ITERATIVE ALGORITHMS FOR SOLVING THE VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS

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ABSTRACT

In this thesis, we introduce the generalized iteration processes and generalized contractive mappings in the setting of partial b-metric spaces and Hilbert spaces. Furthermore, we illustrate weak and strong convergence theorems for the variational inequality problems, fixed point problems and the related problems. Moreover, we present some numerical examples to demonstrate the capability of our iteration processes.

LIST OF CONTENTS

LIST OF TABLES

Table **Page**

LIST OF FIGURES

Figure **Page**

CHAPTER I

INTRODUCTION

A popular tool in nonlinear analysis is the fixed point theory. The knowledge of the existence of fixed points has attracted increasing attention and have been widely investigated by many authors (e.g., [1–4] and the references therein) because these theorems play important roles in mechanics, physics, differential equations, and so on. The Banach contraction principle [5] is the first important result on fixed points for contractive-type mappings, by using the Picard iteration process for approximation of a fixed point. Since then generalizations of the contraction principle in different directions as well as many new fixed point results with applications have been established by different researchers (refer to [6–8]).

The notion of b-metric spaces as a generalization of metric spaces was introduced by Bakhtin [6], which was formally defined by Czerwik [9] in 1993 with a view of generalizing the Banach contraction principle. On the other hand, Matthews [10] presented the notion of partial metric spaces, which is a generalization of metric spaces, and proved the partial metric version of the Banach fixed point theorem and applied it in program verification. There are many authors who have worked in b-metric spaces and partial metric spaces (refer to [11–14] and the references therein). In 2014, Shukla [15] proposed the notion of partial b-metric spaces, which is a generalization of partial metric spaces and b-metric spaces, and established fixed point theorems for Banach contractions and Kannan contractions defined on complete partial b-metric spaces.

The variational inequality problem (VIP) was introduced by Stampacchia [16]. After that, Hartman and Stampacchia [17] suggested the VIP as a tool for the study of partial differential equations. The ideas of the VIP are being applied in many fields including mechanics, nonlinear programming, game theory and economic equilibrium (see [18–22]). Many authors have studied extensively convergence theorem for finding a common element of fixed point set and the solution of VIP (refer to [23–27]). In 2012, Censor et al. [28] proposed a new problem, the so-called split variational inequality problem (SVIP), it constitutes a pair of the VIPs. One of the special cases of the SVIP is the split feasibility problem (SFP), it was first suggested by Censor and Elfving [29] in the finite-dimensional spaces. The SFP arises in many fields in the real world, such as signal processing, image reconstruction, and medical care (see [29–31]). The multiple-sets split feasibility problem (MSSFP) was first proposed by Censor et al. [32], which finds the application in intensity-modulated radiation therapy (IMRT) [33]. The MSSFP arises in many fields in the real world, such as inverse problem of intensity-modulated radiation therapy, image reconstruction and signal processing (see [29, 32, 34] and the references therein). The MSSFP includes the two-set split feasibility problem as its special case, and it is usually called SFP for simplicity.

In [35, 36], the sequence of Picard iterates is a strongly convergent sequence in the solution of VIP. However, Picard iterates cannot use to solve VIP when a function is monotone and Lipschitz continuous, which can be seen from the counterexample in [37]. During the last decade, many authors devoted their attention to study algorithms for solving the VIP. Extragradient method for solving the VIP was studied by Korpelevich [38] in 1976 (see also Facchinei and Pang [39]). After that, Nadezhkina and Takahashi [24] suggested a new modified extragradient method motivated by Korpelevich [38] and Takahashi et al. [40]. They showed that the sequence generated by the mentioned method converges weakly to a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the VIP. In 1974, Ishikawa [41] proposed a new iteration scheme which is called the Ishikawa iteration for constructing fixed points of a nonlinear mapping. Since then, several authors studied the Ishikawa iteration process for solving the equilibrium problems, the variational inequality problems in Hilbert spaces and Banach spaces (see [42–44]). Krasnosel'skii [45] and Mann [46] used the K-M algorithm to solve a fixed point equation. Note that the K-M algorithm is the special case of the Ishiwaka iteration process.

In 2002, Byrne [34] proposed the popular CQ algorithm, which is a special case of the K-M algorithm, to solve the SFP. Afterward, Zhao and Yang [47], studied the convergence of general CQ algorithm, which is a motivation for Zhao and Yang [48] to study more general K-M algorithm in a finite dimensional Hilbert space. In 2006, Xu [49] extended the results of Zhao and Yang [47] from finite dimensional Hilbert spaces to infinite dimensional Banach spaces and presented some projection algorithms for solving the MSSFP in Hilbert spaces. Motivated by the idea of Xu [49], that is CQ algorithm can be viewed as a fixed point algorithm for averaged mappings, Xu [50] applied the K-M algorithm to present the algorithm for solving the SFP. Furthermore, in 2017, Tian and Jiang [51] introduced an iterative method by combining Korpelevich's extragradient method with Byrne's CQ algorithm, for finding an element to solve a class of split variational inequality problems under weaker conditions and get a weak convergence theorem. On the other hand, Buong [52] considered the following algorithms, which is proposed in [53] and [54] for solving the common solution of variational inequality problem and split feasibility problem. Moreover, Buong [52] considered the sequence $\{x_n\}$, which is generated by the following algorithm, which is weakly convergent to a solution of MSSFP.

In the following, we give a description of the contents of this thesis.

Chapter II. We will recall some well-known definitions and useful results that will be used in our main results of this thesis.

Chapter III. In this chapter, we introduce a concept of generalized JS-quasi-contractions and obtain sufficient conditions for the existence of fixed points of such mappings on p_b-complete partial b-metric spaces. Our results extend the results in the literature. In addition, an example is given to illustrate and support our main result.

Chapter IV. In this chapter, we separate two sections of convergence theorems for solving the variational inequality problems and related problems with applications as the following:

Firstly, we establish a new iterative algorithm by combining Nadezhkina and Takahashi's modified extragradient method and Xu's algorithm. The mentioned iterative algorithm presents the common solution of the split variational inequality problems and fixed point problems. We show that the sequence produced by our algorithm is weakly convergent. Finally, we give some applications of the main results.

Secondly, we establish an iterative algorithm by combining Yamada's hybrid steepest descent method and Wang's algorithm for finding the common solutions of variational inequality problems and split feasibility problems. The strong convergence of the sequence generated by our suggested iterative algorithm to such a common solution is proved in the setting of Hilbert spaces under some suitable assumptions imposed on the parameters. Moreover, we propose iterative algorithms for finding the common solutions of variational inequality problems and multiple-sets split feasibility problems. Finally, we also give numerical examples for illustrating our algorithms.

Chapter V. We give the conclusion of this thesis.

CHAPTER II

PRELIMINARIES

In this chapter, we give some definitions, several notations and useful results that will be used in the later chapter.

Throughout this thesis, let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ be the set of all real numbers, the set of all non-negative real numbers, and the set of all natural numbers, respectively. Let C be a closed convex subset of a real Hilbert space H.

A mapping $T : C \to C$ is said to be k-Lipschitz continuous with $k > 0$, if

$$\|Tx - Ty\| \leq k\|x - y\|,$$

for all $x, y \in C$. The mapping T is said to be nonexpansive when $k = 1$. We say that $x \in C$ is a fixed point of T if $Tx = x$ and the set of all fixed points of T is denoted by $F(T)$. It is well-known that if C is nonempty bounded closed convex subset of H and $T : C \to C$ is a nonexpansive, then $F(T) \neq \emptyset$ (see [61]). Moreover, for a fixed $\alpha \in (0, 1)$, a mapping $T : H \to H$ is α-averaged if and only if it can be written as the average of the identity mapping on H and a nonexpansive mapping $S : H \to H$, i.e.,

$$T = (1 - \alpha)I + \alpha S.$$

Recall that a mapping $f : C \to H$ is called η-strongly monotone with $\eta > 0$ if

$$\langle fx - fy, x - y\rangle \geq \eta\|x - y\|^2,$$

for all $x, y \in C$. If $\eta = 0$, then the mapping f is said to be monotone. Further, a mapping f is said to be ν-inverse strongly monotone with $\nu > 0$ (ν-ism) if

$$\langle fx - fy, x - y\rangle \geq \nu\|fx - fy\|^2,$$

for all $x, y \in C$. In [55], we know that a η-strongly monotone mapping f is monotone and a ν-ism mapping f is monotone and $\frac{1}{\nu}$-Lipschitz continuous. Moreover, $I - \gamma f$ is nonexpansive where f is ν-ism with $\gamma \in (0, 2\nu)$, see [50] for more details of averaged and ν-ism mappings.

A mapping $T : H \to H$ is said to be firmly nonexpansive, if

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle,$$

for all $x, y \in H$. Moreover, a firmly nonexpansive mapping is $\frac{1}{2}$-averaged.

In [29], we know that the metric projection $P_C : H \to C$ i.e., for $x \in H$,

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|,$$

is firmly nonexpansive.

We collect some basic properties of averaged and inverse strongly monotone mappings in the following results.

Lemma 2.0.1. [50, 56] *We have:*

(i) *The composite of finitely many averaged mappings is averaged. In particular, if T_i is α_i-averaged, where $\alpha_i \in (0,1)$ for $i = 1, 2$, then the composite T_1T_2 is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$;*

(ii) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$F(T_1T_2 \cdots T_N) = \bigcap_{i=1}^{N} F(T_i).$$

(iii) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*

(iv) *If T is ν-ism and $\gamma > 0$, then γT is $\frac{\nu}{\gamma}$-ism.*

(v) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Proposition 2.0.2. [57] *Let D be a nonempty subset of H, let $m \geq 2$ be an integer, and $\phi : (0,1)^m \to (0,1)$ defined by*

$$\phi(\alpha_1, \ldots, \alpha_m) = \frac{1}{1 + \dfrac{1}{\sum_{i=1}^m \dfrac{\alpha_i}{1-\alpha_i}}}. \tag{2.0.1}$$

For every $i \in \{1, \ldots, m\}$, let $\alpha_i \in (0,1)$ and $T_i : D \to D$ be α_i-averaged. Then $T = T_1 \cdots T_m$ is α-averaged, where $\alpha = \phi(\alpha_1, \ldots, \alpha_m)$.

The following result concerns the averagedness of a convex combination of averaged operators.

Proposition 2.0.3. [57] *Let C be a nonempty subset of H, let $\{T_i\}_{i\in I}$ be a finite family nonexpansive mappings from C to H. Assume that $\{\tilde{\alpha}_i\}_{i\in I} \subset (0,1)$, and $\{\delta_i\}_{i\in I} \subset (0,1]$ such that $\sum_{i\in I}\delta_i = 1$. Suppose that, for every $i \in I$, T_i is $\tilde{\alpha}_i$-averaged, then $T = \sum_{i\in I}\delta_i T_i$ is α-averaged, where $\alpha = \sum_{i\in I}\delta_i\tilde{\alpha}_i$.*

The following lemmas of the nonexpansive mappings are very convenient and helpful to use:

Lemma 2.0.4. [58] *Assume that H_1 and H_2 are Hilbert spaces. Let $A : H_1 \to H_2$ be a linear bounded mapping such that $A \neq 0$ and let $T : H_2 \to H_2$ be a nonexpansive mapping. Then, $A^*(I-T)A$ is $\frac{1}{2\|A\|^2}$-ism. Moreover, for $0 \leq \gamma < 1/\|A\|^2$, $I - \gamma A^*(I-T)A$ is $\gamma\|A\|^2$-averaged.*

Lemma 2.0.5. [59] *Let $T, A, N : H \to H$ be mappings where the T is defined by $T = (1-\alpha)A + \alpha N$, for some $\alpha \in (0,1)$. If A is β-averaged and N is nonexpansive, then T is $\alpha + (1-\alpha)\beta$-averaged.*

The following results play the crucial role in the next section:

Lemma 2.0.6. [60] *Let t be a real number in $(0,1]$. Let $f : H \to H$ be an η-strongly monotone and k-Lipschitz continuous mapping, the mapping $I - t\mu f$, for each fixed point $\mu \in (0, \frac{2\eta}{k^2})$, is contractive with constant $1 - t\tau$, i.e.,*

$$\|(I - t\mu f)x - (I - t\mu f)y\| \leq (1 - t\tau)\|x - y\|,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0,1]$.

2.1 Fixed point theorems for generalized JS-quasi-contractions in complete partial b-metric spaces

We recall the following definitions and preliminary results that will be used in the sequel. Throughout this section, let C be a closed convex subset of a real Hilbert space H.

We begin discussing a basic definition on the setting of metric spaces.

Definition 2.1.1. [61] Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is called a metric if for all $x, y, z \in X$ the following properties hold:

(d_1) $d(x, y) = 0$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$;

(d_3) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a metric space.

Two of the most interesting generalizations of metric spaces are partial metric spaces and b-metric spaces. The notion of b-metric spaces was introduced by Czerwik [9].

Definition 2.1.2. [9] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d_b : X \times X \to \mathbb{R}_+$ is called a b-metric if for all $x, y, z \in X$ the following properties hold:

(d_{b1}) $d_b(x, y) = 0$ if and only if $x = y$;

(d_{b2}) $d_b(x, y) = d_b(y, x)$;

(d_{b3}) $d_b(x, z) \leq s(d_b(x, y) + d_b(y, z))$.

The pair (X, d_b) is called a b-metric space.

The class of b-metric spaces is larger than that of the class of metric spaces, since a b-metric is a metric when $s = 1$.

The following example shows that a b-metric space need not be a metric space.

Example 2.1.3. [62] Let $X = x_1, x_2, x_3, x_4$ and $d_b(x_1, x_2) = k \geq 2$, $d_b(x_1, x_3) = d_b(x_1, x_4) = d_b(x_2, x_3) = d_b(x_2, x_4) = d_b(x_3, x_4) = 1$, $d_b(x_i, x_j) = d_b(x_j, x_i)$ for $i, j = 1, 2, 3, 4$ and $d_b(x_i, x_i) = 0$ for $i = 1, 2, 3, 4$. Then

$$d_b(x_i, x_j) \leq \frac{k}{2}[d_b(x_i, x_n) + d_b(x_n, x_j)],$$

for $i, j = 1, 2, 3, 4$, and if $k > 2$, the ordinary triangle inequality does not hold.

We need the following definition that will be used in the next part.

Definition 2.1.4. [9] Let (X, d_b) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.

(ii) The sequence $\{x_n\}$ in X is said to be b-Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

(iii) The b-metric space (X, d_b) is called b-complete if every b-Cauchy sequence in X is b-convergent.

The following notion of partial metric spaces was introduced by Matthews [10].

Definition 2.1.5. [10] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $p : X \times X \to \mathbb{R}_+$ is called a partial metric if for all $x, y, z \in X$ the following properties hold:

(p_1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;

(p_2) $p(x,x) \leq p(x,y)$;

(p_3) $p(x,y) = p(y,x)$;

(p_4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X, p) is called a partial metric space.

The class of partial metric spaces is larger than the class of metric spaces since a metric space is a special case of a partial metric space with the self distance $p(x,x) = 0$. But the converse does not hold in general (see [10]). A trivial example of a partial metric space is the pair $(\mathbb{R}_+, p)$, where $p : X \times X \to \mathbb{R}_+$ is defined by $p(x,y) = \max\{x, y\}$.

The notion of partial b-metric spaces which introduced by Shukla [15], it is a generalization of partial metric spaces and b-metric spaces.

Definition 2.1.6. [15] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to \mathbb{R}_+$ is called a partial b-metric if for all $x, y, z \in X$ the following properties hold:

(p_{b1}) $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$;

(p_{b2}) $p_b(x,x) \leq p_b(x,y)$;

(p_{b3}) $p_b(x,y) = p_b(y,x)$;

(p_{b4}) $p_b(x,y) \leq s[p_b(x,z) + p_b(y,z)] - p_b(z,z)$.

