ROBUST WEAK SHARP SOLUTIONS AND ROBUST APPROXIMATE SOLUTIONS FOR UNCERTAIN NONSMOOTH OPTIMIZATION PROBLEMS

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ABSTRACT

In this thesis, our aim is to perform study of theoretical side of uncertain optimization problems related to robust weak sharp solutions and robust approximate solutions in uncertain nonsmooth optimizationproblems. Firstly, we introduce robust weak sharp solution to an uncertain convex optimization problem. The characterizations of the sets of all the robust weak sharp solutions are obtained. Moreover, we apply the results to an uncertain convex mutiobjective optimization problem and obtain optimality conditions for robust weak sharp weak efficient solutions in the multi-objective optimization problem. Secondly, we investigate the robust optimization problem involving nonsmooth real-valued functions. Some necessary and sufficient optimality conditions for the robust weak sharp solutions of considered problem under a constraint qualification are established. Thirdly, we move to the investigation of robust approximate solutions for an uncertain convex optimization problem. The notion of an ε-quasi highly robust solution for the uncertain convex optimization problem is introduced. The highly robust approximate optimality theorems for ε-quasi highly robust solutions of the considered problem are established by means of a robust optimization approach. Finally, the highly robust approximate duality theorems in terms of Wolfe

type on ε-quasi highly robust solutions of the uncertain convex optimization problem are obtained. In order to illustrate the obtained results or support this thesis, some examples are presented.

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CHAPTER I

INTRODUCTION

Constrained optimization problems concern the minimization or maximization of functions over some set of conditions called constraints, and play a vital role in many fields of science as diverse as economics, accounting, computer science, engineering and others to select an optimal solution in many decisions based on computational methods. It was realized that constraints in the form of inequalities play a predominant role in modeling real world, and, therefore; it leads to more challenging necessary conditions for global optimality of an optimization problem in terms of a system of inequalities.

As we know, the majority of many practical constrained optimization problems often involve input data that are noisy or uncertain due to modeling, estimation errors, prediction errors as well as measurement errors [1–6]. Therefore, it is imperative to study the optimization problems with data uncertainty. In addition, in many situations often we need to make decisions now before we can know the true values or have better estimations of the parameters, for instance, optimization problems arising in industry or commerce might involve various costs, financial returns, and future demands that might be unknown at the time of the decision. If the uncertainties are ignored while solving the optimization problem, it may lead to solutions which are suboptimal or even infeasible. Consequently, how to explicate mathematical approaches that are capable of treating data uncertainty in constrained optimization has become a critical question in mathematical optimization. As we have seen the problematic situations where a decision based on a model has to be taken here and now, we need naturally to the additional requirement that any feasible vectors must satisfy all constraints including each set of constraints corresponding to a possible realization of the uncertain parameters from the set uncertainty set.

Robust optimization, which is its robust counterpart of an uncertain optimization problem, has emerged as a powerful deterministic approach for studying optimization problems with data uncertainty in the sense that it minimizes the objective function value in the worst case of all scenarios and gets a solution that works well even in the worst-case scenario, but also is immunized against the data uncertainty. Over the years, a great deal of attention has been attracted to treat uncertain optimization problems by using robust optimization methodology. For issues related to optimality conditions and duality properties, see [7–12] and other references therein.

There have been proposed numerous ways to define robust solutions for uncertain programming problems. Among the first one in all such notions is that the so-called *strictly robust solution* also called *minimax robust solution*, which was introduced by Soyster [13]. This concept is to have a solution that is feasible for all possible scenarios and is obtained composed by minimizing the objective function within the worst-case scenario. The notion of the strictly robust solution has been studied extensively from different aspects (see, e.g., [2, 5, 14–16]). Another solution concept is that of a *highly robust solution* which was introduced to study the various uncertain multi-objective programming problems; see, e.g., [17–19]. It is worth noting that the notion of a strictly robust solution coincides therewith of the highly robust solution if the objective function of single-objective programming problems is uncertainty-free; see, e.g., [8,11,20,21]. The notion of a highly robust solution is stricter than that of the strictly robust solution when the objective function is in the face of data uncertainty. Nevertheless, in many cases, it is enough to study the highly robust solution for an uncertain single-objective programming problem; see, e.g., [18, 19, 22].

In the same time, the notion of a *weak sharp solution* or *weak sharp minimizer* in general mathematical programming problems was first introduced in [23]. It is an extension of a sharp minimizer (or equivalently, strongly unique minimizer) in [24] to include the possibility of non-unique solution set. It has been acknowledged that the weak sharp minimizer plays important roles in stability and sensitivity analysis and convergence analysis of a wide range of numerical algorithms in mathematical programming; see, e.g., [25–30] and references therein.

In the context of optimization, much attention has been paid to concerning sufficient and/or necessary conditions for weak sharp solutions and characterizing weak sharp solution sets (of such weak sharp minimizers) in various types of problems. Particularly, the study of characterizations of the weak sharp solution sets covers both single-objective and multi-objective optimization problems; see, e.g., [31–34] and references therein and, recently, is extended to mathematical programs with inequality constraints and semi-infinite programs; see, e.g., [35,36]. As it might be seen, the study of optimality conditions for weak sharp solutions and/or characterizations of the weak sharp solution sets has been popular in many optimization problems. *How about the issue of this study, particularly, in uncertain optimization problems?*

On the other hand, finding a solution of an optimization problem might not be always possible and obviously, neither does finding a weak sharp solution of the problem. Then it leads to the notion of approximate solutions that play a crucial role in the algorithmic study of optimization problems. Among such approximate solutions, the notion of an ε-*quasi solution* first introduced by Loridan [37]. Since then many researchers have studied the approximate solutions in optimization programming problems and approximate necessary conditions under different suitable constrained qualifications have been established, see [11,38–42] and also the references therein, for example.

To the best of our knowledge, there are only a few papers to deal with approximate optimal solutions of optimization problems with data uncertainty in both objective and constraint functions, for example, [43,44]. More precisely, by virtue of the epigraphs of the conjugates of the constraint functions, Sun et. al. [44] obtained some approximate optimality conditions for the robust quasi approximate optimal solution of an uncertain semi-infinite optimization problem. The notion of their obtained approximate solutions is given to approximate the strictly robust solutions to the problems. However, as far as we are concerned, the notion of approximate solutions to approximate the highly robust solutions for uncertain optimization problems has been not presented so far. A natural question is: *"How about the study of approximate optimality conditions and approximate duality theorems for an approximate solution that approximates the highly robust solutions to an uncertain convex optimizaion problem?"*.

Motivated and inspired by all above contributions, in this thesis, our aim is to perform study of theoretical side of uncertain optimization problems related to optimality conditions and characterizations of the robust weak sharp solution sets for uncertain nonsmooth convex optimization problems, optimality conditions for robust weak sharp solutions for uncertain nonnsmooth (not necessarily convex) optimization problems as well as optimality conditions and duality theorems for ε-quasi highly robust solutions in uncertain nonsmooth convex optimization problems.

In the following, we give a description of how is this thesis organized.

Chapter II. We will include several notions and preliminary results in order to make this thesis as self-contained as possible.

Chapter III. We draw our attention to the investigation of robust weak sharp solutions in uncertain (convex) nonsmooth optimization problems. In the first part of the chapter, we introduce robust weak sharp and robust sharp solution to a convex programming with the objective and constraint functions involved uncertainty. The characterizations of the sets of all the robust weak sharp solutions are obtained by means of subdiferentials of convex functions, DC fuctions, Fermat rule and the robust-type subdifferential constraint qualification, which was introduced in X.K. Sun, Z.Y. Peng and X. Le Guo [45]. Moreover, we apply the results to an uncertain convex mutiobjective optimization problem and obtain optimality conditions for robust weak sharp weak efficient solutions in the multi-objective optimization problem. In the second part of the chapter, we investigate the robust optimization problem involving nonsmooth and nonconvex real-valued functions. By means of the generalized Fermat rule, the Mordukhovich subdifferential for maximum functions, the fuzzy sum rule for Fréchet subdifferentials and the sum rule for Mordukhovich subdifferentials, we firstly establish a necessary condition for the local robust weak sharp solution of considered problem under a constraint qualification. These optimality conditions are presented in terms of multipliers and Mordukhovich subdifferentials of the related functions. Then, by employing the robust version of the (KKT) condition, and some appropriate generalized convexity conditions, we also obtain some sufficient conditions for the global robust weak sharp solutions of the problem. In addition, some examples are presented for illustrating or supporting the results.

Chapter IV. We draw our attention to the investigation of ε-quasi-highly robust solutions of uncertain convex optimization problems. In the first part of the chapter, we investigate a convex optimization problem in the face of data uncertainty in both objective and constraint functions. The notion of an ε-quasi highly robust solution (one sort of approximate solutions) for the convex optimization problem with data uncertainty is introduced. The highly robust approximate optimality theorems for ε-quasi highly robust solutions of uncertain convex optimization problem are established by means of a robust optimization approach (worst-case approach). Then, in the second part of the chapter, the highly robust approximate duality theorems in terms of Wolfe type on ε-quasi highly robust solutions for the uncertain convex optimization problem are obtained. Moreover, to illustrate the obtained results or support this study, some examples are presented

Chapter V. We give the conclusion.

CHAPTER II

PRELIMINARIES

In this chapter, we will review the certain notations, basic definitions, and preliminary results that are related to our research.

Throughout this thesis, all spaces under consideration are the n-dimensional Euclidean space $\mathbb{R}^n$. All vectors are considered to be column vectors which can be transposed to be a row vector by the superscript T. For vectors $x := (x_1, x_2, \ldots, x_n)$ and $y := (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, the (usual) inner product of x and y is denoted by $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$, while the norm of x is given by $\|x\| = \sqrt{\langle x, x\rangle}$.

The closed, open, and left closed right open intervals between $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ are denoted by $[\alpha, \beta]$, (α, β), and $[\alpha, \beta)$, respectively. The non-negative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, 2, \ldots, n\}$.

For any two sets $A, C \subseteq \mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}$, sets $A + C$ and αA : are defined by $A + C := \{a + c \in \mathbb{R}^n : a \in A, \in C\}$, and $\alpha A := \{\alpha a \in \mathbb{R}^n : a \in A\}$, respectively. Besides, for any nonempty set $A \subseteq \mathbb{R}^n$, the distance function $d_A : \mathbb{R}^n \to \mathbb{R}$ and the indicator function $\delta_A : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ of A are respectively defined by

$$d_A(x) := \inf_{y \in S} \|x - y\|, \forall x \in \mathbb{R}^n, \text{ and } \delta_A(x) = \begin{cases} 0; & x \in A, \\ +\infty; & x \notin A. \end{cases}$$

2.1 Basic concepts

Definition 2.1.1. A sequence $\{x_k\} \subseteq \mathbb{R}$ is said to be a convergent sequence if there exists $x \in \mathbb{R}$ such that for every $\varepsilon > 0$,

$$|x_k - x| < \varepsilon,\ \forall k \geq k_\varepsilon,$$

for some integer k_ε (that depends on ε).

The scalar x is said to be the limit of $\{x_k\}$, and the sequence $\{x_k\}$ is said to converge to x. Symbolically, it is expressed as $x_k \to x$ or $\lim_{k\to+\infty} x_k = x$.

Definition 2.1.2. A sequence $\{x_k\}$ is said to be bounded from above (resp. bounded from below) if there exists some scalar α such that $x_k \leq \alpha$, (resp. $x_k \geq \alpha$) for all $k \in \mathbb{N}$. In addition, it is said to be bounded if it is bounded above and bounded below.

Definition 2.1.3. A sequence $\{x_k\}$ is said to be monotonically nonincreasing (resp. monotonically nondecreasing) if $x_{k+1} \leq x_k$, (resp. $x_{k+1} \geq x_k$) for all $k \in \mathbb{N}$.

If $x_k \to x$ and $\{x_k\}$ is monotonically nonincreasing (resp. nondecreasing), then we use the notation $x_k \downarrow x$. (resp. $x_k \uparrow x$).

Let $\{x_k\} \subseteq \mathbb{R}$ and $y_r := \sup\{x_k : k \geq r\}$ and $z_r := \inf\{x_k : k \geq r\}$. Observe that the sequences $\{y_r\}$ and $\{z_r\}$ are nonincreasing and nondecreasing, respectively. Therefore, $\{y_r\}$ has a limit whenever $\{x_k\}$ is bounded above while or $\{z_r\}$ has a limit whenever $\{x_k\}$ is bounded below.

Definition 2.1.4. Let a sequence $\{x_k\} \subseteq \mathbb{R}$ be given. The limit of $\{y_r\}$ is denoted by $\limsup_{k\to+\infty} x_k$, and is called the upper limit of $\{x_k\}$. Besides, the limit of $\{z_r\}$ is denoted by $\liminf_{k\to+\infty} x_k$, and is called the lower limit of $\{x_k\}$.

Definition 2.1.5. A sequence $\{x_k\}$ of vectors in $\mathbb{R}^n$ is said to converge to some $x \in \mathbb{R}^n$ if the i-th component of x_k converges to the i-th component of x for every $i = 1, 2, \ldots, n$. We use the notations $x_k \to x$ or $\lim_{k\to+\infty} x_k = x$ to indicate convergence for vector sequences as well.

We say that x is a closure point of a subset A of $\mathbb{R}^n$ if there exists a sequence $\{x_k\} \subseteq A$ such that $x_k \to x$. The closure of A, denoted by cl$A$, is the set of all closure points of A.

Definition 2.1.6. The sequence $\{x_k\} \subseteq \mathbb{R}^n$ is bounded if there exists $M > 0$ such that $\|x_k\| \leq M$ for every $l \in \mathbb{N}$.

Definition 2.1.7. A subsequence of $\{x_k\} \subseteq \mathbb{R}^n$ is a sequence $\{x_{k_j}\}$, $j = 1, 2, \ldots$, where each x_{k_j} is a member of the original sequence and the order of the elements as in the original sequence is maintained.

The symbol $B(x, r)$ denotes an open ball of radius $r > 0$ with center at x, i.e., $B(x, r) := \{y \in \mathbb{R}^n : \|y - x\| < r\}$. We say that x is an interior point of a subset A of $\mathbb{R}^n$ if there exists $r > 0$ such that $B(x, r) \subseteq A$. The interior of A, denoted by int$A$, is the set of all interior points of A.

Definition 2.1.8. A subset A of $\mathbb{R}^n$ is said to be

(i) closed if $A = \text{cl}A$.

(ii) open if its complement, $\mathbb{R}^n \backslash A$, is closed, or equivalently, $A = \text{int}A$.

(iii) bounded if there exists a scalar M such that $\|x\| \leq M$ for all $x \in A$.

(iv) compact if it is closed and bounded.

Generally, we prefer to deal with functions that are real-valued and are defined over $\mathbb{R}^n$. However, in some situations, prominently arising in the context of optimization, we will encounter operations on real-valued functions that produce extended real-valued functions, that is, functions that take values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. As an example, a function of the form

$$f(x) = \sup_{i \in I} f_i(x),$$

where I is an infinite index set, can take the value ∞ even if the functions f_i are real-valued. Most rules with infinity are intuitively clear except possibly $0 \times (+\infty)$ and $\infty - \infty$. Because we will be dealing mainly with minimization problems, we will follow the convention $0 \times (+\infty) = (+\infty) \times 0 = 0$ and $\infty - \infty = 0$.

Definition 2.1.9. An extended real-valued function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be a proper function if $f(x) > -\infty$ for every $x \in \mathbb{R}^n$ and the domain of f,

$$\text{dom}\, f := \{x \in \mathbb{R}^n : f(x) < +\infty\},$$

is nonempty. In addition, the epigraph of the function f is given by

$$\text{epi } f := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq \alpha\}.$$

Now we move on the semicontinuities of a real-valued function, which involve the limit inferior and limit suprerior of the function.

Definition 2.1.10. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be lower semicontinuous (lsc) at $\bar{x} \in \mathbb{R}^n$ if for every sequence $\{x_k\} \subseteq \mathbb{R}^n$ converging to $\bar{x}$,

$$f(\bar{x}) \leq \liminf_{l \to +\infty} f(x_l).$$

Equivalently,

$$f(\bar{x}) \leq \liminf_{x \to \bar{x}} f(x) := \lim_{\delta \downarrow 0} \inf_{x \in \mathbb{B}(\bar{x}, \delta)} f(x).$$

The function f is lsc over a set $A \subseteq \mathbb{R}^n$ if f is lsc at every $\bar{x} \in A$.

Similar to the concept of lower semicontinuity and limit infirior, we next define the upper semicontinuity and the limit supremum of a function.

Definition 2.1.11. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be upper semicontinuous (usc) at $\bar{x} \in \mathbb{R}^n$ if for every sequence $\{x_k\} \subseteq \mathbb{R}^n$ converging to $\bar{x}$,

$$f(\bar{x}) \geq \limsup_{l \to +\infty} f(x_l).$$

Equivalently,

$$f(\bar{x}) \geq \limsup_{x \to \bar{x}} f(x) := \lim_{\delta \downarrow 0} \sup_{x \in \mathbb{B}(\bar{x}, \delta)} f(x).$$

The function f is usc over a set $A \subseteq \mathbb{R}^n$ if f is usc at every $\bar{x} \in A$.

Definition 2.1.12. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be continuous at $\bar{x}$ if it is lsc and usc at $\bar{x}$, that is,

$$\lim_{x \to \bar{x}} f(x) = f(\bar{x}).$$

The next result, a generalization of the classical theorem of Weierstrass, suggests a way that whether an optimal solution exists.

Theorem 2.1.13. [46] *Let A be a nonempty closed subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be lsc over A. Assume that one of the following conditions holds:*

(i) *A is bounded.*

(ii) *Some level set $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is nonempty and bounded.*

(iii) *For every sequence $\{x_k\} \subseteq A$ such that $\|x_l\| \to +\infty$, $\lim_{k\to+\infty} f(x_k) = \infty$.*

Then, the set of all minimizers of f over A, i.e., $\{x \in A : f(x) \leq f(y),\ \forall y \in A\}$, is nonempty and compact.

Next we go through the notions of differentiability for real-valued functions and vector-valued functions. We start by recalling the partial derivative.

Definition 2.1.14. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a given function, and $x \in \mathbb{R}^n$ be fixed. Consider the following expression:

$$\lim_{t\to 0} \frac{f(x + te_i) - f(x)}{t},$$

where e_i is the i-th unit vector (all components are 0 except for the i-th component which is 1).

If the above limit exists, it is called the i-th partial derivative of f at the vector x and it is denoted by $\frac{\partial f(x)}{\partial x_i}$.

Definition 2.1.15. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be given, and let x be a point where f is finite. We say that f is (Fréchet) differentiable at x if and only if there exists a vector ξ (necessarily unique) with the property that

$$\lim_{y\to x} \frac{f(y) - f(x) - \langle \xi, y - x\rangle}{|y - x|} = 0.$$

The vector ξ, if it exists, is called the gradient of f at x and is denoted by $\nabla f(x)$.

Definition 2.1.16. A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be smooth if it is continuously differentiable, i.e., ∇f is continuous, over $\mathbb{R}^n$. On the other hand, if f is not smooth, it is said to be nonsmooth.

Definition 2.1.17. For a proper function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we define the one-sided directional derivative of f at $\bar{x} \in \operatorname{dom} f$ in the direction $d \in \mathbb{R}^n$ to be

$$f'(\bar{x}; d) := \lim_{t \downarrow 0} \frac{f(\bar{x} + td) - f(\bar{x})}{t},$$

provided that $+\infty$ and $-\infty$ are allowed as limits.

Remark 2.1.18. If $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is differentiable at $\bar{x}$, $f(\bar{x}) \in \mathbb{R}$, we have

$$f'(\bar{x}; d) = \langle \nabla f(\bar{x}), d \rangle, \ \forall d \in \mathbb{R}^n.$$

Moreover,

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right).$$

Proof. Suppose that f is differentiable at $\bar{x}$. It then follows from the definition that for any $d \neq 0$,

$$\begin{aligned} 0 &= \lim_{t \downarrow 0} \frac{f(\bar{x} + td) - f(\bar{x}) - \langle \nabla f(\bar{x}), td \rangle}{t\|d\|} \\ &= \frac{f'(\bar{x}; d) - \langle \nabla f(\bar{x}), d \rangle}{\|d\|}. \end{aligned}$$

Therefore, $f'(\bar{x}, d)$ exists and is a linear function of d:

$$f'(\bar{x}; d) = \langle \nabla f(\bar{x}), d \rangle, \ \forall d \in \mathbb{R}^n.$$

In particular, for $i = 1, 2, \ldots, n$,

$$\langle \nabla f(\bar{x}, d), e_i \rangle = \lim_{t \to 0} \frac{f(\bar{x} + td) - f(x)}{t} = \frac{\partial f(x)}{\partial x_i},$$

which in turn implies that $\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right)$. □

The following example presents a differentiable but nonsmooth function.

Example 2.1.19. *Consider the following real-valued function*

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ x^2 \sin(\frac{1}{x}), & \text{otherwise.} \end{cases}$$

It is clear that the function f is differentiable at $x \neq 0$ and its derivative is

$$f'(x) = 2x\sin(1/x) - \cos(1/x), \ \forall x \neq 0.$$

Since $f(0+x) - f(0) = x^2\sin(\frac{1}{x})$ for any $x \neq 0$ and $\lim_{x\to 0} x\sin(1/x) = 0$, the function f is differentiable at $x = 0$ and $f'(0) = 0$. Therefore, f is differentiable on $\mathbb{R}$. However, f is not continuously differentiable because the limit $\lim\limits_{x\to 0} f'(x)$ does not exist.

Definition 2.1.20. A vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^p$ is called differentiable (or smooth) if each component f_j of f, $j = 1, 2, \ldots, p$, is differentiable (or smooth, respectivelty).

Definition 2.1.21. The Jacobian of f, denoted $\nabla f(x)$, is the $p \times n$ matrix and can be expressed as

$$\nabla f(x) := (\nabla f_1(x), \nabla f_2(x), \ldots, \nabla f_p(x))^T.$$

Let $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable vector-valued functions, and let h be their composition, i.e.,

$$h(x) := g(f(x)), \ \forall x \in \mathbb{R}^n.$$

Then, the chain rule for differentiation [46] states that

$$\nabla h(x) = \nabla f(x)^T \nabla g(f(x)), \ \forall x \in \mathbb{R}^n. \tag{2.1.1}$$

A set-valued mapping F from $\mathbb{R}^n$ to $\mathbb{R}^m$ associates every $x \in \mathbb{R}^n$ to a set in $\mathbb{R}^m$; that is, for every $x \in \mathbb{R}^n$, $F(x) \subseteq \mathbb{R}^m$. Symbolically, it is expressed as $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$.

Definition 2.1.22. A set-valued map $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be upper semicontinuous (usc) at $x \in \mathbb{R}^n$ if for any sequences $\{\xi_k\}$ and $\{x_k\}$ tending to ξ and x respectively, and if $\xi_k \in \Phi(x_k)$ for each $k \in \mathbb{N}$, then $\xi \in \Phi(x)$.

2.2 Nonsmooth analysis

Definition 2.2.1. A set $A \subseteq \mathbb{R}^n$ is said to be

(i) convex if $\alpha a_1 + (1-\alpha)a_2 \in A,\ \forall a_1,\ a_2 \in A,\ \forall \alpha \in [0,1]$.

(ii) affine if $\alpha a_1 + (1-\alpha)a_2 \in A,\ \forall a_1,\ a_2 \in A,\ \forall \alpha \in \mathbb{R}$.

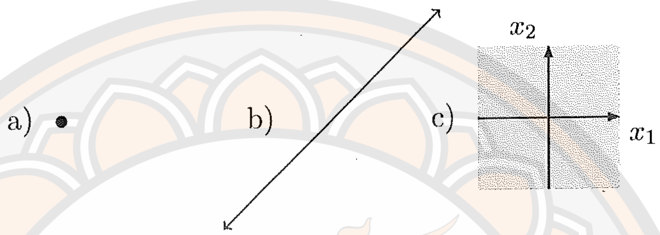


Figure 1: Example of affine sets

Next we state some basic properties of convex sets.

Proposition 2.2.2. [Operations on convex sets]

(i) *The intersection of an arbitrary collection of convex sets is convex.*

(ii) *For two convex sets $A, C \subseteq \mathbb{R}^n$, $A + C$ is convex.*

(iii) *For a convex set $A \subseteq \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$, αA is convex.*

(iv) *For a convex set $A \subseteq \mathbb{R}^n$ and scalar $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$, $(\alpha_1+\alpha_2)A = \alpha_1 A + \alpha_2 C$ which is convex.*

Theorem 2.2.3. [47] *Let A and C be nonempty convex subsets of $\mathbb{R}^n$ with $\operatorname{int} A \neq \emptyset$. Then $\operatorname{int} A \cap C = \emptyset$ if and only if there exist a vector $\xi \in \mathbb{R}^n \setminus \{0\}$ and a real number α with*

$$\langle \xi, a \rangle \leq \alpha \leq \langle \xi, c \rangle \text{ for all } a \in A \text{ and all } c \in C$$

and

$$\langle \xi, a \rangle < \alpha \text{ for all } a \in \operatorname{int} A.$$

Definition 2.2.4. The convex hull of a set $A \subseteq \mathbb{R}^n$, is denoted by co A, is the smallest convex set containing A and can be expressed as

$$\text{co}A = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{m} \alpha_i a_i,\ a_i \in A,\ \alpha_i \geq 0,\ i = 1, 2, \ldots, m,\ \sum_{i=1}^{m} \alpha_i = 1 \right\},$$

for some $m \in \mathbb{N}$.

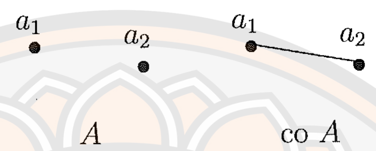


Figure 2: Illustration of a convex hull

Definition 2.2.5. The affine hull of a set $A \subseteq \mathbb{R}^n$, is denoted by aff$A$, is the smallest affine set containing A and can be expressed as

$$\text{aff}A = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{m} \alpha_i a_i,\ a_i \in A,\ \alpha_i \in \mathbb{R},\ i = 1, 2, \ldots, m,\ \sum_{i=1}^{m} \alpha_i = 1 \right\},$$

for some $m \in \mathbb{N}$.

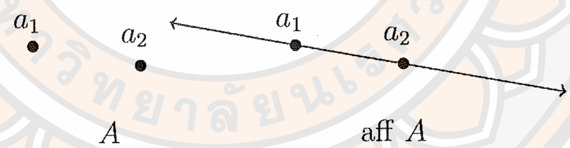


Figure 3: Illustration of an affine hull

Definition 2.2.6. [48] The relative interior of a convex set $A \subseteq \mathbb{R}^n$, ri$A$, is the interior of A relative to the affine hull of A, that is,

$$\text{ri}A := \{x \in A : \exists \varepsilon > 0 \text{ s.t. } B(x, \varepsilon) \cap \text{aff } A \subseteq A\}.$$

For an n-dimensional convex set $A \subseteq \mathbb{R}^n$, i.e., the dimension of a subspace which parallel to aff A, aff $A = \mathbb{R}^n$ and thus ri$A$ = int$A$.

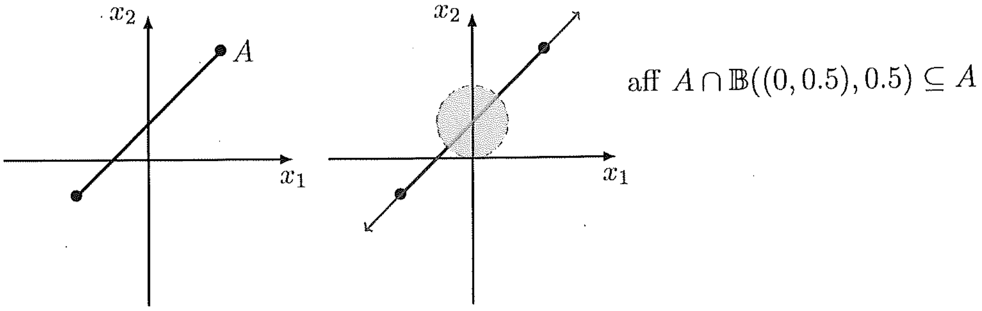


Figure 4: Illustration of a relative interior of a one-dimensional convex set on a two-dimensional space

Proposition 2.2.7. [48] *Consider a nonempty convex set $A \subseteq \mathbb{R}^n$. Then, the following assertions hold:*

(i) $\text{ri}A$ *is nonempty.*

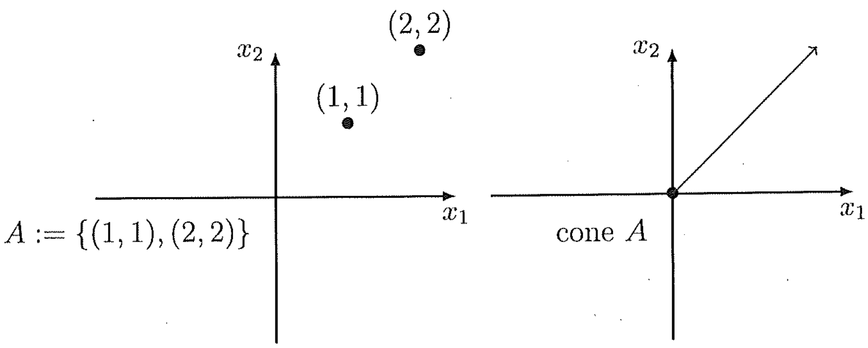
(ii) *Let $x \in \text{ri}A$ and $y \in \text{cl}A$. Then for $\alpha \in [0, 1)$,*

$$(1 - \alpha)x + \alpha y \in \text{ri}A.$$

Definition 2.2.8. A set $A \subseteq \mathbb{R}^n$ is said to be a cone if for every $x \in A$, $\alpha x \in A$ for every $\alpha \geq 0$.

