ITERATIVE APPROXIMATIONS FOR SOLUTIONS OF GENERALIZED MONOTONE EQUILIBRIUM PROBLEMS AND FIXED POINTS OF QUASI-NONEXPANSIVE MAPPINGS

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ABSTRACT

In this thesis, we separate into two parts. Firstly, we present some iterative methods for solving pseudomonotone equilibrium problems and fixed point problems of quasi-nonexpansive mappings, that is, two shrinking extragradient algorithms and two hybrid extragradient algorithms are proposed for solving the considered problem. Secondary, we introduce a new extragradient algorithm for finding a solution to the split equilibrium and fixed point problems for pseudomonotone bifunctions and nonexpansive mappings. Additionally, the application of the split equilibrium and fixed point problems is also discussed. Finally, some numerical experiments and comparisons of the introduced algorithms with well-known algorithms are considered.

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CHAPTER I

INTRODUCTION

The equilibrium problem started to obtain interest after the publication of the paper of Blum and Oettli [1]. It had been used for studying physics, chemistry, engineering, and economics in different mathematical models such as mechanical structures, chemical processes, the distribution of traffic over computer and telecommunication networks. In particular, generalized monotone equilibrium problems are a very useful tool for constructing mathematical models for several problems such as Nash-Cournot equilibrium problems and Nash-Cournot oligopolistic equilibrium problems of electricity markets, see [2, 3]. Additionally, the equilibrium problems include other important problems, such as optimization problems, variational inequality problems, minimax problems, Nash equilibrium problems, saddle point problems, and fixed point problems, see [1, 4, 5, 6], and the references therein. Therefore, iterative methods for approximating solutions of equilibrium problems have been studied by a number of researchers. In the most appeared researches, the proposed method for solving the equilibrium problem is the proximal point method. This method was first introduced by Martinet [7] for solving variational inequality problems and it was extended by Moudafi [8] to monotone equilibrium problems, where a regularized equilibrium problem needs to solved at each iteration. However, the proximal point method cannot be applied to generalized monotone equilibrium problems, namely pseudomonotone equilibrium problems. To overcome this drawback, the extragradient method is introduced for solving pseudomonotone equilibrium problems instead of the proximal point method. The extragradient method was first introduced by Korpelevich [9] for solving saddle point problems and it was extended by Noor [10] to pseudomonotone variational inequality problems. After that, Tran et al. [2] proposed the extragradient method for solving pseudomonotone equilibrium problems. The weak convergence of the proposed method is shown.

On the other hand, fixed point theory plays an important role in engineering, physics, computer science, economics, and telecommunication. Besides, the fixed point problem has applications in many problems, such as null point problem, variational inequality problem, equilibrium problem, and optimization problem, see [11, 12, 13, 14], and the references therein. In particular, the fixed point problems of quasi-nonexpansive and nonexpansive mappings can be applied to another problem such as the network bandwidth allocation problem that can be translated into the convex optimization problem over the fixed point sets of quasi-nonexpansive and nonexpansive mappings, see [15]. Therefore, many researchers devoted their efforts to approximate fixed points by using iterative methods. A famous iterative method for finding fixed points of a nonexpansive mapping is Mann iterative method, see [16]. However, this method has only weak convergence, in general. In order to obtain a strong convergence result for Mann iterative method, Nakajo and Takahashi [17] proposed the hybrid method for finding fixed points of a nonexpansive mapping. Moreover, it was noted that Mann iterative method may not, in general, be applicable for finding fixed points of a Lipschitz pseudocontractive mapping in a Hilbert space. To overcome this drawback, Ishikawa [18] proposed the method, called Ishikawa iterative method, for finding fixed points of a Lipschitz pseudocontractive mapping. By using the idea of Ishikawa iterative method, Takahashi et al. [19] proposed the hybrid method, called the shrinking projection method, which is different from Nakajo and Takahashi's method in [17] for finding fixed points of a nonexpansive mapping. The strong convergence of the proposed method is presented.

Furthermore, the splitting type problem is the problem in which the image of a solution to one problem under a given bounded linear operator is a solution to another problem. The outstanding form of this problem is the split feasibility problem which is finding a point in closed convex sets. Many important problems arising from real-world problems can be formulated as the split feasibility problems which had been used for studying signal processing, medical image reconstruction, intensity-modulated radiation therapy, sensor networks, and data compression, see [20, 21, 22, 23] and the references therein. Also, many researchers have studied and introduced several problems, as generalizations of the split feasibility problems, such as split variational inequality problems, split common fixed point problems, and split equilibrium problems. The popular proposed method for solving the split feasibility problems is the CQ algorithm. This algorithm was first introduced by Byrne [24] for solving the split feasibility problems in finite dimensional Hilbert spaces and it was extended by Xu [25] for solving the split feasibility problems in infinite dimensional Hilbert spaces. Recently, Dinh et al. [26] considered both the split equilibrium problems and the split fixed point problems. They proposed some algorithms and proved convergence theorems for a solution of the considered problems.

Motivated by the significant of these problems, in this thesis, we are going to establish strong convergence theorems for a solution to the problems in real Hilbert spaces. Firstly, we consider the pseudomonotone equilibrium problems and fixed point of quasi-nonexpansive mappings problems. Secondary, we consider the split equilibrium and fixed point problems. Some algorithms will be introduced for finding the solutions of the considered problems. Finally, some numerical examples will be considered and the introduced algorithms will be discussed and compared with well-known algorithms.

In the following, we describe the contents of this thesis.

Chapter I. This Chapter is an introduction to the research problems.

Chapter II. We will present some definitions and properties that will be used subsequently.

Chapter III. In this chapter, we study iterative methods for solving the pseudomonotone equilibrium problems and fixed point of quasi-nonexpansive mappings problems. In section 3.1, we consider the equilibrium problems and fixed point problems. In section 3.2, we present two iterative algorithms for finding a common solution of the pseudomonotone equilibrium problems and fixed point of quasi-nonexpansive mappings problems by using shrinking projection and extragradient methods. Besides, we show the strong convergence theorems of the introduced algorithms. Some numerical experiments are presented to demonstrate the introduced algorithms. In section 3.3, we present two iterative algorithms for finding the closest point to the intersection of the solution set of the pseudomonotone equilibrium problems and fixed point of quasi-nonexpansive mappings problems by using hybrid and extragradient methods. The strong convergence theorems of the introduced algorithms are proved. Finally, we discuss the performance of introduced algorithms and compare them with well-known algorithms via the numerical experiments.

Chapter IV. In this chapter, we study the iterative method for solving the split equilibrium and fixed point problems. In section 4.1, we consider the split equilibrium and fixed point problems. In section 4.2, we present a new iterative algorithm for finding a solution to the split equilibrium and fixed point problems. The strong convergence theorem of the introduced algorithm is shown. We also provide the applications of the considered problems and some numerical experiments.

Chapter V. We give the concluding research.

CHAPTER II

PRELIMINARIES

In this chapter, we will provide some definitions, properties and useful results that will be used in subsequent chapters. From now on, the set of all natural numbers and the set of all real numbers will be denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively.

2.1 Hilbert spaces

In this section, we will present some definitions and theorems which are concerned with Hilbert spaces.

Definition 2.1.1. A *vector space* or *linear space* X over $\mathbb{R}$ is a set X with the operation called *vector addition* defined on $X \times X$ to X given by $(x, y) \to x + y$ and an operation called *scalar multiplication* defined on $\mathbb{R} \times X$ to X given by $(\alpha, x) \to \alpha x$ satisfy the following conditions: for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$,

(1) $x + y = y + x$.

(2) $(x + y) + z = x + (y + z)$.

(3) there exists an element $0 \in X$ called the *zero vector* such that $x + 0 = x$, for all $x \in X$.

(4) for each $x \in X$, there exists an element $-x \in X$ called the *additive inverse* of x such that $x + (-x) = 0$.

(5) $\alpha(x + y) = \alpha x + \alpha y$.

(6) $(\alpha + \beta)x = \alpha x + \beta x$.

(7) $(\alpha\beta)x = \alpha(\beta x)$.

(8) $1x = x$.

The elements of a vector space X are called *vectors* and the elements of $\mathbb{R}$ are called *scalars*.

We now consider the notions of norm by the following definition.

Definition 2.1.2. Let X is a real vector space. A norm on X is a function $\|\cdot\| : X \to \mathbb{R}$ with the following conditions: for all $x, y \in X$ and $\alpha \in \mathbb{R}$,

(1) $\|x\| \geq 0$.

(2) $\|x\| = 0$ if and only if $x = 0$.

(3) $\|\alpha x\| = |\alpha|\|x\|$.

(4) $\|x + y\| \leq \|x\| + \|y\|$.

A real vector space with a norm defined on it is called a *normed space*.

In what follows, we recall some basic definitions which are related to normed space.

Definition 2.1.3. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a normed space X is said to be *bounded* if there exists a positive number M such that $\|x_k\| \leq M$, for all $k \in \mathbb{N}$.

Definition 2.1.4. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a normed space X is said to *converges* (strongly) to $x \in X$ if $\lim_{k\to\infty} \|x_k - x\| = 0$. In this case, we write $\lim_{k\to\infty} x_k = x$ or $x_k \to x$, as $k \to \infty$. The element x is called the limit of the sequence $\{x_k\}_{k\in\mathbb{N}}$.

Definition 2.1.5. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a normed space X is said to be *Cauchy* if $\lim_{m,k\to\infty} \|x_m - x_k\| = 0$.

Definition 2.1.6. A normed space X is said to be *complete* if every Cauchy sequence in X converges to an element of X.

Next, we consider the concepts of inner product by the following definition.

Definition 2.1.7. Let X is a real vector space. An inner product on X is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ with the following conditions: for all $x, y, z \in X$ and $\alpha \in \mathbb{R}$,

(1) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.

(2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

(3) $\langle x, y \rangle = \langle y, x \rangle$.

(4) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

A real vector space with an inner product defined on it is called an *inner product space.*

The fact below confirms that an inner product naturally induces the norm.

For an inner product space X, the function $\|\cdot\| : X \to \mathbb{R}$ defined by

$$\|x\| = \sqrt{\langle x, x \rangle}, \text{ for all } x \in X,$$

is a *norm* on X. Consequently, an inner product space is a normed space. From now on, the notation $\langle \cdot\, , \cdot \rangle$ and $\|\cdot\|$ will be used in this thesis.

The following theorem shows that inner products satisfy an important inequality which is known as Cauchy-Schwarz inequality.

Theorem 2.1.8 (Cauchy-Schwarz inequality). *Let X be an inner product space. Then the following holds:*

$$|\langle x, y \rangle| \leq \|x\| \|y\|, \text{ for all } x, y \in X.$$

Proof. See [27, Lemma 3.2-1]. □

We are in a position to propose the definition of a Hilbert space as follows.

Definition 2.1.9. A complete inner product space is called a *Hilbert space.*

We now recall some definitions and interesting properties of weak convergence in a Hilbert space.

Definition 2.1.10. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a Hilbert space H is said to *converges weakly* to $x \in H$ if for any $y \in H$, $\langle x_k, y\rangle \to \langle x, y\rangle$, as $k \to \infty$. In this case, we write $x_k \rightharpoonup x$, as $k \to \infty$. The element x is called a weak limit of the sequence $\{x_k\}_{k\in\mathbb{N}}$.

Theorem 2.1.11. *A strong convergent sequence in a Hilbert space is weak convergent with the same limit. In particular, a weakly convergent sequence of a finite dimensional Hilbert space is strong convergent with the same limit.*

Proof. See [27, Theorem 4.8-4]. □

A Hilbert space has an important property that is presented in the following theorem.

Theorem 2.1.12. *Every bounded sequence in a Hilbert space possesses a weakly convergent subsequence.*

Proof. See [28, Lemma 2.37]. □

We end this section by recalling some basic facts in the functional analysis which are needed in the sequel. Let H be a Hilbert space and let $x, y \in H$, we know that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \tag{2.1.1}$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \tag{2.1.2}$$

and

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.1.3}$$

see [29, 30].

Theorem 2.1.13. *Let H be a Hilbert space, let x and y be elements in H and let $\lambda \in \mathbb{R}$. Then*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Proof. See [31, Theorem 6.1.2]. □

2.2 Convexity and continuity

In this section, we provide some definitions and properties which are related to the convexity and the continuity. From now on, the real Hilbert space will be denoted by H.

2.2.1 Convex set

We recall the convexity of a set by the following definition.

Definition 2.2.1. A subset C of H is said to be *convex* if $\lambda x + (1-\lambda)y \in C$, for all $x, y \in C$ and for all $\lambda \in (0, 1)$.

The theorem below shows some useful properties for convexity of intersections.

Theorem 2.2.2. *Let $\{C_i : i \in I\}$ be an arbitrary collection of convex sets in H. Then, their intersection $\bigcap_{i\in I} C_i$ is also convex.*

Proof. See [28, Example 3.2(iv)]. □

2.2.2 Convex function

We consider some definitions which are concerned with the convexity of a real-valued function.

Definition 2.2.3. A function $f : H \to \mathbb{R}$ is said to be:

(i) *convex* if

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y),$$

for all $x, y \in H$ and for all $\lambda \in (0, 1)$.

(ii) $\alpha-$*strongly convex*, where $\alpha > 0$ or, shortly, *strongly convex* if

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda)f(y) - \frac{1}{2}\alpha\lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and for all $\lambda \in (0, 1)$.

The following theorem gives some interesting properties of the convex functions.

Theorem 2.2.4. *(see [29]) Let $f_i : H \to \mathbb{R}, i \in I$, be convex functions. Then, the function $f := \max_{i\in I} f_i$ is convex.*

2.2.3 Continuity

Firstly, we will present the semicontinuity of a function on a Hilbert space.

Definition 2.2.5. A function $f : H \to \mathbb{R}$ is said to be *upper semicontinuous* on H if $\{x \in H : f(x) \geq \alpha\}$ is a closed set for all $\alpha \in \mathbb{R}$.

Definition 2.2.6. A function $f : H \to \mathbb{R}$ is said to be *lower semicontinuous* on H if $\{x \in H : f(x) \leq \alpha\}$ is a closed set for all $\alpha \in \mathbb{R}$.

In order to obtain some basic concepts of semicontinuity, we denote the extended real number $[-\infty, +\infty] := \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 2.2.7. [28] Let D be a subset of $[-\infty, +\infty]$. A number $a \in [-\infty, +\infty]$ is the (necessarily unique) *infimum* (or the greatest lower bound) of D if it is a lower bound of D and if, for every lower bound $\bar{a}$ of D, we have $\bar{a} \leq a$. This number is denoted by $\inf(D)$. The *supremum* (or least upper bound) of D is $\sup(D) := -\inf\{-b : b \in D\}$.

Remark 2.2.8. Note that If D is bounded from above in $\mathbb{R}$, we know from the completeness of $\mathbb{R}$ that there exists the supremum $\sup(D)$ of D in $\mathbb{R}$. If D is not bounded from above in $\mathbb{R}$, in this situation, we have $\sup(D) = +\infty$. Similarly, if D is not bounded from below in $\mathbb{R}$, we have the infimum $\inf(D) = -\infty$. In this viewpoint, the set D always admits an infimum and a supremum in $[-\infty, +\infty]$.

Definition 2.2.9. [32] Let $f : H \to \mathbb{R}$ be a function. For a sequence $\{x_k\}_{k\in\mathbb{N}} \in H$, the *limit inferior* of $\{f(x_k)\}_{k\in\mathbb{N}}$ in $[-\infty, +\infty]$ is

$$\liminf_{k\to\infty} f(x_k) := \sup_{k\geq 1} \inf_{n\geq k} f(x_n)$$

and its *limit superior* in $[-\infty, +\infty]$ is

$$\limsup_{k\to\infty} f(x_k) := \inf_{k\geq 1} \sup_{n\geq k} f(x_n).$$

The following theorems show the characterization of lower semicontinuity in the term of limit inferior and the characterization of upper semicontinuity in the term of limit superior.

Theorem 2.2.10. *Let $f : H \to \mathbb{R}$ be a function. Then, f is lower semicontinuous at $x \in H$ if and only if, for every sequence $\{x_k\}_{k\in\mathbb{N}}$ in H,*

$$x_k \to x, \ \ \text{as } k \to \infty \quad \Longrightarrow \quad f(x) \leq \liminf_{k\to\infty} f(x_k).$$

Proof. See [32, Theorem 1.3.2]. □

Theorem 2.2.11. *Let $f : H \to \mathbb{R}$ be a function. Then, f is upper semicontinuous at $x \in H$ if and only if, for every sequence $\{x_k\}_{k\in\mathbb{N}}$ in H,*

$$x_k \to x, \ \ \text{as } k \to \infty \quad \Longrightarrow \quad \limsup_{k\to\infty} f(x_k) \leq f(x).$$

Proof. See [32, Problem 1.3(7)]. □

Definition 2.2.12. A function $f : H \to \mathbb{R}$ is said to be *continuous* at $x \in H$ if, it is lower and upper semicontinuous at x.

The following theorem concerning a sufficient condition for continuity of a convex function.

Theorem 2.2.13. *Assume that H is finite dimensional. A convex function $f : H \to \mathbb{R}$ is continuous.*

Proof. See [33, Theorem 5.23]. □

This section will be closed by recalling the subdifferentiability of a function in Hilbert space.

Definition 2.2.14. Let $f : H \to \mathbb{R}$ be a function. The *subdifferential* of f at $x \in H$ is defined by

$$\partial f(x) = \{w \in H : f(y) - f(x) \geq \langle w, y - x \rangle, \forall y \in H\}.$$

The function f is said to be *subdifferentiable* at x if $\partial f(x) \neq \emptyset$. An element of the subdifferential $\partial f(x)$ is called a *subgradient* of f at x.

In order to guarantee the subdifferentiability of a function, we need both convexity and continuity as the following theorem.

Theorem 2.2.15. *(see [29]) For any $x \in H$, the subdifferential $\partial f(x)$ of a continuous convex function f is a nonempty, weakly closed and bounded convex set.*

The following theorems provide the characterizations of the minimizers of a function which are related to the subdifferentiability of a function.

Theorem 2.2.16. *[34] Let C be a convex subset of H and $f : C \to \mathbb{R}$ be subdifferentiable on C. Then x^* is a solution to the following convex problem:*

$$\min\{f(x) : x \in C\}$$

if and only if $0 \in \partial f(x^*) + N_C(x^*)$, *where* $N_C(x^*) := \{y \in H : \langle y, z - x^* \rangle \leq 0, \forall z \in C\}$ *is the normal cone of C at x^*.*

2.3 Operators

This section will present some definitions and necessary knowledge that will be used subsequently.

2.3.1 Bounded linear operators

In this part, let H_1 and H_2 be real Hilbert spaces. We recall some basic definitions in the functional analysis which are the need for this work.

Definition 2.3.1. An operator $L : H_1 \to H_2$ is said to be:

(i) *linear* if

$$L(\alpha x + \beta y) = \alpha Lx + \beta Ly, \ \ \forall x, y \in H_1, \ \text{ and } \ \forall \alpha, \beta \in \mathbb{R}.$$

(ii) *bounded* if there exists a positive real number M such that

$$\|Lx\|_{H_2} \leq M\|x\|_{H_1}, \ \ \forall x \in H_1,$$

where $\|\cdot\|_{H_1}$ and $\|\cdot\|_{H_2}$ are norms in H_1 and H_2, respectively.

Definition 2.3.2. Let $L : H_1 \to H_2$ be a bounded linear operator. The number

$$\|L\| := \sup_{0 \neq x \in H_1} \frac{\|Lx\|_{H_2}}{\|x\|_{H_1}}$$

is called a *norm* of L.

The following theorems show some useful properties of a bounded linear operator.

Theorem 2.3.3. *Let $L : H_1 \to H_2$ be a bounded linear operator. Then we have*

$$\|Lx\|_{H_2} \leq \|L\|\|x\|_{H_1}, \quad \forall x \in H_1.$$

Proof. See [31, Theorem 4.3.6]. □

Theorem 2.3.4. *Let $L : H_1 \to H_2$ be a bounded linear operator. If $x \in H_1$, and $\{x_k\}_{k\in\mathbb{N}}$ is a sequence in H_1 with $x_k \to x$, then $Lx_k \to Lx$.*

Proof. See [27, Corollary 2.7-10]. □

Now, we state some interesting operators concerning a bounded linear operator.

Definition 2.3.5. Let $L : H_1 \to H_2$ be a bounded linear operator. An operator $L^* : H_2 \to H_1$ is said to be *adjoint operator* of L if

$$\langle Lx, y\rangle_{H_2} = \langle x, L^*y\rangle_{H_1}, \quad \forall x \in H_1, \text{ and } \quad y \in H_2,$$

where $\langle \cdot\,, \cdot\,\rangle_{H_1}$ and $\langle \cdot\,, \cdot\,\rangle_{H_2}$ are inner products in H_1 and H_2, respectively.

We guarantee the well-definedness of the adjoint operator by the following theorem.

Theorem 2.3.6. *Let $L : H_1 \to H_2$ be a bounded linear operator. Then there exists a unique adjoint operator $L^* : H_2 \to H_1$ of L. Furthermore, the adjoint operator L^* is bounded linear operator with norm*

$$\|L^*\| = \|L\|.$$

Proof. See [27, Theorem 3.9-2]. □

The following theorem gives a general property of the adjoint operator which is used frequently.

Theorem 2.3.7. *Let $L : H_1 \to H_2$ be a bounded linear operator. Then we have*

$$\|L^*L\| = \|LL^*\| = \|L\|^2.$$

Proof. See [27, Theorem 3.9-4]. □

2.3.2 Nonlinear operators

We consider some definitions and facts which are related to nonlinear operators.

Definition 2.3.8. Let C be a nonempty closed convex subset of H. An operator $T : C \to C$ is said to be:

(i) *quasi-nonexpansive* if $Fix(T)$ is a nonempty set and

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C, p \in Fix(T),$$

where $Fix(T) := \{x \in C : Tx = x\}$ is the set of fixed points of operator T.