The pair (X, p_b) is called a partial b-metric space.

The class of partial b-metric spaces is larger than the class of partial metric spaces, since a partial metric space is a special case of a partial b-metric space with the coefficient $s = 1$. Also, the class of partial b-metric spaces is larger than the class of b-metric spaces since a b-metric space is a special case of a partial b-metric space with the same coefficient and the self distance $p_b(x,x) = 0$.

The following example shows that a partial b-metric space need not be a partial metric space nor a b-metric space.

Example 2.1.7. [15] Let $X = \mathbb{R}_+$ and $q > 1$ be a constant. Define a function $p_b : X \times X \to \mathbb{R}_+$ by

$$p_b(x,y) = [\max\{x,y\}]^q + |x-y|^q \quad \text{for all } x, y \in X.$$

Then (X, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1} > 1$, but it is neither a partial metric space nor a b-metric space.

Proposition 2.1.8. [15] Let X be a nonempty set, p be a partial metric and d be a b-metric with the coefficient $s \geq 1$ on X. Then the function $p_b : X \times X \to \mathbb{R}_+$, defined by $p_b(x,y) = p(x,y) + d(x,y)$ for all $x, y \in X$, is a partial b-metric with the coefficient s.

Proposition 2.1.9. [15] Let (X, p) be a partial metric space and $q \geq 1$. Then (X, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1}$, where $p_b : X \times X \to \mathbb{R}_+$ is defined by $p_b(x,y) = [p(x,y)]^q$.

Mustafa et al. [63] introduced a modified version of Definition 2.1.6 in order to get that each partial b-metric p_b generates a b-metric d_{p_b}.

Definition 2.1.10. [63] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to \mathbb{R}_+$ is called a partial b-metric if for all $x, y, z \in X$ the following properties hold:

(p_{b1}) $x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$;

(p_{b2}) $p_b(x,x) \leq p_b(x,y)$;

(p_{b3}) $p_b(x,y) = p_b(y,x)$;

($\mathrm{p}_{b4'}$) $p_b(x,y) \leq s(p_b(x,z) + p_b(y,z) - p_b(z,z)) + (\frac{1-s}{2})(p_b(x,x) + p_b(y,y))$.

The pair (X, p_b) is called a partial b-metric space.

Since $s \geq 1$, by ($\mathrm{p}_{b4'}$), we obtain that

$$p_b(x,y) \;\leq\; s(p_b(x,z) + p_b(z,y) - p_b(z,z)) \;\leq\; s(p_b(x,z) + p_b(z,y)) - p_b(z,z).$$

Thus, a partial b-metric in the sense of Definition 2.1.10 is also a partial b-metric in Definition 2.1.6.

In a partial b-metric space (X, p_b), if $p_b(x,y) = 0$ implies $p_b(x,x) = p_b(x,y) = p_b(y,y) = 0$, then $x = y$, but if $x = y$, then $p_b(x,y)$ may not be 0. It is clear that every partial metric space is a partial b-metric space with the coefficient $s = 1$ and every b-metric space is a partial b-metric space with the same coefficient and the self distance $p_b(x,x) = 0$, but the converse of these facts may not hold.

The following example shows that a partial b-metric space (Definition 2.1.10) need not to be a partial metric space nor a b-metric space.

Example 2.1.11. [63] Let (X,d) be a metric space and $p_b : X \times X \to \mathbb{R}_+$ defined by

$$p_b(x,y) = d(x,y)^q + a \quad \text{for all } x, y \in X,$$

where $q > 1$ and $a \geq 0$. Then p_b is a partial b-metric with $s = 2^{q-1}$, but it is neither a partial metric nor a b-metric.

We need the following propositions, definitions and lemmas that will be used in the next part.

Proposition 2.1.12. [63] Every partial b-metric p_b defines a b-metric d_{p_b}, where

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y) \quad \text{for all } x,y \in X.$$

Definition 2.1.13. [63] Let $\{x_n\}$ be a sequence in a partial b-metric space (X, p_b).

(i) A sequence $\{x_n\}$ is p_b-convergent to a point $x \in X$ if

$$\lim_{n\to\infty} p_b(x, x_n) = p_b(x,x).$$

(ii) A sequence $\{x_n\}$ is a p_b-Cauchy sequence if

$$\lim_{n,m\to\infty} p_b(x_n, x_m) \text{ exists (and is finite).}$$

(iii) A partial b-metric space (X, p_b) is said to be p_b-complete if every p_b-Cauchy sequence $\{x_n\}$ in X p_b-converges to a point $x \in X$ such that

$$\lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n\to\infty} p_b(x_n, x) = p_b(x,x).$$

Lemma 2.1.14. [63]

(1) A sequence $\{x_n\}$ is a p_b-Cauchy sequence in a partial b-metric space (X, p_b) if and only if it is a b-Cauchy sequence in the b-metric space (X, d_{p_b}).

(2) A partial b-metric space (X, p_b) is p_b-complete if and only if a b-metric space (X, d_{p_b}) is b-complete. Moreover, $\lim_{n\to\infty} d_{p_b}(x, x_n) = 0$ if and only if

$$\lim_{n\to\infty} p_b(x, x_n) = \lim_{n,m\to\infty} p_b(x_n, x_m) = p_b(x,x).$$

Definition 2.1.15. [63] Let (X, p_b) and (X', p_b') be two partial b-metric spaces and let $f : (X, p_b) \to (X', p_b')$ be a mapping. Then f is said to be p_b-continuous at a point $a \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $p_b(a,x) < \delta + p_b(a,a)$ imply that $p_b'(f(a), f(x)) < \varepsilon + p_b'(f(a), f(a))$. The mapping f is p_b-continuous on X if it is p_b-continuous at all $a \in X$.

Proposition 2.1.16. [63] Let (X, p_b) and (X', p'_b) be two partial b-metric spaces. Then a mapping $f : X \to X$ is p_b-continuous at a point $x \in X$ if and only if it is p_b-sequentially continuous at x, that is, whenever $\{x_n\}$ is p_b-convergent to x, $\{f(x_n)\}$ is p'_b-convergent to $f(x)$.

Recently, Li and Jiang [64] introduced the notion of JS-quasi-contractions and proved some fixed point results for JS-quasi-contractions in complete metric spaces.

Following Hussain et al. [65], Li and Jiang [64] denoted Ψ by the set of all nondecreasing functions $\psi : [0, +\infty) \to [1, +\infty)$ satisfying the following conditions:

(Ψ1) $\psi(t) = 1$ if and only if $t = 0$;

(Ψ2) for each sequence $\{t_n\} \subset (0, +\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

(Ψ3) there exist $r \in (0, 1)$ and $l \in (0, +\infty]$ such that $\lim_{t\to 0^+} \dfrac{\psi(t) - 1}{t^r} = l$;

(Ψ4) $\psi(t + s) \leq \psi(t)\psi(s)$ for all $t, s > 0$.

Li and Jiang [64] have set the following symbols:

Φ_1 $= \{\psi : (0, +\infty) \to (1, +\infty) : \psi$ is a nondecreasing function satisfying (Ψ2) and (Ψ3)$\}$;

Φ_2 $= \{\psi : (0, +\infty) \to (1, +\infty) : \psi$ is a nondecreasing continuous function$\}$;

Φ_3 $= \{\psi : [0, +\infty) \to [1, +\infty) : \psi$ is a nondecreasing continuous function satisfying (Ψ1)$\}$;

Φ_4 $= \{\psi : [0, +\infty) \to [1, +\infty) : \psi$ is a nondecreasing continuous function satisfying (Ψ1) and (Ψ4)$\}$.

Furthermore, in [64], they presented the following examples for illustrating the relationship among the above sets.

Example 2.1.17. [64] Let $f(t) = e^{te^t}$ for $t \geq 0$. Then $f \in \Phi_2 \cap \Phi_3$, but $f \notin \Psi \cup \Phi_1 \cup \Phi_4$ since $\lim_{t\to 0^+} \dfrac{e^{te^t} - 1}{t^r} = 0$ for each $r \in (0, 1)$ and $e^{(t+s)e^{(t+s)}} > e^{se^s} e^{te^t}$ for all $s, t > 0$.

Example 2.1.18. [64] Let $g(t) = e^{t^a}$ for $t \geq 0$, where $a > 0$. When $a \in (0,1)$, $g \in \Psi \cap \Phi_1 \cap \Phi_2 \cap \Phi_3 \cap \Phi_4$. When $a = 1$, $g \in \Phi_2 \cap \Phi_3 \cap \Phi_4$, but $g \notin \Psi \cup \Phi_1$ since $\lim\limits_{t\to 0^+} \dfrac{e^t - 1}{t^r} = 0$ for each $r \in (0,1)$. When $a > 1$, $g \in \Phi_2 \cap \Phi_3$, but $g \notin \Psi \cup \Phi_1 \cup \Phi_4$ since $\lim\limits_{t\to 0^+} \dfrac{e^{t^a} - 1}{t^r} = 0$ for each $r \in (0,1)$ and $e^{(t+s)^a} > e^{t^a} e^{s^a}$ for all $s, t > 0$.

The main aim of [64] is introducing the concept of JS-quasi-contractions and assure the existence of the fixed point theorems for such mappings in complete metric spaces.

Definition 2.1.19. [64] Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a JS-quasi-contraction if there exist a function $\psi : (0,+\infty) \to (1,+\infty)$ and $\lambda \in (0,1)$ such that

$$\psi(d(Tx,Ty)) \leq \psi(M_d(x,y))^\lambda \quad \text{for all } x,y \in X \text{ with } Tx \neq Ty, \tag{2.1.1}$$

where $M_d(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

Remark 2.1.20. [64] Let $T : X \to X$ and $\psi : [0,+\infty) \to [1,+\infty)$ and for all $x, y \in X$ such that

$$\begin{aligned}&\psi(d(Tx,Ty))\\ &\quad\leq \psi(d(x,y))^{k_1}\psi(d(x,Tx))^{k_2}\psi(d(y,Ty))^{k_3}\psi(\frac{d(x,Ty)+d(y,Tx)}{2})^{2k_4},\end{aligned} \tag{2.1.2}$$

where k_1, k_2, k_3, k_4 are nonnegative numbers with $k_1 + k_2 + k_3 + 2k_4 < 1$. Then T is a JS-quasi-contraction with $\lambda = k_1 + k_2 + k_3 + 2k_4$, provided that $(\Psi 2)$ is satisfied.

Theorem 2.1.21. [64] Let (X,d) be a complete metric space and $T : X \to X$ be a JS-quasi-contraction with $\psi \in \Phi_2$. Then T has a unique fixed point in X.

Theorem 2.1.22. [64] Let (X,d) be a complete metric space and $T : X \to X$. Assume that there exist $\psi \in \Phi_3$ and nonnegative numbers k_1, k_2, k_3, k_4 with $k_1 + k_2 + k_3 + 2k_4 < 1$ such that (2.1.2) is satisfied. Then T has a unique fixed point in X.

In this part, we introduce a concept of generalized JS-quasi-contractions and obtain sufficient conditions for the existence of fixed points of such mappings on p_b-complete partial b-metric spaces. Our results extend the results in the literature. In addition, an example is given to illustrate and support our main result.

2.2 Algorithms for the common solution of the split variational inequality and fixed point problems with applications

In this section, we recall the following definitions and preliminary results that will be used in the sequel.

In 2005, Censor et al. [32] introduced the multiple-sets split feasibility problem (MSSFP) which is formulated as follows:

$$\text{Find } x \in \bigcap_{i=1}^{N} C_i \quad \text{such that } Ax \in \bigcap_{j=1}^{M} Q_j, \tag{2.2.1}$$

where $C_i, i = 1, 2, \ldots, N$ and $Q_j, j = 1, 2, \ldots, M$ are nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear mapping. Denote by Ω the set of solutions of MSSFP (2.2.1).

When $N = M = 1$, the MSSFP is known as the split feasibility problem (SFP), it was first introduced by Censor and Elfving [29], which is formulated as follows:

$$\text{Find } x \in C \quad \text{such that } Ax \in Q, \tag{2.2.2}$$

where C and Q are nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Denote by Γ the set of solutions of SFP (2.2.2).

Assuming that the SFP is consistent (i.e., (2.2.2) has a solution). It is well-known that $x \in C$ solves (2.2.2) if and only if it solves the fixed point equation

$$x = Tx, \quad T = P_C(I - \gamma A^*(I - P_Q)A), \quad x \in C, \tag{2.2.3}$$

where γ is a positive constant, A^* is the adjoint of A, P_C and P_Q are the metric projections of H_1 and H_2 onto C and Q, respectively, for more details see [50].

One of the iterative algorithms for solving (2.2.2) is the CQ algorithm, it was presented by Byrne [30, 34], which generates a sequence $\{x_n\}$ by

$$x_{n+1} = P_C(x_n - \gamma A^*(I - P_Q)Ax_n), \; n \geq 1, \tag{2.2.4}$$

where $\gamma \in (0, 2/\|A\|^2)$. Moreover, Byrne [30] proved the weakly convergence result for algorithm (2.2.4) in Hilbert spaces.

Since CQ algorithm can be viewed as a fixed point algorithm for averaged mappings, in 2010, Xu [50] showed the weakly convergence result of the following method by apply K-M algorithm for solving the SFP:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C(x_n - \gamma A^*(I - P_Q)Ax_n). \tag{2.2.5}$$

The split variational inequality problem (SVIP) was first investigated by Censor et al., which is the problem of finding a point

$$\begin{aligned} &x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in C, \\ &\text{and} \\ &y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^* \rangle \geq 0, \text{ for all } y \in Q, \end{aligned} \tag{2.2.6}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are mappings and $A : H_1 \to H_2$ is a bounded linear operator.

The variational inequality problem (VIP) was introduced by Stampacchia [16], which is the problem of finding a point x^* in a subset C of a Hilbert space H such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in C, \tag{2.2.7}$$

where $f : C \to H$ is a mapping and we denote its solution set of (2.2.7) by $VI(C, f)$.

In [36], we see that $x \in C$ solves (2.2.7) if and only if it solves the fixed point equation

$$x = Sx, \quad S = P_C(I - \gamma f), \quad x \in C. \tag{2.2.8}$$

In [35, 36], the following sequence $\{x_n\}$ of Picard iterates is a strongly convergent sequence in $VI(C, f)$ because $P_C(I - \gamma f)$ is a contraction on C, where f is η-strongly monotone and k-Lipschitz continuous, $0 < \gamma < \frac{2\eta}{k^2}$:

$$x_{n+1} = P_C(I - \gamma f)x_n. \tag{2.2.9}$$

However, algorithm (2.2.9) cannot use to solve VIP when f is monotone and k-Lipschitz continuous, which can be seen from the counterexample in [37]. During the last decade, many authors devoted their attention to study algorithms for solving the VIP. One of the methods is the extragradient method which was introduced and studied in 1976 by Korpelevich [38] in the finite dimensional Euclidean space $\mathbb{R}^n$:

$$\begin{aligned} y_n &= P_C(x_n - \gamma f x_n), \\ x_{n+1} &= P_C(x_n - \gamma f y_n), \end{aligned} \tag{2.2.10}$$

when f is monotone and k-Lipschitz continuous. Then sequence $\{x_n\}$ strongly converges to the solution of VIP.