Definition 2.2.9. For any set $A \subseteq \mathbb{R}^n$, the cone generated by A is denoted by cone A and is defined as

$$\text{cone}A := \bigcup_{\alpha \geq 0} \alpha A = \{x \in \mathbb{R}^n : x = \alpha a, \, a \in A, \ \alpha \geq 0\}.$$



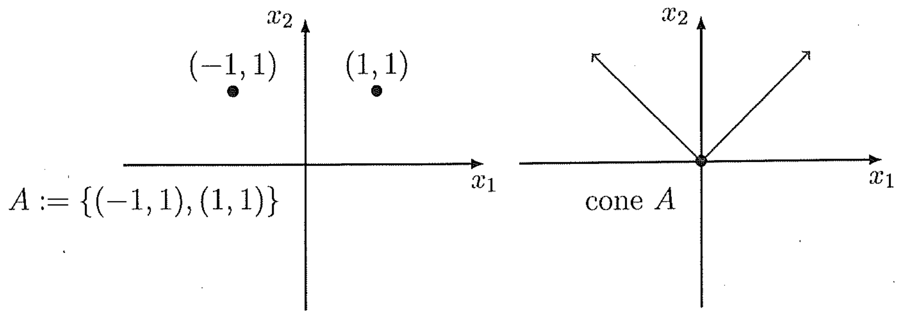


Figure 5: Illustration of the cone generated by a set

We are interested in the convex scenarios, thereby moving on to the notion of the convex cone.

Definition 2.2.10. The set $A \subseteq \mathbb{R}^n$ is said to be convex cone if it is a cone and is convex.

Definition 2.2.11. For any set $A \subseteq \mathbb{R}^n$, the convex cone generated by A is denoted by cone co A and is expressed as

$$\text{cone co } A = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{m} \alpha_i a_i,\ a_i \in A,\ \alpha_i \geq 0,\ i = 1, 2, \ldots, m,\ m \in \mathbb{N} \right\}.$$

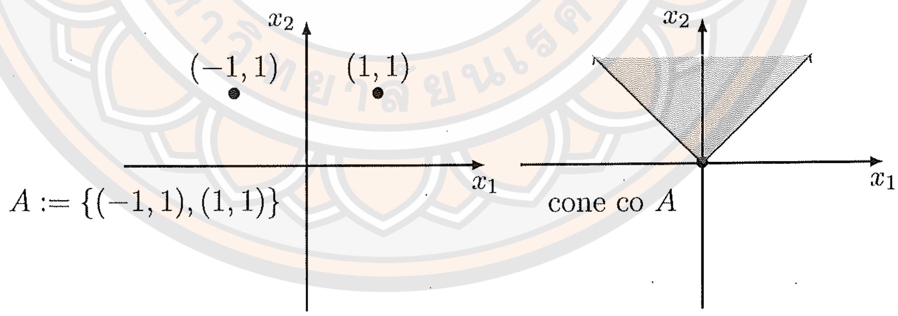


Figure 6: Illustration of a convex cone generated by a set

In addition, for a collection of convex sets $A_i \subseteq \mathbb{R}^n$, $i = 1, 2, \ldots, m$, the convex cone generated by A_i, $i = 1, 2, \ldots, m$, can be shown to be expressed as

$$\text{cone co} \bigcup_{i=1}^{m} A_i = \bigcup_{\substack{\alpha_i \geq 0 \\ i=1,\ldots,m}} \sum_{i=1}^{m} \alpha_i A_i. \tag{2.2.1}$$

See, [49], for more details.

Theorem 2.2.12. *A cone $A \subseteq \mathbb{R}^n$ is convex if and only if $A + A \subseteq A$.*

Definition 2.2.13. Consider a set $A \subseteq \mathbb{R}^n$. The cone defined as

$$A^\circ := \{\xi \in \mathbb{R}^n : \langle \xi, x \rangle \leq 0, \ \forall x \in A\}$$

is called the polar cone of A. Note that the polar cone of the set A is a closed convex cone.

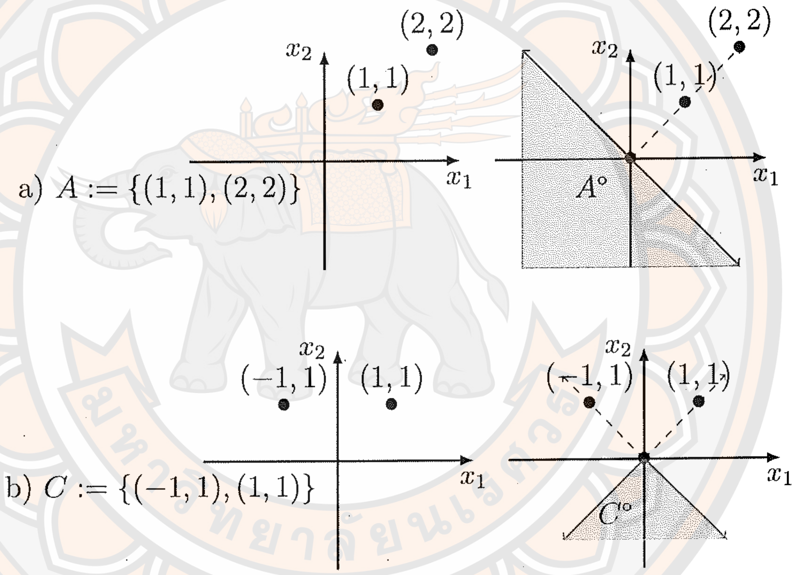


Figure 7: Polar cones of sets

Proposition 2.2.14. [46] *Let $a_1, a_2, \ldots, a_m$ be vectors in $\mathbb{R}^n$. Then, the finitely generated cone*

$$A := \text{cone}\{a_1, a_2, \ldots, a_m\}$$

is closed and its polar cone is the polyhedral cone given by

$$A^\circ = \{d \in \mathbb{R}^n : \langle a_i, d \rangle \leq 0, \ i = 1, 2, \ldots, m\}.$$

Definition 2.2.15. Consider a set $A \subseteq \mathbb{R}^n$. The positive polar cone (dual cone) to the set A is defined by

$$A^* := \{\xi \in \mathbb{R}^n : \langle \xi, x \rangle \geq 0, \ \forall x \in A\}.$$

Observe that $A^* = (-A)^\circ = -A^\circ$.

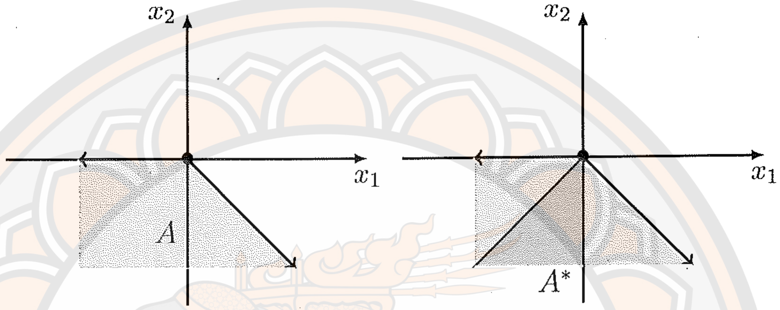


Figure 8: Dual cone of a set

Lemma 2.2.16. [47, Lemma 3.21] *Let A be a convex cone in $\mathbb{R}^p$.*

(i) *If A is closed, then $A = \{x \in \mathbb{R}^p : \langle \xi, x \rangle \geq 0$ for all $\xi \in A^*\}$.*

(ii) *If $\text{int}A \neq \emptyset$, then $\text{int}A = \{x \in \mathbb{R}^p : \langle \xi, x \rangle > 0$ for all $\xi \in A^* \backslash \{0\}\}$.*

We present some properties of polar cones.

Proposition 2.2.17.

(i) *Consider two sets $A,\ C \subseteq \mathbb{R}^n$ such that $A \subseteq C$. Then $C^\circ \subseteq A^\circ$.*

(ii) [The bipolar cone theorem] *Consider a nonempty set $A \subseteq \mathbb{R}^n$. Then $A^{\circ\circ} =$* cl cone co A.

Definition 2.2.18. Consider a set $A \subseteq \mathbb{R}^n$ and $\bar{x} \in A$. The tangent cone to the set A at $\bar{x}$, $T(A, \bar{x})$, is defined by

$$T(A, \bar{x}) := \left\{ d \in \mathbb{R}^n : \exists \{x_k\} \subseteq A,\ x_k \to \bar{x},\ t_k \downarrow 0 \text{ s.t. } \frac{1}{t_l}(x_k - \bar{x}) \to d \text{ as } k \to +\infty \right\}.$$

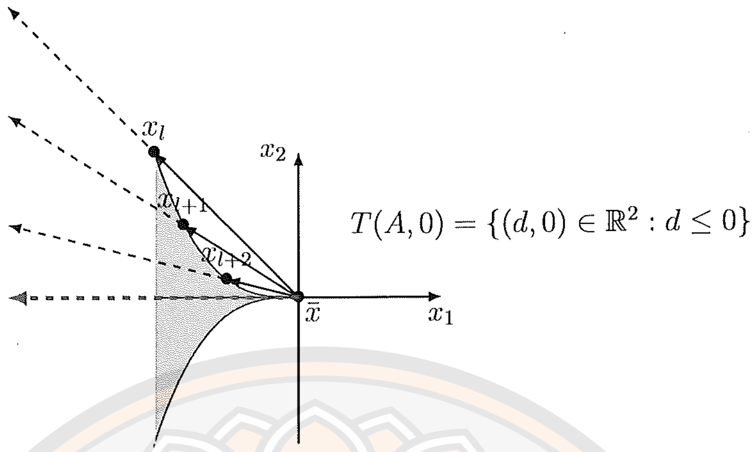


Figure 9: Illustration of the behavior of the vector in a tangent cone

In view of the definition, to construct a tangent cone we consider all the sequences $\{x_k\}$ in A the converge to the given point $\bar{x} \in A$, and then calculate all the directions $d \in \mathbb{R}^n$ that are tangential to the sequences at $\bar{x}$. However, if A is a convex set, then the tangent cone to the set A can be obtained by the following way.

Theorem 2.2.19. *Consider a set $A \subseteq \mathbb{R}^n$ and $\bar{x} \in A$. Then the following hold:*

(i) *$T(A, \bar{x})$ is closed.*

(ii) *If A is convex, then $T(A, \bar{x}) = \text{cl cone}(A - \bar{x})$ and hence $T(A, \bar{x})$ is convex.*

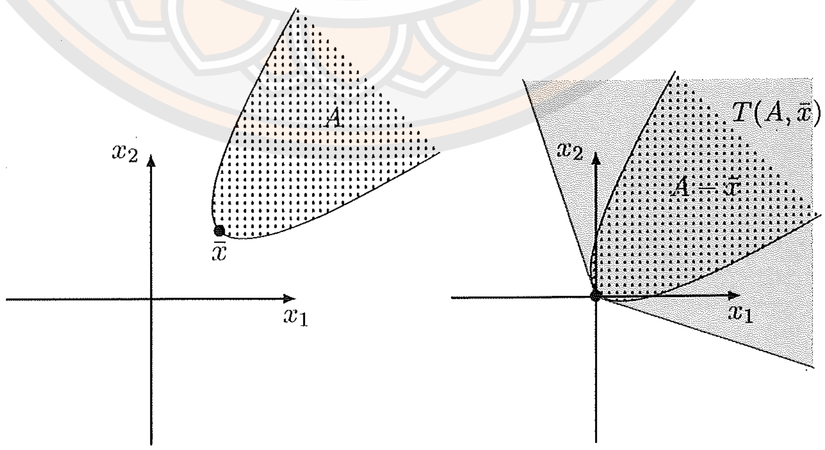


Figure 10: Illustration of a tangent cone of a convex set

Definition 2.2.20. Consider a convex set $A \subseteq \mathbb{R}^n$ and $\bar{x} \in A$. The normal cone to the set A at $\bar{x}$, $N(A, \bar{x})$, is given by

$$N(A, \bar{x}) := \{d \in \mathbb{R}^n : \langle d, x - \bar{x} \rangle \leq 0, \ \forall x \in A\}.$$

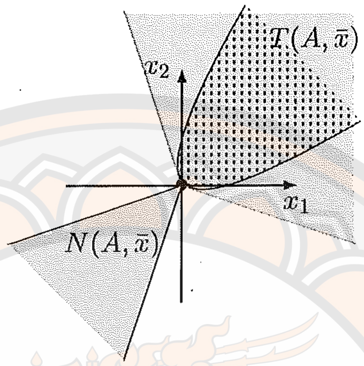


Figure 11: Tangent cone and normal cone of a convex set

Proposition 2.2.21. *Consider a convex set $A \subseteq \mathbb{R}^n$. Then,*

$$N(A, \bar{x}) = (T(A, \bar{x}))^\circ \text{ and } T(A, \bar{x}) = (N(A, \bar{x}))^\circ.$$

Definition 2.2.22. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be convex if for any $x, y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$ we have

$$f((1 - \alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y).$$

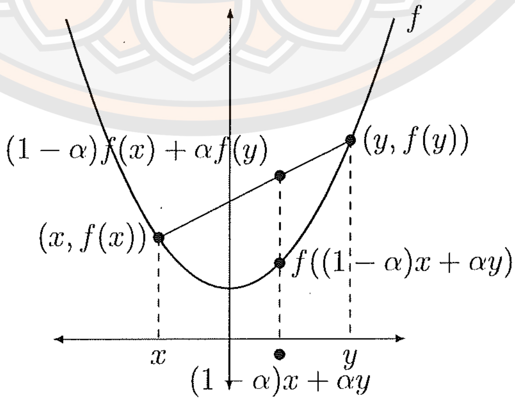


Figure 12: Geometric interpretation of convex functions

In the case of vector valued function, let $g : \mathbb{R}^n \to \mathbb{R}^p$ be a given function and $A \subseteq \mathbb{R}^p$ is a convex set. The function g is said to be A-convex if and only if for any $x, y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$,

$$g(\lambda x + (1-\lambda)y) - \lambda g(x) - (1-\lambda)g(y) \in -A.$$

On the other hand, the function f is said to be a concave function if and only if $-f$ is a convex function. Similarly, the function g is said to be an A-concave function if and only if $-f$ is an A-convex function.

Proposition 2.2.23. *Consider a proper function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$. f is convex if and only if* epi f *is a convex set on $\mathbb{R}^n \times \mathbb{R}$.*

The following proposition presents operations that preserve the convexity.

Proposition 2.2.24. (i) *Consider proper convex functions $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\alpha_i \geq 0$, $i = 1, 2, \ldots, m$. Then $f := \sum_{i=1}^{m} \alpha_i f_i$ is also a convex function.*

(ii) *Consider a family of proper convex functions $f_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, $i \in \mathcal{I}$, where $\mathcal{I}$ is an arbitrary index set. Then $f := \sup_{i \in \mathcal{I}} f_i$ is a convex function.*

Definition 2.2.25. [48] Consider a proper convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\bar{x} \in \operatorname{dom} f$. Then $\xi \in \mathbb{R}^n$ is said to be the subgradient of the function f at $\bar{x}$ if

$$f(x) - f(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle, \ \forall x \in \mathbb{R}^n.$$

The collection of all such vectors is called the subdifferential of f at $\bar{x}$ and is denoted by $\partial f(\bar{x})$.

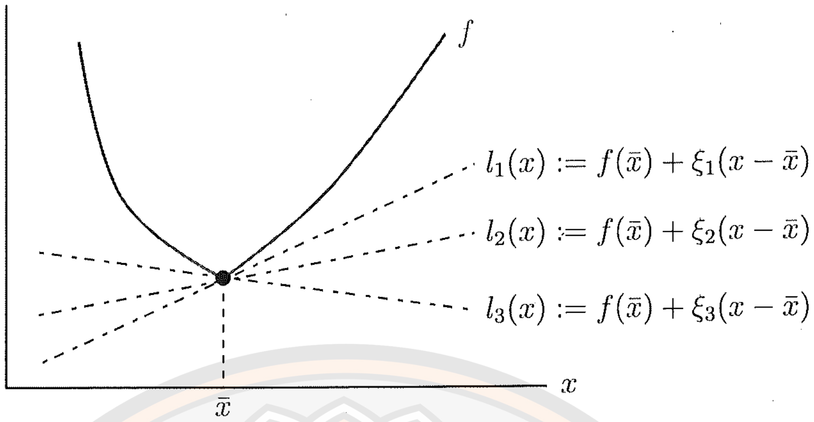


Figure 13: Geometric interpretation of subdifferentials

Theorem 2.2.26. [48] *Consider a proper convex function* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $\bar{x} \in \operatorname{dom} f$. *Then*

$$\partial f(\bar{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \forall x \in \mathbb{R}^n\}.$$

More generally, for each $\varepsilon \geq 0$, the ε-subdifferential of the function f at $\bar{x} \in \operatorname{dom} f$, is defined by

$$\partial_\varepsilon f(\bar{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \varepsilon, \forall x \in \mathbb{R}^n\}.$$

It is obvious that for $\varepsilon \geq \varepsilon'$, we have $\partial_{\varepsilon'} f(\bar{x}) \subseteq \partial_\varepsilon f(\bar{x})$. Specially, if f is a proper lsc convex function, then for every $\bar{x} \in \operatorname{dom} f$, the ε-subdifferential $\partial_\varepsilon f(\bar{x})$ is a nonempty closed convex set and

$$\partial f(\bar{x}) = \bigcap_{\varepsilon > 0} \partial_\varepsilon f(\bar{x}).$$

If $x \notin \operatorname{dom} f$, then we set $\partial f(x) = \emptyset$.

Proposition 2.2.27. [48, Theorem 25.1] *Consider a convex function* $f : \mathbb{R}^n \to \mathbb{R}$ *differentiable at* $\bar{x}$ *with gradient* $\nabla f(\bar{x})$. *Then,* $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$.

Proposition 2.2.28. [48] *Consider a proper convex function* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *and* $\bar{x} \in \operatorname{dom} f$. *Then* $\partial f(\bar{x})$ *is closed and convex. For* $\bar{x} \in \operatorname{ridom} f$, $\partial f(\bar{x}) \neq \emptyset$. *Furthermore, if* $\bar{x} \in \operatorname{intdom} f$, $\partial f(\bar{x})$ *is nonempty and compact. Moreover, if* f *is continuous at* $\bar{x} \in \operatorname{dom} f$, *then* $\partial f(\bar{x})$ *is compact.*

Theorem 2.2.29. [48, Moreau-Rockafellar Sum Rule] *Consider two proper convex function* $f, h : \mathbb{R}^n \to \overline{\mathbb{R}}$. *Suppose that* ri dom $f \cap$ ri dom $h \neq \emptyset$. *Then*

$$\partial(f+h)(x) = \partial f(x) + \partial h(x)$$

for every $x \in \mathrm{dom}(f+h)$.

Remark 2.2.30. [48] For a convex set A, $\partial\delta_A(\cdot) = N(A, \cdot)$.

Now, let us recall some basic concepts dealing with a difference convex (DC) programming problem.

Definition 2.2.31. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be a DC function is it is the difference of two convex functions. The minimization (or maximization) problem of a DC function is called a DC problem, i.e., the DC proplem can be expressed in the following form:

$$\text{Minimize } f(x) := h(x) - \phi(x) \text{ subject to } x \in \mathbb{R}^n,$$

where $h, \phi : \mathbb{R}^n \to \mathbb{R}$ are convex.

Note that the function f is DC and it is not expected to be convex.

It shall be found later that some DC problems are considered and their properties, in particular the following lemma, are employed in Chapter III.

Lemma 2.2.32. [49] *Let* $f, h : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be two proper lsc convex functions. Then*

(i) A point $\bar{x} \in \mathrm{dom}\, f \cap \mathrm{dom}\, h$ *is a (global) minimizer of the DC problem:*

$$\textit{Minimize } f(x) - h(x) \textit{ subject to } x \in \mathbb{R}^n$$

if and only if for any $\varepsilon \geq 0, \partial_\varepsilon h(\bar{x}) \subseteq \partial_\varepsilon f(\bar{x})$.

(ii) If $\bar{x} \in \mathrm{dom}\, f \cap \mathrm{dom}\, h$ *is a local minimizer of the DC problem:*

$$\textit{Minimize } f(x) - h(x) \textit{ subject to } x \in \mathbb{R}^n,$$

then $\partial h(\bar{x}) \subseteq \partial f(\bar{x})$.

Next, let us recall some basic concepts dealing with the conjugate function of a function.

Definition 2.2.33. The Legendre-Fenchel conjugate function of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup_{x \in \mathbb{R}^n} \{\langle x^*, x\rangle - f(x)\}$$

for all $x \in \mathbb{R}^n$.

The function f^* is lsc convex irrespective of the nature of f but for f^* to be proper, we need f to be a proper convex function.

We collect the following propoties of conjugate functions which are useful in later analysis, especially in Chapter IV.

Proposition 2.2.34. [50] *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper lsc convex function and $a \in domf := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. Then*

$$epi\ f^* = \bigcup_{\epsilon \geq 0} \{(v, \langle v, a\rangle + \epsilon - f(a)) : v \in \partial_\epsilon f(a)\}.$$

Proposition 2.2.35. [51] *Let $f, h : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper lsc convex functions. If $domf \cap domh \neq \emptyset$, then*

$$epi\ (f+h)^* = cl(epi\ f^* + epi\ h^*).$$

Moreover, if one of the functions f and h is continuous, then

$$epi\ (f+h)^* = cl(epi\ f^* + epi\ h^*).$$

Proposition 2.2.36. [8] *Let $g_i : R^n \times \mathbb{R}^q \to \mathbb{R}, i = 1, \ldots m$ be continuous functions. Suppose that each $\mathcal{V}_i \subseteq \mathbb{R}^q, i = 1, \ldots, m$, is convex, for all $v_i \in \mathbb{R}^q, g_i(\cdot, v_i)$ is a convex function, and for each $x \in \mathbb{R}^n, g_i(x, \cdot)$ is concave on $\mathcal{V}_i$. Then the cone $\bigcup_{\substack{v_i \in \mathcal{V}_i, \\ \lambda_i \geq 0}} epi \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\right)^*$ is convex.*

Proposition 2.2.37. [8] *Let $g_i : R^n \times \mathbb{R}^q \to \mathbb{R}, i = 1, \dots m$ be continuous functions. Suppose that each $\mathcal{V}_i \subseteq \mathbb{R}^q, i = 1, \dots, m$, is compact and convex, for all $v_i \in \mathbb{R}^q, g_i(\cdot, v_i)$ is a convex function, and there exists $y \in \mathbb{R}^n$ such that $g_i(y, v_i) < 0, \forall v_i \in \mathcal{V}_i, i = 1, \dots, m$. Then the cone $\bigcup_{\substack{v_i \in \mathcal{V}_i, \\ \lambda_i \geq 0}} epi \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \right)^*$ is closed.*

Proposition 2.2.38. [52] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i \in I$ be continuous functions such that for each $v_i \in \mathbb{R}^q, g(\cdot, v_i)$ is convex. Let $\mathcal{V}_i \subseteq \mathbb{R}^q, i \in I$ be compact and let $K := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i \in I\} \neq \emptyset$. Then the following statements are equivalent:*

(i) $K \subseteq \{x \in \mathbb{R}^n : f(x) \geq 0\}$;

(ii) $(0,0) \in epi\ f^* + cl \left(co \bigcup_{\substack{v_i \in \mathcal{V}_i, \\ \lambda_i \geq 0}} epi \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \right)^* \right)$.

As we also deal with a class of nonconvex functions instead of convex functions for constraint functions, we shall need generalized subdifferential to nonconvex function. The notation and definitions including the notations generally used in variational analysis, the Mordukhovich generalized differentiation notions (see more details in [53, 54]) are the main tools for our study dealing with nonconvex functions.

Definition 2.2.39. Let a point $\bar{x} \in A$ be given. The set A is said to be closed around $\bar{x}$ if there is a neighborhood U of $\bar{x}$ such that $A \cap U$ is closed. In addition, the set A is said to be locally closed if it is closed around every $\bar{x} \in A$.

Definition 2.2.40. Given a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the sequential Painlevé-Kuratowski upper/outer limit of F as $x \to \bar{x}$ is denoted by

$$\operatorname*{Lim\,sup}_{x \xrightarrow{A} \bar{x}} F(x) := \left\{ x^* \in \mathbb{R}^n : \exists x_n \xrightarrow{A} \bar{x}, \exists x_n^* \to x^* \text{ with } x_n^* \in F(x_n), \ \forall n \in \mathbb{N} \right\}.$$

Let A be closed around $\bar{x}$. Recall that the tangent (or contingent) cone of A at $\bar{x}$ is denoted by $T(A, \bar{x})$ and defined by

$$T(A, \bar{x}) := \{v \in \mathbb{R}^n : \exists v_n \to v, \exists t_n \downarrow 0 \text{ s.t. } \bar{x} + t_n v_n \in A, \forall n \in \mathbb{N}\}.$$

Definition 2.2.41. Let A be closed around $\bar{x}$. The Fréchet (or regular) normal cone of A at $\bar{x}$, which is a set of all the Fréchet normals, has the form $\widehat{N}(A,\bar{x})$ and is defined by

$$\widehat{N}(A,\bar{x}) := \left\{ x^* \in \mathbb{R}^n : \limsup_{x \xrightarrow{A} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq 0 \right\}.$$

Note that the Fréchet (or regular) normal cone $\widehat{N}(A,\bar{x})$ is a closed convex subset of $\mathbb{R}^n$ and we set $\widehat{N}(A,\bar{x}) = \emptyset$ if $\bar{x} \notin A$.

Definition 2.2.42. Let A be closed around $\bar{x}$. The notation $N(A,\bar{x})$ stands for the Mordukhovich (or basic, limiting) normal cone of A at $\bar{x}$. It is defined by

$$N^M(A,\bar{x}) := \left\{ x^* \in \mathbb{R}^n : \exists x_n \xrightarrow{A} \bar{x}, \exists x_n^* \to x^* \text{ with } x_n^* \in \widehat{N}(A,x_n), \forall n \in \mathbb{N} \right\}.$$

Observe that the Mordukhovich normal cone is obtained by the Fréchet normal cones by taking the sequential Painlevé-Kuratowski upper/outer limit (see [53] for more details) as:

$$N^M(A,\bar{x}) = \operatorname*{Lim\,sup}_{x \xrightarrow{A} \bar{x}} \widehat{N}(A,x).$$

Specially, in the case that A is a convex set, then we obtain

$$\widehat{N}(A,\bar{x}) = N^M(A,\bar{x}) = T(A,\bar{x})^\circ = \{x^* \in \mathbb{R}^n : \langle x^*, x-\bar{x}\rangle \leq 0, \forall x \in A\}.$$

Definition 2.2.43. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an extended real-valued function, $\bar{x} \in \operatorname{dom} f$ and $\varepsilon \geq 0$ be given.

(i) The analytic ε-subdifferential of function f at $\bar{x}$, which has the form $\widehat{\partial}_\varepsilon f(\bar{x})$ is defined by

$$\widehat{\partial}_\varepsilon f(\bar{x}) := \left\{ x^* \in \mathbb{R}^n : \liminf_{\substack{x \to \bar{x}, \\ x \neq \bar{x}}} \frac{f(x) - f(\bar{x}) - \langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \geq -\varepsilon \right\}.$$

(ii) If $\varepsilon = 0$, then the analytic ε-subdifferential $\widehat{\partial}_\varepsilon f(\bar{x})$ of f at $\bar{x}$ reduces to the general Fréchet subdifferential of h at $\bar{x}$, which is denoted by $\widehat{\partial} f(\bar{x})$.

(iii) $\partial^M f(\bar{x})$ denotes the Mordukhovich subdifferential of f at $\bar{x}$. It is defined by

$$\partial^M f(\bar{x}) := \left\{ x^* \in \mathbb{R}^n : \exists x_n \xrightarrow{f} \bar{x}, \exists x_n^* \to x^* \text{ with } x_n^* \in \widehat{\partial} f(x_n), \forall n \in \mathbb{N} \right\},$$

where $x_n \xrightarrow{f} \bar{x}$ means $x_n \to \bar{x}$ and $f(x_n) \to f(x)$.

In the case that $x \notin \text{dom} f$, we set $\widehat{\partial} f(\bar{x}) = \partial^M f(\bar{x}) = \emptyset$.

Remark 2.2.44. It is obvious that for any $x \in \mathbb{R}^n, \widehat{\partial} f(\bar{x}) \subseteq \partial^M h(\bar{x})$. Specially, if f is a convex function, then

$$\widehat{\partial} h(\bar{x}) = \partial^M f(\bar{x}) = \left\{ x^* \in \mathbb{R}^n : \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}), \forall x \in \mathbb{R}^n \right\}.$$

Recall that, the distance function $d_S : \mathbb{R}^n \to \mathbb{R}$ and the indicator function $\delta_A : \mathbb{R}^n \to \overline{\mathbb{R}}$ of A are respectively defined by

$$d_A(x) := \inf_{y \in A} \|x - y\|, \forall x \in \mathbb{R}^n, \text{ and } \delta_A(x) = \begin{cases} 0; & x \in A, \\ +\infty; & x \notin A. \end{cases}$$

By above notations and definitions, we get

$$\widehat{\partial} \delta_A(\bar{x}) = \widehat{N}(A, \bar{x}) \text{ and } \partial^M \delta_A(\bar{x}) = N^M(A, \bar{x}).$$

Simultaneously, one has

$$\widehat{\partial} d_A(\bar{x}) = \mathbb{B} \cap \widehat{N}(A, \bar{x}) \text{ and } \partial^M d_A(\bar{x}) \subseteq \mathbb{B} \cap N^M(A, \bar{x}).$$

Next, we recall some useful and important propositions and definitions dealing with nonsmooth (not necessarily convex) functions.