(ii) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(iii) *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \quad \forall x, y \in C.$$

Remark 2.3.9. A firmly nonexpansive operator is a nonexpansive operator. Besides, a nonexpansive operator with at least one fixed point is a quasi-nonexpansive operator, but the converse is not true, for instance, see [35]. Finally, it is well-known that $Fix(T)$ is closed and convex when T is a quasi-nonexpansive, see [36].

The following definition involving the demiclosedness of an operator in Hilbert space.

Definition 2.3.10. (see [37]) Let C be a nonempty closed convex subset of H. An operator $T : C \to H$ is said to be *demiclosed* at $y \in H$ if for any sequence $\{x_k\}_{k\in\mathbb{N}} \subset C$ with $x_k \rightharpoonup x^* \in C$ and $Tx_k \to y$ imply $Tx^* = y$.

The demiclosedness of the nonexpansive operator is presented by the following theorem.

Theorem 2.3.11. *[38] Let $T : C \to C$ be a nonexpansive operator with $Fix(T) \neq \emptyset$. Then $I - T$ demiclosed at 0.*

We now collect some definitions, which are mentioned in the sequel.

Definition 2.3.12. [39, 40] Let C be a nonempty closed convex subset of H. An operator $T : C \to C$ is said to be:

(i) *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \ \ \forall x, y \in C,$$

where I denotes the identity operator on C.

(ii) *Lipschitz continuous* with constant $\rho > 0$ if

$$\|Tx - Ty\| \leq \rho\|x - y\|, \ \ \forall x, y \in C.$$

In particular, if $\rho < 1$, then T is said to be ρ-contraction or, shortly, contraction.

(iii) $(\alpha, \beta, \gamma, \delta)$-*symmetric generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 \ &+ \ \beta(\|x - Ty\|^2 + \|y - Tx\|^2) + \gamma\|x - y\|^2 \\ &+ \ \delta(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0, \ \ \forall x, y \in C \end{aligned}$$

Remark 2.3.13. Note that if T is an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid satisfies (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta > 0$, and (3) $\delta \geq 0$, then T is quasi-nonexpansive and $I - T$ demiclosed at 0, see [41, 42].

The following definitions are some nonlinear operators which are concerned with this work.

Definition 2.3.14. Let C be a nonempty closed convex subset of H. An operator $T : C \to C$ is said to be:

(i) *monotone* on C if

$$\langle Tx - Ty, x - y\rangle \geq 0, \ \ \forall x, y \in C.$$

(ii) *pseudomonotone* on C if

$$\langle Tx, y-x\rangle \geq 0 \Rightarrow \langle Ty, x-y\rangle \leq 0, \quad \forall x, y \in C.$$

Remark 2.3.15. We observe that a monotone operator is a pseudomonotone operator. However, the converse may not be true.

2.3.3 Projection operators

We recall some definitions of projection operators and calculus concepts in Hilbert space.

Definition 2.3.16. Let C be a nonempty subset of H. For each $x \in H$, we denote the *metric projection* of x onto C by $P_C(x)$, that is

$$\|x - P_C(x)\| \leq \|y - x\|, \quad \forall y \in C.$$

Moreover, if $P_C(x)$ exists and uniquely determined for each $x \in H$, then the operator $P_C : H \to C$ is called the *metric projection* onto C.

We guarantee the well-definedness of the metric projection by the following theorem.

Theorem 2.3.17. *Let C be a nonempty closed and convex subset of H. Then for each $x \in H$ there exists a unique metric projection $P_C(x)$.*

Proof. See [29, Theorem 1.2.3]. □

The theorem below gives the characterization of the metric projection.

Theorem 2.3.18. *(see [29, 35]) Let C be a nonempty closed and convex subset of H. The*

(i) $z = P_C(x)$ *if and only if* $\langle x - z, y - z\rangle \leq 0, \ \forall y \in C$;

(ii) $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|P_C(x) - x + y - P_C(y)\|^2, \quad \forall x, y \in C.$

The following theorem shows some useful properties of metric projection.

Theorem 2.3.19. *Let C be a nonempty closed and convex subset of H. Then $P_C : H \to H$ is a firmly nonexpansive mapping and $Fix(P_C) = C$.*

Proof. See [29, Theorem 2.2.21]. □

The generalization of metric projection is proposed by the following definition.

Definition 2.3.20. Let $f : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semicontinuous function and $\rho > 0$. For each $x \in H$, the operator $\text{prox}_{\rho f} : H \to H$ is given by

$$\text{prox}_{\rho f}(x) = \arg\min\{\rho f(y) + \frac{1}{2}\|x - y\|^2 : y \in H\}$$

is called the *proximal operator* of f with ρ.

Remark 2.3.21. We note that if $f = \iota_C$, where ι_C is the indicator function of a nonempty closed convex subset C of H, i.e.,

$$\iota_C(x) = \begin{cases} 0, & \text{if} \quad x \in C, \\ +\infty, & \text{otherwise}, \end{cases}$$

then $\text{prox}_{\rho f} = P_C$. This means that the proximal operator is a generalization of metric projection.

Now, we present fundamental theorems that are related to the proximal operator.

Theorem 2.3.22. *Let $f : H \to \mathbb{R}\cup\{+\infty\}$ be a proper convex lower semicontinuous function. Let $J_{\rho\partial f}$ be the resolvent of ∂f with $\rho > 0$, i.e., $J_{\rho\partial f} = (I + \rho\partial f)^{-1}$. Then, for each $x \in H$,*

$$J_{\rho\partial f}(x) = \arg\min\{\rho f(y) + \frac{1}{2}\|x - y\|^2 : y \in H\}.$$

Proof. See [31, Theorem 7.5.2]. □

Remark 2.3.23. We see that the proximal operator $\text{prox}_{\rho f} = J_{\rho\partial f}$. Consequently, $\text{prox}_{\rho f}(x)$ exists and uniquely determined for each $x \in H$.

Below, we recall the definition of subgradient projection and its properties.

Definition 2.3.24. Let $f : H \to \mathbb{R}$ be a continuous convex function. Let $z_x \in \partial f(x)$ be a subgradient of f at x, $x \in H$. The *subgradient projection* $P_f : H \to H$ is defined by

$$P_f(x) = \begin{cases} x - \dfrac{f(x)}{\|z_x\|^2} z_x, & \text{if } f(x) > 0, \\ x, & \text{otherwise.} \end{cases}$$

The following theorem gives some important properties of subgradient projection.

Theorem 2.3.25. *(see [12, 14]) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. If there is $x \in \mathbb{R}^n$ such that $f(x) \leq 0$, then the subgradient projection P_f is quasi-nonexpansive with $I - P_f$ demiclosed at 0, and $Fix(P_f) = \{x \in \mathbb{R}^n : f(x) \leq 0\}$.*

2.4 Auxiliary concepts

We provide some definitions and useful results that will be used in the sequel.

Definition 2.4.1. [43, 44, 45] Let C be a nonempty closed convex subset of H. A bifunction $f : C \times C \to \mathbb{R}$ is said to be:

(i) *strongly monotone* on C if there exists a constant $\gamma > 0$ such that

$$f(x,y) + f(y,x) \leq -\gamma\|x-y\|^2, \quad \forall x,y \in C;$$

(ii) *monotone* on C if

$$f(x,y) + f(y,x) \leq 0, \quad \forall x,y \in C;$$

(iii) *pseudomonotone* on C if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0, \quad \forall x,y \in C;$$

(iv) *Lipschitz-type continuous* on C with constants $L_1 > 0$ and $L_2 > 0$ if

$$f(x,y) + f(y,z) \geq f(x,z) - L_1\|x-y\|^2 - L_2\|y-z\|^2, \quad \forall x,y,z \in C.$$

Remark 2.4.2. (i) From Definition 2.4.1, we note that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). However, the converses may not be true, for instance, see [46].

(ii) We observe that if each bifunction f_i, $i = 1, 2, \ldots, N$, is Lipschitz-type continuous on C with constants $L_1^i > 0$ and $L_2^i > 0$, then

$$\begin{aligned} f_i(x,y) + f_i(y,z) &\geq f_i(x,z) - L_1^i\|x-y\|^2 - L_2^i\|y-z\|^2 \\ &\geq f_i(x,z) - L_1\|x-y\|^2 - L_2\|y-z\|^2, \end{aligned}$$

where $L_1 = \max\{L_1^i : i = 1, 2, \ldots, N\}$ and $L_2 = \max\{L_2^i : i = 1, 2, \ldots, N\}$. Consequently, the bifunctions f_i, $i = 1, 2, \ldots, N$, are Lipschitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$.

For a nonempty closed convex subset C of H, the following assumptions on the bifunction $f : C \times C \to \mathbb{R}$ will be considered in this thesis:

$(A1)$ f is weakly continuous on $C \times C$ in the sense that, if $x \in C$, $y \in C$ and $\{x_k\} \subset C$, $\{y_k\} \subset C$ are two sequences converge weakly to x and y respectively, then $f(x_k, y_k)$ converges to $f(x, y)$;

$(A2)$ $f(x, \cdot)$ is convex and subdifferentiable on C for each fixed $x \in C$;

$(A3)$ f is psuedomonotone on C and $f(x, x) = 0$ for each $x \in C$;

$(A4)$ f is Lipschitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$.

Remark 2.4.3. If the bifunction f satisfies the assumptions $(A1) - (A3)$, it is well-known that the set $EP(f, C) := \{x^* \in C : f(x^*, y) \geq 0, \forall y \in C\}$ is closed and convex, see [1, 2, 3] and the references therein.

The following lemma is very important in order to obtain the main results in this thesis.

Lemma 2.4.4. *[47] Let* $f : C \times C \to \mathbb{R}$ *be satisfied* $(A2) - (A4)$. *Assume that* $EP(f, C)$ *is a nonempty set and* $0 < \rho_0 < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$. *Let* $x_0 \in C$, *and construct* y_0 *and* z_0 *by*

$$\begin{cases} y_0 = \arg\min\{\rho_0 f(x_0, y) + \frac{1}{2}\|y - x_0\|^2 : y \in C\}, \\ z_0 = \arg\min\{\rho_0 f(y_0, y) + \frac{1}{2}\|y - x_0\|^2 : y \in C\}. \end{cases}$$

Then,

(i) $\rho_0 \; [f(x_0, y) - f(x_0, y_0)] \geq \langle y_0 - x_0, y_0 - y \rangle, \forall y \in C$;

(ii) $\|z_0 - q\|^2 \leq \|x_0 - q\|^2 - (1 - 2\rho_0 L_1)\|x_0 - y_0\|^2 - (1 - 2\rho_0 L_2)\|y_0 - z_0\|^2$, $\forall q \in EP(f, C)$.

Proof. Let $q \in EP(f,C)$. By the definition of z_0 and Theorem 2.2.16, we have

$$0 \in \partial_2\{\rho_0 f(y_0, z_0) + \frac{1}{2}\|z_0 - x_0\|^2\} + N_C(z_0).$$

Then, there exists $w \in \partial_2 f(y_0, z_0)$ and $\overline{w} \in N_C(z_0)$ such that

$$0 = \rho_0 w + z_0 - x_0 + \overline{w}. \tag{2.4.1}$$

It follows from the subdifferentiability of f that

$$f(y_0, y) - f(y_0, z_0) \geq \langle w, y - z_0\rangle, \forall y \in C. \tag{2.4.2}$$

On the other hand, since $\overline{w} \in N_C(z_0)$, we have

$$\langle \overline{w}, z_0 - y\rangle \geq 0, \forall y \in C.$$

Thus, by using (2.4.1), we get

$$\langle z_0 - x_0, y - z_0\rangle \geq \rho_0\langle w, z_0 - y\rangle, \forall y \in C. \tag{2.4.3}$$

The relations (2.4.2) and (2.4.3) imply that

$$\langle z_0 - x_0, y - z_0\rangle \geq \rho_0[f(y_0, z_0) - f(y_0, y)], \forall y \in C. \tag{2.4.4}$$

Moreover, since $q \in C$, we see that

$$\langle z_0 - x_0, q - z_0\rangle \geq \rho_0[f(y_0, z_0) - f(y_0, q)].$$

It follows from the pseudomonotonic of f that

$$\langle z_0 - x_0, q - z_0\rangle \geq \rho_0 f(y_0, z_0).$$

Thus, by using the Lipshitz-type continuity of f, we have

$$\langle z_0 - x_0, q - z_0\rangle \geq \rho_0[f(x_0, z_0) - f(x_0, y_0) - L_1\|x_0 - y_0\|^2 - L_2\|y_0 - z_0\|^2]. \tag{2.4.5}$$

Similarly, by the definition of y_0 and Theorem 2.2.16, we can show that

$$\rho_0[f(x_0, y) - f(x_0, y_0)] \geq \langle y_0 - x_0, y_0 - y\rangle, \forall y \in C.$$

Note that, since $z_0 \in C$, we have

$$\rho_0[f(x_0, z_0) - f(x_0, y_0)] \geq \langle y_0 - x_0, y_0 - z_0 \rangle. \quad (2.4.6)$$

Next, in view of (2.4.5) and (2.4.6), we consider

$$\begin{aligned} \|x_0 - q\|^2 - \|z_0 - x_0\|^2 - \|q - z_0\|^2 &= 2\langle z_0 - x_0, q - z_0 \rangle \\ &\geq 2\rho_0[f(x_0, z_0) - f(x_0, y_0) - L_1\|x_0 - y_0\|^2 \\ &\quad - L_2\|y_0 - z_0\|^2] \\ &\geq 2\langle y_0 - x_0, y_0 - z_0 \rangle - 2\rho_0 L_1\|x_0 - y_0\|^2 \\ &\quad - 2\rho_0 L_2\|y_0 - z_0\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} \|z_0 - q\|^2 &\leq \|x_0 - q\|^2 - \|z_0 - x_0\|^2 - 2\langle y_0 - x_0, y_0 - z_0 \rangle \\ &\quad + 2\rho_0 L_1\|x_0 - y_0\|^2 + 2\rho_0 L_2\|y_0 - z_0\|^2 \\ &= \|x_0 - q\|^2 - \|z_0 - y_0\|^2 - \|y_0 - x_0\|^2 - 2\langle z_0 - y_0, y_0 - x_0 \rangle \\ &\quad + 2\langle z_0 - y_0, y_0 - x_0 \rangle + 2\rho_0 L_1\|x_0 - y_0\|^2 + 2\rho_0 L_2\|y_0 - z_0\|^2 \\ &= \|x_0 - q\|^2 - (1 - 2\rho_0 L_1)\|x_0 - y_0\|^2 - (1 - 2\rho_0 L_2)\|y_0 - z_0\|^2. \end{aligned}$$

This completes the proof. □

In what follows, we give some theorems needed for the convergence analysis.

Theorem 2.4.5. [48] *Assume that* $\{a_k\}$ *is a sequence of nonnegative numbers such that*

$$a_{k+1} \leq (1 - \gamma_k)a_k + \gamma_k \delta_k, \ \forall k \in \mathbb{N},$$

where $\{\gamma_k\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_k\}$ *is a sequence in* $\mathbb{R}$ *such that* *(i)* $\lim_{k\to\infty} \gamma_k = 0$, $\sum_{k=1}^{\infty} \gamma_k = \infty$,

(ii) $\limsup_{k\to\infty} \delta_k \leq 0$.

Then $\lim_{k\to\infty} a_k = 0$.

Theorem 2.4.6. [49] *Let* $\{a_k\}$ *be a sequence of real numbers such that there exists a subsequence* $\{k_i\}$ *of* $\{k\}$ *such that* $a_{k_i} < a_{k_i+1}$ *for all* $i \in \mathbb{N}$. *Then there exists a nondecreasing sequence* $\{m_n\} \subset \mathbb{N}$ *such that* $m_n \to \infty$ *as* $n \to \infty$ *and the following properties are satisfied by all (sufficiently large) numbers* $n \in \mathbb{N}$*:*

$$a_{m_n} \leq a_{m_n+1} \text{ and } a_n \leq a_{m_n+1}.$$

In fact, $m_n = \max\{j \leq n : a_j < a_{j+1}\}$.

Finally, if we denoted by $\omega_w(x_k)$ the *weak limit set* of the considered sequence $\{x_k\}$, that is, $\omega_w(x_k) = \{x \in H$: there is a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $x_{k_n} \rightharpoonup x\}$, then the following theorem shows some useful properties for the convergence theorems in this thesis.

Theorem 2.4.7. *[50] Let* C *be a nonempty closed convex subset of* H. *Let* $\{x_k\}$ *be a sequence of* H *and* $u \in H$. *If* $\|x_k - u\| \leq \|u - P_C(u)\|, \forall k \in \mathbb{N}$, *and* $\omega_w(x_k) \subset C$, *then* $x_k \to P_C(u)$.

CHAPTER III

ITERATIVE METHODS FOR SOLVING EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

In this chapter, we consider the equilibrium problems and the fixed points problems. And, we present some iterative methods for finding the common solution of the pseudomonotone equilibrium problems and fixed points of quasi-nonexpansive mappings problems in a real Hilbert space. Some numerical experiments and comparisons of the introduced methods with well-known algorithms are shown and discussed.

3.1 Equilibrium problems and fixed point problems

The equilibrium problem and the fixed point problem are very useful tools for studying physics, chemistry, engineering and economics in different mathematical models, for instance, see [51, 52, 53, 54], and the references therein. The equilibrium problem is a problem of finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \forall y \in C, \tag{3.1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, and $f : C \times C \to \mathbb{R}$ is a bifunction. The solution set of the equilibrium problem (3.1.1) will be represented by $EP(f, C)$. In order to solve equilibrium problem (3.1.1), when f is a monotone bifunction, the approximate solutions are frequently based on proximal point method. That is, at each iteration, we need to solve a regularized equilibrium problem:

$$\text{find} \quad x \in C \quad \text{such that} \quad f(x, y) + \frac{1}{r_k}\langle y - x, x - x_k\rangle \geq 0, \forall y \in C, \tag{3.1.2}$$

where $\{r_k\} \subset (0, \infty)$. Note that the existence of the solution of the problem (3.1.2) is guaranteed, see [43, 55]. However, if f satisfies a weaker assumption as pseudomonotone, the proximal point method cannot be applied in this situation. To overcome this drawback, Tran et al. [2] proposed the following extragradient method for solving the equilibrium problem when the bifunction f is pseudomonotone and Lipschitz-type continuous with positive constants L_1 and L_2:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min\{\rho f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ x_{k+1} = \arg\min\{\rho f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \end{cases} \tag{3.1.3}$$

where $0 < \rho < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$. They proved that the sequence $\{x_k\}$ generated by (3.1.3) converges weakly to a solution of the equilibrium problem (3.1.1).

On the other hand, for a nonempty closed convex subset C of H, and a mapping $T : C \to C$, the fixed point problem is a problem of finding a point $x \in C$ such that $Tx = x$. The set of fixed points of the mapping T will be denoted by $Fix(T)$. A famous iterative method for finding fixed points of a nonexpansive mapping T was proposed by Mann [16] as followed:

$$\begin{cases} x_0 \in C, \\ x_{k+1} = (1 - \alpha_k)x_k + \alpha_k T x_k, \end{cases} \tag{3.1.4}$$

where $\{\alpha_k\} \subset (0, 1)$. In [56], the author proved that if T has a fixed point and $\sum_{k=0}^{\infty} \alpha_k(1 - \alpha_k) = \infty$, then the sequence $\{x_k\}$ generated by (3.1.4) converges weakly to a fixed point of T. Besides, Park and Jeong [57] presented that if T is a quasi-nonexpansive mapping with $I - T$ demiclosed at 0, then the sequence which is generated by (3.1.4) also converges weakly to some fixed point of T.

In order to obtain a strong convergence result for Mann iterative method (3.1.4), Nakajo and Takahashi [17] proposed the following hybrid method for finding fixed points of a nonexpansive mapping T:

$$\begin{cases} x_0 \in C, \\ y_k = \alpha_k x_k + (1-\alpha_k)Tx_k, \\ C_k = \{x \in C : \|y_k - x\| \leq \|x_k - x\|\}, \\ Q_k = \{x \in C : \langle x_0 - x_k, x - x_k \rangle \leq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0), \end{cases} \tag{3.1.5}$$

where $\{\alpha_k\} \subset [0,1]$ such that $\alpha_k \leq 1 - \overline{\alpha}$, for some $\overline{\alpha} \in (0,1]$. They proved that the sequence $\{x_k\}$ generated by (3.1.5) converges strongly to $P_{Fix(T)}(x_0)$.

Furthermore, Ishikawa [18] proposed the following method for finding fixed points of a Lipschitz pseudocontractive mapping T:

$$\begin{cases} x_0 \in C, \\ y_k = (1-\alpha_k)x_k + \alpha_k Tx_k, \\ x_{k+1} = (1-\beta_k)x_k + \beta_k Ty_k, \end{cases} \tag{3.1.6}$$

where $0 \leq \beta_k \leq \alpha_k \leq 1$, $\lim_{k\to\infty} \alpha_k = 0$, and $\sum_{k=0}^{\infty} \alpha_k \beta_k = \infty$. If C is a convex compact subset of H, then the sequence $\{x_k\}$ generated by (3.1.6) converges strongly to fixed points of T. It was noted that Mann iterative method may not, in general, be applicable for finding fixed points of a Lipschitz pseudocontractive mapping in a Hilbert space, for instance, see [58].

In 2008, by using Ishikawa iterative method, Takahashi et al. [19] proposed the following hybrid method, called the shrinking projection method, which is different from Nakajo and Takahashi's method [17]:

$$\begin{cases} u_0 \in H, C_1 = C, \\ x_1 = P_{C_1}(u_0), \\ y_k = \alpha_k x_k + (1-\alpha_k)Tx_k, \\ z_k = \beta_k x_k + (1-\beta_k)Ty_k, \\ C_{k+1} = \{x \in C_k : \|z_k - x\| \leq \|x_k - x\|\}, \\ x_{k+1} = P_{C_{k+1}}(x_0), \end{cases} \tag{3.1.7}$$

where $\{\alpha_k\} \subset [\underline{\alpha}, \overline{\alpha}]$ with $0 < \underline{\alpha} \leq \overline{\alpha} < 1$, and $\{\beta_k\} \subset [0, 1-\overline{\beta}]$ for some $\overline{\beta} \in (0,1)$. They proved that if T is a nonexpansive mapping, then the sequence $\{x_k\}$ generated by (3.1.8) converges strongly to $P_{Fix(T)}(x_0)$.