After that, Nadezhkina and Takahashi [24] suggested the following modified extragradient method motivated by the idea of Korpelevich [38] and Takahashi et al. [40]. They showed that the sequence generated by the mentioned method converges weakly to an element in $F(S) \cap VI(C, f)$:

$$\begin{aligned} y_n &= P_C(x_n - \gamma_n f x_n), \\ x_{n+1} &= \lambda_n x_n + (1 - \lambda_n) S P_C(x_n - \gamma_n f y_n), \end{aligned} \tag{2.2.11}$$

when $S : C \to C$ is nonexpansive, and f is monotone and k-Lipschitz continuous. Since then, it has been used to study the problems of finding a common solution of VIP and fixed point problem (see [27] and the references therein).

In 2017, Tian and Jiang [51] considered the following iteration method by combining extragradient method with CQ algorithm for solving the SVIP:

$$\begin{aligned} y_n &= P_C(x_n - \gamma_n A^*(I - P_Q(I - \theta g))A x_n), \\ z_n &= P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} &= P_C(y_n - \lambda_n f(z_n)), \end{aligned} \tag{2.2.12}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $f : C \to H_1$ is a monotone and k-Lipschitz continuous mapping, and $g : H_2 \to H_2$ is a δ-inverse strongly monotone mapping.

Lemma 2.2.1. [66] Given $x \in H$ and $z \in C$. Then the following statements are equivalent:

(i) $z = P_C x$;

(ii) $\langle x - z, z - y \rangle \geq 0,$ for all $y \in C$;

(iii) $\|x - y\|^2 \geq \|x - z\|^2 + \|y - z\|^2,$ for all $y \in C$.

We need the following definitions about set-valued mappings for proving our main results.

Definition 2.2.2. [58] Let $B : H \rightrightarrows H$ be a set-valued mapping with the effective domain $D(B) = \{x \in H : Bx \neq \emptyset\}$.

The set-valued mapping B is said to be monotone if for each $x, y \in D(B),\ u \in Bx,$ and $v \in By$, we have

$$\langle x - y, u - v \rangle \geq 0.$$

Also the monotone set-valued mapping B is said to be maximal if its graph $G(B) = \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone set-valued mappings.

The following property of the maximal monotone mappings is very convenient and helpful to use:

A monotone mapping B is maximal if and only if for $(x, u) \in H \times H$,

$\langle x - y, u - v \rangle \geq 0$ for each $(y, v) \in G(B)$ implies $u \in Bx$.

The fixed point problem for nonexpansive mappings in Hilbert spaces is related to the problem of finding zero points of a maximal monotone operator B on H, then, for each $r > 0$, the resolvent J_r of B defined by

$$J_r := (I + rB)^{-1} : H \to D(B),$$

is a single-valued firmly nonexpansive mapping from H into itself and $F(J_r) = B^{-1}0,$ for all $r > 0$ (see [67]).

Remark 2.2.3. [67] Let r be any positive scalar. A mapping B on H is monotone if and only if its resolvent J_r is firmly nonexpansive.

Let $f : C \to H$ be a monotone and k-Lipschitz continuous mapping. In [68], we know that normal cone to C defined by

$$N_C x = \{z \in H : \langle z, y - x \rangle \leq 0,\ \text{for all } y \in C\},\ \text{for all } x \in C,$$

is a maximal monotone mapping and a resolvent of N_C is P_C.

The following results play the crucial role in the next section.

Lemma 2.2.4. [24] Let $f : C \to H$ be a monotone and k-Lipschitz continuous mapping. Define a set-valued mapping $B : H \rightrightarrows H$ by

$$Bv = \begin{cases} fv + N_C v, & \text{if } v \in C; \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then B is maximal monotone and $0 \in Bv$ if and only if $v \in VI(C, f)$.
In particular, $B^{-1}0 = \{v \in H : 0 \in Bv\}$, called the set of zero points, is closed and convex.

Lemma 2.2.5. [24] Let H_1 and H_2 be real Hilbert spaces. Let $B : H_1 \rightrightarrows H_1$ be a maximal monotone mapping, and J_r be the resolvent of B for $r > 0$. Suppose that $T : H_2 \to H_2$ is a nonexpansive mapping and $A : H_1 \to H_2$ is a bounded linear operator. Assume that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $r, \gamma > 0$ and $z \in H_1$. Then the following statements are equivalent:

(i) $z = J_r(I - \gamma A^*(I - T)A)z$;

(ii) $0 \in A^*(I - T)Az + Bz$;

(iii) $z \in B^{-1}0 \cap A^{-1}F(T)$.

In [51], they obtained the following results by putting $B = N_C$ in Lemma 2.2.5.

Corollary 2.2.6. [51] Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty closed convex subset of H_1. Let $T : H_1 \to H_1$ be a nonexpansive mapping, and $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $C \cap A^{-1}F(T) \neq \emptyset$. Let $\gamma > 0$ and $z \in H_1$. Then the following statements are equivalent:

(i) $z = P_C(I - \gamma A^*(I - T)A)z$;

(ii) $0 \in A^*(I - T)Az + N_C z$;

(iii) $z \in C \cap A^{-1}F(T)$.

We also need the following lemmas.

Lemma 2.2.7. [69] Let H be a real Hilbert space and $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in H converges weakly to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.

Lemma 2.2.8. [70] Let $\{\alpha_n\}$ be a real sequence satisfying $0 < a \leq \alpha_n \leq b < l$ for all $n \geq 0$, and let $\{v_n\}$ and $\{w_n\}$ be two sequences in H such that, for some $\sigma \geq 0$,

$$\limsup_{n\to\infty} \|v_n\| \leq \sigma, \ \limsup_{n\to\infty} \|w_n\| \leq \sigma, \quad \text{and} \ \lim_{n\to\infty} \|\alpha_n v_n + (1-\alpha_n)w_n\| = \sigma.$$

Then

$$\lim_{n\to\infty} \|v_n - w_n\| = 0.$$

Lemma 2.2.9. [56] Let $\{x_n\}$ be a sequence in H satisfying the properties:

(i) $\lim_{n\to\infty} \|x_n - u\|$ exists for each $u \in C$;

(ii) $\omega_w(x_n) \subset C$.

Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.2.10. [40] Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H. Suppose that

$$\|x_{n+1} - u\| \leq \|x_n - u\|, \quad \forall u \in C,$$

for every $n = 0, 1, 2, \ldots$. Then the sequence $\{P_C x_n\}$ converges strongly to a point in C.

Theorem 2.2.11. [24] Let $f : C \to H$ be a monotone and k-Lipschitz continuous mapping. Assume that $S : C \to C$ is a nonexpansive mapping such that $F(T) \cap VI(C,f) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by (2.2.11), where $\{\lambda_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{k})$ and $\{\alpha_n\} \subset [c,d]$ for some $c, d \in (0,1)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $z \in F(T) \cap VI(C,f) \neq \emptyset$, where $z \in \lim_{n\to\infty} P_{F(T)\cap VI(C,f)} x_n$.

Theorem 2.2.12. [50] Assume that the solution set of SFP is consistent and $0 < \gamma < \frac{2}{\|A\|^2}$. Let $\{x_n\}$ be defined by the averaged CQ algorithm (2.2.5) where $\{\alpha_n\}$ is a sequence in $\left[0, \frac{4}{2+\gamma\|A\|^2}\right]$ satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n \left(\frac{4}{2+\gamma\|A\|^2} - \alpha_n \right) = \infty.$$

Then the sequence $\{x_n\}$ is a weakly convergent to a point in the solution set of SFP.

The equilibrium problem is formulated by Blum and Oettli [23] in 1994 for finding a point x^* such that

$$F(x^*, y) \geq 0, \text{ for all } y \in C, \tag{2.2.13}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction. The solution set of equilibrium problem (2.2.13) is denoted by $EP(C, F)$.

In [23], we know that, if F is a bifunction satisfying the following conditions:

(A1) $F(x, x) = 0,$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0,$ for all $x, y \in C$;

(A3) for each $x, y, z \in C, \limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) for each fixed $x \in C, y \mapsto F(x, y)$ is lower semicontinuous and convex,

then there exists $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C,$$

where r is a positive real number and $x \in H$.

From [71], we obtain that, for $r > 0$ and $x \in H$, the resolvent $T_r : H \to C$ of bifunction F which satisfies the conditions (A1)-(A4) is formulated as follows:

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \text{ for all } y \in C\}, \text{ for all } x \in H,$$

has the following properties:

(i) T_r is single-valued and firmly nonexpansive;

(ii) $F(T_r) = EP(C, F)$;

(iii) $EP(C, F)$ is closed and convex.

Let ϕ be a real-valued convex function from C to $\mathbb{R}$, the typical form of constrained convex minimization problem is finding a point $x^* \in C$ satisfying

$$\phi(x^*) = \min_{x\in C} \phi(x). \tag{2.2.14}$$

Denote the solution set of constrained convex minimization problem (2.2.14) by $\arg\min_{x\in C} \phi(x)$.

Lemma 2.2.13. [51] Let ϕ be a convex function of H into $\mathbb{R}$. If ϕ is differentiable, then $z \in \arg\min_{x\in C}$ if and only if $z \in VI(C, \nabla\phi)$.

In this part, we establish a new iterative algorithm by combining Nadezhkina and Takahashi's modified extragradient method (2.2.11) and Xu's algorithm (2.2.5). The mentioned iterative algorithm presents the common solution of the split variational inequality problems and fixed point problems. We show that the sequence produced by our algorithm is weak convergent. Finally, we give some applications of the main results.

2.3 Convergence theorems for the variational inequality problems

Jung [72] studied the common solution of variational inequality problem and split feasibility problem: Find a point

$$x^* \in \Gamma : \langle fx^*, x - x^*\rangle \geq 0, \text{ for all } x \in \Gamma, \tag{2.3.1}$$

where Γ is the solution set of SFP (2.2.2), and $f : H \to H$ is an η-strongly monotone and k-Lipschitz continuous mapping. After that, for solving the problem (2.3.1), Buong [52] considered the following algorithms, which are proposed in [53] and [54], respectively:

$$x_{n+1} = (I - t_n\mu f)Tx_n, \ n \geq 0, \tag{2.3.2}$$

and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)(I - t_n\mu f)Tx_n, \ n \geq 0, \tag{2.3.3}$$

where $T = P_C(I - \gamma A^*(I - P_Q)A)$, and under the following conditions:

(C1) $t_n \in (0,1), t_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} t_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

In 2011, Buong showed the existence of a unique solution of the variational inequality problems.

Theorem 2.3.1. [73] Let f be a k-Lipschitz continuous and η-strongly monotone self-mapping of H. Assume that $\{T_i\}_{i=1}^N$ is nonexpansive self-mappings of H such that $C = \cap_{i=1}^N F(T_i) \neq \emptyset$. Then the sequence $\{x_n\}$ defined by the following algorithm, converges strongly to the unique solution x^* of the variational inequality (2.2.7):

$$x_{n+1} = (1-\beta_n^0)x_n + \beta_n^0(I - t_n\mu f)T_N^n T_{N-1}^n \cdots T_1^n x_n, \quad n \geq 0, \tag{2.3.4}$$

where $\mu \in (0, 2\eta/k^2)$, $T_i^n := (1-\beta_n^i)I + \beta_n^i T_i$, for $i = 1,\ldots,N$, and under the following conditions:

(i) $t_n \in (0,1), t_n \to 0$ as $n \to \infty$ and $\sum_{n=0}^{\infty} t_n = \infty$;

(ii) $\beta_n^i \in (\alpha, \beta)$, for some $\alpha, \beta \in (0,1)$, and $|\beta_{n+1}^i - \beta_n^i| \to 0$ as $n \to \infty$ $(i = 0,\ldots,N)$.

After that, Zhou [74] presented the strong convergence theorem for solving the variational inequality problems.

Theorem 2.3.2. [74] *Let f, C, $\mu, \{\beta_n^i\}_{i=1}^N$, $\{t_n\}$ and $\{T_i\}_{i=1}^N$ be as in Theorem 2.3.1. Then the sequence $\{x_n\}$ defined by the following algorithm:*

$$x_{n+1} = (I - t_n\mu f)T_N^n T_{N-1}^n \cdots T_1^n x_n, \quad n \geq 1, \tag{2.3.5}$$

converges strongly to the unique solution x^* of the variational inequality (2.2.7).

Moreover, Buong [52] considered the sequence $\{x_n\}$, which is generated by the following algorithm, which is weakly convergent to a solution of MSSFP (2.2.1):

$$x_{n+1} = P_1(x_n - \gamma A^*(I - P_2)Ax_n), \tag{2.3.6}$$

where $P_1 = P_{C_1} \cdots P_{C_N}$ and $P_2 = P_{Q_1} \cdots P_{Q_M}$ or $P_1 = \sum_{i=1}^{N} \alpha_i P_{C_i}$ and $P_2 = \sum_{j=1}^{M} \beta_j P_{Q_j}$, α_i and β_j, for $1 \leq i \leq N$ and $1 \leq j \leq M$, are positive real numbers such that $\sum_{i=1}^{N} \alpha_i = \sum_{j=1}^{M} \beta_j = 1$.

In this part, we establish an iterative algorithm by combining algorithms (2.3.2) and (2.3.3) for solving the solution of problem (2.3.1), and obtaining a strongly convergence theorem in Hilbert spaces. Moreover, we propose iterative algorithms for solving the common solutions of variational inequality problems and multiple-sets split feasibility problems. Finally, we also give numerical examples for illustrating our algorithms.

CHAPTER III

FIXED POINT PROBLEMS

In this chapter, we propose the concept of generalized JS-quasi-contractions and obtain sufficient conditions for the existence of fixed points of such mappings on p_b-complete partial b-metric spaces. Our results extend the results in the literature. In addition, an example is given to illustrate and support our main result.

3.1 Fixed point theorems for generalized JS-quasi-contractions in complete partial b-metric spaces

We now introduce the concept of generalized JS-quasi-contractions on partial b-metric spaces.

Definition 3.1.1. Let (X, p_b) be a partial b-metric space with the coefficient $s \geq 1$. We say that a mapping $T : X \to X$ is a generalized JS-quasi-contraction if there exist a function $\psi : (0, +\infty) \to (1, +\infty)$ and $\lambda \in (0, 1)$ such that

$$\psi(sp_b(Tx, Ty)) \leq \psi(M_s(x, y))^\lambda \quad \text{for all } x, y \in X \quad \text{with} \quad Tx \neq Ty, \qquad (3.1.1)$$

where $M_s(x, y) = \max\{p_b(x, y), p_b(x, Tx), p_b(y, Ty), \frac{p_b(x,Ty)+p_b(y,Tx)}{2s}\}$.

The following example shows that a generalized JS-quasi-contraction need not to be p_b-continuous.

Example 3.1.2. Let $X = [0, +\infty)$ with the partial b-metric $p_b : X \times X \to \mathbb{R}_+$ defined by

$$p_b(x, y) = [\max\{x, y\}]^2$$

for all $x, y \in X$. Obviously, (X, p_b) is a p_b-complete partial b-metric space with $s = 2$. Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{2}{3}, & x \in [0, 1), \\ \frac{x-1}{2x}, & \text{otherwise.} \end{cases}$$

We will show that T is a generalized JS-quasi-contraction with $\psi(t) = e^t \in \Phi_2$. In fact, it suffices to show that there exists $\lambda \in (0,1)$ such that, for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{2p_b(Tx,Ty)}{M_s(x,y)} \leq \lambda.$$

Let $x, y \in X$ with $Tx \neq Ty$. Without loss of generality, we may assume that $x < y$. It follows that $1 \leq x < y$. Therefore,

$$p_b(Tx,Ty) = [\max\{\frac{x-1}{2x}, \frac{y-1}{2y}\}]^2 = \frac{y^2-2y+1}{4y^2},$$

and

$$M_s(x,y) = \max\{y^2, x^2, y^2, \frac{[\max\{x, \frac{y-1}{2y}\}]^2 + y^2}{2s}\} = y^2.$$

This implies that

$$\frac{2p_b(Tx,Ty)}{M_s(x,y)} = \frac{y^2-2y+1}{2y^4} \leq \frac{1}{32}.$$

This shows that T is a generalized JS-quasi-contraction with $\psi(t) = e^t \in \Phi_2$ and $\lambda \in [\frac{1}{32}, 1)$.