Definition 2.2.45. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipshitz at $x \in \mathbb{R}^n$, if there exist an open neighborhood U and a constant L such that, for all y and z in U, one has

$$|f(y) - f(z)| \leq L\|y - z\|.$$

If the function f is locally Lipshitz at every point $x \in \mathbb{R}^n$, one says that f is a locally Lipshitz function on $\mathbb{R}^n$.

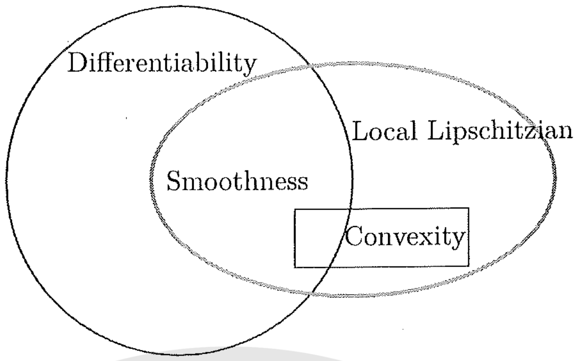


Figure 14: Relationship among differentiability, smoothness, convexity, and local Lipschitzian.

Lemma 2.2.46. [53] *If* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *is locally Lipschitz at* $\bar{x}$, *with modulus* $l > 0$, *then we always have* $\|x^*\| \leq l, \ \forall x^* \in \partial^M f(\bar{x})$.

Theorem 2.2.47. [53, 54, The generalized Fermat rule] *Let* $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be a proper lsc function. If* f *attains a local minimum at* $\bar{x} \in \mathbb{R}^n$, *then* $0 \in \widehat{\partial} f(\bar{x})$, *which implies* $0 \in \partial^M f(\bar{x})$.

Theorem 2.2.48. [53, 54, The fuzzy sum rule for the Fréchet subdifferential and the sum rule for the Mordukhovich subdifferential] *Let* $f, h : \mathbb{R}^n \to \overline{\mathbb{R}}$ *be proper lsc around* $\bar{x} \in \ dom f \cap \ dom h$. *If* f *is Lipschitz continuous around* $\bar{x}$, *then*

1. *for every* $x^* \in \widehat{\partial}(f + h)(\bar{x})$ *and every* $\varepsilon > 0$, *there exist* $x_1, x_2 \in B(\bar{x}, \varepsilon)$ *such that*

$$|f(x_1) - f(\bar{x})| < \varepsilon, |h(x_2) - h(\bar{x})| < \varepsilon \text{ and } x^* \in \widehat{\partial} f(x_1) + \widehat{\partial} h(x_2) + \varepsilon \mathbb{B}.$$

2. $\partial^M (f + h)(\bar{x}) \subseteq \partial^M f(\bar{x}) + \partial^M h(\bar{x})$.

2.3 Weak sharp solutions

Before recalling notions of weak sharp solutions in optimization problems, let us collect problems which will be considered in this thesis. First of all, consider the following optimization problem:

$$\text{Minimize } f(x) \text{ subject to } g_i(x) \leq 0, \ i = 1, \ldots, m, \tag{P}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I := \{1, \ldots, m\}$, are given functions. The feasible set of (P), denoted by K, is defined as

$$K := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I\}.$$

The optimization problem (P) in the face of constraint data uncertainty can be captured by the following optimization problem:

$$\text{Minimize } f(x, u) \text{ subject to } g_i(x, v_i) \leq 0, \; i \in I \tag{UP}$$

where $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathcal{V}_i \to \mathbb{R}$, $i \in I$, are given functions, and u and v_i are the uncertain parameters that are not exactly known, but are only known to reside in certain uncertainty sets $\mathcal{U} \subseteq \mathbb{R}^p$ and $\mathcal{V}_i \subseteq \mathbb{R}^q, i \in I$, respectively.

In general, the robust counterpart of the problem (UP) which, by a parametric reformulation of (UP) (see, [4]), is given by

$$\text{Minimize } \sup_{u \in \mathcal{U}} f(x, u) \text{ subject to } g_i(x, v_i) \leq 0, \; v_i \in \mathcal{V}_i, \; i \in I \tag{RP}$$

where the uncertain constraint are enforced for every possible value of the parameters within their prescribed uncertainty and the global minimizer of the problem (RP) is known as robust optimal solution of the problem (UP).

Now, we are ready to recall the notions of the weak sharp solutions of the optimization problem (P). In order to deal with such notions, we recall the following notion of a sharp minimum, or equivalently, a strongly unique local minimum, which has far reaching consequences for the convergence analysis of many iterative procedures [55–59].

Definition 2.3.1. (i) A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ has a sharp minimum at $\bar{x} \in K \subseteq \mathbb{R}^n$ if there exists a real number $\eta > 0$ such that $f(x) \geq f(\bar{x}) + \eta\|x - \bar{x}\|$ for all $x \in K$.

(ii) A point $\bar{x} \in K$ is said to be a sharp solution (or minima, or minimizer) of (P) if f has a sharp minimum at $\bar{x}$, i.e., there exists $\eta > 0$ such that

$$f(x) - f(\bar{x}) \geq \eta\|x - \bar{x}\|, \forall x \in K.$$

In [30], Burke and Ferris extended the notion of a sharp minimum to include the possibility of a non unique solution set.

Definition 2.3.2. We say that $\overline{S} \subseteq \mathbb{R}^n$ is a set of weak sharp minima for the function f relative to the (feasible) set $K \subseteq \mathbb{R}^n$ where $\overline{S} \subseteq K$ if there is an $\eta > 0$ such that

$$f(x) - f(y) \geq \eta d_{\overline{S}}(x) \tag{2.3.1}$$

for all $x \in K$ and $y \in \overline{S}$.

The constant η and the set $\overline{S}$ are called the modulus and domain of sharpness for f over K, respectively. Clearly, $\overline{S}$ is a set of global minima for f over K. The notion of weak sharp minima is easily localized.

Next, let us focus on the notion of the weak sharp solution for the constrained optimization problem (P) with its feasible set K.

Definition 2.3.3. A point $\bar{x} \in K$ is said to be a local weak sharp solution for (P) if and only if there exist a neighborhood U of $\bar{x}$ and a real number $\eta > 0$ such that

$$f(x) - f(\bar{x}) \geq \eta d_{\overline{S}}(x), \forall x \in K \cap U,$$

where $\overline{S} := \{x \in K : f(x) = f(\bar{x})\} = K \cap f^{-1}(f(\bar{x}))$. Specially, if $U = \mathbb{R}^n$, then $\bar{x}$ is said to be a global weak sharp solution for (P).

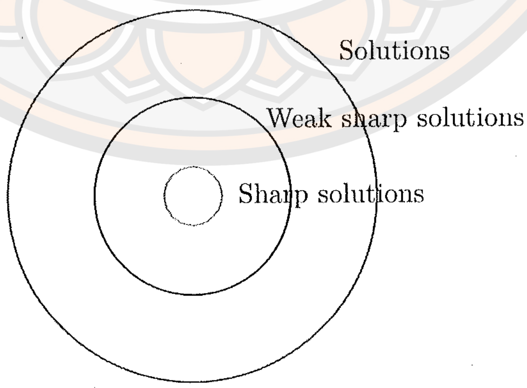


Figure 15: Relationship among sets of all solusions, weak sharp solutions, and sharp solutions.

In this thesis, we are interested in the studies of weak sharp solutions of nonsmooth convex/nonconvex optimization problems with data uncertainty. More clearly, we introduce new concepts of solutions, related the sharpness, for uncertain optimization problems in forms of (UP). The studies are conducted by examing the robust optimization problems in forms of (RP).

With the intention to answer the question - *How about the issue of weak sharp solutions, particularly, in uncertain optimization problems?*, we obtain some main results which are presented in Chapter III.

2.4 Approximate solutions

In this section, we collect the notions of approximate solutions of the optimization problems (P) with its feasible set K. The following three kinds of approximate solutions of the problem (P) were introduced by Loridan [37].

Definition 2.4.1. [37] Let $\varepsilon > 0$, a point $\bar{x} \in K$ is said to be:

(i) an ε-solution for (P) if $f(\bar{x}) \leq f(x) + \varepsilon$ for all $x \in K$,

(ii) an ε-quasi solution for (P) if $f(\bar{x}) \leq f(x) + \sqrt{\varepsilon}\|x - \bar{x}\|$ for all $x \in K$,

(iii) a regular ε-solution for (P) if it is an ε-solution and an ε-quasi solution for (P).

The following example indicates approximate solutions of the optimization problem in form of (P).

Example 2.4.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} -x^{\frac{1}{3}}; & x \geq 0, \\ 0; & x < 0, \end{cases}$$

and let $K = [0, 1]$. For $\bar{x} = 1$, we set $\varepsilon_n := \frac{1}{n^2}, n \in \mathbb{N}$ and get that for each $x \in K$,

$$f(x) - f(\bar{x}) + \frac{1}{n^2} \geq 0 \text{ and } f(x) - f(\bar{x}) + \frac{1}{n}|x - \bar{x}| \geq 0.$$

Therefore $\bar{x}$ is a regular ε_n-solution for (P) *for any $n \in \mathbb{N}$. Besides for $\bar{x}_i = \frac{1}{i}, i = 1, \ldots, 5$, by taking $\varepsilon_n := n, n \in \mathbb{N}$, we obtain*

$$f(x) - f(\bar{x}_i) + n \geq 0 \text{ and } f(x) - f(\bar{x}_i) + \sqrt{n}|x - \bar{x}_i| \geq 0, \forall x \in K.$$

Hence for each $i = 1, \ldots, 5$, one has $\frac{1}{i}$ is a regular ε_n-solution for (P) *for any $n \in \mathbb{N}$.*

Remark 2.4.3. If $\bar{x}$ is an ε-quasi solution for (P), then there exists a ball $B(\bar{x}, \sqrt{\varepsilon})$ such that $f(\bar{x}) \leq f(x) + \varepsilon$ for all $x \in B(\bar{x}, \sqrt{\varepsilon}) \cap K$. In this case, we can say that $\bar{x}$ is a locally ε-solution for (P).

The following example sheds some light on to the fact stated in Remark 2.4.3.

Example 2.4.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 10x^{\frac{1}{2}}; & x \geq 0, \\ 0; & x < 0, \end{cases}$$

and let $K = [0, 2]$. Consider $\bar{x} = \frac{1}{2} \in K$ and $\bar{\varepsilon} = \frac{4}{9}$. We can see that $\bar{x}$ is an $\bar{\varepsilon}$-quasi solution but not an $\bar{\varepsilon}$-solution for (P). *However, the point $\bar{x}$ is a locally $\bar{\varepsilon}$-solution of* (P) *since $f(x) - f(\bar{x}) + \bar{\varepsilon} \geq 0$ for all $x \in [0, \frac{7}{6}] = K \cap B(\bar{x}, \sqrt{\bar{\varepsilon}})$.*

In this thesis, we are interested in the approximate solutions, particularly approximated quasi solutions, that approximate the solutions of uncertain optimization problems in forms of (UP). In order to study our interested solutions, we investigate the problem (UP) by examing the robust optimization problems in forms of (RP).

With the intention to answer the question - *How about the study of approximate optimality conditions and approximate duality theorems for an approximate solution that approximates the (highly) robust solutions to an uncertain convex optimizaion problem?*, we obtain some main results which are presented in Chapter IV.

CHAPTER III

OPTIMALITY CONDITIONS AND CHARACTERIZATIONS FOR ROBUST WEAK SHARP SOLUTIONS

3.1 Uncertain convex optimization problems

In this section, we consider uncertain convex optimization problems involing convex objective functions and D-convex constraint functions. First of all, we introduce the notion of a robust weak sharp solution to an uncertain convex optimization problem. Then, optimality conditions for the robust weak sharp solutions and characterizations of the sets of all the robust weak sharp solutions of the problem are obtained. Finally, we apply the results to an uncertain convex muti-objective optimization problem and obtain optimality conditions for robust weak sharp weakly efficient solutions in the multi-objective optimization problem.

Let $C \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Let $D \subseteq \mathbb{R}^p$ be a nonempty closed convex cone. Consider the following convex optimization problem:

$$\text{Minimize } f(x) \text{ subject to } x \in C, g(x) \in -D, \tag{P$_1$}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a D-convex function. The feasible set of (P_1) is defined by $\{x \in C : g(x) \in -D\}$. The problem (P_1) in the face of data uncertainty both in the objective and constraints can be captured by the following uncertain optimization problem:

$$\text{Minimize } f(x, u) \text{ subject to } x \in C, g(x, v) \in -D, \tag{UP$_1$}$$

where $\mathcal{U} \subseteq \mathbb{R}^p$ and $\mathcal{V} \subseteq \mathbb{R}^q$ are convex and compact uncertainty sets, $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ is a given real-valued function such that, for any uncertain parameter $u \in \mathcal{U}$, $f(\cdot, u)$ is convex as well as $f(x, \cdot)$ is concave for any $x \in \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}^m$ is a vector-valued function such that, for any uncertain parameter $v \in \mathcal{V}$, $g(\cdot, v)$ is D-convex as well as $g(x, \cdot)$ is D-concave for any $x \in \mathbb{R}^n$. The uncertain sets can be apprehended in the sense that the parameter vectors u and v are not known exactly at the time of the decision.

For examining the uncertain optimization problem (UP_1), one usually associates with its robust (worst-case) counterpart, which is the following problem:

$$\text{Minimize } \sup_{u\in\mathcal{U}} f(x,u) \text{ subject to } x \in C, g(x,v) \in -D, \forall v \in \mathcal{V}. \tag{RP$_1$}$$

It is worth observing here that the robust counterpart, which is termed as the robust optimization problem, finds a worst-case possible solution that can be immunized opposed the data uncertainty.

The problem (RP_1) is said to be feasible if the robust feasible set K_1 is nonempty where it is denoted by

$$K_1 := \{x \in C : g(x,v) \in -D, \forall v \in \mathcal{V}\}. \tag{3.1.1}$$

We recall the following concept of solutions, which was introduced in [2].

Definition 3.1.1. [2] A point $\bar{x} \in K_1$ is said to be a robust optimal solution for (UP_1) if it is an optimal solution for (RP_1), i.e., for all $x \in K_1$,

$$\sup_{u\in\mathcal{U}} f(x,u) - \sup_{u\in\mathcal{U}} f(\bar{x},u) \geq 0.$$

The robust optimal solution set of (UP_1) is the set which consists of all robust optimal solutions of (UP_1) and is given by

$$S_1 := \left\{ x \in K_1 : \sup_{u\in\mathcal{U}} f(x,u) \leq \sup_{u\in\mathcal{U}} f(y,u),\ \forall y \in K_1 \right\}.$$

In [68], using the idea of weak sharp solution, and the robust optimal solution, we introduced a new concept of solutions for (UP_1), which related to the sharpness, namely the robust weak sharp solution.

Definition 3.1.2. A point $\bar{x} \in K_1$ is said to be a (or an optimal) weak sharp solution for (RP_1) if there exist a real number $\eta > 0$ such that for all $x \in K_1$,

$$\sup_{u\in\mathcal{U}} f(x,u) - \sup_{u\in\mathcal{U}} f(\bar{x},u) \geq \eta d_{\widetilde{K}_1}(x)$$

where $\widetilde{K}_1 := \left\{ x \in K_1 : \sup_{u \in \mathcal{U}} f(x,u) = \sup_{u \in \mathcal{U}} f(\bar{x},u) \right\}$.

Definition 3.1.3. A point $\bar{x} \in K_1$ is said to be a (or an optimal) robust weak sharp solution for (UP_1) if it is a weak sharp solution for (RP_1). The robust weak sharp solution set of (UP_1) is given by

$$\widetilde{S}_1 := \left\{ \bar{x} \in K_1 : \exists \eta > 0 \text{ s.t. } \sup_{u \in \mathcal{U}} f(y,u) - \sup_{u \in \mathcal{U}} f(\bar{x},u) \geq \eta d_{\widetilde{K}_1}(y), \ \forall y \in K_1 \right\}.$$

Throughout the chapter, we assume that $\widetilde{S}_1$ is nonempty.

Remark 3.1.4. It is worthwhile to be noted that every robust weak sharp solution for (UP_1) is a robust optimal solution. In general, the reverse implication need not to be valid.

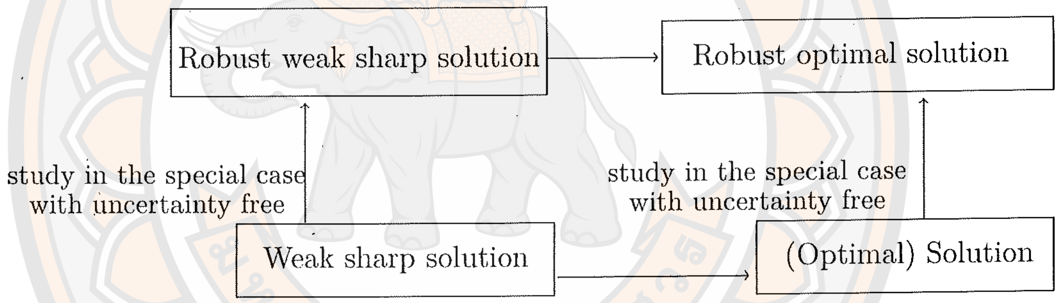


Figure 16: Relationship between sets of all robust optimal solusions and robust weak sharp solutions

3.1.1 Optimality conditions and characterizations for robust weak sharp solutions

We establish some optimality conditions for the robust weak sharp solution in convex uncertain optimization problems as well as obtain characterizations of the robust weak sharp solution sets for the considered problems. For any $\bar{x} \in \mathbb{R}^n$, we use the following notations:

$$\mathcal{U}(\bar{x}) := \left\{ \hat{u} \in \mathcal{U} : f(\bar{x},\hat{u}) = \sup_{u \in \mathcal{U}} f(\bar{x},u) \right\},$$

and

$$\mathcal{V}(\bar{x}) := \left\{ \hat{v} \in \mathcal{V} : g(\bar{x}, \hat{v}) = \sup_{v \in \mathcal{V}} g(\bar{x}, v) \right\}.$$

The following definition, which was introduced in [45], plays a vital role in determining characterizations of robust weak sharp solution sets.

Definition 3.1.5. [45] The robust type subdifferential constraint qualification (RSCQ) is said to be satisfied at $\bar{x} \in K_1$ if

$$\partial \delta_{K_1}(\bar{x}) \subseteq \partial \delta_C(\bar{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\bar{x}, v) = 0}} \partial(\mu g)(\cdot, v)(\bar{x}).$$

Remark 3.1.6. In an excellent work, [45], Sun et. al. introduced the (RSCQ) and then obtained some characterizations of the the robust optimal solution set, for an uncertain convex optimization problem. Although it has been used as a guideline for dealing with the (UP_1), our attention is paid to characterizing the sets containing the robust weak sharp solutions of such problem. Furthermore, the presence of the term $d_{\widetilde{K}}(x)$ has led us to deal with some different tools and methods, for studying this issue, from those in work of Sun et.al.

Lemma 3.1.7. [69] *Let $\mathcal{U} \subseteq \mathbb{R}^p$ be a convex compact set, and $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ be a function such that, $f(\cdot, u)$ is a convex function for any $u \in \mathcal{U}$, and $f(x, \cdot)$ is a concave function for any $x \in \mathbb{R}^n$. Then,*

$$\partial \left(\sup_{u \in \mathcal{U}} f(\cdot, u) \right)(\bar{x}) = \bigcup_{u \in \mathcal{U}(\bar{x})} \partial f(\cdot, u)(\bar{x}).$$

The following theorem presents that the robust type subdifferential constraint qualification (RSCQ) defined in Definition 3.1.5 is fulfilled if and only if optimality conditions for a robust weak sharp solution of (UP_1) are satisfied.

Theorem 3.1.8. *Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^m$ satisfy the following properties :*

(i) *for any* $u \in \mathcal{U}$ *and* $v \in \mathcal{V}$, $f(\cdot, u)$ *is convex and continuous as well as* $g(\cdot, v)$ *is* D*-convex on* $\mathbb{R}^n$;

(ii) *for any* $x \in \mathbb{R}^n$, $f(x, \cdot)$ *is concave on* $\mathcal{U}$ *and* $g(x, \cdot)$ *is* D*-concave on* $\mathcal{V}$.

Then, the following statements are equivalent:

(a) *The (RSCQ) is fulfilled at* $\bar{x} \in K_1$;

(b) $\bar{x} \in \mathbb{R}^n$ *is a robust weak sharp solution of* (UP_1) *if and only if there exists a positive constant* η *such that*

$$\begin{aligned} &N(\widetilde{K}_1, \bar{x}) \cap \eta\mathbb{B} \\ &\subseteq \bigcup_{u \in \mathcal{U}(\bar{x})} \partial f(\cdot, u)(\bar{x}) + \partial \delta_C(\bar{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\bar{x}, v) = 0}} \partial \left((\mu g)(\cdot, v) \right)(\bar{x}). \end{aligned} \tag{3.1.2}$$

Proof. $[(a) \Rightarrow (b)]$ Assume that the (RSCQ) is satisfied at $\bar{x} \in K$. Let $\bar{x}$ be a robust weak sharp solution of (UP_1). Consequently, there exists $\eta > 0$ such that

$$\sup_{u \in \mathcal{U}} f(x, u) - \sup_{u \in \mathcal{U}} f(\bar{x}, u) \geq \eta d_{\widetilde{K}_1}(x). \tag{3.1.3}$$

By (3.1.3), we obtain that for all $x \in K_1$,

$$\begin{aligned} \sup_{u \in \mathcal{U}} f(x, u) + \delta_{K_1}(x) - \eta d_{\widetilde{K}_1}(x) &\geq \sup_{u \in \mathcal{U}} f(\bar{x}, u) \\ &= \sup_{u \in \mathcal{U}} f(\bar{x}, u) + \delta_{K_1}(\bar{x}) - \eta d_{\widetilde{K}_1}(\bar{x}), \end{aligned}$$

thereby implying that, for all $\xi_d \in \partial \eta d_{\widetilde{K}_1}(x)$,

$$\begin{aligned} &\left(\sup_{u \in \mathcal{U}} f(\cdot, u) + \delta_{K_1} \right)(x) - \left(\sup_{u \in \mathcal{U}} f(\cdot, u) + \delta_{K_1} \right)(\bar{x}) \\ &\geq \eta d_{\widetilde{K}_1}(x) - \eta d_{\widetilde{K}_1}(\bar{x}) \\ &\geq \langle \xi_d, x - \bar{x} \rangle. \end{aligned}$$

Thus, $\xi_d \in \partial \left(\sup_{u \in \mathcal{U}} f(\cdot, u) + \delta_{K_1} \right)(\bar{x})$. Hence,

$$\partial(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq \partial \left(\sup_{u \in \mathcal{U}} f(\cdot, u) + \delta_{K_1} \right)(\bar{x}).$$

Since $\sup_{u\in\mathcal{U}} f(\cdot,u)$ is continuous on $\mathbb{R}^n$and δ_{K_1} is proper lsc convex on $\mathbb{R}^n$, we have

$$\partial(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq \partial(\sup_{u\in\mathcal{U}} f(\cdot,u))(\bar{x}) + \partial\delta_{K_1}(\bar{x}).$$

It can be noted that $\partial d_{\widetilde{K}_1}(x) = N(\widetilde{K}_1,(x)) \cap \mathbb{B}$. Since (RSCQ) is satisfied at $\bar{x}$, we have the following:

$$\begin{aligned} N(\widetilde{K}_1,x) \cap \eta\mathbb{B} &= \partial(\eta d_{\widetilde{K}_1})(\bar{x}) \\ &\subseteq \bigcup_{u\in\mathcal{U}(\bar{x})} \partial f(\cdot,u)(\bar{x}) + \partial\delta_C(\bar{x}) + \bigcup_{\substack{\mu\in D^*,v\in\mathcal{V} \\ (\mu g)(\bar{x},v)=0}} \partial\left(\mu g)(\cdot,v)\right)(\bar{x}), \end{aligned}$$

which implies that (3.1.2) holds.

Conversely, assume that there is a positive number η such that (3.1.2) holds. Since $N(\widetilde{K}_1,\bar{x}) \cap \eta\mathbb{B}$ always contains 0, it is a nonempty set and so is $\bigcap_{\varepsilon>0} \partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x})$. Thus, for any $\varepsilon \geq 0$, $\partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x}) \neq \emptyset$. Let $\varepsilon > 0$ be arbitrary and let $\xi \in \partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x})$. Then for any $x \in K_1$,

$$\eta d_{\widetilde{K}_1}(x) - \eta d_{\widetilde{K}_1}(\bar{x}) \geq \langle \xi, x-\bar{x}\rangle - \varepsilon. \tag{3.1.4}$$

Note that $0 \in \partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x})$. It follows that

$$\eta d_{\widetilde{K}_1}(\bar{x}) \leq \inf_{x\in\mathbb{R}^n} \eta d_{\widetilde{K}_1}(x) + \varepsilon \leq \inf_{x\in K_1} \eta d_{\widetilde{K}_1}(x) + \varepsilon.$$

Above inequality and (3.1.4) imply that

$$0 \geq \langle \xi, x-\bar{x}\rangle - \varepsilon. \tag{3.1.5}$$

Simultaneously, there exist $\hat{u} \in \mathcal{U}(\bar{x})$, $\hat{\mu} \in D^*, \hat{v} \in \mathcal{V}(\bar{x})$

$\xi_f \in \partial f(\cdot,\hat{u})(\bar{x}), \xi_\delta \in \partial\delta_C(\bar{x})$, and $\xi_{\hat{\mu}g} \in \partial\left((\hat{\mu}g)(\cdot,\hat{v})\right)(\bar{x})$ such that

$$\xi_f + \xi_\delta + \xi_{\hat{\mu}g} = 0, \tag{3.1.6}$$

and for any $x \in \mathbb{R}^n$, we have

$$f(x,\hat{u}) - f(\bar{x},\hat{u}) \geq \langle \xi_f, x-\bar{x}\rangle,$$

$$\delta_C(x) - \delta_C(\bar{x}) \geq \langle \xi_\delta, x - \bar{x} \rangle, \text{ and}$$

$$(\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\bar{x}, \hat{v}) \geq \langle \xi_{\hat{\mu} g}, x - \bar{x} \rangle.$$

Adding these above inequalities implies that for each $x \in K_1$

$$f(x, \hat{u}) - f(\bar{x}, \hat{u}) + (\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\bar{x}, \hat{v}) \geq \langle 0, x - \bar{x} \rangle = 0.$$

Since $\hat{u}$ belongs to $\mathcal{U}(\bar{x})$, for each $x \in K_1$, above inequality becomes

$$\sup_{u \in \mathcal{U}} f(x, u) - \sup_{u \in \mathcal{U}} f(\bar{x}, u) + (\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\bar{x}, \hat{v}) \geq 0.$$

This along with $(\hat{\mu} g)(x, \hat{v}) \leq 0, (\hat{\mu} g)(\bar{x}, \hat{v}) = 0$, and (3.1.6) imply

$$\sup_{u \in \mathcal{U}} f(x, u) - \sup_{u \in \mathcal{U}} f(\bar{x}, u) \geq 0, \tag{3.1.7}$$

for all $x \in K_1$. Observe that, combining inequalities (3.1.5) and (3.1.7) leads to

$$\sup_{u \in \mathcal{U}} f(x, u) - \sup_{u \in \mathcal{U}} f(\bar{x}, u) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon, \ \forall x \in K_1.$$

This means $\xi \in \partial_\varepsilon(\sup_{u \in \mathcal{U}} f(\cdot, u))(\bar{x})$, and so $\partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq \partial_\varepsilon(\sup_{u \in \mathcal{U}} f(\cdot, u))(\bar{x})$. Since the inclusion holds for arbitrary $\varepsilon \geq 0$, it follows from the Lemma 2.2.32 that $\bar{x}$ is a minimizer of the DC problem: $\inf_{x \in \mathbb{R}^n} \{\sup_{u \in \mathcal{U}} f(x, u) - \eta d_{\widetilde{K}_1}(x)\}$ and hence for any $x \in K_1$

$$\sup_{u \in \mathcal{U}} f(x, u) - \sup_{u \in \mathcal{U}} f(\bar{x}, u) - \left(\eta d_{\widetilde{K}_1}(x) - \eta d_{\widetilde{K}_1}(\bar{x})\right) \geq 0.$$

Therefore, for any $x \in K_1$,

$$\sup_{u \in \mathcal{U}} f(x, u) - \sup_{u \in \mathcal{U}} f(\bar{x}, u) \geq \eta d_{\widetilde{K}_1}(x).$$

This means $\bar{x}$ is a robust weak sharp solution of (UP_1).