In recent years, many algorithms have been proposed for finding a common element of the solution set of the equilibrium problems and the solution set of the fixed point problems, for instance, [13, 15, 47, 59] and the references therein. In 2016, by using the ideas of extragradient and hybrid methods together with Ishikawa iterative method, Dinh and Kim [51] proposed the following algorithm for finding a common element of the set of fixed points of a symmetric generalized hybrid mapping T and the solution set of equilibrium problem, when a bifunction f is pseudomonotone and Lipschitz-type continuous with positive constants L_1, L_2:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min\{\rho_k f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ z_k = \arg\min\{\rho_k f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ t_k = \alpha_k x_k + (1-\alpha_k)Tx_k, \\ u_k = \beta_k t_k + (1-\beta_k)Tz_k, \\ C_k = \{x \in H : \|x - u_k\| \leq \|x - x_k\|\}, \\ Q_k = \{x \in H : \langle x - x_k, x_0 - x_k\rangle \leq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k \cap C}(x_0), \end{cases} \tag{3.1.8}$$

where $\{\rho_k\} \subset [\underline{\rho}, \overline{\rho}]$ with $0 < \underline{\rho} \leq \overline{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0,1]$ such that $\lim_{k\to\infty}\alpha_k = 1$, and $\{\beta_k\} \subset [0, 1-\overline{\beta}]$, for some $\overline{\beta} \in (0,1)$. They proved that the sequence $\{x_k\}$ generated by (3.1.8) converges strongly to $P_{EP(f,C)\cap Fix(T)}(x_0)$.

In 2016, Hieu et al. [53] considered the following problem:

$$\begin{cases} \text{find a point } x^* \in C \quad \text{such that} \quad T_j x^* = x^*, j = 1, \ldots, M, \\ \text{and} \quad f_i(x^*, y) \geq 0, \forall y \in C, i = 1, \ldots, N, \end{cases} \tag{3.1.9}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $T_j : C \to C$, $j = 1, \ldots, M$, are mappings, and $f_i : C \times C \to \mathbb{R}$, $i = 1, \ldots, N$, are bifunctions satisfying $f_i(x,x) = 0$, for each $x \in C$. From now on, the solution set of problem (3.1.9) will be denoted by S. That is:

$$S := (\cap_{j=1}^{M} Fix(T_j)) \cap (\cap_{i=1}^{N} EP(f_i, C)).$$

By using the ideas of extragradient and hybrid methods together with Mann iterative method and parallel splitting-up techniques, see [60, 61], Hieu et al. [53] proposed the following algorithm for finding the solutions of problem (3.1.9), when mappings are nonexpansive, and bifunctions are pseudomonotone and Lipschitz-type continuous with positive constants L_1 and L_2:

$$\begin{cases} x_0 \in C, \\ y_k^i = \arg\min\{\rho f_i(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, i = 1, 2, \ldots, N, \\ z_k^i = \arg\min\{\rho f_i(y_k^i, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, i = 1, 2, \ldots, N, \\ \overline{z}_k = \arg\max\{\|z_k^i - x_k\| : i = 1, 2, \ldots, N\}, \\ u_k^j = \alpha_k x_k + (1 - \alpha_k) T_j \overline{z}_k, j = 1, 2, \ldots, M, \\ \overline{u}_k = \arg\max\{\|u_k^j - x_k\| : j = 1, 2, \ldots, M\}, \\ C_k = \{x \in C : \|x - \overline{u}_k\| \leq \|x - x_k\|\}, \\ Q_k = \{x \in C : \langle x - x_k, x_0 - x_k \rangle \leq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0), \end{cases} \tag{3.1.10}$$

where $0 < \rho < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, and $\{\alpha_k\} \subset (0, 1)$ such that $\limsup_{k\to\infty} \alpha_k < 1$. They proved that the sequence $\{x_k\}$ generated by (3.1.10) converges strongly to $P_S(x_0)$. In this thesis, the algorithm (3.1.10) will be called PHMEM.

3.2 Shrinking extragradient methods for pseudomonotone equilibrium problems and fixed points of quasi-nonexpansive mappings problems

In this section, motivated by the literatures in Section 3.1, we will continue develop methods for finding the solutions of problem (3.1.9). That is, we will introduce two shrinking extragradient algorithms for finding the solutions of problem (3.1.9), when each mapping T_j, $j = 1, 2, \ldots, M$, is quasi-nonexpansive with $I - T_j$ demiclosed at 0, and each bifunction f_i, $i = 1, 2, \ldots, N$, satisfies the assumptions $(A1) - (A4)$. We know that, by Remark 2.4.2 (ii), the bifunctions f_i, $i = 1, 2, \ldots, N$, are Lipschitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$. Besides, the performance of the introduced algorithms will be compared to the performance of the PHMEM algorithm and discussed via the numerical experiments.

3.2.1 Cyclic Shrinking Extragradient Method (CSEM)

In this part, we will consider the strong convergence theorem of CSEM Algorithm. From now on, for each $N \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, a modulo function at k with respect to N will be denoted by $[k]_N$, that is,

$$[k]_N = k(mod\ N) + 1.$$

Now, the CSEM Algorithm is proposed as follows:

CSEM Algorithm. Pick $x_0 \in C =: C_0$, choose parameters $\{\rho_k\}$ with $0 < \inf \rho_k \leq \sup \rho_k < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0, 1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, and $\{\beta_k\} \subset [0, 1)$ with $0 \leq \inf \beta_k \leq \sup \beta_k < 1$.

Step 1. Solve the strongly convex program

$$y_k = \arg\min\{\rho_k f_{[k]_N}(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}.$$

Step 2. Solve the strongly convex program

$$z_k = \arg\min\{\rho_k f_{[k]_N}(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}.$$

Step 3. Compute

$$t_k = \alpha_k x_k + (1 - \alpha_k) T_{[k]_M} x_k,$$

$$u_k = \beta_k t_k + (1 - \beta_k) T_{[k]_M} z_k.$$

Step 4. Construct closed convex subsets of C:

$$C_{k+1} = \{x \in C_k : \|x - u_k\| \leq \|x - x_k\|\}.$$

Step 5. The next approximation x_{k+1} is defined as the projection of x_0 onto C_{k+1}, i.e.,

$$x_{k+1} = P_{C_{k+1}}(x_0).$$

Step 6. Put $k := k+1$ and go to **Step 1**.

Before going to prove the strong convergence of CSEM Algorithm, we need the following lemma.

Lemma 3.2.1. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by CSEM Algorithm is well-defined.*

Proof. To prove the Lemma, it suffices to show that C_k is a nonempty closed and convex subset of H, for each $k \in \mathbb{N} \cup \{0\}$. Firstly, we will show the non-emptiness by showing that $S \subset C_k$, for each $k \in \mathbb{N} \cup \{0\}$. Obviously, $S \subset C_0$.

Now, let $q \in S$. Then, by Lemma 2.4.4 (ii), we have

$$\|z_k - q\|^2 \leq \|x_k - q\|^2 - (1 - 2\rho_k L_1)\|x_k - y_k\|^2 - (1 - 2\rho_k L_2)\|y_k - z_k\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. This implies that

$$\|z_k - q\| \leq \|x_k - q\|, \tag{3.2.1}$$

for each $k \in \mathbb{N} \cup \{0\}$. On the other hand, since $q \in Fix(T_j)$, it follows from the quasi-nonexpansivity of each T_j $(j \in \{1, 2, \ldots, M\})$ and the definitions of t_k, u_k that

$$\begin{aligned}
\|t_k - q\| &\leq \alpha_k\|x_k - q\| + (1 - \alpha_k)\|T_{[k]_M} x_k - q\| \\
&\leq \alpha_k\|x_k - q\| + (1 - \alpha_k)\|x_k - q\| \\
&= \|x_k - q\|, \qquad (3.2.2)
\end{aligned}$$

and

$$\begin{aligned} \|u_k - q\| &\leq \beta_k\|t_k - q\| + (1-\beta_k)\|T_{[k]_M} z_k - q\| \\ &\leq \beta_k\|t_k - q\| + (1-\beta_k)\|z_k - q\|, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. The relations (3.2.1) and (3.2.2) imply that

$$\begin{aligned} \|u_k - q\| &\leq \beta_k\|x_k - q\| + (1-\beta_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{3.2.3}$$

for each $k \in \mathbb{N} \cup \{0\}$. Now, suppose that $S \subset C_k$. Thus, by using (3.2.3), we see that $S \subset C_{k+1}$. So, by induction, we have $S \subset C_k$, for each $k \in \mathbb{N} \cup \{0\}$. Since S is a nonempty set, we obtain that C_k is a nonempty set, for each $k \in \mathbb{N} \cup \{0\}$.

Next, we show that C_k is a closed and convex subset, for each $k \in \mathbb{N} \cup \{0\}$. Note that we already have that C_0 is a closed and convex subset. Now, suppose that C_k is a closed and convex subset, we will show that C_{k+1} is likewise. To do this, let us consider a set $B_k = \{x \in H : \|x - u_k\| \leq \|x - x_k\|\}$. We see that

$$B_k = \left\{x \in H : \langle x_k - u_k, x\rangle \leq \frac{1}{2}(\|x_k\|^2 - \|u_k\|^2)\right\}.$$

This means that B_k is a halfspace and $C_{k+1} = C_k \cap B_k$. Thus, C_{k+1} is a closed and convex subset. Thus, by induction, we can conclude that C_k is a closed and convex subset, for each $k \in \mathbb{N} \cup \{0\}$. Consequently, we can guarantee that $\{x_k\}$ is well-defined. □

Now, we are ready to prove the strong convergence theorem of the sequence $\{x_k\}$ which is generated by the CSEM Algorithm.

Theorem 3.2.2. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by CSEM Algorithm converges strongly to $P_S(x_0)$.*

Proof. Let $q \in S$. By the definition of x_{k+1}, we observe that $x_{k+1} \in C_{k+1} \subset C_k$, for each $k \in \mathbb{N} \cup \{0\}$. Since $x_k = P_{C_k}(x_0)$ and $x_{k+1} \in C_k$, we have

$$\|x_k - x_0\| \leq \|x_{k+1} - x_0\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. This means that $\{\|x_k - x_0\|\}$ is a nondecreasing sequence. Similarly, for each $q \in S \subset C_{k+1}$, we obtain that

$$\|x_{k+1} - x_0\| \leq \|q - x_0\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. By the above inequalities, we get

$$\|x_k - x_0\| \leq \|q - x_0\|, \tag{3.2.4}$$

for each $k \in \mathbb{N} \cup \{0\}$. So $\{\|x_k - x_0\|\}$ is a bounded sequence. Consequently, we can conclude that $\{\|x_k - x_0\|\}$ is a convergent sequence. Moreover, we see that $\{x_k\}$ is bounded. Thus, in view of (3.2.2) and (3.2.3), we get that $\{t_k\}$ and $\{u_k\}$ are also bounded. Suppose $k, j \in \mathbb{N} \cup \{0\}$ such that $k > j$. It follows that $x_k \in C_k \subset C_j$. Then, by Theorem 2.3.18 (ii), we have

$$\|P_{C_j}(x_k) - P_{C_j}(x_0)\|^2 \leq \|x_0 - x_k\|^2 - \|P_{C_j}(x_k) - x_k + x_0 - P_{C_j}(x_0)\|^2.$$

Consequently,

$$\|x_k - x_j\|^2 \leq \|x_0 - x_k\|^2 - \|x_j - x_0\|^2.$$

Thus, by using the existence of $\lim_{k\to\infty} \|x_k - x_0\|$, we get

$$\lim_{k,j\to\infty} \|x_k - x_j\| = 0.$$

That is $\{x_k\}$ is a Cauchy sequence in C. Since C is closed, there exists $p \in C$ such that

$$\lim_{k\to\infty} x_k = p. \tag{3.2.5}$$

By the definition of C_{k+1} and $x_{k+1} \in C_k$, we see that

$$\|x_{k+1} - u_k\| \leq \|x_{k+1} - x_k\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that

$$\begin{aligned} \|u_k - x_k\| &\leq \|u_k - x_{k+1}\| + \|x_{k+1} - x_k\| \\ &\leq \|x_{k+1} - x_k\| + \|x_{k+1} - x_k\| \\ &= 2\|x_{k+1} - x_k\|, \end{aligned} \tag{3.2.6}$$

for each $k \in \mathbb{N} \cup \{0\}$. Since $x_k \to p$ and $x_{k+1} \to p$, as $k \to \infty$, we obtain that

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0.$$

This together with (3.2.6) imply that

$$\lim_{k\to\infty} \|u_k - x_k\| = 0. \tag{3.2.7}$$

Since $\lim_{k\to\infty} \alpha_k = 1$, it follows that

$$\begin{aligned} \lim_{k\to\infty} \|t_k - x_k\| &= \lim_{k\to\infty} \|\alpha_k x_k + (1 - \alpha_k)T_{[k]_M} x_k - x_k\| \\ &= \lim_{k\to\infty} (1 - \alpha_k)\|x_k - T_{[k]_M} x_k\| \\ &= 0. \end{aligned} \tag{3.2.8}$$

Consider,

$$\begin{aligned} \|u_k - q\|^2 &= \|\beta_k(t_k - q) + (1 - \beta_k)(T_{[k]_M} z_k - q)\|^2 \\ &= \beta_k\|t_k - q\|^2 + (1 - \beta_k)\|T_{[k]_M} z_k - q\|^2 - \beta_k(1 - \beta_k)\|t_k - T_{[k]_M} z_k\|^2 \\ &\leq \beta_k\|t_k - q\|^2 + (1 - \beta_k)\|T_{[k]_M} z_k - q\|^2, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. By using (3.2.2) and the quasi-nonexpansivity of each T_j ($j \in \{1, 2, \ldots, M\}$), we obtain

$$\|u_k - q\|^2 \leq \beta_k\|x_k - q\|^2 + (1 - \beta_k)\|z_k - q\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by Lemma 2.4.4 (ii), we have

$$\begin{aligned} \|u_k - q\|^2 &\leq \beta_k\|x_k - q\|^2 + (1-\beta_k)[\|x_k - q\|^2 - (1-2\rho_k L_1)\|x_k - y_k\|^2 \\ &\quad -(1-2\rho_k L_2)\|y_k - z_k\|^2] \\ &\leq \|x_k - q\|^2 - (1-\beta_k)[(1-2\rho_k L_1)\|x_k - y_k\|^2 \\ &\quad +(1-2\rho_k L_2)\|y_k - z_k\|^2], \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that

$$\begin{aligned} &(1-\beta_k)[(1-2\rho_k L_1)\|x_k - y_k\|^2 + (1-2\rho_k L_2)\|y_k - z_k\|^2] \\ &\leq \|x_k - u_k\|(\|x_k - q\| + \|u_k - q\|), \end{aligned} \tag{3.2.9}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, by using (3.2.7) and the choices of $\{\beta_k\}$, $\{\rho_k\}$, we have

$$\lim_{k\to\infty} \|x_k - y_k\| = 0, \tag{3.2.10}$$

and

$$\lim_{k\to\infty} \|y_k - z_k\| = 0. \tag{3.2.11}$$

These imply that

$$\lim_{k\to\infty} \|x_k - z_k\| = 0. \tag{3.2.12}$$

Then, by $\lim_{k\to\infty} x_k = p$, we also have

$$\lim_{k\to\infty} y_k = p, \tag{3.2.13}$$

and

$$\lim_{k\to\infty} z_k = p. \tag{3.2.14}$$

Next, we claim that $p \in S$. From the definition of u_k, we see that

$$\begin{aligned}(1-\beta_k)\|T_{[k]_M}z_k - z_k\| &= \|u_k - z_k - \beta_k(t_k - z_k)\| \\ &\le \|u_k - z_k\| + \beta_k\|t_k - z_k\| \\ &\le \|u_k - x_k\| + \beta_k\|t_k - x_k\| + (1+\beta_k)\|x_k - z_k\|,\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by using (3.2.7), (3.2.8) and (3.2.12), we have

$$\lim_{k\to\infty} \|T_{[k]_M}z_k - z_k\| = 0. \tag{3.2.15}$$

Furthermore, for each fixed $j \in \{1, 2, \ldots, M\}$, we observe that

$$[(j-1) + kM]_M = j,$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, it follows from (3.2.15) that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \|T_{[(j-1)+kM]_M} z_{(j-1)+kM} - z_{(j-1)+kM}\| \\ &= \lim_{k\to\infty} \|T_j z_{(j-1)+kM} - z_{(j-1)+kM}\|, \end{aligned} \tag{3.2.16}$$

for each $j \in \{1, 2, \ldots, M\}$. Since $z_k \to p$, as $k \to \infty$, then for each $j \in \{1, 2, \ldots, M\}$, we get $z_{(j-1)+kM} \to p$, as $k \to \infty$. Combining with (3.2.16), by the demiclosedness at 0 of $I - T_j$, implies that

$$T_j p = p,$$

for each $j = 1, 2, \ldots, M$.

Similarly, for each fixed $i \in \{1, 2, \ldots, N\}$, we note that

$$[(i-1) + kN]_N = i,$$

for each $k \in \mathbb{N} \cup \{0\}$. Since $x_k \to p$ and $y_k \to p$, as $k \to \infty$, then for each $i \in \{1, 2, \ldots, N\}$, we have $x_{(i-1)+kN} \to p$ and $y_{(i-1)+kN} \to p$, as $k \to \infty$. By Lemma 2.4.4 (i), for each $i \in \{1, 2, \ldots, N\}$, we obtain

$$\rho_{(i-1)+kN}[f_{[(i-1)+kN]_N}(x_{(i-1)+kN}, y) - f_{[(i-1)+kN]_N}(x_{(i-1)+kN}, y_{(i-1)+kN})]$$
$$\geq \langle y_{(i-1)+kN} - x_{(i-1)+kN}, y_{(i-1)+kN} - y\rangle, \forall y \in C.$$

It follows that, for each $i \in \{1, 2, \ldots, N\}$, we have

$$f_{[(i-1)+kN]_N}(x_{(i-1)+kN}, y) - f_{[(i-1)+kN]_N}(x_{(i-1)+kN}, y_{(i-1)+kN})$$
$$\geq -\frac{1}{\rho_{(i-1)+kN}}\|y_{(i-1)+kN} - x_{(i-1)+kN}\|\|y_{(i-1)+kN} - y\|, \forall y \in C.$$

By using (3.2.10) and weak continuity of each f_i ($i \in \{1, 2, \ldots, N\}$), we get that

$$f_i(p, y) \geq 0, \forall y \in C,$$

for each $i = 1, 2, \ldots, N$. Then, we had shown that $p \in S$.

Finally, we will show that $p = P_S(x_0)$. In fact, since $P_S(x_0) \in S$, it follows from (3.2.4) that

$$\|x_k - x_0\| \leq \|P_S(x_0) - x_0\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by using the continuity of norm and $\lim_{k\to\infty} x_k = p$, we see that

$$\|p - x_0\| = \lim_{k\to\infty} \|x_k - x_0\| \leq \|P_S(x_0) - x_0\|.$$

Thus, by the definition of $P_S(x_0)$ and $p \in S$, we obtain that $p = P_S(x_0)$. This completes the proof. □

3.2.2 Parallel Shrinking Extragradient Method (PSEM)

In this part, we start by introducing the PSEM Algorithm as follows:

PSEM Algorithm. Pick $x_0 \in C =: C_0$, choose parameters $\{\rho_k^i\}$ with $0 < \inf \rho_k^i \leq \sup \rho_k^i < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}, i = 1, 2, \ldots, N$, $\{\alpha_k\} \subset [0,1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, and $\{\beta_k\} \subset [0,1)$ with $0 \leq \inf \beta_k \leq \sup \beta_k < 1$.

Step 1. Solve N strongly convex programs

$$y_k^i = \arg\min\{\rho_k^i f_i(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, i = 1, 2, \ldots, N.$$

Step 2. Solve N strongly convex programs

$$z_k^i = \arg\min\{\rho_k^i f_i(y_k^i, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, i = 1, 2, \ldots, N.$$

Step 3. Find the farthest element from x_k among z_k^i, $i = 1, 2, \ldots, N$, i.e.,

$$\overline{z}_k = \arg\max\{\|z_k^i - x_k\| : i = 1, 2, \ldots, N\}.$$

Step 4. Compute

$$t_k^j = \alpha_k x_k + (1 - \alpha_k) T_j x_k, j = 1, 2, \ldots, M,$$
$$u_k^j = \beta_k t_k^j + (1 - \beta_k) T_j \overline{z}_k, j = 1, 2, \ldots, M.$$

Step 5. Find the farthest element from x_k among u_k^j, $j = 1, 2, \ldots, M$, i.e.,

$$\overline{u}_k = \arg\max\{\|u_k^j - x_k\| : j = 1, 2, \ldots, M\}.$$

Step 6. Construct closed convex subsets of C:

$$C_{k+1} = \{x \in C_k : \|x - \overline{u}_k\| \leq \|x - x_k\|\}.$$

Step 7. The next approximation x_{k+1} is defined as the projection of x_0 onto C_{k+1}, i.e.,

$$x_{k+1} = P_{C_{k+1}}(x_0).$$

Step 8. Put $k := k+1$ and go to **Step 1**.

Now, we are in a position to present the strong convergence theorem of the sequence $\{x_k\}$ which is generated by the PSEM Algorithm.