On the other hand, T is not p_b-continuous because there exists a sequence $\{\frac{1}{n+1}\}$ such that

$$\lim_{n\to\infty} p_b(1, x_n) = \lim_{n\to\infty} [\max\{1, x_n\}]^2 = 1 = p_b(1,1),$$

but

$$\lim_{n\to\infty} p_b(T1, Tx_n) = [\max\{0, \frac{2}{3}\}]^2 = \frac{4}{9} \neq 0 = p_b(T1, T1).$$

The following example shows that a p_b-continuous mapping need not to be a generalized JS-quasi-contraction.

Example 3.1.3. Let $X = \{0,1,2\}$ with the partial b-metric $p_b : X \times X \to \mathbb{R}_+$ defined by

$$p_b(x,y) = |x-y|^2$$

for all $x, y \in X$. Obviously, (X, p_b) is a p_b-complete partial b-metric space with $s = 2$. Define the mapping $T : X \to X$ by $T0 = T1 = 0$ and $T2 = 1$. Then T is p_b-continuous.

We will show that T is not a generalized JS-quasi-contraction with $\psi(t) = e^{te^t} \in \Phi_2$. In fact, it suffices to show that for all $\lambda \in (0,1)$, there exist $x, y \in X$ with $Tx \neq Ty$ such that

$$\frac{2p_b(Tx,Ty)e^{2p_b(Tx,Ty)-M_s(x,y)}}{M_s(x,y)} > \lambda.$$

Let $\lambda \in (0,1)$, for $x = 1$ and $y = 2$, we have $p_b(T1,T2) = 1$ and $M_s(1,2) = 1$. Therefore,

$$\frac{2p_b(T1,T2)e^{2p_b(T1,T2)-M_s(1,2)}}{M_s(1,2)} = 2(1)e^{2-1} = 2e > \lambda,$$

which implies that T is not a generalized JS-quasi-contraction.

Remark 3.1.4. As in [64], we obtain the following statements in a partial b-metric space (X, p_b) :

(i) Let $T : X \to X$ and $\lambda \in (0,1)$ such that

$$sp_b(Tx,Ty) \leq \lambda M_s(x,y) \quad \text{for all } x,y \in X.$$

Then T is a generalized JS-quasi-contraction with $\psi(t) = e^t$.

(ii) Let $T : X \to X$ and $\psi : (0,+\infty) \to (1,+\infty)$ be such that

$$\psi(sp_b(Tx,Ty)) \leq \psi(p_b(x,y))^{\lambda} \quad \text{for all } x,y \in X \text{ with } Tx \neq Ty, \tag{3.1.2}$$

where $\lambda \in (0,1)$. Then T is a generalized JS-quasi-contraction.

(iii) Let $T : X \to X$ and $\psi : [0,+\infty) \to [1,+\infty)$ be such that

$$\begin{aligned}&\psi(sp_b(Tx,Ty))\\ &\leq \psi(p_b(x,y))^{k_1}\psi(p_b(x,Tx))^{k_2}\psi(p_b(y,Ty))^{k_3}\psi(\frac{p_b(x,Ty)+p_b(y,Tx)}{2s})^{2k_4}\end{aligned} \tag{3.1.3}$$

for all $x, y \in X$, where k_1, k_2, k_3, k_4 are nonnegative numbers with $k_1 + k_2 + k_3 + 2k_4 < 1$. Then T is a generalized JS-quasi-contraction with $\lambda = k_1 + k_2 + k_3 + 2k_4$, provided that $(\Psi1)$ is satisfied.

(iv) Let $T : X \to X$ and $\psi : [0, +\infty) \to [1, +\infty)$ be such that

$$\psi(sp_b(Tx, Ty)) \leq \psi(p_b(x, y))^{k_1} \psi(p_b(x, Tx))^{k_2} \psi(p_b(y, Ty))^{k_3} \psi(\frac{p_b(x, Ty) + p_b(y, Tx)}{s})^{k_4} \tag{3.1.4}$$

for all $x, y \in X$. Suppose that ψ is a nondecreasing function such that $(\Psi 4)$ is satisfied. It follows that

$$\psi(\frac{p_b(x, Ty) + p_b(y, Tx)}{s})^{k_4} \leq \psi(\frac{p_b(x, Ty) + p_b(y, Tx)}{2s})^{2k_4} \quad \text{for all } x, y \in X,$$

and so (3.1.3) holds. Moreover, if $(\Psi 1)$ is satisfied, then it follows from (iii) that T is a generalized JS-quasi-contraction with $\lambda = k_1 + k_2 + k_3 + 2k_4$. Therefore, T is a generalized JS-quasi-contraction with $\psi \in \Phi_4$ or $\psi \in \Psi$.

We now prove the existence of a unique fixed point for a generalized JS-quasi-contraction.

Theorem 3.1.5. *Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$. Let $T : X \to X$ be a generalized JS-quasi-contraction with $\psi \in \Phi_2$ and be p_b-continuous. Then T has a unique fixed point in X.*

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. If there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$, then x_n is a fixed point of T and the proof is finished. So we may assume that for every $n \in \mathbb{N}$,

$$x_n \neq x_{n+1}. \tag{3.1.5}$$

From (3.1.1), (3.1.5), and ψ is nondecreasing, we have

$$\psi(p_b(x_n, x_{n+1})) \leq \psi(sp_b(x_n, x_{n+1})) \leq \psi(M_s(x_{n-1}, x_n))^{\lambda}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M_s(x_{n-1}, x_n) &= \max\{p_b(x_{n-1}, x_n), p_b(x_{n-1}, Tx_{n-1}), p_b(x_n, Tx_n), \\ &\qquad \frac{p_b(x_{n-1}, Tx_n) + p_b(x_n, Tx_{n-1})}{2s}\} \\ &= \max\{p_b(x_{n-1}, x_n), p_b(x_{n-1}, x_n), p_b(x_n, x_{n+1}), \end{aligned}$$

$$\frac{p_b(x_{n-1},x_{n+1})+p_b(x_n,x_n)}{2s}\}$$

$$\le \max\{p_b(x_{n-1},x_n), p_b(x_n,x_{n+1})$$

$$,\frac{s(p_b(x_{n-1},x_n)+p_b(x_n,x_{n+1}))-p_b(x_n,x_n)+p_b(x_n,x_n)}{2s}\}$$

$$= \max\{p_b(x_{n-1},x_n), p_b(x_n,x_{n+1}), \frac{p_b(x_{n-1},x_n)+p_b(x_n,x_{n+1})}{2}\}$$

$$= \max\{p_b(x_{n-1},x_n), p_b(x_n,x_{n+1})\}.$$

This implies that

$$\psi(p_b(x_n,x_{n+1})) \le \psi(sp_b(x_n,x_{n+1})) \le \psi(\max\{p_b(x_{n-1},x_n), p_b(x_n,x_{n+1})\})^\lambda \quad (3.1.6)$$

for all $n \in \mathbb{N}$. If there exists some $n \in \mathbb{N}$ such that $p_b(x_n,x_{n+1}) > p_b(x_{n-1},x_n)$, then

$$\psi(p_b(x_n,x_{n+1})) \le \psi(p_b(x_n,x_{n+1}))^\lambda < \psi(p_b(x_n,x_{n+1})),$$

which is a contradiction. It follows that

$$p_b(x_n,x_{n+1}) \le p_b(x_{n-1},x_n)$$

for all $n \in \mathbb{N}$. So the sequence $\{p_b(x_n,x_{n+1})\}$ is a nonincreasing sequence of real numbers which is bounded from below and thus there exists $\alpha \ge 0$ such that

$$\lim_{n\to\infty} p_b(x_n,x_{n+1}) = \alpha \quad \text{and} \quad p_b(x_n,x_{n+1}) \ge \alpha. \quad (3.1.7)$$

Suppose that $\alpha > 0$. From (3.1.6), (3.1.7), and ψ being nondecreasing, we obtain that

$$1 < \psi(\alpha) \le \psi(p_b(x_n,x_{n+1})) \le \psi(p_b(x_{n-1},x_n))^\lambda \le \cdots \le \psi(p_b(x_0,x_1))^{\lambda^n} \quad (3.1.8)$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (3.1.8), we have $1 < \psi(\alpha) \le 1$, which is a contradiction. Thus $\alpha = 0$ and this yields

$$\lim_{n\to\infty} p_b(x_n,x_{n+1}) = 0. \quad (3.1.9)$$

Now, we show that $\{x_n\}$ is a p_b-Cauchy sequence in (X,p_b) which is equivalent to show that $\{x_n\}$ is a b-Cauchy sequence in (X,d_{p_b}). Suppose not, that is, there exist $\varepsilon > 0$ and two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index with $n_k > m_k > k$ for which

$$d_{p_b}(x_{m_k}, x_{n_k}) \geq \varepsilon \tag{3.1.10}$$

and

$$d_{p_b}(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{3.1.11}$$

This implies that

$$\varepsilon \leq d_{p_b}(x_{m_k}, x_{n_k}) \leq sd_{p_b}(x_{m_k}, x_{n_k-1}) + sd_{p_b}(x_{n_k-1}, x_{n_k}) < s\varepsilon + sd_{p_b}(x_{n_k-1}, x_{n_k}). \tag{3.1.12}$$

Taking the upper limit as $k \to \infty$ in (3.1.11), we get that

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d_{p_b}(x_{m_k}, x_{n_k-1}) \leq \limsup_{k\to\infty} d_{p_b}(x_{m_k}, x_{n_k-1}) \leq \varepsilon. \tag{3.1.13}$$

It follows from (3.1.12) that,

$$\varepsilon \leq \limsup_{k\to\infty} d_{p_b}(x_{m_k}, x_{n_k}) \leq s\varepsilon. \tag{3.1.14}$$

By using the triangular inequality, we have

$$\begin{aligned} d_{p_b}(x_{m_k+1}, x_{n_k}) &\leq sd_{p_b}(x_{m_k+1}, x_{m_k}) + sd_{p_b}(x_{m_k}, x_{n_k}) \\ &\leq sd_{p_b}(x_{m_k+1}, x_{m_k}) + s^2 d_{p_b}(x_{m_k}, x_{n_k-1}) + s^2 d_{p_b}(x_{n_k-1}, x_{n_k}) \\ &\leq sd_{p_b}(x_{m_k+1}, x_{m_k}) + s^2\varepsilon + s^2 d_{p_b}(x_{n_k-1}, x_{n_k}). \end{aligned}$$

Taking the upper limit as $k \to \infty$ in above inequality, we obtain that

$$\limsup_{k\to\infty} d_{p_b}(x_{m_k+1}, x_{n_k}) \leq s^2\varepsilon.$$

Further,

$$\begin{aligned} d_{p_b}(x_{m_k+1}, x_{n_k-1}) &\leq sd_{p_b}(x_{m_k+1}, x_{m_k}) + sd_{p_b}(x_{m_k}, x_{n_k-1}) \\ &\leq sd_{p_b}(x_{m_k+1}, x_{m_k}) + s\varepsilon, \end{aligned}$$

and hence

$$\limsup_{k\to\infty} d_{p_b}(x_{m_k+1}, x_{n_k-1}) \leq s\varepsilon. \tag{3.1.15}$$

By Proposition 2.1.12 and (3.1.9), we deduce that

$$\begin{aligned}
&\limsup_{k\to\infty} d_{p_b}(x_{m_k}, x_{n_k-1}) \\
&= \limsup_{k\to\infty}(2p_b(x_{m_k}, x_{n_k-1}) - p_b(x_{m_k}, x_{m_k}) - p_b(x_{n_k-1}, x_{n_k-1})) \qquad (3.1.16)\\
&= 2\limsup_{k\to\infty} p_b(x_{m_k}, x_{n_k-1}).
\end{aligned}$$

Also, by (3.1.13) and (3.1.16), we get that

$$\frac{\varepsilon}{2s} \leq \liminf_{k\to\infty} p_b(x_{m_k}, x_{n_k-1}) \leq \limsup_{k\to\infty} p_b(x_{m_k}, x_{n_k-1}) \leq \frac{\varepsilon}{2}. \qquad (3.1.17)$$

In analogy to (3.1.16), by (3.1.12), (3.1.14), and (3.1.15), we can prove that

$$\limsup_{k\to\infty} p_b(x_{m_k}, x_{n_k}) \leq \frac{s\varepsilon}{2}, \qquad (3.1.18)$$

$$\frac{\varepsilon}{2s} \leq \limsup_{k\to\infty} p_b(x_{m_k+1}, x_{n_k}),$$

$$\limsup_{k\to\infty} p_b(x_{m_k+1}, x_{n_k-1}) \leq \frac{s\varepsilon}{2}. \qquad (3.1.19)$$

By (3.1.17), (3.1.18), and (3.1.19), we obtain that

$$\begin{aligned}
&\limsup_{k\to\infty} M_s(x_{m_k}, x_{n_k-1}) \\
&= \max\{\limsup_{k\to\infty} p_b(x_{m_k}, x_{n_k-1}), \limsup_{k\to\infty} p_b(x_{m_k}, Tx_{m_k}), \\
&\qquad \limsup_{k\to\infty} p_b(x_{n_k-1}, Tx_{n_k-1}), \limsup_{k\to\infty} \frac{p_b(x_{m_k}, Tx_{n_k-1}) + p_b(x_{n_k-1}, Tx_{m_k})}{2s}\} \\
&\leq \max\{\limsup_{k\to\infty} p_b(x_{m_k}, x_{n_k-1}), 0, 0, \limsup_{k\to\infty} \frac{p_b(x_{m_k}, x_{n_k}) + p_b(x_{n_k-1}, x_{m_k+1})}{2s}\} \\
&= \max\{\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\} = \frac{\varepsilon}{2}.
\end{aligned}$$

We claim that $x_{m_k+1} \neq x_{n_k}$. If $x_{m_k+1} = x_{n_k}$, then $d_{p_b}(x_{m_k+1}, x_{n_k}) = 0$. From (3.1.10) and Proposition 2.1.12, we have

$$\begin{aligned}
\varepsilon \leq d_{p_b}(x_{m_k}, x_{n_k}) &\leq sd_{p_b}(x_{m_k}, x_{m_k+1}) + sd_{p_b}(x_{m_k+1}, x_{n_k}) \\
&= sd_{p_b}(x_{m_k}, x_{m_k+1}) \\
&= s(2p_b(x_{m_k}, x_{m_k+1}) - p_b(x_{m_k}, x_{m_k}) - p_b(x_{m_k+1}, x_{m_k+1})) \\
&\leq 2sp_b(x_{m_k}, x_{m_k+1}).
\end{aligned}$$