[$(b) \Rightarrow (a)$] Let $\xi_\delta \in \partial \delta_{K_1}(\bar{x})$ be given. Then, we have

$$0 = \delta_{K_1}(x) - \delta_{K_1}(\bar{x}) \geq \langle \xi_\delta, x - \bar{x} \rangle$$

holds for all $x \in K_1$. Let $\bar{\eta} > 0$ be given, and then, set $f(x, u) := -\langle \xi_\delta, x \rangle + \bar{\eta} d_{\widetilde{K}_1}(x)$. Thus, for any $x \in K_1$,

$$\sup_{u \in \mathcal{U}} f(x, u) - \bar{\eta} d_{\widetilde{K}_1}(x) = -\langle \xi_\delta, x \rangle$$

$$\geq -\langle \xi_\delta, \bar{x} \rangle + \bar{\eta} d_{\widetilde{K}_1}(\bar{x})$$
$$= \sup_{u \in \mathcal{U}} f(\bar{x}, u).$$

Thus, $\bar{x}$ is a robust weak sharp solution of (UP_1). By hypothesis, there is $\eta := \bar{\eta}$ such that (3.1.2) is fulfilled. Since for any $u \in \mathcal{U}, \partial f(\cdot, u)(\bar{x}) \subseteq \{-\xi_\delta\} + \partial(\eta d_{\widetilde{K}_1})(\bar{x})$, we obtain that for any $x^* \in N(\widetilde{K}_1, \bar{x}) \cap \eta \mathbb{B}$, there exist $\hat{u} \in \mathcal{U}(\bar{x}), \hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that

$$x^* \in \{-\xi_\delta\} + \partial(\eta d_{\widetilde{K}_1})(\bar{x}) + \partial \delta_C(\bar{x}) + \partial\left((\hat{\mu} g)(\cdot, \hat{v})\right)(\bar{x}) \text{ and } (\hat{\mu} g)(\bar{x}, \hat{v}) = 0.$$

As $0 \in N(\widetilde{K}_1, \bar{x}) \cap \eta \mathbb{B}$, we obtain

$$\xi_\delta \in \partial \delta_C(\bar{x}) + \partial\left((\hat{\mu} g)(\cdot, \hat{v})\right)(\bar{x}) \text{ and } (\hat{\mu} g)(\bar{x}, \hat{v}) = 0.$$

It follows that

$$\xi_\delta \in \partial \delta_C(\bar{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\bar{x}, v) = 0}} \partial\left((\mu g)(\cdot, v)\right)(\bar{x}),$$

and so we get the desired inclusion. Therefore, the proof is complete. □

Remark 3.1.9. In [62], the necessary conditions for weak sharp minima in cone constrained optimization problems, which can be captured by weak sharp minima in cone constrained robust optimization problems, were established by means of upper Studniarski or Dini directional derivatives. With the result in Theorem 3.1.8, the mentioned necessary conditions are established by an alternative method different from the referred work.

The following result is established easily by means of the basic concepts of variational analysis.

Corollary 3.1.10. *Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^p$ satisfying the following properties:*

(i) for any $u \in \mathcal{U}$, and $v \in \mathcal{V}$, $f(\cdot, u)$ is convex and continuous as well as $g(\cdot, v)$ is D-convex on $\mathbb{R}^n$;

(ii) for any $x \in \mathbb{R}^n$, $f(x,\cdot)$ *is concave on* $\mathcal{U}$ *and* $g(x,\cdot)$ *is D-concave on* $\mathcal{V}$, *respectively.*

The following two below statements are equivalent:

(a) The (RSCQ) is fulfilled at $\bar{x} \in K_1$;

(b) $\bar{x} \in \mathbb{R}^n$ *is a robust weak sharp solution of* (UP_1) *if and only if there exists a real number* $\eta > 0$ *such that for any* $x^* \in N(\widetilde{K}_1, \bar{x}) \cap \eta\mathbb{B}$, *there exist* $\hat{u} \in \mathcal{U}(\bar{x}), \hat{v} \in \mathcal{V}$ *and* $\hat{\mu} \in D^*$ *yield*

$$x^* \in \partial f(\cdot,\hat{u}) + \partial\delta_C(\bar{x}) + \partial\left((\hat{\mu}g)(\cdot,\hat{v})\right)(\bar{x}), \text{ and } (\hat{\mu}g)(\bar{x},\hat{v}) = 0. \tag{3.1.8}$$

The result, which deals with a special case that $\mathcal{U}$ and $\mathcal{V}$ are singleton sets, can be obtained easily and be presented as follows:

Corollary 3.1.11. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *is convex and continuous and* $g : \mathbb{R}^n \to \mathbb{R}^m$ *is D-convex. The following statements are equivalent:*

(i) The (SCQ) is fulfilled at $\bar{x} \in K_1$

(ii) $\bar{x} \in \mathbb{R}^n$ *is a weak sharp solution of* (P_1) *if and only there exists a real number* $\eta > 0$ *such that for any* $x^* \in N(\widetilde{K}_1, \bar{x}) \cap \eta\mathbb{B}$, *there exist* $\hat{\mu} \in D^*$ *such that*

$$x^* \in \partial f(\bar{x}) + \partial\delta_C(\bar{x}) + \partial(\hat{\mu}g)(\bar{x}) \text{ and } (\hat{\mu}g)(\bar{x}) = 0. \tag{3.1.9}$$

Next, a characterization of robust weak sharp solution sets in terms of a given robust weak sharp solution point of our considered problem is also illustrated in this section.

In order to present the mentioned characterization, we first prove that the Lagrangian-type function associated with fixed Lagrange multiplier and uncertainty parameters corresponding to a robust weak sharp solution is constant on the robust weak sharp solution solution set under suitable conditions.

In what follows, let $u \in \mathcal{U}, v \in \mathcal{V}$ and $\mu \in D^*$. The Lagrangian-type function $\mathcal{L}(\cdot, \mu, u, v)$ is given by

$$\mathcal{L}(x, \mu, u, v) = f(x, u) + (\mu g)(x, v), \ \forall x \in \mathbb{R}^n.$$

Now, we denote by

$$\widetilde{S}_1 := \left\{ x \in K : \exists \eta > 0 \text{ s.t. } \sup_{u \in \mathcal{U}} f(y, u) \geq \sup_{u \in \mathcal{U}} f(x, u) + \eta d_{\widetilde{K}_1}(y), \forall y \in K_1 \right\},$$

the robust weak sharp solution set of (UP_1), and then we prove that the Lagrangian-type function associated with a Lagrange multiplier corresponding to a robust weak sharp solution is constant on the robust weak sharp solution set.

Theorem 3.1.12. Let $\bar{x} \in \widetilde{S}_1$ be given. Suppose that the (RSCQ) is satisfied at $\bar{x}$. Then, there exist uncertainty parameters $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$, and Lagrange multiplier $\hat{\mu} \in D^*$, such that for any $x \in \widetilde{S}_1$,

$$(\hat{\mu} g)(x, \hat{v}) = 0, \ \hat{u} \in \mathcal{U}(x), \ \text{ and } \mathcal{L}(x, \hat{\mu}, \hat{u}, \hat{v}) \text{ is a constant on } \widetilde{S}_1.$$

Proof. Since $\bar{x} \in \widetilde{S}_1$ with the real number $\eta_1 > 0$ and the (RSCQ) is satisfied at this point $\bar{x}$, by Theorem 3.1.8 we have that (3.1.2) holds for $\eta := \eta_1$. Clearly $N(\widetilde{K}_1, \bar{x}) \cap \eta\mathbb{B}$ contains 0, then it is nonempty and so is any $\partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x})$ where $\varepsilon > 0$. Let $\varepsilon > 0$ and $x^* \in \partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x})$ be arbitrary. Again, we obtain that there exist $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that (3.1.2) is fulfilled. Let $x \in \widetilde{S}_1$ be arbitrary, then we have

$$f(x, \hat{u}) - f(\bar{x}, \hat{u}) + (\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\bar{x}, \hat{v}) \geq \langle x^*, x - \bar{x} \rangle,$$

and so

$$f(x, \hat{u}) - f(\bar{x}, \hat{u}) + (\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\bar{x}, \hat{v}) \geq \langle x^*, x - \bar{x} \rangle - \varepsilon. \tag{3.1.10}$$

Since $f(\cdot, u)$ and $g(\cdot, v)$ are convex, for all $u \in \mathcal{U}$ and $v \in \mathcal{V}$ respectively,

$$x^* \in \partial_\varepsilon(f(\cdot, u) + \lambda g(\cdot, v))(\bar{x}).$$

Therefore, we obtain $\partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq \partial_\varepsilon \left(f(\cdot, u) + \lambda g(\cdot, v) \right)(\bar{x})$, and so

$$f(x, \hat{u}) + (\hat{\mu} g)(x, \hat{v}) - \eta d_{\widetilde{K}_1}(x) \geq f(\bar{x}, \hat{u}) = \sup_{u \in \mathcal{U}} f(\bar{x}, u). \tag{3.1.11}$$

Note that, as $x \in \widetilde{S}_1$, there exists $\eta_2 > 0$ such that

$$\sup_{u\in\mathcal{U}} f(y,u) \geq \sup_{u\in\mathcal{U}} f(x,u) + \eta_2 d_{\widetilde{K}_1}(y),\ \forall y \in \widetilde{S}_1,$$

and so

$$\sup_{u\in\mathcal{U}} f(\bar{x},u) \geq \sup_{u\in\mathcal{U}} f(x,u) + \eta_2 d_{\widetilde{K}_1}(\bar{x}) = \sup_{u\in\mathcal{U}} f(x,u). \tag{3.1.12}$$

From $\hat{\mu} \in D^*, g(x,\hat{v}) \in -D$, and (3.1.11), it is not hard to see that

$$(\hat{\mu}g)(x,\hat{v}) = 0. \tag{3.1.13}$$

Then, by (3.1.11) and the positivity of $\eta d_{\widetilde{K}_1}(x)$, we see that

$$\sup_{u\in\mathcal{U}} f(x,u) \geq f(x,\hat{u}) \geq \sup_{u\in\mathcal{U}} f(\bar{x},u) + \eta d_{\widetilde{K}_1}(x) \geq \sup_{u\in\mathcal{U}} f(\bar{x},u), \tag{3.1.14}$$

which together with (3.1.12) leads to

$$\sup_{u\in\mathcal{U}} f(x,u) = f(x,\hat{u}). \tag{3.1.15}$$

It follows that $\mathcal{L}(x,\hat{\mu},\hat{u},\hat{v}) = f(\bar{x},\hat{u})$, which is constant. Since $x \in \widetilde{S}_1$ was arbitrary, we finish the proof. □

Theorem 3.1.13. *For the problem* (UP_1), *let* $\widetilde{S}_1$ *be the robust weak sharp solutions set of* (UP_1) *and* $\bar{x}$ *belongs to it. Suppose that the (RSCQ) is satisfied at* $\bar{x} \in \widetilde{S}$. *Then, there exist uncertain parameters* $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$ *and Lagrange multiplier* $\hat{\mu} \in D^*$ *such that*

$$\begin{aligned}\widetilde{S}_1 =\Big\{ & x \in K : \exists \eta > 0, \exists \xi_f \in \partial_\varepsilon f(\cdot,\hat{u})(\bar{x}) \cap \partial_\varepsilon f(\cdot;\hat{u})(x), \exists \varepsilon > \eta d_{\widetilde{K}_1}(x),\\ & \langle \xi_f, \bar{x} - x\rangle = \eta d_{\widetilde{K}_1}(x), (\mu g)(x,\hat{v}) = 0, \sup_{u\in\mathcal{U}} f(x,u) = f(x,\hat{u})\Big\}.\end{aligned} \tag{3.1.16}$$

Proof. Let $x \in \widetilde{S}_1$ be given. Then there exists $\eta > 0$ such that (3.1.2) holds. Hence, there exist $\xi_f \in \partial f(\cdot,\hat{u})(x), \xi_\delta \in \partial\delta_C(\bar{x})$ and $\xi_{\hat{\mu}g} \in \partial\left((\hat{\mu}g)(\cdot,\hat{v})\right)(\bar{x})$ such that

$$0 = \xi_f + \xi_\delta + \xi_{\hat{\mu}g} \text{ since } 0 \in N(\widetilde{K}_1,\bar{x}) \cap \eta\mathbb{B}, \tag{3.1.17}$$

and

$$(\hat{\mu}g)(\bar{x}, \hat{v}) = 0. \tag{3.1.18}$$

Since $\xi_\delta \in \partial\delta_C(\bar{x})$ and $\xi_{\hat{\mu}g} \in \partial((\hat{\mu}g)(\cdot, \hat{v}))(\bar{x})$,

$$\delta_C(x) - \delta_C(\bar{x}) + (\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\bar{x}, \hat{v}) \geq \langle \xi_\delta + \xi_{\hat{\mu}g}, x - \bar{x} \rangle. \tag{3.1.19}$$

By the same fashion in the proof of Theorem 3.1.8, we have

$$(\hat{\mu}g)(x, \hat{v}) = (\hat{\mu}g)(\bar{x}, \hat{v}) = 0,$$

and

$$\sup_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}).$$

Therefore, it follows from (3.1.19) that

$$0 \geq \langle \xi_\delta + \xi_{\hat{\mu}g}, x - \bar{x} \rangle,$$

and so by (3.1.17), we obtain

$$\eta d_{\widetilde{K}_1}(x) \geq \langle \xi_f, \bar{x} - x \rangle.$$

Simultaneously, since $\xi_f \in \partial f(\cdot, \hat{u})(\bar{x})$, we have

$$\langle \xi_f, \bar{x} - x \rangle \geq f(\bar{x}, \hat{u}) - f(x, \hat{u}).$$

By (3.1.15) in the proof of Theorem 3.1.8, we obtain

$$\langle \xi_f, \bar{x} - x \rangle \geq \sup_{u \in \mathcal{U}} f(\bar{x}, \hat{u}) - \sup_{u \in \mathcal{U}} f(x, u) \geq 0 = \eta d_{\widetilde{K}_1}(x). \tag{3.1.20}$$

Hence, we have that $\langle \xi_f, \bar{x} - x \rangle = \eta d_{\widetilde{K}_1}(x)$. Now, we prove that for $\xi_f \in \partial_\varepsilon f(\cdot, \hat{u})(x)$, there is an $\varepsilon > \eta d_{\widetilde{K}_1}(x) \geq 0$. In fact, we can show that for any $y \in \mathbb{R}^n$,

$$\langle \xi_f, y - x \rangle = \langle \xi_f, y - \bar{x} \rangle + \langle \xi_f, \bar{x} - x \rangle \leq \langle \xi_f, y - \bar{x} \rangle$$

as $\langle \xi_f, \bar{x} - x \rangle \leq 0$. Since $\xi_f \in \partial f(\cdot, \hat{u})(\bar{x})$ and $f(x, \hat{u}) = f(\bar{x}, \hat{u})$ by (3.1.14) and (3.1.12),

$$\langle \xi_f, y - x \rangle \leq f(y, \hat{u}) - f(\bar{x}, \hat{u}) = f(y, \hat{u}) - f(x, \hat{u}),$$

which means $\xi_f \in \partial f(\cdot,\hat{u})(x)$.

Conversely, let

$$x \in \Big\{x \in K_1 : \exists \eta > 0, \exists \xi_f \in \partial_\varepsilon f(\cdot,\hat{u})(\bar{x}) \cap \partial_\varepsilon f(\cdot,\hat{u})(x), \exists \varepsilon > \eta d_{\widetilde{K}_1}(x),$$
$$\langle \xi_f, x-\bar{x}\rangle = \eta d_{\widetilde{K}_1}(x), (\mu g)(x,\hat{v}) = 0, \sup_{u\in\mathcal{U}} f(x,u) = f(x,\hat{u})\Big\}.$$

Since $\bar{x} \in \widetilde{S}_1$, it is clear that $\eta d_{\widetilde{K}_1}(\bar{x}) = 0$. By assumption and $\xi_f \in \partial_\varepsilon f(\cdot,\hat{u})(x)$ for some $\varepsilon > 0$, we get

$$\begin{aligned}
-\eta d_{\widetilde{K}_1}(\bar{x}) &= 0 \\
&= \langle \xi_f, \bar{x}-x\rangle - \eta d_{\widetilde{K}_1}(x) \\
&\leq f(\bar{x},\hat{u}) - f(x,\hat{u}) + \varepsilon - \eta d_{\widetilde{K}_1}(x) \\
&= f(\bar{x},\hat{u}) - f(x,\hat{u}) - \eta d_{\widetilde{K}_1}(x) + \eta d_{\widetilde{K}_1}(x) \\
&= f(\bar{x},\hat{u}) - f(x,\hat{u}).
\end{aligned} \tag{3.1.21}$$

Therefore, we obtain

$$\sup_{u\in\mathcal{U}} f(x,u) \leq \sup_{u\in\mathcal{U}} f(x,u) + \eta d_{\widetilde{K}_1}(\bar{x}).$$

Since $\bar{x} \in \widetilde{S}_1$ and $x \in K_1$, the conclusion that $x \in \widetilde{S}_1$ is satisfied. □

In the case that $D := \mathbb{R}_+$, which is a closed convex (and pointed) cone in $\mathbb{R}$, the problem is reduced to be an inequality constrain problem. Suppose that $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ is a function such that $f(\cdot,u)$ is convex for any $u \in \mathcal{U}$ and $f(x,\cdot)$ is concave for any $x \in \mathbb{R}^n$ as well as $g : \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$ is a function such that $g(\cdot,v)$ is convex for any $v \in \mathcal{V}$ and $g(x,\cdot)$ is concave for any $x \in \mathbb{R}^n$. Here, the problem (UP$_1$) is represented as

$$\text{Minimize } f(x,u) \text{ subject to } g(x,v) \leq 0,$$

and its robust counter part is

$$\text{Minimize } \sup_{u\in\mathcal{U}} f(x,u) \text{ subject to } g(x,v) \leq 0, \ \forall v \in \mathcal{V}.$$

In this case, we can see that robust feasible set K_1 is denoted by

$$K_1 := \{x \in \mathbb{R}^n : g(x,v) \leq 0, \forall v \in \mathcal{V}\}.$$

Corollary 3.1.14. *Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ satisfying the following properties:*

(i) for any $u \in \mathcal{U}$, and $v \in \mathcal{V}$, $f(\cdot, u)$ is convex and continuous as well as $g(\cdot, v)$ is convex on $\mathbb{R}^n$;

(ii) for any $x \in \mathbb{R}^n, f(x, \cdot)$ and $g(x, \cdot)$ are concave on $\mathcal{U}$ and $\mathcal{V}$, respectively.

The following statements are equivalent:

(a) The (RSCQ) is fulfilled at $\bar{x} \in K_1$;

(b) $\bar{x} \in \mathbb{R}^n$ is a robust weak sharp solution of (UP$_1$) *if and only if there exists a real number $\eta > 0$ such that for any $x^* \in N(\widetilde{K}_1, \bar{x}) \cap \eta\mathbb{B}$, there exist $\hat{u} \in \mathcal{U}(\bar{x}), \hat{v} \in \mathcal{V}$ and $\hat{\mu} \geq 0$ yield*

$$x^* \in \partial f(\cdot, \hat{u})(\bar{x}) + \partial \delta_C(\bar{x}) + \partial(\hat{\mu} g)(\cdot, \hat{v})(\bar{x}), \text{ and } (\hat{\mu} g)(\bar{x}, \hat{v}) = 0.$$

Corollary 3.1.15. Let $\bar{x} \in \widetilde{S}_1$ be given. Suppose that the (RSCQ) is satisfied at $\bar{x}$. Then, there exist uncertain parameters $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$, and Lagrange multiplier $\hat{\mu} \geq 0$ such that for any $x \in \widetilde{S}_1$,

$$(\hat{\mu} g)(x, \hat{v}) = 0,\ \hat{u} \in \mathcal{U}(x), \text{ and } \mathcal{L}(x, \hat{\mu}, \hat{u}, \hat{v}) \text{ is constant on } \widetilde{S}_1.$$

Corollary 3.1.16. For the problem (UP$_1$), let $\widetilde{S}_1$ be the robust weak sharp solutions set of (UP$_1$) and $\bar{x}$ belongs to it. Suppose that the (RSCQ) is satisfied at $\bar{x} \in \widetilde{S}_1$. Then, there exist uncertain parameters $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$ and Lagrange multiplier $\hat{\mu} \geq 0$ such that

$$\begin{aligned}\widetilde{S}_1 = \Big\{ & x \in K_1 : \exists \eta > 0, \exists a \in \partial_\varepsilon f(\cdot, \hat{u})(\bar{x}) \cap \partial_\varepsilon f(\cdot, \hat{u})(x), \exists \varepsilon > \eta d_{\widetilde{K}_1}(x), \\ & \langle a, \bar{x} - x \rangle = \eta d_{\widetilde{K}_1}(x), (\mu g)(x, \hat{v}) = 0, \sup_{u \in \mathcal{U}} f(x, u) = f(x, \hat{u}) \Big\}. \end{aligned} \tag{3.1.22}$$

3.1.2 Applications to multi-objective optimization

In this section, in order to apply our general results of the previous section, we investigate the following multi-objective optimization problem

$$\text{Minimize } (f_1(x), f_2(x), \ldots, f_l(x)) \text{ subject to } x \in C, g(x) \in -D, \tag{MP$_1$}$$

where where $C \subseteq \mathbb{R}^n$ is a nonempty convex set,$D \subseteq \mathbb{R}^m$, $f_j : \mathbb{R}^n \to \mathbb{R}$ is a convex function for any $j = 1, \ldots, l$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a D-convex function. The feasible set of (MP$_1$) is defined by $K := \{x \in C : g(x) \in -D\}$.

The problem (MP$_1$) in the face of data uncertainty both in the objective and constraint can be captured by the following multi-objective optimization problem

$$\begin{aligned} &\text{Minimize } (f_1(x, u_1), f_2(x, u_2), \ldots, f_l(x, u_l)) \\ &\text{subject to } x \in C, g(x, v) \in -D, \end{aligned} \tag{UMP$_1$}$$

where $f_j : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, j = 1, \ldots l$, and $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^m$, $u_j, j = 1, \ldots, l$, and v are uncertain parameters, and they belong to the corresponding convex and compact uncertainty sets $\mathcal{U} \subseteq \mathbb{R}^p$, and $\mathcal{V} \subseteq \mathbb{R}^q$. Suppose that for any $u_j \in \mathcal{U}_j, j = 1, \ldots l$, the function $f_j(\cdot, u_j)$ is convex on $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n$, $f_j(x, \cdot)$ is concave on $\mathcal{U}_j, j = 1, \ldots l$. Besides, suppose that for any $v \in \mathcal{V}$, the function $g(\cdot, v)$ is D-convex on $\mathbb{R}^n$ and for any $x \in \mathbb{R}^n, g(x, \cdot)$ is D-concave on $\mathcal{V}$.

Similarly, we obtain some characterizations of the robust weak sharp weakly efficient solutions of (UMP$_1$) by using investigation of its robust (worst case) counterpart:

$$\begin{aligned} &\text{Minimize } \left(\sup_{u_1 \in \mathcal{U}_1} f_1(x, u_1), \ldots, \sup_{u_l \in \mathcal{U}_l} f_l(x, u_l) \right) \\ &\text{subject to } x \in C, g(x, v) \in -D, \forall v \in \mathcal{V} \end{aligned} \tag{RMP$_1$}$$

where the robust feasible set of (UMP$_1$) is also defined by

$$K_1 := \{x \in C : g(x, v) \in -D, \forall v \in \mathcal{V}\}.$$

Now, we recall the following concepts of robust weak sharp weakly efficient solutions in multi-objective optimization, which can be found in the literature; see e.g., [61] and [34].

Definition 3.1.17. [61] A point $\bar{x} \in K_1$ is said to be a robust weakly efficient solution of for (UMP$_1$) if it is a weakly efficient solution solution for (RMP$_1$) i.e., there does not exist $x \in K_1$ such that

$$\sup_{u_i \in \mathcal{U}_i} f_j(x, u_j) < \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j), \forall j = 1, \ldots, l.$$

Definition 3.1.18. [34] A point feasible element $\bar{x}$ is said to be a weak sharp efficient solution for (MP$_1$) if there exists a real number $\eta > 0$ such that for any $x \in K := \{x \in C : g(x) \in -D\}$

$$\max_{1 \le k \le l} \{f_k(x) - f_k(\bar{x})\} \geq \eta d_{\widehat{K}}(x)\}$$

where $\widehat{K} := \{x \in K : f(x) = f(\bar{x})\}$.

Now, we introduce a new concept of solution, which related to the sharpness, namely the robust weak sharp weakly efficient solutions.

Definition 3.1.19. A point $\bar{x} \in K_1$ is said to be a weak sharp weakly efficient solution for (RMP$_1$) if and only if there exist a real number $\eta > 0$ such that there does not exist $y \in K_1 \setminus \{\bar{x}\}$ satisfying

$$\sup_{u_j \in \mathcal{U}_j} f_j(y, u_j) - \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j) < \eta d_{\widetilde{K}_1}(y), \forall j = 1, \ldots, l,$$

or equivalently, for all $x \in K_1$

$$\max_{1 \le j \le l} \left\{ \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) - \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j) \right\} \geq \eta d_{\widetilde{K}_1}(x)$$

where $\widetilde{K}_1 := \left\{ x \in K : \sup_{u \in \mathcal{U}} f_j(x, u) = \sup_{u \in \mathcal{U}} f_j(\bar{x}, u), j = 1, \ldots, l \right\}$.

Definition 3.1.20. A point $\bar{x} \in K_1$ is said to be a robust weak sharp weakly efficient solution for (UMP$_1$) if it is a weakly weak sharp weakly efficient solution for (RMP$_1$).

The following lemma is useful for establishing our later results in this chapter.

Lemma 3.1.21. [43] *Let $\mathcal{U}_1,\ldots,\mathcal{U}_l$ be nonempty convex and compact sets of $\mathbb{R}^p$ and for any $u_j \in \mathcal{U}_j, j=1,\ldots,l$, the function $f_j(\cdot,u_j):\mathbb{R}^n \to \mathbb{R}$ be convex as well as for any $x \in \mathbb{R}^n$, $f_j(x,\cdot):\mathcal{U}_j \to \mathbb{R}$ be concave where $j=1,\ldots,l$. Then, for any $\lambda_j \geq 0, j=1,\ldots,l$,*

$$\partial\left(\sup_{u\in\prod_{j=1}^{l}\mathcal{U}_j(\bar{x})}\sum_{j=1}^{l}\lambda_j f_j(\cdot,u_j)\right)(\bar{x}) = \bigcup_{u\in\prod_{j=1}^{l}\mathcal{U}_j(\bar{x})}\sum_{j=1}^{l}\lambda_j\left(f_j(\cdot,u_j)\right)(\bar{x}),$$

where

$$\prod_{j=1}^{l}\mathcal{U}_j(\bar{x}) := \left\{(\hat{u}_1,\ldots,\hat{u}_l)\in\prod_{j=1}^{l}\mathcal{U}_j\right.$$
$$\left. : \sum_{j=1,\ldots,l}\lambda_j f_j(\bar{x},\hat{u}_j) = \sup_{u\in\prod_{j=1}^{l}\mathcal{U}_j}\sum_{j=1}^{l}\lambda_j f_j(\bar{x},u_j)\right\}$$

Now, by using the similar methods used for the single-objective case, we can characterize the corresponding robust weak sharp weakly efficient solutions of (UMP_1).

Theorem 3.1.22. *Let $f:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}^l$ and $g:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}^m$ satisfying the following properties:*

(i) for any $u_j\in\mathcal{U}_j, j=1,\ldots,l$ and $v\in\mathcal{V}$, $f_j(\cdot,u_j)$ is convex and continuous as well as $g(\cdot,v)$ is D-convex on $\mathbb{R}^n$;

(ii) for any $x\in\mathbb{R}^n$, $f_j(x,\cdot)$ is concave on $\mathcal{U}_j, j\ldots,l$ and $g(x,\cdot)$ is D-concave on $\mathcal{V}$.

