AN APPLICATION OF THE FADDEEV-JACKIW CANONICAL ANALYSIS TO FIELD THEORY

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Title AN APPLICATION OF THE FADDEEV-JACKIW CANONICAL ANALYSIS TO FIELD THEORY

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ABSTRACT

This thesis presents the study of Faddeev-Jackiw formalism on the generalized of Proca field. Purpose of this work is to calculate degree of freedom and to check that the number of degrees of freedom is as epected. An important point is that Lagrangian density of the system is written in first order form. Procedures for this work are canonical momenta, canonical 1-form, symplectic two-form, constraint calculation and inverse of the symplectic 2-form calculation, respectively. According to flowing steps, the degrees of freedom of the system equals to three.

CHAPTER I

INTRODUCTION

1.1 Background and motivation

Scientists try to explain the Universe [1] from the beginning era to the present age. The early Universe is called inflation and at the present is called late time accelerated expansion of the Universe [2]

At the present, there are various models for discussion the late time accelerated expansion of the Universe . In particular we are interested in two models which are Generalized of Proca field [3], and cosmology in case of two barotropic fluids [4].

This work will explore some aspects of the two models. Firstly, for the generalized of Proca field is a constrained system. It is expected to have three degrees of freedom [3]. There are many kinds of methodology to study and to confirm the degrees of freedom of constrained systems for example Dirac formalism [5], and Faddeev-Jackiw formalism [6]. Starting from the first, the Dirac formalism is a method used for constrained systems [5]. We are interested in calculating the number of degrees of freedom of the system. The calculation involves reclassifying all of constraints of the system. In case of there are many constraints and complicated constraints, this process may be complicated and inconvenient. The second is the Faddeev-Jackiw formalism [6] which is also a technique used for a constrained system. If we want to find the degrees of freedom number of the system, one can use the constraints of the system to directly calculate the number. The Faddeev-jackiw formalism is more easier than the Dirac formalism [7] [8] [9].

This work, we use Faddeev-Jackiw formalism on the generalized of Proca field [10]. The reasons that we select this formalism are the generalized of Proca field [3] is a constrained system, Lagrangian density form of the system is too complicated and this formalism is applied on the Proca field [7] [8] [9].

The second model is cosmology in case of two barotropic fluids. In this model we apply by non-linear Schrödinger-type formalism. We want to connect the Ermakov-Pinney equation [11] and the Friedmann equation to study some events of the Universe in the context of cosmology, which is called non-linear Schrödinger equation (NLS) with two barotropic fluids [4].

1.2 Objectives

The aim of this work is to study the generalized of Proca field by using the Faddeev-Jackiw formalism and NLS of scalar field cosmology.

1.3 Frameworks

Scope of this work is to study the generalized of Proca field with derivative self-interactions from $\mathcal{L}_2$ to $\mathcal{L}_5$ [3] and NLS of scalar field with 2-barotropic fluids.

CHAPTER II

THEORIES AND TOOLS

2.1 Classical Mechanics

2.1.1 Lagrangian Formalism in Classical Mechanics

Considering a point particle in d dimensional spaces, action of this system is given by

$$S = \int_{t_1}^{t_2} dt L(\vec{q}, \dot{\vec{q}}, t), \tag{2.1}$$

where $L(\vec{q}, \dot{\vec{q}}, t)$ is Lagrangian of the system, that can be extended as

$$L(\vec{q}, \dot{\vec{q}}, t) \equiv L(q^1, q^2, ..., q^d, \dot{q}^1, \dot{q}^2, \dot{q}^d, ..., t). \tag{2.2}$$

where $\vec{q}$ is coordinate, and $\dot{\vec{q}}$ is velocity of the system. Equations of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0. \tag{2.3}$$

where i runs for 1, ..., d. The equation (2.3) is called Euler-Lagrange equation.

2.1.2 Hamiltonian Formalism in Classical Mechanics

Hamiltonian formalism is defined by using Legendre transformation as

$$H(\vec{p}, \vec{q}; t) \equiv \vec{p} \cdot \dot{\vec{q}} - L(\vec{q}, \dot{\vec{q}}, t). \tag{2.4}$$

In this context, p is conjugate momentum which reads

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{2.5}$$

Considering the equation (2.4), the equations of motion are written as

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \tag{2.6}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \tag{2.7}$$

Phase space function in d dimensional spaces of the system is $g(\vec{p},\vec{q},t)$, time derivative of g is in the form of

$$\begin{aligned}\frac{dg}{dt} &= \frac{\partial g}{\partial t}+\frac{\partial g}{\partial \vec{q}}\frac{d\vec{q}}{dt}+\frac{\partial g}{\partial \vec{p}}\frac{d\vec{p}}{dt} \\ &= \frac{\partial g}{\partial t}+\sum_{i=1}^{d}\Big(\frac{\partial g}{\partial q^i}\dot{q}^i+\frac{\partial g}{\partial p_i}\dot{p}_i\Big). \qquad (2.8)\end{aligned}$$

We are interested in the case in which the phase space functions are not explicit functions of time. Therefore the equation (2.8) becomes

$$\frac{dg}{dt} = \sum_{i=1}^{d}\Big(\frac{\partial g}{\partial q^i}\dot{q}^i+\frac{\partial g}{\partial p_i}\dot{p}_i\Big). \qquad (2.9)$$

Substituting the equation (2.6) and (2.7) into the equation (2.8), one can see that

$$\frac{dg}{dt}=\sum_{i=1}^{d}\Big(\frac{\partial g}{\partial q^i}\frac{\partial H}{\partial p_i}-\frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q^i}\Big). \qquad (2.10)$$

The right hand side of the equation (2.10) is Poisson bracket,

$$\{g,H\}=\sum_{i=1}^{d}\Big(\frac{\partial g}{\partial q^i}\frac{\partial H}{\partial p_i}-\frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q^i}\Big). \qquad (2.11)$$

Considering the equation (2.11), the meaning of the Poisson bracket between the the phase space function g and the Hamiltonian H is time derivative of the phase space function. Using the Poisson bracket relation to calculate the Poisson bracket between phase space function g with the phase space variable q^i, and p_i respectively, one can see that

$$\begin{aligned}\{g,q^i\} &= \sum_{j}^{d}\Big(\frac{\partial g}{\partial q^j}\frac{\partial q^i}{\partial p_j}-\frac{\partial g}{\partial p_j}\frac{\partial q^i}{\partial q^j}\Big) \\ &= -\frac{\partial g}{\partial p_i}, \qquad (2.12)\end{aligned}$$

and

$$\begin{aligned}\{g,p_j\} &= \sum_{j=1}^{d}\Big(\frac{\partial g}{\partial q^i}\frac{\partial p_j}{\partial p_i}-\frac{\partial g}{\partial p_i}\frac{\partial p_j}{\partial q^i}\Big) \\ &= \frac{\partial g}{\partial q^j}. \qquad (2.13)\end{aligned}$$

In generally, the Poisson bracket between phase space variables q and p is in the form of

$$\begin{aligned}\{q^i, p_j\} &= \sum_{k=1}^{d}\Big(\frac{\partial q^i}{\partial q^k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q^i}{\partial p_k}\frac{\partial p_j}{\partial q^k}\Big)\\ &= 1. \end{aligned} \tag{2.14}$$

2.2 Classical Field

2.2.1 Lagrangian Formalism in Classical Field

Considering a field $\phi^a(t,\vec{x})$, the action in d-dimensional space time is written as

$$S = \int d^{d+1}x\mathcal{L}(\phi^a, \partial_\mu\phi^a), \tag{2.15}$$

where $\mathcal{L}(\phi^a, \partial_\mu\phi^a)$ is Lagrangian density. In this context, a is a label, which runs for $1,...,N$, μ is space-time index, $\mu = 0,1,2...,d$, and x^μ=$(t,x^1,x^2,...,x^d)$. From the equation (2.15), the is in the form of

$$L(t) = \int d^dx\mathcal{L}(\phi^a, \partial_\mu\phi^a). \tag{2.16}$$

The equation of motion can be written as

$$\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\Big) - \frac{\partial\mathcal{L}}{\partial\phi^a} = 0. \tag{2.17}$$

The equation (2.17) is called Euler-Lagrange equation.

2.2.2 Hamiltonian Formalism in Classical Field

The Hamiltonian can be written as

$$H = \int d^d\vec{x}\mathcal{H}, \tag{2.18}$$

where $\mathcal{H}$ is Hamiltonian density . From the section 2.1.2 is stated that the Hamiltonian density is defined by using the Legendre transformation as

$$\mathcal{H} = \sum_{a=1}^{N}\Pi_a\dot{\phi}^a - \mathcal{L}, \tag{2.19}$$

where Π_a is defined as conjugate momentum of the field ϕ^a, which is calculated as

$$\Pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^a}. \tag{2.20}$$

Letting G and F are phase space function, which read in the form of

$$\begin{aligned} G &= G(\phi^a, \Pi_a, t), \\ F &= F(\phi^a, \Pi_a, t). \end{aligned} \tag{2.21}$$

Likewise, the Poisson bracket between the phase space function and the Hamiltonian in classical mechanics is shown in the equation (2.10). If we are interested in the classical field, it is written as

$$\{G(\vec{x}), H\} = \sum_{a=1}^{N} \int d^d\vec{z}\Big(\frac{\delta G(\vec{x})}{\delta \phi^a(\vec{z})}\frac{\delta H}{\delta \Pi_a(\vec{z})} - \frac{\delta G(\vec{x})}{\delta \Pi_a(\vec{z})}\frac{\delta H}{\delta \phi^a(\vec{z})}\Big). \tag{2.22}$$

The Poisson bracket between the phase space function and phase space variable are presented in the equation (2.12) and (2.13). In classical field, the Poisson bracket relations are in the form

$$\begin{aligned} \{G(\vec{x}), \phi^a(\vec{y})\} &= \sum_{b=1}^{N} \int d^d\vec{z}\Big(\frac{\delta G(\vec{x})}{\delta \phi^b(\vec{z})}\frac{\delta \phi^a(\vec{y})}{\delta \Pi_b(\vec{z})} - \frac{\delta G(\vec{x})}{\delta \Pi_b(\vec{z})}\frac{\delta \phi^a(\vec{y})}{\delta \phi^b(\vec{z})}\Big), \\ &= -\frac{\delta G(\vec{x})}{\delta \Pi_a(\vec{y})}, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} \{G(\vec{x}), \Pi_b(\vec{y})\} &= \sum_{a=1}^{N} \int d^d\vec{z}\Big(\frac{\delta G(\vec{x})}{\delta \phi^a(\vec{z})}\frac{\delta \Pi_b(\vec{y})}{\delta \Pi_a(\vec{z})} - \frac{\delta G(\vec{x})}{\delta \Pi_a(\vec{z})}\frac{\delta \Pi_b(\vec{y})}{\delta \phi^a(\vec{z})}\Big), \\ &= \frac{\delta G(\vec{x})}{\delta \phi^b(\vec{y})}, \end{aligned} \tag{2.24}$$

where a, b are labels of the phase space variables.

In generally, the Poisson bracket between the phase space function G and F is in the form of

$$\{G(\vec{x}), F(\vec{y})\} = \sum_{a=1}^{N} \int d^d\vec{z}\Big(\frac{\delta G(\vec{x})}{\delta \phi^a(\vec{z})}\frac{\delta F(\vec{y})}{\delta \Pi_a(\vec{z})} - \frac{\delta G(\vec{x})}{\delta \Pi_a(\vec{z})}\frac{\delta F(\vec{y})}{\delta \phi^a(\vec{z})}\Big). \tag{2.25}$$

2.3 Tools

In this thesis, we use various mathematics techniques. Examples of the techniques are Dirac delta function, differential form and differential equation.

2.3.1 Dirac delta function

We use the Dirac delta function for canonical momenta calculating in the process of Faddeev-Jackiw formalism. The Dirac delta function $\delta(x)$ is not a function. For example, the Dirac delta function at any point except $x = 0$ equals to zero, and the Dirac delta function at $x = 0$ equals to infinity as

$$\delta(x) = \begin{cases} 0, & \text{if} \quad x \neq 0 \\ \infty, & \text{if} \quad x = 0 \end{cases}. \tag{2.26}$$

Properties of the Dirac delta function with integration can be written as

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \tag{2.27}$$

and

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0). \tag{2.28}$$

In case of the spike moves from $x = 0$ to point a, the equation (2.26), (2.27) and (2.28) becomes

$$\delta(x-a) = \begin{cases} 0, & \text{if} \quad x \neq a \\ \infty, & \text{if} \quad x = a \end{cases} \tag{2.29}$$

$$\int_{-\infty}^{\infty} \delta(x-a)dx = 1 \tag{2.30}$$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a). \tag{2.31}$$

If we consider the Dirac delta function in d-dimensional spaces, the Dirac delta function is written as

$$\delta^d(\vec{x}) = \delta(x_1)\delta(x_2), ..., \delta(x_d). \tag{2.32}$$

In d-dimensional spaces, the equation (2.30) and (2.31) become

$$\int \delta^d(\vec{x})\delta^d(\vec{x}) = 1 \tag{2.33}$$

and

$$\int f(\vec{x})\delta^d(\vec{x}-\vec{x}_0)\delta^d(\vec{x}) = f(\vec{x}_0). \tag{2.34}$$

In this work, we interested in case of Dirac delta function integration. Especially, in the canonical 1-form momenta calculation of Faddeev-Jackiw formalism. Example calculation for this work is

$$\int \pi^0(x)\delta(x-x')d^3x = \pi^0(x'). \tag{2.35}$$

2.3.2 Differential form

In this section we use differential form by applying with wedge product, interior product and exterior derivative. Firstly, we will start to explain the differential forms.

In a coordinate basis, a differential p-form is written as

$$w_{(p)} = \frac{1}{p!}w_{i_1 \dots i_p}dx^{i_1} \wedge \cdots dx^{i_p}, \tag{2.36}$$

where $\wedge$ is wedge product. The wedge product between p-form and q-form can be written as

$$\alpha_{(p)} \wedge \beta_{(q)} = \frac{1}{p!q!}\alpha_{i_1 \dots i_p}\beta_{j_1 \dots j_q}dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_p}. \tag{2.37}$$

The property of the wedge product is similar to cross product as

$$\alpha_{(p)} \wedge \beta_{(q)} = (-1)^{pq}\beta_{(q)} \wedge \alpha_{(p)}. \tag{2.38}$$

For this work, we usually consider 1-form and 2-form in the process of Faddeev-Jackiw formalism. For example, one can see the 1-form and the 2-form in the process of the canonical 1-form and the symplectic 2-form respectively as

$$\mathcal{A} = \int d^3x\Big[\mathcal{A}_{A_0}\delta A_0(x) + \mathcal{A}_{A_i}\delta A_i(x) + \mathcal{A}_{A_\gamma}\delta\gamma(x)\Big], \quad (2.39)$$

$$\begin{aligned}\mathcal{F} &= \int d^3x\Big[-(\delta\pi_0(x)\wedge\delta A_0(x)) + (\delta\pi_i\wedge\delta A_i(x)) \\ &- (\delta\pi_0(x)\wedge\delta\gamma(x))\Big]. \end{aligned} \quad (2.40)$$

The form of the equation (2.39) is called canonical 1-form, and the equation (2.40) is called symplectic 2-form, which is the wadge product between 1-form.

We use the wedge product between 1-form in the process Faddeev-Jackiw formalism. That process is the symplectic 2-form calculation Next, we use interior product in the process of Faddeev-Jackiw formalism, that processes are contraction with the symplectic 2-form and constraint calculation of the system.the definition of the interior product is in the form of

$$i_z D^p \rightarrow D^{p-1}, \quad (2.41)$$

where i_z is interior product operator, D^p is differential p forms. Considering the equation (2.41), After using the exterior derivative operation, it can be reduce the order of the differential form of the process. Example of the interior product with the symplectic 2-form is written as

$$i_z\mathcal{F} = \int d^3x[(-z^{\pi_0}\delta A_0 + z^{A_0}\delta\pi_0) + (z^{\pi_i}\delta A_i - z^{A_i}\delta\pi_i) + (-z^{\pi_0}\delta\gamma + z^{\gamma}\delta\pi_0]. \quad (2.42)$$

Finally, we use exterior derivative in the process of Faddeev-Jackiw formalism, that processes use for calculating of the symplectic 2-form and constraint. The definition of the exterior derivative reads

$$\delta D^p \rightarrow D^{p+1}, \quad (2.43)$$

where δ is exterior derivative operator. After using the exterior derivative operation, it can be increase the order of the differential form of the process. Example of the exterior derivative in this work is written as

$$\delta\mathcal{L}_v = \int\delta \quad [\quad -\frac{1}{2}\pi_i\pi^i - \pi_i(\partial_i A_0) - \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{4}F_{ij}F^{ij}]dx$$
$$\delta\mathcal{L}_v = \int \quad [\quad -\pi_i\delta\pi^i - (\partial_i A_0)\delta\pi^i - \pi_i(\partial_i\delta A_0)$$
$$- \quad m^2 A^\mu\delta A_\mu - F^{ij}(\partial_i\delta A_j)]dx. \tag{2.44}$$

2.3.3 Linear Ordinary Differential equation and Non-Linear Ordinary Differential equation

Ordinary differential equation (ODE) is differential equation when the derivative $dy/dx, d^2y/dx^2, ...$ are total derivative namely, the solution $y = y(x)$ is only depend on one variable. The term linear is means that taking ordinary derivative is a operator ($\mathcal{L}$). An ODE is called **linear** if the operator $\mathcal{L}$ satisfy linear operator. Considering linear functions $\varphi(x)$ and $\psi(x)$, the functions are able to write down as linear combination

$$\Psi(x) = a\varphi(x) + b\psi \tag{2.45}$$

where a and b are constant coefficients. Taking operator $\mathcal{L}$ to (2.45), the operator is linear (in general) when one satisfy

$$\mathcal{L}\Psi(x) = a\mathcal{L}\varphi(x) + b\mathcal{L}\psi. \tag{2.46}$$

For example first order derivative, $\mathcal{L} = d/dx$, linear operator satisfy

$$\mathcal{L}\Psi(x) = \frac{d(a\varphi(x) + b\psi)}{dx} = a\frac{d\varphi}{dx} + b\frac{d\psi}{dx} \tag{2.47}$$

Thus, linear ODE appear as linear operator equation

$$\mathcal{L}\psi = F \tag{2.48}$$

where ψ is general solution, F is a known function, and $\mathcal{L}$ is a linear combination of derivative operating on ψ. If $F = 0$ called homogeneous, $F \neq 0$ called in-homogeneous. For example, if $F(x)$, $G(x)$, and $P(x)$ are continuous function, linear differential operator is taken to the form $\mathcal{L} = d^2/dx^2 + F(x)dy/dx + G(x)$. (Second order) Differential equation is given by

$$\frac{d^2y}{dx^2} + F(x)\frac{dy}{dx} + G(x)y = P(x), \tag{2.49}$$

where y are general solutions. In this case, we denote y'' is defined as d^2y/dx^2 and y' is also written as dy/dx. Examples of the linear and non-linear differential equation are in the form of

$$y'' + xy' + x^2y = e^{-x}, \tag{2.50}$$

$$y'' + 2y = 0, \tag{2.51}$$

Note, according to linear relation, combination of general solutions y still solution of differential equation.

If operator $\mathcal{L}'$ does not satisfy linear relation eq.(2.46). We will say that this is non-linear differential equation. For example,

$$y''y + y' + xy = 0, \tag{2.52}$$

is a non-linear differential equation because the first term $y''y$ is not follow the linear relation (2.46).

2.4 Basic cosmology

According to general relativity context, gravity can be described as curvature of spacetime influenced by matter as a source. Albert Einstein proposed a set of 10 independent equations which is known as Einstein field equation,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{2.53}$$

where μ, ν run over 0,1,2,3 in 4-dimensional spacetime. Th quantity $G_{\mu\nu}$ is called Einstein tensor. The LHS of equation (2.53) represents curvature of spacetime and RHS of (2.53) describes source (mass) of matter. The quantity $T_{\mu\nu}$ is called energy momentum tensor. Einstein tensor $G_{\mu\nu}$ in According to eq.(2.53) can be constructed from Ricci tensor $R_{\mu\nu}$,

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}, \tag{2.54}$$

where $\Gamma^r_{\mu\nu}$ is Christoffel symbol,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}), \tag{2.55}$$

Ricci scalar R,

$$R = g^{\mu\nu} R_{\mu\nu}, \tag{2.56}$$

and metric tensor, $g_{\mu\nu}$ which is dynamical variable of theory.

