\beta$. For each $r \in R$, we choose $r' \in R$ such that $r = r'\beta$ from $R = R\beta$. Let $r \in R$. Define $\mu_r : r\sigma \to X$ by

$$x\mu_r = \begin{cases} x'\alpha & \text{if } x \in X\beta, \\ r'\alpha & \text{otherwise.} \end{cases}$$

Define $\mu : X \to X$ by $\mu|_{r\sigma} = \mu_r$ for all $r \in R$. Since R is a cross-section of the partition X/σ induced by σ, we have that μ is well-defined. To show that $\mu \in T(X,\sigma)$, let $r \in R$. Then $r = r'\beta$ for some $r' \in R$. We will show that $(r\sigma)\mu \subseteq (r'\alpha)\sigma$. Let $y \in r\sigma$. If $y \notin X\beta$, then $y\mu = y\mu_r = r'\alpha \in (r'\alpha)\sigma$. If $y \in X\beta$, then $y = y'\beta$ for some $y' \in X$. Thus $y' \in s\sigma$ for some $s \in R$. Since $y' \in s\sigma$ and $y'\beta = y \in r\sigma$ by Lemma 2.2.13, we get that $s\beta = r = r'\beta$. By assumption, we have $s\alpha = r'\alpha$. This implies that

$$y\mu = y\mu_r = y'\alpha \in (s\sigma)\alpha \subseteq (s\alpha)\sigma = (r'\alpha)\sigma.$$

Hence $(r\sigma)\mu \subseteq (r'\alpha)\sigma$. It follows from Corollary 2.2.6 that $\mu \in T(X,\sigma)$. Since $R\beta = R$, $r\mu = r'\alpha \in R$ for all $r \in R$. Hence $R\mu \subseteq R$. For the reverse inclusion, let $r \in R$. Then $s\alpha = r$ for some $s \in R$. Thus $s\beta = t$ for some $t \in R$ and so there exists $t' \in R$ such that $s\beta = t = t'\beta$. By assumption, we deduce that

$$r = s\alpha = t'\alpha = t\mu_t = t\mu.$$

It implies that $R \subseteq R\mu$ and hence $\mu \in T_R(X,\sigma)$. Finally, we will show that $\alpha = \beta\mu$. Let $x \in X$. Then $x\beta \in X\beta$ and $x\beta \in r\sigma$ for some $r \in R$ and so $(x\beta)'\beta = x\beta$ for some $(x\beta)' \in X$. Thus $((x\beta)', x) \in \ker\beta$ so that $x\alpha = (x\beta)'\alpha = (x\beta)\mu_r = x\beta\mu$ by assumption. This shows that $\alpha = \beta\mu$. $\square$

Using Theorem 2.1.11 and Lemma 4.3.1, we can establish the next result.

Theorem 4.3.2. *Let $\alpha, \beta \in T_R(X, \sigma)$. Then $(\alpha, \beta) \in \mathcal{R}$ on $T_R(X, \sigma)$ if and only if* $\ker \alpha = \ker \beta$.

Next, we consider the relation $\mathcal{L}$, the following lemma is needed.

Lemma 4.3.3. *Let α and β be regular elements of $T_R(X, \sigma)$. Then the following statements are equivalent.*

(1) *$\alpha = \lambda\beta$ for some $\lambda \in T_R(X, \sigma)$.*

(2) *For every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $A\alpha \subseteq B\beta$.*

(3) *$X\alpha \subseteq X\beta$.*

Proof. (1) $\Rightarrow$ (2) Assume that $\alpha = \lambda\beta$ for some $\lambda \in T_R(X, \sigma)$. Let $r \in R$. By Corollary 2.2.6, there exists $s \in R$ such that $(r\sigma)\lambda \subseteq s\sigma$. By assumption, we have $(r\sigma)\alpha = (r\sigma)\lambda\beta \subseteq (s\sigma)\beta$.

(2) $\Rightarrow$ (3) For each $y \in X\alpha$, there is $x \in X$ such that $y = x\alpha$. If $x \in A$ for some $A \in X/\sigma$, then by (2) we have $y = x\alpha \in A\alpha \subseteq B\beta \subseteq X\beta$ where $B \in X/\sigma$. Hence $X\alpha \subseteq X\beta$.

(3) $\Rightarrow$ (1) Suppose that $X\alpha \subseteq X\beta$. For each $x \in X \setminus R$, we choose and fix an element $x' \in X$ such that $x\alpha = x'\beta$. If $x \in R$, then $x\alpha \in R\alpha = R = R\beta$. Thus we choose and fix an element $x' \in R$ such that $x\alpha = x'\beta$. Define $\lambda : X \to X$ by

$$x\lambda = x' \text{ for all } x \in X.$$

Let $(x, y) \in \sigma$. Then $(x'\beta, y'\beta) = (x\alpha, y\alpha) \in \sigma$ where $x', y' \in X$. If $x' \in r\sigma$ and $y' \in s\sigma$ for some $r, s \in R$, then $r\beta = s\beta$ by Lemma 2.2.13. It follows from Theorem 3.5.1 that $r = s$. Hence $(x\lambda, y\lambda) = (x', y') \in \sigma$. Consequently, $\lambda \in T(X, \sigma)$. Clearly, $R\lambda \subseteq R$. On the other hand, let $r \in R$. Then $r\beta \in R$ and $r\beta = s\alpha = s'\beta$ for some $s, s' \in R$. By Theorem 3.5.1, $r = s'$, hence $r = s' = s\lambda$. Thus $\lambda \in T_R(X, \sigma)$. Finally, if $x \in X$, then $x\lambda\beta = x'\beta = x\alpha$. Hence $\alpha = \lambda\beta$. □

The following theorem is a direct consequence of Theorem 2.1.11 and Lemma 4.3.3.

Theorem 4.3.4. *Let α and β be regular elements of $T_R(X,\sigma)$. Then the following statements are equivalent.*

(1) $(\alpha,\beta)\in\mathcal{L}$.

(2) *For every $A\in X/\sigma$, there exist $B,C\in X/\sigma$ such that $A\alpha\subseteq B\beta$ and $A\beta\subseteq C\alpha$.*

(3) $X\alpha = X\beta$.

We now observe a sufficient condition for two arbitrary regular elements of $T_R(X,\sigma)$ to be $\mathcal{D}$-related.

Theorem 4.3.5. *Let α and β be regular elements of $T_R(X,\sigma)$. Then $(\alpha,\beta)\in\mathcal{D}$ on $T_R(X,\sigma)$ if and only if there is a bijection $\varphi : X\alpha\to X\beta$ satisfying*

(1) *$R\varphi = R$ and*

(2) *for every $A\in X/\sigma$, there exists $B\in X/\sigma$ such that $(A\alpha)\varphi\subseteq B\beta$.*

Proof. Suppose that $(\alpha,\beta)\in\mathcal{D}$ on $T_R(X,\sigma)$. Then there exists $\gamma\in T_R(X,\sigma)$ such that $(\alpha,\gamma)\in\mathcal{R}$ and $(\gamma,\beta)\in\mathcal{L}$ on $T_R(X,\sigma)$.

Next, we shall construct a bijection $\varphi : X\alpha\to X\beta$ such that $R\varphi = R$ and for every $A\in X/\sigma$, there exists $B\in X/\sigma$ such that $(A\alpha)\varphi\subseteq B\beta$. By Theorem 4.3.4, we observe that $X\beta = X\gamma$. For each $x\alpha\in X\alpha$, define $\varphi : X\alpha\to X\gamma$ by $(x\alpha)\varphi = x\gamma$. If $x\alpha = y\alpha$, then $(x,y)\in\ker\alpha$ and so $(x\alpha)\varphi = x\gamma = y\gamma = (y\alpha)\varphi$ from $\ker\alpha\subseteq\ker\gamma$. Hence φ is well-defined. Similarly, since $\ker\gamma\subseteq\ker\alpha$, we can show that φ is an injection. Since $x\gamma = (x\alpha)\varphi$ for all $x\in X$, φ is a surjection. Next, we will show that $R\varphi = R$. Let $r\in R$. Since $R\alpha = R$, $r = s\alpha$ for some $s\in R$. Therefore $r\varphi = (s\alpha)\varphi = s\gamma\in R$. On the other hand, if $r\in R$, then there is $s\in R$ such that $r = s\gamma$. This implies that $s\alpha\in R$ and $(s\alpha)\varphi = s\gamma = r$. Hence $R\varphi = R$. For each $A\in X/\sigma$ and by Theorem 4.3.4, there exists $B\in X/\sigma$ such that $(A\alpha)\varphi = A\gamma\subseteq B\beta$.

Conversely, assume that $\varphi : X\alpha\to X\beta$ is a bijection satisfying (1) and (2). Define $\gamma : X\to X$ by $x\gamma = (x\alpha)\varphi$ for all $x\in X$. Let $r\in R$. Then $r\alpha\in R$ and so $r\gamma = (r\alpha)\varphi\in R\varphi = R$ whence $R\gamma\subseteq R$. For the reverse inclusion, let $r\in R$. Then there exists $s\in R$ such that $r = s\varphi$ and so $s = t\alpha$ for some $t\in R$. Therefore

$t\gamma = (t\alpha)\varphi = s\varphi = r$. This means that $R\gamma = R$. Moreover, if $A \in X/\sigma$, then by (2), there exists $B \in X/\sigma$ such that $(A\alpha)\varphi \subseteq B\beta$. By Corollary 2.2.6, $B\beta \subseteq C$ for some $C \in X/\sigma$. It follows from Corollary 2.2.6 again that $\gamma \in T(X,\sigma)$. Consequently, $\gamma \in T_R(X,\sigma)$. Since φ is injective, for each $x, y \in X$ we have

$$x\gamma = y\gamma \Leftrightarrow (x\alpha)\varphi = (y\alpha)\varphi \Leftrightarrow x\alpha = y\alpha.$$

This shows that $\ker\alpha = \ker\gamma$. Since φ is surjective, $X\gamma = (X\alpha)\varphi = X\beta$. Hence by Theorems 4.3.2 and 4.3.4, $(\alpha,\gamma) \in \mathcal{R}$ and $(\gamma,\beta) \in \mathcal{L}$, which give $(\alpha,\beta) \in \mathcal{D}$. □

In any semigroup S, if D is an arbitrary $\mathcal{D}$-class in S containing a regular element, then every element of D is regular. In the relation $\mathcal{J}$, this result is not true, but in the semigroup $T_R(X,\sigma)$, this result holds for the relation $\mathcal{J}$.

Lemma 4.3.6. *Let $\alpha, \beta, \lambda, \mu \in T_R(X,\sigma)$. If $\alpha = \lambda\beta\mu$ and α is regular, then β is regular.*

Proof. Suppose that $\alpha = \lambda\beta\mu$ and α is regular. Let $r, s \in R$ be such that $r\beta = s\beta$. Then $r\beta\mu = s\beta\mu$. Since $R\lambda = R$, there are $r', s' \in R$ such that $r = r'\lambda$ and $s = s'\lambda$. By assumption, we get that $r'\alpha = r'\lambda\beta\mu = s'\lambda\beta\mu = s'\alpha$. By Theorem 3.5.1, $r' = s'$ and thus $r = s$. Therefore, β is regular by Theorem 3.5.1. □

This translates immediately into the following theorem.

Theorem 4.3.7. *If J is an arbitrary $\mathcal{J}$-class in a semigroup $T_R(X,\sigma)$ containing a regular element, then every element of J is regular.*

Finally, we characterize Green's relation $\mathcal{J}$ for regular elements of $T_R(X,\sigma)$.

Lemma 4.3.8. *Let α and β be regular elements of $T_R(X,\sigma)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in T_R(X,\sigma)$ if and only if there is a mapping $\varphi : X\beta \to X\alpha$ satisfying*

(1) *$\varphi|_R : R \to R$ is a bijection,*

(2) *for every $x, y \in X\beta$, $(x,y) \in \sigma$ implies $(x\varphi, y\varphi) \in \sigma$ and*

(3) *for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $A\alpha \subseteq (B\beta)\varphi$.*

Proof. Assume that $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in T_R(X, \sigma)$. For each $x \in X$, we let $r_x \in R$ with $(x, r_x) \in \sigma$. Define $\varphi : X\beta \to X\alpha$ by

$$x\varphi = \begin{cases} x\mu & \text{if } x \in X\lambda\beta, \\ r_x\mu & \text{otherwise.} \end{cases}$$

Let $x \in X\beta$. If $x \in X\lambda\beta$, then $x = x'\lambda\beta$ for some $x' \in X$. By assumption, we have $x\varphi = x\mu = x'\lambda\beta\mu = x'\alpha \in X\alpha$. If $x \notin X\lambda\beta$, then $r_x \in X\lambda\beta$ and so $r_x = s\lambda\beta$ for some $s \in R$. By assumption, we obtain that $x\varphi = r_x\mu = s\lambda\beta\mu = s\alpha \in X\alpha$. This shows that φ indeed maps $X\beta$ to $X\alpha$. If $r, s \in R$ such that $r\varphi = s\varphi$, then $r\mu = s\mu$ by $R = R\lambda\beta$. Thus $r'\alpha = s'\alpha$ such that $r = r'\lambda\beta$ and $s = s'\lambda\beta$ where $r', s' \in R$. By the regularity of α, $r' = s'$, so that $r = s$. Since $R\mu = R$, we have that $R\varphi = R\mu = R$. Therefore $\varphi|_R : R \to R$ is a bijection. Let $x, y \in X\beta$ be such that $(x, y) \in \sigma$. Then $r_x = r_y$ and $x, y \in r_x\sigma$. By Corollary 2.2.6, there is $A \in X/\sigma$ such that $x\mu, y\mu, r_x\mu \in (r_x\sigma)\mu \subseteq A$. This implies that $x\varphi, y\varphi \in A$, that is, $(x\varphi, y\varphi) \in \sigma$. Finally, let $A \in X/\sigma$. By Corollary 2.2.6, there exists $B \in X/\sigma$ such that $A\lambda \subseteq B$. By assumption and the definition of φ, we then have $A\alpha = A\lambda\beta\mu \subseteq (B\beta \cap X\lambda\beta)\mu = (B\beta \cap X\lambda\beta)\varphi \subseteq (B\beta)\varphi$.

Conversely, assume that $\varphi : X\beta \to X\alpha$ is a mapping satisfying the conditions (1), (2) and (3). Let $r \in R$. By (2), $(r\sigma \cap X\beta)\varphi \subseteq B_r$ for some fix $B_r \in X/\sigma$. Define $\mu_r : r\sigma \to B_r$ by

$$x\mu_r = \begin{cases} x\varphi & \text{if } x \in X\beta, \\ r\varphi & \text{otherwise.} \end{cases}$$

Let $\mu : X \to X$ be defined by $\mu|_{r\sigma} = \mu_r$ for all $r \in R$. Since R is a cross-section of the partition X/σ induced by σ, it follows that μ is well-defined. For each $r \in R$, $(r\sigma)\mu_r \subseteq B_r$ for some $B_r \in X/\sigma$ and by Corollary 2.2.6, we have $\mu \in T(X, \sigma)$. It follows from (1) that $R\mu = R\varphi = R$. Hence $\mu \in T_R(X, \sigma)$.