Then, the following statements are equivalent:

(a) The (RSCQ) is fulfilled at $\bar{x}\in K_1$;

(b) $\bar{x}\in\mathbb{R}^n$ is a robust weak sharp weakly efficient solutions of (UMP_1) if and only if there exists $\eta>0$ such that for any $x^*\in N(\widetilde{K}_1,\bar{x})\cap\eta\mathbb{B}$, there exist $\hat{u}_j\in\mathcal{U}_j(\bar{x}),\sigma_j\geq 0, j=1,\ldots,l$, not all zero, $\hat{v}\in\mathcal{V}$, and $\hat{\mu}\geq 0$ such that

$$0\in\{-x^*\}+\sum_{j=1}^{l}\hat{\sigma}_j\left(\partial f_j(\cdot,\hat{u}_j)(\bar{x})\right)+\partial\delta_C(\bar{x})+\partial\left((\hat{\mu}g)(\cdot,\hat{v})\right)(\bar{x}) \qquad (3.1.23)$$

$$(\hat{\mu}g)(\bar{x}, \hat{v}) = 0, \tag{3.1.24}$$

and

$$\sigma_i f_j(\bar{x}, \hat{u}_j) = \sigma_j \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j), j = 1, \ldots, l. \tag{3.1.25}$$

Proof. $[(a) \Rightarrow (b)]$ Assume that the (RSCQ) is satisfied at $\bar{x} \in \mathbb{R}^n$. Let $\bar{x}$ be a robust weak sharp weakly efficient solutions of (UMP_1) i.e., there exists $\eta > 0$ such that there does not exist $y \in K_1 \setminus \{\bar{x}\}$ satisfying

$$\sup_{u_j \in \mathcal{U}_j} f_j(y, u_j) - \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j) < \eta d_{\widetilde{K}_1}(y), \text{ for all } j = 1, \ldots, l,$$

or equivalently, for any $x \in K_1$,

$$\max_{1 \leq j \leq l} \left\{ \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) - \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j) \right\} \geq \eta d_{\widetilde{K}_1}(x). \tag{3.1.26}$$

By (3.1.26), there is $s \in \{1, \ldots, l\}$ such that for all $x \in K_1$,

$$\begin{aligned} \sup_{u_s \in \mathcal{U}_s} f_s(x, u_s) + \delta_{K_1}(x) - \eta d_{\widetilde{K}_1}(x) &\geq \sup_{u_s \in \mathcal{U}_s} f(\bar{x}, u_s) \\ &= \sup_{u_s \in \mathcal{U}_s} f_s(\bar{x}, u_s) + \delta_{K_1}(\bar{x}) - \eta d_{\widetilde{K}_1}(\bar{x}). \end{aligned} \tag{3.1.27}$$

Besides, according to (3.1.27), we follow the techniques used in Theorem 3.1.8 and obtain that for any $\xi \in \partial \eta d_{\widetilde{K}_1}(x)$,

$$\begin{aligned} &\langle \xi, x - \bar{x} \rangle \\ \leq\ & \sup_{u_s \in \mathcal{U}_s} f_s(x, u_s) + \delta_{K_1}(x) - \sup_{u_s \in \mathcal{U}_s} f_s(\bar{x}, u_s) - \delta_{K_1}(\bar{x}). \end{aligned} \tag{3.1.28}$$

Therefore,

$$\partial(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq \partial \left(\sup_{u_s \in \mathcal{U}_s} f_s(\cdot, u) + \delta_{K_1} \right)(\bar{x}), \tag{3.1.29}$$

Note that the right hand side term of above inclusion is in the subdifferential of the max function:

$$\phi(x) = \max_{1 \leq j \leq l} \phi_j(x) := \max_{1 \leq j \leq l} \left(\sup_{u_j \in \mathcal{U}_j} f_j(\cdot, u_j) + \delta_{K_1} \right)(x).$$

Due to the well-known fact, subdifferential of maximum of functions at x is the convex hull of the union of subdifferentials of the active functions at x, the inclusion (3.1.29) becomes

$$\partial(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq \text{co}\left(\cup\{\partial\phi_j(\bar{x}) : \phi_j(\bar{x}) = \phi(x)\}\right),$$

thereby

$$\partial(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq \sum_{j\in J(\bar{x})} \sigma_j \partial\phi_j(\bar{x}),$$

where $\sigma_j \geq 0, j \in J(\bar{x})$ with $\sum_{j\in J(\bar{x})} \sigma_j = 1$ and $J(\bar{x}) := \{k \in \{1,\ldots,l\} : \phi_k(\bar{x}) = \phi(\bar{x})\}$. Further, setting $\hat{\sigma}_j = \sigma, j \in J(\bar{x})$, and otherwise equals to 0 leads to

$$\partial(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq \sum_{j=1}^{l} \hat{\sigma}_j \partial\phi_j(\bar{x}).$$

By the definition of $\phi_j, i = 1,\ldots,l$, the continuity of $\sup_{u_j\in\mathcal{U}_j} f_j(\cdot,u_j), j = 1,\ldots,l$ and the lower semicontinuity and convexity of δ_{K_1}, we have

$$\partial(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq \sum_{j=1}^{l} \hat{\sigma}_j \partial\left(\sup_{u_j\in\mathcal{U}_j} f_j(\cdot,u_j)\right)(\bar{x}) + \sum_{j=1}^{l} \hat{\sigma}_j \left(\partial\delta_{K_1}(\bar{x})\right).$$

It follows from Lemma 3.1.21 and the hypothesis such (RSCQ) is satisfied at $\bar{x} \in K_1$ that

$$\begin{aligned}\partial(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq & \bigcup_{u\in\prod_{j=1}^{l}\mathcal{U}_j(\bar{x})} \sum_{i=1}^{l} \hat{\sigma}_j \partial f_j(\cdot,u_j)(\bar{x}) + \sum_{j=1}^{l} \hat{\sigma}_j \left(\partial\delta_C(\bar{x})\right) \\ & + \bigcup_{\substack{\mu\in D^*, v\in\mathcal{V} \\ (\mu g)(\bar{x},v)=0}} \partial\left((\mu g)(\cdot,v)\right)(\bar{x}).\end{aligned}$$

Because $\hat{\sigma}_j \geq 0, j = 1,\ldots,l$, all nonzero, thereby

$$\begin{aligned}\partial(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq & \bigcup_{\substack{u=(u_j)_{j=1}^{l}, \\ u\in\prod_{j=1}^{l}\mathcal{U}_j(\bar{x})}} \sum_{j=1}^{l} \hat{\sigma}_j \left(\partial f_j(\cdot,u_j)(\bar{x})\right) + \partial\delta_C(\bar{x}) \\ & + \bigcup_{\substack{\mu\in D^*, v\in\mathcal{V} \\ (\mu g)(\bar{x},v)=0}} \partial\left((\mu g)(\cdot,v)\right)(\bar{x}).\end{aligned}$$

As $\partial d_{\widetilde{K}_1}(x) = N_{\widetilde{K}_1}(x) \cap \mathbb{B}$, we obtain (3.1.23) as desired.

Conversely, assume that there is $\eta > 0$ such that (3.1.23)-(3.1.25) hold. Then, for any $x^* \in N(\widetilde{K}_1, \bar{x}) \cap \eta\mathbb{B}$, there exist $\hat{u} := (\hat{u}_1, \ldots \hat{u}_l) \in \prod_{j=1}^{l} \mathcal{U}_j(\bar{x}), \hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that

$$x^* \in \sum_{j=1}^{l} \hat{\sigma}_j \left(\partial f_j(\cdot, \hat{u}_j)(\bar{x})\right) + \partial\delta_C(\bar{x}) + \partial\left((\hat{\mu}g)(\cdot, \hat{v})\right)(\bar{x}), \text{ and}$$
$$(\hat{\mu}g)(\bar{x}, \hat{v}) = 0. \tag{3.1.30}$$

Since $0 \in N(\widetilde{K}_1, \bar{x}) \cap \eta\mathbb{B} = \bigcap_{\varepsilon>0} \partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x})$, for each positive ε, $\partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x})$ is nonempty. Let $\varepsilon > 0$ and $\xi \in \partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x})$ be arbitrary, then for any $x \in K_1$

$$\eta d_{\widetilde{K}_1}(x) - \eta d_{\widetilde{K}_1}(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle - \varepsilon. \tag{3.1.31}$$

Therefore, we obtain

$$\eta d_{\widetilde{K}_1}(\bar{x}) \leq \inf_{x \in \mathbb{R}^n} \eta d_{\widetilde{K}_1}(x) + \varepsilon \leq \inf_{x \in K_1} \eta d_{\widetilde{K}_1}(x) + \varepsilon.$$

Above inequality and (3.1.31) imply that

$$0 \geq \langle \xi, x - \bar{x} \rangle - \varepsilon. \tag{3.1.32}$$

Further, since $0 \in N(\widetilde{K}_1, \bar{x}) \cap \eta\mathbb{B}$, we have that there exist $\xi_f \in \sum_{i=1}^{l} \hat{\sigma}_j \left(\partial f_j(\cdot, \hat{u}_j)(\bar{x})\right), \xi_\delta \in \partial\delta_C(\bar{x})$, and $\xi_{\hat{\mu}g} \in \partial\left((\hat{\mu}g)(\cdot, \hat{v})\right)(\bar{x})$ such that

$$\xi_f + \xi_\delta + \xi_{\hat{\mu}g} = 0. \tag{3.1.33}$$

Since $\xi_f \in \sum_{j=1}^{l} \hat{\sigma}_i \left(\partial f_j(\cdot, \hat{u}_j)(\bar{x})\right) = \partial \left(\sum_{j=1}^{l} \hat{\sigma}_j f_j(\cdot, \hat{u}_j) \right)(\bar{x}), \xi_\delta \in \partial\delta_C(\bar{x})$ and $\xi_{\hat{\mu}g} \in \partial\left((\hat{\mu}g)(\cdot, \hat{v})\right)(\bar{x})$, we have

$$\sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j) - \sum_{j=1}^{l} \hat{\sigma}_j f_j(\bar{x}, \hat{u}_j) \geq \langle \xi_f, x - \bar{x} \rangle,$$
$$\delta_C(x) - \delta_C(\bar{x}) \geq \langle \xi_\delta, x - \bar{x} \rangle, \text{ and}$$
$$(\hat{\mu}g)(x, \hat{v}) - (\hat{\mu}g)(\bar{x}, \hat{v}) \geq \langle \xi_{\hat{\mu}g}, x - \bar{x} \rangle.$$

Then, adding these inequalities yields

$$\begin{aligned}\langle 0, x-\bar{x}\rangle \leq & \sum_{j=1}^{l} \hat{\sigma}_j f_j(x,\hat{u}_j) - \sum_{j=1}^{l} \hat{\sigma}_j f_j(\bar{x},\hat{u}_j) \\ & + (\hat{\mu}g)(x,\hat{v}) - (\hat{\mu}g)(\bar{x},\hat{v}).\end{aligned}$$

Since $\hat{u}_j$ belongs to $\mathcal{U}_j(\bar{x})$, above inequality becomes the following one:

$$\begin{aligned}0 \leq & \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j\in\mathcal{U}_j} f_j(x,u_j) - \sum_{j=1}^{l} \sup_{u_j\in\mathcal{U}_j} f_j(\bar{x},u_j)) \\ & + (\hat{\mu}g)(x,\hat{v}) - (\hat{\mu}g)(\bar{x},\hat{v}).\end{aligned}$$

This together with $(\hat{\mu}g)(x,\hat{v}) \leq 0, (\hat{\mu}g)(\bar{x},\hat{v}) = 0$, and (3.1.33), for any $x \in K_1$,

$$\sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j\in\mathcal{U}_j} f_j(x,u_j) - \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j\in\mathcal{U}_j} f_j(\bar{x},u_j) \geq \langle 0. \tag{3.1.34}$$

By summing (3.1.34) with (3.1.31), for any $x \in K_1$, we obtain

$$\sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j\in\mathcal{U}_j} f_j(x,u_j) - \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j\in\mathcal{U}_j} f(\bar{x},u_j) \geq \langle \xi, x-\bar{x}\rangle - \varepsilon,$$

which means $\xi \in \partial_\varepsilon \left(\sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j\in\mathcal{U}_j} f_j(\cdot,u_j)\right)(\bar{x})$, and so $\partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq \partial_\varepsilon \left(\sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j\in\mathcal{U}_j} f_j(\cdot,u_j)\right)(\bar{x})$. As $\varepsilon > 0$ was arbitrary, for each $x \in K_1$,

$$\begin{aligned}0 \ \leq \ & \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j\in\mathcal{U}_j} f_j(x,u_j) - \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j\in\mathcal{U}_j} f_j(\bar{x},u_j) \\ & - \left(\eta d_{\widetilde{K}_1}(x) - \eta d_{\widetilde{K}_1}(\bar{x})\right),\end{aligned}$$

which is equivalent to the following inequality: for all $x \in K_1$

$$\sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j\in\mathcal{U}_j} f_j(x,u_j) - \eta d_{\widetilde{K}_1}(x) \geq \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j\in\mathcal{U}_j} f_j(\bar{x},u_j) - \eta d_{\widetilde{K}_1}(\bar{x}).$$

It follows that

$$\sum_{j=1}^{l} \hat{\sigma}_j \left(\sup_{u_j\in\mathcal{U}_j} f_j(x,u_j) - \eta d_{\widetilde{K}_1}(x)\right) \geq \sum_{j=1}^{l} \hat{\sigma}_j \left(\sup_{u_j\in\mathcal{U}_j} f_j(\bar{x},u_j) - \eta d_{\widetilde{K}_1}(\bar{x})\right),$$

for any $x \in K_1$, which yields for any $j = 1, \ldots, l$,

$$\sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) - \eta d_{\widetilde{K}_1}(x) \geq \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j) - \eta d_{\widetilde{K}_1}(\bar{x}), \forall x \in K_1.$$

Therefore, for any $x \in K$

$$\max_{1 \leq j \leq l} \left\{ \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) - \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j) \right\} \geq \eta d_{\widetilde{K}_1}(x).$$

This means $\bar{x}$ is a robust weak sharp weakly efficient solutions of (UMP_1).

$[(b) \Rightarrow (a)]$ Let $\bar{\eta} > 0$ be given. Consider $f_j(x, u_j) := -\langle \xi_\delta, x \rangle + \bar{\eta} d_{\widetilde{K}_1}(x), j = 1, \ldots, l$. Thus, for any $x \in K_1$,

$$\begin{aligned} \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) - \bar{\eta} d_{\widetilde{K}_1}(x) &= -\langle \xi_\delta, x \rangle \\ &\geq -\langle \xi_\delta, \bar{x} \rangle + \bar{\eta} d_{\widetilde{K}_1}(\bar{x}) \\ &= \max_{u_i \in \mathcal{U}_j} f_j(\bar{x}, u_j). \end{aligned}$$

Thus, $\bar{x}$ is a robust weak sharp weakly efficient solutions of (UMP_1). By hypothesis, there is $\eta := \bar{\eta}$ such that (3.1.23) is fulfilled. Since for any $u_j \in \mathcal{U}_j, \partial f_j(\cdot, u_j)(\bar{x}) \subseteq \{-\xi_\delta\} + \partial(\eta d_{\widetilde{K}_1})(\bar{x})$, one has

$$\sum_{j=1}^{l} \hat{\sigma}_j \left(\partial f_j(\cdot, u_j)(\bar{x}) \right) \subseteq \{-\xi_\delta\} + \partial(\eta d_{\widetilde{K}_1})(\bar{x}),$$

where $\hat{\sigma}_j \geq 0, j = 1, \ldots, l$ and all nonzero. Thus, we obtain that for any $x^* \in N(\widetilde{K}_1, \bar{x}) \cap \eta\mathbb{B}$, there exist $\hat{u}_j \in \mathcal{U}_j(\bar{x}), \hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that

$$x^* \in \{-\xi_\delta\} + \partial(\eta d_{\widetilde{K}_1})(\bar{x}) + \partial\delta_C(\bar{x}) + \partial\left((\hat{\mu} g)(\cdot, \hat{v})\right)(\bar{x}) \text{ and } (\hat{\mu} g)(\bar{x}, \hat{v}) = 0.$$

As $0 \in N(\widetilde{K}_1, \bar{x}) \cap \eta\mathbb{B}$, we obtain

$$\xi_\delta \in \partial\delta_C(\bar{x}) + \partial\left((\hat{\mu} g)(\cdot, \hat{v})\right)(\bar{x}) \text{ and } (\hat{\mu} g)(\bar{x}, \hat{v}) = 0.$$

It follows that

$$\xi_\delta \in \partial\delta_C(\bar{x}) + \bigcup_{\substack{\mu \in D^*, v \in \mathcal{V} \\ (\mu g)(\bar{x}, v) = 0}} \partial\left((\mu g)(\cdot, v)\right)(\bar{x}),$$

and so we get the desired inclusion. Therefore, the proof is complete. □

Remark 3.1.23. (i) In [63] and [64], the authors presented the necessary condition for the local sharp efficiency for the semi-infinite vector optimization problem by using the different method with Theorem 3.1.22. In fact, they employed the exact sum rule for Fréchet subdifferentials to obtained their results.

(ii) In [66], the exact sum rule for Mordukhovich subdifferentials was used as a vital tool under some regularity and differentiability assumptions for establishing their results. This means Theorem 3.1.22 use the different medthod from the mentioned work.

Next, by using the similar methods of section 3, a characterization of robust weak sharp weakly efficient solution sets in terms of a given robust weak sharp weakly efficient solution point of the problem is also illustrated in this section.

In order to present the mentioned characterization, we start by deriving constant Lagrangian-type property for robust weak sharp weakly efficient solution sets of (MP_1). In what follows, let $u = (u_1, \ldots, u_l) \in \mathcal{U}_1 \times, \ldots \times \mathcal{U}_l, \sigma = (\sigma_1, \ldots, \sigma_l) \in \mathbb{R}^l_+, v \in \mathcal{V}$ and $\mu \geq 0$. The Lagrangian-type function $\mathcal{L}(\cdot, \sigma\mu, u, v)$ is given by

$$\mathcal{L}(x, \sigma, \mu, u, v) = \sum_{j=1}^{l} \sigma_j f_j(x, u_j) + (\mu g)(x, v), \ \forall x \in \mathbb{R}^n.$$

Theorem 3.1.24. *Let $x \in \widetilde{S}_1$ be given. Suppose that the (RSCQ) is fulfilled at $\bar{x}$. Then, there exist a positive valued vector $\hat{\sigma} := (\hat{\sigma}_1, \ldots, \hat{\sigma}_l) \in \mathbb{R}^l_+, \hat{\sigma}_j, j = 1, \ldots, l$ all nonzero, uncertain parameters $\hat{u} := (u_1, \ldots, u_l) \in \mathcal{U} = \mathcal{U}_1 \times \ldots \times \mathcal{U}_l, \hat{v} \in \mathcal{V}$, and Lagrange multiplier $\hat{\mu} \geq 0$ such that for any $x \in \widetilde{S}_1$,*

$$(\hat{\mu} g)(x, \hat{v}) = 0, \ \hat{u} \in \mathcal{U}(x), \ \text{ and } \mathcal{L}(x, \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v}) \text{ is a constant on } \widetilde{S}_1.$$

Proof. Since $\bar{x} \in \widetilde{S}_1$ with the real number $\eta_1 > 0$ and the (RSCQ) is satisfied at this point $\bar{x}$, by Theorem 3.1.22, (3.1.23) holds for $\eta := \eta_1$. Since $N(\widetilde{K}_1, \bar{x}) \cap \eta\mathbb{B}$ is nonempty we can let $\varepsilon > 0$ be arbitrary and $x^* \in \partial_\varepsilon(\eta d_{\widetilde{K}_1})(\bar{x})$ be given. Besides, there exist $\hat{\sigma} \in \mathbb{R}^l_+$, all nonzero, $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$ and $\hat{\mu} \in D^*$ such that (3.1.23) is fulfilled. Let

$x \in \widetilde{S}_1$ be arbitrary. By the same fashion using in the proof of Theorem 3.1.8 we have

$$\langle x^*, x - \bar{x} \rangle \le \sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j) - \sum_{j=1}^{l} \hat{\sigma}_j f_j(\bar{x}, \hat{u}_j) + (\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\bar{x}, \hat{v}),$$

and so

$$\langle x^*, x - \bar{x} \rangle - \varepsilon \le \sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j) - \sum_{j=1}^{l} \hat{\sigma}_j f_j(\bar{x}, \hat{u}_j) + (\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\bar{x}, \hat{v}), \tag{3.1.35}$$

As $f_j(\cdot, u_j), j = 1, \dots, l$ and $g(\cdot, v)$ are convex, for any $u_i \in \mathcal{U}_i$ and $v \in \mathcal{V}$, we have $x^* \in \partial_\varepsilon \left(\sum_{j=1}^{l} \hat{\sigma}_j (f_j(\cdot, u_j) + \lambda g(\cdot, v)) \right) (\bar{x})$. Hence, one has

$$\partial_\varepsilon (\eta d_{\widetilde{K}_1})(\bar{x}) \subseteq \partial_\varepsilon \left(\sum_{j=1}^{l} \hat{\sigma}_j \left(f_j(\cdot, u_j) + \lambda g(\cdot, v) \right) \right) (\bar{x}),$$

thereby

$$\sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j) + (\hat{\mu} g)(x, \hat{v}) - \eta d_{\widetilde{K}_1}(x) \ge \sum_{j=1}^{l} \hat{\sigma}_j f_j(\bar{x}, \hat{u}_j) = \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_i} f_j(\bar{x}, u_j). \tag{3.1.36}$$

Note that, as $x \in \widetilde{S}_1$, then there exists $\eta_2 > 0$ such that for all $y \in K_1$,

$$\sup_{u_j \in \mathcal{U}_j} f_j(y, u_j) \ge \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) + \eta_2 d_{\widetilde{K}_1}(y),$$

which implies

$$\sum_{j=1}^{l} \hat{\sigma}_i \sup_{u_j \in \mathcal{U}_j} f_j(y, u_j) \ge \sum_{j=1}^{l} \hat{\sigma}_i \left(\sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) + \eta_2 d_{\widetilde{K}_1}(y) \right) = \sum_{j=1}^{l} \hat{\sigma}_i \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) + \eta_2 d_{\widetilde{K}_1}(y)$$

$$= \sum_{j=1}^{l} \hat{\sigma}_i \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j),$$

for all $y \in \widetilde{S}$. Since $\bar{x} \in \widetilde{S}_1$,

$$\sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_i} f_j(\bar{x}, u_j) \geq \sum_{j=1}^{l} \hat{\sigma}_i \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j). \tag{3.1.37}$$

From $\hat{\mu} \geq 0, g(x, \hat{v}) \leq 0$, and (3.1.36), it is not hard to see that

$$(\hat{\mu} g)(x, \hat{v}) = 0. \tag{3.1.38}$$

Moreover, by (3.1.36) and the positivity of $\eta d_{\widetilde{K}_1}(x)$, we see that

$$\begin{aligned} \sum_{j=1}^{l} \hat{\sigma}_i \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) &\geq \sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j) \\ &\geq \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_i} f_j(\bar{x}, u_j) + \eta d_{\widetilde{K}_1}(x) \\ &\geq \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_i} f_j(\bar{x}, u_j). \end{aligned} \tag{3.1.39}$$

This together with (3.1.38) leads to

$$\sum_{j=1}^{l} \hat{\sigma}_i \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) = \sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j). \tag{3.1.40}$$

Thus, $\mathcal{L}(\cdot, \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v})$ is constant on $\widetilde{S}_1$ as follows:

$$\begin{aligned} \mathcal{L}(x, \hat{\sigma}, \hat{\mu}, \hat{u}, \hat{v}) &= \sum_{j=1}^{l} \hat{\sigma}_i f_i(x, u_i) + (\hat{\mu} g)(x, \hat{v}) \\ &= \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) + (\hat{\mu} g)(x, \hat{v}) \\ &= \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j) + (\hat{\mu} g)(x, \hat{v}) \\ &= \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j). \end{aligned}$$

This completes the proof. □

Theorem 3.1.25. *For the problem* (UMP$_1$), *let* $\widetilde{S}_1$ *be the robust weak sharp weakly efficient solution set of* (UMP$_1$) *and* $\bar{x} \in \widetilde{S}_1$. *Suppose that the (RSCQ) is fulfilled at* $\bar{x} \in \widetilde{S}_1$. *Then, there exist* $\hat{\sigma}_j \geq 0, j = 1, \ldots, l$ *all non zero,* $\hat{u} := (\hat{u}_1, \ldots, \hat{u}_l) \in \mathcal{U} = \mathcal{U}_1 \times \ldots, \times \mathcal{U}_l, \hat{v} \in \mathcal{V}$ *and* $\hat{\mu} \geq 0$ *such that*

$$\widetilde{S}_1 = \left\{ x \in K_1 : \exists \eta > 0, \exists a \in \bigcap_{y \in \{x, \bar{x}\}} \partial_\varepsilon \left(\sum_{j=1}^{l} \hat{\sigma}_j f_j(\cdot, \hat{u}_j) \right) (\hat{y}), \right.$$
$$\exists \varepsilon > \eta d_{\widetilde{K}_1}(x), \langle a, \bar{x} - x \rangle = \eta d_{\widetilde{K}_1}(x), (\mu g)(x, \hat{v}) = 0,$$
$$\left. \sup u_j \in \mathcal{U}_j f_j(x, u_j) = f_j(x, \hat{u}_j), j = 1, \ldots, l \right\}.$$

Proof. Let $x \in \widetilde{S}_1$ be given. Then there exists $\eta > 0$ such that (3.1.23) holds. Thus, there exist $\hat{u} \in \mathcal{U}, \hat{v} \in \mathcal{V}$ and $\hat{\mu} \geq 0$ such that (3.1.23) is fulfilled. Hence, we have that there exist $\xi_f \in \sum_{j=1}^{l} \hat{\sigma}_j \left(\partial f_j(\cdot, \hat{u}_j)(x) \right), \xi_\delta \in \partial \delta_C(\bar{x})$ and $\xi_{\hat{\mu} g} \in \partial \left((\hat{\mu} g)(\cdot, \hat{v}) \right)(\bar{x})$ such that

$$0 = \xi_f + \xi_\delta + \xi_{\hat{\mu} g}, \text{ since } 0 \in N(\widetilde{K}_1, \bar{x}) \cap \eta \mathbb{B}, \tag{3.1.41}$$

and

$$(\hat{\mu} g)(\bar{x}, \hat{v}) = 0. \tag{3.1.42}$$

Since $\xi_\delta \in \partial \delta_C(\bar{x})$ and $\xi_{\hat{\mu} g} \in \partial((\hat{\mu} g)(\cdot, \hat{v}))(\bar{x})$,

$$\delta_C(x) - \delta_C(\bar{x}) + (\hat{\mu} g)(x, \hat{v}) - (\hat{\mu} g)(\bar{x}, \hat{v}) \geq \langle \xi_\delta + \xi_{\hat{\mu} g}, x - \bar{x} \rangle. \tag{3.1.43}$$

By the same fashion in the proof of Theorem 3.1.22, we have

$$(\hat{\mu} g)(x, \hat{v}) = (\hat{\mu} g)(\bar{x}, \hat{v}) = 0,$$

and

$$\sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) = \sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j).$$

Therefore, it follows from (3.1.43) that

$$\eta d_{\widetilde{K}_1}(x) = 0 \geq \langle b + c, x - \bar{x} \rangle,$$

and so by (3.1.41), we obtain

$$\eta d_{\widetilde{K}_1}(x) \geq \langle \xi_f, \bar{x} - x \rangle.$$

Simultaneously, since $\xi_f \in \sum_{j=1}^{l} \hat{\sigma}_j \left(\partial f_j(\cdot, \hat{u}_j)(\bar{x})\right) = \partial \left(\sum_{j=1}^{l} \hat{\sigma}_j f_j(\cdot, \hat{u}_j) \right)(\bar{x})$, we have

$$\langle \xi_f, \bar{x} - x \rangle \geq \sum_{j=1}^{l} \hat{\sigma}_j f_j(\bar{x}, \hat{u}_j) - \sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j).$$

By (3.1.34) in the proof of Theorem 3.1.22, we obtain

$$\langle \xi_f, \bar{x} - x \rangle \geq \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, \hat{u}_j) - \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) = \geq 0 = \eta d_{\widetilde{K}_1}(x). \tag{3.1.44}$$

Hence, we have that $\langle \xi_f, \bar{x} - x \rangle = \eta d_{\widetilde{K}_1}(x)$. Next, we shall prove that there is $\varepsilon > \eta d_{\widetilde{K}_1}(x) \geq 0$ such that

$$\xi_f \in \partial_\varepsilon \left(\sum_{j=1}^{l} \hat{\sigma}_j f_j(\cdot, \hat{u}_j) \right)(x).$$

In fact, we can show that $\xi_f \in \partial \left(\sum_{j=1}^{l} \hat{\sigma}_j f_j(\cdot, \hat{u}_j) \right)(x)$. For any $y \in \mathbb{R}^n$,

$$\langle \xi_f, y - x \rangle = \langle \xi_f, y - \bar{x} \rangle + \langle \xi_f, \bar{x} - x \rangle \leq \langle \xi_f, y - \bar{x} \rangle$$

as $\langle \xi_f, \bar{x} - x \rangle \leq 0$. Since $a \in \partial \left(\sum_{j=1}^{l} \hat{\sigma}_j f_j(\cdot, \hat{u}_j) \right)(\bar{x})$ and $f_j(x, \hat{u}_j) = f_j(\bar{x}, \hat{u}_j), j = 1, \ldots, l$,

$$\begin{aligned}\langle \xi_f, y - x \rangle &\leq \sum_{j=1}^{l} \hat{\sigma}_j f_j(y, \hat{u}_j) - \sum_{j=1}^{l} \hat{\sigma}_j f_j(\bar{x}, \hat{u}_j) \\ &= \sum_{j=1}^{l} \hat{\sigma}_j f_j(y, \hat{u}_j) - \sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j),\end{aligned}$$

which means $\xi_f \in \partial \left(\sum_{j=1}^{l} \hat{\sigma}_j f_j(\cdot, \hat{u}_j) \right)(x)$.