Theorem 3.2.3. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by PSEM Algorithm converges strongly to $P_S(x_0)$.*

Proof. Let $q \in S$. By the definition of $\overline{z}_k$, we suppose that $i_k \in \{1, 2, \ldots, N\}$ such that $z_k^{i_k} = \overline{z}_k = \arg\max\{\|z_k^i - x_k\| : i = 1, 2, \ldots, N\}$. Then, by Lemma 2.4.4 (ii), we have

$$\|\overline{z}_k - q\|^2 \leq \|x_k - q\|^2 - (1 - 2\rho_k^{i_k} L_1)\|x_k - y_k^{i_k}\|^2 - (1 - 2\rho_k^{i_k} L_2)\|y_k^{i_k} - \overline{z}_k\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. This implies that

$$\|\overline{z}_k - q\| \leq \|x_k - q\|, \tag{3.2.17}$$

for each $k \in \mathbb{N} \cup \{0\}$. On the other hand, by the definition of t_k^j and the quasi-nonexpansivity of each T_j ($j \in \{1, 2, \ldots, M\}$), we have

$$\begin{aligned} \|t_k^j - q\| &\leq \alpha_k\|x_k - q\| + (1 - \alpha_k)\|T_j x_k - q\| \\ &\leq \alpha_k\|x_k - q\| + (1 - \alpha_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{3.2.18}$$

for each $k \in \mathbb{N} \cup \{0\}$. Additionally, by the definition of $\overline{u}_k$, we suppose that $j_k \in \{1, 2, \ldots, M\}$ such that $u_k^{j_k} = \overline{u}_k = \arg\max\{\|u_k^j - x_k\| : j = 1, 2, \ldots, M\}$. It follows from the quasi-nonexpansivity of each T_j $(j \in \{1, 2, \ldots, M\})$ that

$$\begin{aligned} \|\overline{u}_k - q\| &\leq \beta_k \|t_k^{j_k} - q\| + (1 - \beta_k)\|T_{j_k}\overline{z}_k - q\| \\ &\leq \beta_k \|t_k^{j_k} - q\| + (1 - \beta_k)\|\overline{z}_k - q\|, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. The relations (3.2.17) and (3.2.18) imply that

$$\begin{aligned} \|\overline{u}_k - q\| &\leq \beta_k \|x_k - q\| + (1 - \beta_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{3.2.19}$$

for each $k \in \mathbb{N} \cup \{0\}$. Following the proof of Lemma 3.2.1 and Theorem 3.2.2, we can show that C_k is a closed convex subset of H and $S \subset C_k$, for each $k \in \mathbb{N} \cup \{0\}$. Moreover, we can check that the sequence $\{x_k\}$ is a convergent sequence, say

$$\lim_{k \to \infty} x_k = p, \tag{3.2.20}$$

for some $p \in C$.

By the definition of C_{k+1} and $x_{k+1} \in C_k$, we see that

$$\|x_{k+1} - \overline{u}_k\| \leq \|x_{k+1} - x_k\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that

$$\begin{aligned} \|\overline{u}_k - x_k\| &\leq \|\overline{u}_k - x_{k+1}\| + \|x_{k+1} - x_k\| \\ &\leq \|x_{k+1} - x_k\| + \|x_{k+1} - x_k\| \\ &= 2\|x_{k+1} - x_k\|, \end{aligned} \tag{3.2.21}$$

for each $k \in \mathbb{N} \cup \{0\}$. Since $x_k \to p$ and $x_{k+1} \to p$, as $k \to \infty$, we obtain that

$$\lim_{k \to \infty} \|x_{k+1} - x_k\| = 0.$$

This together with (3.2.21) implies that

$$\lim_{k \to \infty} \|\overline{u}_k - x_k\| = 0.$$

Then, by the definition of $\overline{u}_k$, we have

$$\lim_{k\to\infty} \|u_k^j - x_k\| = 0, \tag{3.2.22}$$

for each $j = 1, 2, \ldots, M$. Since $\lim_{k\to\infty} \alpha_k = 1$, it follows that

$$\begin{aligned}
\lim_{k\to\infty} \|t_k^j - x_k\| &= \lim_{k\to\infty} \|\alpha_k x_k + (1-\alpha_k)T_j x_k - x_k\| \\
&= \lim_{k\to\infty} (1-\alpha_k)\|x_k - T_j x_k\| \\
&= 0,
\end{aligned} \tag{3.2.23}$$

for each $j = 1, 2, \ldots, M$. Beside, by the definition of u_k^j, for each $j = 1, 2, \ldots, M$, we see that

$$\begin{aligned}
\|u_k^j - q\|^2 &= \|\beta_k(t_k^j - q) + (1-\beta_k)(T_j\overline{z}_k - q)\|^2 \\
&= \beta_k\|t_k^j - q\|^2 + (1-\beta_k)\|T_j\overline{z}_k - q\|^2 - \beta_k(1-\beta_k)\|t_k^j - T_j\overline{z}_k\|^2 \\
&\leq \beta_k\|t_k^j - q\|^2 + (1-\beta_k)\|T_j\overline{z}_k - q\|^2,
\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, by using (3.2.18) and the quasi-nonexpansivity of each T_j ($j \in \{1, 2, \ldots, M\}$), we have

$$\|u_k^j - q\|^2 \leq \beta_k\|x_k - q\|^2 + (1-\beta_k)\|\overline{z}_k - q\|^2,$$

for $k \in \mathbb{N} \cup \{0\}$. So, by Lemma 2.4.4 (ii), for each $j = 1, 2, \ldots, M$, we get that

$$\begin{aligned}
\|u_k^j - q\|^2 &\leq \beta_k\|x_k - q\|^2 + (1-\beta_k)[\|x_k - q\|^2 - (1-2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 \\
&\quad -(1-2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2] \\
&= \|x_k - q\|^2 - (1-\beta_k)[(1-2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 \\
&\quad +(1-2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2],
\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that, for each $j = 1, 2, \ldots, M$, we have

$$
\begin{aligned}
& (1-\beta_k)[(1-2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 + (1-2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2] \\
\leq\ & \|x_k - q\|^2 - \|u_k^j - q\|^2 \\
=\ & \|x_k - u_k^j\|(\|x_k - q\| + \|u_k^j - q\|), \qquad (3.2.24)
\end{aligned}
$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, by using (3.2.22) and the choices of $\{\beta_k\}$, $\{\rho_k^i\}$, we see that

$$\lim_{k\to\infty} \|x_k - y_k^{i_k}\| = 0, \qquad (3.2.25)$$

and

$$\lim_{k\to\infty} \|y_k^{i_k} - \overline{z}_k\| = 0. \qquad (3.2.26)$$

These imply that

$$\lim_{k\to\infty} \|x_k - \overline{z}_k\| = 0. \qquad (3.2.27)$$

Then, by the definition of $\overline{z}_k$, we have

$$\lim_{k\to\infty} \|x_k - z_k^i\| = 0, \qquad (3.2.28)$$

for each $i = 1, 2, \ldots, N$. Moreover, by Lemma 2.4.4 (ii), for each $i = 1, 2, \ldots, N$, we get that

$$\|z_k^i - q\|^2 \leq \|x_k - q\|^2 - (1-2\rho_k^i L_1)\|x_k - y_k^i\|^2 - (1-2\rho_k^i L_2)\|y_k^i - z_k^i\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that, for each $i = 1, 2, \ldots, N$, we have

$$
\begin{aligned}
(1-2\rho_k^i L_1)\|x_k - y_k^i\|^2 + (1-2\rho_k^i L_2)\|y_k^i - z_k^i\|^2 \ &\leq\ \|x_k - q\|^2 - \|z_k^i - q\|^2 \\
&=\ \|x_k - z_k^i\|(\|x_k - q\| + \|z_k^i - q\|),
\end{aligned}
$$

for each $k \in \mathbb{N} \cup \{0\}$. Combining with (3.2.28) implies that

$$\lim_{k\to\infty} \|x_k - y_k^i\| = 0, \qquad (3.2.29)$$

and

$$\lim_{k\to\infty} \|y_k^i - z_k^i\| = 0, \tag{3.2.30}$$

for each $i = 1, 2, \ldots, N$. Thus, by using (3.2.27), (3.2.29) and $\lim_{k\to\infty} x_k = p$, we have

$$\lim_{k\to\infty} \overline{z}_k = p, \tag{3.2.31}$$

and

$$\lim_{k\to\infty} y_k^i = p, \tag{3.2.32}$$

for each $i = 1, 2, \ldots, N$.

Next, we claim that $p \in S$. From the definition of u_k^j, for each $j = 1, 2, \ldots, M$, we see that

$$\begin{aligned}(1-\beta_k)\|T_j\overline{z}_k - \overline{z}_k\| &= \|u_k^j - \overline{z}_k - \beta_k(t_k^j - \overline{z}_k)\| \\ &\leq \|u_k^j - \overline{z}_k\| + \beta_k\|t_k^j - \overline{z}_k\| \\ &\leq \|u_k^j - x_k\| + \beta_k\|t_k^j - x_k\| + (1+\beta_k)\|x_k - \overline{z}_k\|,\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, in view of (3.2.22), (3.2.23), and (3.2.27), we get that

$$\lim_{k\to\infty} \|T_j\overline{z}_k - \overline{z}_k\| = 0, \tag{3.2.33}$$

for each $j = 1, 2, \ldots, M$. Combining with (3.2.31), by the demiclosedness at 0 of $I - T_j$, implies that

$$T_j p = p,$$

for each $j = 1, 2, \ldots, M$.

On the other hand, by Lemma 2.4.4 (i), for each $i = 1, 2, \ldots, N$, we see that

$$\rho_k^i[f_i(x_k, y) - f_i(x_k, y_k^i)] \geq \langle y_k^i - x_k, y_k^i - y\rangle, \forall y \in C.$$

It follows that, for each $i = 1, 2, \ldots, N$, we get

$$f_i(x_k, y) - f_i(x_k, y_k^i) \geq -\frac{1}{\rho_k^i}\|y_k^i - x_k\|\|y_k^i - y\|, \forall y \in C.$$

By using (3.2.20), (3.2.29), (3.2.32) and weak continuity of each f_i ($i \in \{1, 2, \ldots, N\}$), we have

$$f_i(p, y) \geq 0, \forall y \in C,$$

for each $i = 1, 2, \ldots, N$. Thus, we can conclude that $p \in S$. The rest of the proof is similar to the arguments in the proof of Theorem 3.2.2, and it leads to the conclusion that the sequence $\{x_k\}$ converges strongly to $P_S(x_0)$. □

Remark 3.2.4. We note that for the PSEM Algorithm we solve y_k^i, z_k^i, $i = 1, 2, \ldots, N$, by using N bifunctions and compute t_k^j, u_k^j, $j = 1, 2, \ldots, M$, by using M mappings. The farthest elements from x_k among all z_k^i and u_k^j are chosen for the next step calculation. However, we solve only y_k, z_k, by using a bifunction and compute only t_k, u_k, by using a mapping for the CSEM Algorithm. After that, we construct closed convex subset C_{k+1}, and the approximation x_{k+1} is the projection of x_0 onto C_{k+1} for both algorithms. We claim that the numbers of iterations of the PSEM Algorithm should be less than the CSEM Algorithm. However, the computational times of the CSEM Algorithm should be less than the PSEM Algorithm for sufficiently large N, M. On the other hand, for the PHMEM Algorithm they solved y_k^i, z_k^i, $i = 1, 2, \ldots, N$, by using N bifunctions, and computed u_k^j, $j = 1, 2, \ldots, M$, by using M mappings. The farthest elements from x_k among all z_k^i and u_k^j are chosen similar to the PSEM Algorithm. However, they constructed two closed convex subsets C_k, Q_k, and the approximation x_{k+1} is the projection of x_0 onto $C_k \cap Q_k$, which is difficult to compute.

3.2.3 A Numerical Experiment

In this part, we will compare the two introduced algorithms, CSEM and PSEM, with the PHMEM Algorithm, which was presented in [53]. The following setting is taken from Hieu et al. [53]. Let $H = \mathbb{R}$ be a Hilbert space with the standard inner product $\langle x, y\rangle = xy$ and the norm $\|x\| = |x|$, for each $x, y \in H$. To be considered here are the nonexpansive self-mappings T_j, $j = 1, 2, \ldots, M$, and the bifunctions f_i, $i = 1, 2, \ldots, N$, which are given on $C = [0, 1]$ by

$$T_j(x) = \frac{x^j \sin^{j-1}(x)}{2j-1}, \quad j = 1, 2, \ldots, M,$$

and

$$f_i(x, y) = B_i(x)(y - x), \quad i = 1, 2, \ldots, N,$$

where $B_i(x) = 0$ if $0 \leq x \leq \xi_i$, and $B_i(x) = e^{x-\xi_i} + \sin(x - \xi_i) - 1$ if $\xi_i < x \leq 1$. Moreover, $0 < \xi_1 < \xi_2 < \ldots < \xi_N < 1$. Then, the bifunctions f_i, $i = 1, 2, \ldots, N$, satisfy conditions $(A1) - (A4)$, see [53]. Indeed, the bifunctions f_i, $i = 1, 2, \ldots, N$, are Lipschitz-type continuous with constants $L_1 = L_2 = 2$. Note that the solution set S is nonempty because $0 \in S$.

The following numerical experiment is considered with these parameters: $\rho_k = \frac{1}{5}$, $\xi_{[k]_N} = \frac{[k]_N}{N+1}$ for the CSEM Algorithm; $\rho_k^i = \frac{1}{5}$, $\xi_i = \frac{i}{N+1}$, $i = 1, 2, \ldots, N$ for the PSEM Algorithm, when $N = 1000$ and $M = 2000$. The following six cases of the parameters α_k and β_k are considered:

Case 1. $\alpha_k = 1 - \dfrac{1}{k+2}$, $\beta_k = \dfrac{1}{k+2}$.

Case 2. $\alpha_k = 1 - \dfrac{1}{k+2}$, $\beta_k = 0.5 + \dfrac{1}{k+3}$.

Case 3. $\alpha_k = 1 - \dfrac{1}{k+2}$, $\beta_k = 0.99 - \dfrac{1}{k+2}$.

Case 4. $\alpha_k = 1$, $\beta_k = \dfrac{1}{k+2}$.

Case 5. $\alpha_k = 1$, $\beta_k = 0.5 + \dfrac{1}{k+3}$.

Case 6. $\alpha_k = 1$, $\beta_k = 0.99 - \dfrac{1}{k+2}$.

The experiment was written in Matlab R2015b and performed on a PC desktop with Intel(R) Core(TM) i3-3240 CPU @ 3.40GHz 3.40GHz and RAM

4.00 GB. The function *fmincon* in Matlab Optimization Toolbox was used to solve vectors y_k, z_k for the CSEM Algorithm; y_k^i, z_k^i, $i = 1, 2, \ldots, N$, for the PSEM Algorithm. The set C_{k+1} was computed by using the function *solve* in Matlab Symbolic Math Toolbox. One can see that the set C_{k+1} is the interval $[a, b]$, where $a, b \in [0, 1]$, $a \leq b$. Consequently, the metric projection of a point x_0 onto the set C_{k+1} was computed by using this form

$$P_{C_{k+1}}(x_0) = \max\{\min\{x_0, b\}, a\},$$

see [29]. The CSEM and PSEM algorithms were tested along with the PHMEM Algorithm by using the stopping criteria $|x_{k+1} - x_k| < 10^{-4}$ and the results below were presented as averages calculated from four starting points: x_0 at 0.01, 0.25, 0.75 and 1.

Table 1 **Numerical results for six different cases of parameters α_k and β_k**

	Average times (sec)			Average iterations		
Cases	CSEM	PSEM	PHMEM	CSEM	PSEM	PHMEM
1	4.905197	165.099794	173.347257	14.25	13.75	14.25
2	7.326055	287.918141	345.025914	25.25	24.25	28.25
3	20.371064	834.001035	2004.693844	91.25	74.25	177
4	5.079676	173.091716	173.347257	14.75	14.25	14.25
5	8.016109	342.870819	345.025914	28.75	28.25	28.25
6	42.035240	1986.147273	2004.693844	200	177	177

Table 1 shows that the parameter $\beta_k = \dfrac{1}{k+2}$ yields faster computational times and fewer computational iterations than other cases. Compare cases 1-3 with each other and cases 4-6 with each other. Meanwhile, the parameter $\alpha_k = 1$, in which the Ishikawa iteration reduces to the Mann iteration, yields slower computational times and more computational iterations than the other case. Compare cases 1 with 4, 2 with 5, and 3 with 6. Moreover, the computational times of the CSEM algorithm are faster than other algorithms, while the computational iterations of the PSEM algorithm are fewer than or equal to other algorithms. Finally, we see that both computational times and iterations of the CSEM and PSEM algorithms are better than or equal to those of the PHMEM Algorithm.

Remark 3.2.5. Let us consider the case of parameters $\alpha_k = 1$ and $\beta_k = 0$, in which the Ishikawa iteration will be reduced to the Picard iteration. We notice that the convergence of the PHMEM Algorithm cannot be guaranteed in this setting. The computational results of the CSEM and PSEM algorithms are shown as follows.

Table 2 **Numerical results for parameters $\alpha_k = 1$ and $\beta_k = 0$**

Average times (sec)		Average iterations	
CSEM	PSEM	CSEM	PSEM
4.657696	137.200812	12.50	11.50

From Table 2, we see that both computational times and iterations are better than all those cases presented in Table 1. However, it should be warned that the Picard iteration method may not always converge to a fixed point of a nonexpansive mapping in general. For example, see [62].

3.3 Hybrid extragradient methods for pseudomonotone equilibrium problems and fixed points of quasi-nonexpansive mappings problems

In this section, motivated by the literatures in Section 3.1, we will still focus to the methods for finding the solutions of problem (3.1.9). That is, we introduce two hybrid extragradient algorithms for finding the closest point to the solution set of problem (3.1.9), when each mapping $T_j : C \to C$, $j = 1, 2, \ldots, M$, is quasi-nonexpansive with $I - T_j$ demiclosed at 0, and each bifunction f_i, $i = 1, 2, \ldots, N$, satisfies the assumptions $(A1) - (A4)$. Again, by Remark 2.4.2 (ii), we know that the bifunctions f_i, $i = 1, 2, \ldots, N$, are Lipschitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$. Besides, we discuss the performance of introduced algorithms and compare it with some appeared algorithms via the numerical experiments.

3.3.1 Cyclic Hybrid Extragradient Method (CHEM)

In this part, we begin by recalling that for each $N \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, a modulo function at k with respect to N is denoted by $[k]_N$, that is,

$$[k]_N = k(mod\ N) + 1.$$

Now, we consider the CHEM Algorithm as follows:

CHEM Algorithm. Choose parameters $\{\rho_k\}$ with $0 < \inf \rho_k \leq \sup \rho_k < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0, 1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, and $\{\beta_k\} \subset [0, 1)$ with $0 \leq \inf \beta_k \leq \sup \beta_k < 1$. Pick $x_0 \in C$.

Step 1. Solve the strongly convex program

$$y_k = \arg\min\{\rho_k f_{[k]_N}(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}.$$

Step 2. Solve the strongly convex program

$$z_k = \arg\min\{\rho_k f_{[k]_N}(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}.$$

Step 3. Compute

$$t_k = \alpha_k x_k + (1 - \alpha_k) T_{[k]_M} x_k,$$
$$u_k = \beta_k t_k + (1 - \beta_k) T_{[k]_M} z_k.$$

Step 4. Construct two closed convex subsets of C

$$C_k = \{x \in C : \|x - u_k\| \leq \|x - x_k\|\},$$
$$Q_k = \{x \in C : \langle x_0 - x_k, x - x_k \rangle \leq 0\}.$$

Step 5. The next approximation x_{k+1} is defined as the projection of x_0 onto $C_k \cap Q_k$, i.e.,

$$x_{k+1} = P_{C_k \cap Q_k}(x_0).$$

Step 6. Put $k := k + 1$ and go to **Step 1**.

Before going to prove the strong convergence of the CHEM Algorithm, we guarantee the well-definedness of the constructed sequence by the following lemma.

Lemma 3.3.1. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by CHEM Algorithm is well-defined.*

Proof. To get the conclusion, it suffices to show that $C_k \cap Q_k$ is a nonempty closed convex subset of H, for each $k \in \mathbb{N} \cup \{0\}$. First, we will assert the non-emptiness by showing that $S \subset C_k \cap Q_k$, for each $k \in \mathbb{N} \cup \{0\}$.

Let $k \in \mathbb{N} \cup \{0\}$ be fixed and let $q \in S$. Then, by Lemma 2.4.4 (ii), we have

$$\|z_k - q\|^2 \leq \|x_k - q\|^2 - (1 - 2\rho_k L_1)\|x_k - y_k\|^2 - (1 - 2\rho_k L_2)\|y_k - z_k\|^2.$$

This implies that

$$\|z_k - q\| \leq \|x_k - q\|. \tag{3.3.1}$$

Since for each $j \in \{1, 2, \ldots, M\}$, we also have $q \in Fix(T_j)$, it follows from the quasi-nonexpansivity of each T_j that

$$\begin{aligned} \|t_k - q\| &\leq \alpha_k \|x_k - q\| + (1 - \alpha_k)\|T_{[k]_M} x_k - q\| \\ &\leq \alpha_k \|x_k - q\| + (1 - \alpha_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{3.3.2}$$

and

$$\begin{aligned} \|u_k - q\| &\leq \beta_k \|t_k - q\| + (1 - \beta_k)\|T_{[k]_M} z_k - q\| \\ &\leq \beta_k \|t_k - q\| + (1 - \beta_k)\|z_k - q\|. \end{aligned}$$

Thus, in view of (3.3.1) and (3.3.2), we get

$$\begin{aligned} \|u_k - q\| &\leq \beta_k \|x_k - q\| + (1 - \beta_k)\|x_k - q\| \\ &= \|x_k - q\|. \end{aligned} \tag{3.3.3}$$

Using this relation, in view of the definition of C_k, we see that $q \in C_k$. Since $k \in \mathbb{N} \cup \{0\}$ is arbitrary, we can conclude that $S \subset C_k$, for each $k \in \mathbb{N} \cup \{0\}$.