For each $r \in R$, by (3) we choose and fix $r' \in R$ such that $(r\sigma)\alpha \subseteq ((r'\sigma)\beta)\varphi$. If $(r'\beta)\varphi = a\alpha$ for some $a \in X$, then since $r'\beta \in R$ and $R\varphi = R$, $a\alpha \in R$ and so $(r\sigma)\alpha \subseteq ((r'\sigma)\beta)\varphi \subseteq (a\alpha)\sigma$. Thus $r\alpha = a\alpha = (r'\beta)\varphi$ by Lemma 2.2.13. Let $x \in r\sigma$. Then we choose and fix an element $b_x \in r'\sigma$ (if $x = r$, we choose $b_x = r'$) such that $x\alpha = (b_x\beta)\varphi$. Define $\lambda : X \to X$ by $x\lambda = b_x$ for all $x \in X$. For each $r \in R$, we get that $(r\sigma)\lambda \subseteq r'\sigma$. By Corollary 2.2.6, we obtain that $\lambda \in T(X, \sigma)$. Obviously, $R\lambda \subseteq R$. On the other hand, let $r \in R$. Then $r\beta \in R$ and so $(r\beta)\varphi = s\alpha$ for some $s \in R$. Thus

$(r\beta)\varphi = s\alpha = (b_s\beta)\varphi$ where $b_s \in s'\sigma$ and $s' \in R$. Since $\varphi|_R$ is injective, $r\beta = b_s\beta$ and by the regularity of β, it follows that $r = b_s$. Hence $s\lambda = b_s = r$, which implies the equality. This prove that $\lambda \in T_R(X, \sigma)$. Furthermore, for $x \in X$,

$$x\lambda\beta\mu = b_x\beta\mu = (b_x\beta)\varphi = x\alpha,$$

which implies that $\alpha = \lambda\beta\mu$. □

By the above lemma, we have the following result immediately.

Theorem 4.3.9. *Let α and β be regular elements of $T_R(X, \sigma)$. Then $(\alpha, \beta) \in \mathcal{J}$ on $T_R(X, \sigma)$ if and only if there exist mappings $\varphi : X\beta \to X\alpha$ and $\psi : X\alpha \to X\beta$ satisfying*

(1) *$\varphi|_R, \psi|_R : R \to R$ are bijections,*

(2) *for every $x, y \in X\beta$, $(x, y) \in \sigma$ implies $(x\varphi, y\varphi) \in \sigma$,*

(3) *for every $x, y \in X\alpha$, $(x, y) \in \sigma$ implies $(x\psi, y\psi) \in \sigma$ and*

(4) *for every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $A\alpha \subseteq (B\beta)\varphi$ and $A\beta \subseteq (C\alpha)\psi$.*

CHAPTER V

SOME ALGEBRAIC STRUCTURES OF TRANSFORMATION PRESERVING AN EQUIVALENCE RELATION

In this chapter, let σ and ρ be equivalence relations on a nonempty set X with $\rho \subseteq \sigma$ and R a cross-section of the partition X/σ induced by σ. We investigate conditions for the semigroups $T(X, \sigma, \rho)$, $T(X, \sigma^*)$, $T(X, \sigma, R)$ and $T_R(X, \sigma)$ which is an inverse semigroup and an E-inversive semigroup. Moreover, we present a characterization of abundant semigroups for $T(X, \sigma, \rho)$, $T(X, \sigma^*)$, $T(X, \sigma, R)$ and $T_R(X, \sigma)$. Also, we prove that the semigroup $T(X, \sigma, \rho)$ can be embeddable in $T(Y, Z)$ for some sets Y, Z with $Z \subseteq Y$ and it does not necessary to isomorphic.

5.1 Inverse semigroups

In this section, we find necessary and sufficient conditions for the semigroups $T(X, \sigma, \rho)$, $T(X, \sigma^*)$, $T(X, \sigma, R)$ and $T_R(X, \sigma)$ which is an inverse semigroup.

The first, we investigate a condition under which two elements of $T(X, \sigma, \rho)$ are inverse of each other. The following lemma is needed.

Lemma 5.1.1. *Let $\alpha, \beta \in T(X, \sigma, \rho)$. Suppose that α and β are inverses of each other. Then the following statements hold.*

(1) *For every $A \in \pi_\rho(\alpha)$, there exists a unique $B \in X/\rho$ such that $A \subseteq B\beta^{-1}$ and $B \subseteq A\alpha^{-1}$.*

(2) *For every $A \in \pi_\sigma(\alpha)$, there exists a unique $B \in X/\sigma$ such that $A \subseteq B\beta^{-1}$ and $B \subseteq A\alpha^{-1}$.*

Proof. (1) Suppose that $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$. Let $A \in \pi_\rho(\alpha)$. Then $A \in X/\rho$ and $A\alpha^{-1} \neq \emptyset$. Thus $\emptyset \neq A\alpha^{-1}\alpha \subseteq A$. By Corollary 2.2.6(1), there is a unique $B \in X/\rho$

such that $A\beta \subseteq B$. Thus $A \subseteq B\beta^{-1}$. By Corollary 2.2.6(1), we have $B\alpha \subseteq C$ for some $C \in X/\rho$. Therefore

$$A\alpha^{-1}\alpha = A\alpha^{-1}\alpha\beta\alpha \subseteq A\beta\alpha \subseteq B\alpha \subseteq C.$$

This implies that $A \cap C \neq \emptyset$, so that $A = C$. Hence $B \subseteq (B\alpha)\alpha^{-1} \subseteq C\alpha^{-1} = A\alpha^{-1}$, as required.

(2) By the symmetry of the proof (1). □

Theorem 5.1.2. *Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then the following statements are equivalent.*

(1) *α and β are inverses of each other.*

(2) *Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\varphi : \pi_\rho(\alpha) \to \pi_\rho(\beta)$ satisfying $U\alpha\beta = U\varphi\beta$ for all $U \in \pi_\rho(\alpha)$ and $V\beta\alpha = V\varphi^{-1}\alpha$ for all $V \in \pi_\rho(\beta)$.*

(3) *Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\psi : \pi_\sigma(\alpha) \to \pi_\sigma(\beta)$ satisfying $U\alpha\beta = U\psi\beta$ for all $U \in \pi_\sigma(\alpha)$ and $V\beta\alpha = V\psi^{-1}\alpha$ for all $V \in \pi_\sigma(\beta)$.*

(4) *For every $A \in X/\rho$ with $A' = A \cap X\alpha \neq \emptyset$, there exists $B \in X/\rho$ with $B' = B \cap X\beta \neq \emptyset$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.*

(5) *For every $A \in X/\sigma$ with $A' = A \cap X\alpha \neq \emptyset$, there exists $B \in X/\sigma$ with $B' = B \cap X\beta \neq \emptyset$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$. Then $\alpha\beta = \alpha\beta\alpha\beta$ and $\beta\alpha = \beta\alpha\beta\alpha$ hence $\alpha\beta$ and $\beta\alpha$ are idempotents. For each $A \in X/\rho$ with $A\alpha^{-1} \neq \emptyset$. By Lemma 5.1.1(1), we let $A' \in X/\rho$ such that $A \subseteq A'\beta^{-1}$ and $A' \subseteq A\alpha^{-1}$. Define $\varphi : \pi_\rho(\alpha) \to \pi_\rho(\beta)$ by

$$(A\alpha^{-1})\varphi = A'\beta^{-1} \text{ for all } A\alpha^{-1} \in \pi_\rho(\alpha).$$

By the uniqueness of A', we get that φ is well-defined. Let $U, V \in \pi_\rho(\alpha)$ be such that $U\varphi = V\varphi$. Then $U = A\alpha^{-1}$ and $V = B\alpha^{-1}$ for some $A, B \in X/\rho$. Thus $A'\beta^{-1} = B'\beta^{-1}$ where $A', B' \in X/\rho$. Therefore $A' = B'$. By the uniqueness, we obtain that $A = B$ and hence $U = V$. We conclude that φ is an injection. Let $V \in \pi_\rho(\beta)$. Then $V = A\beta^{-1}$ for some $A \in X/\rho$ and so $V\beta \subseteq A$. By Lemma 5.1.1(1), there is a unique $B \in X/\rho$ such that $A \subseteq B\alpha^{-1}$ and $B \subseteq A\beta^{-1}$. Since $\emptyset \neq A' \subseteq A\alpha^{-1}$, $A\alpha^{-1} \in \pi_\rho(\alpha)$. By the uniqueness of A', we have $A = A'$. Therefore $(B\alpha^{-1})\varphi = A'\beta^{-1} = A\beta^{-1} = V$. Hence φ is surjective. We conclude that φ is a bijection.

Let $U \in \pi_\rho(\alpha)$. Then $U = A\alpha^{-1}$ for some $A \in X/\rho$ and so $U\alpha \subseteq A$. By Lemma 5.1.1(1), there is a unique $A' \in X/\rho$ such that $A \subseteq A'\beta^{-1}$ and $A' \subseteq A\alpha^{-1}$. Thus

$$U\alpha\beta \subseteq A\beta \subseteq A'\beta^{-1}\beta = (A\alpha^{-1})\varphi\beta = U\varphi\beta.$$

Since $\beta = \beta\alpha\beta$, it follows that

$$U\varphi\beta = (A\alpha^{-1})\varphi\beta = A'\beta^{-1}\beta = A'\beta^{-1}\beta\alpha\beta \subseteq A'\alpha\beta \subseteq A\alpha^{-1}\alpha\beta = U\alpha\beta.$$

This shows that $U\alpha\beta = U\varphi\beta$.

Let $V \in \pi_\rho(\beta)$. Then there exist $A, A' \in X/\rho$ such that $V\beta \subseteq A'$ and $(A\alpha^{-1})\varphi = A'\beta^{-1} = V$. By the definition of φ, we have that $A \subseteq A'\beta^{-1}$ and $A' \subseteq A\alpha^{-1}$. Thus

$$V\beta\alpha \subseteq A'\alpha \subseteq A\alpha^{-1}\alpha = (A'\beta^{-1})\varphi^{-1}\alpha = V\varphi^{-1}\alpha.$$

Since $\alpha = \alpha\beta\alpha$, we have $V\varphi^{-1}\alpha = A\alpha^{-1}\alpha \subseteq A\alpha^{-1}\alpha\beta\alpha \subseteq A\beta\alpha \subseteq A'\beta^{-1}\beta\alpha = V\beta\alpha$. Hence $V\varphi^{-1}\alpha = V\beta\alpha$.

(2) $\Rightarrow$ (4) Suppose that both $\alpha\beta$ and $\beta\alpha$ are idempotents and $\varphi : \pi_\rho(\alpha) \to \pi_\rho(\beta)$ is a bijection satisfying $U\alpha\beta = U\varphi\beta$ for all $U \in \pi_\rho(\alpha)$ and $V\beta\alpha = V\varphi^{-1}\alpha$ for all $V \in \pi_\rho(\beta)$. Let $A \in X/\rho$ with $A' = A \cap X\alpha \neq \emptyset$. Then $A' = A\alpha^{-1}\alpha$ and $A'\alpha^{-1} = A\alpha^{-1} \neq \emptyset$ so $A\alpha^{-1} \in \pi_\rho(\alpha)$. By assumption, there exists $B \in X/\rho$ such that $(A\alpha^{-1})\varphi = B\beta^{-1} \neq \emptyset$ and we then have

$$A\alpha^{-1}\alpha\beta = A\alpha^{-1}\varphi\beta = B\beta^{-1}\beta.$$

Let $B' = B \cap X\beta$. Then $B' = B\beta^{-1}\beta \neq \emptyset$ and so $A'\beta = A\alpha^{-1}\alpha\beta = B'$. Let $x \in A'$. By the definition of φ, $B'\alpha = (B\beta^{-1})\beta\alpha = (B\beta^{-1})\varphi^{-1}\alpha = A\alpha^{-1}\alpha = A'$ and so there is $b \in B'$ such that $x = b\alpha$. Thus $b = a\beta$ for some $a \in A'$. Since $\beta\alpha$ is idempotent, we have

$$x\beta|_{A'}\alpha|_{B'} = x\beta\alpha = b\alpha\beta\alpha = a\beta\alpha\beta\alpha = a\beta\alpha = b\alpha = x.$$

Let $y \in B'$. Then $y = a'\beta$ for some $a' \in A'$ and so $a' = b'\alpha$ for some $b' \in B$. Since $\alpha\beta$ is idempotent, we deduce that $y\alpha|_{B'}\beta|_{A'} = y\alpha\beta = a'\beta\alpha\beta = b'\alpha\beta\alpha\beta = b'\alpha\beta = a'\beta = y$. It follows that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.

(4) $\Rightarrow$ (1) Suppose that (4) holds. Let $x \in X$. Then $x\alpha \in A$ for some $A \in X/\rho$. Thus $x\alpha \in A \cap X\alpha = A'$. By assumption, there exists $B \in X/\rho$ with $B' = B \cap X\beta \neq \emptyset$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections. This implies that $\beta|_{A'}\alpha|_{B'} = i_{A'}$. Therefore $x\alpha\beta\alpha = (x\alpha)\beta|_{A'}\alpha|_{B'} = (x\alpha)i_{A'} = x\alpha$. Hence $\alpha = \alpha\beta\alpha$. Similarly, $\beta = \beta\alpha\beta$.

Furthermore, in the same way we can show the equivalence of (1), (3) and (5). $\square$

From Theorem 5.1.2, we certainly have the following corollary.

Corollary 5.1.3. *Let $\alpha, \beta \in T(X, \sigma)$. Then the following statements are equivalent.*

(1) *α and β are inverses of each other.*

(2) *Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\psi : \pi_\sigma(\alpha) \to \pi_\sigma(\beta)$ satisfying $U\alpha\beta = U\psi\beta$ for all $U \in \pi_\sigma(\alpha)$ and $V\beta\alpha = V\psi^{-1}\alpha$ for all $V \in \pi_\sigma(\beta)$.*

(3) *For every $A \in X/\sigma$ with $A' = A \cap X\alpha \neq \emptyset$, there exists $B \in X/\sigma$ with $B' = B \cap X\beta \neq \emptyset$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.*

Theorem 5.1.2 and Corollary 3.2.4 can be summarized as follows:

Corollary 5.1.4. *Let $\alpha, \beta \in E(X, \sigma)$. Then the following statements are equivalent.*

(1) *α and β are inverses of each other.*

(2) *Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\varphi : \pi_\sigma(\alpha) \to \pi_\sigma(\beta)$ satisfying $U\alpha\beta = U\varphi\beta$ for all $U \in \pi_\sigma(\alpha)$ and $V\beta\alpha = V\varphi^{-1}\alpha$ for all $V \in \pi_\sigma(\beta)$.*

(3) *Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\psi : \pi(\alpha) \to \pi(\beta)$ satisfying $U\alpha\beta = U\psi\beta$ for all $U \in \pi(\alpha)$ and $V\beta\alpha = V\psi^{-1}\alpha$ for all $V \in \pi(\beta)$.*

(4) *For every $A \in X/\sigma$ with $A' = A \cap X\alpha \neq \emptyset$, there exists $B \in X/\sigma$ with $B' = B \cap X\beta \neq \emptyset$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.*

(5) *$\alpha|_{X\alpha} : X\alpha \to X\beta$ and $\beta|_{X\beta} : X\beta \to X\alpha$ are mutually inverse bijections.*

From Corollary 5.1.3 and Lemma 2.2.12, we can conclude the following corollary.

Corollary 5.1.5. *Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements are equivalent.*

(1) *α and β are inverses of each other.*

(2) *Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\psi : X/\sigma \to X/\sigma$ satisfying $U\alpha\beta = U\psi\beta$ and $V\beta\alpha = V\psi^{-1}\alpha$ for all $U, V \in X/\sigma$.*

(3) *For every $A \in X/\sigma$ with $A' = A \cap X\alpha \neq \emptyset$, there exists $B \in X/\sigma$ with $B' = B \cap X\beta \neq \emptyset$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.*

As an immediate consequence of Corollary 5.1.3, we have the following.