For large-scale, the universe can be described as being homogeneous and isotropic. This is called cosmological principle. The line element which corresponds to the cosmology of expanding universe is Friedmann-Lemaitre-Roberson-Walker (FLRW) metric,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{2.57}$$

$$= -c^2\mathrm{d}t^2 + a(t)^2\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right), \tag{2.58}$$

where $a(t)$ is a scale factor, k is $k = -1, 0, 1$ corresponding to open, flat, and closed universe respectively. In this case, non zero FLRW metric element are

$$g_{00} = -1, \quad g_{11} = \frac{a^2}{1-kr^2}, \quad g_{22} = a^2r^2, \quad g_{33} = a^2r^2\sin^2\phi. \tag{2.59}$$

Using FLRW metric tensor one can compute all non zero component of Christoffel symbols $\Gamma^\rho_{\mu\nu}$, Ricci tensor, $R_{\mu\nu}$, Ricci scalar, R. The RHS of (2.53) corresponds to energy momentum tensor which satisfy to cosmological principle is perfect fluid,

$$T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right)u_\mu u_\nu + pg_{\mu\nu} \tag{2.60}$$

where $u_\mu = (-c, 0, 0, 0)$ is a four-velocity, ρ is energy density, and p is pressure of the fluid. Hence using FLRW metric (2.58) and perfect fluid (2.60), the Einstein equation (2.53) gives

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{c^2}\left(\rho + 3\frac{p}{c^2}\right) \tag{2.61}$$

and

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{kc^2}{a^2} = -\frac{4\pi G}{2}(p - \rho c^2) \tag{2.62}$$

Substituting equation (2.61) into (2.61), we have Friedmann equation

$$H^2 = \frac{\kappa^2}{3}\rho - \frac{kc^2}{a^2}, \tag{2.63}$$

where Hubble parameter $H = \dot{a}/a$, and $\kappa^2 = 8\pi G$ is constant. Now we have two equations, (2.63), (2.61) but we have three variable, a, ρ, p. We need more equation to solve the solution. Let us consider conservation law of energy, time component of covariant derivative of energy momentum,

$$\nabla_\mu T^{\mu\nu} = 0 \tag{2.64}$$

or

$$\nabla_\mu \left[\left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu + pg^{\mu\nu}\right] = 0. \tag{2.65}$$

Time component solution is given by

$$\dot{\rho} + 3H\left(\rho + \frac{p}{c^2}\right) = 0. \tag{2.66}$$

But the fluid equation eq.(2.66) is a consequence from Friendmann equation (2.63) and acceleration equation (2.61). This is means that the fluid equation is not independent to Friemann and acceleration equation. Hence, we need to search for more equation which relates energy density and pressure. So, the related equation between ρ and p is equation of state,

$$p = w\rho c^2, \tag{2.67}$$

where w is equation of state parameter. We may classify the ingredients which are contained in the universe as follows

1. Non-relativistic matters or dust have $w = 0$ (pressureles matter)

2. Relativistic matters or radiation have $w = 1/3$

3. Dark energy driving accelerating expansion.

Now we assume the homogeneous and isotropic universe, and energy momentum which correspond to cosmological principle is perfect fluid. According to Friedmann equation (2.63), the energy density is total energy density namely

$$\rho_{\text{tot}} = \rho_m + \rho_r + \rho_{de}, \tag{2.68}$$

where $\rho_m, \rho_r, \rho_{de}$ are energy density of mater, radiation and dark energy respectively. If we add other ingredient to model for example scalar field, holographic dark energy, barotropic fluid it will appear on the Friedmann equation.

According to observation, the universe does not only expand but also acceleratly expand by observing red shift. Let us consider equation (2.61). By using equation of state (2.67), acceleration expansion gives condition

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{c^2}\left(1 + 3w\right)\rho > 0, \tag{2.69}$$

or

$$w < -\frac{1}{3} \tag{2.70}$$

It means that some mysterious matter driving the accelerated expansion which is called dark energy have equation of state parameter less than $-1/3$, or one have negative pressure. There are many dark energy model but one candidate of dark energy is cosmological constant, Λ, which proposed by Albert Einstein. The equation of state parameter of cosmological constant is -1. In addition, interpretation of cosmological constant is vacuum energy. But, there is inconsistency between energy density of cosmological constant and quantum field vacuum energy density. This problems is known as cosmological constant problem. Now, the origin of dark energy is still unknown. There are many theories in order to solve the problem, one of the theories is modified gravity theories. As part of this thesis, we are interested in scalar-tensor theory which represent scalar field as a source of dark energy. The simplest scalar-tensor theory is called quintessence model.

Moreover, we are interested in two-barotropic fluid as a ingredient of the model. Finally, we are able to rearrange Friedmann and acceleration equation as non-linear schrödinger type formalism, the exact solutions are studied.

CHAPTER III

DIRAC & FADDEEV-JACKIW FORMALISM AND APPLICATION ON EM & PROCA FIELD

3.1 Dirac Formalism

Dirac formalism [12] is a technique use for studying constrained system. The aim of Dirac formalism is to find types of constraints and numbers of the degrees of freedom. If the system is a constrained system, there are at lease one constraint, that is primary constraint. In this context, constraints mean relation between phase space variables. Especially, the constrained equations have to write with out time derivative terms of the phase space variables of the system. In case of the first constraint we met, the primary constraint [13], it comes from the conjugate momentum calculation of the system, which is a phase space relation form. Next step is to check that there are others constraints by using Poisson bracket between the primary constraint and the Hamiltonian of the system. That process is time evolution of the primary constraint.

If the Poisson bracket between the primary constraint and the Hamiltonian equals to zero, it presents that the time evolution of the primary constraint remains on the constraint surface. On the other hand, if the result is non-zero, that result is defined as a secondary constraint.

To continue the process of finding other constraints by using the time evolution of the constraint, if it is non-zero, it presents next order of the new constraint in the name of tertiary constraint, quaternary constraint, quinary constraint, etc.

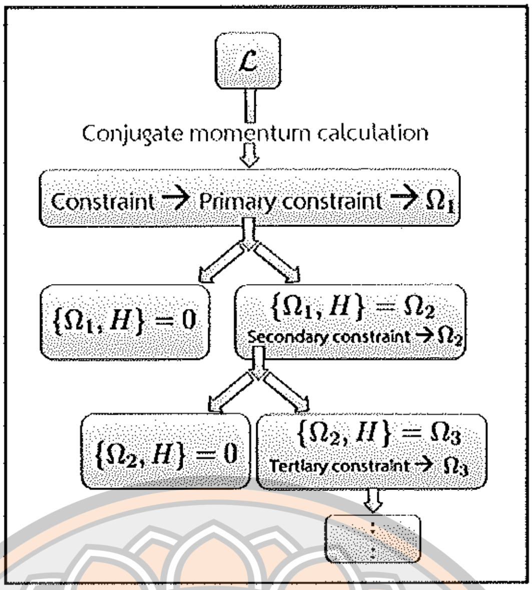

Figure 1 Dirac process

If it equals to zero, it presents that the constraint is on the constraint surface. After finishing the time evolution calculation of the constraints, next step is constraint reclassification by using Poisson bracket between all of constraints. If the Poisson bracket between all of the constraints equal to zero, it shows that all of the constraints are first-class constraints. On the other hand, if the Poisson bracket between the any constraints is non-zero, it means that any constraints are second-class constraints. After finding the constraints of the system, and reclassifying all of constraints, Finally to calculate number degrees of the freedom of the system, which is a purpose of the Dirac formalism by using the formula [10] as

$$DOF = \frac{n_{PS} - 2n_1 - 2n_2}{2}, \tag{3.1}$$

where, n_{PS} is number of the phase space variables, n_1 is number of the first-class constraints and n_2 is number of the second-class constraints.

Dirac Formalism [12] is a well known technique used for constrained systems, the Lagrangian density is in the form of

$$\mathcal{L}\big(\phi_a(x), \partial_\mu \phi_a(x)\big), \tag{3.2}$$

where $a = 1, 2, \cdots, N$. If the determinant of the Hessian

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}_a \partial \dot{\phi}_b} \tag{3.3}$$

is zero, then the system is a constrained system.

Starting from the Lagrangian density of the constrained system $\mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x))$, one defines conjugate momenta as

$$\pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}. \tag{3.4}$$

Another way to check a constrained system or an unconstrained system is proved by conjugate momenta of the system. From equation (3.4), If one can write $\dot{\phi}_a$ in terms of ϕ_a and π^a, it presents that this system is not a constrained system. On the other hand, if the system is a constrained system, one can get primary constraint of the system in terms of ϕ_a and π^a. The constrained systems not consist of dot terms.

Next step is the procedure to check other constraints by calculating Poisson braket. Firstly, to find Hamiltonian density of the system $\mathcal{H}$ by using Legendre transformation.

3.2 Faddeev-Jackiw Formalism

Faddeev-Jackiw formalism is a technique applied for constrained systems [6]. A purpose of Faddeev-Jackiw formalism is to calculate number of the degrees of freedom. The beginning of Faddeev-Jackiw formalism is to find constraint from the conjugate momentum calculation of the system to create first order form of Lagrangian density. The first order form of the Lagrangian density consists of terms with no more than first order derivative in time and constrained terms, each of which is a multiplication between Lagrange multiplier and constraint. After getting first order form of the Lagrangian density, next steps to get constraint of the system are canonical momenta calculation, canonical 1-form, symplectic 2-form, and zero-mode calculation, respectively.

For canonical momenta, they come from partial differential of the first order form of the Lagrangian density with respect to time derivative of semplectic variables, where the variables consist of the phase space variables and the Lagrange multipliers. After that, one then automatically obtain the canonical 1-form. In case of the symplectic 2-form, it comes from taking exterior derivative with the canonical 1-form. Next step is to find the zero-mode by using the interior product with the symplectic 2-form. If the zero mode equals to zero, there is no more constraint. On the other hand, if the zero exists, next step is to find the remain constraints. After that, adding multiplication between Lagrange multiplier and the new constraint in to the first order form of the Lagrangian density, one can get the new first of order form of the Lagrangian density. Next step is to repeat all of process from canonical momenta calculation until vanishing of zero mode calculation.

Recall that the goal of this work is to get the number of degrees of freedom, so to reach that purpose we have to find inverse of symplectic 2-from, which is the Dirac bracket of the system. As a result, using the Dirac bracket and the formula from the section ?? to calculate, one can get number of the degrees of freedom.

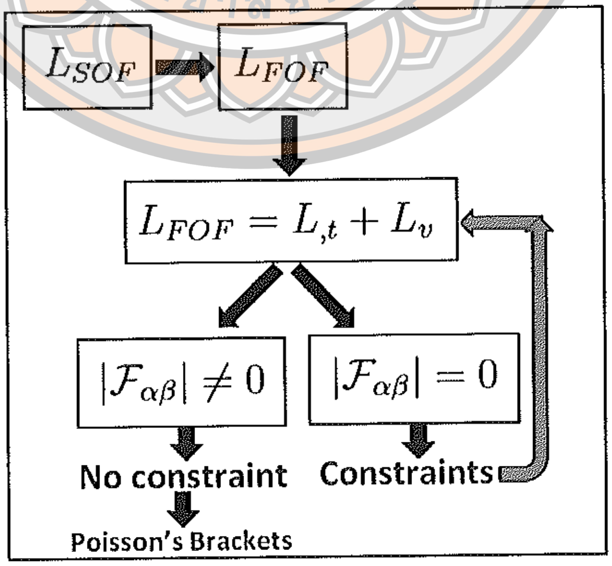

Figure 2 FJ process

The figure 2 shows the steps of the Faddeev-Jackiw formalism from the first to the final, which is first order form of the Lagrangian creation to Dirac bracket calculation.

Technical process of the Fadeev-Jackiw formalism is to create the first order form of the Lagrangian density ($\mathcal{L}_{FOF}$), which is written as

$$\mathcal{L}_{FOF} = \pi^a \dot{\phi}_a - \mathcal{H} + \gamma_k \Omega_k. \tag{3.5}$$

Considering the 1^{st} term of the right hand side of the equation (3.5), π^a is conjugate momentum which is similar to the equation (3.4). If we want to get constraint, one can calculate by using the conjugate momentum. Next parameter, $\dot{\phi}_a$, are time derivative of field and a runs for $1, 2, ..., N$. The 2^{nd} term is Hamiltonian density ($\mathcal{H}$), which can be written as

$$\mathcal{H}(\phi_a, \pi^a) = \pi^a \dot{\phi}_a - \mathcal{L}_{SOF}. \tag{3.6}$$

Recall that, Hamiltonian density in the equation (3.6) comes Legendre transformation. The last term of the equation (3.5) is multiplication between Lagrange multiplier, γ_k, and constraint Ω_k, where $k = 1, 2, ..., N$. This term is called the constraint term. After sustituting all parameters into the equation (3.5), one can get the first order form of the Lagrangian, $\mathcal{L}_{FOF}$.

Next process is canonical 1-form calculation, that reads

$$\mathcal{A} = \int \mathrm{d}^n x \Big[\mathcal{A}_{\xi^I} \delta \xi^I(x) \Big]. \tag{3.7}$$

Paramiter $\mathcal{A}_{\xi^I}$ is canonical momenta, which is calculated by

$$\mathcal{A}_{\xi^I} = \frac{\partial \mathcal{L}_{FOF}}{\partial \dot{\xi}^I}, \tag{3.8}$$

where ξ^I are symplectic variables; $\xi^I = (\phi_a, \pi^a, \gamma_k)$, and $I = 1, 2, ..., 2N + k$. Taking exterior derivative with the canonical 1-form in the equation (3.7), one then obtain symplectic 2-form as

$$\mathcal{F} = \delta \mathcal{A}. \tag{3.9}$$

We want to find zero-mode (z^{ξ^I}), so we have to take interior product (i_z) with the symplectic 2-form and set it to the zero, one can see that

$$i_z \mathcal{F} = 0. \tag{3.10}$$

From the equation (3.10), one then automatically obtain zero-mode (z^{ξ^I}), which means eigen vector of the zero eigen value. if there is no zero-mode, it presents that this system is no more constraint, one can continue to the final process of the faddeev-jackiw formailsm. On the other hand, if the zero-mode exist, it present that this system has more constraints. To calculate others constraints, firstly we have to define $\mathcal{L}_v$. Considering the equation (3.5), $\mathcal{L}_v$ is the first order form of Lagrangian density without time-derivative terms and constraint terms. After that, using the exterior derivative with $\mathcal{L}_v$ and then taking the interior product with the $\mathcal{L}_v$ term, one can see relation that

$$\Omega_k = i_z(\delta \mathcal{L}_v). \tag{3.11}$$

The equation (3.11) shows constraint of the system. After that, we have to add the new term into the the first order form of the the system. Recall that, the new term is multiplication between Lagrange multiplier and the constraint from the equation (3.11). Therefore, the new first order form of the Lagrangian density is in the form of

$$\mathcal{L}_{FOF}(new) = \mathcal{L}_{FOF} + \gamma_2 \Omega_2, \tag{3.12}$$

where γ_2 is the Lagrange multiplier of the constraint Ω_2. To repeat the process of the Faddeev-Jackiw formalism from the canonical 1-form calculation to the zero-mode calculation, if the zero-mode is non-zero we have to calculate the new constraint to create the new Lagrangian density again. On the other hand, if the zero-mode equals to zero, it presents that there is no more constraint.

Next step is to find number of the degrees of freedom by starting from the inverse of the symplectic 2-form calculation ($\mathcal{F}^{-1}$). Recall that, the $\mathcal{F}^{-1}$ is the Dirac's bracket which we want to use for calculation the number of the degrees of freedom of the system, that is the aim of this work.

3.3 Application on Electromagnetic Field

Motivation of studying in this section is the first term of Lagrangian density ($\mathcal{L}$) of the generalised of the Proca field is the electromagnetic field (EM). We will start from basic ideas of the electromagnetic field. After that we will apply the Dirac formalism on the EM field. The last topic of this section is application on the EM field by using the Faddeev-Jackiw formalism.

3.3.1 Electromagnetic Field

This part starts with electromagnetic four-potential, component form of field strength tensor, equation of motion of electromagnetic field, and Maxwell's equations respectively.

Electromagnetic four-potential

Starting from the anti-symmetric field strength tensor $F_{\mu\nu}$ is written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.13}$$

where A_μ are electric four-potential, $A_\mu = (\phi, \vec{A})$, $\mu = 0, 1, 2, 3$, A_0 is an electric scalar potential, $A_i = (\vec{A})_i$ are magnetic vecter potential. The electric field $\vec{E}$ and magnetic field $\vec{B}$ that are associated with the four-potential as

$$\vec{E} = -\vec{\nabla}\phi, \tag{3.14}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}, \tag{3.15}$$

where component form of $\vec{B}$ is written as

$$B_i = \epsilon_{ijk}\partial_j A_k. \tag{3.16}$$

Component form of the field strength tensor

Considering equation (3.13), because $F_{\mu\nu}$ is an anti-symmetric matrix. Therefore, one can see that

$$F_{\mu\nu} = -F_{\nu\mu} \tag{3.17}$$

Using the anti-symmetric property to calculate all components of the matrix $F_{\mu\nu}$

1. In case of $\mu = \nu = 0$, one can see that

$$\begin{aligned} F_{00} &= -F_{00} \\ 2F_{00} &= 0 \\ F_{00} &= 0. \end{aligned} \tag{3.18}$$

2. Likewise, In case of $\mu = \nu = 1$, $\mu = \nu = 2$, and $\mu = \nu = 3$, one can see that

$$F_{11} = F_{22} = F_{33} = 0. \tag{3.19}$$

3. Considering $F_{0j} = E_j$,where $j = 1, 2, 3$, one can see that F_{01}, F_{02}, and $F_{03} = E_1, E_2$, and E_3 respectively.

4. On the other hand, $F_{i0} = -E_i$,where $i = 1, 2, 3$, one can see that F_{10}, F_{20}, and $F_{30} = -E_1, -E_2$, and $-E_3$ respectively.

5. Considering $F_{ij} = \epsilon_{ijk}B_k$,where $i, j, k = 1, 2, 3$, one can see that

$$
\begin{aligned}
F_{12} &= \epsilon_{12K}B_k, \\
F_{12} &= \cancelto{0}{\epsilon_{121}B_1} + \cancelto{0}{\epsilon_{122}B_2} + \epsilon_{123}B_3, \\
F_{12} &= B_3, && (3.20) \\
F_{13} &= \epsilon_{13K}B_k, \\
F_{13} &= \cancelto{0}{\epsilon_{131}B_1} + \epsilon_{132}B_2 + \cancelto{0}{\epsilon_{133}B_3}, \\
F_{13} &= -B_2, && (3.21) \\
F_{23} &= \epsilon_{23K}B_k, \\
F_{23} &= \epsilon_{231}B_1 + \cancelto{0}{\epsilon_{232}B_2} + \cancelto{0}{\epsilon_{233}B_3}, \\
F_{23} &= B_1. && (3.22)
\end{aligned}
$$

6. Using the anti-symmetric property, equation (3.20),(3.21) and (3.22) become

$$F_{21} = -B_3, \tag{3.23}$$

$$F_{31} = B_2, \tag{3.24}$$

$$F_{32} = -B_1. \tag{3.25}$$

To write 4×4 metrics, we can see that

$$F_{\mu\nu} = \begin{bmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{bmatrix} = \begin{bmatrix} F_{00} & F_{0j} \\ F_{i0} & F_{ij3\times 3} \end{bmatrix}. \tag{3.26}$$

Using the equation (3.20) to the equation (3.25), the equation (3.26) becomes

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{bmatrix}, \tag{3.27}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix} \tag{3.28}$$

Considering the equation (3.28), it presents that the inverse matrix of the matrix $F_{\mu\nu}$.

Equation of motion of electromagnetic field

To find the Maxwell's equations, firstly one have to calculate equation of the electromagnetic field. Starting point of calculating is electromagnetic action in the form of

$$S = -\int \frac{1}{4} F^{\mu\nu} F_{\mu\nu} d^4x \tag{3.29}$$

Using variation with the equation (3.29), we see

$$\begin{aligned} \delta S &= -\frac{1}{4}\int d^4x \delta[F^{\mu\nu}F_{\mu\nu}] = 0 \\ &= -\frac{1}{4}\int d^4x [F^{\mu\nu}\delta(F_{\mu\nu})] + [F_{\mu\nu}\delta(\eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta})] = 0 \\ &= -\frac{1}{4}\int d^4x [F^{\mu\nu}\delta(F_{\mu\nu})] + F^{\alpha\beta}\delta(F_{\alpha\beta})] = 0 \\ \delta S &= -\frac{1}{2}\int d^4x [F^{\mu\nu}\delta(F_{\mu\nu})] = 0 \end{aligned} \tag{3.30}$$

Substituting $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ into equation (3.30), we then obtain

$$\begin{aligned}
\delta S &= -\frac{1}{2}\int d^4x[F^{\mu\nu}\delta(\partial_\mu A_\nu - \partial_\nu A_\mu)] = 0 \\
&= -\frac{1}{2}\int d^4x[F^{\mu\nu}\partial_\mu(\delta A_\nu) - F^{\mu\nu}\partial_\nu(\delta A_\mu)] = 0 \\
&= -\frac{1}{2}\int d^4x[F^{\mu\nu}\partial_\mu(\delta A_\nu) - F^{\nu\mu}\partial_\mu(\delta A_\nu)] = 0 \\
&= -\frac{1}{2}\int d^4x[F^{\mu\nu}\partial_\mu(\delta A_\nu) + F^{\mu\nu}\partial_\mu(\delta A_\nu)] = 0 \\
&= -\int d^4x[F^{\mu\nu}\partial_\mu(\delta A_\nu)] = 0 \\
&= -\cancelto{0}{\int d^4x\partial_\mu[F^{\mu\nu}(\delta A_\nu)]} + \int d^4x[(\delta A_\nu)\partial_\mu(F^{\mu\nu})] = 0 \\
\delta S &= \int d^4x[(\delta A_\nu)\partial_\mu(F^{\mu\nu})] = 0 \qquad (3.31)
\end{aligned}$$

Considering equation (3.31),the equation of motion of the electromagnetic field can be written as

$$\partial_\mu F^{\mu\nu} = 0. \qquad (3.32)$$

Maxwell's equations

As we get the equation of motion of the electromagnetic field which is shown in (3.32). In order to get Maxwell's equations, one can calculate these equations by starting from the equation of motion

1. In case of $\nu = 0$, one can see that

$$\begin{aligned}
\partial_\mu F^{\mu 0} &= 0 \\
\partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} &= 0 \qquad (3.33)
\end{aligned}$$

In this work, using $\eta_{\mu\nu} = (-1, 1, 1, 1)$, equation (3.33) becomes

$$\begin{aligned}
\cancelto{0}{\partial_0 F^{00}} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} &= 0 \\
-\partial_1 F_{10} - \partial_2 F_{20} - \partial_3 F_{30} &= 0 \qquad (3.34)
\end{aligned}$$

Considering 4×4 matrices from equation (3.28) and substituting value of each component into equation (3.34), one can see that

$$\begin{aligned} -\partial_1(-E_1) - \partial_2(-E_2) - \partial_3(-E_3) &= 0 \\ \vec{\nabla} \cdot \vec{E} &= 0. \end{aligned} \tag{3.35}$$

2. In case of $\nu = 1$, one can see that

$$\begin{aligned} \partial_\mu F^{\mu 1} &= 0 \\ \partial_0 F^{01} + \cancelto{0}{\partial_1 F^{11}} + \partial_2 F^{21} + \partial_3 F^{31} &= 0 \\ \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} &= 0 \\ \partial_0(-E_1) + \partial_2(-B_3) + \partial_3(B_2) &= 0 \\ -\partial_0 E_1 - (\partial_2 B_3 - \partial_3 B_2) &= 0. \end{aligned} \tag{3.36}$$

3. In case of $\nu = 2$, one can see that

$$\begin{aligned} \partial_\mu F^{\mu 2} &= 0 \\ \partial_0 F^{02} + \partial_1 F^{12} + \cancelto{0}{\partial_2 F^{22}} + \partial_3 F^{32} &= 0. \end{aligned} \tag{3.37}$$

4. Likewise, in case of $\nu = 2$, one can see that

$$\begin{aligned} \partial_\mu F^{\mu 3} &= 0 \\ \partial_0 F^{03} + \partial_1 F^{13} + \partial_2 F^{23} + \cancelto{0}{\partial_3 F^{33}} &= 0. \end{aligned} \tag{3.38}$$

From equation (3.36),(3.37) and (3.38), can be written in the form of

$$\partial_0 \vec{E} - \vec{\nabla} \times \vec{B} = 0. \tag{3.39}$$

3.3.2 Dirac formalism on Electromagnetic Field

Previous topic is basic information of the electromagnetic field. It is well known that the electromagnetic field has 2 degrees of freedom. Dirac formalism is an approach to prove that. At the beginning, Lagrangian density of the electromagnetic field is

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.40}$$

To check that the electromagnetic field is a constrained system, one can see via conjugate momentum calculation. In case of electromagnetic field, the conjugate momentum is in the form of

$$\pi^{\rho} = \frac{\partial \mathcal{L}_{EM}}{\partial \dot{A}_{\rho}}, \tag{3.41}$$

substituting the equation (3.40) into the equation (3.41), one can see

$$\begin{aligned} \pi^{\rho} &= -\frac{1}{4}\frac{\partial (F_{\mu\nu}F^{\mu\nu})}{\partial \dot{A}_{\rho}}, \\ &= F^{\rho 0}. \end{aligned} \tag{3.42}$$

Considering the equation (3.42), in case of $\rho = 0$ and $\rho = i$, one can see

$$\pi^{0} = F^{00} = 0. \tag{3.43}$$

$$\pi^{i} = F^{i0} = -F_{i0}. \tag{3.44}$$

Because of $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, therefore the equation (3.44) becomes

$$\dot{A}_{i} = \pi_{i} + \partial_{i}A_{0}. \tag{3.45}$$

As the results, the equation (3.45) is written in the form of canonical variables, π_i and A_0. Likewise, the equation (3.43) can not be shown as the equation (3.45). Therefore, one can conclude that the equation (3.43) is a constrained equation and π^0 is a primary constraint of the electromagnetic field, which follow by Dirac formalism.