Next, we will show that $S \subset Q_k$, for each $k \in \mathbb{N} \cup \{0\}$, by induction. Let $q \in S$. It is obvious $S \subset Q_0 = C$. Now, suppose that $S \subset Q_k$. Observe that, since $x_{k+1} = P_{C_k \cap Q_k}(x_0)$, by Theorem 2.3.18 (i), we have

$$\langle x_0 - x_{k+1}, x - x_{k+1} \rangle \leq 0, \forall x \in C_k \cap Q_k.$$

It follows that

$$\langle x_0 - x_{k+1}, q - x_{k+1} \rangle \leq 0, \forall q \in S.$$

This implies that $q \in Q_{k+1}$, and so $S \subset Q_{k+1}$. Thus, by induction, we conclude that $S \subset Q_k$, for each $k \in \mathbb{N} \cup \{0\}$. Then, since S is a nonempty set, it follows that $C_k \cap Q_k$ is a nonempty closed convex subset, for each $k \in \mathbb{N} \cup \{0\}$. Consequently, we can guarantee that $\{x_k\}$ is well-defined. □

Now, we are ready to prove the strong convergence theorem of the sequence $\{x_k\}$ which is generated by the CHEM Algorithm.

Theorem 3.3.2. *If the solution set S is nonempty, then the sequence $\{x_k\}$ which is generated by CHEM Algorithm converges strongly to $P_S(x_0)$.*

Proof. Let $q \in S$ be picked. By the definition of Q_k and Theorem 2.3.18 (i), we observe that $x_k = P_{Q_k}(x_0)$, for each $k \in \mathbb{N} \cup \{0\}$. Thus, since $S \subset Q_k$, we have

$$\|x_k - x_0\| \leq \|q - x_0\|, \tag{3.3.4}$$

for each $k \in \mathbb{N} \cup \{0\}$. This implies that the sequence $\{x_k\}$ is bounded. Thus, by the relations (3.3.1), (3.3.2), and (3.3.3), we have $\{z_k\}$, $\{t_k\}$, and $\{u_k\}$ are also bounded.

Next, consider,

$$\begin{aligned} \|x_{k+1} - x_k\|^2 &= \|x_{k+1} - x_0\|^2 + \|x_0 - x_k\|^2 + 2\langle x_{k+1} - x_0, x_0 - x_k \rangle \\ &= \|x_{k+1} - x_0\|^2 + \|x_0 - x_k\|^2 + 2\langle x_{k+1} - x_k, x_0 - x_k \rangle - 2\|x_0 - x_k\|^2 \\ &= \|x_{k+1} - x_0\|^2 - \|x_0 - x_k\|^2 + 2\langle x_{k+1} - x_k, x_0 - x_k \rangle, \qquad (3.3.5) \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Note that, since $x_k = P_{Q_k}(x_0)$ and $x_{k+1} \in Q_k$, we have

$$\langle x_{k+1} - x_k, x_0 - x_k \rangle \leq 0,$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, from (3.3.5), we have

$$\|x_{k+1} - x_k\|^2 \leq \|x_{k+1} - x_0\|^2 - \|x_0 - x_k\|^2, \tag{3.3.6}$$

for each $k \in \mathbb{N} \cup \{0\}$. This implies that

$$\|x_k - x_0\| \leq \|x_{k+1} - x_0\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. This means that $\{\|x_k - x_0\|\}$ is a nondecreasing sequence. Consequently, by using this one together with the boundness property of $\{\|x_k - x_0\|\}$, we can conclude that $\{\|x_k - x_0\|\}$ is a convergent sequence. Thus, in view of (3.3.6), we also have

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.3.7}$$

By the definition of C_k and $x_{k+1} \in C_k$, we see that

$$\|x_{k+1} - u_k\| \leq \|x_{k+1} - x_k\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that

$$\begin{aligned} \|u_k - x_k\| &\leq \|u_k - x_{k+1}\| + \|x_{k+1} - x_k\| \\ &\leq \|x_{k+1} - x_k\| + \|x_{k+1} - x_k\| \\ &= 2\|x_{k+1} - x_k\|, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, by applying (3.3.7) to the above inequality, we get

$$\lim_{k\to\infty} \|u_k - x_k\| = 0. \tag{3.3.8}$$

Next, for each $j \in \{1, 2, \ldots, M\}$, by (3.3.2) and the quasi-nonexpansivity of T_j, we see that

$$\begin{aligned}\|u_k - q\|^2 &= \|\beta_k(t_k - q) + (1-\beta_k)(T_{[k]_M} z_k - q)\|^2 \\ &= \beta_k\|t_k - q\|^2 + (1-\beta_k)\|T_{[k]_M} z_k - q\|^2 - \beta_k(1-\beta_k)\|t_k - T_{[k]_M} z_k\|^2 \\ &\le \beta_k\|t_k - q\|^2 + (1-\beta_k)\|T_{[k]_M} z_k - q\|^2, \\ &\le \beta_k\|x_k - q\|^2 + (1-\beta_k)\|z_k - q\|^2,\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. So, by applying Lemma 2.4.4 (ii) to the vector z_k, we have

$$\begin{aligned}\|u_k - q\|^2 &\le \beta_k\|x_k - q\|^2 + (1-\beta_k)[\|x_k - q\|^2 - (1-2\rho_k L_1)\|x_k - y_k\|^2 \\ &\quad -(1-2\rho_k L_2)\|y_k - z_k\|^2] \\ &\le \|x_k - q\|^2 - (1-\beta_k)[(1-2\rho_k L_1)\|x_k - y_k\|^2 + (1-2\rho_k L_2)\|y_k - z_k\|^2],\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. This means

$$\begin{aligned}&(1-\beta_k)[(1-2\rho_k L_1)\|x_k - y_k\|^2 + (1-2\rho_k L_2)\|y_k - z_k\|^2] \\ &\le \|x_k - u_k\|(\|x_k - q\| + \|u_k - q\|),\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by (3.3.8) and the properties of the control sequences $\{\beta_k\}$, $\{\rho_k\}$, we obtain

$$\lim_{k\to\infty} \|x_k - y_k\| = 0, \tag{3.3.9}$$

and

$$\lim_{k\to\infty} \|y_k - z_k\| = 0. \tag{3.3.10}$$

These imply that

$$\lim_{k\to\infty} \|x_k - z_k\| = 0. \tag{3.3.11}$$

Using this one together with (3.3.7), we have

$$\lim_{k\to\infty} \|z_{k+1} - z_k\| = 0. \tag{3.3.12}$$

Now, for each fixed $j \in \{1, 2, \ldots, M\}$, we consider

$$\|z_{k+j} - z_k\| \leq \|z_{k+j} - z_{k+(j-1)}\| + \|z_{k+(j-1)} - z_{k+(j-2)}\| + \ldots + \|z_{k+1} - z_k\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, by using (3.3.12), we have

$$\lim_{k\to\infty} \|z_{k+j} - z_k\| = 0, \tag{3.3.13}$$

for each fixed $j \in \{1, 2, \ldots, M\}$.

From the definition of u_k, we see that

$$\begin{aligned}
(1-\beta_k)\|T_{[k]_M} z_k - z_k\| &= \|u_k - z_k - \beta_k(t_k - z_k)\| \\
&\leq \|u_k - z_k\| + \beta_k\|t_k - z_k\| \\
&\leq \|u_k - x_k\| + \beta_k\|t_k - x_k\| + (1+\beta_k)\|x_k - z_k\| \\
&= \|u_k - x_k\| + \beta_k\|\alpha_k x_k + (1-\alpha_k)T_{[k]_M} x_k - x_k\| \\
&\quad + (1+\beta_k)\|x_k - z_k\| \\
&= \|u_k - x_k\| + \beta_k(1-\alpha_k)\|x_k - T_{[k]_M} x_k\| \\
&\quad + (1+\beta_k)\|x_k - z_k\|,
\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by using the assumption on $\{\alpha_k\}$ together with (3.3.8), and (3.3.11), we get

$$\lim_{k\to\infty} \|T_{[k]_M} z_k - z_k\| = 0. \tag{3.3.14}$$

Next, since $\{x_k\}$ is a bounded sequence, we can find a subsequence $\{x_{k_m}\}$ of $\{x_k\}$ and $p \in H$ such that $x_{k_m} \rightharpoonup p$, as $m \to \infty$. We now show that $p \in S$.

First, we show that $p \in \cap_{j=1}^{M} Fix(T_j)$. We know that, by using (3.3.11), the subsequence $\{z_{k_m}\}$ of $\{z_k\}$ also weakly converges to p. This together with (3.3.13), for each $j \in \{1, 2, \ldots, M\}$, we have $z_{k_m+j} \rightharpoonup p$, as $m \to \infty$.

Now, let $j \in \{1, 2, \ldots, M\}$ be fixed. For $m = 0$, we see that there is $\triangle_0^j \in \{1, 2, \ldots, M\}$ such that $[k_0 + \triangle_0^j]_M = j$. Put $r_0^j = k_0 + \triangle_0^j$. Again, for $m \geq 1$, there is $\Delta_m^j \in \{1,2,\ldots,M\}$ such that $[k_m + \Delta_m^j]_M = j$. Put $r_m^j = \min A_{m-1}^j$, where $A_{m-1}^j = \{k_l + \Delta_l^j : k_l + \Delta_l^j > r_{m-1}^j \text{ and } l > m-1\}$. Then, for each $j \in \{1,2,\ldots,M\}$, we can choose a subsequence $\{r_m^j\}$ such that $[r_m^j]_M = j$, and $z_{r_m^j} \rightharpoonup p$, as $m \to \infty$. This together with (3.3.14) implies that

$$0 = \lim_{m\to\infty} \|T_{[r_m^j]_M} z_{r_m^j} - z_{r_m^j}\| = \lim_{m\to\infty} \|T_j z_{r_m^j} - z_{r_m^j}\|, \tag{3.3.15}$$

for each $j \in \{1,2,\ldots,M\}$. Combining with $z_{r_m^j} \rightharpoonup p$, as $m \to \infty$, by the demiclosedness at 0 of $I - T_j$, implies that

$$T_j p = p,$$

for each $j = 1,2,\ldots,M$.

Next, we show that $p \in \cap_{i=1}^N EP(f_i, C)$. Similarly, by using (3.3.7), for each fixed $i \in \{1,2,\ldots,N\}$, we get that $\lim_{k\to\infty} \|x_{k+i} - x_k\| = 0$. It follows from $x_{k_m} \rightharpoonup p$, as $m \to \infty$, that for each $i \in \{1,2,\ldots,N\}$, we have $x_{k_m+i} \rightharpoonup p$, as $m \to \infty$. Then, for each $i \in \{1,2,\ldots,N\}$, we can choose a subsequence $\{r_n^i\}$ such that $[r_n^i]_N = i$, and $x_{r_n^i} \rightharpoonup p$, as $n \to \infty$. This together with (3.3.9) implies that for each $i \in \{1,2,\ldots,N\}$, we obtain $y_{r_n^i} \rightharpoonup p$, as $n \to \infty$. By Lemma 2.4.4 (i), for each $i \in \{1,2,\ldots,N\}$, we have

$$\rho_{r_n^i}[f_{[r_n^i]_N}(x_{r_n^i}, y) - f_{[r_n^i]_N}(x_{r_n^i}, y_{r_n^i})] \geq \langle y_{r_n^i} - x_{r_n^i}, y_{r_n^i} - y\rangle, \forall y \in C.$$

This implies that, for each $i \in \{1,2,\ldots,N\}$, we have

$$f_{[r_n^i]_N}(x_{r_n^i}, y) - f_{[r_n^i]_N}(x_{r_n^i}, y_{r_n^i}) \geq -\frac{1}{\rho_{r_n^i}}\|y_{r_n^i} - x_{r_n^i}\|\|y_{r_n^i} - y\|, \forall y \in C.$$

By using (3.3.9) and the weak continuity of each f_i ($i \in \{1,2,\ldots,N\}$), we obtain that

$$f_i(p, y) \geq 0, \forall y \in C,$$

for each $i = 1,2,\ldots,N$. Then, we had shown that $p \in S$, and so $\omega_w(x_k) \subset S$.

Finally, we show that the sequence $\{x_k\}$ converges strongly to $P_S(x_0)$.

In fact, since $x_k = P_{Q_k}(x_0)$, it follows from $P_S(x_0) \in S \subset Q_k$ that

$$\|x_k - x_0\| \leq \|P_S(x_0) - x_0\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by Theorem 2.4.7, we can conclude that the sequence $\{x_k\}$ converges strongly to $P_S(x_0)$. This completes the proof. □

3.3.2 Parallel Hybrid Extragradient Method (PHEM)

In this part, we will show the strong convergence theorem of the PHEM Algorithm. And, we will consider the interesting result in the case $M = N = 1$.

Firstly, we propose the PHEM Algorithm as follows:

PHEM Algorithm. Choose parameters $\{\rho_k^i\}$ with $0 < \inf \rho_k^i \leq \sup \rho_k^i < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}, i = 1, 2, \ldots, N$, $\{\alpha_k\} \subset [0, 1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, and $\{\beta_k\} \subset [0, 1)$ with $0 \leq \inf \beta_k \leq \sup \beta_k < 1$. Pick $x_0 \in C$.

Step 1. Solve N strongly convex programs

$$y_k^i = \arg\min\{\rho_k^i f_i(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, i = 1, 2, \ldots, N.$$

Step 2. Solve N strongly convex programs

$$z_k^i = \arg\min\{\rho_k^i f_i(y_k^i, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, i = 1, 2, \ldots, N.$$

Step 3. Find the farthest element from x_k among z_k^i, $i = 1, 2, \ldots, N$, i.e.,

$$\overline{z}_k = \arg\max\{\|z_k^i - x_k\| : i = 1, 2, \ldots, N\}.$$

Step 4. Compute

$$t_k^j = \alpha_k x_k + (1-\alpha_k)T_j x_k, j = 1,2,\ldots,M,$$
$$u_k^j = \beta_k t_k^j + (1-\beta_k)T_j \overline{z}_k, j = 1,2,\ldots,M.$$

Step 5. Find the farthest element from x_k among u_k^j, $j = 1,2,\ldots,M$, i.e.,

$$\overline{u}_k = \arg\max\{\|u_k^j - x_k\| : j = 1,2,\ldots,M\}.$$

Step 6. Construct two closed convex subsets of C

$$C_k = \{x \in C : \|x - \overline{u}_k\| \leq \|x - x_k\|\},$$
$$Q_k = \{x \in C : \langle x_0 - x_k, x - x_k\rangle \leq 0\}.$$

Step 7. The next approximation x_{k+1} is defined as the projection of x_0 onto $C_k \cap Q_k$, i.e.,

$$x_{k+1} = P_{C_k\cap Q_k}(x_0).$$

Step 8. Put $k := k+1$ and go to **Step 1**.

Now, we are in a position to prove the strong convergence theorem of the sequence $\{x_k\}$ which is generated by the PHEM Algorithm.

Theorem 3.3.3. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by PHEM Algorithm converges strongly to $P_S(x_0)$.*

Proof. Let $q \in S$. By the definition of $\overline{z}_k$, we suppose that $i_k \in \{1,2,\ldots,N\}$ such that $z_k^{i_k} = \overline{z}_k = \arg\max\{\|z_k^i - x_k\| : i = 1,2,\ldots,N\}$. Then, by Lemma 2.4.4 (ii), we have

$$\|\overline{z}_k - q\|^2 \leq \|x_k - q\|^2 - (1 - 2\rho_k^{i_k} L_1)\|x_k - y_k^{i_k}\|^2 - (1 - 2\rho_k^{i_k} L_2)\|y_k^{i_k} - \overline{z}_k\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. This implies that

$$\|\overline{z}_k - q\| \leq \|x_k - q\|, \tag{3.3.16}$$

for each $k \in \mathbb{N} \cup \{0\}$. Since for each $j \in \{1, 2, \ldots, M\}$, we also have $q \in Fix(T_j)$, it follows from the quasi-nonexpansivity of each T_j that

$$\begin{aligned} \|t_k^j - q\| &\leq \alpha_k\|x_k - q\| + (1 - \alpha_k)\|T_j x_k - q\| \\ &\leq \alpha_k\|x_k - q\| + (1 - \alpha_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{3.3.17}$$

for each $k \in \mathbb{N} \cup \{0\}$. Besides, by the definition of $\overline{u}_k$, we suppose that $j_k \in \{1, 2, \ldots, M\}$ such that $u_k^{j_k} = \overline{u}_k = \arg\max\{\|u_k^j - x_k\| : j = 1, 2, \ldots, M\}$. It follows from the quasi-nonexpansivity of each T_j, $j \in \{1, 2, \ldots, M\}$, that

$$\begin{aligned} \|\overline{u}_k - q\| &\leq \beta_k\|t_k^{j_k} - q\| + (1 - \beta_k)\|T_{j_k}\overline{z}_k - q\| \\ &\leq \beta_k\|t_k^{j_k} - q\| + (1 - \beta_k)\|\overline{z}_k - q\|, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, in view of (3.3.16), and (3.3.17), we get

$$\begin{aligned} \|\overline{u}_k - q\| &\leq \beta_k\|x_k - q\| + (1 - \beta_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{3.3.18}$$

for each $k \in \mathbb{N} \cup \{0\}$. Following the proof of Lemma 3.3.1 and Theorem 3.3.2, we can show that $S \subset C_k \cap Q_k$, for each $k \in \mathbb{N} \cup \{0\}$. Moreover, we can check that the sequence $\{x_k\}$ is bounded, and

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.3.19}$$

By the definition of C_k and $x_{k+1} \in C_k$, we see that

$$\|x_{k+1} - \overline{u}_k\| \leq \|x_{k+1} - x_k\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that

$$\begin{aligned} \|\overline{u}_k - x_k\| &\leq \|\overline{u}_k - x_{k+1}\| + \|x_{k+1} - x_k\| \\ &\leq \|x_{k+1} - x_k\| + \|x_{k+1} - x_k\| \\ &= 2\|x_{k+1} - x_k\|, \end{aligned} \tag{3.3.20}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, applying (3.3.19) to the above inequality, we get

$$\lim_{k\to\infty} \|\overline{u}_k - x_k\| = 0.$$

From the definition of $\overline{u}_k$, we have

$$\lim_{k\to\infty} \|u_k^j - x_k\| = 0, \tag{3.3.21}$$

for each $j = 1, 2, \ldots, M$.

Next, for each $j = 1, 2, \ldots, M$, by (3.3.17) and the quasi-nonexpansivity of T_j, we see that

$$\begin{aligned} \|u_k^j - q\|^2 &= \|\beta_k(t_k^j - q) + (1 - \beta_k)(T_j\overline{z}_k - q)\|^2 \\ &= \beta_k\|t_k^j - q\|^2 + (1 - \beta_k)\|T_j\overline{z}_k - q\|^2 - \beta_k(1 - \beta_k)\|t_k^j - T_j\overline{z}_k\|^2 \\ &\leq \beta_k\|t_k^j - q\|^2 + (1 - \beta_k)\|T_j\overline{z}_k - q\|^2 \\ &\leq \beta_k\|x_k - q\|^2 + (1 - \beta_k)\|\overline{z}_k - q\|^2, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. So, by applying Lemma 2.4.4 (ii) to the vector $\overline{z}_k$, we have

$$\begin{aligned} \|u_k^j - q\|^2 &\leq \beta_k\|x_k - q\|^2 + (1 - \beta_k)[\|x_k - q\|^2 - (1 - 2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 \\ &\quad -(1 - 2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2] \\ &= \|x_k - q\|^2 - (1 - \beta_k)[(1 - 2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 + (1 - 2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2], \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. This means

$$(1-\beta_k)[(1-2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 + (1-2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2]$$
$$\leq \|x_k - u_k^j\|(\|x_k - q\| + \|u_k^j - q\|),$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by (3.3.21) and the properties of the control sequences $\{\beta_k\}$, $\{\rho_k^i\}$, we obtain

$$\lim_{k\to\infty} \|x_k - y_k^{i_k}\| = 0, \tag{3.3.22}$$

and

$$\lim_{k\to\infty} \|y_k^{i_k} - \overline{z}_k\| = 0. \tag{3.3.23}$$

These imply that

$$\lim_{k\to\infty} \|x_k - \overline{z}_k\| = 0. \tag{3.3.24}$$

Then, by the definition of $\overline{z}_k$, we have

$$\lim_{k\to\infty} \|x_k - z_k^i\| = 0, \tag{3.3.25}$$

for each $i = 1, 2, \ldots, N$. Moreover, by Lemma 2.4.4 (ii), for each $i = 1, 2, \ldots, N$, we get that

$$\|z_k^i - q\|^2 \leq \|x_k - q\|^2 - (1-2\rho_k^i L_1)\|x_k - y_k^i\|^2 - (1-2\rho_k^i L_2)\|y_k^i - z_k^i\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that, for each $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}(1-2\rho_k^i L_1)\|x_k - y_k^i\|^2 + (1-2\rho_k^i L_2)\|y_k^i - z_k^i\|^2 &\leq \|x_k - q\|^2 - \|z_k^i - q\|^2 \\ &= \|x_k - z_k^i\|(\|x_k - q\| + \|z_k^i - q\|),\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. This together with (3.3.25) implies that

$$\lim_{k\to\infty} \|x_k - y_k^i\| = 0, \tag{3.3.26}$$

and

$$\lim_{k\to\infty} \|y_k^i - z_k^i\| = 0, \tag{3.3.27}$$

for each $i = 1, 2, \ldots, N$. From the definition of u_k^j, for each $j = 1, 2, \ldots, M$, we see that

$$\begin{aligned}
(1-\beta_k)\|T_j\overline{z}_k - \overline{z}_k\| &= \|u_k^j - \overline{z}_k - \beta_k(t_k^j - \overline{z}_k)\| \\
&\leq \|u_k^j - \overline{z}_k\| + \beta_k\|t_k^j - \overline{z}_k\| \\
&\leq \|u_k^j - x_k\| + \beta_k\|t_k^j - x_k\| + (1+\beta_k)\|x_k - \overline{z}_k\| \\
&= \|u_k^j - x_k\| + \beta_k\|\alpha_k x_k + (1-\alpha_k)T_j x_k - x_k\| \\
&\quad +(1+\beta_k)\|x_k - \overline{z}_k\| \\
&= \|u_k^j - x_k\| + \beta_k(1-\alpha_k)\|x_k - T_j x_k\| + (1+\beta_k)\|x_k - \overline{z}_k\|,
\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by using the assumption on $\{\alpha_k\}$ together with (3.3.21) and (3.3.24), we get

$$\lim_{k\to\infty} \|T_j\overline{z}_k - \overline{z}_k\| = 0, \tag{3.3.28}$$

for each $j = 1, 2, \ldots, M$.