Corollary 5.1.6. *Let $\alpha, \beta \in T(X, \sigma, R)$. Then the following statements are equivalent.*

(1) *α and β are inverses of each other.*

(2) *Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\varphi : \pi_\sigma(\alpha) \to \pi_\sigma(\beta)$ satisfying $U\alpha\beta = U\varphi\beta$ for all $U \in \pi_\sigma(\alpha)$ and $V\beta\alpha = V\varphi^{-1}\alpha$ for all $V \in \pi_\sigma(\beta)$.*

(3) *For every $A \in X/\sigma$ with $A' = A \cap X\alpha \neq \emptyset$, there exists $B \in X/\sigma$ with $B' = B \cap X\beta \neq \emptyset$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.*

As an immediate consequence of Corollary 5.1.3 and Theorem 3.5.1, we have the following.

Corollary 5.1.7. *Let $\alpha, \beta \in T_R(X, \sigma)$. Then the following statements are equivalent.*

(1) *α and β are inverses of each other.*

(2) *Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\psi : X/\sigma \to X/\sigma$ satisfying $U\alpha\beta = U\psi\beta$ and $V\beta\alpha = V\psi^{-1}\alpha$ for all $U, V \in X/\sigma$.*

(3) *For every $A \in X/\sigma$ with $A' = A \cap X\alpha$, there exists $B \in X/\sigma$ with $B' = B \cap X\beta$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.*

The second, we find a necessary and sufficient condition for the semigroups $T(X,\sigma,\rho)$, $T(X,\sigma)$, $E(X,\sigma)$, $T(X,\sigma^*)$, $T(X,\sigma,R)$ and $T_R(X,\sigma)$ which is an inverse semigroup.

Now, we investigate the condition under which the semigroup $T(X,\sigma,\rho)$ is an inverse semigroup.

Theorem 5.1.8. *$T(X,\sigma,\rho)$ is an inverse semigroup if and only if $|X| = 1$.*

Proof. Suppose that $T(X,\sigma,\rho)$ is inverse. Let $a, b \in X$. Define $\alpha, \beta : X \to X$ by

$$x\alpha = a \text{ for all } x \in X \text{ and } y\beta = b \text{ for all } y \in X.$$

Then $\alpha, \beta \in T(X,\sigma,\rho)$, $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$. Thus α and β are inverses of each other. Since α is idempotent, α is also an inverse of α. By assumption, $\alpha = \beta$. Hence $a = b$. We conclude that $|X| = 1$.

For the converse, if $|X| = 1$, then $T(X,\sigma,\rho)$ contains only one element. It is clear that $T(X,\sigma,\rho)$ is an inverse semigroup. □

Theorem 5.1.8 can be summarized as follows:

Corollary 5.1.9. *The following statements hold.*

(1) *$T(X,\sigma)$ is an inverse semigroup if and only if $|X| = 1$.*

(2) *$E(X,\sigma)$ is an inverse semigroup if and only if $|X| = 1$.*

According to next result, the class of semigroups $T(X,\sigma^*)$ is an inverse semigroup.

Theorem 5.1.10. *$T(X,\sigma^*)$ is an inverse semigroup if and only if X/σ is finite and $\sigma = I_X$.*

Proof. Suppose that $T(X,\sigma^*)$ is an inverse semigroup. Then $T(X,\sigma^*)$ is a regular semigroup. By Theorem 3.3.6, we have that X/σ is finite. We will show that $\sigma = I_X$. Let $a, b \in X$ such that $(a,b) \in \sigma$. Thus $a, b \in A$ for some $A \in X/\sigma$. Define $\alpha, \beta : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A, \\ x & \text{otherwise} \end{cases}$$

and

$$x\beta = \begin{cases} b & \text{if } x \in A, \\ x & \text{otherwise.} \end{cases}$$

Then $B\alpha \subseteq B$ and $C\beta \subseteq C$ for all $B, C \in X/\sigma$. This implies that $\alpha, \beta \in T(X, \sigma^*)$. For each $x \in X$, if $x \in A$, then $x\alpha\alpha = a\alpha = a = x\alpha$ and if $x \notin A$, then $x\alpha\alpha = x\alpha$. Therefore α is idempotent. Let $B \in X/\sigma$ with $B' = B \cap X\alpha \neq \emptyset$. For each $x \in B'$. If $B = A$, then $B' = \{a\}$ and

$$x\beta|_{B'}\alpha|_{B'} = a\beta|_{B'}\alpha|_{B'} = a\beta\alpha = b\alpha = a = x.$$

If $B \neq A$, then $B' = B$ and

$$x\beta|_{B'}\alpha|_{B'} = x\beta\alpha = x\alpha = x.$$

Hence α and β satisfying the condition (3) of Corollary 5.1.5 which implies that α and β are inverse of each other. It follows that α and β are inverses of α. By assumption, we obtain that $\alpha = \beta$. Hence $a = a\alpha = a\beta = b$. We conclude that $\sigma = I_X$.

Conversely, assume that X/σ is finite and σ is the identity relation. By Theorem 3.3.6, we obtain that $T(X, \sigma^*)$ is regular. Thus every element of $T(X, \sigma^*)$ has an inverse. Let $\alpha \in T(X, \sigma^*)$. We will show that α is a bijection. Let $x, y \in X$ be such that $x\alpha = y\alpha$. Then $(x\alpha, y\alpha) \in \sigma$. Since $\alpha \in T(X, \sigma^*)$, $(x, y) \in \sigma$. By assumption of σ, $x = y$. Hence α is an injection. Since X/σ is finite and σ is the identity relation, it follows that X is finite. Using Theorem 2.1.18, we get that α is also surjective. Thus we conclude that α is a bijection. Hence α has a unique inverse. We conclude that $T(X, \sigma^*)$ is an inverse semigroup. □

Now, we give a characterization for the semigroup $T(X, \sigma, R)$ is an inverse semigroup in terms of the cardinalities of X and a cross-section R.

Theorem 5.1.11. *$T(X, \sigma, R)$ is an inverse semigroup if and only if $|X| \leq 2$ and $|R| = 1$.*

Proof. Assume that $T(X, \sigma, R)$ is an inverse semigroup. To show that $|R| = 1$, let $r, s \in R$. Define $\alpha, \beta : X \to X$ by

$$x\alpha = r \text{ and } y\beta = s \text{ for all } x, y \in X.$$

Clearly, $\alpha, \beta \in T(X, \sigma, R)$ and α, β are idempotents. By Corollary 2.2.8, we get that $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$. This implies that α and β are inverses of α. By assumption, $\alpha = \beta$ and hence $r = s$. Therefore $|R| = 1$. Suppose that $|X| \neq 1$. Let $r \in R$. To show that $|X| = 2$, let $a, b \in X \setminus \{r\}$. Define $\lambda, \mu : X \to X$ by

$$x\lambda = \begin{cases} r & \text{if } x = r, \\ a & \text{otherwise} \end{cases}$$

and

$$x\mu = \begin{cases} r & \text{if } x = r, \\ b & \text{otherwise.} \end{cases}$$

Since $|R| = 1$ and by Corollary 2.2.6, we have $\lambda, \mu \in T(X, \sigma)$. Note that $r\lambda = r$ and $r\mu = r$. This implies that $\lambda, \mu \in T(X, \sigma, R)$. For each $x \in X$, if $x = r$, then $x\lambda\lambda = r\lambda = r = x\lambda$ and if $x \neq r$, then $x\lambda\lambda = a\lambda = a = x\lambda$. Therefore λ is idempotents and thus λ is an inverse of itself. Let $A' = \{a, r\}$ and $B' = \{b, r\}$. Then $b\lambda\mu = a\mu = b$ and $r\lambda\mu = r\mu = r$. Similarly, $a\mu\lambda = a$ and $r\mu\lambda = r$. Therefore λ and μ satisfy the condition (3) of Corollary 5.1.6. Hence λ and μ are inverses of each other. By assumption, $\lambda = \beta$. Therefore $a = a\lambda = a\mu = b$. Hence $|X| = 2$.

Conversely, suppose that $|X| \leq 2$ and $|R| = 1$. Then $\sigma = X \times X$. If $|X| = 1$, then $T(X, \sigma, R)$ contains only one element and so $T(X, \sigma, R)$ is an inverse semigroup. Assume that $|X| = 2$, say $X = \{a, b\}$ and $a \in R$. Then

$$T(X, \sigma, R) = \left\{ \begin{pmatrix} a & b \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ a & b \end{pmatrix} \right\}.$$

It easy to check that $T(X, \sigma, R)$ is an inverse semigroup. □

Finally, we characterize the equivalence relations σ on X and a cross-section R for which the semigroup $T_R(X, \sigma)$ is inverse. The following lemma is required.

Lemma 5.1.12. *Let $\alpha \in T_R(X, \sigma)$. If α is idempotent, then $(r\sigma)\alpha \subseteq r\sigma$ for all $r \in R$.*

Proof. Suppose that α is idempotent. Let $r \in R$. Then $r = s\alpha$ for some $s \in R$. By assumption, $r\alpha = s\alpha\alpha = s\alpha$. By Theorem 3.5.1, $r = s$. If $a \in r\sigma$, then $a\alpha \in r\alpha = r\sigma$. Hence $(r\sigma)\alpha \subseteq r\sigma$. □

Theorem 5.1.13. *$T_R(X,\sigma)$ is an inverse semigroup if and only if R is finite and σ is 2-bounded.*

Proof. Assume that $T_R(X,\sigma)$ is an inverse semigroup. Then $T_R(X,\sigma)$ is a regular semigroup. By Theorem 3.5.7, we have that R is a finite set. To show that σ is 2-bounded, let $r \in R$ and $a, b \in r\sigma \setminus \{r\}$. Define $\alpha, \beta : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in r\sigma \setminus \{r\}, \\ x & \text{otherwise} \end{cases}$$

and

$$x\beta = \begin{cases} b & \text{if } x \in r\sigma \setminus \{r\}, \\ x & \text{otherwise.} \end{cases}$$

Then $(s\sigma)\alpha \subseteq s\sigma$, $s\alpha = s$, $(t\sigma)\beta \subseteq t\sigma$ and $t\beta = t$ for all $s, t \in R$. By Corollary 2.2.6, we obtain $\alpha, \beta \in T_R(X,\sigma)$. For each $x \in X$, if $x \in r\sigma \setminus \{r\}$, then $x\alpha\alpha = a\alpha = a = x\alpha$ and if $x \notin r\sigma \setminus \{r\}$, then $x\alpha\alpha = x\alpha$. Hence α is idempotent and thus α is an inverse of itself. It easy to check that α and β satisfy the condition (3) of Corollary 5.1.7 which implies that α and β are inverse of each other. It follows from assumption that $\alpha = \beta$. This implies that $a = a\alpha = a\beta = b$. We conclude that σ is 2-bounded.

Conversely, assume that R is finite and σ is 2-bounded. By Theorem 3.5.7, we have that $T_R(X,\sigma)$ is a regular semigroup. Let $\alpha, \beta \in T_R(X,\sigma)$ be idempotents. Let $x \in X$. Since σ is 2-bounded and by Lemma 5.1.12, we consider two cases as follows.

Case 1. $x\alpha = x$. If $x\beta = x$, then $x\alpha\beta = x\beta\alpha$. If $x\beta \neq x$, then $x\beta = r$ for some $r \in R$. Thus $x \in r\sigma$ and $r\alpha = r\beta = r$. Therefore $x\alpha\beta = x\beta = r\alpha = x\beta\alpha$.

Case 2. $x\alpha \neq x$. Then $x\alpha = r$ for some $r \in R$ and so $x \in r\sigma$. If $x\beta = r$, then done. If $x\beta \neq r$, then $x\beta = x$. Thus $r\beta = r$. Hence $x\alpha\beta = r\beta = r = r\alpha = x\beta\alpha$.

From the two cases, we conclude that $\alpha\beta = \beta\alpha$. By Theorem 2.1.4, $T_R(X,\sigma)$ is an inverse semigroup, as required. □

5.2 E-inversive semigroups

In this section, we present the characterization of E-inversive for elements of the semigroups $T(X,\sigma,\rho)$, $T(X,\sigma^*)$, $T(X,\sigma,R)$ and $T_R(X,\sigma)$ and give a necessary and sufficient condition for the semigroups $T(X,\sigma,\rho)$, $T(X,\sigma^*)$, $T(X,\sigma,R)$ and $T_R(X,\sigma)$ to be E-inversive.

If S is any one of the semigroups $T(X,\sigma,\rho)$ and $T(X,\sigma,R)$. Then S contains a constant mapping. By Proposition 2.2.7 and Corollary 2.2.8, we have S contains a right zero element. Hence the following results follow directly from Lemma 2.1.6.

Theorem 5.2.1. *Let S be any one of the semigroups $T(X,\sigma,\rho)$ and $T(X,\sigma,R)$. Then S is an E-inversive semigroup. Consequently, $T(X,\sigma)$ and $E(X,\sigma)$ are E-inversive semigroups.*

We have mentioned that every regular element is E-inversive. But there exists an E-inversive element of a semigroup S which is not regular as shown in the following example.

Example 5.2.2. Let $X = \{1,2,3,4,5,6,7\}$ and $X/\sigma = \{\{1,2,3\},\{4,5\},\{6,7\}\}$. Define $\alpha \in T(X)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 6 & 3 & 3 & 2 & 1 \end{pmatrix}.$$

Then $\alpha \in T(X,\sigma)$, hence α is E-inversive. Let $A = \{1,2,3\}$. Then $A\alpha^{-1} = \{4,5,6,7\}$. Since $\{4,5\}\alpha = \{3\}$ and $\{6,7\}\alpha = \{1,2\}$, $A \cap X\alpha \nsubseteq B\alpha$ for all $B \in X/\sigma$. By Corollary 3.2.3, α is not a regular element of $T(X,\sigma)$.

Next, the following lemma is needed for proving the next theorem.

Lemma 5.2.3. *Let $\alpha \in T(X,\sigma^*)$. If α is idempotent, then $A\alpha \subseteq A$ for all $A \in X/\sigma$.*

Proof. Suppose that α is idempotent. Then $\alpha^2 = \alpha$. Let $A \in X/\sigma$ and $a \in A$. Then $a\alpha^2 = a\alpha$ and hence $(a\alpha,(a\alpha)\alpha) \in \sigma$. Since $\alpha \in T(X,\sigma^*)$, it follows that $(a,a\alpha) \in \sigma$. From $a \in A$, we deduce that $a\alpha \in A$. Therefore $A\alpha \subseteq A$. □

Theorem 5.2.4. *Let $\alpha \in T(X,\sigma^*)$. Then α is E-inversive if and only if $A \cap X\alpha \neq \emptyset$ for all $A \in X/\sigma$.*

Proof. Suppose that α is E-inversive. Then there exists $\beta \in T(X,\sigma^*)$ such that $\alpha\beta$ is idempotent. Let $A \in X/\sigma$. Then $A\beta \subseteq B$ for some $B \in X/\sigma$. By Lemma 5.2.3, we deduce that $B\alpha\beta \subseteq B$. Let $b \in B$. Then $b\alpha\beta \in B$. If $a \in A$, then $a\beta \in B$ and so $(b\alpha\beta, a\beta) \in \sigma$. Since $\beta \in T(X, \sigma^*)$, it follows that $(b\alpha, a) \in \sigma$. Thus $b\alpha \in A$. Hence $B\alpha \subseteq A$. Consequently, $A \cap X\alpha \neq \emptyset$.