Conversely, let

$$x \in \Big\{ x \in K_1 : \exists \eta > 0, \exists \xi_f \in \bigcap_{y \in \{x, \bar{x}\}} \partial_\varepsilon \left(\sum_{j=1}^{l} \hat{\sigma}_j f_j(\cdot, \hat{u}_j) \right)(\bar{x}), \exists \varepsilon > \eta d_{\widetilde{K}_1}(x),$$
$$\langle \xi_f, x - \bar{x} \rangle = \eta d_{\widetilde{K}_1}(x), (\mu g)(x, \hat{v}) = 0, \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) = f_i(x, \hat{u}_i) \Big\}.$$

Since $\bar{x} \in \widetilde{S}_1$, $\eta d_{\widetilde{K}_1}(\bar{x}) = 0$ and so the assumption dealing with ξ_f lead to

$$\begin{aligned} -\eta d_{\widetilde{K}_1}(\bar{x}) &= 0 \\ &= \langle \xi_f, \bar{x} - x \rangle - \eta d_{\widetilde{K}_1}(x) \\ &\leq \sum_{j=1}^{l} \hat{\sigma}_j f_j(\bar{x}, \hat{u}_j) - \sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j) - \eta d_{\widetilde{K}_1}(x) + \varepsilon \\ &= \sum_{j=1}^{l} \hat{\sigma}_j f_j(\bar{x}, \hat{u}_j) - \sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j) - \eta d_{\widetilde{K}_1}(x) + \eta d_{\widetilde{K}_1}(x) \\ &= \sum_{j=1}^{l} \hat{\sigma}_j f_j(\bar{x}, \hat{u}_j) - \sum_{j=1}^{l} \hat{\sigma}_j f_j(x, \hat{u}_j), \end{aligned} \tag{3.1.45}$$

for any $\hat{\sigma}_j \geq 0, j = 1, \ldots, l$, all nonzero. Therefore, we obtain

$$\begin{aligned} \sum_{j=1}^{l} \hat{\sigma}_i \sup_{u_j \in \mathcal{U}_j} f_j(x, u_j) &\leq \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_i} f_j(\bar{x}, u_j) + \eta d_{\widetilde{K}_1}(\bar{x}) \\ &= \sum_{j=1}^{l} \hat{\sigma}_j \sup_{u_j \in \mathcal{U}_j} f_j(\bar{x}, u_j). \end{aligned}$$

Since $\bar{x} \in \widetilde{S}_1$ and $x \in K_1$, the conclusion that $x \in \widetilde{S}_1$ is satisfied. □

3.2 Uncertain nonconvex optimization problems

In this section, we investigate an uncertain nonsmooth optimization problem involving nonsmooth real-valued functions. Firstly, we introduce the notion of a robust weak sharp solution to the considereed problem. Then, some necessary optimality condition for the robust weak sharp solutions of the problem under a constraint qualification are established. Finally, by mean of the robust version of (KKT) conditions, which are introduced here, sufficient optimality conditions for robust weak sharp solutions of the considered uncertain optimization problem are obtained. Moreover, some examples are presented for illustating the results.

Let Ω be a nonempty locally closed subset of $\mathbb{R}^n$. For $p, q \in \mathbb{N}$, let $\mathcal{U} \subseteq \mathbb{R}^p$ and $\mathcal{V}_i \subseteq \mathbb{R}^q, i \in I$ be nonempty compact sets. We consider the following uncertain optimization problem:

$$\text{Minimize } f(x,u) \text{ subject to } g_i(x,v_i) \leq 0,\ i \in I, x \in \Omega, \tag{UP$_2$}$$

where $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ and $g_i : \mathbb{R}^n \times \mathcal{V}_i \to \mathbb{R}$, $i \in I$ are given real-valued functions, x is the vector of decision variable, u and $v_i, i \in I$ are uncertain parameters belonging to the specified compact uncertainty sets $\mathcal{U}$ and $\mathcal{V}_i, i \in I$, respectively. In fact, the uncertainty sets can be apprehended in the sense that the parameter vectors u and all v_i are not known exactly at the time of the decision. For examining the uncertain optimization problem (UP$_2$), one usually associates with it, namely robust counterpart, is the following problem:

$$\text{Minimize } \sup_{u \in \mathcal{U}} f(x,u) \text{ subject to } g_i(x,v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i \in I, x \in \Omega. \tag{RP$_2$}$$

The robust feasible set K_2 is denoted by

$$K_2 := \{x \in \Omega : g_i(x,v_i) \leq 0, \forall v_i \in \mathcal{V}_i,\ i \in I\}.$$

The following concept of robust solutions can be found in the literature; see e.g., [69].

Definition 3.2.1. A point $\bar{x} \in K_2$ is said to be a local robust solution for (UP$_2$) if it is a local solution for (RP$_2$) i.e., if there exists a neighborhood U of $\bar{x}$ such that

$$\sup_{u \in \mathcal{U}} f(x,u) - \sup_{u \in \mathcal{U}} f(\bar{x},u) \geq 0,\ \forall x \in K_2 \cap U.$$

In addition, if $U = \mathbb{R}^n$, then $\bar{x} \in K_2$ is said to be a global robust solution for (UP$_2$).

In [68], a new concept of a solution, which is related to the weak sharpness, namely the (local/global) robust weak sharp solution was introduced.

Definition 3.2.2. A point $\bar{x} \in K_2$ is said to be a local robust weak sharp solution for (UP_2) if it is a local weak sharp solution for (RP_2) i.e., there exist a neighborhood U of $\bar{x}$ and a real number $\eta > 0$ such that

$$\sup_{u \in \mathcal{U}} f(x,u) - \sup_{u \in \mathcal{U}} f(\bar{x},u) \geq \eta d_{\widetilde{K}_2}(x), \ \forall x \in K_2 \cap U, \tag{3.2.1}$$

where $\widetilde{K}_2 := \left\{ x \in K_2 : \sup_{u \in \mathcal{U}} f(x,u) = \sup_{u \in \mathcal{U}} f(\bar{x},u) \right\}$. Specially, if $U = \mathbb{R}^n$, then $\bar{x} \in K_2$ is said to be a global robust weak sharp solution for (UP_2).

It is simple to see that every (local) robust weak sharp solution must be also a (local) robust solution. In contrast, the converse implication need not to be true.

Example 3.2.3. *Let $f : \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{V} \to \mathbb{R}$ be defined by*

$$f(x,u) = x^2 + u \text{ and } g(x,v) = \min\{x, 0\} + v,$$

where $x \in \mathbb{R}, u \in \mathcal{V} := [-1,0]$ and $v \in \mathcal{V} := [-1,0]$, and let $\Omega := [-1,1]$. We can see that the robust feasible set is $K_2 = [-1,1]$. Consider $\bar{x} := 0 \in K_2$, then observe that $\bar{x}$ is a globle robust solution of (UP_2). To show that $\bar{x}$ is not a local robust weak sharp solution of (UP_2), we assume on the contrary. Then there exist $\eta, \varepsilon > 0$

$$\sup_{u \in \mathcal{U}} (x^2 + u) - \sup_{u \in \mathcal{U}} u - \eta d_{\widetilde{K}_2}(x) \geq 0, \ \forall x \in K_2 \cap (-\varepsilon, \varepsilon).$$

It can be seen that $\widetilde{K}_2 = \{0\}$ and then above inequality deduces to

$$x^2 \geq \eta |x|, \forall x \in K_2 \cap (-\varepsilon, \varepsilon),$$

which is clearly impossible.

3.2.1 Necessary optimality conditions for robust weak sharp solutions

In this section, we focus our attention on establishing some necessary optimality conditions for local (global) robust weak sharp solutions in uncertain optimization problems in terms of the advanced tools of variational analysis and generalized differentiation. Given arbitrary $\bar{x} \in \Omega$, we set

$$\mathcal{U}(\bar{x}) := \left\{ u^* \in \mathcal{U} : f(\bar{x}, u^*) = \sup_{u \in \mathcal{U}} f(\bar{x}, u) \right\},$$
$$\mathcal{V}_i(\bar{x}) := \left\{ v_i^* \in \mathcal{V}_i : g_i(\bar{x}, v_i^*) = \max_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i) \right\},$$
$$\mathcal{I}(\bar{x}) := \{ i \in I : g_i(\bar{x}, v_i) = 0, \forall v_i \in \mathcal{V}_i \}.$$

In what follows, throughout this section, we assume $g_i : \mathbb{R}^n \times \mathcal{V}_i \to \mathbb{R}$ is a function such that for each fixed $v_i \in \mathcal{V}_i$, $i \in I, g_i(\cdot, v_i)$ is locally Lipschitz continuous and assume function $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$ satisfies the following conditions:

(C1) For a fixed $\bar{x} \in \Omega$, there exists $r_{\bar{x}} > 0$ such that the function $f(x, \cdot) : \mathcal{U} \to \mathbb{R}$ is usc for all $x \in B(\bar{x}, r_{\bar{x}})$ and $f(\cdot, u)$ is Lipschitz continuous in x, uniformly for $u \in \mathcal{U}$; i.e., for some real number $l > 0$, for all $x, y \in \Omega$ and $u \in \mathcal{U}$, one has $\|f(x, u) - f(y, u)\| \leq l\|x - y\|$.

(C2) The valued of multifunction $\partial^M f(\cdot, u) : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is closed at $(\bar{x}, u)$ for each $u \in \mathcal{U}(\bar{x})$.

In order to obtain the necessary and sufficient optimality condition for local robust weak sharp solutions of (UP_2), we now state a constraint qualification for the uncertain optimization problem with the feasible set K_2 defined.

Definition 3.2.4. Given arbitrary $\bar{x} \in \Omega$, the constraint qualification (CQ) is said to be satisfied at $\bar{x}$ if there do not exist $\mu_i \geq 0$ and $v_i \in \mathcal{V}_i, i \in \mathcal{I}(\bar{x})$ such that $\sum_{i \in \mathcal{I}(\bar{x})} \mu_i \neq 0$ and $0_{\mathbb{R}^n} \in \sum_{i \in \mathcal{I}(\bar{x})} \mu_i \partial^M g_i(\cdot, v_i)(\bar{x}) + N^M(\Omega, \bar{x})$.

Remark 3.2.5. We can see that the (CQ) defined in Definition 3.2.4 reduces to the constraint qualification defined in [70, Definition 3.2] when $\Omega = \mathbb{R}^n$. As well as, it is not hard to verify that this (CQ) reduces to the extended Mangasarian-Fromovitz constraint qualification (see [71]) .in the smooth setting when $\Omega = \mathbb{R}^n$.

The following necessary optimality condition for local robust weak sharp solutions of (UP_2) is obtained under the (CQ).

Theorem 3.2.6. *Let $\bar{x} \in K_2$ and the constraint qualification (CQ), defined in Definition 3.2.4, be satisfied at $\bar{x}$. If $\bar{x}$ is a local robust weak sharp solution for* (UP_2)*, then there exists a real number $\eta > 0$ such that*

$$\eta\mathbb{B}\cap\widehat{N}(K_2,\bar{x}) \subseteq co\left(\bigcup_{u\in\mathcal{U}(\bar{x})} \partial^M f(\cdot,u)(\bar{x})\right) + \bigcup_{\mu_i\in M_i(\bar{x})}\left(\sum_{i=1}^{m}\mu_i\partial^M g_i(\cdot,v_i)(\bar{x})\right) + N^M(\Omega,\bar{x}), \tag{3.2.2}$$

where $M_i(\bar{x}) = \{\mu_i \geq 0 : \mu_i g_i(\bar{x},v_i) = 0, v_i \in \mathcal{V}_i\}$ for all $i \in I$.

Proof. Suppose that $\bar{x}$ is a local robust sharp solution for (UP_2). Then, there exist real numbers $\eta, r_1 > 0$ such that

$$\sup_{u\in\mathcal{U}} f(x,u) - \sup_{u\in\mathcal{U}} f(\bar{x},u) \geq \eta d_{\widetilde{K}_2}(x), \forall x \in K_2 \cap B(\bar{x},r_1). \tag{3.2.3}$$

Let $x^* \in \mathbb{B} \cap \widehat{N}(K_2,\bar{x})$ be given. It follows from $\widehat{\partial} d_{K_2}(\bar{x}) = \mathbb{B} \cap \widehat{N}(K_2,\bar{x})$ that $x^* \in \widehat{\partial} d_{K_2}(\bar{x})$. By the definition of $\widehat{\partial} d_{K_2}(\cdot)$, for any $\varepsilon > 0$, there exists $r_2 \in (0, \frac{1}{2}r_1)$ such that

$$\langle x^*, x-\bar{x}\rangle \leq d_{\widetilde{K}_2}(x) + \varepsilon\|x-\bar{x}\| \tag{3.2.4}$$

for all $x \in B(\bar{x},r_2)$. It is obvious that $B(\bar{x},r_2) \subseteq B(\bar{x},r_1)$, so from (3.2.3) and (3.2.4), we obtain

$$\sup_{u\in\mathcal{U}} f(x,u) - \sup_{u\in\mathcal{U}} f(\bar{x},u) + \eta\varepsilon\|x-\bar{x}\| \geq \eta\langle x^*, x-\bar{x}\rangle \tag{3.2.5}$$

for all $x \in K_2 \cap B(\bar{x},r_2)$. Consider the following function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\varphi(x) := -\eta\langle x^*, x-\bar{x}\rangle + \phi(x) + \eta\varepsilon\|x-\bar{x}\| + \delta_{K_2}(x), \forall x \in \mathbb{R}^n$$

where $\phi(x) := \sup_{u\in\mathcal{U}} f(x,u) - \sup_{u\in\mathcal{U}} f(\bar{x},u)$ for all $x \in \mathbb{R}^n$. Observe that for each $x \in K_2 \cap B(\bar{x},r_2), \varphi(x) \geq 0$, while $\varphi(\bar{x}) = 0$. This means the function φ attains its local minimum point at $\bar{x}$. Further, we can get by the properties of $f(\cdot,u)$, $\|\cdot-\bar{x}\|$ and $\delta_{K_2}(\cdot)$

that the function φ is lsc around $\bar{x}$. Therefore, it follows from Theorem 2.2.47 that $0_{\mathbb{R}^n} \in \widehat{\partial}\varphi(\bar{x})$. Moreover, from Theorem 2.2.48 (i), for each $y \in \widehat{\partial}\varphi(\bar{x})$ and each $\varepsilon > 0$, there exist $x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon \in B(\bar{x}, \varepsilon)$, such that

$$|\phi(x_1^\varepsilon)| < \varepsilon, \quad \eta\varepsilon\|x_2^\varepsilon - \bar{x}\| < \varepsilon, \quad \delta_{K_2}(x_3^\varepsilon) < \varepsilon$$

and

$$y \in \widehat{\partial}\phi(x_1^\varepsilon) + \eta\varepsilon\widehat{\partial}\|x_2^\varepsilon - \bar{x}\| + \widehat{\partial}\delta_{K_2}(x_3^\varepsilon) + \varepsilon\mathbb{B}.$$

Since $0_{\mathbb{R}^n} \in \widehat{\partial}\varphi(\bar{x}), \eta x^* \in \widehat{\partial}\phi(x_1^\varepsilon) + \eta\varepsilon\widehat{\partial}\|x_2^\varepsilon - \bar{x}\| + \widehat{\partial}\delta_{K_2}(x_3^\varepsilon) + \varepsilon\mathbb{B}$. Observe that $x_3^\varepsilon \in K_2$ and $\widehat{\partial}\delta_{K_2}(x_3^\varepsilon) = \widehat{N}(K_2, x_3^\varepsilon)$. It follows from the definition of ϕ that it is Lipschitz continuous around $\bar{x}$ with a constant l. So, due to [53, Proposition 1.85], for all sufficiently small $\varepsilon > 0$, one has $\widehat{\partial}\phi(x_1^\varepsilon) \subseteq l\mathbb{B}$. Similarly, we also get $\widehat{\partial}(\| \cdot - \bar{x}\|)(x_2^\varepsilon) \subseteq \mathbb{B}$. According to these inclusions, the compactness of $\mathbb{B}$, and $x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon \in B(\bar{x}, \varepsilon)$ yields $x_1 \xrightarrow{\phi} \bar{x}, \quad x_2^\varepsilon \xrightarrow{\|\cdot - \bar{x}\|} \bar{x}, \quad x_3^\varepsilon \xrightarrow{K_2} \bar{x}$, as $\varepsilon \downarrow 0$. It follows that

$$\eta x^* \in \partial^M\phi(\bar{x}) + N^M(K_2, \bar{x}). \tag{3.2.6}$$

As f satisfies (C1) and (C2), by the same fashion of proof in Theorem 3.3 of [21], we obtain

$$\partial^M\phi(\bar{x}) \subseteq \text{co}\left(\bigcup_{u \in \mathcal{U}(\bar{x})} \partial^M f(\cdot, u)(\bar{x})\right). \tag{3.2.7}$$

On the other hand, $\Pi := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, v_i \in \mathcal{V}_i, i \in \mathcal{I}\}$. Hence, $K_2 = \Omega \cap \Pi$. As $0_{\mathbb{R}^n} \in N(\Omega, \bar{x})$, the following inclusion always holds:

$$\bigcup_{\mu_i \in M_i(\bar{x})} \left(\sum_{i \in \mathcal{I}(\bar{x})} \mu_i \partial^M g_i(\cdot, v_i)(\bar{x})\right) \subseteq \bigcup_{\mu_i \in M_i(\bar{x})} \left(\sum_{i \in \mathcal{I}(\bar{x})} \mu_i \partial^M g_i(\cdot, v_i)(\bar{x})\right) + N^M(\Omega, \bar{x}).$$

Since the (CQ) is satisfied at $\bar{x}$, there do not exist $\mu_i \geq 0$ and $v_i \in \mathcal{V}_i, i \in \mathcal{I}(\bar{x})$ such that $\sum_{i \in \mathcal{I}(\bar{x})} \mu_i \neq 0$ and $0_{\mathbb{R}^n} \in \sum_{i \in \mathcal{I}(\bar{x})} \mu_i \partial^M g_i(\cdot, v_i)(\bar{x}) + N(\Omega, \bar{x})$. Applying [53, Corollary 4.36], we have

$$N^M(\Pi, \bar{x}) \subseteq \bigcup_{\mu_i \in M_i(\bar{x})} \left(\sum_{i \in \mathcal{I}(\bar{x})} \mu_i \partial^M g_i(\cdot, v_i)(\bar{x})\right). \tag{3.2.8}$$

It follows from [53, Corollary 3.37] that

$$N^M(K_2,\bar{x}) = N^M(\Omega\cap\Pi,\bar{x}) \subseteq N^M(\Omega,\bar{x}) + N^M(\Pi,\bar{x}). \tag{3.2.9}$$

Setting $\mu_i = 0$ for every $i \in \mathcal{I}\setminus\mathcal{I}(\bar{x})$, by (3.2.8) and (3.2.9), we arrive the following inclusion:

$$N^M(K_2,\bar{x}) \subseteq \bigcup_{\mu_i\in M_i(\bar{x})}\left(\sum_{i\in\mathcal{I}}\mu_i\partial^M g_i(\cdot,v_i)(\bar{x})\right) + N^M(\Omega,\bar{x}). \tag{3.2.10}$$

As $x^* \in \mathbb{B}\cap\widehat{N}(K_2,\bar{x})$ was arbitrary, we verify (3.2.2) by combining (3.2.6), (3.2.7) and (3.2.10). □

The following example shows that the (CQ) being satisfied around $\bar{x}\in K_2$ is essential for Theorem 3.2.6.

Example 3.2.7. *Let $f:\mathbb{R}\times\mathcal{U}\to\mathbb{R}$ and $g:\mathbb{R}\times\mathcal{V}\to\mathbb{R}$ be defined by*

$$f(x,u) = \begin{cases} -u; & x=0,\\ x^{-2}-u; & \text{otherwise,}\end{cases}$$

and

$$g(x,v) := v - x^3,$$

where $x\in\mathbb{R}, u\in\mathcal{U}:=[0,1]$ and $v\in\mathcal{V}:=[-1,0]$. Take $\Omega:=[-1,1]$ and consider the problem (UP$_2$). *It is not hard to see that f satisfies (C1) and (C2), and the robust feasible set is $K_2=[0,1]$. Consider $\bar{x}:=0\in K$ with its neighborhood $U=(-\frac{1}{2},\frac{1}{2})$. By choosing a positive real number $\eta=1>0$, we can verify that $\bar{x}$ is a local robust weak sharp solution of the problem* (UP$_2$). *Simultaneously, we get from direct calculating that*

$$\partial^M f(\cdot,u)(\bar{x}) = \{0\},\ \forall u\in\mathcal{U},\ \partial^M g(\cdot,v)(\bar{x}) = \{0\},\ \forall v\in\mathcal{V},$$

$$N^M(\Omega,\bar{x}) = \{0\},\ \text{ and } N^M(K_2,\bar{x}) = -\mathbb{R}_+.$$

It follows that the (CQ) is not satisfied at $\bar{x}$. Furthermore; we get

$$\eta\mathbb{B}_{\mathbb{R}^2}\cap\widehat{N}(K_2,\bar{x}) = [-\eta,0]$$

while

$$co\left(\bigcup_{u\in\mathcal{U}(\bar{x})}\partial^M f(\cdot,u)(\bar{x})\right)+\bigcup_{\mu\in M(\bar{x})}\mu\partial^M g(\cdot,v)(\bar{x})+N(\Omega,\bar{x})=\{0\}.$$

This shows that (3.2.2) does not hold for every $\eta,\delta>0$. Hence, the assumption that (CQ) being satisfied is essential.

Observe that the functions $f(\cdot,u)$ and $g(\cdot,v)$ is not convex. Indeed, choose $x_1=\frac{1}{2}, x_2=0$, and $\lambda=\frac{1}{2}\in[0,1]$, the convexities of these two functions are not satisfied. Therefore, [68, Theorem 3.2] is not applicable for this example.

3.2.2 Sufficient optimality conditions for robust weak sharp solutions

In this section, we focus on sufficient optimality conditions for robust weak sharp solutions of problem (UP_2).

Now, we state a type of the robust version of Karush-Kuhn-Tucker (KKT) conditions as the following definition.

Definition 3.2.8. A point $\bar{x}\in K_2$ is said to be satisfied the robust version of the (KKT) condition if there exist $\lambda>0$ and $\mu\in\mathbb{R}^m_+$ such that $\lambda+\sum_{i=1}^m\mu_i=1$,

$$0\in\lambda\,\mathrm{co}\left(\bigcup_{u\in\mathcal{U}(\bar{x})}\partial^M f(\cdot,u)(\bar{x})\right)+\sum_{i=1}^m\mu_i\,\mathrm{co}\left(\bigcup_{v_i\in\mathcal{V}_i(\bar{x})}\partial^M g_i(\cdot,v_i)(\bar{x})\right),\ \text{and}$$

$$\mu_i\sup_{v_i\in\mathcal{V}_i}g_i(\bar{x},v_i)=0,\ i\in\mathcal{I}.$$

The following example illustrates that only satisfying the robust version of (KKT) condition is not sufficient for a point to be a (local) robust weak sharp solution of problem (UP_2).

Example 3.2.9. *Let $f:\mathbb{R}\times\mathcal{U}\to\mathbb{R}$ and $g:\mathbb{R}\times\mathcal{V}\to\mathbb{R}$ be defined by*

$$f(x,u)=-x^2-u,\ \ \text{and}\ \ g(x,v)=v\max\{x,0\},$$

where $x \in \mathbb{R}, u \in \mathcal{U} := [0,1]$ *and* $v \in \mathcal{V} := [-1,0]$. *By taking* $\Omega = \mathbb{R}$ *and consider the problem* (UP_2) *and the robust feasible set is* $K_2 = \mathbb{R}$. *Consider* $\bar{x} := 0 \in K_2$, *then we have*

$$\partial^M f(\cdot,u)(\bar{x}) = \{0\}, \forall u \in \mathcal{U}, \partial^M g(\cdot,v)(\bar{x}) = \{0\}, \forall v \in \mathcal{V}.$$

The robust version of the (KKT) condition is satisfied at $\bar{x}$ *since there exist* $\lambda = \mu = \frac{1}{2} > 0$, *such that* $\lambda + \mu = 1$,

$$0 = \frac{1}{2}(0) + \frac{1}{2}(0) \in \lambda \ co \left(\bigcup_{u \in \mathcal{U}(\bar{x})} \partial^M f(\cdot,u)(\bar{x}) \right) + \mu \ co \left(\bigcup_{v \in \mathcal{V}(\bar{x})} \partial^M g(\cdot,v)(\bar{x}) \right)$$

and $\mu \sup_{v \in \mathcal{V}} g(\bar{x},v) = \mu(0) = 0$. *However, this* $\bar{x}$ *is not a (local) robust weak sharp solution of our considered problem since there is no* $\eta > 0$ *satisfy*

$$\sup_{u \in \mathcal{U}} f(x,u) - \sup_{u \in \mathcal{U}} f(\bar{x},u) = -x^2 \geq \eta d_{\widetilde{K}_2}(x), \ \forall x \in \mathbb{R}.$$

In order to formulate the sufficient optimality conditions, we need to introduce concepts of generalized convexity at a given point for a family of real-valued functions. We set $g := (g_1, \ldots, g_m)$ for convenience in the sequel.

Definition 3.2.10. (f,g) is said to be generalized convex at $\bar{x} \in \mathbb{R}^n$ if for any $x \in \mathbb{R}^n, z_u^* \in \partial^M f(\cdot,u)(\bar{x}), u \in \mathcal{U}(\bar{x})$, and $x_{v_i}^* \in \partial g_i(\cdot,v_i)(\bar{x}), v_i \in \mathcal{V}_i(\bar{x}), i \in I$, there exists $w \in \mathbb{R}^n$ such that

$$f(x,u) - f(\bar{x},u) \geq \langle z_u^*, w \rangle, \ \ g_i(x,v_i) - g_i(\bar{x},v_i) \geq \langle x_{v_i}^*, w \rangle.$$

Remark 3.2.11. If $f(\cdot,u), u \in \mathcal{U}$ and $g_i(\cdot,v), v \in \mathcal{V}_i, i \in I$ are convex, then (f,g) is generalized convex at any $\bar{x} \in \mathbb{R}^n$ with $w := x - \bar{x}$ for each $x \in \mathbb{R}^n$.

The following example demonstrates, the class of generalized convex functions at a given point is properly wider than the one of convex functions.

Example 3.2.12. *Let* $f : \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ *and* $g : \mathbb{R} \times \mathcal{V} \to \mathbb{R}$ *be defined by*

$$f(x,u) = 2x + u,$$

and

$$g(x,v) = \begin{cases} vx; & x \geq 0, \\ x - v; & \text{otherwise} \end{cases}$$

where $x \in \mathbb{R}, u \in \mathcal{U} := [0,1] \subseteq \mathbb{R}$, *and* $v \in \mathcal{V} := [-1,1] \subseteq \mathbb{R}$. *Consider* $\bar{x} := 0 \in \mathbb{R}$. *Observe that* $\partial f(\cdot,u)(\bar{x}) = \{2\}$ *for all* $u \in \mathcal{U}, \partial^M g(\cdot,v)(\bar{x}) = \{v,1\}$ *for all* $v \in \mathcal{V}$. *We see that* (f,g) *is generalized convex at* $\bar{x} = 0 \in \mathbb{R}$ *as follows:*

Case I: *If* $x \geq 0$, *then there exists* $w := x \in \mathbb{R}$ *such that*

$$f(x,u) - f(\bar{x},u) = 2x = \langle 2, x\rangle \text{ and } g(x,v) - g(\bar{x},v) = vx = \langle v, x\rangle.$$

Case II: *If* $x < 0$, *then there exists* $w = x + v \in \mathbb{R}$ *such that*

$$f(x,u) - f(\bar{x},u) = 2x \geq \langle 2, x - v\rangle, \text{ and } g(x,v) - g(\bar{x},v) = x - v = \langle 1, x - v\rangle.$$

However, $g(\cdot,0)$ *is not a convex function as follows: let* $x_1 = 1, x_2 = -1 \in \mathbb{R}$, *and choose* $\lambda = \frac{1}{2} \in [0,1]$, *we have*

$$g(\lambda x_1 + (1-\lambda)x_2, 0) = g(0,0) = 0 > \frac{1}{2}(0) + \frac{1}{2}(-1+0) = \lambda g(x_1,0) + (1-\lambda)g(x_2,0).$$

By means of the robust version of the (KKT) condition and the generalized convexity, we established the following sufficient optimality conditions for robust weak sharp solutions for the problem (UP_2) .

Theorem 3.2.13. *Let* $\bar{x} \in K_2$ *and the robust version of the (KKT) condition be satisfied at* $\bar{x}$. *If* (f,g) *is generalized convex at* $\bar{x}$, *then* $\bar{x}$ *is a robust weak sharp solution for the problem* (UP_2).