Next step of the Dirac formalism is to find secondary constraint by using Poisson bracket between the primary constraint from the equation (3.43) and Hamiltonian density of the electromagnetic field. At the beginning of this calculation is to find the Hamiltonian density of the field as

$$\mathcal{H}(A_{\rho}, \pi^{\rho}) = \pi^{\rho}\dot{A}_{\rho} - \mathcal{L}_{EM}, \tag{3.46}$$

where ρ runs for 0 and $i = 1, 2, 3$. Substituting the equation (3.40) into the equation (3.46), one can see

$$\begin{aligned}\mathcal{H}(A_\rho, \pi^\rho) &= \pi^0 \dot{A}_0 + \pi^i \dot{A}_i + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \\ &= \pi^0 \dot{A}_0 + \pi^i \dot{A}_i + \frac{1}{4}\left(F_{00}F^{00} + F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right). \end{aligned} \tag{3.47}$$

$F_{\mu\nu}$ is an anti-symmetric matrix, therefore the equation (3.47) becomes

$$\begin{aligned}\mathcal{H}(A_\rho, \pi^\rho) &= \pi^0 \dot{A}_0 + \pi^i \dot{A}_i + \frac{1}{4}\left(F_{00}F^{00} + (-F_{i0})(-F^{i0}) + F_{i0}F^{i0} + F_{ij}F^{ij}\right), \\ &= \pi^0 \dot{A}_0 + \pi^i \dot{A}_i + \frac{1}{4}\left(F_{00}F^{00} + 2F_{i0}F^{i0} + F_{ij}F^{ij}\right). \end{aligned} \tag{3.48}$$

Substituting the equation (3.43) and (3.45) into the equation (3.48), Hamiltonian density of the electromagnetic field can be written as

$$\begin{aligned}\mathcal{H}(A_\rho, \pi^\rho) &= \pi^i[\pi^i + \partial_i A_0] - \frac{1}{2}[\pi_i][\pi^i] + \frac{1}{4} F_{ij}F^{ij}, \\ &= \frac{1}{2}\pi_i \pi^i + \pi^i(\partial_i A_0) + \frac{1}{4} F_{ij}F^{ij}. \end{aligned} \tag{3.49}$$

The equation (3.49) is the Hamiltonian density equation of the electromagnetic field. Next step is Poisson bracket between the primary constraint and the Hamiltonian of the system as

$$\{\pi^0(\vec{x}, t), H(t)\} = \int d^3y \Big[\frac{\partial \pi^0(\vec{x})}{\partial A_\rho(\vec{y})}\frac{\partial H(t)}{\partial \pi^\rho(\vec{y})} - \frac{\partial \pi^0(\vec{x})}{\partial \pi^\rho(\vec{y})}\frac{\partial \mathcal{H}(t)}{\partial A_\rho(\vec{y})}\Big]. \tag{3.50}$$

To consider the equation (3.50), the 1^{st} term vanishes and the 2^{nd} term is in the form of

$$\frac{\partial \pi^0(\vec{x})}{\partial \pi^\rho(\vec{y})}\frac{\partial H(t)}{\partial A_\rho(\vec{y})} = \delta^{(3)}(\vec{x} - \vec{y})\Big(- \partial_i \pi_i(\vec{y})\Big). \tag{3.51}$$

Substituting the equation (3.51) into the equation (3.50), one can see that

$$\begin{aligned}\{\pi^0(\vec{x}, t), H(t)\} &= \int d^3y \Big[0 - \delta^{(3)}(\vec{x} - \vec{y})\Big(- \partial_i \pi_i(\vec{y})\Big)\Big], \\ &= \partial_i \pi_i(\vec{x}). \end{aligned} \tag{3.52}$$

For the electromagnetic field, there are one primary constraint and one secondary constraint which are $\pi^0(\vec{x})$ and $\partial_i \pi_i(\vec{x})$ respectively. Next process is reclassification the constraints by using the Poisson bracket between the primary constraint and the secondary constraint as

$$\{\pi^0(\vec{x},t), \partial_i\pi_i(\vec{x})\} = \int d^3y \Big[\frac{\partial\pi^0(\vec{x})}{\partial A_\rho(\vec{y})}\frac{\partial_i\pi_i(\vec{x})}{\partial\pi^\rho(\vec{y})} - \frac{\partial\pi^0(\vec{x})}{\partial\pi^\rho(\vec{y})}\frac{\partial_i\pi_i(\vec{x})}{\partial A_\rho(\vec{y})}\Big] = 0. \tag{3.53}$$

From the equation (3.53), the result equals to zero, which presents that both $\pi^0(\vec{x})$ and $\partial_i\pi_i(\vec{x})$ are the first-class constraints of the electromagnetic field. On the contrary, if the Poisson bracket exists, which means that both $\pi^0(\vec{x})$ and $\partial_i\pi_i(\vec{x})$ are the second-class constraints.

In conclusion of this part, Dirac formalism on the electromagnetic field in 4-dimensional space time, there is a primary constraint, therefore the electromagnetic field is a constrained system. After that, using the Poisson bracket between the primary constraint and Hamiltonian density of the field, there is a secondary constraint. Results of Poisson bracket between all of constraints show that there are two 1-class constraints. Finally, the formula for finding number ofthe degrees of freedom is

$$D.O.F = \frac{1}{2}\Big[N - 2(F) - S\Big], \tag{3.54}$$

where N represents number of phase space, F means number of first-class constraint, and S presents number of second-class constraint. As a result, using the equation (3.54), one can see that number of the degrees of freedom of the electromagnetic field equals to 2, that we expected.

3.3.3 Faddeev-Jackiw formalism on Electromagnetic Field

The second order form of the Lagrangian is written as

$$\mathcal{L}_{SOF} = \frac{-1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.55}$$

The first order form of (3.55) is written as

$$\mathcal{L}_{FOF} = \dot{A}_i\pi^i - \frac{1}{2}\pi_i\pi^i - \frac{1}{4}F_{ij}F^{ij} + A_0(\partial_i\pi^i). \tag{3.56}$$

The canonical one form of the system are

$$\mathcal{A}_{A_\nu(x)} = \delta_i^\nu \pi^i(x) + \delta_0^\nu \partial_i \pi^i(x), \tag{3.57}$$

$$\mathcal{A}_{\pi^j(x)} = 0. \tag{3.58}$$

The symplectic two-form is

$$\mathcal{F}_{\alpha\beta} = \begin{pmatrix} 0 & -(\delta_j^\mu + \delta_0^\mu \partial_j) \\ (\delta_i^\nu + \delta_0^\nu \partial_i') & 0 \end{pmatrix} \delta(x - x'). \tag{3.59}$$

$$\Omega_I = Z_I^\alpha \frac{\partial L_v}{\partial \xi^\alpha}, \tag{3.60}$$

Since $|\mathcal{F}_{\alpha\beta}| = 0$, next step, we have to find the constraints Ω_I of the system by using

$$\Omega_I = Z_I^\alpha \frac{\partial L_v}{\partial \xi^\alpha}, \tag{3.61}$$

where Z_I^α are the zero modes and $L_v(x) = -\frac{1}{2}\pi_i(x)\pi^i(x) - \frac{1}{4}F_{ij}(x)F^{ij}(x))$, which gives

$$\Omega = -Z_{A_0(x)} \partial_j [\partial_i F^{ij}(x)], \tag{3.62}$$

so the constraint is

$$-\partial_j [\partial_i F^{ij}(x)] = 0. \tag{3.63}$$

$$-\partial_j [\partial_i F^{ij}(x)] = 0. \tag{3.64}$$

Because the left hand side equals to zero so, by using gauge fixing the constraint of the system is in fact

$$\partial_i [A^i(x)] = 0. \tag{3.65}$$

The constraint of the system is $\partial_i [A^i(x)]$. The new first order form of the Lagrangian is written as

$$\mathcal{L}_{FOF} = \dot{A}_i \pi^i - \frac{1}{2}\pi_i \pi^i - \frac{1}{4}F_{ij}F^{ij} + A_0(\partial_1 \pi^i) + \left(\partial_i A^i\right). \tag{3.66}$$

The canonical 1-form are

$$\mathcal{A}_{A_\nu(x)} = \delta_i^\nu \pi^i(x) + \delta_0^\nu \partial_i \pi^i(x), \tag{3.67}$$

$$\mathcal{A}_{\pi^j(x)} = 0, \tag{3.68}$$

$$\mathcal{A}_{(x)} = \partial_j A^j(x). \tag{3.69}$$

The symplectic two-form is expressed as

$$\mathcal{F}_{\alpha\beta} = \begin{pmatrix} 0 & -(\delta_j^\mu + \delta_0^\mu \partial_j) & (\partial'^\mu - \delta_0^\mu \partial'^0) \\ (\delta_i^\nu + \delta_0^\nu \partial_i') & 0 & 0 \\ (-\partial^\nu + \delta_0^\nu \partial^0) & 0 & 0 \end{pmatrix} \delta(x - x'). \tag{3.70}$$

The determinant of (3.70) is $\partial_i \partial^i [\partial^j \partial_j'] \delta(x - x')$. The inverse of (3.70) is called the "Dirac's Brakets" of this constrained system .

3.4 Application on Proca Field

3.4.1 Proca Field

The Proca theory is the theory describing a massive vector field, which propagates the corresponding three polarizations. It is one such simple modification of Maxwell theory. The Lagrange density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu, \tag{3.71}$$

we can be written the standard Proca action

$$S_{Proca} = \int d^4x \Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \Big]. \tag{3.72}$$

Then, we obtain the equation of motion

$$(\partial^2 + m^2)A_\mu = 0. \tag{3.73}$$

Introduction of the mass m of the vector field A_μ allows the propagation in the longitudinal direction due to the breaking of $U(1)$ gauge invariance .

In the Horndeski theory, what happens for a vector field instead of a scalar field. There is Maxwell field which massless spin 1 particle. Its Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{3.74}$$

There are two transverse polarizations, namely, electric and magnetic fields. This lead to 2 degrees of freedom. However, there is Proca field which massive spin 1 and its Lagrangian can be written

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu. \tag{3.75}$$

There are 2 transverse and 1 longitudinal, namely, 3 degrees of freedom. Introduction of the mass m of the vector field A_μ allows the propagation in the longitudinal direction due to the breaking of $U(1)$ gauge invariance.

3.5 U(1) gauge transformation

Then, we show U(1) gauge transformation and the mass term break $U(1)$ gauge invariance. In the general non-abelian gauge transformation is given by

$$A_\mu(x) \to U(x)A_\mu(x)U(x)^{-1} + i\partial_\mu U(x)U(x)^{-1}. \tag{3.76}$$

By using $U(x) = e^{-i\alpha(x)}$, we obtain U(1) gauge transformation

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha(x). \tag{3.77}$$

Let us consider Proca Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu. \tag{3.78}$$

Then, we consider

$$\begin{aligned} F'_{\mu\nu}F'^{\mu\nu} &= (\partial_\mu A'_\nu - \partial_\nu A'_\mu)(\partial^\mu A'^\nu - \partial^\nu A'^\mu) \\ &= F_{\mu\nu}F^{\mu\nu}, \end{aligned} \tag{3.79}$$

and

$$\begin{aligned} A'_\mu A'^\mu &= (A_\mu + \partial_\mu \alpha)(A^\mu + \partial^\mu \alpha) \\ &= A_\mu A^\mu + \underline{2A_\mu \partial^\mu \alpha + \partial_\mu \alpha \partial^\mu \alpha}, \end{aligned} \tag{3.80}$$

these underline terms break $U(1)$ gauge invariance.

Proca field is the a system which combines with massless Maxwell field and massive spin-1 field. Lagrangian density of the field can be written as

$$\mathcal{L}_{Proca} = -\frac{1}{4}F^2_{\mu\nu} - \frac{1}{2}m^2(A)^2, \tag{3.81}$$

where $F^2_{\mu\nu}$ is defined as $F_{\mu\nu}F^{\mu\nu}$. In this case $F_{\mu\nu}$ is $\partial_\mu A^\nu - \partial_\nu A^\mu$ and $(A)^2$ is in the form of $A_\mu A^\nu$. Because this field is an Electromagnetic field, as a result the field is automatically transverse wave which shows that the Electric field is perpendicular with magnetic field.

Moreover, both fields are perpendicular with velocity of the system. Feature of this field is vibration of the both fields which the directions are similar to the velocity.

3.5.1 Dirac formalism on Proca Field

From the previous topic shows the general features of the Proca field, which it has 3 degrees of fredom. To check that , this part also applies the Dirac formalism on the Proca field similar to the section 3.1. This proof is beginning with the Lagrangian density of the Proca field is in the form of

$$\mathcal{L}_{Proca} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu \tag{3.82}$$

To confirm that this system is a constrained system, we use conjugate momentum calculation to prove it.

$$\pi^\rho = \frac{\partial \mathcal{L}_{Proca}}{\partial \dot{A}_\rho}, \tag{3.83}$$

To substitute the equation (3.82) into the equation (3.83), one can see

$$\begin{aligned}\pi^\rho &= -\frac{1}{4}\frac{\partial(F_{\mu\nu}F^{\mu\nu})}{\partial \dot{A}_\rho} - \frac{1}{2}m^2\frac{\partial(A_\mu A^\mu)}{\partial \dot{A}_\rho} \\ &= F^{\rho 0}.\end{aligned} \tag{3.84}$$

The equation (3.84) shows the conjugate momentum of the Proca field. Considering the section 3.3.1, The conjugate momentum of the Proca field is similar to the conjugate momentum of electromagnetic field. From the equation (3.84), if $\rho = i$ one can see that $\pi^i = F^{i0} = -F_{i0}$. If $\rho = 0$, it is $\pi^0 = F^{00} = 0$. In this case, ρ^0 is a primary constraint of the Proca field, which exactly like the primary constraint of the electromagnetic field. As a result, because the Proca field has a primary constraint, therefore this field is a constrained system. Next step is to find Hamiltonian density of the Proca system by using the quation (3.46). Where ρ runs for 0 and $i = 1, 2, 3$, the Hamiltonian density of the Proca field is

$$\mathcal{H}(A_\rho,\pi^\rho) = \pi^0\dot{A}_0 + \pi^i\dot{A}_i + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu,$$
$$= \pi^0\dot{A}_0 + \pi^i\dot{A}_i$$
$$+ \frac{1}{4}\left(F_{00}F^{00} + F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right)$$
$$+ \frac{1}{2}m^2\left(A_0A^0 + A_iA^i\right). \quad (3.85)$$

Substituting $\pi^0 = 0$ and $\dot{A}_i = \pi_i + \partial_i A_0$, the equation (3.85) becomes

$$\mathcal{H}(A_\rho,\pi^\rho) = \pi^i[\pi^i + \partial_i A_0] - \frac{1}{2}[\pi_i][\pi^i] + \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}m^2\Big(-{A_0}^2 + {A_i}^2\Big)$$
$$= \frac{1}{2}\pi_i\pi^i + \pi^i(\partial_i A_0) + \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}m^2\Big(-{A_0}^2 + {A_i}^2\Big). \quad (3.86)$$

The equation (3.86) is the Hamiltonian equation of the Proca system. The reason that we calculate the Hamiltonian density of the Proca system is we want to find the Poisson bracket. Next step is the Poisson bracket between the Primary constraint and Hamiltonian of the Proca system. Substituting the primary constraint and the Hamiltonian into the equation (3.50), one can see

$$\{\pi^0(\vec{x},t), H(t)\} = \int d^3y\Big[\frac{\partial\pi^0(\vec{x})}{\partial A_\rho(\vec{y})}\frac{\partial H(t)}{\partial\pi^\rho(\vec{y})} - \frac{\partial\pi^0(\vec{x})}{\partial\pi^\rho(\vec{y})}\frac{\partial H(t)}{\partial A_\rho(\vec{y})}\Big]. \quad (3.87)$$

The First term of the equation (3.87) vanishes. For the 2^{nd} term, one can prof that

$$\frac{\partial\pi^0(\vec{x})}{\partial\pi^\rho(\vec{y})}\frac{\partial H(t)}{\partial A_\rho(\vec{y})} = \delta^{(3)}(\vec{x}-\vec{y})\delta^0_\rho\Big(-\partial_i\pi_i(\vec{y})\delta^\rho_0 + m^2[-A_0(\vec{y})\delta^\rho_0 + A_i(\vec{y})]\delta^\rho_i\Big). \quad (3.88)$$

Substituting the equation (3.88) into the equation (3.87), therefore the Poisson bracket between the primary constraint and the Hamiltonian of the Proca system is written as

$$\begin{aligned}\{\pi^0(\vec{x},t), H(t)\} &= \int d^3y\Big[-\delta^{(3)}(\vec{x}-\vec{y})\delta^0_\rho\Big(-\partial_i\pi_i(\vec{y})\delta^\rho_0 + m^2[-A_0(\vec{y})\delta^\rho_0 + A_i(\vec{y})\delta^\rho_i]\Big)\Big] \\ &= \int d^3y\,\delta^{(3)}(\vec{x}-\vec{y})\Big[\partial_i\pi_i(\vec{y})\delta^\rho_\rho - m^2[-A_0(\vec{y})\delta^\rho_\rho + A_i(\vec{y})\delta^0_i]\Big] \\ &= \partial_i\pi_i(\vec{x}) - m^2A_0(\vec{x}) \end{aligned} \quad (3.89)$$

The result from the equation (3.89) shows that $\partial_i\pi_i(\vec{x}) - m^2A_0(\vec{x})$ is the secondary constraint of the Proca system. Now, we have one the primary constraint and one secondary constraint. Next step is reclassifying all of the constraints into 1^{st}-class and 2^{st}-class constraints by using the Poisson bracket between each of all constraints of the Proca system. Because in the Proca field there are 2 constraints, therefore the Poisson bracket between the primary constraint and secondary constraint can be written as

$$\{P,S\}=\int d^3y\Big[\frac{\partial P}{\partial A_\rho(\vec{y})}\frac{\partial S}{\partial \pi^\rho(\vec{y})}-\frac{\partial P}{\partial \pi^\rho(\vec{y})}\frac{\partial S}{\partial A_\rho(\vec{y})}\Big], \tag{3.90}$$

where P and S represent the primary and the secondary constraint, respectively.In case of the primary constraint and the secondary constraint equals to $\pi^0(\vec{x})$ and $\partial_i\pi_i(\vec{x}) - m^2A_0(\vec{x})$ respectively, the result of the equation (3.90) becomes

$$\{P,S\}=\int d^3y\Big[\frac{\partial\Big(\pi^0(\vec{x})\Big)}{\partial A_\rho(\vec{y})}\frac{\partial\Big(\partial_i\pi_i(\vec{x})-m^2A_0(\vec{x})\Big)}{\partial \pi^\rho(\vec{y})} - \frac{\partial\Big(\pi^0(\vec{x})\Big)}{\partial \pi^\rho(\vec{y})}\frac{\partial\Big(\partial_i\pi_i(\vec{x})-m^2A_0(\vec{x})\Big)}{\partial A_\rho(\vec{y})}\Big]. \tag{3.91}$$

From the equation (3.91), the result is $\partial_i\pi_i(\vec{x})+m^2A_0(\vec{x})$. It is called 2^{nd}-class constraint, because it comes from the Poisson bracket between the primary constraint and the secondary constraint. As a result, using the equation (3.54) due to the Proca system there are one primary constraint, one secondary constraint and one second class constraint, therefore number of the degrees of freedom calculation is in the form of

$$\begin{aligned} D.O.F &= \frac{1}{2}\Big[N-2(F)-S\Big] \\ &= \frac{1}{2}\Big[8-2(0)-2\Big] \\ &= 3 \end{aligned} \tag{3.92}$$

Summary of this part, using the Dirac formalism on the Proca field in 4-dimensional space time, there is only one primary constraint, as a result the Proca field is a constrained system. Next, After using the Poisson bracket between the primary constraint and Hamiltonian density of the field, there is one secondary constraint. The Poisson bracket between the primary constraint and the secondary constraint exists. The result presents that, this theory consists of 1 primary constraint and one secondary constraint.