The converse follows from Theorems 3.3.1 and 2.1.5. □

The next result follows immediately from Theorem 3.3.1 and Theorem 5.2.4.

Corollary 5.2.5. *$T(X, \sigma^*)$ is an E-inversive semigroup if and only if it is a regular semigroup.*

Finally, we describe E-inversive for elements and semigroups on $T_R(X, \sigma)$.

Theorem 5.2.6. *Let $\alpha \in T_R(X, \sigma)$. Then α is E-inversive if and only if $\alpha|_R$ is an injection.*

Proof. Suppose that α is E-inversive. Then there exists $\beta \in T_R(X, \sigma)$ such that $\alpha\beta$ is idempotent. Let $r, s \in R$ be such that $r\alpha = s\alpha$. Then $r\alpha\beta = s\alpha\beta$. Since $\alpha\beta$ is idempotent, it is regular. By Theorem 3.2.1, we deduce that $(\alpha\beta)|_R$ is injective and hence $r = s$. Thereby $\alpha|_R$ is an injection.

Conversely, if $\alpha|_R$ is injective, then α is regular by Theorem 3.5.1. Therefore α is E-inversive. □

As a consequence of Theorem 5.2.6 and Theorem 3.5.1, the following result follows readily.

Corollary 5.2.7. *$T_R(X, \sigma)$ is an E-inversive semigroup if and only if it is a regular semigroup.*

5.3 Abundant semigroups

In this section, we describe Green's *-relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on the semigroups $T(X, \sigma^*)$, $T(X, \sigma, \rho)$, $T(X, \sigma, R)$ and $T_R(X, \sigma)$. Moreover, we present a necessary and sufficient condition under which the semigroups $T(X, \sigma^*)$, $T(X, \sigma, \rho)$, $T(X, \sigma, R)$ and $T_R(X, \sigma)$ are abundant.

Firstly, the following results are quoted from [34] Theorems 3.1, 3.2, 4.1, 4.3 and 4.6, respectively.

Theorem 5.3.1. [34] *Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements hold.*

(1) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $X\alpha = X\beta$.

(2) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\ker \alpha = \ker \beta$.

Theorem 5.3.2. [34] *The following statements hold.*

(1) *The semigroup $T(X, \sigma^*)$ is left abundant if and only if it is regular.*

(2) *The semigroup $T(X, \sigma^*)$ is right abundant.*

(3) *The semigroup $T(X, \sigma^*)$ is abundant if and only if it is regular.*

Secondly, we investigate the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ for the semigroup $T(X, \sigma, \rho)$ and then we study the condition for the equivalence relations σ and ρ under which $T(X, \sigma, \rho)$ is abundant.

We begin with the $\mathcal{L}^*$-relation for elements of the semigroup $T(X, \sigma, \rho)$.

For every $Y \subseteq X$, we denote

$$\overline{Y} = \{A \in X/\sigma : A \cap Y \neq \emptyset\}.$$

From [36], Sun and Wang described the relation $\mathcal{L}^*$ for elements of the semigroup $T(X, \sigma, \rho)$ where $\rho = I_X$, as follows:

Theorem 5.3.3. [36] *Suppose $\rho = I_X$. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $(\alpha, \beta) \in \mathcal{L}^*$ on $T(X, \sigma, \rho)$ if and only if either of the following statements holds:*

(1) *α, β are not regular on X and* $\overline{X\alpha} = \overline{X\beta}$.

(2) $X\alpha = X\beta$.

Theorem 5.3.4. *Suppose $\rho \neq I_X$. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $(\alpha, \beta) \in \mathcal{L}^*$ on $T(X, \sigma, \rho)$ if and only if $X\alpha = X\beta$.*

Proof. The necessity follows from Theorem 4.1.1(1). To prove the sufficiency, we suppose that $(\alpha,\beta) \in \mathcal{L}^*$. Let $y \notin X\alpha$. We can asset that $y \notin X\beta$. Indeed, if there is some $y' \in X$ such that $y'\beta = y$. Since $\rho \neq I_X$, there exists $x_0 \in X$ such that $|x_0\rho| \geq 2$. If $x_0 \in y\rho$, then $(y, x_0) \in \rho$ and so $y\rho \setminus \{x_0\} \neq \emptyset$. If $x_0 \notin y\rho$, then $y\rho \setminus \{x_0\} \neq \emptyset$. For each $A \in X/\sigma$, if $A = y\sigma$, then we choose and fix an element $x_A \in y\rho \setminus \{x_0\}$. Otherwise, we choose and fix an element $x_A \in A$. Define $\gamma_1, \gamma_2 : X \to X$ by

$$x\gamma_1 = x_{x\sigma} \text{ for all } x \in X$$

and

$$x\gamma_2 = \begin{cases} x_0 & \text{if } x = y, \\ x\gamma_1 & \text{otherwise.} \end{cases}$$

Then $\gamma_1 \in T(X,\sigma,\rho)$. Since $(x_{y\sigma}, y) \in \rho$, we deduce that $\gamma_2 \in T(X,\sigma,\rho)$. Let $x \in X$. Since $y \notin X\alpha$, we have $x\alpha \neq y$ and so $x\alpha\gamma_2 = x\alpha\gamma_1$. Therefore $\alpha\gamma_1 = \alpha\gamma_2$. By assumption and Lemma 2.1.12, we deduce that $\beta\gamma_1 = \beta\gamma_2$. This implies that

$$x_{y\sigma} = y\gamma_1 = y'\beta\gamma_1 = y'\beta\gamma_2 = y\gamma_2 = x_0,$$

which leads to a contradiction. Hence $y \notin X\beta$ and thus $X\beta \subseteq X\alpha$. By symmetry, we have $X\alpha \subseteq X\beta$. Consequently, we have $X\alpha = X\beta$, as required. □

Next, we investigate some conditions under which the semigroup $T(X,\sigma,\rho)$ is left abundant. Pei and Zhou [31] proved that $T(X,\sigma)$ is a left abundant semigroup. We then have the following.

Theorem 5.3.5. *The semigroup $T(X,\sigma,\rho)$ is left abundant if and only if $\sigma = \rho$ or $\sigma = X \times X$.*

Proof. Assume that $T(X,\sigma,\rho)$ is a left abundant semigroup and $\sigma \neq \rho$. Then there exist $a, b \in X$ such that $(a,b) \in \sigma$ and $(a,b) \notin \rho$. Suppose that $\sigma \neq X \times X$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } (x,a) \in \sigma, \\ b & \text{otherwise.} \end{cases}$$

Then by Lemma 2.2.5, we have $\alpha \in T(X,\sigma,\rho)$. By assumption, there exists idempotent $\mu \in T(X,\sigma,\rho)$ such that $(\alpha,\mu) \in \mathcal{L}^*$. Since μ is regular and by Theorems 5.3.3 and 5.3.4, it follows that $X\alpha = X\mu$. From $\sigma \neq X \times X$, we get that $a, b \in X\alpha = X\mu$. Therefore $a = a'\mu$ and $b = b'\mu$ for some $a', b' \in X$. Thus

$$(a, b) = (a'\mu, b'\mu) = (a'\mu\mu, b'\mu\mu) = (a\mu, b\mu) \in \rho.$$

This is a contradiction. Hence $\sigma = X \times X$.

Conversely, assume that $\sigma = \rho$ or $\sigma = X \times X$. If $\sigma = \rho$, then done. Suppose that $\sigma = X \times X$. Let $\alpha \in T(X, \sigma, \rho)$. Fix $z \in X\alpha$. Define $\mu : X \to X$ by

$$x\mu = \begin{cases} x & \text{if } x \in X\alpha, \\ z & \text{otherwise.} \end{cases}$$

Then μ is well-defined and $X\alpha = X\mu$. Since $\sigma = X \times X$, $X/\sigma = \{X\}$ and by Lemma 2.2.5, we have $X\mu = X\alpha \subseteq A$ for some $A \in X/\rho$. Therefore $\mu \in T(X, \sigma, \rho)$. For each $x \in X$, we have $x\mu \in X\mu = X\alpha$ and thus $x\mu\mu = x\mu$. Therefore μ is idempotent and regular. By Theorems 5.3.3 and 5.3.4, we obtain that $(\alpha, \mu) \in \mathcal{L}^*$. Hence $T(X, \sigma, \rho)$ is left abundant. □

Next we consider the relation $\mathcal{R}^*$.

Theorem 5.3.6. *Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $(\alpha, \beta) \in \mathcal{R}^*$ on $T(X, \sigma, \rho)$ if and only if $\ker \alpha = \ker \beta$.*

Proof. The necessity is clear from Theorem 4.1.1(2). To prove the sufficiency, we assume that $(\alpha, \beta) \in \mathcal{R}^*$. Let $a, b \in X$ such that $(a, b) \in \ker \alpha$. Fix $z \in X$. Define $\gamma_1, \gamma_2 : X \to X$ by

$$x\gamma_1 = \begin{cases} a & \text{if } x \in a\sigma \cup b\sigma, \\ z & \text{otherwise,} \end{cases}$$

and

$$x\gamma_2 = \begin{cases} b & \text{if } x \in a\sigma \cup b\sigma, \\ z & \text{otherwise.} \end{cases}$$

Then by Lemma 2.2.5, we get $\gamma_1, \gamma_2 \in T(X, \sigma, \rho)$. Let $x \in X$. If $x \in a\sigma \cup b\sigma$, then since $(a, b) \in \ker \alpha$, we have that $x\gamma_1\alpha = a\alpha = b\alpha = x\gamma_2\alpha$. If $x \notin a\sigma \cup b\sigma$, then we have $x\gamma_1\alpha = z\alpha = x\gamma_2\alpha$. It follows that $\gamma_1\alpha = \gamma_2\alpha$. Applying the characterization of $\mathcal{R}^*$ from Lemma 2.1.13, we obtain that $\gamma_1\beta = \gamma_2\beta$. Therefore

$$a\beta = a\gamma_1\beta = a\gamma_2\beta = b\beta.$$

This implies that $(a,b) \in \ker\beta$. Therefore $\ker\alpha \subseteq \ker\beta$. By symmetry, $\ker\beta \subseteq \ker\alpha$. Hence $\ker\alpha = \ker\beta$. □

Now, we describe right abundant for the semigroup $T(X,\sigma,\rho)$. The following results are quoted from [36] Corollary 2.4 and [31] Theorem 2.15, respectively.

Theorem 5.3.7. [36] *$E(X,\sigma)$ is right abundant.*

Theorem 5.3.8. [31] *$T(X,\sigma)$ is right abundant if and only if σ is one of the following three cases.*

(1) $\sigma = X \times X$.

(2) *σ is 2-bounded.*

(3) *σ is a T-relation.*

Theorem 5.3.9. *The semigroup $T(X,\sigma,\rho)$ is right abundant if and only if $T(X,\sigma,\rho) = E(X,\sigma)$ or $T(X,\sigma,\rho) = T(X)$ or the following statements hold:*

(1) *$T(X,\sigma,\rho) = T(X,\sigma)$ and*

(2) *σ is a T-relation or 2-bounded.*

Proof. The necessity follows from Theorems 5.3.7, 5.3.8, 3.1.1 and 2.1.14. To prove the sufficiency, we assume that $T(X,\sigma,\rho)$ is right abundant and $T(X,\sigma,\rho)$ is neither $E(X,\sigma)$ nor $T(X)$. By (1) and (6) of Theorem 2.3.1, we get that $\rho \notin \{I_X, X \times X\}$ and $\sigma \neq I_X$. Suppose that $\rho \neq \sigma$. Then there exist $y,z \in X$ such that $(y,z) \in \sigma$ and $(y,z) \notin \rho$. Since $\rho \neq I_X$, there exist distinct elements $a,b \in X$ such that $(a,b) \in \rho$. Define $\alpha : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } (x,y) \in \rho, \\ b & \text{otherwise.} \end{cases}$$

Since $(a,b) \in \rho$, we deduce that $\alpha \in T(X,\sigma,\rho)$. By the hypothesis, there exists idempotent $\mu \in T(X,\sigma,\rho)$ such that $(\alpha,\mu) \in \mathcal{R}^*$. From Theorem 5.3.6, $\ker\alpha = \ker\mu$. Since $(y,z) \notin \rho$, $\pi(\alpha) = \pi(\mu) = \{y\rho, X \setminus y\rho\}$. Thus $y\rho = c\mu^{-1}$ and $X \setminus y\rho = d\mu^{-1}$ for some $c,d \in X$. Since $(y,z) \in \sigma$ and $(y,z) \notin \rho$, we deduce that $(c,d) = (y\mu, z\mu) \in \rho$. Therefore

$$c\mu = y\mu\mu = y\mu = c \text{ and } d\mu = z\mu\mu = z\mu = d,$$

so that $c \in c\mu^{-1} = y\rho$ and $d \in d\mu^{-1} = X \setminus y\rho$. This is a contradiction with $(c, d) \in \rho$. Hence $\sigma = \rho$. Accordingly, $T(X, \sigma, \rho) = T(X, \sigma)$. Since $\sigma = \rho \notin \{I_X, X \times X\}$ and by Theorem 5.3.8, we have σ is a T-relation or 2-bounded, as required. □

The following conclusion readily follows from Theorems 5.3.9 and 5.3.5.

Theorem 5.3.10. *The semigroup $T(X, \sigma, \rho)$ is abundant if and only if one of the following statements holds.*

(1) $T(X, \sigma, \rho) = E(X, X \times X)$.

(2) $T(X, \sigma, \rho) = T(X, \sigma)$ *and σ is a T-relation.*

(3) $T(X, \sigma, \rho) = T(X, \sigma)$ *and σ is 2-bounded.*

(4) $T(X, \sigma, \rho) = T(X)$.

Next, we characterize Green's *-relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on the semigroup $T(X, \sigma, R)$. Furthermore, we present a necessary and sufficient condition under which the semigroup $T(X, \sigma, R)$ is left abundant, right abundant and abundant.

We begin with the $\mathcal{L}^*$-relation. For every $x \in X$, we let $r_x \in R$ be such that $(x, r_x) \in \sigma$.

Theorem 5.3.11. *Let $\alpha, \beta \in T(X, \sigma, R)$. Then $(\alpha, \beta) \in \mathcal{L}^*$ on $T(X, \sigma, R)$ if and only if $X\alpha = X\beta$.*

Proof. The necessity follows from Theorem 4.1.1(1). To prove the sufficiency, we suppose that $(\alpha, \beta) \in \mathcal{L}^*$ on $T(X, \sigma, R)$. Fix $r' \in R$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} x & \text{if } x \in X\alpha, \\ r_x & \text{if } x \notin X\alpha \text{ and } r_x\sigma \cap X\alpha \neq \emptyset, \\ r'\alpha & \text{otherwise.} \end{cases}$$

Then γ is well-defined. For each $r \in R$ with $r\sigma \cap X\alpha \neq \emptyset$. Then $y \in r\sigma \cap X\alpha$ for some $y \in X$. Thus $y = y'\alpha$. By Lemma 2.2.13, we have $r = r_{y'}\alpha$. Therefore $r \in X\alpha$. This implies that $X\alpha = X\gamma$. Let $x, y \in X$ be such that $(x, y) \in \sigma$. Then $x, y, r_x, r_y \in r_x\sigma$.