Letting $k \to \infty$ and using (3.1.9), we deduce that

$$\frac{\varepsilon}{2s} \leq \lim_{k\to\infty} p_b(x_{m_k}, x_{m_k+1}) = 0,$$

which is a contradiction. It follows from (3.1.1) that

$$\begin{aligned}\psi(\frac{\varepsilon}{2}) \ \leq \ \psi(s\limsup_{k\to\infty} p_b(x_{m_k+1}, x_{n_k})) &= \limsup_{k\to\infty} \psi(sp_b(x_{m_k+1}, x_{n_k})) \\ &\leq \limsup_{k\to\infty} \psi(M_s(x_{m_k}, x_{n_k-1}))^{\lambda} \leq \psi(\frac{\varepsilon}{2})^{\lambda} \\ &< \psi(\frac{\varepsilon}{2}),\end{aligned}$$

which is a contradiction. Thus $\{x_n\}$ is a b-Cauchy in b-metric space (X, d_{p_b}). Since (X, p_b) is p_b-complete, then (X, d_{p_b}) is a b-complete b-metric space. So, there exists $z \in X$ such that $\lim_{n\to\infty} d_{p_b}(x_n, z) = 0$. By Lemma 2.1.14, we get that

$$\lim_{n\to\infty} p_b(z, x_n) = p_b(z, z). \tag{3.1.20}$$

By Proposition 2.1.12, (3.1.9), (3.1.20), and condition (p_{b2}), we have

$$\lim_{n\to\infty} p_b(z, x_n) = \lim_{n\to\infty} p_b(x_n, x_n) = 0. \tag{3.1.21}$$

Suppose that $z \neq Tz$ implies that $p_b(z, Tz) > 0$ and $d_{p_b}(z, Tz) > 0$. It follows from (3.1.9) and (3.1.21) that there exists a positive integer n_0 such that

$$p_b(x_n, z) \leq \frac{p_b(z, Tz)}{3} \quad \text{and} \quad p_b(x_n, x_{n+1}) \leq \frac{p_b(z, Tz)}{3}$$

for all $n \geq n_0$. This implies that

$$\begin{aligned}M_s(x_n, z) &= \max\{p_b(x_n, z), p_b(x_n, x_{n+1}), p_b(z, Tz), \frac{p_b(x_n, Tz) + p_b(z, x_{n+1})}{2s}\} \\ &\leq \max\{\frac{p_b(z, Tz)}{3}, \frac{p_b(z, Tz)}{3}, p_b(z, Tz), p_b(z, Tz)\} \\ &= p_b(z, Tz)\end{aligned} \tag{3.1.22}$$

for all $n \geq n_0$. Since T is p_b-continuous and (3.1.20), we obtain that

$$\lim_{n\to\infty} p_b(x_{n+1}, Tz) = p_b(Tz, Tz). \tag{3.1.23}$$

By the triangle inequality, we deduce that

$$p_b(z, Tz) \leq sp_b(z, x_{n+1}) + sp_b(x_{n+1}, Tz)$$

for all $n \in \mathbb{N}$. So by taking limit as $n \to \infty$ and using (3.1.23), we have

$$p_b(z, Tz) \leq s\lim_{n\to\infty} p_b(z, x_{n+1}) + s\lim_{n\to\infty} p_b(x_{n+1}, Tz) = sp_b(Tz, Tz). \tag{3.1.24}$$

If there are infinitely many $n \in \mathbb{N}$ such that $x_{n+1} = Tz$, then $d_{p_b}(x_{n+1}, Tz) = 0$. This implies that

$$d_{p_b}(z, Tz) \leq sd_{p_b}(z, x_{n+1}) + sd_{p_b}(x_{n+1}, Tz) = sd_{p_b}(z, x_{n+1}).$$

Letting $n \to \infty$, we get that $d_{p_b}(z, Tz) \leq s \lim_{n\to\infty} d_{p_b}(z, x_{n+1}) = 0$, which is a contradiction. This implies that there exists $n_1 \in \mathbb{N}$ such that $x_{n+1} \neq Tz$ for all $n \geq n_1$. Choose $N = \max\{n_0, n_1\}$. Thus, by (3.1.1) and (3.1.22), for each $n \geq N$, we get that

$$\psi(sp_b(x_{n+1}, Tz)) \leq \psi(M_s(x_n, z))^{\lambda} \leq \psi(p_b(z, Tz))^{\lambda}.$$

Letting $n \to \infty$ in this inequality, using the continuity of ψ, (3.1.23), and (3.1.24), we obtain that

$$\begin{aligned}\psi(p_b(z, Tz)) \leq \psi(sp_b(Tz, Tz)) &= \lim_{n\to\infty} \psi(sp_b(x_{n+1}, Tz)) \\ &\leq \psi(p_b(z, Tz))^{\lambda} < \psi(p_b(z, Tz)),\end{aligned}$$

which is a contradiction. Hence $Tz = z$. Thus z is a fixed point of T. Let x be another fixed point of T with $x \neq z$. It follows from (3.1.1) that

$$\begin{aligned}\psi(p_b(x, z)) &\leq \psi(sp_b(Tx, Tz)) \\ &\leq p_b(M_s(x, z))^{\lambda} \\ &= \psi(\max\{p_b(x, z), p_b(x, Tx), p_b(z, Tz), \frac{p_b(x, Tz) + p_b(z, Tx)}{2s}\})^{\lambda} \\ &= \psi(\max\{p_b(x, z), p_b(x, x), p_b(z, z), \frac{p_b(x, z) + p_b(z, x)}{2s}\})^{\lambda} \\ &= \psi(\max\{p_b(x, z), p_b(x, x), p_b(z, z), \frac{p_b(x, z)}{s}\})^{\lambda} \\ &\leq \psi(\max\{p_b(x, z), p_b(x, z), p_b(x, z), \frac{p_b(x, z)}{s}\})^{\lambda} \\ &= \psi(p_b(x, z))^{\lambda} \\ &< \psi(p_b(x, z)),\end{aligned}$$

which is a contradiction. So $x = z$. Hence T has a unique fixed point. □

We illustrate the following example for supporting our result.

Example 3.1.6. Let $X = \{0, 1, 2\}$ with the partial b-metric $p_b : X \times X \to \mathbb{R}_+$ defined by

$$p_b(x, y) = [\max\{x, y\}]^2$$

for all $x, y \in X$. Obviously, (X, p_b) is a p_b-complete partial b-metric space with $s = 2$, but it is not a metric on X. To see this, let $x = y = 2$ then $p_b(2, 2) = [\max\{2, 2\}]^2 = 4 \neq 0$. Define the mapping $T : X \to X$ by $T0 = T1 = 0$ and $T2 = 1$.

We will show that T is a generalized JS-quasi-contraction with $\psi(t) = e^{te^t}$. In fact, it suffices to prove that there exists $\lambda \in (0, 1)$ such that, for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{2p_b(Tx, Ty)e^{2p_b(Tx,Ty)-M_s(x,y)}}{M_s(x, y)} \leq \lambda.$$

Let $x, y \in X$ with $Tx \neq Ty$. Therefore, $x = 0, y = 2$ or $x = 1, y = 2$. For both cases, we get $p_b(T0, T2) = p_b(T1, T2) = 1$ and $M_s(0, 2) = M_s(1, 2) = 4$. This implies that

$$\frac{2p_b(T0, T2)e^{2p_b(T0,T2)-M_s(0,2)}}{M_s(0, 2)} = \frac{2p_b(T1, T2)e^{2p_b(T1,T2)-M_s(1,2)}}{M_s(1, 2)} = \frac{e^{-2}}{2}.$$

This shows that T is a generalized JS-quasi-contraction with $\psi(t) = e^{te^t}$ and $\lambda \in [\frac{e^{-2}}{2}, 1)$. By Example 2.1.17, we know that $e^{te^t} \in \Phi_2$. Therefore, the conclusion immediately follows from Theorem 3.1.5 to obtain that T has a unique fixed point which is $x = 0$.

Theorem 3.1.7. *Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$. Let $T : X \to X$ be a p_b-continuous mapping. Assume that there exist a function $\psi \in \Phi_3$ and nonnegative numbers k_1, k_2, k_3, k_4 with $k_1+k_2+k_3+2k_4 < 1$ such that* (3.1.3) *is satisfied. Then T has a unique fixed point in X.*

Proof. From Remark 3.1.4 (iii), we get that T is a generalized JS-quasi-contraction with $\lambda = k_1 + k_2 + k_3 + 2k_4$. In case of $0 < \lambda < 1$, by Theorem 3.1.5, the proof is completed. In case of $\lambda = 0$, by (3.1.3) we have

$$\begin{aligned}
&\psi(sp_b(Tx, Ty)) \\
&\quad \leq \psi(p_b(x, y))^{k_1}\psi(p_b(x, Tx))^{k_2}\psi(p_b(y, Ty))^{k_3}\psi(\frac{p_b(x, Ty) + p_b(y, Tx)}{2s})^{2k_4}
\end{aligned}$$

$$\leq \psi(M_s(x,y))^0 = 1, \quad \text{for all } x,y \in X.$$

Further, by ($\Psi 1$) we deduce that $p_b(Tx,Ty) = 0$ for all $x,y \in X$. Thus, for $y = Tx$, we have $p_b(Tx, T(Tx)) = 0$. It follows that $y = Tx$ is a fixed point of T. Let z be another fixed point of T. Then

$$p_b(y,z) = p_b(Ty,Tz) = 0.$$

Therefore, $y = z$ and so T has a unique fixed point. □

By applying Theorem 3.1.5 and Remark 3.1.4 (ii), we get the following result.

Corollary 3.1.8. *Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$ and $T : X \to X$ be a p_b-continuous mapping. Assume that there exist $\psi \in \Phi_2$ and nonnegative real numbers k_1, k_2, k_3, k_4 with $k_1 + k_2 + k_3 + 2k_4 < 1$ such that* (3.1.2) *is satisfied. Then T has a unique fixed point in X.*

By applying Theorem 3.1.7 and Remark 3.1.4, we get the following results.

Corollary 3.1.9. *Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$ and $T : X \to X$ be a p_b-continuous mapping. Assume that there exist $\psi \in \Phi_4$ and nonnegative real numbers k_1, k_2, k_3, k_4 with $k_1 + k_2 + k_3 + 2k_4 < 1$ such that* (3.1.4) *is satisfied. Then T has a unique fixed point in X.*

Corollary 3.1.10. *Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$, and $T : X \to X$ be p_b-continuous. Assume that there exist $a > 0$ and nonnegative numbers k_1, k_2, k_3, k_4 with $k_1 + k_2 + k_3 + 2k_4 < 1$ such that*

$$\begin{aligned}(sp_b(Tx,Ty))^a \leq{} & k_1 p_b(x,y)^a + k_2 p_b(x,Tx)^a + k_3 p_b(y,Ty)^a \\ & + 2k_4\left(\frac{p_b(x,Ty) + p_b(y,Tx)}{2s}\right)^a \end{aligned} \tag{3.1.25}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. From Example 2.1.18, we have $\psi(t) = e^{t^a} \in \Phi_3$, and so (3.1.3) immediately follows from (3.1.25). Thus, by Theorem 3.1.7, T has a unique fixed point. □

By applying Corollary 3.1.10, we get the following result.

Corollary 3.1.11. *Let (X, p_b) be a p_b-complete partial b-metric space with the coefficient $s \geq 1$, and $T : X \to X$ be p_b-continuous. Assume that there exist nonnegative numbers k_1, k_2, k_3, k_4 with $k_1 + k_2 + k_3 + 2k_4 < 1$ such that*

$$\begin{aligned}(sp_b(Tx, Ty))^a \leq k_1 p_b(x, y)^a + k_2 p_b(x, Tx)^a + k_3 p_b(y, Ty)^a \\ + k_4\left(\frac{p_b(x, Ty) + p_b(y, Tx)}{s}\right)^a \end{aligned} \qquad (3.1.26)$$

for all $x, y \in X$, where $a = \frac{1}{n}$. Then T has a unique fixed point in X.

Proof. For each $a \in (0, 1]$, we obtain that

$$\left(\frac{p_b(x, Ty) + p_b(y, Tx)}{s}\right)^a \leq 2\left(\frac{p_b(x, Ty) + p_b(y, Tx)}{2s}\right)^a.$$

Then (3.1.26) implies (3.1.25). Thus, Corollary 3.1.11 immediately follows from Corollary 3.1.10. This implies that T has a unique fixed point. □

CHAPTER IV

VARIATIONAL INEQUALITY PROBLEMS AND RELATED PROBLEMS

In this chapter, we propose the iteration algorithms and use the suitable conditions for establish convergence theorems for solving the variational inequality problems and the related problems in the setting of Hilbert spaces. We have divided into two sections as the following:

4.1 Algorithms for the common solution of the split variational inequality and fixed point problems with applications

Our aim in this section is to consider an iterative method by combining Nadezhkina and Takahashi's modified extragradient method and Xu's algorithm. The mentioned iterative algorithm presents the common solution of the split variational inequality problems and fixed point problems. We show that the sequence produced by our algorithm is weak convergent. Finally, we give some applications of the main results.

Throughout this section, unless otherwise is stated, we assume that C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Suppose that $A : H_1 \to H_2$ is a nonzero bounded linear operator, $f : C \to H_1$ is a monotone and k-Lipschitz continuous mapping and $g : H_2 \to H_2$ is a δ-inverse strongly monotone mapping. Suppose that $T : H_2 \to H_2$ and $S : C \to C$ are nonexpansive. Let $\{\mu_n\}, \{\alpha_n\} \subset (0, 1), \{\gamma_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\|A\|^2})$ and $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, \frac{1}{k})$.

Firstly, we present an algorithm for solving the variational inequality problems and split common fixed point problems, that is, finding a point x^* such that

$$x^* \in VI(C, f) \cap F(S) \quad \text{and} \quad Ax^* \in F(T). \tag{4.1.1}$$

Theorem 4.1.1. *Setting* $\Theta = \{z \in VI(C,f) \cap F(S) : Az \in F(T)\}$ *and assume that* $\Theta \neq \emptyset$. *Let the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be generated by* $x_1 = x \in C$ *and*

$$\begin{aligned} y_n &= \mu_n x_n + (1-\mu_n)P_C(x_n - \gamma_n A^*(I-T)Ax_n), \\ z_n &= P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} &= \alpha_n y_n + (1-\alpha_n)SP_C(y_n - \lambda_n f(z_n)), \end{aligned} \tag{4.1.2}$$

for each $n \in \mathbb{N}$. *Then the sequence* $\{x_n\}$ *converges weakly to a point* $z \in \Theta$, *where* $z = \lim_{n\to\infty} P_\Theta x_n$.

Proof. It follows form Lemma 2.0.1 and Lemma 2.0.4 that $P_C(I - \gamma_n A^*(I-T)A)$ is $\frac{1+\gamma_n\|A\|^2}{2}$-averaged. From Lemma 2.0.5, we obtain that $\mu_n I + (1-\mu_n)P_C(I - \gamma_n A^*(I-T)A)$ is $\mu_n + (1-\mu_n)\frac{1+\gamma_n\|A\|^2}{2}$-averaged. So, y_n can be rewritten as

$$y_n = (1-\beta_n)x_n + \beta_n V_n x_n, \tag{4.1.3}$$

where $\beta_n = \mu_n + (1-\mu_n)\frac{1+\gamma_n\|A\|^2}{2}$ and V_n is a nonexpansive mapping for each $n \in \mathbb{N}$.