After reclassifying the primary and the secondary constraint, there are two second class constraints. As a result, using the equation (3.92), one can see that number of the degrees of freedom of the Proca field equals to 3, that we expected.

3.5.2 Faddeev-Jackiw formalism on Proca Field

Because this work relates with electromagnetic field, so the details of this section are consisted of the electromagnetic four-potential, component form of the field strength tensor, equation of motion of electromagnetic field and Maxwell's equations.

Lagrangian density of Proca field in 4 dimensional space times is written as

$$\mathcal{L}_{Proca} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}m^2(A)^2, \tag{3.93}$$

where $F_{\mu\nu}^2$ is $F_{\mu\nu}F^{\mu\nu}$; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. $(A)^2$ is $A_\mu A^\mu$ and m plays the role as mass of A. In this case, μ, ν runs for 0,1,2,3.

Beginning point of the Faddeev-Jackiw formalism is the Lagrangian density of the system is in the first order from as

$$\mathcal{L}_{FOF} = \pi^\rho \dot{A}_\rho(\pi^\rho) - \mathcal{H}(A_\rho, \pi^\rho), \tag{3.94}$$

where π^ρ is conjugate momentum of the system, $\dot{A}_\rho$ is time derivative of vector A and $\mathcal{H}$ is Hamiltonian density of the system. ρ runs for $0, i$, in this case $i = 1, 2, 3$. Next step is calculation of Hamiltonian density of the system, firstly one can find the conjugate momentum in the form of

$$\pi^\rho = \frac{\partial \mathcal{L}_{Proca}}{\partial \dot{A}_\rho}. \tag{3.95}$$

Substituting the equation (3.93) into the equation (3.95), one can get the conjugate momentum of the Proca field as

$$\pi^\rho = F^{\rho 0}. \tag{3.96}$$

Considering equation (3.94), we want $\pi^0, \dot{A}_0, \pi^i, \dot{A}_i$ respectively, so we calculate those things by using the equation (3.96).

1. In case of $\rho = 0$, one can see that

$$\pi^0 = F^{00} = 0. \tag{3.97}$$

2. In case of $\rho = i$, one can see that

$$\begin{aligned} \pi^i &= F^{i0} = -F_{i0}, \\ &= -\partial_i A_0 + \partial_0 A_i, \\ \pi^i &= \dot{A}_i - \partial_i A_0. \end{aligned} \tag{3.98}$$

Using the result from equation (3.98), one can get

$$\dot{A}_i = \pi^i + \partial_i A_0. \tag{3.99}$$

Conclusion of the results of equation (3.97, 3.98) and (3.99) is $\pi^0 = 0$, which presents the constrained equation of the Proca system. Afterwards, we try to find the Hamiltonian density of the system in the form of

$$\mathcal{H}(A_\rho, \pi^\rho) = \pi^\rho \dot{A}_\rho - \mathcal{L}_{Proca}. \tag{3.100}$$

Substituting $\pi^0, \pi^i, \dot{A}_i$ and the Lagrangian density of the Proca system from the equation (3.93) into the equation (3.100), one then obtain

$$\mathcal{H}(A_\rho, \pi^\rho) = \frac{1}{2}\pi_i^2 + \pi_i(\partial_i A_0) + \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{4}F_{ij}^2. \tag{3.101}$$

Next, one can find first order form of the Lagrangian density of the Proca system by substituting the equation (3.101) into the equation (3.94), one can see that

$$\mathcal{L}_{FOF} = \pi^\rho \dot{A}_\rho - \frac{1}{2}\pi_i^2 - \pi_i(\partial_i A_0) - \frac{1}{2}m^2(A)^2 - \frac{1}{4}F_{ij}^2, \tag{3.102}$$

or one can write as

$$\mathcal{L}_{FOF} = \pi^0 \dot{A}_0 + \pi^i \dot{A}_i - \frac{1}{2}\pi_i \pi^i - \pi_i(\partial_i A_0) - \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{4}F_{ij}F^{ij} + \gamma\pi^0. \tag{3.103}$$

The first order form of the Lagrangian density of the Proca system shows symplectic variables of the system which are $\xi^{(0)} = (A_0, \pi^0, A_i, \pi^i, \gamma)$. After that, one can calculate canonical momenta; $a^{(0)}_{\xi^{(0)}} = \mathcal{A}_{\xi^{(0)}}$ of the system.

1. In case of $\xi^{(0)} = A_0$, one can see that

$$\begin{aligned} \mathcal{A}_{A_0} = a^{(0)}_{A_0} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{A}_0(x')} \\ &= \frac{\partial}{\partial \dot{A}_0(x')} \int d^3x[\pi^0(x)\dot{A}_0(x)] \\ &= \int \pi^0(x)\delta(x-x')d^3x \\ \mathcal{A}_{A_0} &= \pi^0(x') \end{aligned} \tag{3.104}$$

2. In case of $\xi^{(0)} = \pi^0$, one can see that

$$\begin{aligned} \mathcal{A}_{\pi^0} = a^{(0)}_{\pi^0} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{\pi}^0(x')} \\ &= \frac{\partial}{\partial \dot{\pi}^0(x')} \int 0 d^3(x) \\ \mathcal{A}_{\pi^0} &= 0 \end{aligned} \tag{3.105}$$

3. In case of $\xi^{(0)} = A_i$, one can see that

$$\begin{aligned} \mathcal{A}_{A_i} = a^{(0)}_{A_i} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{A}_i(x')} \\ &= \frac{\partial}{\partial \dot{A}_i(x')} \int d^3x[\pi^i(x)\dot{A}_i(x)] \\ &= \int \pi^i(x)\delta(x-x')d^3x \\ \mathcal{A}_{A_i} &= \pi_i(x') \end{aligned} \tag{3.106}$$

4. In case of $\xi^{(0)} = \pi^i$, one can see that

$$\begin{aligned} \mathcal{A}_{\pi^i} = a^{(0)}_{\pi^i} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{\pi}^i(x')} \\ &= \frac{\partial}{\partial \dot{\pi}^i(x')} \int 0 d^3x \\ \mathcal{A}_{\pi^i} &= 0 \end{aligned} \tag{3.107}$$

5. In case of $\xi^{(0)} = \pi^i$, one can see that

$$\begin{aligned} \mathcal{A}_{\pi^i} = a^{(0)}_{\pi^i} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{\pi}^i(x')} \\ &= \frac{\partial}{\partial \dot{\pi}^i(x')} \int 0 d^3(x) \\ \mathcal{A}_{\pi^i} &= 0 \end{aligned} \tag{3.108}$$

6. In case of $\xi^{(0)} = \gamma$, one can see that

$$\begin{aligned} \mathcal{A}_\gamma = a_\gamma^{(0)} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \gamma(x')} \\ &= \frac{\partial}{\partial \gamma(x')} \int \pi^0(x) d^3(x) \\ \mathcal{A}_\gamma &= -\pi_0(x') \end{aligned} \tag{3.109}$$

The canonical 1-form of the Proca system is in the form of

$$\mathcal{A} = \int d^3x [\mathcal{A}_{A_0} \delta A_0(x) + \mathcal{A}_{A_i} \delta A_i(x) + \mathcal{A}_{A_\gamma} \delta A_\gamma(x)]. \tag{3.110}$$

Substituting the equations (3.104),(3.105), (3.106) and (3.109) into the equation (3.110), one can see that

$$\mathcal{A} = \int d^3x [(-\pi_0(x) \delta A_0(x)) + (\pi_i \delta A_i(x)) - (\pi_0(x) \delta A_\gamma(x))]. \tag{3.111}$$

Next step is calculation of symplectic 2-form of the Proca system by using variation with the canonical 1-form of the Proca system as

$$\mathcal{F} = \delta \mathcal{A} = \int d^3x [-(\delta \pi_0(x) \wedge \delta A_0(x)) + (\delta \pi_i \delta \wedge A_i(x)) - (\delta \pi_0(x) \delta \wedge A_\gamma(x))] \tag{3.112}$$

Using interior derivative with the symplectic 2-form of the Proca system in equation (3.112), one can see that

$$i\mathcal{F} = \int d^3x [(-z^{\pi_0} \delta A_0 + z^{A_0} \delta \pi_0) + (z^{\pi_i} \delta A_i - z^{A_i} \delta \pi_i) + (-z^{\pi_0} \delta \gamma + z^\gamma \delta \pi_0]. \tag{3.113}$$

The result of the equation (3.113) is written as $z^{\xi^{(0)}} = 0$ except $z^{A_0} = -z^\gamma$. Because some of $z^{\xi^{(0)}}$ are non-zero, so they present that the zero mode of the system exit. If the zero-mode of the system is non-zero, one can find constraint of the system by using

$$\Omega = i \delta \mathcal{L}_v. \tag{3.114}$$

Where $\mathcal{L}_v$ of the system is a part of the first order form of the Lagrangian density of the system, which except time derivative terms and Lagrange terms. Consequently, $\mathcal{L}_v$ is written as

$$\mathcal{L}_v = -\frac{1}{2}\pi_i \pi^i - \pi_i(\partial_i A_0) - \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{4} F_{ij} F^{ij}. \tag{3.115}$$

If we want to find the constraint of the system, firstly we have to use variation with $\mathcal{L}_v$ in the equation (3.115), we see

$$\begin{aligned}\delta\mathcal{L}_v &= \int\delta\quad\left[\quad -\frac{1}{2}\pi_i\pi^i - \pi_i(\partial_i A_0) - \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{4}F_{ij}F^{ij}\right]dx\\ \delta\mathcal{L}_v &= \int\quad\left[\quad -\pi_i\delta\pi^i - (\partial_i A_0)\delta\pi^i - \pi_i(\partial_i\delta A_0)\right.\\ &\quad -\quad \left. m^2 A^\mu\delta A_\mu - F^{ij}(\partial_i\delta A_j)\right]dx. \qquad (3.116)\end{aligned}$$

After that, we want to calculate the zero mode of the system by using interior derivative with $\delta\mathcal{L}_v$ in equation (3.116), one can see that

$$\begin{aligned}i\delta\mathcal{L}_v &= \int\quad\left[\quad -\pi_i\delta\pi^i - (\partial_i A_0)\delta\pi^i - \pi_i(\partial_i\delta A_0) - m^2 A^\mu\delta A_\mu - F^{ij}(\partial_i\delta A_j)\right]dx\\ i\delta\mathcal{L}_v &= \int\quad\left[\quad -\pi_i z^{\pi_i} - (\partial_i A_0)z^{\pi_i} - \pi_i(\partial_i z^{A_0}) - m^2 A^\mu z^{A_\mu} - F^{ij}(\partial_i z^{A_j})\right]dx \qquad (3.117)\end{aligned}$$

Using the result from the equation (3.113), the equation (3.117) becomes

$$i\delta\mathcal{L}_v = \int[(\partial_i\pi_i) - m^2 A^0]z^{A_0}dx. \qquad (3.118)$$

As a result, the constraint of the system can be written as

$$\Omega = \partial_i\pi_i - m^2 A^0. \qquad (3.119)$$

Next step, to find the new first order form of the Lagrangian of the Proca by multiplying new Lagrange multiplier (γ_1) with the constraint from the equation (3.119) and adding that result into the first order form of the Lagrangian from the equation (3.103), one can see the new first order form of the Proca system as

$$\begin{aligned}\mathcal{L}_{FOF} &= \pi^0\dot{A}_0 + \pi^i\dot{A}_i - \frac{1}{2}\pi_i\pi^i - \pi_i(\partial_i A_0) - \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{4}F_{ij}F^{ij} + \gamma\pi^0\\ &+ \gamma_1(\partial_i\pi_i - m^2 A^0). \qquad (3.120)\end{aligned}$$

The equation (3.120) is the new first order form of the Lagrangian density of the Proca system. The last term of the equation is the new Lagrange multiplier term, which consists of the Lagrange multiplier (γ_1) and its constraint ($\partial_i\pi_i - m^2 A^0$). Next process is calculation of the canonical momenta, canonical 1-form, symplectic 2-form and calculation of its zero-mode of the system, respectively.

considering the new first order form of the system in the equation (3.120), the symplectic variables are $\xi^{(1)} = (A_0, \pi^0, A_i, \pi^i, \gamma, \gamma_1)$. The semplectic number of the old one is 5, but the new one is 6. The reason for increasing of the symplectic number is the new term in the first order form of the Lagrangian density of the Proca system.

1. In case of $\xi^{(1)} = A_0$, one can see that

$$\begin{aligned} \mathcal{A}_{A_0} = a^{(1)}_{A_0} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{A}_0(x')} \\ &= \frac{\partial}{\partial \dot{A}_0(x')} \int d^3x[\pi^0(x)\dot{A}_0(x)] \\ &= \int \pi^0(x)\delta(x-x')d^3x \\ \mathcal{A}_{A_0} &= \pi^0(x') \end{aligned} \tag{3.121}$$

2. In case of $\xi^{(1)} = \pi^0$, one can see that

$$\begin{aligned} \mathcal{A}_{\pi^0} = a^{(1)}_{\pi^0} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{\pi}^0(x')} \\ &= \frac{\partial}{\partial \dot{\pi}^0(x')} \int 0 d^3(x) \\ \mathcal{A}_{\pi^0} &= 0 \end{aligned} \tag{3.122}$$

3. In case of $\xi^{(1)} = A_i$, one can see that

$$\begin{aligned} \mathcal{A}_{A_i} = a^{(1)}_{A_i} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{A}_i(x')} \\ &= \frac{\partial}{\partial \dot{A}_i(x')} \int d^3x[\pi^i(x)\dot{A}_i(x)] \\ &= \int \pi^i(x)\delta(x-x')d^3x \\ \mathcal{A}_{A_i} &= \pi_i(x') \end{aligned} \tag{3.123}$$

4. In case of $\xi^{(1)} = \pi^i$, one can see that

$$\begin{aligned} \mathcal{A}_{\pi^i} = a^{(1)}_{\pi^i} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{\pi}^i(x')} \\ &= \frac{\partial}{\partial \dot{\pi}^i(x')} \int 0 d^3x \\ \mathcal{A}_{\pi^i} &= 0 \end{aligned} \tag{3.124}$$

5. In case of $\xi^{(1)} = \gamma$, one can see that

$$\begin{aligned} \mathcal{A}_\gamma = a_\gamma^{(1)} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \gamma(x')} \\ &= \frac{\partial}{\partial \gamma(x')} \int \pi^0(x) d^3(x) \\ \mathcal{A}_\gamma &= -\pi_0(x') \end{aligned} \tag{3.125}$$

6. In case of $\xi^{(1)} = \gamma_1$, one can see that

$$\begin{aligned} \mathcal{A}_{\gamma)_1} = a_\gamma^{(1)} &= \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \gamma(x')} \\ &= \frac{\partial}{\partial \gamma(x')} \int (\partial_i \pi_i(x) - m^2 A^0(x)) d^3(x) \\ \mathcal{A}_{\gamma_1} &= \partial_i' \pi_i(x') - m^2 A^0(x') \end{aligned} \tag{3.126}$$

The canonical 1-form of the system is written as

$$\mathcal{A} = \int d^3x [\mathcal{A}_{A_0} \delta A_0(x) + \mathcal{A}_{A_i} \delta A_i(x) + \mathcal{A}_{A_\gamma} \delta\gamma(x) + \mathcal{A}_{A_{\gamma_1}} \delta\gamma_1(x)]. \tag{3.127}$$

Using the results from the equation (3.121), (3.122), (3.123), (3.124), (3.125) and (3.127) and substituting into the equation (3.127), one then get

$$\begin{aligned} \mathcal{A} = \int d^3x \quad [& (-\pi_0(x)\delta A_0(x)) + (\pi_i \delta A_i(x)) - (\pi_0(x)\delta\gamma(x)) \\ + & (\partial_i \pi_i(x) + m^2 A_0(x))\delta\gamma_1]. \end{aligned} \tag{3.128}$$

Next step is calculation of the semplectic 2-form of the system by using variation with the canonical 1-form in the equation (3.128). It can be shown that

$$\begin{aligned} \mathcal{F} = \delta\mathcal{A} = \int d^3x \quad [& -(\delta\pi_0(x) \wedge \delta A_0(x)) + (\delta\pi_i \wedge \delta A_i(x)) - (\delta\pi_0(x) \wedge \delta\gamma(x)) \\ + & (\partial_i \delta\pi_i \wedge \delta\gamma_1) + m^2 \delta A_0 \wedge \delta\gamma_1(x)]. \end{aligned} \tag{3.129}$$

Next calculation is to find the zero-mode of the system by using the interior derivative with the canonical 1-form, one can see

$$\begin{aligned} i_z\mathcal{F} = \int d^3x \quad [& (-z^{\pi_0}\delta A_0 + z^{A_0}\delta\pi_0) + (z^{\pi_i}\delta A_i - z^{A_i}\delta\pi_i) + (-z^{\pi_0}\delta\gamma + z^{\gamma}\delta\pi_0 \\ + & (\partial_i z^{\pi_i}\delta\gamma_1) - (z^{\gamma_1}\partial_i\delta\pi_i) + (m^2 z^{A_0}\delta\gamma_1) - (m^2 z^{\gamma_1}\delta A_0)]. \end{aligned} \tag{3.130}$$

As the result, it presents that the zero-mode of the system is equal to zero. Therefore, there is no constraint. The process of the system reaches at the final part.

CHAPTER IV

APPLICATION ON GENERALIZED OF PROCA FIELD: FADDEEV-JACKIW FORMALISM

4.1 Generalized of Proca Field

The Lagrangian for the generalized Proca vector field with derivative self-interactions is given by

$$\mathcal{L}_{gen.Proca} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{n=2}^{5}\alpha_n\mathcal{L}_n, \tag{4.1}$$

where the self-interactions of the vector field.

The simplest modification of the Proca action is promoting the mass term and the potential interactions for the vector field to an arbitrary function f_2,

$$\mathcal{L}_2 = f_2. \tag{4.2}$$

This trivially does not modify the number of degrees of freedom. This function can also contain gauge invariant interactions which are invariant under the $U(1)$ transformations and terms which do not contain any dynamics for the temporal component of the vector field

$$f_2 = f_2(F^2, FF^*, A^2F^2, A^2FF^*, A_\mu A_\nu F^{\rho\mu}F^{\nu}_{\rho}, ...). \tag{4.3}$$

The first term that we can have to the next order in the vector field is simply

$$\mathcal{L}_3 = f_3\partial\cdot A \tag{4.4}$$

with f_3 an arbitrary function of the vector field norm $f_3(A^2)$. It is a trivial observation that the temporal component of the vector field A_0 does not propagate, even if we include the Maxwell kinetic term. The presence of the function f_3 is crucial since if it was simply a constant. Then, one considers $\mathcal{L}_4$ which is given by

$$\mathcal{L}_4 = f_4[c_1(\partial\cdot A)^2 + c_2\partial_\rho A_\sigma\partial^\rho A^\sigma + c_3\partial_\rho A_\sigma\partial^\sigma A^\rho] \tag{4.5}$$

with a priori free parameters c_1, c_2 and c_3 and f_4 an arbitrary function depending on $f_4(A^2)$. One need to fix the parameters such that only three physical degrees of freedom propagate. To eliminate one propagating degree of freedom, the determinant of the Hessian matrix vanishes

$$H^{\mu\nu}_{\mathcal{L}_4} = \frac{\partial^2 \mathcal{L}_4}{\partial \dot{A}_\mu \partial \dot{A}_\nu} = f_4 \begin{pmatrix} 2(c_1 + c_2 + c_3) & 0 & 0 & 0 \\ 0 & -2c_2 & 0 & 0 \\ 0 & 0 & -2c_2 & 0 \\ 0 & 0 & 0 & -2c_2 \end{pmatrix}. \quad (4.6)$$

One chooses $c_1 + c_2 + c_3 = 0$, $c_1 = 1$, and $c_3 = -(1 + c_2)$. Therefore one obtains

$$\mathcal{L}_4 = f_4[(\partial \cdot A)^2 + c_2 \partial_\rho A_\sigma \partial^\rho A^\sigma - (1 + c_2)\partial_\rho A_\sigma \partial^\sigma A^\rho]. \quad (4.7)$$

If Hessian determinant of a system is zero, the system is constrained,

$$\det(H^{\mu\nu}_{\mathcal{L}_4}) = 0. \quad (4.8)$$

To find the expression for the constraint, we have to compute the conjugate momentum $\Pi^\mu_{\mathcal{L}_4}$,

$$\Pi^\mu_{\mathcal{L}_4} = \frac{\partial \mathcal{L}_4}{\partial \dot{A}_\mu}. \quad (4.9)$$

The zero component of the conjugate momentum is given by

$$\Pi^0_{\mathcal{L}_4} = -2 f_4 \nabla A. \quad (4.10)$$

This equation does not contain any time derivative yielding the constraint equation. If an equation contains only generalised coordinates and conjugate momenta, but not generalised velocities, then such equation is called a constrained equation, which defines a constraint surface. The constraint equation is given by

$$\varphi_1 = \Pi^0_{\mathcal{L}_4} + 2 f_4 \nabla A, \quad (4.11)$$

this constraint φ_1 is a primary constraint. This primary constraint (φ_1) will generate a secondary constraint (φ_2) given by

$$\begin{aligned} \dot{\varphi}_1 &= \{H, \varphi_1\} = \frac{\partial H}{\partial A_\mu}\frac{\partial \varphi_1}{\partial \Pi^\mu} - \frac{\partial H}{\partial \Pi^\mu}\frac{\partial \varphi_1}{\partial A_\mu} \\ &= \varphi_2. \end{aligned} \quad (4.12)$$

Then, consider the time evolution of the secondary constraint

$$\{H, \varphi_2\} = 0. \tag{4.13}$$

Hence, there are two constraints φ_1 and φ_2. we reclassify them into first-class and second-class constraints. By definition, a first-class constraint weakly commutes with all other constraints while a second-class constraint does not. One computes

$$\{\varphi_1, \varphi_2\} \neq 0. \tag{4.14}$$

So, φ_1, φ_2 are the second-class constraints. The canonical variables are $\Pi^{\mu}_{\mathcal{L}_4}$, A_μ, the second-class constraints are φ_1, φ_2 and there is no first-class constraint. So the number of degrees of freedom is

$$\begin{aligned}(\text{\#d.o.f}) &= \frac{(\text{\#canonical variables}) - 2\times(\text{\#1st class}) - (\text{\#2nd class})}{2}\\ &= \frac{8-2\times(0)-2}{2}\\ &= 3.\end{aligned} \tag{4.15}$$

This agrees with the fact that a massive particle spin-1 has three polarisations. There are 2 transverse and 1 longitudinal.