If $r_x\sigma \cap X\alpha \neq \emptyset$, then $(x\gamma, y\gamma) \in \sigma$. If $r_x\sigma \cap X\alpha = \emptyset$, then done. Hence $\gamma \in T(X,\sigma)$. Let $r \in R$. If $r\sigma \cap X\alpha \neq \emptyset$, then by above fact, we have $r \in X\alpha$ and so $r\gamma = r \in R$. If $r\sigma \cap X\alpha = \emptyset$, then $r\gamma = r'\alpha \in R\alpha \subseteq R$. Consequently, $\gamma \in T(X,\sigma,R)$. Let $x \in X$. Then we have $x\alpha\gamma = x\alpha$ and so $\alpha\gamma = \alpha$. By assumption, we deduce that $\beta\gamma = \beta$. For each $x \in X$, we have $x\beta = x\beta\gamma \in X\gamma = X\alpha$. Hence $X\beta \subseteq X\alpha$. By symmetry, we have $X\alpha \subseteq X\beta$. Consequently, we have $X\alpha = X\beta$. □

The next result shows that there exists idempotent in each $\mathcal{L}^*$-class of $T(X,\sigma,R)$.

Theorem 5.3.12. *Every $\mathcal{L}^*$-class of $T(X,\sigma,R)$ contains idempotent. Consequently, the semigroup $T(X,\sigma,R)$ is left abundant.*

Proof. Let $\alpha \in T(X,\sigma,R)$. Define $\gamma \in T(X,\sigma,R)$ as in the same proof of Theorem 5.3.11. Then $X\alpha = X\gamma$. By Theorem 5.3.11, $(\alpha,\gamma) \in \mathcal{L}^*$. For each $x \in X$, we have $x\gamma \in X\gamma = X\alpha$ and so $x\gamma\gamma = x\gamma$. Hence γ is idempotent, as required. □

Theorem 5.3.13. *Let $\alpha, \beta \in T(X,\sigma,R)$. Then $(\alpha,\beta) \in \mathcal{R}^*$ on $T(X,\sigma,R)$ if and only if* $\ker\alpha = \ker\beta$.

Proof. The necessity follows from Theorem 4.1.1(2). To prove the sufficiency, we assume that $(\alpha,\beta) \in \mathcal{R}^*$. Let $a, b \in X$ such that $(a,b) \in \ker\alpha$. Define $\gamma_1, \gamma_2 : X \to X$ by

$$x\gamma_1 = \begin{cases} a & \text{if } x \in r_a\sigma \setminus \{r_a\}, \\ b & \text{if } x \in r_b\sigma \setminus \{r_b\}, \\ x & \text{otherwise,} \end{cases}$$

and

$$x\gamma_2 = \begin{cases} b & \text{if } x \in r_a\sigma \setminus \{r_a\}, \\ a & \text{if } x \in r_b\sigma \setminus \{r_b\}, \\ r_b & \text{if } x = r_a, \\ r_a & \text{if } x = r_b, \\ x & \text{otherwise.} \end{cases}$$

Then γ_1 and γ_2 are well-defined and $(r\sigma)\gamma_1 \subseteq r\sigma$ for all $r \in R$ and $(r_a\sigma)\gamma_2 \subseteq r_b\sigma$, $(r_b\sigma)\gamma_2 \subseteq r_a\sigma$ and $(r\sigma)\gamma_2 \subseteq r\sigma$ for all $r \in R \setminus \{r_a, r_b\}$. By Corollary 2.2.6, we have that $\gamma_1, \gamma_2 \in T(X,\sigma)$. As a result $r\gamma_1 = r$ for all $r \in R$, we then have $R\gamma_1 = R$ and hence

$\gamma_1 \in T(X,\sigma,R)$. Since $r_a\gamma_1 = r_b$, $r_b\gamma_2 = r_a$ and $r\gamma_2 = r$ for all $r \in R \setminus \{r_a, r_b\}$, we get that $R\gamma_2 = R$. Consequently, $\gamma_2 \in T(X,\sigma,R)$, as required. Since $a\alpha = b\alpha$ and by Lemma 2.2.13, $r_a\alpha = r_b\alpha$ which implies that $x\gamma_1\alpha = x\gamma_2\alpha$ for all $x \in X$. It follows that $\gamma_1\alpha = \gamma_2\alpha$. Applying the characterization of $\mathcal{R}^*$ from Lemma 2.1.13, we have $\gamma_1\beta = \gamma_2\beta$. Next, we will show that $a\beta = b\beta$. We distinguish four cases as follows.

Case 1. $a = r_a$ and $b = r_b$. Then $a\beta = a\gamma_1\beta = a\gamma_2\beta = b\beta$.

Case 2. $a = r_a$ and $b \neq r_b$. Then $b\beta = b\gamma_1\beta = b\gamma_2\beta = a\beta$.

Case 3. $a \neq r_a$ and $b = r_b$. Then $a\beta = a\gamma_1\beta = a\gamma_2\beta = b\beta$.

Case 4. $a \neq r_a$ and $b \neq r_b$. Then $b\beta = b\gamma_1\beta = b\gamma_2\beta = a\beta$.

From the four cases, it follows that $(a,b) \in \ker\beta$. Therefore $\ker\alpha \subseteq \ker\beta$. By symmetry, $\ker\beta \subseteq \ker\alpha$. Hence $\ker\alpha = \ker\beta$. □

Theorem 5.3.14. *The semigroup $T(X,\sigma,R)$ is right abundant if and only if it is regular.*

Proof. The necessity follows from Theorem 2.1.14. To prove the sufficiency, we assume that $T(X,\sigma,R)$ is right abundant. Let $\alpha \in T(X,\sigma,R)$. Then there exists idempotent $\mu \in T(X,\sigma,R)$ such that $(\alpha,\mu) \in \mathcal{R}^*$. By Theorem 5.3.13, we have $\ker\alpha = \ker\mu$. It follows from Theorem 4.1.7 that $(\alpha,\mu) \in \mathcal{R}$. Since μ is regular and by Theorem 2.1.10, we deduce that α is also regular. Hence $T(X,\sigma,R)$ is a regular semigroup. □

The following conclusion readily follows from Theorems 5.3.14 and 5.3.12.

Theorem 5.3.15. [42] *The semigroup $T(X,\sigma,R)$ is abundant if and only if it is regular.*

Finally, we investigate the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ for the semigroup $T_R(X,\sigma)$ and then we show that regular and abundant in the semigroup $T_R(X,\sigma)$ coincided.

We begin with the $\mathcal{L}^*$-relation.

Theorem 5.3.16. *Let $\alpha,\beta \in T_R(X,\sigma)$. Then $(\alpha,\beta) \in \mathcal{L}^*$ on $T_R(X,\sigma)$ if and only if $X\alpha = X\beta$.*

Proof. The necessity follows from Theorem 4.1.1(1). To prove the sufficiency, we suppose that $(\alpha,\beta) \in \mathcal{L}^*$ on $T_R(X,\sigma)$. Define $\gamma : X \to X$ by

$$x\gamma = \begin{cases} x & \text{if } x \in X\alpha, \\ r_x & \text{otherwise.} \end{cases}$$

Then γ is well-defined. Since $R = R\alpha \subseteq X\alpha$, $X\alpha = X\gamma$ and $r\gamma = r$ for all $r \in R$. Therefore $R\gamma = R$. Let $x, y \in X$ be such that $(x, y) \in \sigma$. Then $x, y, r_x, r_y \in r_x\sigma$ and so $(x\gamma, y\gamma) \in \sigma$. Hence $\gamma \in T_R(X, \sigma)$. Let $x \in X$. Then we have $x\alpha\gamma = x\alpha$ and so $\alpha\gamma = \alpha$. By assumption, we deduce that $\beta\gamma = \beta$. For each $x \in X$, we have $x\beta = x\beta\gamma \in X\gamma = X\alpha$. Hence $X\beta \subseteq X\alpha$. By symmetry, we have $X\alpha \subseteq X\beta$. Consequently, we have $X\alpha = X\beta$. □

The next result shows that there exists idempotent in each $\mathcal{L}^*$-class of $T_R(X, \sigma)$.

Theorem 5.3.17. *Every $\mathcal{L}^*$-class of $T_R(X, \sigma)$ contains idempotent. Consequently, the semigroup $T_R(X, \sigma)$ is left abundant.*

Proof. Let $\alpha \in T_R(X, \sigma)$. Define $\gamma \in T_R(X, \sigma)$ as in the same proof of Theorem 5.3.16. Hence $(\alpha, \gamma) \in \mathcal{L}^*$ by Theorem 5.3.16. For each $x \in X$, we have $x\gamma \in X\gamma = X\alpha$ and so $x\gamma\gamma = x\gamma$. Hence γ is idempotent, as required. □

Next, we consider the relation $\mathcal{R}^*$.

Theorem 5.3.18. *Let $\alpha, \beta \in T_R(X, \sigma)$. Then $(\alpha, \beta) \in \mathcal{R}^*$ on $T_R(X, \sigma)$ if and only if* $\ker \alpha = \ker \beta$.

Proof. The proof is similar to Theorem 5.3.13. □

Now, we investigate some conditions under which the semigroup $T_R(X, \sigma)$ is right abundant.

Lemma 5.3.19. *Let $\alpha, \beta \in T_R(X, \sigma)$. If α is regular and $(\alpha, \beta) \in \mathcal{R}^*$ on $T_R(X, \sigma)$, then β is regular.*

Proof. Suppose that α is regular and $(\alpha, \beta) \in \mathcal{R}^*$. Let $r, s \in R$ be such that $r\beta = s\beta$. Then $(r, s) \in \ker \beta$. By Theorem 5.3.18, we have $r\alpha = s\alpha$. From Theorem 3.5.1, we obtain that $\alpha|_R$ is an injection. This implies that $r = s$. Therefore $\beta|_R$ is an injection. Hence by Theorem 3.5.1, we have β is regular. □

The result of Lemma 5.3.19 is not true for the relation $\mathcal{L}^*$, for example

Example 5.3.20. Let X be the set of all positive integers, $X/\sigma = \{\{3n+1, 3n+2, 3n+3\} : n \in \mathbb{N} \cup \{0\}\}$ and $R = \{1, 4, 7, \ldots\}$. Define $\alpha, \beta \in T_R(X, \sigma)$ by

$$x\alpha = \begin{cases} x & \text{if } x \in \{1,2,3\}, \\ r_x & \text{otherwise,} \end{cases}$$

and

$$x\beta = \begin{cases} 1 & \text{if } x \in \{1,4\}, \\ 2 & \text{if } x \in \{2,3\}, \\ 3 & \text{if } x \in \{5,6\}, \\ r_x - 3 & \text{otherwise.} \end{cases}$$

Then $X\alpha = X\beta$. By Theorem 5.3.16, we have $(\alpha, \beta) \in \mathcal{L}^*$. By the definition of α, it easy to verify that α is regular by using Theorem 3.5.1. Since $1, 4 \in R$ and $1\beta = 4\beta$, $\beta|_R$ is not an injection. From Theorem 3.5.1, we get that β is not regular.

Theorem 5.3.21. *The semigroup $T_R(X, \sigma)$ is right abundant if and only if it is regular.*

The following conclusion readily follows from Theorems 5.3.21 and 5.3.17.

Theorem 5.3.22. *The semigroup $T_R(X, \sigma)$ is abundant if and only if it is regular.*

5.4 Embedding

In this section, we begin recall that

$$T(X, Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$$

where Y is a fixed nonempty subset of X. The purpose of this section, we show that the semigroup $T(X, \sigma, \rho)$ is embedded in the semigroup $T(Y, Z)$ for some sets Y, Z with $Z \subseteq Y$ and it does not necessary to isomorphic.

Theorem 5.4.1. *The semigroup $T(X, \sigma, \rho)$ can be embeddable in $T(Y, Z)$ for some sets Y, Z with $Z \subseteq Y$.*

Proof. Let $Y = \sigma$ and $Z = \rho$. Then $Z \subseteq Y$. For each $\alpha \in T(X, \sigma, \rho)$, we define $\beta_\alpha \in T(Y)$ by

$$(x, y)\beta_\alpha = (x\alpha, y\alpha) \text{ for all } (x, y) \in Y.$$

Since $\alpha \in T(X,\sigma,\rho)$, it then follows that $Y\beta_\alpha \subseteq Z$. Hence β_α is well-defined. Define $\phi : T(X,\sigma,\rho) \to T(Y,Z)$ by

$$\alpha\phi = \beta_\alpha \text{ for all } \alpha \in T(X,\sigma,\rho).$$

Let α_1, $\alpha_2 \in T(X,\sigma,\rho)$ be such that $\alpha_1\phi = \alpha_2\phi$. Then $\beta_{\alpha_1} = \beta_{\alpha_2}$. If $x \in X$, then $(x,x) \in Y$ and

$$(x\alpha_1, x\alpha_1) = (x,x)\beta_{\alpha_1} = (x,x)\beta_{\alpha_2} = (x\alpha_2, x\alpha_2).$$

Hence $x\alpha_1 = x\alpha_2$ for all $x \in X$ and so $\alpha_1 = \alpha_2$. This shows that ϕ is injective. Next, claim that $\beta_{\alpha_1\alpha_2} = \beta_{\alpha_1}\beta_{\alpha_2}$. If $(x,y) \in Y$, then

$$(x,y)\beta_{\alpha_1\alpha_2} = (x\alpha_1\alpha_2, y\alpha_1\alpha_2) = (x\alpha_1, y\alpha_1)\beta_{\alpha_2} = (x,y)\beta_{\alpha_1}\beta_{\alpha_2},$$

as required. □

Nenthein and Kemprasit [39] proved that $T(X,Y)$ is a BQ-semigroup. As a consequence of Theorem 2.1.19, the following result follows readily.

Corollary 5.4.2. *If $T(X,\sigma,\rho) \cong T(Y,Z)$ for some sets Y, Z with $Z \subseteq Y$, then $T(X,\sigma,\rho)$ is a BQ-semigroup.*

Secondly, we characterize when $T(X,\sigma,\rho)$ is a BQ-semigroup in terms of equivalence. The following lemmas are needed.

Lemma 5.4.3. [35] *$E(X,\sigma)$ is a right ideal of $T(X)$.*

From Lemmas 5.4.3 and 2.1.17 yields the following result.