Next, since $\{x_k\}$ is a bounded sequence, we can find a subsequence $\{x_{k_m}\}$ of $\{x_k\}$ and $p \in H$ such that $x_{k_m} \rightharpoonup p$, as $m \to \infty$. We now show that $p \in S$.

We know that, by using (3.3.24), the subsequence $\{\overline{z}_{k_m}\}$ of $\{\overline{z}_k\}$ also weakly converges to p. This together with (3.3.28), by the demiclosedness at 0 of $I - T_j$, implies that

$$T_j p = p,$$

for each $j = 1, 2, \ldots, M$.

On the other hand, by using (3.3.26), for each $i \in \{1, 2, \ldots, N\}$, we get that $y_{k_m}^i \rightharpoonup p$, as $m \to \infty$. Thus, by Lemma 2.4.4 (i), for each $i \in \{1, 2, \ldots, N\}$, we have

$$\rho_{k_m}^i[f_i(x_{k_m}, y) - f_i(x_{k_m}, y_{k_m}^i)] \geq \langle y_{k_m}^i - x_{k_m}, y_{k_m}^i - y\rangle, \forall y \in C.$$

This implies that, for each $i = 1, 2, \ldots, N$, we get

$$f_i(x_{k_m}, y) - f_i(x_{k_m}, y_{k_m}^i) \geq -\frac{1}{\rho_{k_m}^i}\|y_{k_m}^i - x_{k_m}\|\|y_{k_m}^i - y\|, \forall y \in C.$$

It follows from (3.3.26) and the weak continuity of each f_i ($i \in \{1, 2, \ldots, N\}$) that

$$f_i(p, y) \geq 0, \forall y \in C,$$

for each $i = 1, 2, \ldots, N$. Then, we had shown that $p \in S$, and so $\omega_w(x_k) \subset S$.

The rest of the proof is similar to the arguments in the proof of Theorem 3.3.2, and it leads to the conclusion that the sequence $\{x_k\}$ converges strongly to $P_S(x_0)$. □

Remark 3.3.4. We observe that if $\alpha_k = 1$, for each $k \in \mathbb{N} \cup \{0\}$, then the PHEM Algorithm reduces to the PHMEM Algorithm, which was presented in [53]. We point out that, by Remark 2.3.9, we know that the class of quasi-nonexpansive mapping is larger than the class of nonexpansive mapping. The PHEM Algorithm can solve quasi-nonexpansive mappings meanwhile the PHMEM Algorithm may not be applied in this situation.

The next result is an improvement version of the Algorithm (3.1.8) in the reference [51]. Notice that, in this section, we consider the class of quasi-nonexpansive mapping while in [51] the authors considered the class of symmetric generalized hybrid mapping.

Corollary 3.3.5. *Let T be a quasi-nonexpansive self-mapping on C with $I - T$ demiclosed at 0 and let f be a bifunction satisfies the assumptions $(A1)-(A4)$. Suppose that the solution set $S = EP(f, C) \cap Fix(T)$ is nonempty. Pick $x_0 \in C$, choose parameters $\{\rho_k\}$ with $0 < \inf \rho_k \leq \sup \rho_k < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0, 1]$*

such that $\lim_{k\to\infty}\alpha_k = 1$, $\{\beta_k\} \subset [0,1)$ *with* $0 \le \inf\beta_k \le \sup\beta_k < 1$, *and the sequences* $\{x_k\}$, $\{y_k\}$, $\{z_k\}$, $\{t_k\}$, $\{u_k\}$ *are defined by*

$$\begin{cases} y_k = \arg\min\{\rho_k f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\ z_k = \arg\min\{\rho_k f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\ t_k = \alpha_k x_k + (1-\alpha_k)Tx_k, \\ u_k = \beta_k t_k + (1-\beta_k)Tz_k, \\ C_k = \{x \in C : \|x - u_k\| \le \|x - x_k\|\}, \\ Q_k = \{x \in C : \langle x_0 - x_k, x - x_k\rangle \le 0\}, \\ x_{k+1} = P_{C_k\cap Q_k}(x_0). \end{cases} \tag{3.3.29}$$

Then, the sequence $\{x_k\}$ *converges strongly to* $P_S(x_0)$.

From now on, the algorithm (3.3.29) will be called Hybrid Extragradient Method (HEM).

3.3.3 Numerical experiments

In this part, we consider some examples and numerical results to support the main theorems. Additionally, we will compare the two introduced algorithms, CHEM and PHEM, with the PHMEM Algorithm, which was presented in [53]. In the case $M = N = 1$, we will compare the HEM Algorithm (5.2.1) with the algorithm that was presented in [52]. The numerical experiments are written in Matlab R2015b and performed on a Desktop with AMD Dual Core R3-2200U CPU @ 2.50GHz and RAM 4.00 GB.

Example 3.3.6 Consider a real Hilbert space $H = \mathbb{R}^n$, and $C = H$. The bifunctions f_i, $i = 1, 2, \ldots, N$, which are given by the form of Nash-Cournot equilibrium model [2], are defined by

$$f_i(x, y) = \langle P_i x + Q_i y, y - x\rangle,\ \forall x, y \in \mathbb{R}^n,\ i = 1, 2, \ldots, N,$$

where $P_i \in \mathbb{R}^{n\times n}$, and $Q_i \in \mathbb{R}^{n\times n}$ are symmetric positive semidefinite matrices such that $P_i - Q_i$ are also positive semidefinite matrices. We know that the bifunctions f_i, $i = 1, 2, \ldots, N$, satisfy conditions $(A1) - (A4)$, see [2]. Notice that the bifunctions f_i, $i = 1, 2, \ldots, N$, are Lipschitz-type continuous with constants $L_1^i = L_2^i = \frac{1}{2}\|P_i - Q_i\|$. Choose $L_1 = L_2 = \max\{L_1^i : i = 1, 2, \ldots, N\}$. Then, the bifunctions f_i, $i = 1, 2, \ldots, N$, are Lipschitz-type continuous with constants L_1 and L_2. On the other hand, for the boxes D_j, $j = 1, 2, \ldots, M$, which are given by

$$D_j = \{x \in \mathbb{R}^n : -d_j \leq x_l \leq d_j, \forall l = 1, 2, \ldots, n\},\ j = 1, 2, \ldots, M,$$

where d_j are the positive real numbers, we will consider the nonexpansive mappings T_j, $j = 1, 2, \ldots, M$, which are defined by

$$T_j = P_{D_j},\ j = 1, 2, \ldots, M.$$

The numerical experiment is considered under the following setting: for each $i = 1, 2, \ldots, N$, the matrices P_i, and Q_i are randomly chosen from the interval $[-5, 5]$ such that they satisfy the above required properties. Besides, for each $j = 1, 2, \ldots, M$, the real numbers d_j are randomly chosen from the interval $(0, 3)$. We will concern with these parameters: $\rho_k = \frac{0.49}{L_1}$, for the CHEM Algorithm, and $\rho_k^i = \frac{0.49}{L_1}$, $i = 1, 2, \ldots, N$, for the PHEM Algorithm, when $n = 10$, $N = 10$, and $M = 20$. The following five cases of the parameters α_k and β_k are considered:

Case 1. $\alpha_k = 1 - \dfrac{1}{\ln(k+3)}$, $\beta_k = \dfrac{1}{k+2}$.

Case 2. $\alpha_k = 1 - \dfrac{1}{\ln(k+3)}$, $\beta_k = 0.5 + \dfrac{1}{k+3}$.

Case 3. $\alpha_k = 1$, $\beta_k = \dfrac{1}{k+2}$.

Case 4. $\alpha_k = 1$, $\beta_k = 0.5 + \dfrac{1}{k+3}$.

Case 5. $\alpha_k = 1$, $\beta_k = 0$.

The function *quadprog* in Matlab Optimization Toolbox was used to solve vectors y_k, z_k, for the CHEM Algorithm; y_k^i, z_k^i, $i = 1, 2, \ldots, N$, for the PHEM Algorithm. Note that the solution set S is nonempty because of $0 \in S$. The PHMEM Algorithm was tested by using the starting point x_0 as $(1, 1, \ldots, 1)^T \in \mathbb{R}^n$, and the stopping criteria $\|x_{k+1} - x_k\| < 10^{-4}$ for approximating solution $x^* \in S$. After that, the CHEM and PHEM algorithms were tested along with the PHMEM Algorithm by using the starting point x_0 as $(1, 1, \ldots, 1)^T \in \mathbb{R}^n$, and the stopping criteria $\|x_k - x^*\| < 10^{-4}$. Notice that the metric projection of a point x_0 onto the set $C_k \cap Q_k$ was computed by using the explicit formula as in [63].

Table 3 **The numerical results for five different cases parameters α_k and β_k**

	CPU times (sec)			Number of iterations		
Cases	CHEM	PHEM	PHMEM	CHEM	PHEM	PHMEM
1	147.8594	218.0156	216.6094	13255	5824	6154
2	471.0000	777.5313	818.4531	43265	21811	21577
3	108.0156	216.6094	216.6094	13631	6154	6154
4	343.0000	818.4531	818.4531	42921	21577	21577
5	108.7500	217.9375	-	13481	5925	-

Table 3 shows that the number of iterations of the PHEM Algorithm in case 1 is better than other all considered cases. Meanwhile, the CPU times of the CHEM Algorithm in case 3 is better than other all considered cases. We would like to remind that we solve y_k^i, z_k^i, $i = 1, 2, \ldots, N$, by using N bifunctions and compute t_k^j, u_k^j, $j = 1, 2, \ldots, M$, by using M mappings for the PHEM Algorithm. On the other hand, we solve only y_k, z_k, by using a bifunction and compute only t_k, u_k, by using a mapping for the CHEM Algorithm. This should be a reason for the results that the number of iterations of the PHEM Algorithm is better than the CHEM Algorithm, while the CPU times of the CHEM Algorithm is better than the PHEM Algorithm in all considered cases.

Example 3.3.7 In the case $M = N = 1$, we will compare the HEM Algorithm (5.2.1) with the following algorithm (3.3.30), which was presented by Hieu [52], when T is a quasi-nonexpansive mapping and f is a pseudomonotone and Lipschitz-type continuous bifunction with positive constants L_1, L_2:

$$\begin{cases} x_0 \in H, \\ y_k = \arg\min\{\rho_k f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ z_k = \arg\min\{\rho_k f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ x_{k+1} = (1 - \alpha_k - \beta_k) z_k + \beta_k T z_k, \end{cases} \tag{3.3.30}$$

where $\{\rho_k\} \subset [\underline{\rho}, \overline{\rho}]$ with $0 < \underline{\rho} \leq \overline{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0,1]$ such that $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = +\infty$, and $\{\beta_k\} \subset [\underline{\beta}, \overline{\beta}] \subset (0,1)$, for some $\overline{\beta} > \underline{\beta} > 0$. Hieu [52] proved that the sequence $\{x_k\}$ generated by (3.3.30) converges strongly to an element in the solution set $S = EP(f, C) \cap Fix(T)$. In this thesis, the algorithm (3.3.30) will be called NH Algorithm.

Consider a real Hilbert space $H = \mathbb{R}^n$, and $C = H$. Recall that the quadratic function $h : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h(x) = \frac{1}{2} x^T Q x + b^T x,$$

where $b \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive semidefinite matrix. Here, we will focus on the case $(I_n - QQ^+)b = 0$, when I_n is the identity matrix, and $Q^+ \in \mathbb{R}^{n\times n}$ is a pseudoinverse matrix of Q. We consider the bifunction f, which is defined by

$$f(x, y) = h(y) - h(x), \ \forall x, y \in \mathbb{R}^n.$$

It is clear that

$$f(x, y) + f(y, x) = 0, \ \forall x, y \in \mathbb{R}^n.$$

Thus, the bifunction f is monotone, and so is pseudomonotone. Moreover, it is easy to see that

$$f(x,y) + f(y,z) - f(x,z) \geq -\|x-y\|^2 - \|y-z\|^2, \ \forall x,y,z \in \mathbb{R}^n.$$

Then, the bifunction f is Lipschitz-type continuous with constants $L_1 = L_2 = 1$.

On the other hand, for a convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that there is $x \in \mathbb{R}^n$ satisfied $g(x) \leq 0$, we consider a mapping $T : \mathbb{R}^n \to \mathbb{R}^n$, which is defined by

$$T = P_g.$$

Then, by Theorem 2.3.25, we know that T is a quasi-nonexpansive mapping with $I - T$ demiclosed at 0, and $Fix(T) = \{x \in \mathbb{R}^n : g(x) \leq 0\}$.

The numerical experiment is considered under the following setting: $Q_1 \in \mathbb{R}^{n\times n}$ is an orthogonal matrix and its entries are randomly chosen from the interval $(0,5)$. The matrix $Q_2 = (q_{ij}) \in \mathbb{R}^{n\times n}$ is defined by $q_{ij} = a$, if $i = j = 1$; $q_{ij} = 0$, otherwise, where the real number a is randomly chosen from the interval $(4,5)$. The positive semidefinite matrix Q is constructed by $Q = Q_1Q_2Q_1^T$. Besides, we consider $g(x) = \max\{0, \langle c, x\rangle + d\}$, where the real number d is randomly chosen from the interval $(-2,-3)$, and the vector $c \in \mathbb{R}^n$ is randomly chosen from the interval $(0,2)$. Note that the solution set S is nonempty because of $-Q^+b \in S$. We will concern with these parameters: $\rho_k = \frac{1}{5}$, $\alpha_k = 1 - \frac{1}{k+2}$, and $\beta_k = 0.5 + \frac{1}{k+3}$, when $n = 10$. The function *quadprog* in Matlab Optimization Toolbox was used to solve vectors y_k, and z_k. Again, the metric projection of a point x_0 onto the set $C_k \cap Q_k$ was computed by using the explicit formula as in [63]. The HEM Algorithm is compared with the NH Algorithm by using the starting point x_0 as $(0,0,\ldots,0)^T \in \mathbb{R}^n$, and the stopping criteria $\|x_{k+1} - x_k\| < 10^{-6}$. The following results were presented as averages calculated from 10 tested problems.

Table 4 The numerical results for $N = 1$ and $M = 1$

Average CPU Times (sec)		Average Iterations	
HEM	NH	HEM	NH
0.2953	2.1360	91.8	805.3

Table 4 shows that the HEM Algorithm yields better both the CPU times and the number of iterations than the NH Algorithm. We notice that, in this experiment, the starting point $x_0 := 0 \in \mathbb{R}^n$ means that the solution $P_S(0)$ has the minimum norm over the set S. Furthermore, we observe that, if the starting point $x_0 \in S$, the HEM Algorithm will be stopped at the iteration x_1, but the NH Algorithm may not.

CHAPTER IV

ITERATIVE METHOD FOR SOLVING SPLIT EQUILIBRIUM AND FIXED POINT PROBLEMS

In this chapter, we consider the split equilibrium and fixed point problems. Some iterative methods for finding a solution to the split equilibrium and fixed point problems are introduced in real Hilbert spaces. We also apply the obtained main result for the problem of finding a solution to the split variational inequality and fixed point problems. Some numerical examples are considered and the introduced methods are discussed and compared with the well-known algorithm.

4.1 Split equilibrium and fixed point problems

The split feasibility problem was proposed by Censer and Elfving [23] as followed:

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad Lx^* \in Q, \tag{4.1.1}$$

where C and Q are two nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2, respectively, and $L : H_1 \to H_2$ is a bounded linear operator. In finite dimensional Hilbert spaces $H_1 = \mathbb{R}^n$ and $H_2 = \mathbb{R}^m$, Byrne [24] proposed the following CQ method for solving the split feasibility problem:

$$x_{k+1} = P_C(x_k + \delta L^T(P_Q - I)Lx_k), \forall k \in \mathbb{N}, \tag{4.1.2}$$

where $\delta \in (0, 2/\|L\|^2)$, and L is a real $m \times n$ matrix. Byrne proved that the sequence $\{x_k\}$ genenated by (4.1.2) converges strongly to a solution of the split feasibility problem (4.1.1). Later, Xu [25] considered the split feasibility problem in the setting of infinite dimensional Hilbert spaces. In this case, the CQ method becomes

$$x_{k+1} = P_C(x_k + \delta L^*(P_Q - I)Lx_k), \forall k \in \mathbb{N}, \tag{4.1.3}$$

where $\delta \in (0, 2/\|L\|^2)$, and L^* is the adjoint operator of L. Xu proved that the sequence $\{x_k\}$ genenated by (4.1.3) converges weakly to a solution of the split feasibility problem (4.1.1).

Moudafi [64] (see also He [65]) introduced the split equilibrium problem, as a generalization of the split feasibility problem, as follows:

$$\begin{cases} \text{Find} \quad x^* \in C \quad \text{such that} \quad f(x^*, y) \geq 0, \forall y \in C, \\ \text{and} \quad u^* = Lx^* \in Q \quad \text{solves} \quad g(u^*, v) \geq 0, \forall v \in Q, \end{cases} \tag{4.1.4}$$

where C, Q are two nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2, respectively, $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ are bifunctions, and $L : H_1 \to H_2$ is a bounded linear operator. By using the extragradient method, Kim and Dinh [66] proposed the following algorithm for finding a solution of the split equilibrium problem when the bifunctions f and g are pseudomonotone and Lipschitz-type continuous with positive constants c_1 and c_2:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min\{\lambda_k f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\ z_k = \arg\min\{\lambda_k f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\ u_k = \arg\min\{\mu_k g(Lz_k, u) + \frac{1}{2}\|u - Lz_k\|^2 : u \in Q\}, \\ v_k = \arg\min\{\mu_k g(u_k, u) + \frac{1}{2}\|u - Lz_k\|^2 : u \in Q\}, \\ x_{k+1} = P_C(z_k + \delta L^*(v_k - Lz_k)), \end{cases} \tag{4.1.5}$$

where $\{\lambda_k\}, \{\mu_k\} \subset [\underline{\rho}, \overline{\rho}]$ with $0 < \underline{\rho} \leq \overline{\rho} < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$, and L^* is the adjoint operator of L. They proved that the sequence $\{x_k\}$ genenated by (4.1.5) converges weakly to a solution of the split equilibrium problem (4.1.4). In this thesis, the algorithm (4.1.5) will be called PEA Algorithm.

In 2016, Dinh et al. [67] considered the split equilibrium and fixed point problems as follows:

$$\begin{cases} \text{Find} \quad x^* \in C \quad \text{such that} \quad Tx^* = x^*, \quad f(x^*, y) \geq 0, \forall y \in C, \\ \text{and} \quad u^* = Lx^* \in Q \quad \text{solves} \quad Su^* = u^*, \quad g(u^*, v) \geq 0, \forall v \in Q. \end{cases} \tag{4.1.6}$$

where C and Q are two nonempty closed convex subsets of the real Hilbert spaces H_1 and H_2, respectively, $f : C \times C \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ are bifunctions, $T: C \to C$ and $S: Q \to Q$ are mappings, and $L : H_1 \to H_2$ is a bounded linear operator. From now on, the solution set of the split equilibrium and fixed point problems (4.1.6) will be denoted by Ω. That is:

$$\Omega := \{p \in EP(f, C) \cap Fix(T) : \; Lp \in EP(g, Q) \cap Fix(S)\}.$$

By using both proximal point and extragradient methods together with Mann iterative method, Dinh et al. proposed the following algorithm for finding a solution of the split equilibrium and fixed point problems (4.1.6), when S and T are nonexpansive mappings, g is a monotone bifunction, f is a pseudomonotone and Lipschitz-type continuous bifunction with positive constants c_1 and c_2 :

$$\begin{cases} x_1 \in C, \\ y_k = \arg\min\{\lambda_k f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ z_k = \arg\min\{\lambda_k f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ t_k = (1-\alpha)z_k + \alpha S z_k, \\ u_k = T^g_{r_k} L t_k, \\ x_{k+1} = P_C(t_k + \delta L^*(T u_k - L t_k)), \end{cases} \tag{4.1.7}$$

where $\{\lambda_k\} \subset [\underline{\lambda}, \overline{\lambda}]$ with $0 < \underline{\lambda} \leq \overline{\lambda} < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}$, $\{r_k\} \subset (0, +\infty)$ such that $\liminf_{k\to\infty} r_k > 0$, $\alpha \in (0,1)$, $\delta \in (0, 1/\|L\|^2)$, L^* is the adjoint operator of L, and $T^g_{r_k} L t_k := \{u \in Q : g(u, v) + \frac{1}{r_k}\langle v - u, u - Lt_k\rangle \geq 0, \forall v \in Q\}$. They proved that the sequence $\{x_k\}$ genenated by (4.1.7) converges weakly to a solution of the split equilibrium and fixed point problems (4.1.6).

4.2 A new extragradient method for split equilibrium and fixed point problems

In this section, motivated by the literatures in Section 4.1, we introduce a new extragradient algorithm for finding a solution of the split equilibrium and fixed point problems (4.1.6), when f and g are pseudomonotone and Lipschitz-type continuous bifunctions, and S and T are nonexpansive mappings.

4.2.1 Strong convergence theorem

Let H_1 and H_2 be two real Hilbert spaces and C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Suppose that $f\colon C \times C \to \mathbb{R}$ and $g\colon Q \times Q \to \mathbb{R}$ are bifunctions which satisfy $(A1)-(A4)$ with some positive constants $\{c_1, c_2\}$ and $\{d_1, d_2\}$, respectively. Let $T\colon C \to C$ and $S\colon Q \to Q$ be nonexpansive mappings, $h\colon C \to C$ be a ρ-contraction mapping, and $L\colon H_1 \to H_2$ be a bounded linear operator with its adjoint L^*. We introduce the following algorithm for solving the split equilibrium and fixed point problems (4.1.6).