Then, we consider $\mathcal{L}_5$. In $\mathcal{L}_5$, one write all the possible contractions between the derivative self-interactions

$$\begin{aligned}\mathcal{L}_5 &= f_5\,[d_1(\partial\cdot A)^3 - 3d_2(\partial\cdot A)\partial_\rho A_\sigma\partial^\rho A^\sigma - 3d_3(\partial\cdot A)\partial_\rho A_\sigma\partial^\sigma A^\rho\\ &\quad +2d_4\partial_\rho A_\sigma\partial^\gamma A^\rho\partial^\sigma A_\gamma + 2d_5\partial_\rho A_\sigma\partial^\gamma A^\rho\partial_\gamma A^\rho]\end{aligned} \tag{4.16}$$

with a priori the arbitrary parameters d_1, d_2, d_3, d_4 and d_5 and function f_5 depending only on A^2. In order to have only three propagating degrees of freedom the parameters need to fulfilled some conditions. Finally, the quintic Lagrangian is given by

$$\begin{aligned}\mathcal{L}_5 &= f_5\,[(\partial\cdot A)^3 - 3d_2(\partial\cdot A)\partial_\rho A_\sigma\partial^\rho A^\sigma - 3(1-d_2)(\partial\cdot A)\partial_\rho A_\sigma\partial^\sigma A^\rho\\ &\quad +2\Big(1-\frac{3d_2}{2}\Big)\partial_\rho A_\sigma\partial^\gamma A^\rho\partial^\sigma A_\gamma + 2\Big(\frac{3d_2}{2}\Big)\partial_\rho A_\sigma\partial^\gamma A^\rho\partial_\gamma A^\rho].\end{aligned} \tag{4.17}$$

The Hessian matrix with this chosen parameters then becomes

$$H^{\mu\nu}_{\mathcal{L}_5} = f_4(A^2)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -6d_2(A_{z,z}+A_{y,y}) & 3d_2(A_{x,y}+A_{y,x} & 3d_2(A_{x,z}+A_{z,x} \\ 0 & 3d_2(A_{x,y}+A_{y,x} & -6d_2(A_{z,z}+A_{x,x} & 3d_2(A_{y,z}+A_{z,y} \\ 0 & 3d_2(A_{x,z}+A_{z,x} & 3d_2(A_{y,z}+A_{z,y} & -6d_2(A_{y,y}+A_{x,x} \end{pmatrix}. \quad (4.18)$$

The vanishing of the determinant of the Hessian matrix guaranties the existence of a constraint

$$\det(H^{\mu\nu}_{\mathcal{L}_5}) = 0. \quad (4.19)$$

To find the expression for the constraint, we have to compute the conjugate momentum $\Pi^{\mu}_{\mathcal{L}_4}$

$$\Pi^{\mu}_{\mathcal{L}_5} = \frac{\partial \mathcal{L}_5}{\partial \dot{A}_{\mu}}. \quad (4.20)$$

The Hessian matrix only contains one vanishing eingenvalue and hence only one propagating constraint which is again given by the corresponding zero component of the conjugate momentum

$$\begin{aligned} \Pi^{0}_{\mathcal{L}_5} = & -3f_5(A^2)(d_2(A^2_{x,z}+A^2_{y,z}+A^2_{x,y}) - 2A_{z,z}A_{y,z} - 2(-1+d_2)A_{y,z}A_{z,y} \\ & +d_2A^2_{z,y} + d_2A^2_{z,x} - 2(A_{z,z}+A_{y,y})A_{x,x} + 2A_{x,y}A_{y,x} - 2d_2A_{x,y}A_{y,x} \\ & +d_2A^2_{y,x} - 2(-1+d_2)A_{x,z}A_{z,x}). \end{aligned} \quad (4.21)$$

There is no time derivatives appearing in the expression of the zero component of the conjugate momentum, representing the constraint equation. Associated to this constraint, there will be a secondary constraint guarenting the propagation of the constraint equation and removing the unphysical degree of freedom

In this work $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, $\mathcal{L}_5$ is in the form of

$$\begin{aligned}
\mathcal{L}_2 &= f_2 \\
\mathcal{L}_3 &= f_3\, \partial \cdot A \\
\mathcal{L}_4 &= f_4 \left[(\partial \cdot A)^2 + c_2 \partial_\rho A_\sigma \partial^\pi A^\sigma - (1+c_2)\partial_\rho A_\sigma \partial^\sigma A^\rho \right] \\
\mathcal{L}_5 &= f_5 \left[(\partial \cdot A)^3 - 3d_2(\partial \cdot A)\partial_\rho A_\sigma \partial^\rho A^\sigma - 3(1-d_2)(\partial \cdot A)\partial_\rho A_\sigma \partial^\sigma A^\rho \right. \\
&\quad \left. +2\Big(1-\frac{3d_2}{2}\Big)\partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma + 2\Big(\frac{3d_2}{2}\Big)\partial_\rho A_\sigma \partial^\gamma A^\rho \partial_\gamma A^\rho \right]
\end{aligned} \tag{4.22}$$

with $\partial \cdot A = \partial_\mu A^\mu$ and the functions $f_{2,3,4,5}$ are arbitrary functions [3]. The interactions can be also expressed in terms of the Levi-Civita tensors

$$\begin{aligned}
\mathcal{L}_2 &= -\frac{f_2}{24}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\alpha\beta} = f_2 \\
\mathcal{L}_3 &= -\frac{f_3}{6}\varepsilon_{\mu\nu\alpha\beta}\varepsilon^{\rho}{}_{\nu\alpha\beta}\,\partial_\mu A_\rho = f_3\,\partial \cdot A \\
\mathcal{L}_4 &= -\frac{f_4}{2}(\varepsilon^{\mu\nu\alpha\beta}\varepsilon^{\rho\sigma}{}_{\alpha\beta}\partial_\mu A_\rho \partial_\nu A_\sigma + c_2 \varepsilon^{\mu\nu\alpha\beta}\varepsilon^{\rho\sigma}{}_{\alpha\beta}\partial_\mu A_\nu \partial_\rho A_\sigma) \\
&= f_4 \left[(\partial \cdot A)^2 + c_2 \partial_\rho A_\sigma \partial^\pi A^\sigma - (1+c_2)\partial_\rho A_\sigma \partial^\sigma A^\rho \right] \\
\mathcal{L}_5 &= -f_5\Big((1-\frac{3}{2}d_2)\varepsilon^{\mu\nu\alpha\beta}\varepsilon^{\rho\sigma\gamma}{}_{\beta}\partial_\mu A_\rho \partial_\nu A_\sigma \partial_\alpha A_\gamma + \frac{3}{2}d_2 \varepsilon^{\mu\nu\alpha\beta}\varepsilon^{\rho\sigma\gamma}{}_{\beta}\partial_\mu A_\rho \partial_\nu A_\sigma \partial_\gamma A_\alpha \Big) \\
&= f_5 \left[(\partial \cdot A)^3 - 3d_2(\partial \cdot A)\partial_\rho A_\sigma \partial^\rho A^\sigma - 3(1-d_2)(\partial \cdot A)\partial_\rho A_\sigma \partial^\sigma A^\rho \right. \\
&\quad \left. +2\Big(1-\frac{3d_2}{2}\Big)\partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma + 2\Big(\frac{3d_2}{2}\Big)\partial_\rho A_\sigma \partial^\gamma A^\rho \partial_\gamma A^\rho \right]
\end{aligned} \tag{4.23}$$

where

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise.} \end{cases}$$

Higher order interactions beyond the quintic order are trivial in four dimensions, being just total derivatives, hence the series stops here. Expressed in terms of the Levi-Civita tensors this means that we run out of the indices. Lagrangian density of the generalized of the Proca field is in the form of

$$
\begin{aligned}
\mathcal{L}_{\text{GenPro}} =& -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}m^2(A^2) + \frac{3}{2}\alpha_3(A^2)(\partial\cdot A) \\
&+ 2\alpha_4(A^2)(\partial\cdot A)^2 - 2\alpha_4(A^2)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho) \\
&- \frac{5}{2}\alpha_5(A^2)(\partial\cdot A)^3 + \frac{15}{2}\alpha_4(A^2)(\partial\cdot A)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho) \\
&- 5(A^2)(\partial_\rho A_\sigma)(\partial^\gamma A^\rho)(\partial^\sigma A_\gamma)
\end{aligned}
\tag{4.24}
$$

Letting $\frac{3}{2}\alpha_3 = \alpha_3 =$, $2\alpha_4 = \alpha_4$ and $-\frac{5}{2}\alpha_5 = \alpha_5$, so the generalized of the Proca field reads

$$
\begin{aligned}
\mathcal{L}_{\text{GenPro}} =& -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}m^2(A^2) + \alpha_3(A^2)(\partial\cdot A) \\
&+ \alpha_4(A^2)(\partial\cdot A)^2 - \alpha_4(A^2)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho) \\
&+ \alpha_5(A^2)(\partial\cdot A)^3 - 3\alpha_5(A^2)(\partial\cdot A)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho) \\
&+ 2\alpha_5(A^2)(\partial_\rho A_\sigma)(\partial^\gamma A^\rho)(\partial^\sigma A_\gamma)
\end{aligned}
\tag{4.25}
$$

There are construction of general derivative self-interactions for a massive Proca field in more terms. The construction of the most general generalised Proca theories remains an open question. In principle, a possible way to do this is by following the idea of the original construction of generalised Proca theories, that is by starting from demanding that Hessian is degenerate.

$$
\mathcal{L}_{genProca} = -\frac{1}{4}F_{\mu\nu}^2 + \sum_{n=2}^{5}\alpha_n\mathcal{L}_n,
\tag{4.26}
$$

where $\mathcal{L}_n$ are self-interactions of the vector fields in form of

$$
\begin{aligned}
\mathcal{L}_2 &= f_2, \\
\mathcal{L}_3 &= f_3(\partial\cdot A), \\
\mathcal{L}_4 &= f_4[(\partial\cdot A)^2 + c_2\partial_\rho A_\sigma\partial^\rho A^\sigma - (1+c_2)\partial_\rho A_\sigma\partial^\sigma A^\rho] \\
\mathcal{L}_5 &= f_5[(\partial\cdot A)^3 - 3d_2(\partial\cdot A)\partial_\rho A_\sigma\partial^\rho A^\sigma - 3(1-d_2)(\partial\cdot A)\partial_\rho A_\sigma\partial^\sigma A^\rho \\
&+ 2\Big(1-\frac{3d_2}{2}\Big)\partial_\rho A_\sigma\partial^\gamma A^\rho\partial^\sigma A_\gamma + 2\Big(\frac{3d_2}{2}\Big)\partial_\rho A_\sigma\partial^\gamma A^\rho\partial_\gamma A^\sigma].
\end{aligned}
\tag{4.27}
$$

In this work, we are interested in $f_2 = f_3 = f_4 = f_5 = A^2$. Therefor, the equation (4.27) becomes

$$\begin{aligned}
\mathcal{L}_2 &= A^2, \\
\mathcal{L}_3 &= A^2(\partial \cdot A), \\
\mathcal{L}_4 &= A^2[(\partial \cdot A)^2 + c_2 \partial_\rho A_\sigma \partial^\rho A^\sigma - (1 + c_2)\partial_\rho A_\sigma \partial^\sigma A^\rho] \\
\mathcal{L}_5 &= A^2[(\partial \cdot A)^3 - 3d_2(\partial \cdot A)\partial_\rho A_\sigma \partial^\rho A^\sigma - 3(1 - d_2)(\partial \cdot A)\partial_\rho A_\sigma \partial^\sigma A^\rho \\
&+ 2\Big(1 - \frac{3d_2}{2}\Big)\partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma + 2\Big(\frac{3d_2}{2}\Big)\partial_\rho A_\sigma \partial^\gamma A^\rho \partial_\gamma A^\sigma]. \qquad (4.28)
\end{aligned}$$

Considering the equation (3.81) (4.26) and (4.28), if $c_2 = 1, d_2 = 1$ one can see that the Lagrangian of generalized Proca vector field with derivative self-interactions is in the form of

$$\begin{aligned}
\mathcal{L}_{genProca} &= -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}m^2A^2 + \alpha_3\Big[A^2(\partial \cdot A) - A^\mu A^\nu \partial_\nu A_\mu\Big] \\
&+ \alpha_4\Big[A^2[(\partial \cdot A)^2 - \partial_\rho A_\sigma \partial^\sigma A^\rho] - 2A^\mu A^\nu \partial_\nu A_\mu(\partial \cdot A) + 2A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\mu\Big] \\
&+ \alpha_5\Big[A^2[-(\partial \cdot A)^3 + 3\partial_\rho A_\sigma \partial^\sigma A^\rho - 2\partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma] \\
&+ 3A^\mu A^\nu \partial_\nu A_\mu(\partial \cdot A)^2 - 6A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\mu(\partial \cdot A) \\
&+ 6A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\gamma \partial^\gamma A_\mu - 3A^\mu A^\nu \partial_\nu A_\mu \partial_\rho A_\sigma \partial^\sigma A^\rho\Big] \qquad (4.29)
\end{aligned}$$

Considering the equation (4.29), Hessian matrix of generalized Proca vector field with derivative self-interactions equals to zero, therefore this field is "a constrained system".

We now consider the 3^{rd} term of the right hand side of the equation (4.29), it is

$$3^{rd} = \Big[A^2(\partial \cdot A) - A^\mu A^\nu \partial_\nu A_\mu\Big]. \qquad (4.30)$$

Using by part with the 2^{nd} term of the right hand side of the equation (4.30), one can prove that

$$\begin{aligned}
A^\mu A^\nu \partial_\nu A_\mu &= -\frac{1}{2}A_\mu A^\mu(\partial_\nu A^\nu) \\
&= -\frac{1}{2}(A^2)(\partial \cdot A). \qquad (4.31)
\end{aligned}$$

Therefore, the 3^{rd} term is written as

$$3^{rd}_{new} = A_\mu A^\mu(\partial_\nu A^\nu). \tag{4.32}$$

The 4^{th} term of the of the right hand side of the equation (4.29) is in the form of

$$4^{th} = \Big[A^2[(\partial\cdot A)^2 - \partial_\rho A_\sigma \partial^\sigma A^\rho] - 2A^\mu A^\nu \partial_\nu A_\mu(\partial\cdot A) + 2A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\mu\Big]. \tag{4.33}$$

Using by part with the 3^{rd} term of the rigth of the (4.33), one can calculate that

$$-2A^\mu A^\nu(\partial\cdot A)\partial_\nu A_\mu = -A_\mu A^\mu(\partial_\nu A^\nu)(\partial\cdot A) - A_\mu A^\mu A^\nu \partial_\nu(\partial_\rho A^\rho), \tag{4.34}$$

and applying the same technique with the 4^{th} term of the right hand side of the equation (4.33), we see

$$2A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\mu = -A_\mu A^\mu(\partial^\rho A^\nu)(\partial_\nu A_\rho) - A_\mu A^\mu A^\nu \partial^\rho(\partial_\nu A_\rho). \tag{4.35}$$

Therefor, the 4^{rd} term can be written as

$$4^{th}_{new} = A_\mu A^\mu(\partial_\rho A^\rho)(\partial_\sigma A^\sigma) - A_\mu A^\mu(\partial_\rho A_\sigma)(\partial^\sigma A^\rho). \tag{4.36}$$

From the equation (4.29), the 5^{th} term of the right of equation is written as

$$\begin{aligned} 5^{th} &= \Big[A^2[-(\partial\cdot A)^3 + 3\partial_\rho A_\sigma \partial^\sigma A^\rho - 2\partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma] \\ &+ 3A^\mu A^\nu \partial_\nu A_\mu(\partial\cdot A)^2 - 6A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\mu(\partial\cdot A) \\ &+ 6A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\gamma \partial^\gamma A_\mu - 3A^\mu A^\nu \partial_\nu A_\mu \partial_\rho A_\sigma \partial^\sigma A^\rho\Big] \end{aligned} \tag{4.37}$$

Using by part with the 4^{th} term of the right hand side of the equation (4.37), one can calculate that

$$3A^\mu A^\nu(\partial\cdot A)^2\partial_\nu A_\mu = -\frac{3}{2}(A^2)(\partial\cdot A)^3 - 3(A^2)A^\nu)(\partial\cdot A)\partial_\nu(\partial\cdot A). \tag{4.38}$$

Do the same technique with the 5^{th} term of the right hand side of the equation, we see

$$\begin{aligned} 6A^\mu A^\nu \partial_\nu A_\rho(\partial\cdot A)\partial^\rho A_\mu = &- 3(A^2)(\partial^\rho A^\nu)(\partial_\nu A_\rho)(\partial\cdot A) \\ &- 3(A^2)A^\nu \partial_\nu(\partial\cdot A)(\partial\cdot A) \\ &- 3(A^2)A^\nu(\partial_\nu A_\rho)\partial^\rho(\partial\cdot A). \end{aligned} \tag{4.39}$$

Taking by part with the 6^{th} of the right hand side of the (4.37), one can see

$$\begin{aligned} 6A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\gamma \partial^\gamma A_\mu = & - 3(A^2)(\partial^\gamma A^\nu)(\partial_\nu A_\rho)(\partial^\rho A_\gamma) \\ & - 3(A^2)A^\nu \partial^\gamma (\partial_\nu A_\rho)(\partial^\rho A_\gamma) \\ & - 3(A^2)A^\nu (\partial_\nu A_\rho)\partial^\gamma (\partial^\rho A^\gamma). \end{aligned} \tag{4.40}$$

We now use by part with the last term of the right of the equation (4.37), one can calculate that

$$\begin{aligned} 3A^\mu A^\nu (\partial_\rho A_\sigma)(\partial^\sigma A^\rho)(\partial_\nu A_\mu) = & - \frac{3}{2}(A^2)(\partial \cdot A)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho) \\ & - \frac{3}{2}(A^2)A^\nu \partial_\nu (\partial_\rho A_\sigma)(\partial^\sigma A^\rho) \\ & - \frac{3}{2}(A^2)A^\nu (\partial_\rho A_\sigma)\partial_\nu (\partial^\sigma A^\rho). \end{aligned} \tag{4.41}$$

Therefore, the 5^{th} term of the right hand side of the equation (4.29) is written as

$$\begin{aligned} 5^{th}_{new} = & \quad (A^2)(\partial \cdot A)^3 \\ & - 3(A^2)(\partial \cdot A)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho) \\ & + (A^2)(\partial_\rho A_\sigma)(\partial^\gamma A^\rho)(\partial^\sigma A_\gamma). \end{aligned} \tag{4.42}$$

Substituting the 3^{rd}, 4^{th} and the 5^{th} term from the equation (4.32),(4.36) and (4.42) into the equation (4.29), one can see that

$$\begin{aligned} \mathcal{L}_{genProca(new)} = & - \frac{1}{4}F^2_{\mu\nu} - \frac{1}{2}m^2 A^2 + \frac{3}{2}\alpha_3 A^2(\partial \cdot A) \\ & + 2\alpha_4 \Big[(A^2)(\partial \cdot A)^2 - (A^2)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho)\Big] \\ & - \frac{5}{2}\alpha_5 \Big[A^2(\partial \cdot A)^3 - 3(A^2)(\partial \cdot A)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho) \\ & - 2(A^2)(\partial_\rho A_\sigma)(\partial^\gamma A^\rho)(\partial^\sigma A_\gamma)\Big]. \end{aligned} \tag{4.43}$$

Letting $\frac{3}{2}\alpha_3 = \alpha_3$, $2\alpha_4 = \alpha_4$ and $-\frac{5}{2}\alpha_5 = \alpha_5$, the equation (4.43) becomes

$$\begin{aligned} \mathcal{L}_{genProca(new)} = & - \frac{1}{4}F^2_{\mu\nu} - \frac{1}{2}m^2 A^2 + \alpha_3 A^2(\partial \cdot A) \\ & + \alpha_4 \Big[(A^2)(\partial \cdot A)^2 - (A^2)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho)\Big] \\ & + \alpha_5 \Big[A^2(\partial \cdot A)^3 - 3(A^2)(\partial \cdot A)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho) \\ & + 2(A^2)(\partial_\rho A_\sigma)(\partial^\gamma A^\rho)(\partial^\sigma A_\gamma)\Big]. \end{aligned} \tag{4.44}$$

The derivatives applied on the vector field were partial derivatives in flat space-time become covariant derivatives in curve space-time

$$\mathcal{L}^{curved}_{gen.Proca} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{n=2}^{5}\beta_n\mathcal{L}_n, \tag{4.45}$$

where the self-interactions are encoded in the Lagrangian

$$\begin{aligned}
\mathcal{L}_2 &= G_2(X) \\
\mathcal{L}_3 &= G_3(X)(D_\mu A^\mu) \\
\mathcal{L}_4 &= G_4(X)R + G_{4,X}[(D_\mu A^\mu)^2 + c_2 D_\rho A_\sigma D^\sigma A^\rho - (1+c_2)D_\rho A_\sigma D^\sigma A^\rho] \\
\mathcal{L}_5 &= G_5(X)G_{\mu\nu}D^\mu A^\nu - \frac{1}{6}G_{5,X}[(D_\mu A^\mu)^3 - 3d_2(D_\mu A^\mu)D_\rho A_\sigma D^\rho A^\sigma \\
&\quad -3(1-d_2)(D_\mu A^\mu)D_\rho A_\sigma D^\sigma A^\rho + 2\left(1-\frac{3d_2}{2}\right)D_\rho A_\sigma D^\gamma A^\rho D^\sigma A_\gamma \\
&\quad +2\left(\frac{3d_2}{2}\right)D_\rho A_\sigma D^\gamma A^\rho D_\gamma A^\sigma]
\end{aligned} \tag{4.46}$$

with $X = -\frac{1}{2}A_\mu^2$. The two free parameters c_2 and d_2 as in flat space-time case. All these interactions give rise to three propagating degrees of freedom.