Proof. Since the robust version of the (KKT) condition is satisfied at $\bar{x}$, there exist $\lambda_1 \geq 0, \lambda_{1_k} \geq 0, z^*_{1_k} \in \partial f(\cdot, u_{1_k})(\bar{x}), u_{1_k} \in \mathcal{U}(\bar{x}), \sum_{k=1}^{k_1} \lambda_{1_k} = 1, k = 1, \ldots, k_1, k_1 \in \mathbb{N}$, and $\mu \in \mathbb{R}^m_+, \mu_{i_j} \geq 0, x^*_{i_j} \in \partial g_i(\cdot, v_{i_j})(\bar{x}), v_{i_j} \in \mathcal{V}_i(\bar{x}), \sum_{j=1}^{j_i} \mu_{i_j} = 1, j = 1, \ldots, j_i, j_i \in \mathbb{N}$, such that $\lambda_1 + \sum_{i=1}^m \mu_i = 1$ and

$$0 = \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} z^*_{1_k} \right) + \sum_{i=1}^{m} \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} x^*_{i_j} \right), \tag{3.2.11}$$

$$\mu_i \sup_{v_i \in \mathcal{V}_i} g_i(\bar{x}) = 0, i \in I. \tag{3.2.12}$$

Assume on the contrary that $\bar{x}$ is not a robust weak sharp solution for the problem (UP_2). Then, there exists $\tilde{x} \in K_2$ such that for all $\eta > 0$

$$\sup_{u \in \mathcal{U}} f(\tilde{x}, u) - \sup_{u \in \mathcal{U}} f(\bar{x}, u) < \eta d_{\widetilde{K}_2}(\tilde{x}). \tag{3.2.13}$$

It follows from the generalized convexity of (f, g) and (3.2.11) that there exists $w \in \mathbb{R}^n$ such that

$$\begin{aligned}
0 &= \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} \langle z_{1_k}^*, w \rangle \right) + \sum_{i=1}^{m} \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} \langle x_{i_j}^*, w \rangle \right) \\
&\leq \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} [f(\tilde{x}, u_{1_k}) - f(\bar{x}, u_{1_k})] \right) \\
&\quad + \sum_{i=1}^{m} \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} [g_i(\tilde{x}, v_{i_j}) - g_i(\bar{x}, v_{i_j})] \right).
\end{aligned} \tag{3.2.14}$$

Therefore,

$$\begin{aligned}
&\lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} f(\bar{x}, u_{1_k}) \right) + \sum_{i=1}^{m} \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} g_i(\bar{x}, v_{i_j}) \right) \\
&\leq \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} f(\tilde{x}, u_{1_k}) \right) \\
&\quad + \sum_{i=1}^{m} \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} g_i(\tilde{x}, v_{i_j}) \right)
\end{aligned} \tag{3.2.15}$$

Since $v_{i_j} \in \mathcal{V}_i(\bar{x}), g_i(\bar{x}, v_{i_j}) = \sup_{v_i \in \mathcal{V}_i} g_i(\bar{x}, v_i), \forall i \in I$, for all $j = 1, \ldots, j_i$. From (3.2.12), we have $\mu_i g_i(\bar{x}, v_{i_j}) = 0$ for $i \in I$ and $j = 1, \ldots, j_i$. Furthermore, for each $\tilde{x} \in K_2, \mu_i g_i(\tilde{x}, v_{i_j}) \leq 0$ for $i \in I$ and $j = 1, \ldots, j_i$. Hence, by (3.2.15) we have

$$\begin{aligned}
&\lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} f(\bar{x}, u_{1_k}) \right) \\
&= \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} f(\bar{x}, u_{1_k}) \right) + \sum_{i=1}^{m} \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} g_i(\bar{x}, v_{i_j}) \right) \\
&\leq \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} f(\tilde{x}, u_{1_k}) \right) + \sum_{i=1}^{m} \mu_i \left(\sum_{j=1}^{j_i} \mu_{i_j} g_i(\tilde{x}, v_{i_j}) \right)
\end{aligned}$$

$$\leq \lambda_1 \left(\sum_{k=1}^{k_1} \lambda_{1_k} f(\tilde{x}, u_{1_k}) \right).$$

This together with $u_{1_k} \in \mathcal{U}(\bar{x})$ imply

$$\sum_{k=1}^{k_1} \lambda_{1_k} \sup_{u \in \mathcal{U}} f(\bar{x}, u) \leq \sum_{k=1}^{k_1} \lambda_{1_k} f(\tilde{x}, u_{1_k}) \leq \sum_{k=1}^{k_1} \lambda_{1_k} \sup_{u \in \mathcal{U}} f(\tilde{x}, u),$$

which yields $\sup_{u\in\mathcal{U}} f(\bar{x}, u) - \sup_{u\in\mathcal{U}} f(\tilde{x}, u) \leq \eta d_{\widetilde{K}_2}(\tilde{x})$ for all $\eta > 0$. Thus, for any $\eta > 0, \sup_{u\in\mathcal{U}} f(\tilde{x}, u) - \sup_{u \in \mathcal{U}} f(\bar{x}, u) \geq \eta d_{\widetilde{K}_2}(\tilde{x})$. This contradicts (3.2.13) and hence $\bar{x}$ is a robust weak sharp solution of (UP_2). □

Remark 3.2.14. In Theorem 3.2.13, the sufficient optimality conditions for robust weak sharp solutions are established while the assumptions of the convexity of objective and constraint functions and the convexity of parameter uncertain sets are dropped. However, these assumptions are employed to obtain several results on optimality conditions of robust optimal solutions and/or characterizations of robust optimal solution sets of uncertain optimization problems obtained in recent literature (see, e.g., [12, 45, 65, 67–69]).

The next example assert the importance of the generalized convexity of (f, g) imposed in Theorem 3.2.13.

Example 3.2.15. *Let $f : \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{V} \to \mathbb{R}$ be defined by*

$$f(x, u) = x^3 + u \text{ and } g(x, v) = 1 - (v + x^4),$$

where $x \in \mathbb{R}, u \in \mathcal{U} := [-1, 0], v \in \mathcal{V} := [1, 2]$ and let $\Omega := [-2, 2]$. It can be seen that conditions (C1) and (C2) are satisfied and the robust feasible set is $K_2 = \mathbb{R}$. By taking $\bar{x} := 0 \in K_2$, we see that

$$\partial^M f(\cdot, u)(\bar{x}) = \{0\}, \forall u \in \mathcal{U} \text{ and } \partial^M g(\cdot, v)(\bar{x}) = \{0\}, \forall v \in \mathcal{V}.$$

By the same way in Example 3.2.9, we have that the robust version of the (KKT) condition is satisfied at $\bar{x}$. In fact, for any real numbers $\lambda, \mu > 0$ with $\lambda + \mu = 1$, we have

$$0 = \frac{1}{2}(0) + \frac{1}{2}(0) \in \lambda \ co \left\{ \partial^M f(\cdot, u)(\bar{x}) : u \in \mathcal{U}(\bar{x}) \right\} + \mu \ co \left\{ \partial^M g(\cdot, v)(\bar{x}) : v \in \mathcal{V}(\bar{x}) \right\}$$

and

$$\mu \sup_{v\in\mathcal{V}} g(\bar{x}, v) = \mu(0) = 0.$$

However, the generalized convexity of (f, g) *is not satisfied at* $\bar{x}$*. Indeed, there exists* $z = -\frac{1}{2} \in K_2$ *such that for each* $w \in \mathbb{R}$,

$$f(z, u) - f(\bar{x}, u) = \left(-\frac{1}{2}\right)^3 < 0 = \langle 0, w\rangle,$$

and

$$g(z, v) - g(\bar{x}, v) = 1 - \left(v + (-\frac{1}{2})^4\right) < 0 = \langle 0, w\rangle.$$

Notice that $\bar{x}$ *is not a (local) robust weak sharp solution of* (UP_2) *as there is no* $\eta > 0$ *satisfy*

$$\sup_{u\in\mathcal{U}} f(x, u) - \sup_{u\in\mathcal{U}} f(\bar{x}, u) = x^3 \geq \eta d_{\widetilde{K}_2}(x), \ \forall x \in \mathbb{R}.$$

Therefore, the conclusion of the Theorem 3.2.13 may fail if the generalized convexity has been dropped.

It is not hard to see that the functions $f(\cdot, u)$ *and* $g(\cdot, v)$ *are not convex. In fact, the convexities of them are not satisfied when* $x_1 = -\frac{1}{2}, x_2 = 0$*, and* $\lambda = \frac{1}{2}$*. Therefore, this problem cannot be solved by [68, Theorem 3.2].*

CHAPTER IV

OPTIMALITY CONDITIONS AND DUALITY THEOREMS FOR ROBUST APPROXIMATE SOLUTIONS

In this chapter, we recall concepts of robust solutions of a convex optimization problem with data uncertainty as well as introduce a new concept of approximate solution for the highly robust solutions of the problem. Firstly, we begin by recalling the following deterministic convex program:

$$\text{Minimize } f(x) \text{ subject to } g_i(x) \leq 0, i = 1, \ldots, m \tag{P_3}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}, i \in I := \{1, \ldots, m\}$ are convex functions. The following parameterized convex program is an analogue of the deterministic convex program (P_3) if the objective as well as the constraints are uncertain:

$$\text{Minimize } f(x, u) \text{ subject to } g_i(x, v_i) \leq 0, i \in I. \tag{UP_3}$$

Here u is an uncertain parameter belonging to a compact convex uncertainty set $\mathcal{U} \subseteq \mathbb{R}^p$, for each $u \in \mathcal{U}, f(\cdot, u) : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and for each $i \in I, v_i$ belongs to a compact convex set $\mathcal{V}_i \subseteq \mathbb{R}^q, g_i(\cdot, v_i)$ is convex. By enforcing the constraints for all possible uncertainty within $\mathcal{V}_i, i \in I$, the problem (UP_3) becomes an uncertain convex semi-infinite program:

$$\text{Minimize } f(x, u) \text{ subject to } g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i \in I. \tag{RP_3}$$

In other words, we study the uncertain convex programming problem (UP_3) by examining its robust (worst-case) counterpart. Let

$$K_3 := \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i \in I\},$$

then it is termed as the robust feasible set of (UP_3). To avoid triviality in (RP_3), we always assume that $K_3 \neq \emptyset$.

In the literature, there are multiple ways of defining robust solutions for (UP_3). In the following, we recall two concepts of the robust solutions of the uncertain program

(UP_3). The first notion commonly referred to as strictly robust solution or robust minimax solution, can be found in [1, 13]. This concept has been studied extensively by many authors, see, e.g., [2, 19, 60, 69].

Definition 4.0.16. A feasible point $\bar{x} \in K_3$ is said to be a strictly robust solution for (UP_3) if for each $x \in K_3$,

$$\sup_{u \in \mathcal{U}} f(x,u) \geq \sup_{u \in \mathcal{U}} f(\bar{x},u).$$

The second one called highly robust solution can be found in Bitran [17]. This concept was also investigated for different uncertain multi-objective optimization problems, see, e.g., [18, 19].

Definition 4.0.17. A feasible point $\bar{x} \in K_3$ is said to be a highly robust solution for (UP_3) if for each $u \in \mathcal{U}$ and $x \in K_3$,

$$f(x,u) \geq f(\bar{x},u).$$

The following notion is a concept of approximate solution that approximates the strictly robust solutions. It was investigated in a few papers, see, e.g. [44].

Definition 4.0.18. Let $\varepsilon \geq 0$ be given. A feasible point $\bar{x} \in K_3$ is said to be an ε-quasi strictly robust solution (or a robust quasi ε optimal solution) for (UP_3) if for each $x \in K_3$,

$$\sup_{u \in \mathcal{U}} f(x,u) + \sqrt{\varepsilon}\|x - \bar{x}\| \geq \sup_{u \in \mathcal{U}} f(\bar{x},u).$$

Clearly, if $\varepsilon = 0$, then an ε-quasi strictly robust solution for (UP_3) reduces to be a strictly robust solution for (UP_3).

Now, we introduce a new concept of solution to approximate the highly robust solutions for (UP_3).

Definition 4.0.19. Let $\varepsilon \geq 0$ be given. A feasible point $\bar{x} \in K_3$ is said to be an ε-quasi highly robust solution (or a highly robust quasi ε-optimal solution) for (UP_3)

if for each $u \in \mathcal{U}$ and $x \in K_3$,

$$f(x,u) + \sqrt{\varepsilon}\|x - \bar{x}\| \geq f(\bar{x}, u).$$

Clearly, if $\varepsilon = 0$, then an ε-quasi highly robust solution for (UP_3) reduces to be a highly robust solution for (UP_3).

Remark 4.0.20. (i) It is evident from Definition 4.0.16 and Definition 4.0.17 that a highly robust solution for (UP_3) is a strictly robust solution for (UP_3), but the converse does not hold. This means the highly robust solution is more immune to data uncertainty than the strictly robust solution.

(ii) Also, it is evident from Definition 4.0.18 and Definition 4.0.19 that an ε-highly robust solution for (UP_3) is an ε-strictly robust solution for (UP_3), but the converse does not hold. Hence, the ε-quasi highly robust solution is more immune to data uncertainty than the ε-quasi strictly robust solution.

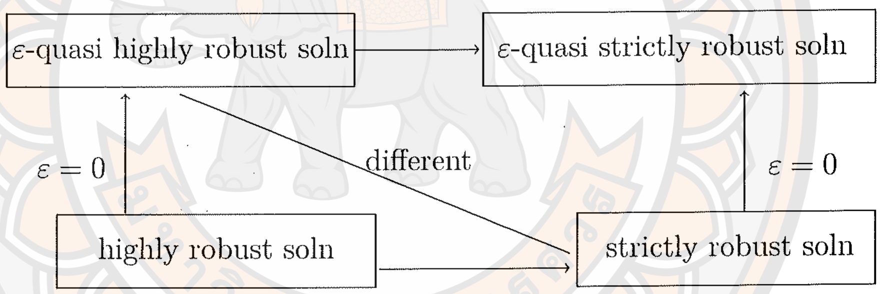


Figure 17: Relationship among strictly robust solutions, highly robust solutions, and approximate solutions which approximate them

The highly robust solution is more immune to data uncertainty than the strictly robust solution and the ε-quasi highly robust solution can reduce to be the highly robust solution. Therefore, the ε-quasi highly robust solution, which is more immune to data uncertainty than the ε-quasi strictly robust solution, is different from the strictly robust solution. The following example sheds some light onto this fact.

Example 4.0.21. *Consider an uncertain convex program with an uncertain objective and uncertainty-free constraints:*

$$\textit{Minimize } ux + |x+1| \quad \textit{subject to} \quad x \in \mathbb{R}, \tag{4.0.16}$$

where $u \in [-1,1]$. *Following the robust optimization approach of [1], the robust counterpart of* (4.0.16) *reads*

$$\text{Minimize} \sup_{u\in[-1,1]} ux + \min|x+1| \quad \text{subject to} \quad x \in \mathbb{R},$$

which is equivalent to $\min\{|x| + |x+1| : x \in \mathbb{R}\}$. *Then it is easy to check that the set of strictly robust solutions for* (4.0.16), *denoted by* S^{SR}, *is*

$$S^{SR} = [-1, 0],$$

while the set of solutions for (4.0.16), *denoted by* S, *is*

$$S = \begin{cases} (-\infty, -1]; & u = 1, \\ \{-1\}; & u \in (-1, 1), \\ [-1, \infty); & u = -1, \end{cases}$$

So, the set of highly robust solutions for (4.0.16), *denoted by* S^{HR}, *is*

$$S^{HR} = \{-1\}.$$

Consider $\bar{x} := -\frac{3}{2} \in (-\infty, -1]$ *with* $\bar{\varepsilon} := 4 > 0$. *We can see that for any* $x \in \mathbb{R}$ *and* $u \in [-1, 1]$,

$$ux + |x+1| + \sqrt{\bar{\varepsilon}}|x - \bar{x}| \geq u\bar{x} + |\bar{x} - 1|.$$

Thus, $\bar{x} = -\frac{3}{2}$ *is an* $\bar{\varepsilon}$*-quasi highly robust solution of* (4.0.16).

Notice that $\bar{x} \notin [-1, 0] = S^{SR}$, *so, the* ε*-quasi highly robust solution for* (4.0.16) *is different from strictly robust solutions of* (4.0.16), *making it valuable to study the* ε*-quasi highly robust solutions.*

4.1 ε-quasi highly robust solutions for robust convex optimization problems

In this section, we consider (UP_3), which is an uncertain convex optimization problem with data uncertainty in both objective and constraint functions. Our aim is to investigate robust approximate solutions for the uncertain convex problem (UP_3). First of all, the notion of an ε-quasi highly robust solution for the uncertain convex optimization problem is introduced. Then, the highly robust approximate optimality theorems for ε-quasi highly robust solutions of the considered problem are established by means of a robust optimization approach.

The following constraint qualification, which was introduced in [8], plays a key role in obtaining results in this section.

Definition 4.1.1. [8] Let $g_i : R^n \times \mathbb{R}^q \to \mathbb{R}, i = 1, \dots m$ be functions such that for all $v_i \in \mathbb{R}^q, g_i(\cdot, v_i)$ is convex. Then the robust characteristic cone constraint qualification (RCCCQ) is satisfied if the cone $\bigcup_{\substack{v_i \in \mathcal{V}_i, \\ \lambda_i \geq 0}} \text{epi} \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \right)^*$ is closed and convex.

Lemma 4.1.2. *Let $\bar{x} \in K$ and let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i \in I$ be continuous functions such that for each $u \in \mathbb{R}^p, f(\cdot, u)$ is convex on $\mathbb{R}^n$ and for each $v_i \in \mathbb{R}^q, g_i(\cdot, v_i)$ is convex on $\mathbb{R}^n$ and let $A := \bigcup_{\substack{v_i \in \mathcal{V}_i, \\ \lambda_i \geq 0}} \text{epi} \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \right)^*$. Suppose that the constraint qualification (RCCCQ), defined in Definition 4.1.1, holds. Then the following statements are equivalent:*

(i) $\bar{x}$ is an ε-quasi highly robust solution for (UP_3)*;*

(ii) there exist $\hat{\lambda}_i \geq 0$ and $\hat{v}_i \in \mathcal{V}_i, i \in I$ such that for any $x \in \mathbb{R}^n$ and $u \in \mathcal{U}$,

$$f(\bar{x}, u) \leq f(x, u) + \sum_{i=1}^{m} \hat{\lambda}_i g_i(x, \hat{v}_i) + \sqrt{\varepsilon} \|x - \bar{x}\|.$$

Proof. [(i) $\Rightarrow$ (ii)] Assume that $\bar{x}$ is an ε-quasi highly robust solution for (UP_3). So for any $x \in K_3, f(x, u) + \sqrt{\varepsilon}\|x - \bar{x}\| \geq f(\bar{x}, u)$ for all $u \in \mathcal{U}$. Hence we obtain the inclusion $K_3 \subseteq \{x \in \mathbb{R}^n : h(x, u) \geq 0\}$ where $h(x, u) = f(x, u) + \sqrt{\varepsilon}\|x - \bar{x}\| - f(\bar{x}, u)$ for $u \in \mathcal{U}$. Due to the Lemma 2.2.38,

$$(0, 0) \in \text{epi } h^*(\cdot, u) + \text{cl (co } A), \text{ where } u \in \mathcal{U}.$$

Since A is closed and convex,

$$(0,0) \in \text{epi } h^*(\cdot,u) + A, \text{ where } u \in \mathcal{U}.$$

Hence, there exist $\bar{\lambda}_i \geq 0, \bar{v}_i \in \mathcal{V}_i$ such that

$$(0,0) \in \text{epi } h^*(\cdot,u) + \text{epi } \left(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot,\bar{v}_i)\right)^*, \text{ where } u \in \mathcal{U}. \tag{4.1.1}$$

Let us prove that for any $u \in \mathcal{U}$,

$$\text{epi } h^*(\cdot,u) = \text{epi } f^*(\cdot,u) + \sqrt{\varepsilon}\mathbb{B} + \Big[f(\bar{x},u) + \|\bar{x}\|, +\infty\Big). \tag{4.1.2}$$

From Proposition 2.2.35, we have

$$\begin{aligned} \text{epi } h^*(\cdot,u) &= \text{epi } \Big[f(\cdot,u) + \sqrt{\varepsilon}\|\cdot - \bar{x}\| - f(\bar{x},u)\Big]^* \\ &= \text{epi } f^*(\cdot,u) + \text{epi } \Big[\sqrt{\varepsilon}\|\cdot - \bar{x}\| - f(\bar{x},u)\Big]^*, \end{aligned} \tag{4.1.3}$$

where $u \in \mathbb{R}^p$. Observe that for any $u \in \mathcal{U}$,

$$\Big[\sqrt{\varepsilon}\|\cdot - \bar{x}\| - f(\bar{x},u)\Big]^*(z) = \begin{cases} f(\bar{x},u) + \sqrt{\varepsilon}\|\bar{x}\|; & \|z\| \leq \sqrt{\varepsilon}, \\ +\infty; & \|z\| > \sqrt{\varepsilon}. \end{cases}$$

By dealing with (4.1.3), we obtain

$$\text{epi } h^*(\cdot,u) = \text{epi } f^*(\cdot,u) + \sqrt{\varepsilon}\mathbb{B} \times \Big[f(\bar{x},u) + \sqrt{\varepsilon}\|\bar{x}\|, +\infty\Big),$$

where $u \in \mathcal{U}$. Hence, it follows from (4.1.1) that for $u \in \mathcal{U}$,

$$(0,0) \in \text{epi } f^*(\cdot,u) + \sqrt{\varepsilon}\mathbb{B} \times \Big[f(\bar{x},u) + \sqrt{\varepsilon}\|\bar{x}\|, +\infty\Big) + \text{epi } \left(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot,\bar{v}_i)\right)^*.$$

This yields

$$(0, -f(\bar{x},u) - \sqrt{\varepsilon}\|\bar{x}\|) \in \text{epi } f^*(\cdot,u) + \text{epi } \left(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot,\bar{v}_i)\right)^* + \sqrt{\varepsilon}\mathbb{B} \times \mathbb{R}_+,$$

where $u \in \mathcal{U}$. Therefore, for each $u \in \mathcal{U}$, there exist $u^* \in \mathbb{R}^n, \alpha \geq 0, v_i^* \in \mathbb{R}^n, \beta_i \geq 0, i \in I, w^* \in \mathbb{B}$ and $\eta \in \mathbb{R}_+$ such that

$$(0, -f(\bar{x},u) - \sqrt{\varepsilon}\|\bar{x}\|) \in (u^*, f^*(u^*,u) + \alpha) + \left(\sum_{i=1}^{m} \bar{\lambda}_i (v_i^*, g_i^*(v_i^*,\bar{v}_i) + \beta_i\right)$$

$$+\langle\sqrt{\varepsilon}w^*,\eta\rangle.$$

Hence, we obtain

$$0=u^*+\sum_{i=1}^m\bar{\lambda}_i v_i^*+\sqrt{\varepsilon}w^*, \text{ and}$$

$$-f(\bar{x},u)-\sqrt{\varepsilon}\|\bar{x}\|=f^*(u^*,u)+\alpha+\sum_{i=1}^m\bar{\lambda}_i(g_i^*(v_i^*,\bar{v}_i)+\beta_i)+\eta,\ u\in\mathcal{U}.$$

Thus, for any $x\in\mathbb{R}^n, u\in\mathcal{U}$,

$$\begin{aligned}
f(\bar{x},u)&=-f^*(u^*,u)-\alpha-\sum_{i=1}^m\bar{\lambda}_i(g_i^*(v_i^*,\bar{v}_i)+\beta_i)-\eta-\sqrt{\varepsilon}\|\bar{x}\|\\
&\le-\Big[\langle u^*,x\rangle-f(x,u)\Big]-\sum_{i=1}^m\bar{\lambda}_i g_i^*(v_i^*,\bar{v}_i)-\sqrt{\varepsilon}\|\bar{x}\|\\
&=\Big\langle\sum_{i=1}^m\bar{\lambda}_i v_i^*+\sqrt{\varepsilon}w^*,x\Big\rangle+f(x,u)-\sum_{i=1}^m\bar{\lambda}_i g_i^*(v_i^*,\bar{v}_i)-\sqrt{\varepsilon}\|\bar{x}\|\\
&=\Big\langle\sum_{i=1}^m\bar{\lambda}_i v_i^*,x\Big\rangle+\langle\sqrt{\varepsilon}w^*,x\rangle+f(x,u)-\sum_{i=1}^m\bar{\lambda}_i g_i^*(v_i^*,\bar{v}_i)-\sqrt{\varepsilon}\|\bar{x}\|\\
&\le\Big\langle\sum_{i=1}^m\bar{\lambda}_i v_i^*,x\Big\rangle+\sqrt{\varepsilon}\|w^*\|\|x-\bar{x}+\bar{x}\|+f(x,u)\\
&\quad-\sum_{i=1}^m\bar{\lambda}_i g_i^*(v_i^*,\bar{v}_i)-\sqrt{\varepsilon}\|\bar{x}\|\\
&\le\Big\langle\sum_{i=1}^m\bar{\lambda}_i v_i^*,x\Big\rangle+\sqrt{\varepsilon}\|x-\bar{x}\|+f(x,u)-\sum_{i=1}^m\bar{\lambda}_i g_i^*(v_i^*,\bar{v}_i)\\
&\le\Big\langle\sum_{i=1}^m\bar{\lambda}_i v_i^*,x\Big\rangle+\sqrt{\varepsilon}\|x-\bar{x}\|+f(x,u)\\
&\quad-\Big[\Big\langle\sum_{i=1}^m\bar{\lambda}_i g_i(x,v_i),x\Big\rangle-\sum_{i=1}^m\bar{\lambda}_i g_i(x,\bar{v}_i)\Big]\\
&=f(x,u)+\sqrt{\varepsilon}\|x-\bar{x}\|+\sum_{i=1}^m\bar{\lambda}_i g_i(x,\bar{v}_i).
\end{aligned}$$

Thus, the statement (ii) is satisfied.

[(ii) $\Rightarrow$ (i)] Suppose that there exist $\bar{\lambda}_i,\ge 0, v_i\in\mathcal{V}_i, i\in I$ such that for any $x\in\mathbb{R}^n$ and $u\in\mathcal{U}$,

$$f(x,u)+\sqrt{\varepsilon}\|x-\bar{x}\|+\sum_{i=1}^m\bar{\lambda}_i g_i(x,\bar{v}_i)\ge f(\bar{x},u).$$

So, for any feasible point $x \in K_3$ and $u \in \mathcal{U}$,

$$f(\bar{x}, u) \leq f(x, u) + \sqrt{\varepsilon}\|x - \bar{x}\| + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i)$$
$$\leq f(x, u) + \sqrt{\varepsilon}\|x - \bar{x}\|.$$

Therefore, $\bar{x}$ is an ε-quasi highly robust solution of (UP$_3$). □

Lemma 4.1.3. *Let all assumptions of Lemma 4.1.3 be satisfied. Then, the following statements are equivalent:*

(i) $\bar{x}$ is an ε-quasi highly robust solution for (UP$_3$)*;*

(ii) for any $u \in \mathcal{U}$,

$$(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|) \in \textit{epi}\ f^*(\cdot, u) + \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \textit{epi} \left(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot, \bar{v}_i)\right)^*$$
$$+ \sqrt{\varepsilon}\mathbb{B} \times \mathbb{R}_+.$$

Proof. Clearly, (i) ⇒ (ii) is true by the proof of Lemma 4.1.2. Let us show (ii) ⇒ (i) now. Suppose that for any $u \in \mathcal{U}$,

$$(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|) \in \text{ epi } f^*(\cdot, u) + A + \sqrt{\varepsilon}\mathbb{B} \times \mathbb{R}_+.$$

Then, for any $u \in \mathcal{U}$, we obtain

$$(0, 0) \in \text{ epi } f^*(\cdot, u) + A + \sqrt{\varepsilon}\mathbb{B} \times \Big[f(\bar{x}, u) + \sqrt{\varepsilon}\|\bar{x}\|, +\infty\Big).$$

From the proof of Theorem 4.1.2, we knew that for any $u \in \mathcal{U}$, epi $f^*(\cdot, u) + \sqrt{\varepsilon}\mathbb{B} \times \Big[f(\bar{x}, u) + \sqrt{\varepsilon}\|\bar{x}\|, +\infty\Big) = \text{epi } \Big(f(\cdot, u) + \sqrt{\varepsilon}\|\cdot - \bar{x}\| - f(\bar{x}, u)\Big)^*$. So, for any $u \in \mathcal{U}$, one has

$$(0, 0) \in \text{epi } \Big(f(\cdot, u) + \sqrt{\varepsilon}\|\cdot - \bar{x}\| - f(\bar{x}, u)\Big)^* + A$$
$$= \text{epi } \Big(f(\cdot, u) + \sqrt{\varepsilon}\|\cdot - \bar{x}\| - f(\bar{x}, u)\Big)^* + \text{cl}\,(\text{co } A).$$

Using the Lemma 2.2.38, for any $u \in \mathcal{U}$, we arrive

$$K \subseteq \Big\{x \in \mathbb{R}^n : \Big(f(\cdot, u) + \sqrt{\varepsilon}\|\cdot - \bar{x}\| - f(\bar{x}, u)\Big)(x) \geq 0\Big\}.$$

Thus, for any $u \in \mathcal{U}$ and $x \in K_3$,

$$f(x, u) + \sqrt{\varepsilon}\|x - \bar{x}\| - f(\bar{x}, u) \geq 0.$$

Hence, for any $x \in K_3$ and $u \in \mathcal{U}$,

$$f(\bar{x}, u) \leq f(x, u) + \sqrt{\varepsilon}\|x - \bar{x}\|,$$

which means $\bar{x}$ is an ε-quasi highly robust solution of (UP_3). □

Theorem 4.1.4 (Highly robust approximate optimality theorem). *Let $\bar{x} \in K_3$ and let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i \in I$ be continuous functions such that for each $u \in \mathbb{R}^p, f(\cdot, u)$ is convex on $\mathbb{R}^n$ and for each $v_i \in \mathbb{R}^q, g_i(\cdot, v_i)$ is convex on $\mathbb{R}^n$. Suppose that the constraint qualification (RCCCQ), defined in Definition 4.1.1, holds. Then the following statements are equivalent:*

(i) $\bar{x}$ *is an* ε*-quasi highly robust solution for* (UP_3);

(ii) for any $u \in \mathcal{U}$

$$(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|) \in epi\ f^*(\cdot, u) + \bigcup_{\substack{v_i \in \mathcal{V}_i,\\ \lambda_i \geq 0}} epi\ \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\right)^* + \sqrt{\varepsilon}\mathbb{B} \times \mathbb{R}_+.$$