Algorithm 4.2.1. *Choose $x_1 \in H_1$. The control parameters $\lambda_k, \mu_k, \alpha_k, \beta_k, \delta_k$ satisfy the following conditions*

$$0 < \underline{\lambda} \leq \lambda_k \leq \overline{\lambda} < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\},\ 0 < \underline{\mu} \leq \mu_k \leq \overline{\mu} < \min\left\{\frac{1}{2d_1}, \frac{1}{2d_2}\right\},$$

$$\beta_k \in (0,1),\ 0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1,\ 0 < \underline{\delta} \leq \delta_k \leq \overline{\delta} < \frac{1}{\|L\|^2},$$

$$\alpha_k \in (0, \frac{1}{2-\rho}), \lim_{k\to\infty} \alpha_k = 0, \sum_{k=1}^{\infty} \alpha_k = \infty.$$

Let $\{x_k\}$ be a sequence generated by

$$\begin{cases} u_k = \arg\min\left\{\mu_k g(P_Q(Lx_k), u) + \frac{1}{2}\|u - P_Q(Lx_k)\|^2 \colon u \in Q\right\}, \\ v_k = \arg\min\left\{\mu_k g(u_k, u) + \frac{1}{2}\|u - P_Q(Lx_k)\|^2 \colon u \in Q\right\}, \\ y_k = P_C\left(x_k + \delta_k L^*\left(Sv_k - Lx_k\right)\right), \\ t_k = \arg\min\left\{\lambda_k f(y_k, y) + \frac{1}{2}\|y - y_k\|^2 \colon y \in C\right\}, \\ z_k = \arg\min\left\{\lambda_k f(t_k, y) + \frac{1}{2}\|y - y_k\|^2 \colon y \in C\right\}, \\ x_{k+1} = \alpha_k h(x_k) + (1-\alpha_k)(\beta_k x_k + (1-\beta_k)Tz_k). \end{cases}$$

Theorem 4.2.2. *Suppose that the solution set Ω is nonempty. Then, the sequence $\{x_k\}$ which is generated by Algorithm 4.2.1 converges strongly to $q = P_\Omega h(q)$.*

Proof. Let $p \in \Omega$. So, $p \in EP(f, C) \cap Fix(T) \subset C$ and $Lp \in EP(g, Q) \cap Fix(S) \subset Q$. Since P_Q is firmly nonexpansive, we get

$$\begin{aligned} \|P_Q(Lx_k) - Lp\|^2 &= \|P_Q(Lx_k) - P_Q(Lp)\|^2 \\ &\leq \langle P_Q(Lx_k) - P_Q(Lp), Lx_k - Lp\rangle \\ &= \langle P_Q(Lx_k) - Lp, Lx_k - Lp\rangle \\ &= \frac{1}{2}\left[\|P_Q(Lx_k) - Lp\|^2 + \|Lx_k - Lp\|^2 - \|P_Q(Lx_k) - Lx_k\|^2\right], \end{aligned}$$

and hence,

$$\|P_Q(Lx_k) - Lp\|^2 \leq \|Lx_k - Lp\|^2 - \|P_Q(Lx_k) - Lx_k\|^2. \tag{4.2.1}$$

Since S is nonexpansive, $Lp \in Fix(S)$ and using Lemma 2.4.4(ii) and definition to u_k and v_k, we have

$$\begin{aligned} \|Sv_k - Lp\|^2 &= \|Sv_k - S(Lp)\|^2 \\ &\leq \|v_k - Lp\|^2 \\ &\leq \|P_Q(Lx_k) - Lp\|^2 - (1 - 2\mu_k d_1)\|P_Q(Lx_k) - u_k\|^2 \\ &\quad -(1 - 2\mu_k d_2)\|u_k - v_k\|^2, \end{aligned} \tag{4.2.2}$$

for each $k \in \mathbb{N}$. From (4.2.1), (4.2.2) and assumptions, we obtain

$$\|Sv_k - Lp\|^2 \leq \|Lx_k - Lp\|^2 - \|P_Q(Lx_k) - Lx_k\|^2. \tag{4.2.3}$$

By (4.2.3), we get

$$\begin{aligned}
\langle L(x_k - p), Sv_k - Lx_k \rangle &= \langle Sv_k - Lp, Sv_k - Lx_k \rangle - \|Sv_k - Lx_k\|^2 \\
&= \frac{1}{2}\left[\|Sv_k - Lp\|^2 - \|Lx_k - Lp\|^2 - \|Sv_k - Lx_k\|^2\right] \\
&\leq -\frac{1}{2}\|P_Q(Lx_k) - Lx_k\|^2 - \frac{1}{2}\|Sv_k - Lx_k\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
2\delta_k \langle L(x_k - p), Sv_k - Lx_k \rangle &\leq -\delta_k \|P_Q(Lx_k) - Lx_k\|^2 \\
&\quad -\delta_k \|Sv_k - Lx_k\|^2.
\end{aligned} \tag{4.2.4}$$

Since P_C is nonexpansive and by (4.2.4), we obtain

$$\begin{aligned}
\|y_k - p\|^2 &= \|P_C(x_k + \delta_k L^*(Sv_k - Lx_k)) - P_C(p)\|^2 \\
&\leq \|(x_k - p) + \delta_k L^*(Sv_k - Lx_k)\|^2 \\
&= \|x_k - p\|^2 + \delta_k^2 \|L^*(Sv_k - Lx_k)\|^2 + 2\delta_k \langle x_k - p, L^*(Sv_k - Lx_k) \rangle \\
&\leq \|x_k - p\|^2 + \delta_k^2 \|L\|^2 \|Sv_k - Lx_k\|^2 - \delta_k \|P_Q(Lx_k) - Lx_k\|^2 \\
&\quad -\delta_k \|Sv_k - Lx_k\|^2 \\
&= \|x_k - p\|^2 - \delta_k(1 - \delta_k \|L\|^2)\|Sv_k - Lx_k\|^2 \\
&\quad -\delta_k \|P_Q(Lx_k) - Lx_k\|^2,
\end{aligned} \tag{4.2.5}$$

then, we obtain

$$\|y_k - p\| \leq \|x_k - p\|. \tag{4.2.6}$$

By Lemma 2.4.4(ii), definition of t_k and z_k and assumptions we have

$$\|z_k - p\| \leq \|y_k - p\|, \tag{4.2.7}$$

for each $k \in \mathbb{N}$. From (4.2.6) and (4.2.7), we get

$$\|z_k - p\| \leq \|x_k - p\|. \tag{4.2.8}$$

Set $q_k = \beta_k x_k + (1-\beta_k)Tz_k$. It follows from (4.2.8) that

$$\begin{aligned}\|q_k - p\| &\leq \beta_k\|x_k - p\| + (1-\beta_k)\|Tz_k - p\| \\ &\leq \beta_k\|x_k - p\| + (1-\beta_k)\|z_k - p\| \\ &\leq \|x_k - p\|.\end{aligned} \tag{4.2.9}$$

By definition of x_{k+1} and (4.2.9), we obtain

$$\begin{aligned}\|x_{k+1} - p\| &\leq \alpha_k\|h(x_k) - p\| + (1-\alpha_k)\|q_k - p\| \\ &\leq \alpha_k\|h(x_k) - h(p)\| + \alpha_k\|h(p) - p\| + (1-\alpha_k)\|x_k - p\| \\ &\leq \alpha_k\rho\|x_k - p\| + \alpha_k\|h(p) - p\| + (1-\alpha_k)\|x_k - p\| \\ &\leq (1-\alpha_k(1-\rho))\|x_k - p\| + \alpha_k(1-\rho)\frac{\|h(p) - p\|}{1-\rho} \\ &\leq \max\left\{\|x_k - p\|, \frac{\|h(p) - p\|}{1-\rho}\right\} \\ &\vdots \\ &\leq \max\left\{\|x_1 - p\|, \frac{\|h(p) - p\|}{1-\rho}\right\}.\end{aligned}$$

This implies that the sequence $\{x_k\}$ is bounded. By (4.2.6) and (4.2.8), the sequences $\{y_k\}$ and $\{z_k\}$ are bounded too.

By Lemma 2.4.4(ii), (4.2.6), the definition of q_k and assumptions on β_k and δ_k, we get

$$\begin{aligned}\|q_k - p\|^2 &\leq \beta_k\|x_k - p\|^2 + (1-\beta_k)\|Tz_k - p\|^2 \\ &\leq \beta_k\|x_k - p\|^2 + (1-\beta_k)\|z_k - p\|^2 \\ &\leq \beta_k\|x_k - p\|^2 + (1-\beta_k)[\|y_k - p\|^2 - (1-2\lambda_k c_1)\|y_k - t_k\|^2 \\ &\quad -(1-2\lambda_k c_2)\|t_k - z_k\|^2] \\ &\leq \beta_k\|x_k - p\|^2 + (1-\beta_k)[\|x_k - p\|^2 - (1-2\lambda_k c_1)\|y_k - t_k\|^2 \\ &\quad -(1-2\lambda_k c_2)\|t_k - z_k\|^2] \\ &= \|x_k - p\|^2 - (1-\beta_k)\left[(1-2\lambda_k c_1)\|y_k - t_k\|^2 + (1-2\lambda_k c_2)\|t_k - z_k\|^2\right].\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_{k+1} - p\|^2 &\leq \alpha_k\|h(x_k) - p\|^2 + (1-\alpha_k)\|q_k - p\|^2 \\ &\leq \alpha_k\|h(x_k) - p\|^2 + (1-\alpha_k)\{\|x_k - p\|^2 \\ &\quad -(1-\beta_k)[(1-2\lambda_k c_1)\|y_k - t_k\|^2 + (1-2\lambda_k c_2)\|t_k - z_k\|^2]\},\end{aligned}$$

and hence,

$$\begin{aligned}&(1-\beta_k)\left[(1-2\lambda_k c_1)\|y_k - t_k\|^2 + (1-2\lambda_k c_2)\|t_k - z_k\|^2\right] \\ &\leq \|x_k - p\|^2 - \|x_{k+1} - p\|^2 + \alpha_k M, \qquad (4.2.10)\end{aligned}$$

where

$$\begin{aligned}M &= \sup\{\left|\|h(x_k) - p\|^2 - \|x_k - p\|^2\right| + (1-\beta_k)[(1-2\lambda_k c_1)\|y_k - t_k\|^2 \\ &\quad +(1-2\lambda_k c_2)\|t_k - z_k\|^2], k \in \mathbb{N}\}.\end{aligned}$$

By (4.2.9), we have

$$\begin{aligned}
\|x_{k+1}-p\|^2 &= \|\alpha_k(h(x_k)-p)+(1-\alpha_k)(q_k-p)\|^2 \\
&\le (1-\alpha_k)^2\|q_k-p\|^2+2\alpha_k\langle h(x_k)-p, x_{k+1}-p\rangle \qquad (4.2.11)\\
&\le (1-\alpha_k)^2\|x_k-p\|^2+2\alpha_k\langle h(x_k)-h(p), x_{k+1}-p\rangle \\
&\quad +2\alpha_k\langle h(p)-p, x_{k+1}-p\rangle \\
&\le (1-\alpha_k)^2\|x_k-p\|^2+2\alpha_k\rho\|x_k-p\|\|x_{k+1}-p\| \\
&\quad +2\alpha_k\langle h(p)-p, x_{k+1}-p\rangle \\
&\le (1-\alpha_k)^2\|x_k-p\|^2+\alpha_k\rho\left(\|x_k-p\|^2+\|x_{k+1}-p\|^2\right) \\
&\quad +2\alpha_k\langle h(p)-p, x_{k+1}-p\rangle \\
&= \left((1-\alpha_k)^2+\alpha_k\rho\right)\|x_k-p\|^2+\alpha_k\rho\|x_{k+1}-p\|^2 \\
&\quad +2\alpha_k\langle h(p)-p, x_{k+1}-p\rangle.
\end{aligned}$$

So, we get

$$\begin{aligned}
\|x_{k+1}-p\|^2 &\le \left(1-\frac{2(1-\rho)\alpha_k}{1-\alpha_k\rho}\right)\|x_k-p\|^2 \\
&\quad +\frac{2(1-\rho)\alpha_k}{1-\alpha_k\rho}\left(\frac{\alpha_k M_0}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle h(p)-p, x_{k+1}-p\rangle\right) \\
&= (1-\gamma_k)\|x_k-p\|^2 \\
&\quad +\gamma_k\left(\frac{\alpha_k M_0}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle h(p)-p, x_{k+1}-p\rangle\right), \qquad (4.2.12)
\end{aligned}$$

where $M_0 = \sup\{\|x_k-p\|^2,\ k\in\mathbb{N}\}$. Put $\gamma_k = \frac{2(1-\rho)\alpha_k}{1-\alpha_k\rho}$, for each $k\in\mathbb{N}$. By assumption on α_k, we have

$$\lim_{k\to\infty}\gamma_k = 0, \quad \text{and} \quad \sum_{k=1}^{\infty}\gamma_k=\infty. \qquad (4.2.13)$$

Since $P_\Omega h$ is a contraction on C, there exists $q\in\Omega$ such that $q = P_\Omega h(q)$. We prove that the sequence $\{x_k\}$ converges strongly to $q = P_\Omega h(q)$. In order to prove it, let us consider two cases.

Case 1. Suppose that there exists $k_0 \in \mathbb{N}$ such that $\{\|x_k - q\|\}_{k=k_0}^{\infty}$ is nonincreasing. In this case, the limit of $\{\|x_k - q\|\}$ exists. This together with assumptions on $\{\alpha_k\}$, $\{\beta_k\}$, $\{\lambda_k\}$ and (4.2.10) implies that

$$\lim_{k\to\infty} \|y_k - t_k\| = \lim_{k\to\infty} \|t_k - z_k\| = 0. \tag{4.2.14}$$

On the other hands, from definition of x_{k+1} and (4.2.8), we get

$$\begin{aligned}
\|x_{k+1} - q\|^2 &\leq \alpha_k\|h(x_k) - q\|^2 + (1-\alpha_k)\|\beta_k x_k + (1-\beta_k)Tz_k - q\|^2 \\
&= \alpha_k\|h(x_k) - q\|^2 + (1-\alpha_k)[\beta_k\|x_k - q\|^2 + (1-\beta_k)\|Tz_k - q\|^2 \\
&\quad -\beta_k(1-\beta_k)\|x_k - Tz_k\|^2] \\
&\leq \alpha_k\|h(x_k) - q\|^2 + (1-\alpha_k)[\beta_k\|x_k - q\|^2 + (1-\beta_k)\|x_k - q\|^2 \\
&\quad -\beta_k(1-\beta_k)\|x_k - Tz_k\|^2] \\
&= \alpha_k\|h(x_k) - q\|^2 + (1-\alpha_k)[\|x_k - q\|^2 - \beta_k(1-\beta_k)\|x_k - Tz_k\|^2],
\end{aligned}$$

and hence,

$$\begin{aligned}
\beta_k(1-\beta_k)(1-\alpha_k)\|x_k - Tz_k\|^2 &\leq \alpha_k\|h(x_k) - q\|^2 + \|x_k - q\|^2 \\
&\quad -\|x_{k+1} - q\|^2.
\end{aligned} \tag{4.2.15}$$

Since the limit of $\{\|x_k - q\|\}$ exists and by assumptions on $\{\alpha_k\}$ and $\{\beta_k\}$, we obtain

$$\lim_{k\to\infty} \|x_k - Tz_k\| = 0. \tag{4.2.16}$$

From (4.2.9) and (4.2.11), we have

$$\begin{aligned}
\|x_{k+1} - q\|^2 - \|x_k - q\|^2 - 2\alpha_k\langle h(x_k) - q, x_{k+1} - q\rangle &\leq \|q_k - q\|^2 - \|x_k - q\|^2 \\
&\leq 0.
\end{aligned} \tag{4.2.17}$$

Again, since the limit of $\{\|x_k - q\|\}$ exists and $\alpha_k \to 0$, it follows that

$$\lim_{k\to\infty} \left(\|q_k - q\|^2 - \|x_k - q\|^2\right) = 0$$

and hence,

$$\lim_{k\to\infty} \|q_k - q\| = \lim_{k\to\infty} \|x_k - q\|,$$

and by (4.2.9), we get

$$\lim_{k\to\infty} \|x_k - q\| = \lim_{k\to\infty} \|z_k - q\|. \tag{4.2.18}$$

We also get from (4.2.6), (4.2.7) and (4.2.18)

$$\lim_{k\to\infty} \|x_k - q\| = \lim_{k\to\infty} \|y_k - q\|. \tag{4.2.19}$$

By (4.2.5) and (4.2.19),

$$\lim_{k\to\infty} \|Sv_k - Lx_k\| = \lim_{k\to\infty} \|P_Q(Lx_k) - Lx_k\| = 0 \tag{4.2.20}$$

which implies that

$$\lim_{k\to\infty} \|Sv_k - P_Q(Lx_k)\| = 0. \tag{4.2.21}$$

It follows from (4.2.2) that

$$\begin{aligned}
&(1 - 2\mu_k d_1)\|P_Q(Lx_k) - u_k\|^2 + (1 - 2\mu_k d_2)\|u_k - v_k\|^2 \\
&\leq \|P_Q(Lx_k) - Lp\|^2 - \|Sv_k - Lp\|^2 \\
&= \left(\|P_Q(Lx_k) - Lp\| + \|Sv_k - Lp\|\right)\left(\|P_Q(Lx_k) - Lp\| - \|Sv_k - Lp\|\right) \\
&= \left(\|P_Q(Lx_k) - Lp\| + \|Sv_k - Lp\|\right)\|P_Q(Lx_k) - Sv_k\|.
\end{aligned}$$

So,

$$\lim_{k\to\infty} \|P_Q(Lx_k) - u_k\| = \lim_{k\to\infty} \|u_k - v_k\| = 0, \tag{4.2.22}$$

and hence,

$$\lim_{k\to\infty} \|P_Q(Lx_k) - v_k\| = 0. \tag{4.2.23}$$

From (4.2.20) and (4.2.23), we get

$$\lim_{k\to\infty} \|Lx_k - v_k\| = 0. \tag{4.2.24}$$

It follows from $x_k \in C$, the definition of y_k and (4.2.20) that

$$\begin{aligned} \|y_k - x_k\| &= \|P_C(x_k + \delta_k L^*(Sv_k - Lx_k)) - P_C(x_k)\| \\ &\leq \|x_k + \delta_k L^*(Sv_k - Lx_k) - x_k\| \\ &\leq \delta_k \|L\| \|Sv_k - Lx_k\| \to 0. \end{aligned} \tag{4.2.25}$$

Because $\{x_k\}$ is bounded, there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $\{x_{k_n}\}$ converges weakly to some $\bar{x}$, as $n \to \infty$ and

$$\begin{aligned} \limsup_{k\to\infty} \langle x_k - q, h(q) - q\rangle &= \lim_{n\to\infty} \langle x_{k_n} - q, h(q) - q\rangle \\ &= \langle \bar{x} - q, h(q) - q\rangle. \end{aligned} \tag{4.2.26}$$

Consequently, $\{Lx_{k_n}\}$ converges weakly to $L\bar{x}$. By (4.2.24), $\{v_{k_n}\}$ converges weakly to $L\bar{x}$. We show that $\bar{x} \in \Omega$. We know that $x_k \in C$ and $v_k \in Q$, for each $k \in \mathbb{N}$. Since C and Q are closed and convex sets, so C and Q are weakly closed, therefore, $\bar{x} \in C$ and $L\bar{x} \in Q$. From (4.2.25) and (4.2.14), we get that $\{y_{k_n}\}$, $\{t_{k_n}\}$ and $\{z_{k_n}\}$ converge weakly to $\bar{x}$. By (4.2.22) and (4.2.23), we also get that $\{u_{k_n}\}$ and $\{P_Q(Lx_{k_n})\}$ converge weakly to $L\bar{x}$. The Algorithm 4.2.1 and assertion (i) in

Lemma 2.4.4 imply that

$$\begin{aligned}\lambda_{k_n}\left(f(y_{k_n},y)-f(y_{k_n},t_{k_n})\right) &\geq \langle t_{k_n}-y_{k_n},t_{k_n}-y\rangle\\ &\geq -\|t_{k_n}-y_{k_n}\|\|t_{k_n}-y\|,\forall y\in C,\end{aligned}$$

and

$$\begin{aligned}\mu_{k_n}\left(g(P_Q(Lx_{k_n}),u)-g(P_Q(Lx_{k_n}),u_{k_n})\right) &\geq \langle u_{k_n}-P_Q(Lx_{k_n}),u_{k_n}-u\rangle\\ &\geq -\|u_{k_n}-P_Q(Lx_{k_n})\|\|u_{k_n}-u\|,\forall u\in Q.\end{aligned}$$

Hence, it follows that

$$f(y_{k_n},y)-f(y_{k_n},t_{k_n})+\frac{1}{\lambda_{k_n}}\|t_{k_n}-y_{k_n}\|\|t_{k_n}-y\|\geq 0,\forall y\in C,$$

and

$$g(P_Q(Lx_{k_n}),u)-g(P_Q(Lx_{k_n}),u_{k_n})+\frac{1}{\mu_{k_n}}\|u_{k_n}-P_Q(Lx_{k_n})\|\|u_{k_n}-u\|\geq 0,\forall u\in Q.$$

Letting $n\to\infty$, by the hypothesis on $\{\lambda_k\}$, $\{\mu_k\}$, (4.2.14), (4.2.22) and the weak continuity of f and g, we obtain that

$$f(\bar{x},y)\geq 0,\forall y\in C,\text{ and } g(L\bar{x},u)\geq 0,\forall u\in Q.$$

This means that $\bar{x}\in EP(f,C)$ and $L\bar{x}\in EP(g,Q)$. It follows from (4.2.14), (4.2.16) and (4.2.25) that

$$\|z_k-Tz_k\|\leq\|z_k-t_k\|+\|t_k-y_k\|+\|y_k-x_k\|+\|x_k-Tz_k\|\to 0.$$

This together with Theorem 2.3.11 implies that $\bar{x}\in Fix(T)$. On the other hand, from (4.2.21) and (4.2.23), we get

$$\|v_k-Sv_k\|\leq\|v_k-P_Q(Lx_k)\|+\|P_Q(Lx_k)-Sv_k\|\to 0,$$

and using again Theorem 2.3.11, we obtain $L\bar{x} \in Fix(S)$. Then, we proved that $\bar{x} \in EP(f, C) \cap Fix(T)$ and $L\bar{x} \in EP(g, Q) \cap Fix(S)$, that is $\bar{x} \in \Omega$. By Theorem 2.3.18(i), $\bar{x} \in \Omega$ and (4.2.26), we get

$$\limsup_{k\to\infty} \langle x_k - q, h(q) - q\rangle = \langle \bar{x} - q, h(q) - q\rangle \leq 0. \tag{4.2.27}$$

Finally, from (4.2.12), (4.2.13), (4.2.27) and Theorem 2.4.5, we imply that the sequence $\{x_k\}$ converges strongly to q.