The generalized Proca theories have been applied extensively to different phenomenological scenarios, which include the construction of inflationary cosmological models, the analysis of de Sitter solutions relevant to dark energy models, the study of their cosmological implications in the presence of matter, the analysis of the strong lensing and time delay effects around black holes, and the construction of static and spherically symmetric solutions for black holes and neutron stars.

The generalized Proca theory is the vector field version of the Horndeski theory satisfies a necessary condition required to avoid the Ostrogradsky's instability. One has constructed the generalized Proca action for a vector field with derivative self-interactions with only three propagating degrees of freedom.

The resulting theory is simple and constitutes four Lagrangians for the self-interactions of the vector field. The constrained coefficients yield the necessary propagating constraint in order to remove the unphysical degree of freedom.

However, after the discovery of the Gravitational Wave GW170187 event, the higher order terms in Proca theories have been ruled out. By considering

$$c_t^2 = \frac{2G_4 + \phi^2\dot{\phi}G_{5,X}}{2G_4 - 2\phi^2 G_{4,X} + H\phi^3 G_{5,X}}, \tag{4.47}$$

and demanding that $c_t^2 \simeq 1$, we obtain

$$\begin{aligned}
\mathcal{L}_2 &= G_2(X) \\
\mathcal{L}_3 &= G_3(X)(D_\mu A^\mu) \\
\mathcal{L}_4 &= G_4(X)R + G_{4,X}[(D_\mu A^\mu)^2 + c_2 D_\rho A_\sigma D^\sigma A^\rho - (1+c_2)D_\rho A_\sigma D^\sigma A^\rho] \\
\mathcal{L}_5 &= G_5(X)G_{\mu\nu}D^\mu A^\nu - \frac{1}{6}G_{5,X}[(D_\mu A^\mu)^3 - 3d_2(D_\mu A^\mu)D_\rho A_\sigma D^\rho A^\sigma \\
&\quad -3(1-d_2)(D_\mu A^\mu)D_\rho A_\sigma D^\sigma A^\rho + 2\Big(1-\frac{3d_2}{2}\Big)D_\rho A_\sigma D^\gamma A^\rho D^\sigma A_\gamma \\
&\quad +2\Big(\frac{3d_2}{2}\Big)D_\rho A_\sigma D^\gamma A^\rho D_\gamma A^\sigma] - g_5(X)\tilde{F}^{\alpha\mu}\tilde{F}^\beta{}_\mu D_\alpha A_\beta \\
\mathcal{L}_6 &= G_6(X)L^{\mu\nu\alpha\beta}D_\mu A_\nu D_\alpha A_\beta + \frac{1}{2}G_{6,X}(X)\tilde{F}^{\alpha\beta}\tilde{F}^{\mu\nu}D_\alpha A_\mu D_\beta A_\nu,
\end{aligned} \tag{4.48}$$

$G_4(X)$ and $G_5(X)$ need to be constant.

Although, the tems in $\mathcal{L}_4, \mathcal{L}_5$ was vanish, we can construct more high-order action. They can write the complete expression of the generalized Abelian Proca theory in curved spacetime, which reads

$$\mathcal{L}_{gen} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}^{\text{Curv}} + \sum_{n\geq 2}\mathcal{L}_n + \sum_{n\geq 5}\mathcal{L}_n^{\epsilon}, \tag{4.49}$$

where the complete expression of the Lagrangians

$$
\begin{aligned}
\mathcal{L}^{\text{Curv}} &= f_1^{\text{Curv}} G_{\mu\nu} A^\mu A^\nu + f_2^{\text{Curv}}(X) L_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}, \\
\mathcal{L}_2 &= f_2(A_\mu, F_{\mu\nu} \tilde{F}_{\mu\nu}), \\
\mathcal{L}_3 &= f_3^{\text{Gal}}(X) \mathcal{L}_3^{\text{Gal}}, \\
\mathcal{L}_4 &= = f_4^{\text{Gal}}(X) R - 2 f_{4,X}^{\text{Gal}}, \\
\mathcal{L}_5 &= f_5^{\text{Gal}}(X) G_{\mu\nu} \nabla^\mu A^\nu + 3 f_{5,X}^{\text{Gal}}(X) \mathcal{L}_5^{\text{Gal}} + f_5^{\text{Perm}}(X) \mathcal{L}_5^{\text{Perm}}, \\
\mathcal{L}_6 &= f_6^{\text{Perm}}(X) \mathcal{L}_6^{\text{Perm}}, \\
\mathcal{L}_7 &= f_7^{\text{Perm},1}(X) \mathcal{L}_7^{\text{Perm},1}(X) + f_7^{\text{Perm},2}(X) \mathcal{L}_7^{\text{Perm},2}, \\
\mathcal{L}_{n\geq 8} &= \sum_i f_n^{\text{Perm},i}(X) \mathcal{L}_n^{\text{Perm},i}, \\
\mathcal{L}_n^{\epsilon} &= \sum_i g_n^{\epsilon,i}(X) \mathcal{L}_n^{\epsilon,i},
\end{aligned}
\tag{4.50}
$$

all f and g being arbitrary functions of X, except f_1^{Curv} which is a constant, and f_2 which is an arbitrary function of A_μ, $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$.

4.2 Faddeev-Jackiw Formalism on Generalized of Proca Field

Because of the beginning of the Faddeev-Jackiw method is starting from first order form of the Lagrangian density. So, in this section we use the result of the section 4.1. Therefor, to calculate the first order form of the Lagrangian density of the generalized of the Proca field, firstly we have to find Hamiltonian density of the system by using Legendre transformation.From section 4.1, we know that Lagrangian density of Generalized Proca field [3] in case of $n = 5$ is written as

$$\mathcal{L}_{GenPro} = \int d^3x[-\frac{1}{4}F^2_{\mu\nu} + \sum_{n=2}^{5} \alpha_n \mathcal{L}_n], \tag{4.51}$$

where $F^2_{\mu\nu} = F_{\mu\nu}F^{\mu\nu}$, α_n is any constant and $\mathcal{L}$ are self-interaction of vector fields.

$$\begin{aligned}
\mathcal{L}_{GenPro} &= \int d^3x\Big[-\frac{1}{4}F^2_{\mu\nu} - \frac{1}{2}m^2(A^2) \\
&+ \alpha_3\Big(A^2(\partial{\cdot}A) - A^\mu A^\nu \partial_\nu A_\mu\Big) \\
&+ \alpha_4\Big(A^2[(\partial{\cdot}A)^2 - \partial_\rho A_\sigma \partial^\sigma A^\rho] \\
&- 2A^\mu A^\nu \partial_\nu A_\mu(\partial{\cdot}A) + 2A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\mu\Big) \\
&+ \alpha_5\Big(A^2[-(\partial{\cdot}A)^3 + 3(\partial{\cdot}A)\partial_\rho A_\sigma \partial^\sigma A^\rho - 2\partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma] \\
&+ 3A^\mu A^\nu \partial_\nu A\mu(\partial\cdot A)^2 - 6A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\mu(\partial\cdot A) \\
&+ 6A^\mu A^\nu \partial_\nu A_\rho \partial^\rho A_\gamma \partial^\gamma A_\mu - 3A^\mu A^\nu \partial_\nu A_\mu \partial_\rho A_\sigma \partial^\sigma A^\rho\Big)\Big],
\end{aligned} \tag{4.52}$$

where $F^2_{\mu\nu} = F_{\mu\nu}F^{\mu\nu}$, $A^2 = A_\mu A^\mu$, $(\partial{\cdot}A) = \partial_\mu A^\mu$ and $\alpha_3, \alpha_4, \alpha_5$ are arbitrary constants. Rearranging some terms in equation (4.52) and seeing more details of this calculation in appendix, the equation (4.52) becomes

$$\begin{aligned}
\mathcal{L}_{GenProca} &= \int d^3x\Big[-\frac{1}{4}F^2_{\mu\nu} - \frac{1}{2}m^2(A^2) + \alpha_3(A^2)(\partial{\cdot}A) \\
&+ \alpha_4(A^2)(\partial{\cdot}A) + \alpha_4(A^2)(\partial{\cdot}A)^2 \\
&+ \alpha_5(A^2)(\partial{\cdot}A)^3 - 3\alpha_5(A^2)(\partial{\cdot}A)\partial_\rho A_\sigma \partial^\sigma A^\rho + 2\alpha_5(A^2)\partial_\rho A_\sigma \partial^\gamma A^\rho \partial^\sigma A_\gamma\Big].
\end{aligned} \tag{4.53}$$

In the context of the Faddeev-Jackiw formalism, the starting point is "first order form of the Lagrangian density" in the form of $\mathcal{L}_{FOF} = \pi^\xi \dot{A}^\xi - \mathcal{H}(A_\xi, \pi^\xi)$. The equation consists of 2 terms, the first is time-derivative term and the second is the Hamiltonian density term. Therefore, the first calculation is conjugate momentum as

$$\begin{aligned}
\pi^{\xi} &= \frac{\partial \mathcal{L}_{GenPro}}{\partial \dot{A}^{\xi}}, \\
&= F^{\xi 0} + \alpha_3 A_\mu A^\mu \eta^{0\xi} + 2\alpha_4 A_\mu A^\mu \left(\partial_\rho A^\rho \eta^{0\xi} - \partial^\xi A^0\right) \\
&+ 3\alpha_5 A_\mu A^\mu (\partial_\rho A^\rho)(\partial_\sigma A^\sigma)\eta^{0\xi} - 3\alpha_5 A_\mu A^\mu (\partial_\rho A_\sigma)(\partial^\sigma A^\rho)\eta^{0\xi} \\
&- 6\alpha_5 A_\mu A^\mu (\partial_\nu A^\nu)(\partial^\xi A^0) - 6\alpha_5 A_\mu A^\mu (\partial^\rho A_0)(\partial^\xi A_\rho). \qquad (4.54)
\end{aligned}$$

The equation (4.54) is the conjugate momentum of the Generalized of the Proca field.

1. In case of $\xi = 0$, one can see that

$$\begin{aligned}
\pi^0 &= -\alpha_3 A_\mu A^\mu - 2\alpha_4 A_\mu A^\mu (\partial_i A_i) \\
&- 3\alpha_5 A_\mu A^\mu (\partial_i A_i)(\partial_j A_j) + 3\alpha_5 A_\mu A^\mu (\partial_i A_j)(\partial_j A_i). \qquad (4.55)
\end{aligned}$$

2. In case of $\xi = i$, one can see that

$$\begin{aligned}
\pi^i &= F^{i0} + 2\alpha_4 A_\mu A^\mu (\partial_i A_0) \\
&+ 6\alpha_5 A_\mu A^\mu (\partial_i A_0)(\partial_j A_j) - 6\alpha_5 A_\mu A^\mu (\partial_j A_0)(\partial_i A_j). \qquad (4.56)
\end{aligned}$$

In this case $\eta_{\mu\nu} = (-,+,+,+)$, using $F^{i0} = -F_{i0} = -\partial_i A_0 + \partial_0 A_i$ and rearranging the equation (4.56), one can see

$$\begin{aligned}
\dot{A} &= \pi_i + (\partial_i)A_0 - 2\alpha_4 A_\mu A^\mu (\partial_i A_0) \\
&- 6\alpha_5 A_\mu A^\mu (\partial_i A_0)(\partial_j A_j) + 6\alpha_5 A_\mu A^\mu (\partial_i A_0)(\partial_j A_j). \qquad (4.57)
\end{aligned}$$

Because the second part of the first order form of the Lagrangian density is the Hamiltonian density of its system, so the Hamiltonian density of the Generalized of the Proca field is in the form of

$$\begin{aligned}
\mathcal{H}_{GenProca} &= \mathcal{H}(A_\xi, \pi^\xi) \\
&= \pi^0 \dot{A}_0 + \pi^i \dot{A}_i - \mathcal{L}_{n=5}. \qquad (4.58)
\end{aligned}$$

Substituting π^0, $\dot{A}_i$ and $\mathcal{L}_{n=5}$ into the equation (4.58), one can see that

Because the first part of the first order form of the Lagrangian density is time-derivative term as

$$\pi^\xi \dot{A}^\xi = \pi^0 \dot{A}_0 + \pi^i \dot{A}_i, \tag{4.59}$$

so, the equation (4.59) becomes

$$\begin{aligned}
\pi^\xi \dot{A}^\xi &= \Big(-\alpha_3 A_\mu A^\mu - 2\alpha_4 A_\mu A^\mu (\partial_i A_i) \\
&- 3\alpha_5 A_\mu A^\mu (\partial_\rho A^\rho)(\partial_\sigma A^\sigma) + 3\alpha_5 A_\mu A^\mu (\partial_\rho A_\sigma)(\partial^\sigma A^\rho) \\
&- 6\alpha_5 A_\mu A^\mu (\partial_\nu A^\nu)\dot{A}_0 + 6\alpha_5 A_\mu A^\mu (\partial^\rho A_0)\dot{A}_\rho \Big) \dot{A}_0 \\
&+ \Big(F^{i0} + 2\alpha_4 A_\mu A^\mu (\partial_i A_0) \\
&+ 6\alpha_5 A_\mu A^\mu (\partial_\nu A^\nu)(\partial_i A_0) - 6\alpha_5 A_\mu A^\mu (\partial^\rho A_0)(\partial_i A_\rho) \Big) \dot{A}_i
\end{aligned} \tag{4.60}$$

In this case $\eta_{\mu\nu} = (-,+,+,+)$, so $F_{i0} = \partial_i A_0 - \dot{A}_i$. From the equation (4.60), one can see that $\dot{A}_i$ is written as

$$\begin{aligned}
\dot{A}_i &= \pi_i + \partial_i A_0 - 2\alpha_4 A_\mu A^\mu (\partial_i A_0) \\
&- 6\alpha_5 A_\mu A^\mu (\partial_i A_0)(\partial_j A_j) + 6\alpha_5 A_\mu A^\mu (\partial_j A_0)(\partial_i A_j)
\end{aligned} \tag{4.61}$$

The first order form of the Lagrangian density is in the form of

$$
\begin{aligned}
\mathcal{L}_{GenPro_{(FOF)}} &= \pi^0\dot{A}_0 + \pi_i\dot{A}_i \\
&- \frac{1}{2}\pi_i\pi^i - \pi_i(\partial_i A_0) - \frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}m^2 A_\mu A^\mu \\
&+ \alpha_3 A_\mu A^\mu(\partial_i A^i) - 2{\alpha_4}^2 A_\mu A^\mu A_\nu A^\nu(\partial_i A_0)^2 \\
&+ \alpha_4 A_\mu A^\mu(\partial_i A_i)(\partial_j A_j) - \alpha_4 A_\mu A^\mu(\partial_i A_j)(\partial_j A_i) \\
&+ 2\alpha_4 A_\mu A^\mu(\partial_i A_0)\pi_i + 2\alpha_4 A_\mu A^\mu(\partial_i A_0)^2 \\
&- 18{\alpha_5}^2 A_\mu A^\mu A_\nu A^\nu(\partial_i A_0)^2(\partial_j A_j)(\partial_k A_k) \\
&+ 12{\alpha_5}^2 A_\mu A^\mu A_\nu A^\nu(\partial_i A_0)(\partial_k A^k)(\partial_j A_0)(\partial_i A_j)(\partial_i A_j) \\
&+ 6{\alpha_5}^2 A_\mu A^\mu A_\nu A^\nu(\partial_j A_0)(\partial_i A_j)(\partial_k A_0)(\partial_i A_k) \\
&+ \alpha_5 A_\mu A^\mu(\partial_i A_i)(\partial_j A_j)(\partial_k A_k) + 6\alpha_5 A_\mu A^\mu(\partial_i A_0)(\partial_k A_k)\pi_i \\
&+ 6\alpha_5 A_\mu A^\mu(\partial_i A_0)^2(\partial_k A_k) - 3\alpha_5 A_\mu A^\mu(\partial_i A_j)(\partial_j A_i)(\partial_k A_k) \\
&- 2\alpha_5 A_\mu A^\mu(\partial_j A_0)(\partial_i A_j)\pi_i - 2\alpha_5 A_\mu A^\mu(\partial_i A_0)(\partial_j A_0)(\partial_i A_j) \\
&+ 2\alpha_5 A_\mu A^\mu(\partial_i A_j)(\partial_k A_i)(\partial_j A_k) \\
&- 12\alpha_4\alpha_5 A_\mu A^\mu A_\nu A^\nu(\partial_i A_0)^2(\partial_k A_k) \\
&+ 4\alpha_4\alpha_5 A_\mu A^\mu A_\nu A^\nu(\partial_i A_0)(\partial_i A_0)(\partial_j A_0)(\partial_i A_j) \\
&+ \gamma_1[\pi_0 - \alpha_3 A_\mu A^\mu - 2\alpha_4 A_\mu A^\mu(\partial_i A_i) \\
&- 3\alpha_5 A_\mu A^\mu(\partial_i A_i)(\partial_j A_j) + 3\alpha_5 A_\mu A^\mu(\partial_i A_j)(\partial_j A_i)] \qquad (4.62)
\end{aligned}
$$

Considering the equation (4.62)the symplectic variables are $\xi^{(0)} = (A_0, \pi^0, A_i, \pi_i, \gamma_1)$. Next step is to calculate the canonical momenta $a_\xi^{(0)}$ for the symplectic variables $\xi^{(0)}$. When $\xi^{(0)} = A_0$, we see

$$
\begin{aligned}
\mathcal{A}_{A_0} = a_{A_0}^{(0)} = \frac{\partial\mathcal{L}_{FOF}(x)}{\partial\dot{A}_0(x')} &= \frac{\partial}{\partial\dot{A}_0(x')}\int d^3x\left\{\pi^0(x)\dot{A}_0(x)\right\}, \qquad (4.63)\\
&= \int \pi^0(x')\delta(x-x')d^3x,
\end{aligned}
$$

$$
\mathcal{A}_{A_0} = a_{A_0}^{(0)} = \frac{\partial\mathcal{L}_{FOF}(x)}{\partial\dot{A}_0(x')} = \pi^0(x') = -\pi_0(x'). \qquad (4.64)
$$

When $\xi^{(0)} = \pi_0$, we see

$$\mathcal{A}_{\pi_0} = a^{(0)}_{\pi_0} = \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{\pi}_0(x')} = \frac{\partial}{\partial \dot{\pi}_0(x')} \int d^3x \, \{0\} = 0. \tag{4.65}$$

when $\xi^{(0)} = A_i$, we see

$$\begin{aligned} \mathcal{A}_{A_i} = a^{(0)}_{A_i} = \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{A}_i(x')} &= \frac{\partial}{\partial \dot{A}_i(x')} \int d^3x \left\{ \pi_i(x) \dot{A}_i(x) \right\}, \\ &= \int \pi_i(x') \delta(x - x') d^3x, \end{aligned} \tag{4.66}$$

$$\mathcal{A}_{A_i} = a^{(0)}_{A_i} = \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{A}_i(x')} = \pi_i(x'). \tag{4.67}$$

when $\xi^{(0)} = \pi_i$, we see

$$\mathcal{A}_{\pi_i} = a^{(0)}_{\pi_i} = \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \dot{\pi}_i(x')} = \frac{\partial}{\partial \dot{\pi}_i(x')} \int d^3x \, \{0\} = 0. \tag{4.68}$$

when $\xi^{(0)} = \gamma_1$, we see

$$\begin{aligned} \mathcal{A}_{\gamma_1} &= a^{(0)}_{\gamma_1} = \frac{\partial \mathcal{L}_{FOF}(x)}{\partial \gamma_1(x')} \qquad (4.69) \\ &= \frac{\partial}{\partial \gamma_1(x')} \int d^3x \gamma_1 \Big(\pi_0 - \alpha_3 A_\mu A^\mu - 2\alpha_4 A_\mu A^\mu (\partial_i A_i) \\ &- 3\alpha_5 A_\mu A^\mu (\partial_i A_i)(\partial_j A_j) + 3\alpha_5 A_\mu A^\mu (\partial_i A_j)(\partial_j A_i) \Big) \\ \mathcal{A}_{\gamma_1} &= \pi_0(x') - \alpha_3 A_\mu(x') A^\mu(x') - 2\alpha_4 A_\mu(x') A^\mu(x') \partial_i A_i(x') \\ &- 3\alpha_5 A_\mu(x') A^\mu(x') \partial_i A_i(x') \partial_j A_j(x') \\ &+ 3\alpha_5 A_\mu(x') A^\mu(x') \partial_i A_j(x') \partial_j A_i(x'). \end{aligned} \tag{4.70}$$

Because $\mathcal{A}_{A_0}, \mathcal{A}_{A_i}, \mathcal{A}_{\gamma_1} \neq 0$, so the canonical 1-form of the system is in the form of

$$\mathcal{A} = \int d^3x \Big(\mathcal{A}_{A_0} \delta A_0(x) + \mathcal{A}_{A_i} \delta A_i(x) + \mathcal{A}_{A_{\gamma_1}} \delta \gamma_1(x) \Big). \tag{4.71}$$