(iii) there exist $\bar{v}_i \in \mathcal{V}_i$ *and* $\bar{\lambda}_i \geq 0, i \in I$ *such that for any* $u \in \mathcal{U}$,

$$0 \in \partial f(\cdot, u)(\bar{x}) + \sum_{i=1}^{m} \partial(\bar{\lambda}_i g_i(\cdot, v_i))(\bar{x}) + \sqrt{\varepsilon}\mathbb{B} \quad and \quad \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0.$$

Proof. By Lemma 4.1.2 and Lemma 4.1.3, the statement [(i) ⇔ (ii)] is proved.
[(ii) ⇒ (iii)] Suppose that the statement (ii) holds, i.e., for any $u \in \mathcal{U}$,

$$(0, -f(\bar{x}, u) - \sqrt{\varepsilon}\|\bar{x}\|)$$

$$\in \text{ epi } f^*(\cdot,u) + \bigcup_{v_i\in\mathcal{V}_i,\lambda_i\geq 0} \text{epi } \left(\sum_{i=1}^m \lambda_i g_i(\cdot,\bar{v}_i)\right)^* + \sqrt{\varepsilon}\mathbb{B}\times\mathbb{R}_+.$$

Therefore, for any $u\in\mathcal{U}$ there exist $\bar{\lambda}_i\geq 0, \bar{v}_i\in\mathcal{V}_i, i\in I$ such that

$$(0,-f(\bar{x},u)-\sqrt{\varepsilon}\|\bar{x}\|)\in \text{epi } f^*(\cdot,u)+\text{epi } \left(\sum_{i=1}^m \bar{\lambda}_i g_i(\cdot,\bar{v}_i)\right)^* + \sqrt{\varepsilon}\mathbb{B}\times\mathbb{R}_+.$$

By the continuity of $g_i(\cdot,v_i), i\in I$ and Proposition 2.2.35, equivalently, for any $u\in\mathcal{U}$ there exist $\bar{\lambda}_i\geq 0, \bar{v}_i\in\mathcal{V}_i, i\in I$ such that

$$(0,-f(\bar{x},u)-\sqrt{\varepsilon}\|\bar{x}\|)\in \text{epi } f^*(\cdot,u)+\sum_{i=1}^m \text{epi } \left(\bar{\lambda}_i g_i(\cdot,\bar{v}_i)\right)^* + \sqrt{\varepsilon}\mathbb{B}\times\mathbb{R}_+.$$

By Proposition 2.2.34, equivalently, for any $u\in\mathcal{U}$ there exist $\bar{\lambda}_i\geq 0, \bar{v}_i\in\mathcal{V}_i, i\in I$ and $\varepsilon_i\geq 0, i=0,1,\ldots,m$ such that

$$\begin{aligned}
&(0,-f(\bar{x},u)-\sqrt{\varepsilon}\|\bar{x}\|)\\
&\in \bigcup_{\varepsilon_0\geq 0}\left\{(w_0,\langle w_0,\bar{x}\rangle+\varepsilon_0-f(\bar{x},u)) : w_0\in\partial_{\varepsilon_0}f(\cdot,u)(\bar{x})\right\}\\
&\quad+\sum_{i=1}^m\bigcup_{\varepsilon_i\geq 0}\left\{(w_i,\langle w_i,\bar{x}\rangle+\varepsilon_i-\bar{\lambda}_i g_i(\bar{x},\bar{v}_i)) : w_i\in\partial_{\varepsilon_i}\bar{\lambda}_i g_i(\cdot,\bar{v}_i)(\bar{x})\right\}\\
&\quad+\sqrt{\varepsilon}\mathbb{B}\times\mathbb{R}_+.
\end{aligned}$$

Hence, for any $u\in\mathcal{U}$, there exist $\bar{\lambda}_i\geq 0, \bar{v}_i\in\mathcal{V}_i, w_i\in\partial_{\varepsilon_i}\bar{\lambda}_i g_i(\cdot,\bar{v}_i)(\bar{x}), i\in I, w_0\in\partial_{\varepsilon_0}f(\cdot,u)(\bar{x}), w^*\in\mathbb{B}, \eta\in\mathbb{R}_+$ and $\varepsilon_i\geq 0, i=0,1,\ldots,m$ such that

$$\begin{aligned}
&(0,-f(\bar{x},u)-\sqrt{\varepsilon}\|\bar{x}\|)\\
&=(w_0,\langle w_0,\bar{x}\rangle+\varepsilon_0-f(\bar{x},u))+\sum_{i=1}^m(w_i,\langle w_i,\bar{x}\rangle+\varepsilon_i-\bar{\lambda}_i g_i(\bar{x},\bar{v}_i))\\
&\quad+(\sqrt{\varepsilon}w^*,\eta).
\end{aligned}$$

It follows that, for any $u\in\mathcal{U}$, there exist $\bar{\lambda}_i\geq 0, \bar{v}_i\in\mathcal{V}_i, w_i\in\partial_{\varepsilon_i}\bar{\lambda}_i g_i(\cdot,\bar{v}_i)(\bar{x}), i\in I, w_0\in\partial_{\varepsilon_0}f(\cdot,u)(\bar{x}), w^*\in\mathbb{B}, \eta\in\mathbb{R}_+$ and $\varepsilon_i\geq 0, i=0,1,\ldots,m$ such that

$$0=\sum_{i=0}^m w_i+\sqrt{\varepsilon}w^* \quad\text{and}$$

$$-\sqrt{\varepsilon}\|\bar{x} - f(\bar{x},u) = \sum_{i=0}^{m}(\langle w_i,\bar{x}\rangle + \varepsilon_i) - f(\bar{x},u) - \sum_{i=1}^{m}\bar{\lambda}_i g_i(\bar{x},\bar{v}_i) + \eta.$$

Equivalently, for any $u \in \mathcal{U}$, there exist $\bar{\lambda}_i \geq 0, \bar{v}_i \in \mathcal{V}_i, w_i \in \partial_{\varepsilon_i}\bar{\lambda}_i g_i(\cdot,\bar{v}_i)(\bar{x}), i \in I, w_0 \in \partial_{\varepsilon_0} f(\cdot,u)(\bar{x}), w^* \in \mathbb{B}, \eta \in \mathbb{R}_+$ and $\varepsilon_i \geq 0, i = 0,1,\ldots,m$ such that

$$\begin{aligned}
0 &\geq \sum_{i=1}^{m}\bar{\lambda}_i g_i(\bar{x},\bar{v}_i)\\
&= \sqrt{\varepsilon}\|\bar{x}\| + \sum_{i=0}^{m}(\langle w_i,\bar{x}\rangle + \varepsilon_i) + \eta\\
&= \sqrt{\varepsilon}\|\bar{x}\| + \sum_{i=0}^{m}\varepsilon_i - \langle\sqrt{\varepsilon}w^*,\bar{x}\rangle + \eta\\
&\geq \sqrt{\varepsilon}\|\bar{x}\| + \sum_{i=0}^{m}\varepsilon_i - \sqrt{\varepsilon}\|w^*\|\|\bar{x}\| + \eta\\
&\geq \sum_{i=0}^{m}\varepsilon_i \geq 0.
\end{aligned}$$

Hence, the statement (iii) holds.

[(iii) $\Rightarrow$ (ii)] Suppose that the statement (iii) holds. Then for any $u \in \mathcal{U}$, there exist $\bar{\lambda}_i \geq 0, \bar{v}_i \in \mathcal{V}_i, w_i \in \partial(\bar{\lambda}_i g_i)(\cdot,\bar{v}_i)(\bar{x}), i \in I, w_0 \in \partial f(\cdot,u)(\bar{x})$ and $w^* \in \mathbb{B}$ such that

$$0 = w_0 + \sum_{i=1}^{m}\bar{\lambda}_i w_i + \sqrt{\varepsilon}w^* \quad\text{and}\quad \sum_{i=1}^{m}\bar{\lambda}_i g_i(\bar{x},\bar{v}_i) = 0.$$

This, together with the definition of the subdifferential of $f(\cdot,u)$, yields that for any $x \in K_3, u \in \mathcal{U}$,

$$\begin{aligned}
f(x,u) - f(\bar{x},u) &\geq \langle w_0, x-\bar{x}\rangle\\
&= \left\langle -\sum_{i=1}^{m}\bar{\lambda}_i w_i - \sqrt{\varepsilon}w^*, x-\bar{x}\right\rangle\\
&= -\left\langle \sum_{i=1}^{m}\bar{\lambda}_i w_i, x-\bar{x}\right\rangle - \langle\sqrt{\varepsilon}w^*, x-\bar{x}\rangle\\
&\geq -\sum_{i=1}^{m}\bar{\lambda}_i g_i(x,\bar{v}_i) + \sum_{i=1}^{m}\bar{\lambda}_i g_i(\bar{x},\bar{v}_i) - \sqrt{\varepsilon}\|w^*\|\|x-\bar{x}\|
\end{aligned}$$

$$\geq -\sqrt{\varepsilon}\|x-\bar{x}\|.$$

Therefore, for any $x \in K_3, u \in \mathcal{U}$,

$$f(\bar{x},u) \leq f(x,u) + \sqrt{\varepsilon}\|x-\bar{x}\|,$$

which means $\bar{x}$ is an ε-quasi highly robust solution of (UP_3). Thus, by Lemma 4.1.3, the statement (ii) holds. □

Corollary 4.1.5. *Let $\bar{x} \in K_3$ and let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i \in I$ be continuous functions such that for each $u \in \mathbb{R}^p, f(\cdot,u)$ is convex on $\mathbb{R}^n$ and for each $v_i \in \mathbb{R}^q, g_i(\cdot,v_i)$ is convex on $\mathbb{R}^n$. Suppose that for each $x \in \mathbb{R}^n, g_i(x,\cdot)$ is concave on $\mathcal{V}_i, i \in I$ and there exists $y \in \mathbb{R}^n$ such that $g_i(y,v_i) < 0, \forall v_i \in \mathcal{V}_i, i \in I$. Then the following statements are equivalent:*

(i) $\bar{x}$ *is an ε-quasi highly robust solution for* (UP_3)*;*

(ii) for any $u \in \mathcal{U}$,

$$(0, -f(\bar{x},u) - \sqrt{\varepsilon}\|\bar{x}\|) \in epi\ f^*(\cdot,u) + \bigcup_{\substack{v_i \in \mathcal{V}_i,\\ \lambda_i \geq 0}} epi\ \left(\sum_{i=1}^{m} \lambda_i g_i(\cdot,v_i)\right)^* + \sqrt{\varepsilon}\mathbb{B} \times \mathbb{R}_+.$$

(iii) there exist $\bar{v}_i \in \mathcal{V}_i$ and $\bar{\lambda}_i \geq 0, i \in I$ such that for any $u \in \mathcal{U}$,

$$0 \in \partial f(\cdot,u)(\bar{x}) + \sum_{i=1}^{m} \partial(\bar{\lambda}_i g_i(\cdot,v_i))(\bar{x}) + \sqrt{\varepsilon}\mathbb{B} \quad and \quad \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x},\bar{v}_i) = 0.$$

Proof. It follows from Proposition (2.2.37) and Proposition (2.2.36) that the constraint qualification (RCCCQ), defined in Definition 4.1.1, holds. Then, all conditions of Theorem 4.1.4 are satisfied and so we finish this proof. □

4.2 Duality theorems for ε-quasi highly robust solutions

In this section, we formulate a Wolfe type dual problem (UD) for the primal uncertain convex optimization problem (UP_3). Then we propose a highly robust approximate weak duality theorem and a highly robust approximate strong duality between the primal problem and its Wolfe type dual problem. In addition, an example is given for supporting and illustrating the results.

Now we formulate a Wolfe dual problem (UD) for (UP_3) as follows:

$$\begin{aligned} &\text{Maximize} \quad f(y,u)+\sum_{i=1}^{m}\lambda_i g_i(y,v_i) \\ &\text{subject to} \quad 0\in\partial f(\cdot,u)(y)+\sum_{i=1}^{m}\partial\left(\lambda_i g_i\right)(\cdot,v_i)(y)+\sqrt{\varepsilon}\mathbb{B}, \\ &\qquad\qquad\quad u\in\mathcal{U},\lambda_i\geq 0,v_i\in\mathcal{V}_i,i\in I,\varepsilon\geq 0. \end{aligned} \tag{UD}$$

Let $K_D := \Big\{(y,v,\lambda)\in\mathbb{R}^n\times\mathcal{V}\times\mathbb{R}^m_+ : 0\in\partial f(\cdot,u)(y)+\sum_{i=1}^{m}\partial\left(\lambda_i g_i\right)(\cdot,v_i)(y)+\sqrt{\varepsilon}\mathbb{B},\lambda_i\geq 0,v_i\in\mathcal{V}_i,i\in I\Big\}$, then it is termed as the robust feasible set of the dual problem (UD).

Definition 4.2.1. Let $\varepsilon\geq 0$ be given, then $(\bar{y},\bar{\lambda},\bar{v})$ is said to be an ε-quasi highly robust solution of the dual problem (UD) if for any robust feasible solution $(y,v,\lambda)\in K_D$ and $u\in\mathcal{U}$,

$$f(\bar{y},u)+\sum_{i=1}^{m}\bar{\lambda}_i g_i(\bar{y},\bar{v}_i)\geq f(y,u)+\sum_{i=1}^{m}\lambda_i g_i(y,v_i)-\sqrt{\varepsilon}\|\bar{y}-y\|.$$

Let us move on the highly robust approximate weak duality theorem and the highly robust approximate strong duality theorem for highly robust solutions. The following theorem proposes a highly robust approximate weak duality between the primal problem and its Wolfe type dual problem.

Theorem 4.2.2 (Highly robust approximate weak duality theorem). *Let $\varepsilon\geq 0$ be given. For any $(x,u)\in K_3\times\mathcal{U}$ and any $(y,v,\lambda)\in K_D$,*

$$f(x,u)\geq f(y,u)+\sum_{i=1}^{m}\lambda_i g_i(y,v_i)-\sqrt{\varepsilon}\|x-y\|.$$

Proof. Let $(x,u)\in K_3\times\mathcal{U}$ and $(y,v,\lambda)\in K_D$, be arbitrary. Then, there exist $w_0\in\partial f(\cdot,u)(y), w_i\in\partial\left(\lambda_i g_i(\cdot,v_i)\right)(y), i\in I$ and $w^*\in\mathbb{B}$ such that $w_0+\sum_{i=1}^{m}w_i+\sqrt{\varepsilon}w^*=0$.

Hence, we obtain

$$
\begin{aligned}
& f(x,u) - f(y,u) - \sum_{i=1}^{m} \lambda_i g_i(y,v_i) \\
& \geq \langle w_0, x-y \rangle - \sum_{i=1}^{m} \lambda_i g_i(y,v_i) \\
& = \Big\langle -\sum_{i=1}^{m} w_i - \sqrt{\varepsilon} w^*, x-y \Big\rangle - \sum_{i=1}^{m} \lambda_i g_i(y,v_i) \\
& = -\Big\langle \sum_{i=1}^{m} w_i, x-y \Big\rangle - \langle \sqrt{\varepsilon} w^*, x-y \rangle - \sum_{i=1}^{m} \lambda_i g_i(y,v_i) \\
& \geq -\sum_{i=1}^{m} \lambda_i g_i(x,v_i) + \sum_{i=1}^{m} \lambda_i g_i(y,v_i) - \langle \sqrt{\varepsilon} w^*, x-y \rangle - \sum_{i=1}^{m} \lambda_i g_i(y,v_i) \\
& = -\sum_{i=1}^{m} \lambda_i g_i(x,v_i) - \sqrt{\varepsilon} \|x-y\| \\
& \geq -\sqrt{\varepsilon} \|x-y\|.
\end{aligned}
$$

Thus, one has $f(x,u) \geq f(y,u) + \sum_{i=1}^{m} \lambda_i g_i(y,v_i) - \sqrt{\varepsilon}\|x-y\|$ as desired. □

The following highly robust approximate strong duality theorem holds under the constraint qualification (RCCCQ).

Theorem 4.2.3 (Highly robust approximate strong duality theorem). *Let* $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}, g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, i \in I$ *be continuous functions such that for each* $u \in \mathbb{R}^p, f(\cdot, u)$ *is convex on* $\mathbb{R}^n$ *and for each* $v_i \in \mathbb{R}^q, g_i(\cdot, v_i)$ *is convex on* $\mathbb{R}^n$. *Suppose that the constraint qualification (RCCCQ), defined in Definition 4.1.1, holds. If* $\bar{x} \in K_3$ *is an* ε*-quasi highly robust solution of the primal problem* (UP$_3$)*, then there exist* $\bar{\lambda} \in \mathbb{R}^m_+$ *and* $\bar{v} \in \mathbb{R}^q$ *such that* $(\bar{x}, \bar{v}, \bar{\lambda})$ *is an* ε*-quasi highly robust solution of the dual problem* (UD).

Proof. Let $\bar{x} \in K_3$ be an ε-quasi highly robust solution of (UP$_3$). Hence, by Theorem 4.1.4, for any $u \in \mathcal{U}$, there exist $\bar{v}_i \in \mathcal{V}_i, \bar{\lambda}_i \geq 0, i \in I$ such that

$$
0 \in \partial f(\cdot, u)(\bar{x}) + \sum_{i=1}^{m} \partial(\bar{\lambda}_i g_i(\cdot, v_i))(\bar{x}) + \sqrt{\varepsilon}\mathbb{B} \quad \text{and} \quad \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = 0.
$$

This means $(\bar{x}, \bar{v}, \bar{\lambda})$ is a feasible solution of (UD), i.e., $(\bar{x}, \bar{v}, \bar{\lambda}) \in K_D$. By Theorem 4.2.2, for any $u \in \mathcal{U}$ and $(y, v, \lambda) \in K_D$, we have

$$f(\bar{x}, u) \geq f(y, u) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\varepsilon}\|x - y\|.$$

It follows that for any $u \in \mathcal{U}$ and $(y, v, \lambda) \in K_D$,

$$\begin{aligned} & f(\bar{x}, u) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) - \Big[f(y, u) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i)\Big] \\ & \geq -\sqrt{\varepsilon}\|x - y\| + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \\ & = -\sqrt{\varepsilon}\|x - y\|. \end{aligned}$$

It yields, for any $u \in \mathcal{U}$ and $(y, v, \lambda) \in K_D$,

$$f(\bar{x}, u) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \geq f(y, u) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\varepsilon}\|x - y\|.$$

Therefore, $(\bar{x}, \bar{v}, \bar{\lambda})$ is an ε-quasi highly robust solution of (UD) as desired. □

The following example illustrates Theorem 4.2.2 and Theorem 4.2.3.

Example 4.2.4. *Let $f : \mathbb{R}^2 \times \mathcal{U} \to \mathbb{R}$ and $g : \mathbb{R}^2 \times \mathcal{V} \to \mathbb{R}$ be defined by*

$$f(x, u) = ux_1 + x_2^2 \quad \text{and} \quad g(x, v) = x_1^2 - vx_1,$$

where $\mathcal{U} := [-1, 1]$ and $\mathcal{V} := \mathbb{R}$. Consider the following convex optimization problem with uncertainty:

$$\text{Minimize } f(x, u) \quad \text{subject to} \quad g(x, v) \leq 0, \; v \in \mathcal{V}. \tag{4.2.1}$$

Observe that the robust feasible set of (4.2.1) is the set

$$\begin{aligned} K_3 : &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - vx_1 \leq 0, \, v \in \mathcal{V}\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \in \mathbb{R}\}, \end{aligned}$$

while the set of all ε-quasi highly solution of (4.2.1) is

$$S^{HR} := \Big\{(x_1, x_2) \in K_3 : ux_1 + x_2^2 \leq uy_1 + y_2^2 + \sqrt{\varepsilon}\|(y_1, y_2) - (x_1, x_2)\|,$$

$$
\begin{aligned}
&\quad (y_1, y_2) \in K_3, u \in \mathcal{U}\Big\} \\
&= \Big\{(x_1, x_2) \in K_3 : u(0) + x_2^2 \leq u(0) + y_2^2 \\
&\quad + \sqrt{\varepsilon}\|(0, y_2) - (0, x_2)\|,\ y_2 \in \mathbb{R}, u \in \mathcal{U}\Big\} \\
&= \Big\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, \frac{-\sqrt{\varepsilon}}{2} \leq x_2 \leq \frac{\sqrt{\varepsilon}}{2}\Big\}.
\end{aligned}
$$

We can prove that the (RCCCQ) holds for (4.2.1). *To show the cone*

$$
\bigcup_{v \in \mathcal{V}, \lambda \geq 0} epi\ (\lambda g(\cdot, v))^*
$$

is closed and convex, let $v \in \mathcal{V}$ *and* $\lambda \geq 0$ *be given. Then, we have*

$$
(\lambda g(\cdot, v))^*(x^*) = \begin{cases} 0; & \lambda = 0, \\ \dfrac{(x^* + \lambda v)^2}{4\lambda}; & \lambda > 0. \end{cases}
$$

So, it can be seen that

$$
\begin{aligned}
\bigcup_{v \in \mathcal{V}, \lambda \geq 0} epi\ (\lambda g(\cdot, v))^* &= \Big(\{0\} \times \mathbb{R}_+\Big) \cup \bigcup_{v \in \mathcal{V}, \lambda > 0} \Big\{(x^*, \alpha) : \alpha \geq \frac{(x^* + \lambda v)^2}{4\lambda}\Big\} \\
&= \mathbb{R} \times \mathbb{R}_+.
\end{aligned}
$$

Next, we formulate a dual problem for (4.2.1) *as follows:*

$$
\begin{aligned}
&Maximize\ f(y_1, y_2, u) + \lambda g(y_1, y_2, v) \\
&\ subjet\ to\ 0 \in \partial f(\cdot, u)(y_1, y_2) + \partial(\lambda g(\cdot, v))(y_1, y_2) + \sqrt{\varepsilon}\mathbb{B}, \\
&\qquad u \in \mathcal{U}, \lambda \geq 0, v \in \mathcal{V}, \varepsilon \geq 0.
\end{aligned} \tag{4.2.2}
$$

Then the set

$$
\begin{aligned}
K_D := \Big\{&((y_1, y_2), v, \lambda) : y_1 \in \mathbb{R}, (0,0) \in \partial f(\cdot, u)(y_1, y_2) + \partial(\lambda g(\cdot, v))(y_1, y_2) \\
&+ \sqrt{\varepsilon}\mathbb{B}, u \in [-1, 1], \lambda \geq 0, v \in \mathbb{R}, \varepsilon \geq 0\Big\}
\end{aligned}
$$

is the robust feasible set of (4.2.2). *We can calculate the robust feasible set* K_D *as follows:*

$$
K_D := \Big\{((y_1, y_2), v, \lambda) : (0,0) \in \partial f(\cdot, u)(y_1, y_2) + \partial(\lambda g(\cdot, v))(y_1, y_2)
$$

$$+\sqrt{\varepsilon}\mathbb{B}, u\in[-1,1], \lambda\geq 0, v\in\mathbb{R}, \varepsilon\geq 0\Big\}$$

$$=\Big\{((y_1,y_2),v,\lambda) : y_1\in\mathbb{R}, u+2\lambda y_1-\lambda v+\sqrt{\varepsilon}w_1=0,$$

$$y_2=-\frac{\sqrt{\varepsilon}}{2}, w_1^2+w_2^2\leq 1, u\in[-1,1], \lambda\geq 0, v\in\mathbb{R}, \varepsilon\geq 0\Big\}.$$

Observe that for any $u\in\mathcal{U}, (x_1,x_2)\in K_3$ *and* $(y_1,y_2,v,\lambda)\in K_D$,

$$\begin{aligned}
& f(x_1,x_2,u)-\Big[f(y_1,y_2,u)+\lambda g(y_1,y_2,v)-\sqrt{\varepsilon}\|(x_1,x_2)-(y_1,y_2)\|\Big] \\
&= x_2^2-\Big[uy_1+y_2^2+\lambda y_1^2-\lambda v y_1-\sqrt{\varepsilon}\sqrt{y_1^2+(x_2-y_2)^2}\Big] \\
&= x_2^2-y_2^2-\lambda y_1^2+(\lambda v-u)y_1+\sqrt{\varepsilon}\sqrt{y_1^2+(x_2-y_2)^2} \\
&= x_2^2-\frac{\varepsilon}{4}w_2^2+\lambda y_1^2+\sqrt{\varepsilon}w_1y_1+\sqrt{\varepsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2} \\
&= (x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2+\lambda y_1^2+\sqrt{\varepsilon}\Big[w_1y_1-w_2(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)\Big] \\
&\quad +\sqrt{\varepsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2} \\
&\geq \sqrt{\varepsilon}\Big[w_1y_1-w_2(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)\Big]+\sqrt{\varepsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2} \\
&\geq -\sqrt{\varepsilon}\sqrt{(-w_1)^2+w_2^2}\sqrt{y_1^2+(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2}+\sqrt{\varepsilon}\sqrt{y_1^2+(x_2+\frac{\sqrt{\varepsilon}}{2}w_2)^2} \\
&\geq 0.
\end{aligned}$$

Hence, for any $u\in\mathcal{U}, (x_1,x_2)\in K_3$ *and* $(y_1,y_2,v,\lambda)\in K_D$,

$$f(x_1,x_2,u)\geq f(y_1,y_2,u)+\lambda g(y_1,y_2,v)-\sqrt{\varepsilon}\|(x_1,x_2)-(y_1,y_2)\|,$$

and so the conclusion of Theorem 4.2.2(The highly robust approximate weak duality theorem) holds. Let $(\bar{x}_1,\bar{x}_2)\in K$ *be an* ε*-quasi highly robust solution for* (UP$_3$)*. So,* $\bar{x}_1=0$ *and* $-\frac{\sqrt{\varepsilon}}{2}\leq\bar{x}_2\leq\frac{\sqrt{\varepsilon}}{2}$*. By taking* $\bar{\lambda}:=\sqrt{\varepsilon}$ *and* $\bar{v}=\frac{u}{\sqrt{\varepsilon}}+w_1$*, we can see that* $((\bar{x}_1,\bar{x}_2),\bar{v},\bar{\lambda})\in K_D$*. Indeed,* $\bar{\lambda}\geq 0, \bar{v}\in\mathbb{R}$ *and*

$$u+2\bar{\lambda}\bar{x}_1-\bar{\lambda}\bar{v}+\sqrt{\varepsilon}w_1=u-\sqrt{\varepsilon}\left(\frac{u}{\sqrt{\varepsilon}}+w_1\right)+\sqrt{\varepsilon}w_1=0.$$

Besides, for any $u\in\mathcal{U}$ *and* $(y_1,y_2,v,\lambda)\in K_D$,

$$f(\bar{x}_1,\bar{x}_2,u)+\bar{\lambda}g(\bar{x}_1,\bar{x}_2,\bar{v})-\Big[f(y_1,y_2,u)+\lambda g(y_1,y_2,v)\Big]$$

$$\geq -\sqrt{\varepsilon}\|(x_1, x_2) - (y_1, y_2)\| + \bar{\lambda} g(\bar{x}_1, \bar{x}_2, \bar{v})$$
$$= -\sqrt{\varepsilon}\|(x_1, x_2) - (y_1, y_2)\|.$$

Therefore, $(\bar{x}_1, \bar{x}_2)$ *is an* ε*-quasi highly robust solution of* (4.2.2)*, and then the conclusion of Theorem 4.2.3 (The highly robust approximate strong duality theorem) holds.*

CHAPTER V

CONCLUSION

In this chapter, we conclude again that what we get from the results.

5.1 Robust weak sharp solutions in uncertain convex optimization problems

In the section, we considered uncertain convex optimization problems involing convex objective functions and D-convex constraint functions. First of all, we introduced the notion of a robust weak sharp solution to an uncertain convex optimization problem. Then, optimality conditions for the robust weak sharp solutions and characterizations of the sets of all the robust weak sharp solutions of the problem were obtained. Finally, we applied the results to an uncertain convex muti-objective optimization problem and obtain optimality conditions for robust weak sharp weakly efficient solutions in the multi-objective optimization problem.

5.2 Robust weak sharp solutions in uncertain nonconvex optimization problems

In the section, we investigated an uncertain nonsmooth optimization problem involving nonsmooth real-valued functions. Firstly, we introduced the notion of a robust weak sharp solution to the considereed problem. Then, some necessary optimality condition for the robust weak sharp solutions of the problem under a constraint qualification were established. Finally, by mean of the robust version of (KKT) conditions, which were introduced here, sufficient optimality conditions for robust weak sharp solutions of the considered uncertain optimization problem were obtained. Moreover, some examples were presented for illustating the results.

5.3 Optimality conditions for ε-quasi-highly robust solutions in uncertain convex optimization problems

In the section, we considered an uncertain convex optimization problem with data uncertainty in both objective and constraint functions. The robust approximate solutions for the uncertain convex problem were investigated. First of all, the notion of an ε-quasi highly robust solution for the uncertain convex optimization problem was introduced. Then, the highly robust approximate optimality theorems for ε-quasi highly robust solutions of the considered problem were established by means of a robust optimization approach.

5.4 Duality theorems for ε-quasi-highly robust solutions in uncertain convex optimization problems

In the section, we investigated the duality theorems for approximate robust solution in uncertain convex optimization problems. Firstly, we formulated a Wolfe type dual problem for the primal uncertain convex optimization problem, which was studied in previous. Secondly, we proposed a highly robust approximate weak duality theorem between the primal and its Wolfe type dual problem. Finally, a highly robust approximate strong duality between those problems were presented as well. In addition, an example was given for supporting and illustraing the results.

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