Corollary 5.4.4. *If $\rho = I_X$, then $T(X,\sigma,\rho)$ is a BQ-semigroup.*

Lemma 5.4.5. *Let $\alpha \in T(X,\sigma,\rho)$. If for every $A \in X/\sigma$ there exists $B \in X/\sigma$ such that $A \cap X\alpha \subseteq B\alpha$, then $(\alpha)_b = (\alpha)_q$.*

Proof. Suppose that for each $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $A \cap X\alpha \subseteq B\alpha$. Let $\beta \in (\alpha)_q$. If $\beta = \alpha$, then done. Assume that $\beta \neq \alpha$. Then $\beta = \alpha\gamma = \lambda\alpha$ for some γ, $\lambda \in T(X,\sigma,\rho)$. Let $A \in X/\sigma$ be such that $A \cap X\alpha \neq \emptyset$. Then $A \cap X\alpha \subseteq A'$ for some fix

$A' \in X/\sigma$. For each $y \in A \cap X\alpha$, we choose and fix an element $y' \in A'$ such that $y'\alpha = y$. For fix $b_A \in A'$ and define $\mu_A : A \to X$ by

$$x\mu_A = \begin{cases} x'\lambda & \text{if } x \in X\alpha, \\ b_A\lambda & \text{otherwise.} \end{cases}$$

Let $\mu : X \to X$ be defined by

$$\mu|_A = \begin{cases} \mu_A & \text{if } A \cap X\alpha \neq \emptyset, \\ C_A & \text{otherwise} \end{cases}$$

for all $A \in X/\sigma$ and C_A is a constant map from A into X. Since X/σ is a partition of X, μ is well-defined. For each $A \in X/\sigma$ with $A \cap X\alpha \neq \emptyset$, by Lemma 2.2.5 we have that $A\mu_A \subseteq A'\lambda \subseteq C$ for some $C \in X/\rho$. It follows from Lemma 2.2.5 that $\mu \in T(X, \sigma, \rho)$. Let $x \in X$. Then $x\alpha \in A$ for some $A \in X/\sigma$. Since $\beta = \alpha\gamma = \lambda\alpha$, we deduce that

$$x\alpha\mu\alpha = x\alpha\mu_A\alpha = (x\alpha)'\lambda\alpha = (x\alpha)'\alpha\gamma = x\alpha\gamma = x\beta.$$

This means that $\beta = \alpha\mu\alpha$ and so $\beta \in (\alpha)_b$. Hence $(\alpha)_q \subseteq (\alpha)_b$. We conclude that $(\alpha)_q = (\alpha)_b$. □

As a consequence of Lemma 5.4.5, the following result follows readily.

Corollary 5.4.6. *If $\sigma = X \times X$, then $(\alpha)_b = (\alpha)_q$ for all $\alpha \in T(X, \sigma, \rho)$.*

The following theorem for which characterizes when $T(X, \sigma, \rho)$ is a BQ-semigroup.

Theorem 5.4.7. *$T(X, \sigma, \rho)$ is a BQ-semigroup if and only if one of the following statements holds.*

(1) $\sigma = X \times X$.

(2) $\sigma = I_X$.

(3) $\rho = X \times X$.

(4) $\rho = I_X$.

Proof. Suppose that σ, $\rho \notin \{X \times X, I_X\}$. Since $\rho \neq I_X$, there exist distinct elements a, $b \in X$ such that $(a,b) \in \rho$. It follows from $\rho \subseteq \sigma$ that a, $b \in A$ for some $A \in X/\sigma$. Since $\sigma \neq X \times X$, there is $B \in X/\sigma$ such that $A \neq B$. Let $c \in B$. Define α, β, $\gamma : X \to X$ by

$$x\alpha = \begin{cases} a & \text{if } x \in A, \\ b & \text{otherwise,} \end{cases}$$

$$x\beta = \begin{cases} a & \text{if } x = b, \\ b & \text{otherwise,} \end{cases}$$

and

$$x\gamma = \begin{cases} c & \text{if } x \in A, \\ b & \text{otherwise.} \end{cases}$$

By Lemma 2.2.5, it follows that α, $\gamma \in T(X,\sigma,\rho)$. Since $(a,b) \in \rho$, $\beta \in T(X,\sigma,\rho)$. We will show that $\alpha\beta = \gamma\alpha$. Let $x \in X$. If $x \in A$, then $x\alpha\beta = a\beta = b = c\alpha = x\gamma\alpha$. If $x \notin A$, then $x\alpha\beta = b\beta = a = b\alpha = x\gamma\alpha$. This means that $\alpha\beta = \gamma\alpha \in (\alpha)_q$. Suppose that $(\alpha)_q = (\alpha)_b$. Since $a\alpha\beta = a\beta = b \neq a = a\alpha$ and $a\alpha\beta = b \neq a = a\alpha = a\alpha\alpha$, it follows that $\alpha\beta \neq \alpha$ and $\alpha\beta \neq \alpha^2$. By Lemma 2.1.7, $(\alpha)_b = \{\alpha, \alpha^2\} \cup \alpha T(X,\sigma,\rho)\alpha$ and then there exists $\mu \in T(X,\sigma,\rho)$ such that $\alpha\beta = \alpha\mu\alpha$. Therefore $b = a\alpha\beta = a\alpha\mu\alpha = a\mu\alpha$ and $a = c\alpha\beta = c\alpha\mu\alpha = b\mu\alpha$. It follows that

$$a\mu \in b\alpha^{-1} \text{ and } b\mu \in a\alpha^{-1}.$$

Since $\mu \in T(X,\sigma,\rho)$ and $(a,b) \in \rho \subseteq \sigma$, we deduce that $(a\mu, b\mu) \in \rho$. Then there is $C \in X/\rho$ such that $a\mu$, $b\mu \in C$. Thus $C \cap a\alpha^{-1} \neq \emptyset$ and $C \cap b\alpha^{-1} \neq \emptyset$. Therefore a, $b \in C\alpha$. This is a contradiction. Hence $(\alpha)_q \neq (\alpha)_b$. By Proposition 2.1.16, we conclude that $T(X,\sigma,\rho)$ is not a BQ-semigroup.

Conversely, assume that the converse conditions hold. If $\sigma = I_X$ or $\rho = X \times X$, then by Theorems 2.3.1 and 3.1.1, we have $T(X,\sigma,\rho) = T(X)$ is a regular semigroup. Thus $T(X,\sigma,\rho)$ is a BQ-semigroup. If $\rho = I_X$, then $T(X,\sigma,\rho)$ is a BQ-semigroup by Corollary 5.4.4.

Suppose that $\sigma = X \times X$. Let α, $\beta \in T(X,\sigma,\rho)$. If $\alpha = \beta$, then by Corollary 5.4.6 we have $(\alpha)_b = (\alpha)_q$. Assume that $\alpha \neq \beta$. Let $\gamma \in (\{\alpha,\beta\})_q$. We consider four cases as follows.

Case 1. $\gamma \in \alpha T(X,\sigma,\rho) \cap T(X,\sigma,\rho)\alpha$. Then by Proposition 2.1.15, we have $\gamma \in (\alpha)_q$. Since $\sigma = X \times X$ by Corollary 5.4.6, $\gamma \in (\alpha)_q = (\alpha)_b$. By minimality of $(\alpha)_b$, we deduce that $\gamma \in (\{\alpha,\beta\})_b$.

Case 2. $\gamma \in \beta T(X,\sigma,\rho) \cap T(X,\sigma,\rho)\beta$. Then $\gamma \in (\beta)_q$. Since $\sigma = X \times X$ by Corollary 5.4.6, $\gamma \in (\beta)_q = (\beta)_b$. It follows that $\gamma \in (\{\alpha,\beta\})_b$.

Case 3. $\gamma \in \alpha T(X,\sigma,\rho) \cap T(X,\sigma,\rho)\beta$. Then $\gamma = \alpha\alpha' = \beta'\beta$ for some α', $\beta' \in T(X,\sigma,\rho)$. For each $y \in X\alpha$, we choose and fix $y' \in X$ such that $y = y'\alpha$. Define $\mu : X \to X$ by

$$x\mu = \begin{cases} x'\beta' & \text{if } x \in X\alpha, \\ x\beta' & \text{otherwise.} \end{cases}$$

Since $\sigma = X \times X$ and $\beta' \in T(X,\sigma,\rho)$, we have that $\mu \in T(X,\sigma,\rho)$. Let $x \in X$. Since $\gamma = \alpha\alpha' = \beta'\beta$, we deduce that

$$x\alpha\mu\beta = (x\alpha)'\beta'\beta = (x\alpha)'\alpha\alpha' = x\alpha\alpha' = x\gamma.$$

Therefore $\gamma = \alpha\mu\beta \in \alpha T(X,\sigma,\rho)\beta \subseteq (\{\alpha,\beta\})_b$.

Case 4. $\gamma \in \beta T(X,\sigma,\rho) \cap T(X,\sigma,\rho)\alpha$. Then $\gamma = \alpha'\alpha = \beta\beta'$ for some α', $\beta' \in T(X,\sigma,\rho)$. For each $y \in X\beta$, we choose and fix $y' \in X$ such that $y = y'\beta$. Define $\mu : X \to X$ by

$$x\mu = \begin{cases} x'\alpha' & \text{if } x \in X\alpha, \\ x\alpha' & \text{otherwise.} \end{cases}$$

Since $X/\sigma = \{X\}$ and $\alpha' \in T(X,\sigma,\rho)$, we deduce that $\mu \in T(X,\sigma,\rho)$. Let $x \in X$. Since $\gamma = \alpha'\alpha = \beta\beta'$, we obtain that $x\beta\mu\alpha = (x\beta)'\alpha'\alpha = (x\beta)'\beta\beta' = x\beta\beta' = x\gamma$. Then $\gamma = \beta\mu\alpha \in \beta T(X,\sigma,\rho)\alpha \subseteq (\{\alpha,\beta\})_b$.

From the above discussions, we deduce that $(\{\alpha,\beta\})_b = (\{\alpha,\beta\})_q$, it follows from Proposition 2.1.16 that $T(X,\sigma,\rho)$ is a BQ-semigroup. □

The following result follows immediately from Corollary 5.4.2 and Theorem 5.4.7.

Corollary 5.4.8. *If $T(X,\sigma,\rho) \cong T(Y,Z)$ for some sets Y, Z with $Z \subseteq Y$, then*

(1) $\sigma = X \times X$ *or*

(2) $\sigma = I_X$ *or*

(3) $\rho = X \times X$ *or*

(4) $\rho = I_X$.

Finally, we give a necessary condition for the semigroups $T(X,\sigma,\rho)$ and $T(Y,Z)$ to be isomorphic.

Theorem 5.4.9. [45] *$T(X) \cong T(Y)$ if and only if $|X| = |Y|$.*

Proposition 5.4.10. *If $\sigma = I_X$ or $\rho = X \times X$, then $T(X,\sigma,\rho) \cong T(Y,Z)$ for some sets Y, Z with $Z \subseteq Y$.*

Proof. Suppose that $\sigma = I_X$ or $\rho = X \times X$.

Case 1. $\sigma = I_X$. Then $\sigma = \rho$. By Theorem 2.3.1, we obtain $T(X,\sigma,\rho) = T(X)$. Let $Y = Z = \sigma$. Then $T(Y,Z) = T(Y)$. Since $\sigma = I_X$, we deduce that $|X| = |I_X| = |\sigma| = |Y|$. This implies that $T(X) \cong T(Y)$ by Theorem 5.4.9.

Case 2. $\rho = X \times X$. Then $\sigma = \rho$. Thus $T(X,\sigma,\rho) = T(X)$. Let $Y = Z = I_X$. Then $T(Y,Z) = T(Y)$. Since $Y = I_X$, $|X| = |I_X| = |Y|$. Hence $T(X) \cong T(Y)$. □

From the proof Theorem 5.4.1, if $Y = \sigma = X \times X$ or $Z = \rho = I_X$, then $T(X,\sigma,\rho)$ does not necessarily isomorphic to $T(Y,Z)$. As we see in the next example.

Example 5.4.11. Let $X = \{1,2,3\}$, $\sigma = X \times X$ and $X/\rho = \{\{1,2\},\{3\}\}$. Then

$$T(X,\sigma,\rho) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} \right\}.$$

If $Y = \sigma$ and $Z = \rho$, then $|Y| = 9$ and $|Z| = 5$. It easy to check that $|T(Y,Z)| \geq 10$. Hence $T(X,\sigma,\rho) \not\cong T(Y,Z)$.

Example 5.4.12. Let $X = \{1, 2, 3\}$, $X/\sigma = \{\{1, 2\}, \{3\}\}$ and $\rho = I_X$. Then

$$T(X,\sigma,\rho) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} \right\}.$$

If $Y = \sigma$ and $Z = \rho$, then $|Y| = 5$ and $|Z| = 3$. It easy to check that $|T(Y, Z)| \geq 10$. Hence $T(X, \sigma, \rho) \not\cong T(Y, Z)$.

CHAPTER VI

CONCLUSIONS

In this thesis we found that:

6.1 Generalization of semigroups of transformations preserving equivalence relations

In this section, let σ and ρ be equivalence relations on a nonempty set X with $\rho \subseteq \sigma$. We define a subsemigroup of $T(X)$ as follows:

$$T(X, \sigma, \rho) = \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \rho\}.$$

Then we found that:

Lemma 1. Let $\alpha \in T(X)$. Then $\alpha \in T(X, \sigma, \rho)$ if and only if for every $B \in X/\sigma$, there exists $B' \in X/\rho$ such that $B\alpha \subseteq B'$. Consequently, for each $A \in X/\sigma$, the set $A\alpha^{-1}$ is either $\emptyset$ or a union of some σ-classes.

Proposition 2. Let $\alpha \in T(X, \sigma, \rho)$. Then α is a right zero element of $T(X, \sigma, \rho)$ if and only if α is constant.

Corollary 3. Let σ and ρ be equivalence relations on X with $\rho \subseteq \sigma$. Then $T(X, \sigma, \rho)$ is a right zero semigroup if and only if $\sigma = X \times X$ and $\rho = I_X$.

Theorem 4. Let $\alpha \in T(X, \sigma, \rho)$. Then α is regular of $T(X, \sigma, \rho)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\rho$ such that $A \cap X\alpha \subseteq B\alpha$.

Corollary 5. Let α be a regular element of $T(X, \sigma, \rho)$. Then the following statements hold.

(1) For every $A \in X/\rho$, there exists $B \in X/\rho$ such that $A \cap X\alpha \subseteq B\alpha$.

(2) For every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $A \cap X\alpha \subseteq B\alpha$.

Theorem 6. Let $\alpha \in T(X, \sigma, \rho)$. Then α is left regular of $T(X, \sigma, \rho)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\rho$ such that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$.

Theorem 7. Let $\alpha \in T(X, \sigma, \rho)$. Then α is right regular of $T(X, \sigma, \rho)$ if and only if the following statements hold.

(1) $\alpha|_{X\alpha}$ is an injection.

(2) For every $x, y \in X\alpha$, $(x\alpha, y\alpha) \in \sigma$ implies $(x, y) \in \rho$.

Theorem 8. Let $\alpha \in T(X, \sigma, \rho)$. Then α is completely regular of $T(X, \sigma, \rho)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\rho$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A(\alpha)$.

Theorem 9. $T(X, \sigma, \rho)$ is a regular semigroup if and only if one of the following statements holds.

(1) $\sigma = I_X$.

(2) $\rho = X \times X$.

(3) $\sigma = X \times X$ and $\rho = I_X$.

Theorem 10. $T(X, \sigma, \rho)$ is a left regular semigroup if and only if one of the following statements holds.

(1) $|X| \leq 2$.

(2) $\sigma = X \times X$ and $\rho = I_X$.

Theorem 11. $T(X, \sigma, \rho)$ is a right regular semigroup if and only if one of the following statements holds.

(1) $|X| \leq 2$.

(2) $\sigma = X \times X$ and $\rho = I_X$.

Corollary 12. $T(X, \sigma, \rho)$ is a completely regular semigroup if and only if one of the following statements holds.

(1) $|X| \leq 2$.

(2) $\sigma = X \times X$.

Lemma 13. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then the following statements are equivalent.

(1) $\alpha = \lambda\beta$ for some $\lambda \in T(X, \sigma, \rho)$.

(2) For every $A \in X/\sigma$, there exists $B \in X/\rho$ such that $A\alpha \subseteq B\beta$.

(3) There exists $\sigma\rho$-admissible $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.

Theorem 14. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then the following statements are equivalent.

(1) $(\alpha, \beta) \in \mathcal{L}$.