Substituting equation (4.64), (4.67) and (4.70) into equation (4.71), we can see that

$$\begin{aligned}\mathcal{A} = \int d^3x \Big([&- \pi_0(x)]\delta A_0(x) + [\pi_i(x)]\delta A_i(x) \\ &+ [\pi_0(x) - \alpha_3 A_\mu(x)A^\mu(x) - 2\alpha_4 A_\mu(x)A^\mu(x)\partial_i A_i(x) \\ &- 3\alpha_5 A_\mu(x)A^\mu(x)\partial_i A_i(x)\partial_j A_j(x) \\ &+ 3\alpha_5 A_\mu(x)A^\mu(x)\partial_i A_j(x)\partial_j A_i(x)]\delta\gamma_1(x)\Big). \end{aligned} \tag{4.72}$$

After we get canonical 1-form ($\mathcal{A}$), Using vary with the canonical 1-form one can get Symplectic 2-form ($\mathcal{F}$) of the system in terms of wedge products

$$\begin{aligned}\mathcal{F} = \int d^3x \Big(&- [\pi_0(x) \wedge \delta A_0(x)] + [\pi_i(x) \wedge \delta A_i(x)] \\ &+ [\pi_0(x) \wedge \delta\gamma_1(x)] - 2\alpha_3 A^\mu(x)[\delta A_\mu(x) \wedge \delta\gamma_1(x)] \\ &- 4\alpha_4 A^\mu(x)\partial_i A_i(x)[A_\mu(x) \wedge \delta\gamma_1(x)] \\ &- 2\alpha_4 A_\mu(x)A^\mu(x)[\partial_i \delta A_i(x) \wedge \delta\gamma_1(x)] \\ &- 6\alpha_5 A^\mu(x)\partial_i A_i(x)\partial_j A_j(x)[\delta A_\mu(x) \wedge \delta\gamma_1(x)] \\ &- 6\alpha_5 A_\mu(x)A^\mu(x)\partial_j A_j(x)[\partial_i \delta A_i(x) \wedge \delta\gamma_1(x)] \\ &+ 6\alpha^5 A^\mu(x)\partial_i A_j(x)\partial_j A_i(x)[\delta A_\mu(x) \wedge \delta\gamma_1(x)] \\ &+ 6\alpha_5 A_\mu(x)A^\mu(x)\partial_i A_j(x)[\partial_j \delta A_i(x)]\delta\gamma_1(x)\Big). \end{aligned} \tag{4.73}$$

Considering equation (4.73) and using interior derivative with the symplectic 2-form ($\mathcal{F}$), we then obtain

$$
\begin{aligned}
\mathcal{F} = \int d^3x \quad \Big(& [-z^{\pi_0(x)}\delta A_0(x) + z^{A_0(x)}\delta\pi_0(x)] + [z^{\pi_i(x)}\delta A_i(x) - z^{A_i(x)}\delta\pi_i(x)] \\
+ & [z^{\pi_0(x)}\delta\gamma_1(x) - z^{\gamma_1(x)}\delta\pi_0(x)] \\
+ & [-2\alpha_3 A^\mu(x) z^{A_\mu(x)}\delta\gamma_1(x) + 2\alpha_3 A^\mu(x) z^{\gamma_1(x)}\delta A_\mu(x)] \\
+ & [-4\alpha_4 A^\mu(x)\partial_i A_i(x) z^{A_\mu(x)}\delta\gamma_1(x) + 4\alpha_4 A^\mu(x)\partial_i A_i(x) z^{\gamma_1(x)}\delta A_\mu(x)] \\
+ & [-2\alpha_4 A_\mu(x) A^\mu(x)\partial_i z^{A_i(x)}\delta\gamma_1(x) + \underline{2\alpha_4 A_\mu(x) A^\mu(x) z^{\gamma_1(x)}\partial_i\delta A_i(x)}] \\
+ & [-6\alpha_5 A^\mu(x)\partial_i A_i(x)\partial_j A_j(x) z^{A_\mu(x)}\delta\gamma_1(x) \\
+ & 6\alpha_5 A^\mu(x)\partial_i A_i(x)\partial_j A_j(x) z^{\gamma_1(x)}\delta A_\mu(x)] \\
+ & [-6\alpha_5 A_\mu(x) A^\mu(x)\partial_j A_j(x)\partial_i z^{A_i(x)}\delta\gamma_1(x) \\
+ & \underline{\underline{6\alpha_5 A_\mu(x) A^\mu(x)\partial_j A_j(x) z^{\gamma_1(x)}\partial_i\delta A_i(x)]}} \\
+ & [6\alpha_5 A^\mu(x)\partial_i A_j(x)\partial_j A_i(x) z^{A_\mu(x)}\delta\gamma_1 \\
- & 6\alpha_5 A^\mu(x)\partial_i A_j(x)\partial_j A_i(x) z^{\gamma_1}\delta A_\mu(x)] \\
+ & [6\alpha_5 A_\mu(x) A^\mu(x)\partial_i A_j(x)\partial_j z^{A_i(x)}\delta\gamma_1(x) \\
- & \underline{\underline{6\alpha_5 A_\mu(x) A^\mu(x)\partial_i A_j(x) z^{\gamma_1(x)}\partial_j\delta A_i(x)]}}\Big).
\end{aligned} \tag{4.74}
$$

Considering equation (4.74) and using integrate by part with the underline terms, so equation (4.74) is written as

$$
\begin{aligned}
\mathcal{F} = \int d^3x \quad \Big(& -z^{\pi_0}\delta A_0 + z^{A_0}\delta\pi_0 + z^{\pi_i}\delta A_i - z^{A_i}\delta\pi_i \\
+ \; & z^{\pi_0}\delta\gamma_1 - z^{\gamma_1}\delta\pi_0 \\
- \; & 2\alpha_3 A^\mu z^{A_\mu}\delta\gamma_1 + 2\alpha_3 A^\mu z^{\gamma_1}\delta A_\mu \\
- \; & 4\alpha_4 A^\mu \partial_i A_i z^{A_\mu}\delta\gamma_1 + 4\alpha_4 A^\mu \partial_i A_i z^{\gamma_1}\delta A_\mu \\
- \; & 2\alpha_4 A_\mu A^\mu \partial_i z^{A_i}\delta\gamma_1 \underline{- 4\alpha_4 A^\mu \partial_i A_\mu z^{\gamma_1}\delta A_i} \\
- \; & \underline{2\alpha_4 A_\mu A^\mu \partial_i z^{\gamma_1}\delta A_i} \\
- \; & 6\alpha_5 A^\mu \partial_i A_i \partial_j A_j z^{A_\mu}\delta\gamma_1 \\
+ \; & 6\alpha_5 A^\mu \partial_i A_i \partial_j A_j z^{\gamma_1}\delta A_\mu \\
- \; & 6\alpha_5 A_\mu A^\mu \partial_j A_j \partial_i z^{A_i}\delta\gamma_1 \\
- \; & \underline{\underline{12\alpha_5 A^\mu \partial_i A_\mu \partial_j A_j z^{\gamma_1}\delta A_i}} \\
- \; & \underline{6\alpha_5 A_\mu A^\mu \partial_i \partial_j A_j z^{\gamma_1}\delta A_i} \\
- \; & \underline{\underline{6\alpha_5 A_\mu A^\mu \partial_j A_j \partial_i z^{\gamma_1}\delta A_i}} \\
+ \; & 6\alpha_5 A^\mu \partial_i A_j \partial_j A_i z^{A_\mu}\delta\gamma_1 \\
- \; & 6\alpha_5 A^\mu \partial_i A_j \partial_j A_i z^{\gamma_1}\delta A_\mu \\
+ \; & 6\alpha_5 A_\mu A^\mu \partial_i A_j \partial_j z^{A_i}\delta\gamma_1 \\
+ \; & \underline{\underline{12\alpha_5 A^\mu \partial_j A_\mu \partial_i A_j z^{\gamma_1}\delta A_i}} \\
+ \; & \underline{\underline{6\alpha_5 A_\mu A^\mu \partial_i \partial_j A_j z^{\gamma_1}\delta A_i}} \\
+ \; & \underline{\underline{\underline{6\alpha_5 A_\mu A^\mu \partial_i A_j \partial_j z^{\gamma_1}\delta A_i}}}\Big).
\end{aligned}
\tag{4.75}
$$

In this work, index $\mu = 0, i$ where $i = 1, 2, 3$, to expand equation (4.75) by using index in 4 dimensions, equation (4.75) becomes

$$
\begin{aligned}
\mathcal{F} = \int d^3x \quad \Big(& -z^{\pi_0}\delta A_0 + z^{A_0}\delta\pi_0 + z^{\pi_i}\delta A_i - z^{A_i}\delta\pi_i \\
+ & \; z^{\pi_0}\delta\gamma_1 - z^{\gamma_1}\delta\pi_0 \\
- & \; 2\alpha_3 A^0 z^{A_0}\delta\gamma_1 - 2\alpha_3 A^i z^{A_i}\delta\gamma_1 + 2\alpha_3 A^0 z^{\gamma_1}\delta A_0 + 2\alpha_3 A^i z^{\gamma_1}\delta A_i \\
- & \; 4\alpha_4 A^0 \partial_i A_i z^{A_0}\delta\gamma_1 - 4\alpha_4 A^j \partial_i A_i z^{A_i}\delta\gamma_1 \\
+ & \; 4\alpha_4 A^0 \partial_i A_i z^{\gamma_1}\delta A_0 + 4\alpha_4 A^j \partial_i A_i z^{\gamma_1}\delta A_j \\
- & \; 2\alpha_4 A_\mu A^\mu \partial_i z^{A_i}\delta\gamma_1 \underline{- 4\alpha_4 A^\mu \partial_i A_\mu z^{\gamma_1}\delta A_i} \\
- & \; \underline{2\alpha_4 A_\mu A^\mu \partial_i z^{\gamma_1}\delta A_i} \\
- & \; 6\alpha_5 A^\mu \partial_i A_i \partial_j A_j z^{A_\mu}\delta\gamma_1 \\
+ & \; 6\alpha_5 A^\mu \partial_i A_i \partial_j A_j z^{\gamma_1}\delta A_\mu \\
- & \; 6\alpha_5 A_\mu A^\mu \partial_j A_j \partial_i z^{A_i}\delta\gamma_1 \\
- & \; \underline{\underline{12\alpha_5 A^\mu \partial_i A_\mu \partial_j A_j z^{\gamma_1}\delta A_i}} \\
- & \; \underline{\underline{6\alpha_5 A_\mu A^\mu \partial_i \partial_j A_j z^{\gamma_1}\delta A_i}} \\
- & \; \underline{\underline{6\alpha_5 A_\mu A^\mu \partial_j A_j \partial_i z^{\gamma_1}\delta A_i}} \\
+ & \; 6\alpha_5 A^\mu \partial_i A_j \partial_j A_i z^{A_\mu}\delta\gamma_1 \\
- & \; 6\alpha_5 A^\mu \partial_i A_j \partial_j A_i z^{\gamma_1}\delta A_\mu \\
+ & \; 6\alpha_5 A_\mu A^\mu \partial_i A_j \partial_j z^{A_i}\delta\gamma_1 \\
+ & \; \underline{\underline{\underline{12\alpha_5 A^\mu \partial_j A_\mu \partial_i A_j z^{\gamma_1}\delta A_i}}} \\
+ & \; \underline{\underline{\underline{6\alpha_5 A_\mu A^\mu \partial_i \partial_j A_j z^{\gamma_1}\delta A_i}}} \\
+ & \; \underline{\underline{\underline{6\alpha_5 A_\mu A^\mu \partial_i A_j \partial_j z^{\gamma_1}\delta A_i}}} \Big).
\end{aligned}
\tag{4.76}
$$

After we use the Faddeev-Jackiw formalism on th generalized of the Proca field, we then obtain 2 constraints as

$$
\begin{aligned}
\Omega_1 &= \pi^0 + \alpha_3 f_3(A^2) + 2\alpha_4 f_4(A^2)(\vec{\nabla}\cdot\vec{A}) \\
&+ 3\alpha_5 f_5(A^2)\Big((\vec{\nabla}\cdot\vec{A})^2 - \partial_i A_j \partial_j A_i\Big),
\end{aligned} \tag{4.77}
$$

and

$$
\begin{aligned}
\Omega_2 &= \vec{\nabla}\cdot\vec{\pi} - 2\alpha_2 f_2'(A^2)A_0 + 2\alpha_3 f_3'(A^2)\Big[\vec{A}\cdot(\vec{\pi} + \vec{\nabla}A_0) - A_0(\vec{\nabla}\cdot\vec{A})\Big] \\
&+ \alpha_4\Big[4f_4'(A^2)\Big(2A^{[i}(\pi_i + \partial_i A_0)\partial^{j]}A_j - A_\mu \partial_j A^\mu \partial_j A_0 \\
&+ A_0 \partial^{[i} A_j \partial^{j]} A_i\Big) - 2f_4\Big(A^2\Big)\nabla^2 A_0\Big] \\
&- 4\alpha_3\alpha_4 f_3'\Big(A^2\Big) f_4\Big(A^2\Big)\vec{A}\cdot\vec{\nabla}A_0 + 16{\alpha_4}^2 f_4\Big(A^2\Big) f_4'\Big(A^2\Big)A^{[i}\partial_j A_0 \partial^{j]} A_i \\
&+ 12\alpha_5\Big(f_5'\Big(A^2\Big)\Big(3A_i(\pi^{[i} + \partial^{[i}A_0)\partial_j A^j \partial_k A^{k]} + 2A_\mu \partial_j A^\mu \partial^{[j} A_k \partial^{k]} A_0 \\
&- A_0 \partial^{[i} A_i \partial^j A_j \partial^{k]} A_k\Big) + f_5\Big(A^2\Big)\partial^{[i}\Big(\partial_j A_0 \partial_i A^{j]}\Big)\Big) \\
&+ 24\alpha_3\alpha_5 f_3'\Big(A^2\Big) f_5\Big(A^2\Big) A_i \partial^{[j} A_0 \partial^{i]} A_j \\
&- 24\alpha_4\alpha_5\Big(4f_4'\Big(A^2\Big) f_5\Big(A^2\Big) A^{[i}\partial^{k]} A_k \partial^{[i} A_0 \partial^{j]} A_j \\
&+ 3f_4\Big(A^2\Big) f_5'\Big(A^2\Big) A_i \partial^{[i} A_0 \partial_j A^j \partial_k A^{k]}\Big) \\
&- 432(\alpha_5)^2 f_5\Big(A^2\Big) f_5'\Big(A^2\Big) A^{[i}\partial^j A_j \partial^{k]} A_k \partial_{[i|} A_0 \partial_{|l]} A^l.
\end{aligned} \tag{4.78}
$$

CHAPTER V

RESULTS AND DISCUSSIONS

5.1 On constrained analysis and diffeomorphism invariance on generalised Proca theories

In this section, we will review and conclude on the paper [10]. For this paper, we are interested in the Faddeev-Jackiw formalism on the generalised of the Proca field.

$$\begin{aligned}\mathcal{L}_{\text{genProca(new)}} &= -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}m^2A^2 + \alpha_3 A^2(\partial\cdot A)\\ &+ \alpha_4\Big[(A^2)(\partial\cdot A)^2 - (A^2)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho)\Big]\\ &+ \alpha_5\Big[A^2(\partial\cdot A)^3 - 3(A^2)(\partial\cdot A)(\partial_\rho A_\sigma)(\partial^\sigma A^\rho)\\ &+ 2(A^2)(\partial_\rho A_\sigma)(\partial^\gamma A^\rho)(\partial^\sigma A_\gamma)\Big]. \qquad (5.1)\end{aligned}$$

In this theory, there are 2 constraints [10]. The constraints can be written as

$$\begin{aligned}\Omega_1 &= \pi^0 + \alpha_3 f_3(A^2) + 2\alpha_4 f_4(A^2)(\vec{\nabla}\cdot\vec{A})\\ &+ 3\alpha_5 f_5(A^2)\Big((\vec{\nabla}\cdot\vec{A})^2 - \partial_i A_j \partial_j A_i\Big), \qquad (5.2)\end{aligned}$$

and

$$\begin{aligned}\Omega_2 &= \vec{\nabla}\cdot\vec{\pi} - 2\alpha_2 f_2'(A^2)A_0 + 2\alpha_3 f_3'(A^2)\Big[\vec{A}\cdot(\vec{\pi} + \vec{\nabla}A_0) - A_0(\vec{\nabla}\cdot\vec{A})\Big]\\ &+ \alpha_4\Big[4f_4'(A^2)\Big(2A^{[i}(\pi_i + \partial_i A_0)\partial^{j]}A_j - A_\mu\partial_j A^\mu\partial_j A_0\\ &+ A_0\partial^{[i}A_j\partial^{j]}A_i\Big) - 2f_4(A^2)\nabla^2 A_0\Big]\\ &- 4\alpha_3\alpha_4 f_3'(A^2)f_4(A^2)\vec{A}\cdot\vec{\nabla}A_0 + 16{\alpha_4}^2 f_4(A^2)f_4'(A^2)A^{[i}\partial_j A_0\partial^{j]}A_i\\ &+ 12\alpha_5{\alpha_4}^2 f_4(A^2)f_4'(A^2) + A^{[i}\partial_j A_0\partial^{j]}A_i\\ &- A_0A_0\partial^{[i}A_j\partial^{j]}A_i\Big) - 2f_4(A^2)\nabla^2 A_0\\ &+ 24\alpha_3\alpha_5 f_4(A^2)f_4'(A^2) + A^{[i}\partial_j A_0\partial^{j]}A_i\\ &+ 3f_4(A^2)A_\mu\partial_j A^\mu\partial_j A^0\\ &+ -423(\alpha_5)^2 f_5(A^2)f_5'(A^2) \qquad (5.3)\end{aligned}$$

5.2 Non-linear Schrödinger-type formulation of scalar field cosmology: two barotropic fluids and exact solutions

Phenomena in physic can often be described as solutions to same differential equation. Ermakov-Pinney system is a non-linear 2^{nd} order ordinary diferrentail equation, that is in the form

$$\ddot{b}(t) + Q(t)b(t) = \frac{\lambda}{b^3(t)} \tag{5.4}$$

In this work, we want to use the Ermakov-Pinney equation to study some events of the Universe in the context of cosmology. Ermakov-Pinney equation can be related to equations describing cosmology by using,

$$b(t) \equiv u^{-1}(t) = a^{n/2}(t), \tag{5.5}$$

where $a(t)$ is scale factor. From , Q and λ are

$$Q(t) = \frac{\kappa^2 n\dot{\phi}^2}{4}, \tag{5.6}$$

$$\lambda = -\frac{Dn^2\kappa^2}{12}. \tag{5.7}$$

If λ vanishes, the equation (5.4) reduces to homogeneous second order ordinary differential equation

$$\ddot{b}(t) + Q(t)b(t) = 0. \tag{5.8}$$

These system is relate to flat FLRW cosmology where Fridmann and Klein-Gordon equations are

$$H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi) + \frac{D}{a^n}\right), \tag{5.9}$$

$$\epsilon(\ddot{\phi} + 3H\dot{\phi}) = -\frac{\mathrm{d}V}{\mathrm{d}\phi}. \tag{5.10}$$

Considering Ermakov-Pinney system eq.(5.4), this system can be reparametrized by $\dot{x}(t) = u(x)$. The eq.(5.4) becomes 1-dimensional linear Schrödinger equation as

$$u''(x) + [E - P(x)]u(x) = 0. \tag{5.11}$$

In this case,

$$P(x) = \frac{\kappa^2 n\epsilon\phi'^2}{4}, \quad (5.12)$$
$$E = -\frac{Dn^2\kappa^2}{12}. \quad (5.13)$$

Where $\epsilon = 1$ and $\epsilon = -1$ mean canonical scalar field and phantom field respectively. In case of linear ordinary differential equation, one can use the equation (5.8) and (5.11) to connect and study together. In this work we want to study in context of cosmology, therefore we will start from Friedmann equation and acceleration equation as

$$H^2 = \frac{\kappa^2}{3}\rho_{tot} - \frac{k}{a^2}, \quad (5.14)$$
$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}\Big(\rho_{tot} + 3p_{tot}\Big), \quad (5.15)$$

where H is Hubble parameter, κ is constant (Gravitational constant), k is curvature, a is scale factor, ρ_{tot} and p_{tot} is total energy density and total pressure respectively. According to scalar-tensor theory, the energy density and pressure of scalar field is given by

$$\rho_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi), \quad (5.16)$$
$$p_\phi = \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi). \quad (5.17)$$

In the case of barotropic fluid, the energy density and pressure can be written as

$$\rho_\gamma = D_\gamma/a^n, \quad (5.18)$$

$$p_\gamma = w_\gamma\rho_\gamma, \quad (5.19)$$

where $n = 3(1+w)$. Consider FLRW universe which has non-interacting two barotropic and minimally coupled scalar field, ϕ, as a sources. The density and pressure of two barotropic fluid read

$$\rho_1 = \frac{D_1}{a^n}, \quad \rho_2 = \frac{D_2}{a^m} \quad (5.20)$$

and

$$p_1 = \left(\frac{n-3}{3}\right)\frac{D_1}{a^n}, \quad p_2 = \left(\frac{m-3}{3}\right)\frac{D_2}{a^m} \quad (5.21)$$

where n and m implies types of 1st and 2nd fluids respectively. In this case study, the Friedmann equation (5.14) can be expressed as

$$\begin{aligned} H^2 &= \frac{\kappa^2}{3}(\rho_1+\rho_2+\rho_\phi) - \frac{k}{a^2} \\ &= \frac{\kappa^2}{3}\left[\frac{1}{2}\epsilon\dot{\phi}^2 + V(\phi) + \frac{D_1}{a^n} + \frac{D_2}{a^m}\right] - \frac{k}{a^2}, \end{aligned} \tag{5.22}$$

and acceleration equation (5.15) yields

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{\kappa^2}{6}\Big(\rho_1+\rho_2+\rho_\phi+3[p_1+p_2+p_\phi]\Big), \\ &= -\frac{\kappa^2}{6}\Big(\Big[\frac{D_1}{a^n}+\frac{D_2}{a^m}+\frac{1}{2}\epsilon\dot{\phi}^2+V(\phi)\Big]+3\Big[\frac{(n-3)}{3}\frac{D_1}{a^n}+\frac{(m-3)}{3}\frac{D_2}{a^m} \\ &\quad +\frac{1}{2}\epsilon\dot{\phi}^2-V(\phi)\Big]\Big), \\ &= -\frac{\kappa^2}{6}\left(2\epsilon\dot{\phi}^2-2V(\phi)+(n-2)\frac{D_1}{a^n}+(m-2)\frac{D_2}{a^m}\right). \end{aligned} \tag{5.23}$$