Let $u \in \Theta$, we get that

$$\begin{aligned} \|y_n - u\|^2 &= \|(1-\beta_n)(x_n - u) + \beta_n(V_n x_n - u)\|^2 \\ &= (1-\beta_n)\|x_n - u\|^2 + \beta_n\|V_n x_n - u\|^2 \\ &\quad -\beta_n(1-\beta_n)\|x_n - V_n x_n\|^2 \\ &\leq \|x_n - u\|^2 - \beta_n(1-\beta_n)\|x_n - V_n x_n\|^2 \\ &\leq \|x_n - u\|^2. \end{aligned} \tag{4.1.4}$$

Thus

$$\beta_n(1-\beta_n)\|x_n - V_n x_n\|^2 \leq \|x_n - u\|^2 - \|y_n - u\|^2. \tag{4.1.5}$$

Setting $t_n = P_C(y_n - \lambda_n f z_n)$ for all $n \geq 0$. It follows from Lemma 2.2.1 that

$$\begin{aligned} \|t_n - u\|^2 &\leq \|y_n - \lambda_n f(z_n) - u\|^2 - \|y_n - \lambda_n f(z_n) - t_n\|^2 \\ &\leq \|y_n - u\|^2 - \|y_n - t_n\|^2 + 2\lambda_n\langle f(z_n), u - t_n\rangle \end{aligned}$$

$$
\begin{aligned}
&= \|y_n - u\|^2 - \|y_n - t_n\|^2 + 2\lambda_n(\langle f(z_n) - f(u), u - z_n\rangle \\
&\quad +\langle f(u), u - z_n\rangle + \langle f(z_n), z_n - t_n\rangle) \\
&\leq \|y_n - u\|^2 - \|y_n - t_n\|^2 + 2\lambda_n\langle f(z_n), z_n - t_n\rangle \\
&= \|y_n - u\|^2 - \|y_n - z_n\|^2 - 2\langle y_n - z_n, z_n - t_n\rangle - \|z_n - t_n\|^2 \\
&\quad +2\lambda_n\langle f(z_n), z_n - t_n\rangle \\
&= \|y_n - u\|^2 - \|y_n - z_n\|^2 - \|z_n - t_n\|^2 \\
&\quad +2\langle y_n - \lambda_n f(z_n) - z_n, t_n - z_n\rangle.
\end{aligned}
$$

Using Lemma 2.2.1 again, this yields:

$$
\begin{aligned}
\langle y_n - \lambda_n f(z_n) - z_n, t_n - z_n\rangle &= \langle y_n - \lambda_n f(y_n) - z_n, t_n - z_n\rangle \\
&\quad +\langle \lambda_n f(y_n) - \lambda_n f(z_n), t_n - z_n\rangle \\
&\leq \langle \lambda_n f(y_n) - \lambda_n f(z_n), t_n - z_n\rangle \\
&\leq \lambda_n k\|y_n - z_n\|\|t_n - z_n\|,
\end{aligned}
$$

and so

$$
\|t_n - u\|^2 \leq \|y_n - u\|^2 - \|y_n - z_n\|^2 - \|z_n - t_n\|^2 + 2\lambda_n k\|y_n - z_n\|\|t_n - z_n\|.
$$

For each $n \in \mathbb{N}$, we obtain that

$$
\begin{aligned}
0 &\leq (\|t_n - z_n\| - \lambda_n k\|y_n - z_n\|)^2 \\
&= \|t_n - z_n\|^2 - 2\lambda_n k\|t_n - z_n\|\|y_n - z_n\| + \lambda_n^2 k^2\|y_n - z_n\|^2.
\end{aligned}
$$

That is,

$$
2\lambda_n k\|t_n - z_n\|\|y_n - z_n\| \leq \|t_n - z_n\|^2 + \lambda_n^2 k^2\|y_n - z_n\|^2.
$$

So,

$$
\begin{aligned}
\|t_n - u\|^2 &\leq \|y_n - u\|^2 - \|y_n - z_n\|^2 - \|z_n - t_n\|^2 + \|t_n - z_n\|^2 \\
&\quad +\lambda_n^2 k^2\|y_n - z_n\|^2 \\
&= \|y_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|y_n - z_n\|^2 \\
&\leq \|y_n - u\|^2. \qquad (4.1.6)
\end{aligned}
$$

By the convexity of the norm and (4.1.6), we have

$$
\begin{aligned}
\|x_{n+1}-u\|^2 &= \|\alpha_n y_n + (1-\alpha_n)S(t_n) - u\|^2 \\
&= \|\alpha_n(y_n - u) + (1-\alpha_n)(S(t_n)-u)\|^2 \\
&= \alpha_n\|y_n - u\|^2 + (1-\alpha_n)\|S(t_n)-u\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|y_n - u - (S(t_n)-u)\|^2 \\
&\leq \alpha_n\|y_n - u\|^2 + (1-\alpha_n)\|S(t_n) - S(u)\|^2 \\
&\leq \alpha_n\|y_n - u\|^2 + (1-\alpha_n)\|t_n - u\|^2 \\
&\leq \alpha_n\|y_n - u\|^2 + (1-\alpha_n)[\|y_n - u\|^2 \\
&\quad +(\lambda_n^2k^2-1)\|y_n - z_n\|^2] \\
&= \|y_n - u\|^2 + (1-\alpha_n)(\lambda_n^2k^2-1)\|y_n - z_n\|^2 \\
&\leq \|y_n - u\|^2 \leq \|x_n - u\|^2. \qquad (4.1.7)
\end{aligned}
$$

Hence, there exists $c \geq 0$ such that

$$\lim_{n\to\infty}\|x_n - u\| = c, \qquad (4.1.8)$$

and then $\{x_n\}$ is bounded. This implies that $\{y_n\}$ and $\{t_n\}$ are also bounded. From (4.1.5) and (4.1.7), we deduce that

$$\beta_n(1-\beta_n)\|x_n - V_nx_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1}-u\|^2.$$

Therefore, it follows from (4.1.8) that

$$x_n - V_nx_n \to 0, \quad \text{as} \quad n\to\infty.$$

By (4.1.3), we get that

$$x_n - y_n = \beta_n(x_n - V_nx_n) \to 0, \quad \text{as} \quad n\to\infty. \qquad (4.1.9)$$

The relation (4.1.7) implies

$$(1-\alpha_n)(1-\lambda_n^2k^2)\|y_n - z_n\|^2 \;\leq\; \|y_n - u\|^2 - \|x_{n+1}-u\|^2,$$

and so

$$y_n - z_n \to 0, \quad \text{as} \quad n\to\infty. \qquad (4.1.10)$$

Moreover, by the definition of z_n, we have

$$\begin{aligned}\|z_n - t_n\|^2 &= \|P_C(y_n - \lambda_n f(y_n)) - P_C(y_n - \lambda_n f(z_n))\|^2 \\ &\leq \|(y_n - \lambda_n f(y_n)) - (y_n - \lambda_n f(z_n))\|^2 \\ &= \|\lambda_n f(z_n) - \lambda_n f(y_n)\|^2 \\ &\leq \lambda_n^2 k^2 \|z_n - y_n\|^2.\end{aligned}$$

Hence

$$z_n - t_n \to 0, \quad \text{as} \quad n \to \infty. \tag{4.1.11}$$

Using the triangle inequality, we see that

$$\|y_n - t_n\| \leq \|y_n - z_n\| + \|z_n - t_n\|,$$

and

$$\|x_n - z_n\| \leq \|x_n - y_n\| + \|y_n - z_n\|.$$

This implies that

$$y_n - t_n \to 0 \quad \text{and} \quad x_n - z_n \to 0, \quad \text{as} \quad n \to \infty. \tag{4.1.12}$$

The definition of y_n implies

$$(1 - \mu_n)(x_n - P_C(x_n - \gamma_n A^*(I - T)Ax_n)) = x_n - y_n.$$

Thus

$$x_n - P_C(x_n - \gamma_n A^*(I - T)Ax_n) \to 0, \quad \text{as} \quad n \to \infty. \tag{4.1.13}$$

Let $z \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z. We obtain that $\{A^*(I - T)Ax_{n_i}\}$ is bounded because $A^*(I - T)A$ is $\frac{1}{2\|A\|^2}$-inverse strongly monotone. Without loss of generality, we may assume that $\gamma_{n_i} \to \hat{\gamma} \in (0, \frac{1}{\|A\|^2})$. By the nonexpansiveness of P_C, we see that

$$\begin{aligned}&\|P_C(I - \gamma_{n_i} A^*(I - T)A)x_{n_i} - P_C(I - \hat{\gamma} A^*(I - T)A)x_{n_i}\| \\ &\leq |\gamma_{n_i} - \hat{\gamma}| \|A^*(I - T)Ax_{n_i}\|,\end{aligned}$$

and so

$$P_C(I-\gamma_{n_i}A^*(I-T)A)x_{n_i}-P_C(I-\hat{\gamma}A^*(I-T)A)x_{n_i}\to 0,\ i\to\infty. \quad (4.1.14)$$

From (4.1.13), (4.1.14) and

$$\begin{aligned}&\|x_{n_i}-P_C(I-\hat{\gamma}A^*(I-T)A)x_{n_i}\|\\ &\quad\le\ \|x_{n_i}-P_C(I-\gamma_{n_i}A^*(I-T)A)x_{n_i}\|\\ &\qquad+\|P_C(I-\gamma_{n_i}A^*(I-T)A)x_{n_i}-P_C(I-\hat{\gamma}A^*(I-T)A)x_{n_i}\|,\end{aligned}$$

we have

$$x_{n_i}-P_C(I-\hat{\gamma}A^*(I-T)A)x_{n_i}\to 0,\quad \text{as}\ \ i\to\infty. \quad (4.1.15)$$

By the demiclosedness principle, Lemma 2.2.7, we have

$$z\in F(P_C(I-\hat{\gamma}A^*(I-T)A)).$$

Corollary 2.2.6, this yields:

$$z\in C\cap A^{-1}F(T). \quad (4.1.16)$$

Next, we claim that $z\in VI(C,f)$. From (4.1.9), (4.1.10) and (4.1.11), we know that $y_{n_i}\rightharpoonup z, z_{n_i}\rightharpoonup z$ and $t_{n_i}\rightharpoonup z$. Define the set-valued mapping $B:H\rightrightarrows H$ by

$$Bv=\begin{cases} f(v)+N_Cv, & \text{if } v\in C;\\ \emptyset, & \text{if } v\notin C.\end{cases}$$

From Lemma 2.2.4, we obtain that B is maximal monotone and we have $0\in Bv$ iff $v\in VI(C,f)$. If $(v,w)\in G(B)$, then $w\in Bv=f(v)+N_Cv$ and so $w-f(v)\in N_Cv$. Thus, for any $p\in C$, we get

$$\langle v-p,w-f(v)\rangle\ge 0. \quad (4.1.17)$$

Since $v\in C$, it follows from the definition of z_n and Lemma 2.2.1 that

$$\langle y_n-\lambda_nfy_n-z_n,z_n-v\rangle\ge 0.$$

Consequently

$$\langle\frac{z_n-y_n}{\lambda_n}+f(y_n),v-z_n\rangle\ge 0.$$

By using (4.1.17) with $\{z_{n_i}\}$, we obtain

$$\langle w - f(v), v - z_{n_i}\rangle \geq 0.$$

Thus

$$\begin{aligned}
\langle w, v - z_{n_i}\rangle &\geq \langle f(v), v - z_{n_i}\rangle \\
&\geq \langle f(v), v - z_{n_i}\rangle - \langle \frac{z_{n_i} - y_{n_i}}{\lambda_{n_i}} + f(y_{n_i}), v - z_{n_i}\rangle \\
&= \langle f(v) - f(z_{n_i}), v - z_{n_i}\rangle + \langle f(z_{n_i}) - f(y_{n_i}), v - z_{n_i}\rangle \\
&\quad - \langle \frac{z_{n_i} - y_{n_i}}{\lambda_{n_i}}, v - z_{n_i}\rangle \\
&\geq \langle f(z_{n_i}) - f(y_{n_i}), v - z_{n_i}\rangle - \langle \frac{z_{n_i} - y_{n_i}}{\lambda_{n_i}}, v - z_{n_i}\rangle.
\end{aligned}$$

By taking $i \to \infty$ in the above inequality, we deduce

$$\langle w, v - z\rangle \geq 0.$$

By the maximal monotonicity of B, we get $0 \in Bz$ and so $z \in VI(C, f)$. Now, we will show that $z \in F(S)$. Since S is nonexpansive, it follows from (4.1.4) and (4.1.6) that

$$\|S(t_n) - u\| = \|S(t_n) - S(u)\| \leq \|t_n - u\| \leq \|y_n - u\| \leq \|x_n - u\|,$$

and by taking limit superior in the above inequalities and using (4.1.8), we obtain

$$\limsup_{n\to\infty} \|S(t_n) - u\| \leq c \quad \text{and} \quad \limsup_{n\to\infty} \|y_n - u\| \leq c.$$

Further,

$$\begin{aligned}
\lim_{n\to\infty} \|\alpha_n(y_n - u) + (1 - \alpha_n)(S(t_n) - u)\| &= \lim_{n\to\infty} \|\alpha_n y_n + (1 - \alpha_n)S(t_n) - u\| \\
&= \lim_{n\to\infty} \|x_{n+1} - u\| \\
&= c,
\end{aligned}$$

and so, by Lemma 2.2.8 implies

$$\lim_{n\to\infty} \|S(t_n) - y_n\| = 0. \qquad (4.1.18)$$

From (4.1.12), (4.1.18) and

$$\begin{aligned}\|S(y_n) - y_n\| &= \|S(y_n) - S(t_n) + S(t_n) - y_n\| \\ &\leq \|S(y_n) - S(t_n)\| + \|S(t_n) - y_n\| \\ &\leq \|y_n - t_n\| + \|S(t_n) - y_n\|,\end{aligned}$$

we have

$$\lim_{n\to\infty} \|S(y_n) - y_n\| = 0.$$

This implies that

$$\lim_{i\to\infty} \|(I - S)y_{n_i}\| = \lim_{i\to\infty} \|y_{n_i} - S(y_{n_i})\| = 0.$$

Now, by the demiclosedness principle, Lemma 2.2.7, we have $z \in F(S)$. Consequently $\omega_w(x_n) \subset \Theta$. By Lemma 2.2.9, the sequence $\{x_n\}$ is weakly convergent to a point z in Θ.

Setting $u_n = P_\Theta x_n$. We will show that $z = \lim_{n\to\infty} u_n$. By Lemma 2.2.1 and $z \in \Theta$, we get that

$$\langle x_n - u_n, u_n - z\rangle \geq 0.$$

It follows from Lemma 2.2.10 that $\{u_n\}$ converges strongly to some $z_0 \in \Theta$. Thus

$$\langle z - z_0, z_0 - z\rangle \geq 0.$$

Hence $z = z_0$, this implies that $z = \lim_{n\to\infty} P_\Theta x_n$. □

Remark 4.1.2. We can obtain the following cases:

(*i*) If $f = 0$, $T = P_Q$ and $S = I$, then the problem (4.1.1) coincides with the SFP, and if $\alpha_n = 0$, we obtain that the algorithm (4.1.2) reduces to algorithm (2.2.5) for solving the SFP.

(*ii*) If $T = I$, then the problem (4.1.1) coincides with the VIP and FPP, and the algorithm (4.1.2) reduces to algorithm (2.2.11) for solving the VIP and FPP.

(*iii*) If $S = I$, then the problem (4.1.1) coincides with the problem 3.1 in [51] and if $\alpha_n, \mu_n = 0$, we obtain that the algorithm (4.1.2) reduces to algorithm 3.2 in [51].

The following result provides suitable conditions in order to guarantee the existence of a common solution of the split variational inequality problems and fixed point problems, that is finding a point x^* such that

$$x^* \in VI(C,f) \cap F(S) \quad \text{and} \quad Ax^* \in VI(Q,g). \tag{4.1.19}$$

Theorem 4.1.3. *Setting $\Theta = \{z \in VI(C,f) \cap F(S) : Az \in VI(Q,g)\}$ and assume that $\Theta \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by $x_1 = x \in C$ and*

$$\begin{aligned} y_n &= \mu_n x_n + (1-\mu_n)P_C(x_n - \gamma_n A^*(I - P_Q(I-\theta g))Ax_n), \\ z_n &= P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} &= \alpha_n y_n + (1-\alpha_n)SP_C(y_n - \lambda_n f(z_n)), \end{aligned} \tag{4.1.20}$$

for each $n \in N$, where $\theta \in (0, 2\delta)$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in \Theta$, where $z = \lim_{n\to\infty} P_\Theta x_n$.