(2) Either $\alpha = \beta$ or for every $A \in X/\sigma$, there exist $B, C \in X/\rho$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

(3) Either $\alpha = \beta$ or there exists a $(\sigma\rho)^*$-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.

Corollary 15. Let $\alpha, \beta \in T(X, \sigma, \rho)$ be such that $(\alpha, \beta) \in \mathcal{L}$. Then the following statements hold.

(1) For every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

(2) For every $A \in X/\rho$, there exist $B, C \in X/\rho$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

(3) There is a σ^*-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.

(4) There is a ρ^*-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \varphi\beta_*$.

(5) $X\alpha = X\beta$.

Lemma 16. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $\alpha = \beta\mu$ for some $\mu \in T(X, \sigma, \rho)$ if and only if the following statements hold.

(1) $\ker \beta \subseteq \ker \alpha$.

(2) For every $x, y \in X$, $(x\beta, y\beta) \in \sigma$ implies $(x\alpha, y\alpha) \in \rho$.

Theorem 17. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $(\alpha, \beta) \in \mathcal{R}$ on $T(X, \sigma, \rho)$ if and only if either $\alpha = \beta$ or the following statements hold.

(1) $\ker \beta = \ker \alpha$.

(2) For every $x, y \in X$, $(x\beta, y\beta) \in \sigma$ implies $(x\alpha, y\alpha) \in \rho$.

(3) For every $x, y \in X$, $(x\alpha, y\alpha) \in \sigma$ implies $(x\beta, y\beta) \in \rho$.

Lemma 18. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $\alpha = \beta\mu$ for some $\mu \in T(X, \sigma, \rho)$ if and only if there exists a mapping $\varphi : X\beta \to X\alpha$ satisfying

(1) $\alpha = \beta\varphi$ and

(2) for every $x, y \in X\beta$, $(x, y) \in \sigma$ implies $(x\varphi, y\varphi) \in \rho$.

Theorem 19. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $(\alpha, \beta) \in \mathcal{R}$ on $T(X, \sigma, \rho)$ if and only if either $\alpha = \beta$ or there exists a bijection $\varphi : X\beta \to X\alpha$ satisfying

(1) $\alpha = \beta\varphi$,

(2) for every $x, y \in X\beta$, $(x, y) \in \sigma$ implies $(x\varphi, y\varphi) \in \rho$ and

(3) for every $x, y \in X\alpha$, $(x, y) \in \sigma$ implies $(x\varphi^{-1}, y\varphi^{-1}) \in \rho$.

Lemma 20. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in T(X, \sigma, \rho)$ if and only if there exists $\varphi : X\beta \to X$ satisfying

(1) for every $x, y \in X\beta$, $(x, y) \in \sigma$ implies $(x\varphi, y\varphi) \in \rho$ and

(2) for every $A \in X/\sigma$, there exists $B \in X/\rho$ such that $A\alpha \subseteq (B\beta)\varphi$.

Theorem 21. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $(\alpha, \beta) \in \mathcal{J}$ on $T(X, \sigma, \rho)$ if and only if either $\alpha = \beta$ or there exist mappings $\varphi : X\beta \to X$ and $\psi : X\alpha \to X$ satisfying

(1) for every $x, y \in X\beta$, $(x, y) \in \sigma$ implies $(x\varphi, y\varphi) \in \rho$,

(2) for every $x, y \in X\alpha$, $(x, y) \in \sigma$ implies $(x\psi, y\psi) \in \rho$ and

(3) for every $A \in X/\sigma$, there exist $B, C \in X/\rho$ such that $A\alpha \subseteq (B\beta)\varphi$ and $A\beta \subseteq (C\alpha)\psi$.

Theorem 22. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then $(\alpha, \beta) \in \mathcal{D}$ on $T(X, \sigma, \rho)$ if and only if either $\alpha = \beta$ or there exist a $(\sigma\rho)^*$-admissible bijection $\varphi : \pi(\alpha) \to \pi(\beta)$ and a bijection $\psi : X\alpha \to X\beta$ satisfying

(1) for every $x, y \in X\alpha$, $(x, y) \in \sigma$ implies $(x\psi, y\psi) \in \rho$,

(2) for every $x, y \in X\beta$, $(x, y) \in \sigma$ implies $(x\psi^{-1}, y\psi^{-1}) \in \rho$ and

(3) $\alpha_*\psi = \varphi\beta_*$.

Lemma 23. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Suppose that α and β are inverses of each other. Then the following statements hold.

(1) For every $A \in \pi_\rho(\alpha)$, there exists a unique $B \in X/\rho$ such that $A \subseteq B\beta^{-1}$ and $B \subseteq A\alpha^{-1}$.

(2) For every $A \in \pi_\sigma(\alpha)$, there exists a unique $B \in X/\sigma$ such that $A \subseteq B\beta^{-1}$ and $B \subseteq A\alpha^{-1}$.

Theorem 24. Let $\alpha, \beta \in T(X, \sigma, \rho)$. Then the following statements are equivalent.

(1) α and β are inverses of each other.

(2) Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\varphi : \pi_\rho(\alpha) \to \pi_\rho(\beta)$ satisfying $U\alpha\beta = U\varphi\beta$ for all $U \in \pi_\rho(\alpha)$ and $V\beta\alpha = V\varphi^{-1}\alpha$ for all $V \in \pi_\rho(\beta)$.

(3) Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\psi : \pi_\sigma(\alpha) \to \pi_\sigma(\beta)$ satisfying $U\alpha\beta = U\psi\beta$ for all $U \in \pi_\sigma(\alpha)$ and $V\beta\alpha = V\psi^{-1}\alpha$ for all $V \in \pi_\sigma(\beta)$.

(4) For every $A \in X/\rho$ with $A' = A \cap X\alpha \neq \emptyset$, there exists $B \in X/\rho$ with $B' = B \cap X\beta \neq \emptyset$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.

(5) For every $A \in X/\sigma$ with $A' = A \cap X\alpha \neq \emptyset$, there exists $B \in X/\sigma$ with $B' = B \cap X\beta \neq \emptyset$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.

Theorem 25. $T(X, \sigma, \rho)$ is an inverse semigroup if and only if $|X| = 1$.

Theorem 26. $T(X,\sigma,\rho)$ is an E-inversive semigroup.

Theorem 27. [36] Suppose $\rho = I_X$. Let $\alpha, \beta \in T(X,\sigma,\rho)$. Then $(\alpha,\beta) \in \mathcal{L}^*$ on $T(X,\sigma,\rho)$ if and only if either of the following statements holds:

(1) α, β are not regular on X and $\overline{X\alpha} = \overline{X\beta}$.

(2) $X\alpha = X\beta$.

Theorem 28. Suppose $\rho \neq I_X$. Let $\alpha, \beta \in T(X,\sigma,\rho)$. Then $(\alpha,\beta) \in \mathcal{L}^*$ on $T(X,\sigma,\rho)$ if and only if $X\alpha = X\beta$.

Theorem 29. The semigroup $T(X,\sigma,\rho)$ is left abundant if and only if $\sigma = \rho$ or $\sigma = X \times X$.

Theorem 30. Let $\alpha, \beta \in T(X,\sigma,\rho)$. Then $(\alpha,\beta) \in \mathcal{R}^*$ on $T(X,\sigma,\rho)$ if and only if $\ker \alpha = \ker \beta$.

Theorem 31. The semigroup $T(X,\sigma,\rho)$ is right abundant if and only if $T(X,\sigma,\rho) = E(X,\sigma)$ or $T(X,\sigma,\rho) = T(X)$ or the following statements hold:

(1) $T(X,\sigma,\rho) = T(X,\sigma)$ and

(2) σ is a T-relation or 2-bounded.

Theorem 32. The semigroup $T(X,\sigma,\rho)$ is abundant if and only if one of the following statements holds.

(1) $T(X,\sigma,\rho) = E(X, X \times X)$.

(2) $T(X,\sigma,\rho) = T(X,\sigma)$ and σ is a T-relation.

(3) $T(X,\sigma,\rho) = T(X,\sigma)$ and σ is 2-bounded.

(4) $T(X,\sigma,\rho) = T(X)$.

Theorem 33. The semigroup $T(X,\sigma,\rho)$ can be embeddable in $T(Y,Z)$ for some sets Y, Z with $Z \subseteq Y$.

Corollary 34. If $T(X,\sigma,\rho) \cong T(Y,Z)$ for some sets Y, Z with $Z \subseteq Y$, then $T(X,\sigma,\rho)$ is a BQ-semigroup.

Lemma 35. Let $\alpha \in T(X, \sigma, \rho)$. If for each $A \in X/\sigma$ there exists $B \in X/\sigma$ such that $A \cap X\alpha \subseteq B\alpha$, then $(\alpha)_b = (\alpha)_q$.

Corollary 36. If $\sigma = X \times X$, then $(\alpha)_b = (\alpha)_q$ for all $\alpha \in T(X, \sigma, \rho)$.

Theorem 37. $T(X, \sigma, \rho)$ is a BQ-semigroup if and only if one of the following statements holds.

(1) $\sigma = X \times X$.

(2) $\sigma = I_X$.

(3) $\rho = X \times X$.

(4) $\rho = I_X$.

Corollary 38. If $T(X, \sigma, \rho) \cong T(Y, Z)$ for some sets Y, Z with $Z \subseteq Y$, then

(1) $\sigma = X \times X$ or

(2) $\sigma = I_X$ or

(3) $\rho = X \times X$ or

(4) $\rho = I_X$.

Proposition 39. If $\sigma = I_X$ or $\rho = X \times X$, then $T(X, \sigma, \rho) \cong T(Y, Z)$ for some sets Y, Z with $Z \subseteq Y$.

6.2 Semigroups of transformations that preserve double direction equivalence

In this section, we let σ be an equivalence on a nonempty set X. Deng, Zeng and Xu [32] introduced a subsemigroup of $T(X)$ defined by

$$T(X, \sigma^*) := \{\alpha \in T(X) : \forall x, y \in X, (x, y) \in \sigma \text{ if and only if } (x\alpha, y\alpha) \in \sigma\}.$$

Then we found that:

Theorem 40. [32] Let $\alpha \in T(X, \sigma^*)$. Then α is regular of $T(X, \sigma^*)$ if and only if for every $A \in X/\sigma$, $A \cap X\alpha \neq \emptyset$.

Theorem 41. [50] Let $\alpha \in T(X, \sigma^*)$. Then α is left regular of $T(X, \sigma^*)$ if and only if for every $P \in \pi(\alpha)$, $P \cap X\alpha \neq \emptyset$.

Theorem 42. [50] Let $\alpha \in T(X, \sigma^*)$. Then α is right regular of $T(X, \sigma^*)$ if and only if the following statements hold.

(1) $\alpha|_{X\alpha}$ is an injection.

(2) If there exists $A \in X/\sigma$ such that $A \cap X\alpha^2 = \emptyset$, then there exists an injection $\varphi : \{A \in X/\sigma : A \cap X\alpha^2 = \emptyset\} \to \{A \in X/\sigma : A \cap X\alpha = \emptyset\}$.

Theorem 43. [32] Let $\alpha \in T(X, \sigma^*)$. Then α is completely regular of $T(X, \sigma^*)$ if and only if for every $P \in \pi(\alpha)$, $|P \cap X\alpha| = 1$.

Theorem 44. [32] $T(X, \sigma^*)$ is a regular semigroup if and only if X/σ is finite.

Theorem 45. $T(X, \sigma^*)$ is a left regular semigroup if and only if X/σ is finite and σ is both a T-relation and 2-bounded.

Theorem 46. $T(X, \sigma^*)$ is a right regular semigroup if and only if X/σ is finite and σ is both a T-relation and 2-bounded.

Corollary 47. $T(X, \sigma^*)$ is a completely regular semigroup if and only if X/σ is finite and σ is both a T-relation and 2-bounded.

Theorem 48. [32] Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements are equivalent.

(1) $(\alpha, \beta) \in \mathcal{L}$ on $T(X, \sigma^*)$.

(2) $X\alpha = X\beta$.

(3) There exists a σ^*-admissible bijection $\phi : \pi(\alpha) \to \pi(\beta)$ such that $\alpha_* = \phi\beta_*$.

Theorem 49. [32] Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements are equivalent.

(1) $(\alpha, \beta) \in \mathcal{R}$ on $T(X, \sigma^*)$.

(2) $\ker \alpha = \ker \beta$ and $Z(\alpha) = Z(\beta)$.

(3) There exists $\lambda \in T(X, \sigma^*)$ such that $\lambda|_{X\alpha} : X\alpha \to X\beta$ is a bijection and $\beta = \alpha\lambda$ and there exists $\mu \in T(X, \sigma^*)$ such that $\mu|_{X\beta} : X\beta \to X\alpha$ is a bijection and $\alpha = \beta\mu$.

Theorem 50. [32] Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements are equivalent.

(1) $(\alpha, \beta) \in \mathcal{D}$ on $T(X, \sigma^*)$.

(2) $|Z(\alpha)| = |Z(\beta)|$ and there exists $\lambda \in T(X, \sigma^*)$ such that $\lambda|_{X\alpha} : X\alpha \to X\beta$ is a bijection.

Theorem 51. [32] Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements are equivalent.

(1) $(\alpha, \beta) \in \mathcal{J}$ on $T(X, \sigma^*)$.

(2) $|X\alpha| = |X\beta|$ and there exist $\lambda, \mu \in T(X, \sigma^*)$, for every $A \in X/\sigma$, $A\alpha \subseteq B\beta\lambda$ and $A\beta \subseteq C\alpha\mu$ for some $B, C \in X/\sigma$.

Corollary 52. Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements are equivalent.

(1) α and β are inverses of each other.

(2) Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\psi : X/\sigma \to X/\sigma$ satisfying $U\alpha\beta = U\psi\beta$ and $V\beta\alpha = V\psi^{-1}\alpha$ for all $U, V \in X/\sigma$.

(3) For every $A \in X/\sigma$ with $A' = A \cap X\alpha \neq \emptyset$, there exists $B \in X/\sigma$ with $B' = B \cap X\beta \neq \emptyset$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.

Theorem 53. $T(X, \sigma^*)$ is an inverse semigroup if and only if X/σ is finite and $\sigma = I_X$.

Lemma 54. Let $\alpha \in T(X, \sigma^*)$. If α is idempotent, then $A\alpha \subseteq A$ for all $A \in X/\sigma$.

Theorem 55. Let $\alpha \in T(X, \sigma^*)$. Then α is E-inversive if and only if $A \cap X\alpha \neq \emptyset$ for all $A \in X/\sigma$.

Corollary 56. Let $\alpha \in T(X, \sigma^*)$. Then $T(X, \sigma^*)$ is an E-inversive semigroup if and only if $T(X, \sigma^*)$ is a regular semigroup.

Theorem 57. [34] Let $\alpha, \beta \in T(X, \sigma^*)$. Then the following statements hold.

(1) $(\alpha, \beta) \in \mathcal{L}^*$ if and only if $X\alpha = X\beta$.

(2) $(\alpha, \beta) \in \mathcal{R}^*$ if and only if $\ker \alpha = \ker \beta$.

Theorem 58. [34] The following statements hold.

(1) The semigroup $T(X, \sigma^*)$ is left abundant if and only if it is regular.

(2) The semigroup $T(X, \sigma^*)$ is right abundant.

(3) The semigroup $T(X, \sigma^*)$ is abundant if and only if it is regular.