Recall that

$$H = \frac{\dot{a}}{a}. \tag{5.24}$$

So time derivative of the Hubble parameter is written as

$$\dot{H} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2, \tag{5.25}$$

therefore

$$\frac{\ddot{a}}{a} = \dot{H} + H^2. \tag{5.26}$$

Substituting into eq.(5.23) this gives

$$\dot{H} + H^2 = -\frac{\kappa^2}{6}\left(2\epsilon\dot{\phi}^2-2V(\phi)+(n-2)\frac{D_1}{a^n}+(m-2)\frac{D_2}{a^m}\right) \tag{5.27}$$

According to Friedmann equation (5.22), substituting H^2 into (5.27) this gives

$$\begin{aligned} \frac{6}{\kappa^2}\left(\dot{H}-\frac{k}{a^2}\right)+\epsilon\dot{\phi}^2+\frac{2D_1}{a^n}+\frac{2D_2}{a^m} &= -2\epsilon\dot{\phi}^2-(n-2)\frac{D_1}{a^n}-(m-2)\frac{D_2}{a^m} \\ \frac{6}{\kappa^2}\left(\dot{H}-\frac{k}{a^2}\right)+3\epsilon\dot{\phi}^2+\frac{2D_1}{a^n}+\frac{2D_2}{a^m} &= -n\frac{D_1}{a^n}+\frac{2D_1}{a^n}-m\frac{D_2}{a^m}+\frac{2D_1}{a^n} \\ \epsilon\dot{\phi}^2 &= -\frac{2}{\kappa^2}\left(\dot{H}-\frac{k}{a^2}\right)-\frac{nD_1}{3a^n}-\frac{mD_2}{3a^m} \end{aligned} \tag{5.28}$$

Substituting (5.28) into Friedmann equation (5.22), the potential, $V(\phi)$, can be obtain as

$$
\begin{aligned}
H^2 + \frac{\dot{H}}{3} &= -\frac{k}{3a^2} + \frac{\kappa^2}{3}\left[-\frac{nD_1}{6a^n} - \frac{mD_2}{6a^m} + V(\phi) + \frac{D_1}{a^n} + \frac{D_2}{a^m}\right] - \frac{k}{a^2} \\
H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2} &= \frac{\kappa^2}{3}\left[\left(\frac{6-n}{6}\right)\frac{D_1}{a^n} + \left(\frac{6-m}{6}\right)\frac{D_2}{a^m} + V(\phi)\right] \\
\frac{3}{\kappa^2}\left(H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right) &= \left(\frac{6-n}{6}\right)\frac{D_1}{a^n} + \left(\frac{6-m}{6}\right)\frac{D_2}{a^m} + V(\phi) \\
V(\phi) &= \frac{3}{\kappa^2}\left(H^2 + \frac{\dot{H}}{3} + \frac{2k}{3a^2}\right) + \left(\frac{n-6}{6}\right)\frac{D_1}{a^n} + \left(\frac{m-6}{6}\right)\frac{D_2}{a^m}
\end{aligned}
\tag{5.29}
$$

Note that it is sufficient to consider Friendmann and acceleration equations because Klien-Gordon equation is a consequence of these two equation. The value of n or m determine types of fluids i.e. $n = 0$ for $w = -1$ (cosmological constant), $n = 2$ for $w = -1/3$, $n = 3$ for $w = 0$ (non-relativistic matter), $n = 4$ for $w = 1/3$ (radiation), and $n = 6$ for $w = 1$ (stiff fluid).

Now, we accomplish to construct Fridmann equation and acceleration one for scalar field and two barotropic fluids as a sources. Then, we need to construct Schrödinger formalism (5.11) which associate with two barotropic fluid. Let us define

$$
u(x) \equiv a(t)^{-n/2}, \tag{5.30}
$$

By using $\dot{x}(t) = u(x)$, the Schrödinger formalism (5.11) can be calculated as follows. First, let us take derivative with respect to x to equation (5.30),

$$
\begin{aligned}
u'(x) &= \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-n}{2u}a^{-n/2}\frac{\dot{a}}{a} \\
H &= \frac{-2}{n}u'(x) \qquad (5.31) \\
u''(x) &= \frac{-n}{2}\dot{H}\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{-n}{2}\frac{\dot{H}}{u} \\
\dot{H} &= \frac{-2}{n}u''(x)u(x) \qquad (5.32)
\end{aligned}
$$

Furthermore, from eq.(5.30), we obtain the following useful relations

$$
a = u^{-2/n}, \quad a^n = u^{-2}, \quad a^m = u^{-2m/n}. \tag{5.33}
$$

Then, substituting H and $\dot{H}$ into Friedmann equation and using potential equation (5.29), this gives

$$\begin{aligned}
\frac{4}{n^2}(u')^2 &= \frac{\kappa^2}{3}\left[\frac{1}{2}\epsilon\dot{\phi}^2 + \frac{3}{\kappa^2}\left(\frac{4}{n^2}(u')^2 + \frac{-2}{3n}u''(x)u(x) + \frac{2k}{3a^2}\right)\right.\\
&\left. + \left(\frac{n-6}{6}\right)\frac{D_1}{a^n} + \left(\frac{m-6}{6}\right)\frac{D_2}{a^m} + \frac{D_1}{a^n} + \frac{D_2}{a^m}\right] - \frac{k}{a^2},\\
\frac{12}{\kappa^2 n^2}(u')^2 &= \frac{1}{2}\epsilon\dot{\phi}^2 + \frac{12}{\kappa^2 n^2}(u')^2 + \frac{-6}{3\kappa^2 n}u''(x)u(x) + \frac{2k}{\kappa^2 a^2}\\
&+ \left(\frac{n-6}{6}\right)\frac{D_1}{a^n} + \left(\frac{m-6}{6}\right)\frac{D_2}{a^m} + \frac{D_1}{a^n} + \frac{D_2}{a^m} - \frac{3k}{\kappa^2 a^2},
\end{aligned}$$

$$\begin{aligned}
\frac{2}{\kappa^2 n}u''(x)u(x) &= \frac{1}{2}\epsilon\dot{\phi}^2\frac{u^2(x)}{u^2(x)} + \frac{nD}{6}u^2(x) + \frac{mD_2}{6}a^{n-m}u^2(x) - \frac{k}{\kappa^2 a^2}\frac{u(x)}{u(x)},\\
\frac{2}{\kappa^2 n}u''(x) &= \frac{1}{2}\epsilon\dot{\phi}^2 a^n(t)u(x) + \frac{nD}{6}u(x) + \frac{mD_2}{6}a^{n-m}u(x) - \frac{k}{\kappa^2 a^2}\frac{1}{u(x)},\\
u''(x) &= \underbrace{\frac{\kappa^2 n^2 D}{12}}_{-E}u(x) + \left(\underbrace{\frac{\kappa^2 n}{4}\epsilon\dot{\phi}^2 a^n(t) + \frac{\kappa^2 mD_2}{12}na^{n-m}}_{P(x)}\right)u(x)\\
&- \frac{nk}{2}u(x)^{(4-n)/n}.
\end{aligned} \tag{5.34}$$

Finally, the equation (5.28) and (5.29) can be obtained as,

$$u''(x) + [E - P(x)]u(x) = -\frac{nk}{2}u(x)^{(4-n)/n}, \tag{5.35}$$

where

$$E \equiv -\frac{\kappa^2 n^2}{12}D_1, \tag{5.36}$$

$$P(x) \equiv \frac{\kappa^2 n}{4}a(t)^n\epsilon\dot{\phi}(t)^2 + \frac{mD_2}{12}\kappa^2 na^{n-m}. \tag{5.37}$$

We encounter non-linearity equation (5.35) which is called non-linear Schrödinger equation (NLS). We can express kinetic term, $\epsilon\dot{\phi}^2(t)$, and potential, $V(\phi)$, and other cosmological quantities as a function of $u(x)$. According to kinetic term eq.(5.28) and potential

(5.29), using (5.30), (5.31), (5.32), (5.33) and (5.36) this gives

$$\begin{aligned}
\epsilon\dot{\phi}(t)^2 &= -\frac{2}{\kappa^2}\left(\frac{-2}{n}u''u - ku^{4/n}\right) - \frac{nD_1u^2}{3} - \frac{mD_2u^{2m/n}}{3}; \\
&= \frac{4}{n\kappa^2}u''u + \frac{2ku^{4/n}}{\kappa^2} + \frac{4Eu^2}{n\kappa^2} - \frac{mD_2u^{2m/n}}{3},
\end{aligned} \tag{5.38}$$

and

$$\begin{aligned}
V(\phi) &= \frac{3}{\kappa^2}\left(\frac{4(u')^2}{n^2} - \frac{2u''u}{3n} + \frac{2ku^{4/n}}{3}\right) + \left(\frac{n-6}{6}\right)D_1u^2 + \left(\frac{m-6}{6}\right)D_2u^{2m/n} \\
&= \frac{12(u')^2}{n^2\kappa^2} - \frac{2u''u}{n\kappa^2} + \frac{2ku^{4/n}}{\kappa^2} - \frac{2Eu^2}{n\kappa^2} + \frac{12Eu^2}{n^2\kappa^2} + \left(\frac{m-6}{6}\right)D_2u^{2m/n}
\end{aligned} \tag{5.39}$$

Let us consider eq.(5.37). We are able to rearrange this term as

$$\begin{aligned}
P(x) &= \frac{\kappa^2 n}{4}u^{-2}\left(\frac{4}{n\kappa^2}u''u + \frac{2ku^{4/n}}{\kappa^2} + \frac{4Eu^2}{n\kappa^2} - \frac{mD_2u^{2m/n}}{3}\right) \\
&\quad + \frac{mD_2}{12}\kappa^2 n u^{-2}u^{2m/n}, \\
-\frac{2P(x)u^2}{\kappa^2 n} &= -\frac{2}{n\kappa^2}u''u + \frac{2ku^{4/n}}{\kappa^2} - \frac{2Eu^2}{n\kappa^2} - \frac{3ku^{4/n}}{\kappa^2}.
\end{aligned} \tag{5.40}$$

Substituting into equation (5.39), we obtain the expression of potential as function of u

$$V(\phi) = \frac{12(u')^2}{n^2\kappa^2} - \frac{2P(x)u^2}{\kappa^2 n} + \frac{3ku^{4/n}}{\kappa^2} + \frac{12Eu^2}{n^2\kappa^2} + \left(\frac{m-6}{6}\right)D_2u^{2m/n}. \tag{5.41}$$

The scalar field energy density and pressure can be expressed in terms of u by using kinetic term, eq(5.38), and potential, eq.(5.41),

$$\begin{aligned}
\rho_\phi &= \underbrace{\frac{2}{n\kappa^2}u''u + \frac{ku^{4/n}}{\kappa^2} + \frac{2Eu^2}{n\kappa^2}}_{\cancel{\frac{2P(x)u^2}{\kappa^2 n}}} - \cancel{\frac{mD_2u^{2m/n}}{6}} + \frac{12(u')^2}{n^2\kappa^2} - \cancel{\frac{2P(x)u^2}{\kappa^2 n}} + \frac{3ku^{4/n}}{\kappa^2} \\
&\quad + \frac{12Eu^2}{n^2\kappa^2} + \cancel{\frac{m}{6}D_2u^{2m/n}} - D_2u^{2m/n}, \\
&= \frac{12(u')^2}{n^2\kappa^2} + \frac{3ku^{4/n}}{\kappa^2} + \frac{12Eu^2}{n^2\kappa^2} - D_2u^{2m/n}, \\
&= \frac{12(u')^2}{n^2\kappa^2} + \frac{3ku^{4/n}}{\kappa^2} - D_1u^2 - D_2u^{2m/n},
\end{aligned} \tag{5.42}$$

and

$$\begin{aligned}
p_\phi &= \frac{1}{2}\epsilon\dot{\phi}^2 - V(\phi), \\
&= \rho_\phi - 2V(\phi), \\
&= \frac{12(u')^2}{n^2\kappa^2} + \frac{3ku^{4/n}}{\kappa^2} - D_1u^2 - D_2u^{2m/n} - \frac{6}{\kappa^2}\left(\frac{4(u')^2}{n^2} - \frac{2u''u}{3n} + \frac{2ku^{4/n}}{3}\right) \\
&\quad - \left(\frac{n-6}{3}\right)D_1u^2 - \left(\frac{m-6}{3}\right)D_2u^{2m/n}, \\
&= \frac{12(u')^2}{n^2\kappa^2} + \frac{3ku^{4/n}}{\kappa^2} - \frac{24(u')^2}{n^2\kappa^2} + \frac{4u''u}{n\kappa^2} - \frac{4ku^{4/n}}{\kappa^2} \\
&\quad - \left(\frac{n-3}{3}\right)D_1u^2 - \left(\frac{m-3}{3}\right)D_2u^{2m/n}, \\
&= -\frac{12(u')^2}{n^2\kappa^2} - \frac{ku^{4/n}}{\kappa^2} + \frac{4u''u}{n\kappa^2} - \left(\frac{n-3}{3}\right)D_1u^2 - \left(\frac{m-3}{3}\right)D_2u^{2m/n},
\end{aligned} \tag{5.43}$$

In this case, the total energy density and pressure (scalar field + two barotropic fluids) can be expressed as function of u

$$\begin{aligned}
\rho_{\text{tot}} &= \rho_\phi + \rho_1 + \rho_2, \\
&= \frac{12(u')^2}{n^2\kappa^2} + \frac{3ku^{4/n}}{\kappa^2} - D_1u^2 - D_2u^{2m/n} + D_1u^2 + D_2u^{2m/n}, \\
&= \frac{12(u')^2}{n^2\kappa^2} + \frac{3ku^{4/n}}{\kappa^2}.
\end{aligned} \tag{5.44}$$

$$\begin{aligned}
p_{\text{tot}} &= p_\phi + p_1 + p_2, \\
&= -\frac{12(u')^2}{n^2\kappa^2} - \frac{ku^{4/n}}{\kappa^2} + \frac{4u''u}{n\kappa^2} - \left(\frac{n-3}{3}\right)D_1u^2 - \left(\frac{m-3}{3}\right)D_2u^{2m/n} \\
&\quad \left(\frac{n-3}{3}\right)D_1u^2 + \left(\frac{m-3}{3}\right)D_2u^{2m/n}, \\
&= -\frac{12(u')^2}{n^2\kappa^2} - \frac{ku^{4/n}}{\kappa^2} + \frac{4u''u}{n\kappa^2}
\end{aligned} \tag{5.45}$$

Let us consider the NLS equation (5.35). We define constant value as

$$F = -\frac{nk}{2}, \quad C = \frac{n-4}{n} \tag{5.46}$$

and the NLS equation becomes

$$u''(x) + [E - P(x)]u(x) = \frac{F}{u(x)^C}. \tag{5.47}$$

Here we encounter NLS equation as the same form as single barotropic case. The contribution of second barotropic fluid is expressed in $P(x)$ as seen in the equation (5.37). D'Ambroise [ref] accomplish to find seven exact solutions of NLS for single barotropic fluid as follows

$$u_1(x) = e_0x^2 + b_0x + c_0, \quad u_2(x) = e_0\cos^2(b_0x), \quad u_3(x) = e_0\tanh(b_0x),$$
$$u_4(x) = e_0e^{-x\sqrt{-c_0}} - b_0e^{x\sqrt{-c_0}}, \quad u_5(x) = (e_0/x)e^{c_0x^2/2}, \quad u_6(x) = -e_0\cosh^2(b_0x),$$
$$u_7(x) = e_0/x^{b_0}.$$

Now, in this work, we will show the first solution in detail and explain some cosmological interpretation. The first solution is

$$u(x) = \dot{x} = e_0x^2 + b_0x + c_0, \quad \text{(polynomial solution)}. \tag{5.48}$$

Taking derivative with respect to x to the equation (5.48) this gives

$$u'(x) = 2e_0x + b_0, \quad u''(x) = 2e_0. \tag{5.49}$$

Substituting into (5.47) we obtain

$$2e_0 + E(e_0x^2 + b_0x + c_0) - P(x)(e_0x^2 + b_0x + c_0) = \frac{F}{(e_0x^2 + b_0x + c_0)^C}$$

In this case, setting $E = 0$, $F = -d_0$ and $C = 0$, we obtain

$$2e_0 - P(x)(e_0x^2 + b_0x + c_0) = -d_0. \tag{5.50}$$

This imply

$$D_1 = -\frac{12\cancelto{0}{E}}{\kappa^2n^2} = 0, \quad n = \frac{-4}{\cancelto{0}{C} - 1} = 4, \quad k = \frac{-2\cancelto{-d_0}{F}}{n} = \frac{d_0}{2}. \tag{5.51}$$

Therefore the equation (5.40) is reduced to

$$P(x) = \frac{2e_0 + d_0}{e_0x^2 + b_0x + c_0}. \tag{5.52}$$

The relation $P(x)$ correspond to equation (5.50). So, under the conditions $E = 0$, $F = -d_0$, and $C = 0$, the solution (5.48) satisfies the NLS equation eq.(5.47) which in this case is

$$u''(x) - P(x)u(x) = -d_0 \tag{5.53}$$

In this case we obtain $n = 4$ which refer to radiation for D_1. But, there is no radiation for this solution because of $D_1 = 0$, so there is only D_2 fluid and curvature $k = d_0/2$. Let us consider the solution eq.(5.48), there are two conditions for this one namely $e_0 = 0$ and $e_0 \neq 0$. Case one $e_0 = 0$ the solution (5.48) reduce to $u(x) = \dot{x} = b_0 x + c$, and the solution is given

$$x(t) = \frac{c_0}{b_0}\left(e^{b_0(t-t_0)} - 1\right) \tag{5.54}$$

for $b_0 \neq 0$, by definition of scale factor $a(t) = u(x)^{-2/n}$, where $u(x) = \dot{x}(t)$ can be obtained by taking time derivative to equation (5.54),

$$\dot{x}(t) = c_0 e^{b_0(t-t_0)}. \tag{5.55}$$

Hence, the scale factor is given by

$$\begin{aligned} a(t) &= u(x)^{-2/n}, \\ &= c_0^{-2/n} e^{-2b_0(t-t_0)/n}. \end{aligned} \tag{5.56}$$

According to equation (5.31), in this case the Hubble parameter can be expressed as a constant denoted by H_0 namely,

$$\begin{aligned} H &= \frac{-2}{n} u'(x), \\ H_0 &= \frac{-2}{n} b_0. \end{aligned} \tag{5.57}$$

Since $n = 4$ this gives $b_0 = -2H_0$ and scale factor reads

$$a(t) = c_0^{-1/2} e^{H_0(t-t_0)}. \tag{5.58}$$

The expansion of the universe is de Sitter type.

Next, Let us consider to another case $b_0 \neq 0$. This give quadratic equation and solution yields

$$x(t) = \frac{1}{2e_0}\left[\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{2}(t-t_0)\right) - b_0\right], \tag{5.59}$$

where $\Delta = b_0^2 - 4e_0c_0 < 0$. According to definition of scale factor $u(x) = \dot{x}(t) = a(t)^{-n/2}$ this gives

$$u(x) = \dot{x} = -\frac{\Delta}{4e_0}\sec^2\left[\frac{\sqrt{-\Delta}}{2}(t-t_0)\right] \tag{5.60}$$

and scale reads

$$a(t) = u(x)^{-2/n} = \left[-\frac{4e_0}{\Delta}\cos^2\left(\frac{\sqrt{-\Delta}}{2}(t-t_0)\right)\right]^{2/n} \tag{5.61}$$

In this case, we encounter the periodic solution for scale factor. Both scale factor solutions are obtained. However,there is zero energy density for first fluid, $D_1 = 0$. But, the appearance of radiation $n = 4$ make no sense. The other solutions and cosmological analysis show in[4]

REFERENCES

REFERENCES

[1] Burdyuzha VV. From the Early Universe to the Modern Universe. Symmetry. 2020;12(3):382.

[2] Banerjee N, Das S. A Late time acceleration of the universe with two scalar fields: Many possibilities. Mod Phys Lett. 2006;A21:2663–2670.

[3] Heisenberg L. Generalization of the Proca Action. JCAP. 2014;1405:015.

[4] Kritpetch C, Sanongkhun J, Vanichchapongjaroen P, Gumjudpai B. Non-linear Schrödinger-type formulation of scalar field cosmology: two barotropic fluids and exact solutions. 2019.

[5] Chaichian M, Louis Martinez D, Lusanna L. Dirac's constrained systems: The Classification of second class currents. Annals Phys. 1994;232:40–60.

[6] Faddeev LD, Jackiw R. Hamiltonian Reduction of Unconstrained and Constrained Systems. Phys Rev Lett. 1988;60:1692–1694.

[7] Escalante A, Zárate M. Dirac and Faddeev–Jackiw quantization of a five-dimensional Stüeckelberg theory with a compact dimension. Annals Phys. 2014;353:163–178.

[8] Kulshreshtha DS, Muller-Kirsten HJW. Faddeev-Jackiw quantization of selfdual fields. Phys Rev. 1992;D45:393–397.

[9] Garcia JA, Pons JM. Equivalence of Faddeev-Jackiw and Dirac approaches for gauge theories. Int J Mod Phys. 1997;A12:451–464.

[10] Sanongkhun J, Vanichchapongjaroen P. On constrained analysis and diffeomorphism invariance of generalised Proca theories. Gen Rel Grav. 2020;52(3):26.

[11] Hawkins RM, Lidsey JE. The Ermakov-Pinney equation in scalar field cosmologies. Phys Rev. 2002;D66:023523.

[12] Diaz NL, Rossignoli R. History state formalism for Dirac's theory. Phys Rev. 2019;D99(4):045008.

[13] Martínez Pascual E. Dirac quantisation, BRST, Hilbert space, Constraints. Nottingham U.; 2012. Available from: http://eprints.nottingham.ac.uk/12433/.