Case 2. Suppose that there exists a subsequence $\{k_i\}$ of $\{k\}$ such that

$$\|x_{k_i} - q\| < \|x_{k_i+1} - q\|, \ \forall i \in \mathbb{N}.$$

According to Theorem 2.4.6, there exists a nondecreasing sequence $\{m_n\} \subset \mathbb{N}$ such that $m_n \to \infty$,

$$\|x_{m_n} - q\| \leq \|x_{m_n+1} - q\| \text{ and } \|x_n - q\| \leq \|x_{m_n+1} - q\|, \forall n \in \mathbb{N}. \tag{4.2.28}$$

From this and (4.2.10), we get

$$\begin{aligned} &(1 - \beta_{m_n})\left[(1 - 2\lambda_{m_n}c_1)\|y_{m_n} - t_{m_n}\|^2 + (1 - 2\lambda_{m_n}c_2)\|t_{m_n} - z_{m_n}\|^2\right] \\ &\leq \alpha_{m_n}M + \|x_{m_n} - q\|^2 - \|x_{m_n+1} - q\|^2 \\ &\leq \alpha_{m_n}M. \end{aligned}$$

This together with assumptions on $\{\alpha_k\}$, $\{\beta_k\}$ and $\{\lambda_k\}$ implies that

$$\lim_{n\to\infty} \|y_{m_n} - t_{m_n}\| = 0, \lim_{n\to\infty} \|t_{m_n} - z_{m_n}\| = 0 \text{ and } \lim_{n\to\infty} \|y_{m_n} - z_{m_n}\| = 0.$$

From (4.2.15), we have

$$\begin{aligned}\beta_{m_n}(1-\beta_{m_n})(1-\alpha_{m_n})\|x_{m_n}-Tz_{m_n}\|^2 &\le \alpha_{m_n}\|h(x_{m_n})-q\|^2+\|x_{m_n}-q\|^2\\ &\quad -\|x_{m_n+1}-q\|^2\\ &\le \alpha_{m_n}\|h(x_{m_n})-q\|^2.\end{aligned}$$

By hypothesis on $\{\alpha_k\}$ and $\{\beta_k\}$, we have

$$\lim_{n\to\infty}\|x_{m_n}-Tz_{m_n}\|=0.$$

By (4.2.17), we get

$$\begin{aligned}-2\alpha_{m_n}\langle h(x_{m_n})-q, x_{m_n+1}-q\rangle &\le \|x_{m_n+1}-q\|^2-\|x_{m_n}-q\|^2\\ &\quad -2\alpha_{m_n}\langle h(x_{m_n})-q, x_{m_n+1}-q\rangle\\ &\le \|q_{m_n}-q\|^2-\|x_{m_n}-q\|^2\\ &\le 0.\end{aligned}$$

Since the sequence $\{x_k\}$ is bounded and $\alpha_k\to 0$, we obtain

$$\lim_{n\to\infty}\|q_{m_n}-q\|=\lim_{n\to\infty}\|x_{m_n}-q\|.$$

By the same argument as Case 1, we have

$$\limsup_{n\to\infty}\langle x_{m_n}-q, h(q)-q\rangle\le 0.$$

It follows from (4.2.12) and (4.2.28) that

$$\begin{aligned}\|x_{m_n+1}-q\|^2 &\le (1-\gamma_{m_n})\|x_{m_n}-q\|^2\\ &\quad +\gamma_{m_n}\left(\frac{\alpha_{m_n}M_0}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle h(q)-q, x_{m_n+1}-q\rangle\right)\\ &\le (1-\gamma_{m_n})\|x_{m_n+1}-q\|^2\\ &\quad +\gamma_{m_n}\left(\frac{\alpha_{m_n}M_0}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle h(q)-q, x_{m_n+1}-q\rangle\right),\end{aligned}$$

and hence,

$$\gamma_{m_n}\|x_{m_n+1}-q\|^2 \leq \gamma_{m_n}\left(\frac{\alpha_{m_n}M_0}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle h(q)-q, x_{m_n+1}-q\rangle\right).$$

Since $\gamma_{m_n}>0$ and using (4.2.28) we get

$$\|x_n-q\|^2 \leq \|x_{m_n+1}-q\|^2 \leq \left(\frac{\alpha_{m_n}M_0}{2(1-\rho)}+\frac{1}{(1-\rho)}\langle h(q)-q, x_{m_n+1}-q\rangle\right).$$

Taking the limit in the above inequality as $n\to\infty$, we conclude that the sequence $\{x_n\}$ converges strongly to $q=P_\Omega h(q)$. □

4.2.2 Application to variational inequality problems

In this part, we apply Theorem 4.2.2 for finding a solution of variational inequality problems for pseudomonotone and Lipschitz continuous mappings.

Firstly, we consider the variational inequality problem which is a problem of finding a point $x^*\in C$ such that

$$\langle Ax^*, x-x^*\rangle \geq 0, \forall x\in C, \tag{4.2.29}$$

where C is a nonempty closed convex subset of a real Hilbert space H and $A\colon C\to C$ is a mapping. We observe that if $f(x,y)=\langle Ax, y-x\rangle$, for each $x,y\in C$, then the equilibrium problem (3.1.1) become the variational inequality problem (4.2.29). The set of solutions of the variational inequality problem (4.2.29) will be denoted by $VI(A,C)$.

Now, for a nonempty closed convex subset C of H and a mapping $A: C\to C$, we are concerned with the following assumptions:

$(B1)$ A is pseudomonotone on C;

$(B2)$ A is weak to strong continuous on C that is, $Ax_k \to Ax$ for each sequence $\{x_k\} \subset C$ converging weakly to x;

$(B3)$ A is Lipschitz continuous on C with constant $L_1 > 0$.

Next, let H_1 and H_2 be two real Hilbert spaces and C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Suppose that $A\colon C \to C$ and $B\colon Q \to Q$ are mappings which satisfy $B1 - B3$ with some positive constants L_1 and L_2, respectively. Let $T\colon C \to C$ and $S\colon Q \to Q$ be nonexpansive mappings, $h\colon C \to C$ be a ρ-contraction mapping, and $L\colon H_1 \to H_2$ be a bounded linear operator with its adjoint L^*. We present the following algorithm for solving the split variational inequality and fixed point problems.

Algorithm 4.2.3. *Choose $x_1 \in H_1$. The control parameters $\lambda_k, \mu_k, \alpha_k, \beta_k, \delta_k$ satisfy the following conditions*

$$0 < \underline{\lambda} \le \lambda_k \le \overline{\lambda} < L_1,\ \ 0 < \underline{\mu} \le \mu_k \le \overline{\mu} < L_2,\ \ \beta_k \in (0,1),\ \ 0 < \liminf_{k\to\infty} \beta_k \le \limsup_{k\to\infty} \beta_k < 1,\ \ 0 < \underline{\delta} \le \delta_k \le \overline{\delta} < \frac{1}{\|L\|^2}, \alpha_k \in (0, \frac{1}{2-\rho}), \lim_{k\to\infty} \alpha_k = 0, \sum_{k=1}^{\infty} \alpha_k = \infty.$$

Let $\{x_k\}$ be a sequence generated by

$$\begin{cases} u_k = P_Q\left(P_Q(Lx_k) - \mu_k B\left(P_Q(Lx_k)\right)\right), \\ v_k = P_Q\left(P_Q(Lx_k) - \mu_k B\left(u_k\right)\right), \\ y_k = P_C\left(x_k + \delta_k L^*\left(Sv_k - Lx_k\right)\right), \\ t_k = P_C\left(y_k - \lambda_k A y_k\right), \\ z_k = P_C\left(y_k - \lambda_k A t_k\right), \\ x_{k+1} = \alpha_k h(x_k) + (1-\alpha_k)(\beta_k x_k + (1-\beta_k) T z_k). \end{cases}$$

Theorem 4.2.4. *Suppose that the solution set* $\Omega := \{p \in VI(A,C) \cap Fix(T) : Lp \in VI(B,Q) \cap Fix(S)\} \neq \emptyset$. *Then, the sequence* $\{x_k\}$ *which is generated by Algorithm 4.2.3 converges strongly to* $q = P_\Omega h(q)$.

Proof. It is known that the bifunction $f(x,y) = \langle Ax, y - x\rangle$ satisfies conditions (A1)–(A3). Since A is L_1-Lipschitz continuous on C, it follows that

$$\begin{aligned} f(x,y) + f(y,z) - f(x,z) &= \langle Ax - Ay, y - z\rangle \\ &\geq -\|Ax - Ay\|\|y - z\| \\ &\geq -L_1\|x - y\|\|y - z\| \\ &\geq -\frac{L_1}{2}\|x - y\|^2 - \frac{L_1}{2}\|y - z\|^2, \quad \forall x, y, z \in C. \end{aligned}$$

Then, f is Lipschitz-type continuous on C with $c_1 = c_2 = \frac{L_1}{2}$, and hence f satisfies condition (A4).

It follows from the definitions of f and y_n that

$$\begin{aligned} t_k &= \arg\min\left\{\lambda_k\langle Ay_k, y - y_k\rangle + \frac{1}{2}\|y - y_k\|^2 : y \in C\right\} \\ &= \arg\min\left\{\frac{1}{2}\|y - (y_k - \lambda_k Ay_k)\|^2 : y \in C\right\} \\ &= P_C(y_k - \lambda_k Ay_k), \end{aligned}$$

and similarly, we can get $u_k = P_Q(P_Q(Lx_k) - \mu_n B(P_Q(Lx_k)))$, $v_k = P_Q(P_Q(Lx_k) - \mu_k B(u_k)))$, and $z_k = P_C(y_k - \lambda_k At_k)$. Then, the Algorithm 4.2.1 reduces to the Algorithm 4.2.3 and we get the conclusion from and Theorem 4.2.2. □

4.2.3 Numerical experiments

In this part, we consider examples and numerical results to support Theorem 4.2.2. In addition, we compare the introduced algorithm with the PEA Algorithm, which was presented in [66].

We consider the bifunctions f and g which given by the form of Nash-Cournot oligopolistic equilibrium models of electricity markets, see [68, 69],

$$f(x,y) = (Px + Dy)^T(y - x), \quad \forall x, y \in \mathbb{R}^n, \tag{4.2.30}$$

$$g(u,v) = (Uu + Vv)^T(v - u), \quad \forall u, v \in \mathbb{R}^m, \tag{4.2.31}$$

where P, $D \in \mathbb{R}^{n\times n}$ and U, $V \in \mathbb{R}^{m\times m}$ are symmetric positive semidefinite matrices such that $P - D$ and $U - V$ are positive semidefinite matrices. We know that the bifunctions f and g satisfy conditions (A1)–(A4), see [70]. Notice that f and g are Lipschitz-type continuous with constants $c_1 = c_2 = \frac{1}{2}\|P - D\|$ and $d_1 = d_2 = \frac{1}{2}\|U - V\|$, respectively. Choose $b_1 = \max\{c_1, d_1\}$, and $b_2 = \max\{c_2, d_2\}$. Then, both bifunctions f and g are Lipschitz-type continuous with constants b_1 and b_2.

The following numerical experiments are written in Matlab R2015b and performed on a Desktop with Intel(R) Core(TM) i3 CPU M 390 @ 2.67GHz 2.67GHz and RAM 4.00 GB.

Example 4.2.5. Let the bifunctions f and g be given as (4.2.30) and (4.2.31), respectively. We will concern with the following boxes: $C = \prod_{i=1}^{n}[-5,5]$, $Q = \prod_{j=1}^{m}[-20,20]$, $\overline{C} = \prod_{i=1}^{n}[-3,3]$ and $\overline{Q} = \prod_{j=1}^{m}[-10,10]$. The nonexpansive mappings $T : C \to C$ and $S : Q \to Q$ are given by $T = P_{\overline{C}}$ and $S = P_{\overline{Q}}$, respectively. The contraction mapping $h : C \to C$ is a $n \times n$ matrix such that $\|h\| < 1$, while the linear operator $L : \mathbb{R}^n \to \mathbb{R}^m$ is a $m \times n$ matrix.

In this numerical experiment, the matrices P, D, U, and V are randomly generated in the interval $[-5, 5]$ such that they satisfy above required properties. Besides, the matrices h and L are randomly generated in the interval $(0, \frac{1}{n})$ and $[-2, 2]$, respectively. We randomly generated starting point $x_1 \in \mathbb{R}^n$ in the interval $[-20, 20]$ with the following control parameters: $\delta_k = \dfrac{1}{2\|L\|^2}$, $\alpha_k = \dfrac{1}{k+2}$ and $\mu_k = \lambda_k = \dfrac{1}{4\max\{b_1, b_2\}}$. The following 3 cases of the control parameter β_n are considered:

Case 1. $\beta_k = 10^{-10} + \dfrac{1}{k+1}$.

Case 2. $\beta_k = 0.5$.

Case 3. $\beta_k = 0.99 - \dfrac{1}{k+1}$.

Note that to obtain the vector u_k, in the Algorithm 4.2.1, we need to solve the following optimization problem

$$\arg\min \left\{ \mu_k g(P_Q(Lx_k), u) + \frac{1}{2}\|u - P_Q(Lx_k)\|^2 : u \in Q \right\},$$

which is equivalent to the following convex quadratic problem

$$\arg\min \left\{ \frac{1}{2}u^T J u + K^T u : u \in Q \right\}, \tag{4.2.32}$$

where $J = 2\mu_k V + I_m$ and $K = \mu_k U P_Q(Lx_k) - \mu_k V P_Q(Lx_k) - P_Q(Lx_k)$, see [66].

On the other hand, in order to obtain the vector v_k, we need to solve the following convex quadratic problem

$$\arg\min \left\{ \frac{1}{2}u^T \overline{J} u + \overline{K}^T u : u \in Q \right\}, \tag{4.2.33}$$

where $\overline{J} = J$ and $\overline{K} = \mu_k U u_k - \mu_k V u_k - P_Q(Lx_k)$. Similarly, to obtain the vectors t_k and z_k, we have to consider the convex quadratic problems in the same way as in (4.2.32) and (4.2.33), respectively. We use the Matlab Optimization Toolbox to solve vectors u_k, v_k, t_k and z_k. Observe that the solution set Ω is nonempty because of $0 \in \Omega$. The Algorithm 4.2.1 is tested by using the stopping criteria $\|x_{k+1} - x_k\| < 10^{-3}$. In Table 5, we randomly 10 starting points and the presented results are in average.

Table 5 **The numerical results for different parameter β_k of Example 4.2.5**

Size		Average times (sec)			Average iterations		
n	m	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
5	10	1.399695	1.957304	6.356185	37	54	171
10	5	2.168317	2.916557	6.551182	56	75	179
20	50	2.834138	3.785376	8.711813	58	80	186
50	20	5.292192	6.570650	10.418191	111	138	220

From Table 5, we may suggest that a smallest size of parameter β_k, as $\beta_k = 10^{-10} + \dfrac{1}{k+1}$, provides better computational times and iterations than other cases.

Example 4.2.6. We consider the split equilibrium and fixed point problems (4.1.6) when $T = I_{\mathbb{R}^n}$ and $S = I_{\mathbb{R}^m}$ are identity mappings on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. It follows that the split equilibrium and fixed point problems (4.1.6) become the split equilibrium problem (4.1.4). In this case, we compare the Algorithm 4.2.1 with the PEA Algorithm, which was presented in [66]. For this numerical experiment, we consider the problem setting and the control parameters as in Example 4.2.5, but only for the case of parameter β_k is $10^{-10} + \dfrac{1}{k+1}$. Note that the solution set Ω is nonempty because of $0 \in \Omega$. The starting point $x_1 \in \mathbb{R}^n$ is randomly generated in the interval $[-5, 5]$. We compare Algorithm 4.2.1 with PEA by using the stopping criteria $\|x_{k+1} - x_k\| < 10^{-3}$. In Table 6, we randomly 10 starting points and presented results are in average.

Table 6 The numerical results for the split equilibrium problem of Example 4.2.6

Size		Average times (sec)		Average iterations	
n	m	Algorithm 4.2.1	PEA	Algorithm 4.2.1	PEA
5	10	0.862125	0.983111	31	44
10	5	1.037650	1.991282	36	83
20	50	1.607701	2.618173	44	85
50	20	2.937581	7.926821	80	258

From Table 6, we see that both computational times and iterations of Algorithm 4.2.1 are better than those of PEA.

CHAPTER V

CONCLUSION

This chapter is all the results of this thesis including corollary, lemmas, and theorems. We conclude again that what we get from the results.

5.1 Shrinking extragradient methods for pseudomonotone equilibrium problems and fixed points of quasi-nonexpansive mappings problems

In this section, we presented two shrinking extragradient methods, CSEM and PSEM, for finding a common element of the set of fixed points of a finite family for quasi-nonexpansive mappings and the solution set of equilibrium problems of a finite family for pseudomonotone bifunctions in a real Hilbert space. Under some constraint qualifications of the scalar sequences, we obtained one Lemma and two Theorems. Lemma 5.1.1 is an important tool to prove two main theorems. The strong convergence theorems of the CSEM Algorithm and the PSEM Algorithm are considered in Theorem 5.1.2 and Theorem 5.1.3, respectively.

Lemma 5.1.1. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by CSEM Algorithm is well-defined.*

Theorem 5.1.2. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by CSEM Algorithm converges strongly to $P_S(x_0)$.*

Theorem 5.1.3. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by PSEM Algorithm converges strongly to $P_S(x_0)$.*

5.2 Hybrid extragradient methods for pseudomonotone equilibrium problems and fixed points of quasi-nonexpansive mappings problems

For this section, we proposed two hybrid extragradient methods, CHEM and PHEM, for finding the closest point to the intersection of the set of fixed points of a finite family for quasi-nonexpansive mappings and the solution set of equilibrium problems of a finite family for pseudomonotone bifunctions in a real Hilbert space. By supposing that some control conditions hold, one Lemma, two Theorems, and one Corollary were presented. Lemma 5.2.1 is a useful instrument for proving two main theorems. We showed the strong convergence theorems of the CHEM Algorithm and the PHEM Algorithm in Theorem 5.2.2 and Theorem 5.2.3, respectively. Furthermore, Corollary 5.2.4 is an improvement version of Algorithm (3.1.8) in the reference [51].

Lemma 5.2.1. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by CHEM Algorithm is well-defined.*

Theorem 5.2.2. *If the solution set S is nonempty, then the sequence $\{x_k\}$ which is generated by CHEM Algorithm converges strongly to $P_S(x_0)$.*

Theorem 5.2.3. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by PHEM Algorithm converges strongly to $P_S(x_0)$.*

Corollary 5.2.4. *Let T be a quasi-nonexpansive self-mapping on C with $I - T$ demiclosed at 0 and let f be a bifunction satisfies the assumptions $(A1) - (A4)$. Suppose that the solution set $S = EP(f, C) \cap Fix(T)$ is nonempty. Pick $x_0 \in C$, choose parameters $\{\rho_k\}$ with $0 < \inf \rho_k \leq \sup \rho_k < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0, 1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, $\{\beta_k\} \subset [0, 1)$ with $0 \leq \inf \beta_k \leq \sup \beta_k < 1$, and the* *sequences* $\{x_k\}$, $\{y_k\}$, $\{z_k\}$, $\{t_k\}$, $\{u_k\}$ *are defined by*

$$\begin{cases} y_k = \arg\min\{\rho_k f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\ z_k = \arg\min\{\rho_k f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\ t_k = \alpha_k x_k + (1 - \alpha_k)Tx_k, \\ u_k = \beta_k t_k + (1 - \beta_k)Tz_k, \\ C_k = \{x \in C : \|x - u_k\| \leq \|x - x_k\|\}, \\ Q_k = \{x \in C : \langle x_0 - x_k, x - x_k\rangle \leq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0). \end{cases} \tag{5.2.1}$$

Then, the sequence $\{x_k\}$ *converges strongly to* $P_S(x_0)$.

5.3 A new extragradient method for split equilibrium and fixed point problems

This section is the final section of this thesis. We introduced a new extragradient algorithm for finding a solution to the split equilibrium and fixed point problems involving pseudomonotone bifunctions and nonexpansive mappings in real Hilbert spaces. Under the properties of the control sequences, the strong convergence theorem of the introduced algorithm is presented in Theorem 5.3.1. We also applied the obtained theorem to the problem of the split variational inequality and fixed point problems and considered its convergence in Theorem 5.3.2.

Theorem 5.3.1. *Suppose that the solution set* Ω *is nonempty. Then, the sequence* $\{x_n\}$ *generated by Algorithm 4.2.1 converges strongly to* $q = P_\Omega h(q)$.

Theorem 5.3.2. *Suppose that the solution set* $\Omega := \{p \in VI(C, A) \cap Fix(T) : Lp \in VI(Q, B) \cap Fix(S)\} \neq \emptyset$. *Then the sequence* $\{x_k\}$ *generated by Algorithm 4.2.3 converges strongly to* $q = P_\Omega h(q)$.

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