6.3 Semigroups of transformations preserving an equivalence relation and a cross-section

In this section, let σ be an equivalence relation on X and R a cross-section of the partition X/σ induced by σ. Araújo and Konieczny [41] introduced a subsemigroup of $T(X)$ defined by

$$T(X, \sigma, R) = \{\alpha \in T(X) : R\alpha \subseteq R \text{ and } \forall x, y \in X, (x, y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \sigma\}.$$

Then we found that:

Theorem 59. [42] Let $\alpha \in T(X, \sigma, R)$. Then α is regular of $T(X, \sigma, R)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $A \cap X\alpha \subseteq B\alpha$.

Theorem 60. Let $\alpha \in T(X, \sigma, R)$. Then α is left regular of $T(X, \sigma, R)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that for each $P \in \pi_A(\alpha)$, $x\alpha \in P$ for some $x \in B$.

Theorem 61. Let $\alpha \in T(X, \sigma, R)$. Then α is right regular of $T(X, \sigma, R)$ if and only if the following statements hold.

(1) $\alpha|_{X\alpha}$ is an injection.

(2) For every $x, y \in X\alpha$, $(x\alpha, y\alpha) \in \sigma$ implies $(x, y) \in \sigma$.

Corollary 62. Let $\alpha \in T(X, \sigma, R)$. Then α is completely regular of $T(X, \sigma, R)$ if and only if for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $|P \cap B\alpha| = |P \cap X\alpha| = 1$ for all $P \in \pi_A(\alpha)$.

Theorem 63. [42] $T(X, \sigma, R)$ is a regular semigroup if and only if σ is 2-bounded or a T-relation.

Theorem 64. $T(X, \sigma, R)$ is a left regular semigroup if and only if one of the following statements holds.

(1) $|X| \leq 2$.

(2) $|X| = 3$ and $|R| = 2$.

Theorem 65. $T(X, \sigma, R)$ is a right regular semigroup if and only if one of the following statements holds.

(1) $|X| \leq 2$.

(2) $|X| = 3$ and $|R| = 2$.

Corollary 66. $T(X, \sigma, R)$ is a completely regular semigroup if and only if one of the following statements holds.

(1) $|X| \leq 2$.

(2) $|X| = 3$ and $|R| = 2$.

Theorem 67. [42] Let $\alpha, \beta \in T(X, \sigma, R)$. Then $(\alpha, \beta) \in \mathcal{L}$ on $T(X, \sigma, R)$ if and only if for every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

Theorem 68. [42] Let $\alpha, \beta \in T(X, \sigma, R)$. Then $(\alpha, \beta) \in \mathcal{R}$ on $T(X, \sigma, R)$ if and only if $\ker \alpha = \ker \beta$.

Theorem 69. [42] Let $\alpha, \beta \in T(X, \sigma, R)$. Then $(\alpha, \beta) \in \mathcal{D}$ on $T(X, \sigma, R)$ if and only if there exist a bijection $\varphi : X\alpha \to X\beta$ satisfying

(1) $(X\alpha \cap R)\varphi \subseteq R$ and

(2) for every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $(A\alpha)\varphi \subseteq B\beta$ and $A\beta \subseteq (C\alpha)\varphi$.

Theorem 70. [42] Let $\alpha, \beta \in T(X, \sigma, R)$. Then $(\alpha, \beta) \in \mathcal{J}$ on $T(X, \sigma, R)$ if and only if there exist mappings $\varphi : X\alpha \to X\beta$ and $\psi : X\beta \to X\alpha$ satisfying

(1) $(X\alpha \cap R)\varphi \subseteq R$,

(2) for every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $A\beta \subseteq (B\alpha)\varphi \subseteq C$,

(3) $(X\alpha \cap R)\psi \subseteq R$ and

(4) for every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $A\alpha \subseteq (B\beta)\psi \subseteq C$.

Corollary 71. Let $\alpha, \beta \in T(X, \sigma, R)$. Then the following statements are equivalent.

(1) α and β are inverses of each other.

(2) Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\varphi : \pi_\sigma(\alpha) \to \pi_\sigma(\beta)$ satisfying $U\alpha\beta = U\varphi\beta$ for all $U \in \pi_\sigma(\alpha)$ and $V\beta\alpha = V\varphi^{-1}\alpha$ for all $V \in \pi_\sigma(\beta)$.

(3) For every $A \in X/\sigma$ with $A' = A \cap X\alpha \neq \emptyset$, there exists $B \in X/\sigma$ with $B' = B \cap X\beta \neq \emptyset$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.

Theorem 72. $T(X, \sigma, R)$ is an inverse semigroup if and only if $|X| \leq 2$ and $|R| = 1$.

Theorem 73. $T(X, \sigma, R)$ is an E-inversive semigroup.

Theorem 74. Let $\alpha, \beta \in T(X, \sigma, R)$. Then $(\alpha, \beta) \in \mathcal{L}^*$ on $T(X, \sigma, R)$ if and only if $X\alpha = X\beta$.

Theorem 75. Every $\mathcal{L}^*$-class of $T(X, \sigma, R)$ contains idempotent. Consequently, the semigroup $T(X, \sigma, R)$ is left abundant.

Theorem 76. Let $\alpha, \beta \in T(X, \sigma, R)$. Then $(\alpha, \beta) \in \mathcal{R}^*$ on $T(X, \sigma, R)$ if and only if $\ker \alpha = \ker \beta$.

Theorem 77. The semigroup $T(X, \sigma, R)$ is right abundant if and only if it is regular.

Theorem 78. [42] The semigroup $T(X, \sigma, R)$ is abundant if and only if it is regular.

6.4 Semigroups of transformations preserving an equivalence relation and fix a cross-section

In this section, let σ be an equivalence relation on X and R a cross-section of the partition X/σ induced by σ. We define a new subsemigroup of $T(X, \sigma)$ as follows:

$$T_R(X, \sigma) = \{\alpha \in T(X) : R\alpha = R \text{ and } \forall x, y \in X, (x, y) \in \sigma \text{ implies } (x\alpha, y\alpha) \in \sigma\}.$$

Then we found that:

Theorem 79. Let $\alpha \in T_R(X, \sigma)$. Then α is regular of $T_R(X, \sigma)$ if and only if $\alpha|_R$ is an injection.

Theorem 80. Let $\alpha \in T_R(X, \sigma)$ be such that $\alpha|_R$ is an injection. Then α is left regular of $T_R(X, \sigma)$ if and only if for every $P \in \pi(\alpha)$, $P \cap X\alpha \neq \emptyset$.

Theorem 81. Let $\alpha \in T_R(X, \sigma)$. Then α is right regular of $T_R(X, \sigma)$ if and only if $\alpha|_{X\alpha}$ is an injection.

Corollary 82. Let $\alpha \in T_R(X, \sigma)$. Then α is completely regular of $T_R(X, \sigma)$ if and only if for every $P \in \pi(\alpha)$, $|P \cap X\alpha| = 1$.

Theorem 83. $T_R(X, \sigma)$ is a regular semigroup if and only if R is finite.

Theorem 84. Let R be a finite set. Then $T_R(X, \sigma)$ is a left regular semigroup if and only if σ is a T-relation and 2-bounded.

Theorem 85. $T_R(X, \sigma)$ is a right regular semigroup if and only if R is finite and σ is a T-relation and 2-bounded.

Corollary 86. $T_R(X,\sigma)$ is a completely regular semigroup if and only if R is finite and σ is a T-relation and 2-bounded.

Lemma 87. Let $\alpha, \beta \in T_R(X,\sigma)$. Then $\alpha = \beta\mu$ for some $\mu \in T_R(X,\sigma)$ if and only if $\ker\beta \subseteq \ker\alpha$.

Theorem 88. Let $\alpha, \beta \in T_R(X,\sigma)$. Then $(\alpha,\beta) \in \mathcal{R}$ on $T_R(X,\sigma)$ if and only if $\ker\alpha = \ker\beta$.

Lemma 89. Let α and β be regular elements of $T_R(X,\sigma)$. Then the following statements are equivalent.

(1) $\alpha = \lambda\beta$ for some $\lambda \in T_R(X,\sigma)$.

(2) For every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $A\alpha \subseteq B\beta$.

(3) $X\alpha \subseteq X\beta$.

Theorem 90. Let α and β be regular elements of $T_R(X,\sigma)$. Then the following statements are equivalent.

(1) $(\alpha,\beta) \in \mathcal{L}$.

(2) For every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $A\alpha \subseteq B\beta$ and $A\beta \subseteq C\alpha$.

(3) $X\alpha = X\beta$.

Theorem 91. Let α and β be regular elements of $T_R(X,\sigma)$. Then $(\alpha,\beta) \in \mathcal{D}$ on $T_R(X,\sigma)$ if and only if there is a bijection $\varphi : X\alpha \to X\beta$ satisfying

(1) $R\varphi = R$ and

(2) for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $(A\alpha)\varphi \subseteq B\beta$.

Theorem 92. If J is an arbitrary $\mathcal{J}$-class in a semigroup $T_R(X,\sigma)$ containing a regular element, then every element of J is regular.

Lemma 93. Let α and β be regular elements of $T_R(X,\sigma)$. Then $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in T_R(X,\sigma)$ if and only if there is a mapping $\varphi : X\beta \to X\alpha$ satisfying

(1) $\varphi|_R : R \to R$ is a bijection,

(2) for every $x, y \in X\beta$, $(x,y) \in \sigma$ implies $(x\varphi, y\varphi) \in \sigma$ and

(3) for every $A \in X/\sigma$, there exists $B \in X/\sigma$ such that $A\alpha \subseteq (B\beta)\varphi$.

Theorem 94. Let α and β be regular elements of $T_R(X,\sigma)$. Then $(\alpha,\beta) \in \mathcal{J}$ on $T_R(X,\sigma)$ if and only if there exist mappings $\varphi : X\beta \to X\alpha$ and $\psi : X\alpha \to X\beta$ satisfying

(1) $\varphi|_R, \psi|_R : R \to R$ are bijections,

(2) for every $x, y \in X\beta$, $(x,y) \in \sigma$ implies $(x\varphi, y\varphi) \in \sigma$,

(3) for every $x, y \in X\alpha$, $(x,y) \in \sigma$ implies $(x\psi, y\psi) \in \sigma$ and

(4) for every $A \in X/\sigma$, there exist $B, C \in X/\sigma$ such that $A\alpha \subseteq (B\beta)\varphi$ and $A\beta \subseteq (C\alpha)\psi$.

Corollary 95. Let $\alpha, \beta \in T_R(X,\sigma)$. Then the following statements are equivalent.

(1) α and β are inverses of each other.

(2) Both $\alpha\beta$ and $\beta\alpha$ are idempotents and there is a bijection $\psi : X/\sigma \to X/\sigma$ satisfying $U\alpha\beta = U\psi\beta$ and $V\beta\alpha = V\psi^{-1}\alpha$ for all $U, V \in X/\sigma$.

(3) For every $A \in X/\sigma$ with $A' = A \cap X\alpha$, there exists $B \in X/\sigma$ with $B' = B \cap X\beta$ such that $\beta|_{A'} : A' \to B'$ and $\alpha|_{B'} : B' \to A'$ are mutually inverse bijections.

Theorem 96. $T_R(X,\sigma)$ is an inverse semigroup if and only if R is finite and σ is 2-bounded.

Theorem 97. Let $\alpha \in T_R(X,\sigma)$. Then α is E-inversive if and only if $\alpha|_R$ is an injection.

Corollary 98. $T_R(X,\sigma)$ is an E-inversive semigroup if and only if $T_R(X,\sigma)$ is a regular semigroup.

Theorem 99. Let $\alpha, \beta \in T_R(X,\sigma)$. Then $(\alpha,\beta) \in \mathcal{L}^*$ on $T_R(X,\sigma)$ if and only if $X\alpha = X\beta$.

Theorem 100. Every $\mathcal{L}^*$-class of $T_R(X,\sigma)$ contains idempotent. Consequently, the semigroup $T_R(X,\sigma)$ is left abundant.

Theorem 101. Let $\alpha, \beta \in T_R(X,\sigma)$. Then $(\alpha,\beta) \in \mathcal{R}^*$ on $T_R(X,\sigma)$ if and only if $\ker\alpha = \ker\beta$.

Lemma 102. Let $\alpha, \beta \in T_R(X, \sigma)$. If α is regular and $(\alpha, \beta) \in \mathcal{R}^*$ on $T_R(X, \sigma)$, then β is regular.

Theorem 103. The semigroup $T_R(X, \sigma)$ is right abundant if and only if $T_R(X, \sigma)$ is regular.

Theorem 104. The semigroup $T_R(X, \sigma)$ is abundant if and only if $T_R(X, \sigma)$ is regular.

6.5 Relationships between some subsemigroups of the transformation semigroups

In this section, we found that

Theorem 105. Let σ and ρ be equivalence relations on a nonempty set X with $\rho \subseteq \sigma$ and R a cross-section of the partition X/σ induced by σ. Then the following statements hold.

(1) $T(X, \sigma, \rho) = E(X, \sigma)$ if and only if $\rho = I_X$.

(2) $T(X, \sigma, \rho) = T(X, \sigma)$ if and only if $\sigma = \rho$.

(3) $T(X, \sigma, \rho) = T(X, \sigma^*)$ if and only if $\rho = X \times X$.

(4) $T(X, \sigma, \rho) = T(X, \sigma, R)$ if and only if $\sigma = I_X$.

(5) $T(X, \sigma, \rho) = T_R(X, \sigma)$ if and only if $|X| = 1$.

(6) $T(X, \sigma, \rho) = T(X)$ if and only if $\sigma = I_X$ or $\rho = X \times X$.

Corollary 106. Let σ be an equivalence relation on a nonempty set X and R a cross-section of the partition X/σ induced by σ. Then the following statements hold.

(1) $T(X, \sigma) = E(X, \sigma)$ if and only if $\sigma = I_X$.

(2) $T(X, \sigma) = T(X, \sigma^*)$ if and only if $\sigma = X \times X$.

(3) $T(X, \sigma) = T(X, \sigma, R)$ if and only if $\sigma = I_X$.

(4) $T(X, \sigma) = T_R(X, \sigma)$ if and only if $|X| = 1$.

(5) $T(X, \sigma) = T(X)$ if and only if $\sigma = I_X$ or $\sigma = X \times X$.

(6) $E(X, \sigma) = T(X, \sigma^*)$ if and only if $|X| = 1$.

(7) $E(X, \sigma) = T(X, \sigma, R)$ if and only if $\sigma = I_X$.

(8) $E(X, \sigma) = T_R(X, \sigma)$ if and only if $|X| = 1$.

(9) $E(X, \sigma) = T(X)$ if and only if $\sigma = I_X$.

Theorem 107. Let σ be an equivalence relation on a nonempty set X and R a cross-section of the partition X/σ induced by σ. Then the following statements hold.

(1) $T(X, \sigma^*) = T(X, \sigma, R)$ if and only if $|X| = 1$.

(2) $T(X, \sigma^*) = T_R(X, \sigma)$ if and only if R is finite and $\sigma = I_X$.

(3) $T(X, \sigma^*) = T(X)$ if and only if $\sigma = X \times X$.

Theorem 108. Let σ be an equivalence relation on a nonempty set X and R a cross-section of the partition X/σ induced by σ. Then the following statements hold.

(1) $T(X, \sigma, R) = T_R(X, \sigma)$ if and only if $\sigma = X \times X$.

(2) $T(X, \sigma, R) = T(X)$ if and only if $\sigma = I_X$.

(3) $T_R(X, \sigma) = T(X)$ if and only if $|X| = 1$.

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