Proof. It is clear that from δ-inverse strongly monotonicity of g that it is $\frac{1}{\delta}$-Lipschitz continuous and, for $\theta \in (0, 2\delta)$, we obtain that $I - \theta g$ is nonexpansive. Since P_Q is nonexpansive, then $P_Q(I-\theta g)$ is nonexpansive. By taking $T = P_Q(I-\theta g)$ in Theorem 4.1.1, we obtain $z \in VI(C,f) \cap F(S)$ and $Az \in F(P_Q(I-\theta g))$. It follows from $Az = P_Q(I-\theta g)Az$ and Lemma 2.2.1 that $Az \in VI(Q,g)$. This completes the proof. □

Remark 4.1.4. We can obtain the following cases:

(i) If $f = 0$, $g = 0$ and $S = I$, then the problem (4.1.19) coincides with the SFP, and if $\alpha_n = 0$, we obtain that the algorithm (4.1.20) reduces to algorithm (2.2.5) for solving the SFP.

(ii) If $g = 0$ and $Q = H_2$, then the problem (4.1.19) coincides with the VIP and FPP, and the algorithm (4.1.20) reduces to algorithm (2.2.11) for solving the VIP and FPP.

(iii) If $S = I$, then the problem (4.1.19) coincides with the SVIP, and if $\alpha_n, \mu_n = 0$, then the algorithm (4.1.20) reduces to algorithm (2.2.12).

4.1.1 Applications

In this section, by using the main results, we present we give some applications to the weak convergence of the produced algorithms for the equilibrium problem, zero point problem and convex minimization problem.

The following result is related to the equilibrium problems by applying Theorem 4.1.1.

Theorem 4.1.5. *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)-(A4). *Setting $\Theta = \{z \in VI(C, f) \cap F(S) : Az \in EP(C, F)\}$ and suppose that $\Theta \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \mu_n x_n + (1 - \mu_n) P_C(x_n - \gamma_n A^*(I - T_r)Ax_n), \\ z_n = P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} = \alpha_n y_n + (1 - \alpha_n) S P_C(y_n - \lambda_n f(z_n)), \end{cases} \tag{4.1.21}$$

for each $n \in \mathbb{N}$, where T_r is a resolvent of F for $r > 0$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in \Theta$, where $z = \lim_{n\to\infty} P_\Theta x_n$.

Proof. Since T_r is nonexpansive, the proof follows from Theorem 4.1.1 by taking $T = T_r$. □

The following results are the application of Theorem 4.1.1 to the zero point problem.

Theorem 4.1.6. *Let $B : H_2 \rightrightarrows H_2$ be a maximal monotone mapping with $D(B) \neq \emptyset$. Setting $\Theta = \{z \in VI(C, f) \cap F(S) : Az \in B^{-1}0\}$ and assume that $\Theta \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \mu_n x_n + (1 - \mu_n) P_C(x_n - \gamma_n A^*(I - J_r)Ax_n), \\ z_n = P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} = \alpha_n y_n + (1 - \alpha_n) S P_C(y_n - \lambda_n f(z_n)), \end{cases} \tag{4.1.22}$$

for each $n \in \mathbb{N}$, where J_r is a resolvent of B for $r > 0$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in \Theta$, where $z = \lim_{n\to\infty} P_\Theta x_n$.

Proof. Since J_r is nonexpansive and $F(J_r) = B^{-1}0$, the proof follows from Theorem 4.1.1 by taking $T = J_r$. □

Theorem 4.1.7. *Let $B : H_2 \rightrightarrows H_2$ be a maximal monotone mapping with $D(B) \neq \emptyset$ and $F_\delta : H_2 \to H_2$ be a δ-inverse strongly monotone mapping. Setting $\Theta = \{z \in VI(C, f) \cap F(S) : Az \in (B + F_\delta)^{-1}0, \}$, and assume that $\Theta \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \mu_n x_n + (1 - \mu_n) P_C(x_n - \gamma_n A^*(I - J_r(I - rF_\delta))Ax_n), \\ z_n = P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} = \alpha_n y_n + (1 - \alpha_n) S P_C(y_n - \lambda_n f(z_n)), \end{cases} \tag{4.1.23}$$

for each $n \in \mathbb{N}$, where J_r is a resolvent of B for $r \in (0, 2\delta)$. Then the sequence $\{x_n\}$ converges weakly to a point $z \in \Theta$, where $z = \lim_{n\to\infty} P_\Theta x_n$.

Proof. Since F_δ is δ-inverse strongly monotone, then $I - rF_\delta$ is nonexpansive. By the nonexpansiveness of J_r, we obtain that $J_r(I - rF_\delta)$ is also nonexpansive. We know that $z \in (B + F_\delta)^{-1}0$ if and only if $z = J_r(I - rF_\delta)z$. Thus the proof follows from Theorem 4.1.1 by taking $T = J_r(I - rF_\delta)$. □

By applying Theorem 4.1.3 and Lemma 2.2.13, we get the following result, which is related to constrained convex minimization problem.

Theorem 4.1.8. *Let $\phi : H_2 \to \mathbb{R}$ be a differentiable convex function and suppose that $\nabla\phi$ a δ-inverse strongly monotone mapping. Setting $\Theta = \{z \in VI(C, f) \cap F(S) : Az \in \arg\min_{y \in Q} \phi(y)\}$ and assume that $\Theta \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \mu_n x_n + (1 - \mu_n) P_C(x_n - \gamma_n A^*(I - P_Q(I - \theta\nabla\phi))Ax_n), \\ z_n = P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} = \alpha_n y_n + (1 - \alpha_n) S P_C(y_n - \lambda_n f(z_n)), \end{cases} \tag{4.1.24}$$

for each $n \in \mathbb{N}$, where $\theta \in (0, 2\delta)$. Then the sequence $\{x_n\}$ converges weakly to a point in Θ, where $z = \lim_{n\to\infty} P_\Theta x_n$.

Proof. By Lemma 2.2.13 and taking $g = \nabla\phi$, the proof follows from Theorem 4.1.3. $\square$

We obtain the following result for solving the split minimization problems and fixed point problems by applying Theorem 4.1.3 and Lemma 2.2.13.

Theorem 4.1.9. *Let $\phi_1 : H_1 \to \mathbb{R}$ and $\phi_2 : H_2 \to \mathbb{R}$ be differentiable convex functions. Suppose that $\nabla\phi_1$ is a k-Lipschitz continuous mapping and $\nabla\phi_2$ is δ-inverse strongly monotone. Setting $\Theta = \{z \in \arg\min_{x\in C}\phi_1(x) \cap F(S) : Az \in \arg\min_{y\in Q}\phi_2(y)\}$ and assume that $\Theta \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \mu_n x_n + (1-\mu_n)P_C(x_n - \gamma_n A^*(I - P_Q(I - \theta\nabla\phi_2))Ax_n), \\ z_n = P_C(y_n - \lambda_n\nabla\phi_1(y_n)), \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)SP_C(y_n - \lambda_n\nabla\phi_1(z_n)), \end{cases} \tag{4.1.25}$$

for each $n \in \mathbb{N}$, where $\theta \in (0, 2\delta)$. Then the sequence $\{x_n\}$ converges weakly to a point in Θ, where $z = \lim_{n\to\infty} P_\Theta x_n$.

Proof. Since ϕ is convex, for each $x, y \in C$, we have

$$\phi(x + \lambda(z - x)) \leq (1-\lambda)\phi(x) + \lambda\phi(z), \text{ for all } \lambda \in (0,1).$$

It follows that $\langle\nabla\phi(x), x-z\rangle \geq \phi(x)-\phi(z) \geq \langle\nabla\phi(z), x-z\rangle$. Thus $\nabla\phi$ is monotone. The result follows from Lemma 2.2.13 by taking $f = \nabla\phi_1$ and $g = \nabla\phi_2$ in Theorem 4.1.3. $\square$

4.2 Convergence theorems for the variational inequality problems

In this section, we consider the following iterative algorithm by combining Yamada's hybrid steepest descent method [53] and Wang's algorithm [54] for solving the problem (2.3.1):

$$y_n = (1-\alpha_n)x_n + \alpha_n(I - t_n\mu f)Tx_n,$$

$$x_{n+1} = (I - t_n \mu f)Ty_n, \tag{4.2.1}$$

where $T = P_C(I - \gamma A^*(I - P_Q)A)$.

Throughout this section, unless otherwise is stated, we assume that H_1 and H_2 are two real Hilbert spaces and $A : H_1 \to H_2$ is a linear bounded mapping. Let f be an η-strongly monotone and k-Lipschitz continuous mapping on H_1 with some positive constants η and k. Assume that $\mu \in (0, 2\eta/k^2)$ is a fixed number.

Theorem 4.2.1. *Let C and Q be two closed convex subsets in H_1 and H_2, respectively. Then, as $n \to \infty$, the sequence $\{x_n\}$ defined by* (4.2.1), *$\{t_n\}$ and $\{\alpha_n\}$ satisfy conditions* (C1) *and* (C2), *respectively, converges strongly to the solution of* (2.3.1).

Proof. From Lemma 2.0.4, we have $I - \gamma A^*(I - P_Q)A$ is $\gamma\|A\|^2$-averaged. Since $T = P_C(I - \gamma A^*(I - P_Q)A)$ and Lemma 2.0.1 (i), we get that T is λ-averaged where $\lambda = \frac{1+\gamma\|A\|^2}{2}$. Moreover, we obtain that $z \in \Gamma$ if and only if $z \in F(T)$. It follows from Definition of λ-averaged mapping T that $T = (1-\lambda)I + \lambda S$, where S is nonexpansive. Then, the iterative algorithm (4.2.1) can be rewritten as follows:

$$x_{n+1} = (I - t_n \mu f)T\tilde{T}x_n, \tag{4.2.2}$$

where $\tilde{T} = (1 - \alpha_n)I + \alpha_n(I - t_n\mu f)T$ and $T = (1 - \lambda)I + \lambda S$. By Lemma 2.0.6, we obtain that $I - t_n\mu f$ is contractive. Since $(1 - \lambda)I + \lambda S$ and $I - t_n\mu f$ are nonexpansive, then $(I - t_n\mu f)T$ is also nonexpansive. Therefore, the strong convergence of (4.2.1) to the element x^* in (2.3.1) is followed by Theorem 2.3.2. □

In [75], Miao and Li showed the convergence results of the sequence $\{x_n\}$, which is generated by the following algorithm:

$$\begin{aligned} y_n &= (1 - \alpha_n)x_n + \alpha_n(I - t_n\mu f)Tx_n, \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n(I - t_n\mu f)Ty_n, \end{aligned} \tag{4.2.3}$$

where $\{t_n\} \subset [0, 1)$ satisfies condition (C3) $\sum_{n=1}^{\infty} t_n < +\infty$. Next, we will show the strong convergence for (4.2.3) with $\{t_n\}$ satisfies the condition $(C1)$.

Theorem 4.2.2. *Let C and Q be two closed convex subsets in H_1 and H_2, respectively. Then, as $n \to \infty$, the sequence $\{x_n\}$ defined by* (4.2.3), *$\{t_n\}$ satisfies condition* (C1), *$\{\beta_n\}$ and $\{\alpha_n\}$ satisfy condition* (C2), *converges strongly to the solution of* (2.3.1).

Proof. In the proof of Theorem 4.2.1, one can rewrite the iterative algorithm (4.2.3) as follows:

$$x_{n+1} = (1-\beta_n)x_n + \beta_n(I - t_n\mu f)T\tilde{T}x_n, \tag{4.2.4}$$

where $\tilde{T} = (1-\alpha_n)I + \alpha_n(I - t_n\mu f)T$ and $T = (1-\lambda)I + \lambda S$. Since $(I - t_n\mu f)T$ is nonexpansive, then the strong convergence of (4.2.3) to the element x^* in (2.3.1) is followed by Theorem 2.3.1. □

Moreover, we obtain the following results which are solving the common solution of variational inequality problem and multiple-sets split feasibility problem, i.e., find a point

$$x^* \in \Omega : \langle fx^*, x - x^* \rangle \geq 0, \text{ for all } x \in \Omega, \tag{4.2.5}$$

where Ω is solution set of (2.2.1), and $f : H_1 \to H_1$ is an η-strongly monotone and k-Lipschitz continuous mapping. This problem has been studied in [52].

Theorem 4.2.3. *Let $\{C_i\}_{i=1}^N$ and $\{Q_j\}_{j=1}^M$ be two finite families of closed convex subsets in H_1 and H_2, respectively. Assume that $\gamma \in (0, 1/\|A\|^2)$, $\{t_n\}$ and $\{\alpha_n\}$ satisfy conditions* (C1) *and* (C2), *respectively, and the parameters $\{\delta_n\}$ and $\{\zeta_n\}$ satisfy the following conditions:*

(a) $\delta_i > 0$ *for* $1 \leq i \leq N$ *such that* $\sum_{i=1}^N \delta_i = 1$;

(b) $\zeta_j > 0$ *for* $1 \leq j \leq M$ *such that* $\sum_{i=1}^N \zeta_j = 1$.

Then, as $n \to \infty$, the sequence $\{x_n\}$, defined by

$$\begin{aligned} y_n &= (1-\alpha_n)x_n + \alpha_n(I - t_n\mu f)P_1(I - \gamma A(I - P_2)A)x_n, \\ x_{n+1} &= (I - t_n\mu f)P_1(I - \gamma A(I - P_2)A)y_n, \end{aligned} \tag{4.2.6}$$

with one of the following cases:

(A1) $P_1 = P_{C_1} \cdots P_{C_N}$ *and* $P_2 = P_{Q_1} \cdots P_{Q_M}$*;*

(A2) $P_1 = \sum_{i=1}^{N} \delta_i P_{C_i}$ *and* $P_2 = \sum_{j=1}^{M} \zeta_j P_{Q_j}$*;*

(A3) $P_1 = P_{C_1} \cdots P_{C_N}$ *and* $P_2 = \sum_{j=1}^{M} \zeta_j P_{Q_j}$*;*

(A4) $P_1 = \sum_{i=1}^{N} \delta_i P_{C_i}$ *and* $P_2 = P_{Q_1} \cdots P_{Q_M}$*,*

converges to the element x^* *in the solution of* (4.2.5).

Proof. Let $T = P_1(I - \gamma A^*(I - P_2)A)$. We will show that T is averaged.

In the case of (A1), that is $P_1 = P_{C_1} \cdots P_{C_N}$ and $P_2 = P_{Q_1} \cdots P_{Q_M}$. Since P_{C_i} is $\frac{1}{2}$-averaged for all $i = 1, \ldots, N$, by Proposition 2.0.2, we get that P_1 is λ_1-averaged, where $\lambda_1 = N/(N+1)$. Similary, we have P_2 is also averaged and so P_2 is nonexpansive. By using Lemma 2.0.4, we deduce that $I - \gamma A^*(I - P_2)A$ is λ_2-averaged, where $\lambda_2 = \gamma\|A\|^2$. It follows from Lemma 2.0.1 (i) that T is λ-averaged with $\lambda = N/(N+1) + \gamma\|A\|^2 - (N/(N+1))\gamma\|A\|^2$.

In the case of (A2), that is $P_1 = \sum_{i=1}^{N} \delta_i P_{C_i}$ and $P_2 = \sum_{j=1}^{M} \zeta_j P_{Q_j}$. By using Proposition 2.0.3 and condition (a), we obtain that P_1 is $\frac{1}{2}$-averaged. From condition (b) and P_{Q_j} is nonexpansive, for all $j = 1, \ldots, M$, we have P_2 is also nonexpansive. It follows from Lemma 2.0.4 that $I - \gamma A^*(I - P_2)A$ is $\gamma\|A\|^2$-averaged. Thus T is λ-averaged with $\lambda = (1 + \gamma\|A\|^2)/2$.

The cases (A3) and (A4) are similar. This implies that $T := (1-\lambda)I + \lambda S$, where S is nonexpansive. Moerover, by Lemma 2.0.1, we get that

$$\begin{aligned} F(T) &= F(P_1) \cap F(I - \gamma A^*(I - P_2)A) = F(P_1) \cap A^{-1}F(P_2) \\ &= \cap_{i=1}^{N} C_i \cap A^{-1}(\cap_{j=1}^{M} Q_j) = \Omega. \end{aligned}$$

Then, the iterative algorithm (4.2.6) can be rewritten as follows:

$$x_{n+1} = (I - t_n \mu f) T \tilde{T} x_n, \tag{4.2.7}$$

where $\tilde{T} = (1 - \alpha_n)I + \alpha_n(I - t_n\mu f)T$ and $T = (1-\lambda)I + \lambda S$. By Lemma 2.0.6, we obtain that $I - t_n\mu f$ is contractive. Since $(1-\lambda)I + \lambda S$ and $I - t_n\mu f$ are nonexpansive, then $(I - t_n\mu f)T$ is nonexpansive. Thus, the strong convergence of (4.2.6) to the element x^* in (4.2.5) is followed by Theorem 2.3.2. □

Theorem 4.2.4. *Let* $\{C_i\}_{i=1}^N$, $\{Q_j\}_{j=1}^M, \gamma, \{t_n\}, \{\delta_n\}$ *and* $\{\zeta_n\}$ *be as in Theorem 4.2.3. Then, as* $n \to \infty$, *the sequence* $\{x_n\}$, *defined by*

$$y_n = (1-\alpha_n)x_n + \alpha_n(I - t_n\mu f)P_1(I - \gamma A(I - P_2)A)x_n,$$
$$x_{n+1} = (1-\beta_n)x_n + \beta_n(I - t_n\mu f)P_1(I - \gamma A(I - P_2)A)y_n, \tag{4.2.8}$$

with one of the cases (A1)-(A4), *converges strongly to* (4.2.5).

Proof. In the proof of Theorem 4.2.3, one can rewrite the iterative algorithm (4.2.8) as follows:

$$x_{n+1} = (1-\beta_n)x_n + \beta_n(I - t_n\mu f)T\tilde{T}x_n, \tag{4.2.9}$$

where $\tilde{T} = (1-\alpha_n)I + \alpha_n(I - t_n\mu f)T$ and $T = (1-\lambda)I + \lambda S$. Since $(I - t_n\mu f)T$ is nonexpansive, the strong convergence of (4.2.8) to the element x^* in (4.2.5) is followed by Theorem 2.3.1. □

4.2.1 Numerical example

In this section, we give the numerical example comparing the algorithm (2.3.3) which is given by Buong [52] and algorithm (4.2.1) (New method) to solve the following test problem in [52]: Find an element $x^* \in \Omega = \cap_{i=1}^N C_i \cap A^{-1} \cap_{j=1}^M Q_j$ such that

$$\varphi(x^*) = \min_{x\in\Omega}, \tag{4.2.10}$$

where φ is a convex function. Then, the derivative $\varphi'(x)$ is Lipschitz continuous and strongly monotone on the Euclidian space $\mathbb{E}^n$, where

$$C_i = \{x \in \mathbb{E}^n : a_1^i x_1 + a_2^i x_2 + \cdots + a_n^i x_n \leq b_i\}, \tag{4.2.11}$$

$a_k^i, b_i \in (-\infty, +\infty)$, for $1 \leq k \leq n$ and $1 \leq i \leq N$,

$$Q_j = \left\{ y \in \mathbb{E}^m : \sum_{l=1}^m (y_l - a_l^j)^2 \leq R_j^2 \right\}, R_j > 0, \tag{4.2.12}$$

$a_l^j \in (-\infty, +\infty)$, for $1 \leq l \leq m$ and $1 \leq j \leq M$, and A is an upper triangular $n \times m$-matrix. By the same argument as in the proof of Theorem 4.2.3, we obtain that T is λ-averaged where $\lambda = \frac{1+\gamma\|A\|^2}{2}$ and $T = P_1(I - \gamma A^*(I - P_2)A)$. Moreover, we also obtain that $F(T) = \cap_{i=1}^N C_i \cap A^{-1}(\cap_{j=1}^M Q_j) = \Omega$. Thus $\Omega \neq \emptyset$.

Example 4.2.5. We consider the test problem (4.2.10), where $N = M = 1$, $n = m = 2$, $\varphi(x) = (1-a)\|x\|^2/2$ for some fixed $a \in (0,1)$, and

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

So, we have $f := \varphi' = (1-a)I$ is a k-Lipschitz continuous and η-strongly monotone mapping with $k = \eta = (1-a)$. For each algorithm, we set $a^i = (1/i, -1), b_i = 0$, for all $i = 1, \ldots, N$, and $a^j = (1/j, 0), R_j = 1$, for all $j = 1, \ldots, M$. Taking $a = 0.5, \gamma = 0.3$ and the stopping criterion is defined by $E_n = \|x_{n+1} - x_n\| < \varepsilon$ where $\varepsilon = 10^{-4}$ and 10^{-6}. The numerical results are listed in Table 1 with different initial points x^1, where n is the number of iterations and s is the CPU time in seconds. In Figures 1 and 2, we plot the stopping criterion with respect to the number of iterations for both methods follow by Table 1 with the different initial points.

Table 1: Computational results for Example 4.2.5 with different methods.

Initial point		10^{-4}		10^{-6}	
		n	s	n	s
	Buong method	29461	0.364595	2946204	31.362283
$(-2,1)$	New method	11784	0.241371	1178481	23.411679
	Buong method	30632	0.565431	3063343	33.468210
$(1,3)$	New method	12252	0.324808	1225336	25.570356

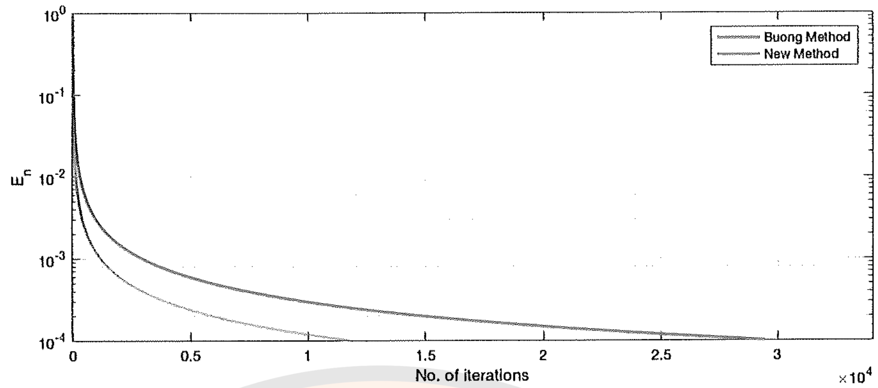

Figure 1: The convergence behavior of E_n for Example 4.2.5 with the intial point $(-2, 1)$.

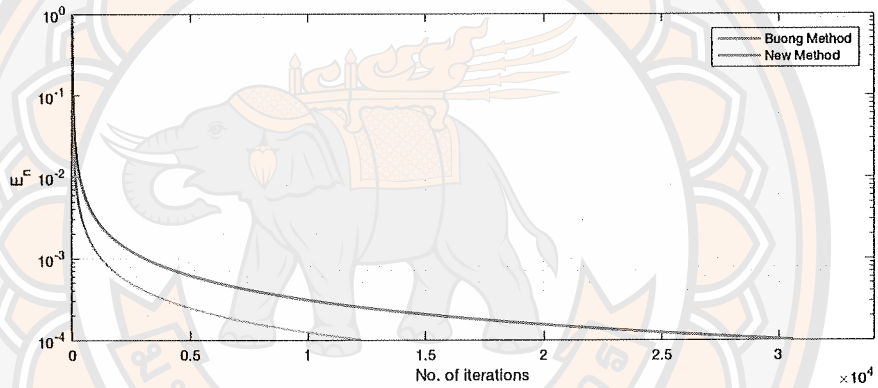

Figure 2: The convergence behavior of E_n for Example 4.2.5 with the intial point $(1, 3)$.

Remark 4.2.6. By the numerical analysis of our results in Table 1 with Fig. 1 and 2, we get that the algorithm (4.2.1) (new method) has lesser number of iterations and faster convergence than algorithm (2.3.3) (Buong method).

Example 4.2.7. In this example, we consider the alogorithm (4.2.8) for solving the test problem (4.2.10), where $N = 5$ and $M = 4$. Setting $\{C_i\}_{i=1}^N$, $\{Q_j\}_{j=1}^M$, φ, a and A be as in Example 4.2.5. In the numerical experiment, we took the stopping criterion $E_n < 10^{-4}$. The numerical results are listed in Table 2 with different cases of P_1 and P_2. In Figures 3 and 4, we plot the stopping criterion with respect to the number of iterations for all cases of P_1 and P_2 follow by Table 2 with the different initial points. Moreover, Table 3 shows that the effect of different choices of γ.

Table 2: Computational results for Example 4.2.7 with different methods.

Initial point		A1	A2	A3	A4
	n	28577	24264	28577	24264
$(-2, 1)$	s	1.491225	1.355074	1.534414	1.282528
	n	33407	31438	33407	31438
$(1, 3)$	s	1.746868	1.693069	1.816897	1.690618

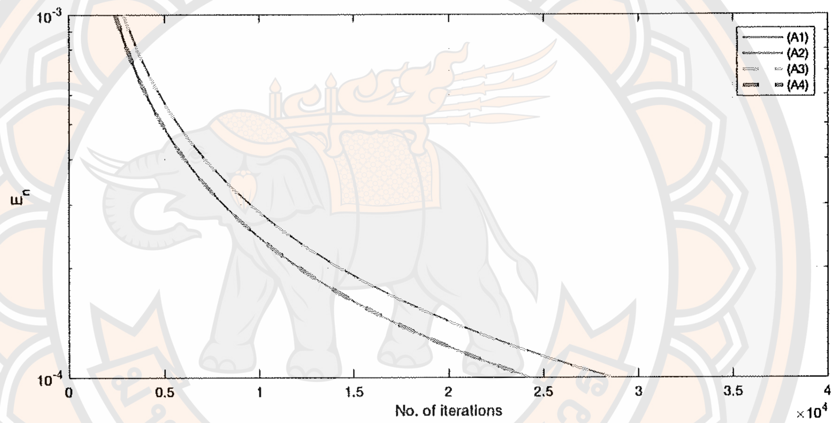

Figure 3: The convergence behavior of E_n for Example 4.2.7 with the intial point $(-2, 1)$.

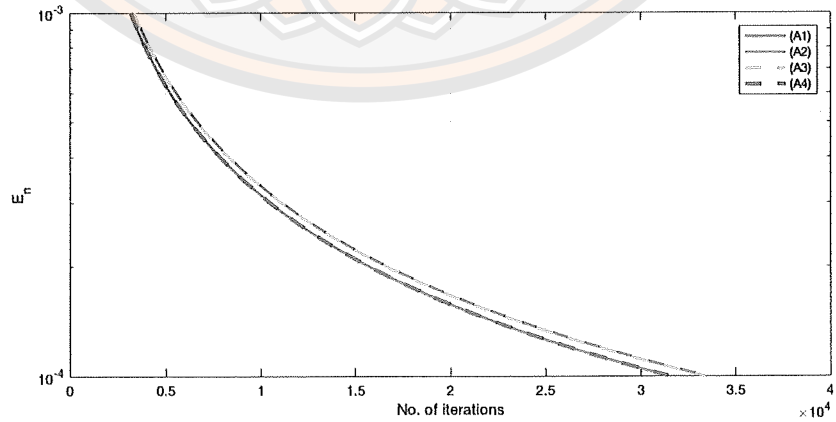

Figure 4: The convergence behavior of E_n for Example 4.2.7 with the intial point $(1, 3)$.

Table 3: Computational results for Example 4.2.7 with different γ.

	γ	0.1	0.2	0.3
	n	9675	19200	28577
$(-2, 1)$	s	0.669508	1.245136	1.666702
	n	11311	22447	33407
$(1, 3)$	s	0.764536	1.372600	1.958486

Remark 4.2.8. We observe from the numerical analysis of Table 2 that the algorithm (4.2.8) has the fastest convergence when P_1 and P_2 satisfy $A4$ and the slowest convergence when P_1 and P_2 satisfy $A3$. Moreover, form Table 3 we require less iteration steps and CPU times for convergence when γ is chosen very small and closed to zero.

CHAPTER V

CONCLUSION

In this thesis, we introduce the generalized iteration processes and generalized contractive mappings in the setting of partial b-metric spaces and Hilbert spaces. Furthermore, we illustrate weak and strong convergence theorems for the variational inequality problems, fixed point problems and the related problems. Moreover, we present some numerical examples to demonstrate the capability of our iteration processes.

5.1 Fixed point theorems for generalized JS-quasi-contractions in complete partial b-metric spaces

In this section, we begin with considering fixed point theorem of the JS-quasi-contractions mapping in complete metric spaces and focused on the convergence theorems on partial b-metric spaces. First, we present the generalized JS-quasi-contractions and prove the convergence theorems of such mapping in the setting of p_b-complete partial b-metric spaces. Our space is a partial b-metric space which is a natural generalization of a metric space. Since our mappings are more general than mappings of Li and Jiang [64]. The obtained results improve and extend those results that have been presented in previous literature. In addition, an example is given to illustrate and support our main result, which is showed that our main result only works when the space is p_b-complete partial b-metric spaces.

5.2 Algorithms for the common solution of the split variational inequality and fixed point problems with applications

In this section, we begin with considering an iterative method which was introduced by Tian and Jiang [51], by combining Korpelevich's extragradient method with Byrne's CQ algorithm, for finding an element to solve a class of split variational inequality problems and get a weak convergence theorem. First, we present the an iterative method by combining Nadezhkina and Takahashi's modified extragradient method and Xu's algorithm. The mentioned iterative algorithm presents the common solution of the split variational inequality problems and fixed point problems. We show that the sequence produced by our algorithm is weak convergent. Since our algorithm is more general than iterative method in Tian and Jiang [51], our results improve results in Tian and Jiang [51]. Finally, by using the main results, we give some applications to the weak convergence of the produced algorithms for the equilibrium problem, zero point problem and convex minimization problem.

5.3 Convergence theorems for the variational inequality problems

In this section, we begin with considering the algorithms in [52], which is first proposed in [53] and [54], for solving the common solution of variational inequality problem and split feasibility problem. Moreover, Buong [52] considered the sequence which is weakly convergent to a solution of multiple-sets split feasibility problems. First, we establish an iterative algorithm by combining Yamada's hybrid steepest descent method and Wang's algorithm for finding the common solutions of variational inequality problems and split feasibility problems. The convergence of the sequence generated by our suggested iterative algorithm to such a common solution is proved in the setting of Hilbert spaces under some suitable assumptions imposed on the parameters. Moreover, we propose iterative algorithms for finding the common solutions of variational inequality problems and multiple-sets split feasibility problems. Finally, the numerical example for supporting our main result is also presented. By the numerical analysis of our results, we get that the algorithm (4.2.1) has lesser number of iterations and faster convergence than algorithm (2.3.3), our result improves result of Buong [52].

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